The Bottom-Up Top-Down Puzzle Solved

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In [2], we introduce a common shock model of portfolio credit risk where one can build a consistent picture of bottom up defaults that are also manageable in a top down aggregate loss space. In this sense this model solves the bottom-up top-down puzzle [4], which the CDO industry had been trying to do for a long time and basically failed. Then the CDO market died and the problem remained standing.

In our model, defaults are the consequence of some "shocks" associated with groups of obligors. We define the following pre-specified set of groups

$$\mathcal{Y} = \{\{1\}, \dots, \{n\}, I_1, \dots, I_m\},$$

where $I_1, \dots, I_m$ are subsets of $N = \{1, \dots, n\}$, and each group I_j contains at least two obligors or more. The shocks are divided in two categories: the "idiosyncratic" shocks associated with singletons $\{1\}, \dots, \{n\}$ can only trigger the default of name $1, \dots, n$ individually, while the "systemic" shocks associated with multi-name groups $I_1, \dots, I_m$ may simultaneously trigger the default of all names in these groups. Note that several groups I_j may contain a given name i, so that only the shock occurring first effectively triggers the default of that name. As a result, when a shock associated with a specific group occurs at time t, it only triggers the default of names that are still alive in that group at time t. In the following, the elements Y of $\mathcal{Y}$ will be used to designate shocks and we let $\mathcal{I} = (I_l)_{1 \le l \le m}$ denote the pre-specified set of multi-name groups of obligors. Shock intensities $\lambda_Y(t, \mathbf{X}_t)$ will be specified later in terms of a Markovian factor process $\mathbf{X}_t$. Letting $\Lambda_t^Y = \int_0^t \lambda_Y(s, \mathbf{X}_s) ds$, we define

$$\tau_Y = \inf\{t > 0; \Lambda_t^Y > E_Y\}, \tag{1}$$

where the random variables E_Y are i.i.d. and exponentially distributed with parameter 1. For every obligor i we let

$$\tau_i = \min_{\{Y \in \mathcal{Y};\, i \in Y\}} \tau_Y, \tag{2}$$

which defines the default time of obligor i in the common shocks model. The model filtration is given as $\mathbb{F} = \mathbb{X} \vee \mathbb{H}$, the filtration generated by the factor process $\mathbf{X}$ and the point process $\mathbf{H} = (H^i)_{1 \le i \le n}$ with $H_t^i = \mathbb{1}_{\tau_i \le t}$.

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This model can be viewed as a doubly stochastic (via the stochastic intensities Λ^Y) and dynamized (via the introduction of the filtration $\mathbb{F}$) generalization of the Marshall-Olkin model [8]. The purpose of the factor process $\mathbf{X}$ is to more realistically model diffusive randomness of credit spreads. Figure 1 shows one possible defaults path in our model with $n = 5$ and

$$\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4, 5\}, \{2, 3, 4\}, \{1, 2\}\}.$$

The inner oval shows which common-shock happened and caused the observed default scenarios at successive default times. At the first instant, default of name 2 is observed as the consequence of the idiosyncratic shock $\{2\}$. At the second instant, names 4 and 5 have defaulted simultaneously as a consequence of the systemic shock $\{4, 5\}$. At the fourth instant, the systemic shock $\{2, 3, 4\}$ triggers the default of name 3 alone as name 2 and 4 have already defaulted. At the fifth instant, default of name 1 alone is observed as the consequence of the systemic shock $\{1, 2\}$.

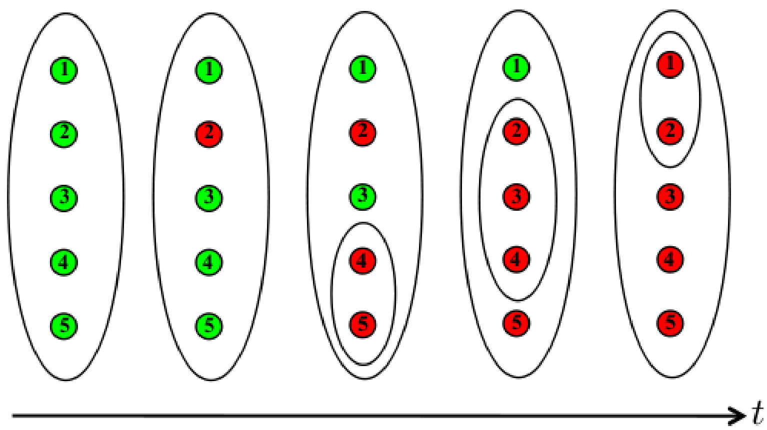

Figure 1: One possible defaults path in a model with $n = 5$ and $\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4, 5\}, \{2, 3, 4\}, \{1, 2\}\}$.

Table 1 summarizes the calibration results obtained with this model (using piecewise-constant intensities), for two different quotation dates and two different CDS indices under the constraint that the model perfectly reproduces each individual CDS curve of the corresponding index at these two dates. Even better fits can be obtained by resorting to random recoveries specifications (see [3]). The calibrated model can then be used for any bottom-up dynamical portfolio credit purpose, in particular, valuation and hedging of counterparty risk on credit derivatives (see [1]).

In this regard, first note that by using suitable stochastic specifications of the shock intensities, the model can generate very significant levels of CDS implied volatilities (see the left panel of Figure 2). The right panel of Figure 2 shows the value of the CVA on a payer CDS in the common shock model, with a stochastic default intensity thus specified, as a function of a Gaussian copula correlation ϱ between the counterparty (protection seller with credit spread κ_2) and the reference firm of the CDS (with credit spread $\kappa_1 = 84$ basis points). Observe that the CVA increases monotonically in ϱ, including at the highest values of the latter, whereas comonotonic pathologies would alter this monotonicity in simplistic models of counterparty credit risk—at least in the case $\kappa_2 = 50bps < 84bps = \kappa_1$ (blue curve on the figure) for which, in a comotonic model at high ϱ, the reference would always default before the counterparty, hence it would be a zero CVA.

Finally Table 2 shows the CVA on stylized $[0-5]\%$, $[5-35]\%$ and $[35-100]\%$ CDO tranches in a common shock model of 100 obligors, including the counterparty of the CDO, without ("naked") and with "continuous" collateralization (collateral continuously updated to track at every time the left-limit of the mark-to-market of the CDO tranche, the most extreme case of collateralization with the left-limit reflecting an "infinitesimal" cure period). As clear from the table, collateralization has

Table 1: CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The $[0, 3]$ spread is quoted in %. All maturities are for five years.

CDX 2007-12-17

CDO tranche	$[0,3]$	$[3,7]$	$[7,10]$	$[10,15]$	$[15,30]$
Market spread	48.07	254.0	124.0	61.00	41.00
Model spread	48.07	254.0	124.0	61.00	38.94
Absolute error in bp	0.010	0.000	0.000	0.000	2.061
Relative error in %	0.0001	0.000	0.000	0.000	5.027

iTraxx Europe 2008-03-3

CDO tranche	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$
Market spread	40.15	479.5	309.5	215.1	109.4
Model spread	41.68	429.7	309.4	215.1	103.7
Absolute error in bp	153.1	49.81	0.0441	0.0331	5.711
Relative error in %	3.812	10.39	0.0142	0.0154	5.218

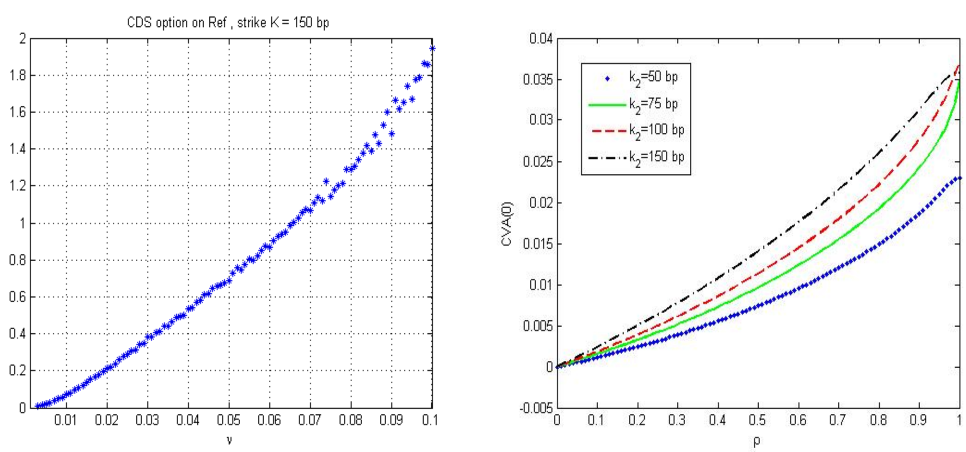

Figure 2: *Left*: Implied volatility of a CDS option on an individual name with respect to the volatility ν of the driving noise of the default intensity (see [5]). *Right*: Time-0 CVA on a CDS with respect to the Gaussian correlation ϱ between the counterparty (protection seller) and the reference firm in a common shock model of the two names (see [6]).

little impact in this case, particularly on the senior tranches, which conveys the important message that due to wrong-way risk (represented in this model by the possibility of joint defaults which are "missed" by the collateral due to the cure period), counterparty risk on credit derivatives may be scarcely collateralizable.

	Naked			Collateralized		
Tranche	0-5	5-35	35+	0-5	5-35	35+
CVA	4.78	2.96	2.44	3.41	2.73	2.26

Table 2: Naked versus collateralized CVA (see [7]).

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