CONVERTIBLE BONDS IN A DEFAULTABLE DIFFUSION MODEL

Tomasz R. Bielecki*
Department of Applied Mathematics
Illinois Institute of Technology
Chicago, IL 60616, USA

Stéphane Crépey†
Département de Mathématiques
Université d'Évry Val d'Essonne
91025 Évry Cedex, France

Monique Jeanblanc‡
Département de Mathématiques
Université d'Évry Val d'Essonne
91025 Évry Cedex, France
and
Europlace Institute of Finance

Marek Rutkowski§
School of Mathematics
University of New South Wales
Sydney, NSW 2052, Australia
and
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-661 Warszawa, Poland

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1 Introduction

In [4], working in an abstract set-up, we characterized arbitrage prices of generic *convertible securities* (CS), such as *convertible bonds* (CB), and we provided a rigorous decomposition of a CB into a straight bond component and a game option component, in order to give a definite meaning to commonly used terms of 'CB spread' and 'CB implied volatility.' Moreover, in [5], we showed that in the hazard process set-up, the theoretical problem of pricing and hedging CS can essentially be reduced to a problem of solving a related doubly reflected Backward Stochastic Differential Equation (BSDE for short). Finally, in [6], we established a formal connection between this BSDE and the corresponding variational inequalities with double obstacle in a generic Markovian intensity model.

In this paper, we study CSs (in particular, CBs) in a specific market set-up. We consider a primary market model consisting of: a savings account, a stock underlying a convertible security, and an associated credit default swap (CDS, or, alternatively to the latter, a *rolling CDS* more realistically used as an hedging instrument, see Section 2.3.1 and Bielecki et al. [7]). The dynamics of these three securities are modeled in terms of Markovian diffusion set-up with default (Section 2). For this particular model, we give explicit conditions, obtained by applying general results of Crépey [13], which ensure that the BSDE related to a convertible security has a unique solution (Proposition 4.2) and we provide the associated (super-)hedging strategy for a convertible security (Proposition 4.1). Moreover, we characterize the pricing function of a convertible security in terms of the viscosity solution to associated variational inequalities (Proposition 5.1) and we prove the convergence to this pricing function of suitable approximation schemes (Proposition 5.2). We then specify these results to a convertible bond and its decomposition into straight bond and option components (Section 6).

The above-mentioned model appears as the simplest equity-to-credit reduced form model one may think of (the connection between equity and credit in the model being materialized by the fact that the *default intensity* γ depends on the stock level S), and it is thus widely used in the industry for dealing with defaultable convertible bonds. This was the first motivation for the present study. The second motivation was the fact that all assumptions that we postulated in our previous theoretical works [4, 5, 6] are satisfied within this set-up; in this sense, the model is consistent with our theory of convertible securities. In particular, we worked in [4, 6] under the assumption that the value U_t^{cb} of a convertible bond upon a call at time t yields, as a function of time, a well-defined process satisfying some natural conditions. In the specific framework of this paper, using uniqueness of arbitrage prices (Propositions 2.1 and 3.1) and a form of *continuous aggregation* property of the value U_t^{cb} of a convertible bond upon a call at time t (Proposition 6.7), we are actually able to prove that this assumption is satisfied, and we also give ways to compute U_t^{cb} (Propositions 6.6 and 6.8).

2 Market Model

In this section, we introduce a simple specification of the generic Markovian default intensity set-up of [6]. More precisely, we consider a *defaultable diffusion model* with time- and stock-dependent *local default intensity* and *local volatility* (see also [2, 1, 17, 19, 28, 11]). We denote by $\int_0^t$ the integrals over $(0, t]$.

2.1 Default Time

Let us be given a standard stochastic basis $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{Q})$, over $[0, \Theta]$ for some fixed $\Theta \in \mathbb{R}_+$, endowed with the following objects:

- a non-negative random variable[1] $\widetilde{S}_0$ with finite moments of every order $p \in [2, +\infty)$;
- a standard Brownian motion $(W_t,\, t \in [0, \Theta])$ independent of $\widetilde{S}_0$.

We assume that $\mathbb{F}$ is the filtration generated by W and $\widetilde{S}_0$. So, in particular, $(\mathbb{F}, \mathbb{Q}; W)$ has the predictable representation property for ($\mathbb{F}$-)local martingales.

The underlying probability measure $\mathbb{Q}$ is devoted to represent a risk-neutral probability measure

[1] We will only need to deal with a non-constant initial condition in Section 6.5.

on a financial market model that we are now going to construct. To start with, we define the *pre-default factor process* $\widetilde{S}$ (to be interpreted later as the *pre-default stock price* of the firm underlying a convertible security) as the diffusion with initial condition $\widetilde{S}_0$ and the dynamics over $[0, \Theta]$ given as

$$d\widetilde{S}_t = \widetilde{S}_t \Big(\big(r(t) - q(t) + \eta\gamma(t, \widetilde{S}_t) \big)\, dt + \sigma(t, \widetilde{S}_t)\, dW_t \Big) \tag{1}$$

with related generator

$$\mathcal{L} \equiv \partial_t + (r - q + \eta\gamma) S \partial_S + \frac{\sigma^2 S^2}{2} \partial^2_{S^2}. \tag{2}$$

Assumption 2.1 **(i)** The riskless short interest rate $r(t)$, the equity dividend yield $q(t)$, and the local default intensity $\gamma(t, S) \geq 0$ are bounded, Borel-measurable functions and $\eta \leq 1$ is a real constant, to be interpreted later as the *fractional loss upon default* on the stock price.
(ii) The local volatility $\sigma(t, S)$ is a positively bounded, Borel-measurable function, so in particular $\sigma(t, S) \geq \underline{\sigma} > 0$ for some constant $\underline{\sigma}$.
(iii) The functions $\gamma(t, S)S$ and $\sigma(t, S)S$ are Lipschitz continuous in S, uniformly in t.

Note that we authorize negative values of r and q, in order, for instance, to possibly account for *repo rates* in the model. Under Assumption 2.1, the SDE (1) admits a unique strong solution $\widetilde{S}$, which is non-negative over $[0, \Theta]$. Moreover, the following (standard) a priori estimate is available, for any $p \in [2, +\infty)$

$$\mathbb{E}_{\mathbb{Q}} \Big(\sup_{t \in [0,\Theta]} |\widetilde{S}_t|^p \,\Big|\, \mathcal{G}_0 \Big) \leq C \left(1 + |\widetilde{S}_0|^p \right), \quad \text{a.s.} \tag{3}$$

In the next step, we define the $[0, \Theta] \cup \{+\infty\}$-valued *default time* τ_d, using the so-called *canonical construction* [8]. Specifically, we set (with, by convention, $\inf \emptyset = \infty$)

$$\tau_d = \inf \Big\{ t \in [0, \Theta];\ \int_0^t \gamma(u, \widetilde{S}_u)\, du \geq \varepsilon \Big\}, \tag{4}$$

where ε is a unit exponential random variable on $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{Q})$ independent of $\mathbb{F}$. Because of our construction of τ_d, the process $G_t := \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t)$ satisfies, for every $t \in [0, \Theta]$,

$$G_t = e^{-\int_0^t \gamma(u, \widetilde{S}_u)\, du}$$

and thus it is continuous and non-increasing. This also means that the process $\gamma(t, \widetilde{S}_t)$ is the $\mathbb{F}$-*intensity* of τ_d (see [5, 6]). The fact that the default intensity γ may depend on S is crucial, since this dependence actually conveys all the 'equity-to-credit' information in the model. A natural choice for γ is a decreasing (e.g., negative power) function of $\widetilde{S}$ capped when $\widetilde{S}$ is close to zero. A possible refinement is to positively floor γ. The lower bound on γ would then represent the pure default risk, as opposed to equity-related default risk.

Let $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$ be the *default indicator process* and let the process $(M^d_t,\, t \in [0, \Theta])$ be given by the formula

$$M^d_t = H_t - \int_0^t (1 - H_u) \gamma(u, \widetilde{S}_u)\, du.$$

We denote by $\mathbb{H}$ the filtration generated by the process H and by $\mathbb{G}$ the filtration given as $\mathbb{F} \vee \mathbb{H}$. Then the process M^d is known to be a $\mathbb{G}$-martingale, called the *compensated jump martingale*. Moreover, the filtration $\mathbb{F}$ is *immersed* in $\mathbb{G}$, in the sense that all $\mathbb{F}$-martingales are $\mathbb{G}$-martingales (this property is commonly referred to as Hypothesis (H)). This implies, in particular, that the $\mathbb{F}$-Brownian motion W is also a $\mathbb{G}$-Brownian motion under $\mathbb{Q}$.

2.2 Primary Traded Assets

We are now in a position to define the prices of primary traded assets in our market model. Assuming that τ_d is the default time of a reference entity (firm), we fix $0 < T \leq \Theta$ and we consider on the

time interval $[0,T]$ a continuous-time market composed of three primary assets:
• the savings account evolving according to the deterministic short-term interest rate r; we denote by β the *discount factor process* (the inverse of the savings account), so that $\beta_t = e^{-\int_0^t r(u)\,du}$;
• the stock of the reference entity with the pre-default price process given as $\widetilde{S}$ above and the *fractional loss upon default* determined by a constant $\eta \leq 1$;
• a CDS contract written at time 0 on the reference entity, with maturity Θ, the *protection payment* given as a Borel-measurable, bounded function $\nu : [0,\Theta] \to \mathbb{R}$ and the fixed *CDS spread* $\bar{\nu}$.

The *stock price* process $(S_t,\, t \in [0,T])$ is formally defined by setting, for every $t \in [0,T]$,

$$dS_t = S_{t-}\Big(\big(r(t) - q(t)\big)\,dt + \sigma(t,S_t)\,dW_t - \eta\,dM_t^d\Big), \quad S_0 = \widetilde{S}_0, \tag{5}$$

so that, as required, $(1-H_t)S_t = (1-H_t)\widetilde{S}_t$ for every $t \in [0,T]$. Note that estimate (3) enforces the following moment condition on the process S

$$\mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,T\wedge\tau_d]} S_t \,\Big|\, \mathcal{G}_0\Big) < \infty, \quad \text{a.s.} \tag{6}$$

We define the *discounted cumulative stock price* $\beta\widehat{S}$ by the expression, for every $t \in [0,T]$

$$\beta_t\widehat{S}_t = \beta_t(1-H_t)\widetilde{S}_t + \int_0^{t\wedge\tau_d} \beta_u\big((1-\eta)\widetilde{S}_u\,dH_u + q(u)\widetilde{S}_u\,du\big)$$

or equivalently, in term of S,

$$\beta_t\widehat{S}_t = \beta_{t\wedge\tau_d}S_{t\wedge\tau_d} + \int_0^{t\wedge\tau_d} \beta_u q(u) S_u\,du.$$

Note that the process $\widehat{S}$ is stopped at τ_d, since we will not need to consider the behavior of the stock price after default. Indeed, we will postulate throughout that all trading activities are stopped at the random time $\tau_d \wedge T$.

Let us now examine the valuation in the present model of a CDS written on the reference entity. We take the perspective of the credit protection buyer. Consistently with arbitrage requirements (cf. [6]), we assume that the *pre-default CDS price* $(\widetilde{B}_t,\, t \in [0,T])$ is given as $\widetilde{B}_t = \widetilde{B}(t,\widetilde{S}_t)$, where the *pre-default CDS pricing function* $\widetilde{B}(t,S)$ is the unique (classical) solution to the following PDE

$$\mathcal{L}\widetilde{B}(t,S) + \delta(t,S) - \mu(t,S)\widetilde{B}(t,S) = 0, \quad \widetilde{B}(\Theta,S) = 0, \tag{7}$$

where
• the operator $\mathcal{L}$ given by (2),
• $\delta(t,S) = \nu(t)\gamma(t,S) - \bar{\nu}$ is the *pre-default dividend function* of the CDS,
• $\mu(t,S) = r(t) + \gamma(t,S)$ is the *credit-risk adjusted interest rate*.

The *discounted cumulative CDS price* $\beta\widehat{B}$ equals, for every $t \in [0,T]$,

$$\beta_t\widehat{B}_t = \beta_t(1-H_t)\widetilde{B}_t + \int_0^{t\wedge\tau_d} \beta_u\big(\nu(u)\,dH_u - \bar{\nu}\,du\big).$$

2.3 Model Completeness

Since $\beta\widehat{S}$ and $\beta\widehat{B}$ are manifestly locally bounded processes, a *risk-neutral measure* on our primary market model is defined as any probability measure $\widetilde{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ such that the discounted cumulative prices $\beta\widehat{S}$ and $\beta\widehat{B}$ are $(\mathbb{G},\widetilde{\mathbb{Q}})$-local martingales (see, e.g., [6]). In particular, we note that the underlying probability measure $\mathbb{Q}$ is a risk-neutral measure on our primary market model. The following lemma can be easily proved using the Itô formula.

Lemma 2.1 *Let us denote* $\widehat{X}_t = \left[\begin{array}{c} \widehat{S}_t \\ \widehat{B}_t \end{array}\right]$. *We have, for every* $t \in [0,T]$,

$$d(\beta_t \widehat{X}_t) = d\left[\begin{array}{c} \beta_t \widehat{S}_t \\ \beta_t \widehat{B}_t \end{array}\right] = \mathbb{1}_{\{t \le \tau_d\}} \beta_t \Sigma_t \, d \left[\begin{array}{c} W_t \\ M^d_t \end{array}\right], \tag{8}$$

where the $\mathbb{F}$*-predictable* dispersion matrix *process* Σ *is given by the formula*

$$\Sigma_t = \left[\begin{array}{cc} \sigma(t,\widetilde{S}_t)\widetilde{S}_t & -\eta \widetilde{S}_t \\ \sigma(t,\widetilde{S}_t)\widetilde{S}_t \partial_S \widetilde{B}(t,\widetilde{S}_t) & \nu(t) - \widetilde{B}_t \end{array}\right]. \tag{9}$$

We work in the sequel under the following standing assumption.

Assumption 2.2 The matrix-valued process Σ is invertible on $[0, \tau_d \wedge T]$.

Proposition 2.1 suggests that, under Assumption 2.2, our market model is complete with respect to defaultable claims maturing at $\tau_d \wedge T$.

Proposition 2.1 *For any risk-neutral measure* $\widetilde{\mathbb{Q}}$ *on the primary market, we have that the Radon-Nikodym density* $Z_t := \mathbb{E}_{\mathbb{Q}}\left(\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} \,\middle|\, \mathcal{G}_t\right) = 1$ *on* $[0, \tau_d \wedge T]$.

Proof. For any probability measure $\widetilde{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$, the Radon-Nikodym density process Z_t, $t \in [0,T]$, is a strictly positive $(\mathbb{G}, \mathbb{Q})$-martingale. Therefore, by the predictable representation theorem due to Kusuoka [27], there exist two $\mathbb{G}$-predictable processes, φ and φ^d say, such that

$$dZ_t = Z_{t-}\left(\varphi_t \, dW_t + \varphi^d_t \, dM^d_t\right), \quad t \in [0,T]. \tag{10}$$

A probability measure $\widetilde{\mathbb{Q}}$ is then a risk-neutral measure whenever the process $\beta \widehat{X}$ is a $(\mathbb{G}, \widetilde{\mathbb{Q}})$-local martingale or, equivalently, whenever the process $\beta \widehat{X} Z$ is a $(\mathbb{G}, \mathbb{Q})$-local martingale. The latter condition is satisfied if and only if

$$\Sigma_t \left[\begin{array}{c} \varphi_t \\ \gamma(t,\widetilde{S}_t)\varphi^d_t \end{array}\right] = 0. \tag{11}$$

The unique solution to (11) on $[0, \tau_d \wedge T]$ is $\varphi = \varphi^d = 0$. We conclude that $Z = 1$ on $[0, \tau_d \wedge T]$. □

2.3.1 Rolling CDS

In practice traders typically use a *rolling CDS* (see [7]) as hedging instrument, rather than a plain CDS contract as considered above. The rolling CDS is defined as the wealth process of a self-financing trading strategy that amounts to continuously rolling one unit of long CDS contracts indexed by their *inception date* $t \in [0,T]$, with respective maturities $\Theta(t) \in [t, \Theta]$, where $\Theta(\cdot)$ is an increasing piecewise constant time-functional (for details, see [7]). We shall denote such contracts as $CDS(t, \Theta(t))$.

Intuitively, the above mentioned strategy amounts to holding at every time $t \in [0,T]$ one unit of the $CDS(t, \Theta(t))$. At time $t + dt$ the unit position in the $CDS(t, \Theta(t))$ is unwounded, the proceeds (which may be positive or negative depending on the evolution of the market between t and $t + dt$) are reinvested in the savings account, and a freshly issued $CDS(t + dt, \Theta(t + dt))$ is entered into at no cost. This procedure is carried on in continuous time (practically speaking, on a daily basis) until the hedging horizon T.

In the case of a rolling CDS, the entry $\beta \widehat{B}$ in (8) is then to be understood as the discounted cumulative value process of this strategy and the only modification with respect to the case of a standard CDS is that the dispersion matrix Σ in (9) needs to be changed into (see Appendix A)

$$\Sigma_t = \left[\begin{array}{cc} \sigma(t,\widetilde{S}_t)\widetilde{S}_t & -\eta \widetilde{S}_t \\ \sigma(t,\widetilde{S}_t)\widetilde{S}_t \partial_S \widetilde{P}^t(t,\widetilde{S}_t) - \bar{\nu}(t,\widetilde{S}_t)\sigma(t,\widetilde{S}_t)\widetilde{S}_t \partial_S \widetilde{F}^t(t,\widetilde{S}_t) & \nu(t) \end{array}\right]. \tag{12}$$

Here, the functions $\widetilde{P}^t$ and $\widetilde{F}^t$ are the pre-default pricing functions of the protection leg and the fee leg, respectively, of $CDS(t, \Theta(t))$, and the quantity

$$\bar{\nu}(t, \widetilde{S}_t) = \frac{\widetilde{P}^t(t, \widetilde{S}_t)}{\widetilde{F}^t(t, \widetilde{S}_t)}$$

represents the related CDS spread. As shown in Appendix A, the functions $\widetilde{P}^t$ and $\widetilde{F}^t$ are characterized as the solutions of PDEs of the form (7) on $[t, \Theta(t)]$ with functions δ therein respectively given by $\delta^1(u, S) = \nu(u)\gamma(u, S)$ and $\delta^2(u, S) = 1$.

3 Convertible Securities

We now specify to the present model the notion of a *convertible security* (CS), as formally defined in [4]. Let 0 (resp. T) stand for the *inception date* (resp. the *maturity date*) of a CS with the underlying asset S. For any $t \in [0, T]$, we write $\mathcal{F}_T^t$ (resp. $\mathcal{G}_T^t$) to denote the set of all $\mathbb{F}$-stopping times (resp. $\mathbb{G}$-stopping times) with values in $[t, T]$. Given the *time of lifting of a call protection of a CS*, $\bar{\tau} \in \mathcal{G}_T^0$, let $\bar{\mathcal{G}}_T^t$ stand for $\{\vartheta \in \mathcal{G}_T^t\,;\, \vartheta \wedge \tau_d \geq \bar{\tau} \wedge \tau_d\}$. Let finally τ denote $\tau_p \wedge \tau_c$, for any $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$.

Definition 3.1 A *convertible security* with the underlying S (cf. (5)) is a *game option* (see [4, 5, 6, 26, 25]) with the *ex-dividend cumulative discounted cash flows* $\pi(t; \tau_p, \tau_c)$ given by the formula, for any $t \in [0, T]$ and $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$,

$$\beta_t \pi(t; \tau_p, \tau_c) = \int_t^\tau \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau_c < \tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau = T\}} \xi \Big),$$

where:

• the *dividend process* $D = (D_t)_{t \in [0,T]}$ equals

$$D_t = \int_{[0,t]} (1 - H_u) \, dC_u + \int_{[0,t]} R_u \, dH_u$$

for some *coupon process* $C = (C_t)_{t \in [0,T]}$, which is a $\mathbb{G}$-adapted, càdlàg process with bounded variation, and some real-valued, $\mathbb{G}$-adapted *recovery process* $R = (R_t)_{t \in [0,T]}$,

• the *put/conversion payment* L is given as a $\mathbb{G}$-adapted, real-valued, càdlàg process on $[0, T]$,

• the *call payment* U is a $\mathbb{G}$-adapted, real-valued, càdlàg process on $[0, T]$, such that $L_t \leq U_t$ on $[\tau_d \wedge \bar{\tau}, \tau_d \wedge T)$,

• the *payment at maturity* ξ is a $\mathcal{G}_T$-measurable real random variable,

• the processes R, L and the random variable ξ are assumed to satisfy the following inequalities, for some positive constant c:

$$\begin{aligned} -c \leq R_t &\leq c\,(1 \vee S_t)\,, \quad t \in [0, T], \\ -c \leq L_t &\leq c\,(1 \vee S_t)\,, \quad t \in [0, T], \\ -c \leq \xi &\leq c\,(1 \vee S_T)\,. \end{aligned} \tag{13}$$

3.1 Valuation of a CS

The notion of an arbitrage price of a CS referred to below is a suitable extension to game options (Definition 2.6 in Kallsen and Kühn [25], see also [4]) of the *No Free Lunch with Vanishing Risk* (NFLVR) condition of Delbaen and Schachermayer [18].

Proposition 3.1 *If the $\mathbb{Q}$-Dynkin game related to the CS admits a value Π, in the sense that*

$$\begin{aligned} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}} \big(\pi(t; \tau_p, \tau_c) \,\big|\, \mathcal{G}_t \big) &= \Pi_t \\ = \text{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}} \big(\pi(t; \tau_p, \tau_c) \,\big|\, \mathcal{G}_t \big), &\quad t \in [0, T], \end{aligned} \tag{14}$$

and Π is a $\mathbb{G}$-semimartingale, then Π is the unique arbitrage (ex-dividend) price of the CS.

Proof. Except for the uniqueness statement, this follows by applying the general results in [4]. To verify the uniqueness property, we first note that for any risk-neutral measure $\widetilde{\mathbb{Q}}$, we have that $Z_t = \mathbb{E}_{\mathbb{Q}}\left(\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\,\Big|\,\mathcal{G}_t\right) = 1$ on $[0, \tau_d \wedge T]$, by Proposition 2.1. In view of the estimate (6) on $\sup_{t\in[0,T\wedge\tau_d]} S_t$, and since $\sup_{t\in[0,T\wedge\tau_d]} S_t$ is a $\mathcal{G}_{\tau_d\wedge T}$-measurable random variable, this implies that, for any risk-neutral measure $\widetilde{\mathbb{Q}}$,

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\sup_{t\in[0,T\wedge\tau_d]} S_t\,\Big|\,\mathcal{G}_0\Big) = \mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,T\wedge\tau_d]} S_t\,\Big|\,\mathcal{G}_0\Big) < \infty, \quad \text{a.s.} \tag{15}$$

Therefore, taking the essential supremum over the set $\mathcal{M}$ of all risk-neutral measures $\widetilde{\mathbb{Q}}$,

$$\text{esssup}_{\widetilde{\mathbb{Q}}\in\mathcal{M}}\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\sup_{t\in[0,T\wedge\tau_d]} S_t\,\Big|\,\mathcal{G}_0\Big) < \infty \quad \text{a.s.} \tag{16}$$

Under condition (16), any arbitrage price of a CS with underlying S is then given by the value of the related Dynkin game for some risk-neutral measure $\widetilde{\mathbb{Q}}$, by the general results of [4]. Furthermore, $\pi(t;\tau_p,\tau_c)$ is a $\mathcal{G}_{\tau_d\wedge T}$-measurable random variable. Therefore, for any $t \in [0,T], \tau_p \in \mathcal{G}_T^t, \tau_c \in \bar{\mathcal{G}}_T^t$,

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big). \tag{17}$$

In conclusion, the $\widetilde{\mathbb{Q}}$-Dynkin game has value Π, for any risk-neutral measure $\widetilde{\mathbb{Q}}$. □

We now define special cases of CSs, corresponding to American- and European-style CSs.

Definition 3.2 A (purely) *puttable security* (as opposed to puttable *and* callable, in the case of a general convertible security) is a convertible security with $\bar{\tau} = T$. An *elementary security* is a puttable security with *bounded variation* dividend process D over $[0,T]$, *bounded* payment at maturity ξ, and such that

$$\int_{[0,t]} \beta_u\,dD_u + \mathbb{1}_{\{\tau_d>t\}}\beta_t L_t \;\leq\; \int_{[0,T]} \beta_u\,dD_u + \mathbb{1}_{\{\tau_d>T\}}\beta_T\xi, \quad t\in[0,T). \tag{18}$$

By Definition 3.2, puttable and elementary securities are special cases of convertible securities. Note that, given Proposition 3.1, a puttable (resp. elementary) security can be redefined equivalently as a financial product with ex-dividend cumulative discounted cash flows $\bar{\pi}(t;\tau_p)$ (resp. $\phi(t)$) given as, for $t\in[0,T]$ and $\tau_p \in \mathcal{G}_T^t$,

$$\beta_t\bar{\pi}(t;\tau_p) = \int_t^{\tau_p} \beta_u\,dD_u + \mathbb{1}_{\{\tau_d>\tau_p\}}\beta_{\tau_p}\big(\mathbb{1}_{\{\tau_p<T\}}L_{\tau_p} + \mathbb{1}_{\{\tau_p=T\}}\xi\big)$$

(resp. $\beta_t\phi(t) = \int_t^T \beta_u\,dD_u + \mathbb{1}_{\{\tau_d>T\}}\beta_T\xi$ for every $t\in[0,T]$).

3.2 Hedging of a CS

The following definition is standard, accounting for the dividends on the primary market.

Definition 3.3 By a (self-financing) *primary strategy*, we mean a pair (V_0,ζ) such that:
- V_0 is a $\mathcal{G}_0$-measurable real-valued random variable representing the *initial wealth*,
- ζ is an $\mathbb{R}^{1\otimes 2}$-valued (bi-dimensional row vector), $\beta\widehat{X}$-integrable process representing holdings (number of units held) in primary risky assets.

The *wealth process* V of a primary strategy (V_0,ζ) is given by

$$\beta_t V_t = V_0 + \int_0^t \zeta_u\,d(\beta_u\widehat{X}_u), \quad t\in[0,T].$$

In the set-up of this paper, the notions of *issuer (super)hedge* and *holder (super)hedge* introduced in [5, 6] take the following form. Recall that we denote $\tau = \tau_p \wedge \tau_c$.

Definition 3.4 Given a CS with ex-dividend cumulative discounted cash flows $\pi(t;\tau_p,\tau_c)$ (cf. (13)):
(i) An *issuer hedge* for a CS is represented by a triplet (V_0,ζ,τ_c) such that:
• (V_0,ζ) is a primary strategy with the wealth process V,
• the call time τ_c belongs to $\bar{\mathcal{G}}^0_T$,
• the following inequality is valid, for every put time $\tau_p\in\mathcal{G}^0_T$,

$$\beta_\tau V_\tau \geq \beta_0\pi(0;\tau_p,\tau_c), \quad \text{a.s.} \tag{19}$$

(ii) A *holder hedge* for a CS is a triplet (V_0,ζ,τ_p) such that:
• (V_0,ζ) is a primary strategy with the wealth process V,
• the put time τ_p belongs to $\mathcal{G}^0_T$,
• the following inequality is valid, for every call time $\tau_c\in\bar{\mathcal{G}}^0_T$,

$$\beta_\tau V_\tau \geq -\beta_0\pi(0;\tau_p,\tau_c), \quad \text{a.s.} \tag{20}$$

Definition 3.4 can be easily extended to hedges that start at any initial date $t\in[0,T]$, and specified to the special case of puttable or elementary securities (see [5, 6]).

4 Doubly Reflected BSDEs Approach

4.1 Technical Assumptions and Definitions

In order to deal with the doubly reflected BSDE associated with a convertible security, we need to impose some technical assumptions. We refer the reader to section 6 for concrete examples.

Assumption 4.1 We postulate that:
• the coupon process C satisfies

$$C_t = C(t) := \int_0^t c(u)\,du + \sum_{0\leq T_i\leq t} c^i,$$

for a bounded, Borel-measurable *continuous-time coupon rate function* $c(\cdot)$ and deterministic *discrete times* and *coupons* T_i and c^i, respectively; we take the tenor of the discrete coupons as $T_0=0<T_1<\cdots<T_{I-1}<T_I$ with $T_{I-1}<T\leq T_I$ (where the latter inequality may be strict for reasons that will become clear in Section 6.5);
• the recovery process R_t is of the form $R(t,S_{t-})$ for a Borel-measurable function R;
• $L_t=L(t,S_t)$, $U_t=U(t,S_t)$, $\xi=\xi(S_T)$ for some Borel-measurable functions L,U and ξ such that, for any t,S, we have

$$L(t,S)\leq U(t,S), \quad L(T,S)\leq\xi(S)\leq U(T,S);$$

• the call protection time $\bar{\tau}\in\mathcal{F}^0_T$.

The *accrued interest* at time t is given by

$$A_t = \frac{t-T_{i_t-1}}{T_{i_t}-T_{i_t-1}}\,c^{i_t}, \tag{21}$$

where i_t is the integer satisfying $T_{i_t-1}\leq t<T_{i_t}$. On open intervals between the discrete coupon dates we thus have $dA_t=a(t)\,dt$ with $a(t)=\frac{c^{i_t}}{T_{i_t}-T_{i_t-1}}$.

To a CS with data (functions) C,R,ξ,L,U and lifting time of call protection $\bar{\tau}$, we associate the Borel-measurable functions $f(t,S,x)$ (for x real), $g(S)$, $\ell(t,S)$ and $h(t,S)$ defined by

$$g(S)=\xi(S)-A_T, \quad \ell(t,S)=L(t,S)-A_t, \quad h(t,S)=U(t,S)-A_t, \tag{22}$$

and (recall that $\mu(t,S)=r(t)+\gamma(t,S)$)

$$f(t,S,x)=\gamma(t,S)R(t,S)+\Gamma(t,S)-\mu(t,S)x, \tag{23}$$

where we set $\Gamma(t,S) = c(t) + a(t) - \mu(t,S)A_t$. In the case of a puttable security, the process U is irrelevant and thus we redefine $h(t,S) = +\infty$. Moreover, in the case of an elementary security, the process L plays no role either, and we redefine further $\ell(t,S) = -\infty$. We define the *processes and random variables associated to a CS* (parameterized by $x \in \mathbb{R}$, regarding f) as

$$f_t(x) = f(t, \widetilde{S}_t, x), \quad g = g(\widetilde{S}_T), \quad \ell_t = \ell(t, \widetilde{S}_t), \quad h_t = h(t, \widetilde{S}_t), \quad \bar{h}_t = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\geq\bar{\tau}\}}h_t$$

with the convention that $0 \times \infty = 0$ in the last identity. We finally introduce

$$\gamma_t = \gamma(t, \widetilde{S}_t)\,, \;\; \mu_t = \mu(t, \widetilde{S}_t)\,, \;\; \alpha_t = e^{-\int_0^t \mu_u\,du} \tag{24}$$

where it will become apparent later that α_t can be interpreted later as the *credit-risk adjusted discount factor*.

Let us now introduce some spaces:

$\mathcal{H}^2$ – the set of $\mathbb{R}$-valued, $\mathbb{F}$-predictable processes Π such that $\mathbb{E}_{\mathbb{Q}}\Big(\int_0^T \Pi_t^2\,dt\,\Big|\,\mathcal{F}_0\Big) < \infty$, a.s.

$\mathcal{S}^2$ – the set of $\mathbb{R}$-valued, $\mathbb{F}$-adapted, continuous processes Π such that $\mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,T]}\Pi_t^2\,\Big|\,\mathcal{F}_0\Big) < \infty$, a.s.

$\mathcal{A}^2$ – the space of finite variation continuous processes K with (continuous and non decreasing) Jordan components $K^{\pm} \in \mathcal{S}^2$ null at time 0,

$\mathcal{A}_i^2$ – the space of non-decreasing processes in $\mathcal{A}^2$.

For any $K \in \mathcal{A}^2$, we thus have that $K = K^+ - K^-$, where $K^{\pm} \in \mathcal{A}_i^2$ define mutually singular measures on $\mathbb{R}^+$.

Given a CS with data $C, R, \xi, L, U, \bar{\tau}$ and the associated processes and random variables $(f, g, \ell, h, \bar{h})$ (cf. (22)–(23)), we introduce the following doubly reflected Backward Stochastic Differential Equation $(\mathcal{E})$ with data $(f, g, \ell, \bar{h})$ (BSDE for short, see [5, 6, 13]), such that almost surely, for $t \in [0,T)$:

$$\begin{cases} -d\Pi_t = f_t(\Pi_t)\,dt + dK_t - Z_t\,dW_t \\ \ell_t \leq \Pi_t \leq \bar{h}_t \\ (\Pi_t - \ell_t)\,dK_t^+ = (\bar{h}_t - \Pi_t)\,dK_t^- = 0 \end{cases}$$

supplemented by the terminal condition $\Pi_T = g$, almost surely.

Definition 4.1 **(i)** By a *solution* to $(\mathcal{E})$, we mean a triple of processes $(\Pi, Z, K) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{A}^2$ satisfying all conditions in $(\mathcal{E})$. In particular, K, hence Π, have to be continuous processes.
(ii) In the case of a puttable security, so $\bar{\tau} = T$, we have $K^- = 0$ in any solution (Π, Z, K) to $(\mathcal{E})$, and $(\mathcal{E})$ reduces to a reflected BSDE with data (f, g, ℓ) and $K \in \mathcal{A}_i^2$ in the solution.
(iii) In the special case of an elementary security, we have $K = 0$ in any solution (Π, Z, K) to $(\mathcal{E})$, so that $(\mathcal{E})$ reduces to a standard BSDE with data (f, g).

In order to establish the well-posedness of the BSDEs introduced in Definition 4.1, as well as their connection with the formally related obstacles problems examined in the next section, we work henceforth under the following

Assumption 4.2 The functions r, q, γ, σ, c, R, g, h, ℓ are continuous.

4.2 Connection with Hedging

By applying the general results of [5, 6], we have the following (super-)hedging result.

Proposition 4.1 *Let $(\widehat{\Pi}, Z, K)$ be a solution to $(\mathcal{E})$, assumed to exist, and let Π_t denote $\mathbb{1}_{\{t<\tau_d\}}\widetilde{\Pi}_t$ with $\widetilde{\Pi} := \widehat{\Pi} + A$. Then Π is the unique arbitrage price process of the CS.*
(i) *For any $t \in [0,T]$, an issuer hedge with initial wealth Π_t is furnished by*

$$\tau_c^* = \inf\Big\{\, u \in [\bar{\tau} \vee t, T];\, \widehat{\Pi}_u = h_u \Big\} \wedge T$$

and

$$\zeta^*_u := \mathbb{1}_{\{u \le \tau_d\}} \big[Z_u \,,\, R_u - \widetilde{\Pi}_{u-} \big] \Lambda_u, \quad t \le u \le T, \tag{25}$$

where $[Z_u, R_u - \widetilde{\Pi}_{u-}]$ *denotes the concatenation of* Z_u *and* $R_u - \widetilde{\Pi}_{u-}$ *and where* Λ *denotes the left-inverse of the dispersion matrix* Σ *over* $[0, \tau_d \wedge T]$ *(cf. Assumption 2.2). Moreover,* Π_t *is the smallest initial wealth of an issuer hedge.*
(ii) *For any* $t \in [0, T]$*, a holder hedge with initial wealth* $-\Pi_t$ *is furnished by*

$$\tau^*_p = \inf \Big\{ u \in [t, T] \,;\, \widehat{\Pi}_u = \ell_u \Big\} \wedge T$$

and $\zeta = -\zeta^*$ *above. Moreover,* $-\Pi_t$ *is the smallest initial wealth of a holder hedge.*

Proof. In view of the general results of [5, 6], we see that the process Π introduced in the statement of the proposition satisfies all the assumptions for the process Π introduced in Proposition 3.1. Hence it is the unique arbitrage price process of the CS. As for statements (i) and (ii), they are rather straightforward consequences of the general results of [5, 6]. □

Note that in the case of an elementary security, there are no stopping times involved and process K is equal to 0, so that (Π_t, ζ^*) in fact defines a (self-financing) *replication strategy* (see [5]).

We thus see that in the present set-up a CS has a *bilateral hedging price*, in the sense that the price Π_t ensures super-hedging (or replication, in the case of an elementary security) to both its issuer and holder, starting from the initial wealth Π_t for the former and $-\Pi_t$ for the latter, where process Π is also the unique arbitrage price. Of course, this conclusion hinges on our temporary assumption that the related BSDE $(\mathcal{E})$ has a solution.

4.3 Solution of the BSDEs

Let $\mathcal{P}$ be the class of functions Π of the real variable S bounded by $C(1 + |S|^p)$ for some real C and integer p that may depend on Π. By a slight abuse of terminology, we shall say that a function $\Pi(S, \dots)$ is of class $\mathcal{P}$ if it has polynomial growth in S, uniformly in any other arguments. We postulate henceforth the following additional

Assumption 4.3 The functions R, g, h, ℓ associated to a CS are of class $\mathcal{P}$ (or $h = +\infty$, in the case of a puttable security, and $\ell = -\infty$, in the case of an elementary security), and $\bar{\tau}$ is given as

$$\bar{\tau} = \inf\{ t > 0 \,;\, \widetilde{S}_t \ge \bar{S} \} \wedge \bar{T} \tag{26}$$

for some constants $\bar{T} \in [0, T]$ and $\bar{S} \in \mathbb{R}_+ \cup \{+\infty\}$ (so, in particular, $\bar{\tau} = 0$ in case $\bar{S} = 0$, and $\bar{\tau} = \bar{T}$ in case $\bar{S} = +\infty$). As for ℓ, it satisfies, more specifically, the following *structure condition*: $\ell(t, S) = \lambda(t, S) \vee c$ for some constant $c \in \mathbb{R} \cup \{-\infty\}$, and a function λ of class $\mathcal{C}^{1,2}$ with

$$\lambda,\, \partial_t \lambda,\, S\partial_S \lambda,\, S^2 \partial^2_{S^2} \lambda \in \mathcal{P} \tag{27}$$

(or $\ell = -\infty$, in the case of an elementary security).

Example 4.1 The standing example for the function $\lambda(t, S)$ in (27) is $\lambda(t, S) = S$. In that case, ℓ corresponds to the payoff function of a call option (or, more precisely, to the lower payoff function of a convertible bond, see Section 6).

By an application of the general results of [6, 13], we then have the following

Proposition 4.2 *The BSDE* $(\mathcal{E})$ *admits a unique solution* $(\widehat{\Pi}, Z, K)$*.* □

In the foregoing sections, we will give analytical characterizations of the so-called *pre-default clean prices* (i.e., pre-default price less accrued interest, which corresponds to the state-process $\widehat{\Pi}$ in a solution to $(\mathcal{E})$; see Proposition 4.1 and [6]) in terms of viscosity solutions to associated variational inequalities. ***In this context, unless explicitly stated otherwise, by a 'price' of a security we mean henceforth its 'pre-default clean price.'*** To get the corresponding pre-default price, it suffices to add to the clean price process the related accrued interest process (if there are any discrete coupons involved in the product under consideration).

5 Variational Inequalities Approach

The goal of this section is to study the variational inequalities approach to convertible securities in the present set-up and the link between variational inequalities and doubly reflected BSDEs.

5.1 No Protection, Protection and Post-protection Prices

For any $\bar{\tau} \in \mathcal{F}_T^0$, the associated price coincides on $[\bar{\tau}, T]$ with the price corresponding to a lifting time of call protection that would be given by $\bar{\tau}^0 := 0$. This follows from the general results in [5], using also the fact that the BSDEs related to the problems with lifting times of call protection $\bar{\tau}$ and $\bar{\tau}^0$ both have solutions, under the standing assumptions.

Then the *no-protection prices* (i.e., prices obtained for the lifting time of call protection $\bar{\tau}^0 = 0$) can also be interpreted as post-protection prices for an arbitrary stopping time $\bar{\tau} \in \mathcal{F}_T^0$, where by the *post-protection price* we mean price restricted to the random time interval $[\bar{\tau}, T]$. Likewise, we define the *protection prices* as prices restricted to the random time interval $[0, \bar{\tau}]$.

5.2 Technical Assumptions and Definitions

Given a closed domain $\mathcal{D} \subseteq [0,T] \times \mathbb{R}$, let $\mathrm{Int}_p\mathcal{D}$ and $\partial_p\mathcal{D}$ stand for the *parabolic interior* and the *parabolic boundary* of $\mathcal{D}$, respectively.

Example 5.1 If $\mathcal{D} = [0,t] \times (-\infty, x] =: \mathcal{D}(t,x)$ for some $x \in \mathbb{R}$, then

$$\mathrm{Int}_p\mathcal{D} = [0,t) \times (-\infty, x)\,, \;\; \partial_p\mathcal{D} = \big([0,t] \times \{x\}\big) \cup \big(\{t\} \times (-\infty, x)\big)\,. \tag{28}$$

In case $\mathcal{D} = [0,t] \times \mathbb{R} =: \mathcal{D}(t, +\infty)$ for some $t \in [0,T]$, then

$$\mathrm{Int}_p\mathcal{D} = [0,t) \times \mathbb{R}\,, \;\; \partial_p\mathcal{D} = \{t\} \times \mathbb{R}\,. \tag{29}$$

Given a continuous boundary condition b of class $\mathcal{P}$ on $\partial_p\mathcal{D}$, we introduce the following *obstacles problem* $(\mathcal{VI})$ on $\mathcal{D}$ ($\mathcal{L}$ and f were defined in (2) and (23), respectively)

$$\max\Big(\min\big(-\mathcal{L}\Pi(t,S) - f(t,S,\Pi(t,S)),\, \Pi(t,S) - \ell(t,S)\big),\, \Pi(t,S) - h(t,S)\Big) = 0 \text{ on } \mathrm{Int}_p\mathcal{D}\,,$$

supplemented by the boundary condition $\Pi = b$ on $\partial_p\mathcal{D}$. Note that in the case of a puttable security with $h = +\infty$, $(\mathcal{VI})$ reduces to, on $\mathrm{Int}_p\mathcal{D}$:

$$\min\big(-\mathcal{L}\Pi(t,S) - f(t,S,\Pi(t,S)),\, \Pi(t,S) - \ell(t,S)\big) = 0$$

which reduces further in the case of an elementary security with also $\ell = -\infty$ to

$$-\mathcal{L}\Pi(t,S) - f(t,S,\Pi(t,S)) = 0\,.$$

So, in the case of a puttable security and an elementary security, the general double obstacle problem $(\mathcal{VI})$ reduces to a simple obstacle problem and to a linear parabolic PDE, respectively.

Also note that the problem $(\mathcal{VI})$ is defined over a domain in space variable S ranging to $-\infty$, although only the positive part of this domain is meaningful for the financial purposes. Had we decided instead to pose problems $(\mathcal{VI})$ over bounded spatial domains then, in order to get a well-posed problem, we would need to impose some appropriate non-trivial boundary condition at the lower space boundary.

We refer the reader to Appendix B for the definition of viscosity solutions that is relevant to cope with the time-discontinuities of f at the T_is (due to the discrete coupons c^is, if any). Building upon Definition B.1, we introduce the following definition of $\mathcal{P}$*-(semi-)solutions* to $(\mathcal{VI})$ on $\mathcal{D}$.

Definition 5.1 By a *$\mathcal{P}$-subsolution, resp. supersolution, resp. solution* Π of $(\mathcal{VI})$ on $\mathcal{D}$ for the boundary condition b, we mean a function of class $\mathcal{P}$ on $\mathrm{Int}_p\mathcal{D}$, which is a viscosity subsolution, resp. supersolution, resp. solution of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$, and such that $\Pi \le b$, resp. $\Pi \ge b$, resp. $\Pi = b$, pointwise on $\partial_p\mathcal{D}$.

5.3 Resolution of the Variational Inequalities and Connection with BSDEs

In the following results, the process $\widehat{\Pi}$ represents the state-process of the solution to the doubly reflected BSDE $(\mathcal{E})$ in Proposition 4.2. It thus depends, in particular, on the stopping time $\bar{\tau}$.

Lemma 5.1 (No-protection price) *Assume that $\bar{\tau} := \bar{\tau}^0 = 0$. Then the related process $\widehat{\Pi}$ can be written as $\widehat{\Pi}_t = \widehat{\Pi}(t, \widetilde{S}_t)$, where the function $\widehat{\Pi}$ is a $\mathcal{P}$-solution of $(\mathcal{VI})$ on $[0,T] \times \mathbb{R}$, with terminal condition g at T.*

Proof. This follows by the application of the results of Crépey [13]. □

Proposition 5.1 *Let $\bar{\tau}$ be given by (26) for some constants $\bar{T} \in [0,T]$ and $\bar{S} \in \mathbb{R}_+ \cup \{+\infty\}$.*
(i) ***Post-protection price.*** *On $[\bar{\tau}, T]$, the related process $\widehat{\Pi}$ can be written as $\widehat{\Pi}(t, \widetilde{S}_t)$, where $\widehat{\Pi}$ is the function defined in Lemma 5.1;*
(ii) ***Protection price.*** *On $[0, \bar{\tau}]$, the related process $\widehat{\Pi}$ can be written as $\bar{\Pi}(t, \widetilde{S}_t)$, where the function $\bar{\Pi}$ is a $\mathcal{P}$-solution of the problem $(\mathcal{VI})$ on $\mathcal{D} = \mathcal{D}(\bar{T}, \bar{S})$, with the function h therein redefined as $+\infty$ and with the boundary condition $\widehat{\Pi}$ on $\partial_p \mathcal{D}$, where the function $\widehat{\Pi}$ is as in part (i).*

Proof. In view of the observations made in Section 5.1, Lemma 5.1 immediately implies (i). In particular, we then have that $\widehat{\Pi}_{\bar{\tau}} = \widehat{\Pi}(\bar{\tau}, \widetilde{S}_{\bar{\tau}})$, where the restriction of $\widehat{\Pi}$ to $\partial_p \mathcal{D}$ defines a continuous function of class $\mathcal{P}$ over $\partial_p \mathcal{D}$. Part (ii) then follows by the application of the results of Crépey [13]. □

We are in a position to state the following corollary to Propositions 4.1 and 5.1.

Corollary 5.1 **(i)** ***Post-protection optimal policies.*** *The pair of* post-protection optimal stopping times (τ_p^*, τ_c^*) *after time $t \in [0,T]$ for the CS is given by*

$$\tau_p^* = \inf \left\{ u \in [t,T] \, ; \, (u, \widetilde{S}_u) \in \mathcal{E}_p \right\} \wedge T,$$
$$\tau_c^* = \inf \left\{ u \in [t,T] \, ; \, (u, \widetilde{S}_u) \in \mathcal{E}_c \right\} \wedge T,$$

where

$$\mathcal{E}_p = \left\{ (u,S) \in [0,T] \times \mathbb{R} \, ; \, \widehat{\Pi}(u,S) = \ell(u,S) \right\},$$
$$\mathcal{E}_c = \left\{ (u,S) \in [0,T] \times \mathbb{R} \, ; \, \widehat{\Pi}(u,S) = h(u,S) \right\},$$

are the post-protection put region *and the* post-protection call region*, respectively.*
(ii) ***Protection optimal policy.*** *The* protection optimal stopping time τ_p^* *after time $t \in [0,T]$ for the CS is given by*

$$\tau_p^* = \inf \left\{ u \in [t, \bar{\tau}] \, ; \, (u, \widetilde{S}_u) \in \bar{\mathcal{E}}_p \right\} \wedge \bar{\tau},$$

where

$$\bar{\mathcal{E}}_p = \left\{ (u,S) \in [0,T] \times \mathbb{R} \, ; \, \bar{\Pi}(u,S) = \ell(u,S) \right\}$$

is the protection put region*.* □

Assuming that the call protection *has not been lifted yet* ($t < \bar{\tau}$) and that the CS is still alive at time t, an optimal strategy for the holder of the CS is to put the CS as soon as $\widetilde{S}$ hits $\bar{\mathcal{E}}_p$ for the first time after t, if this event actually happens before $\tau_d \wedge \bar{\tau}$.

If we assume instead that the call protection *has been lifted* ($t \geq \bar{\tau}$) and that the CS is still alive at time t:

- an optimal call time for the issuer of the CS is given by the first hitting time of $\mathcal{E}_c$ by $\widetilde{S}$ after t, provided this hitting time is realized before $T \wedge \tau_d$;

• an optimal put policy for the holder of the CS consists in putting when $\widetilde{S}$ hits $\mathcal{E}_p$ for the first time after t, if this event occurs before $T \wedge \tau_d$.

We now come to the issues of uniqueness and approximation of solutions for $(\mathcal{VI})$. For this, we make the following additional standing

Assumption 5.1 The functions r, q, γ, σ are locally Lipschitz continuous.

We refer the reader to Barles and Souganidis [3] (see also Crépey [13]) for the definition of *stable, monotone* and *consistent* approximation schemes to $(\mathcal{VI})$ and for the related notion of *convergence* of the scheme, involved in the following

Proposition 5.2 (i) ***Post-protection price.*** *In the situation of Proposition 5.1(i), the function $\widehat{\Pi}$ therein (defined in Lemma 5.1) is the* unique *$\mathcal{P}$-solution, the* maximal $\mathcal{P}$-subsolution*, and the* minimal $\mathcal{P}$-supersolution *of the related problem $(\mathcal{VI})$ on $\mathcal{D} = [0,T] \times \mathbb{R}$. Let $(\widehat{\Pi}_h)_{h>0}$ denote a stable, monotone and consistent approximation scheme for the function $\widehat{\Pi}$. Then $\widehat{\Pi}_h \to \widehat{\Pi}$ locally uniformly on $\mathcal{D}$ as $h \to 0^+$.*
(ii) ***Protection price.*** *In the situation of Proposition 5.1(ii), the function $\bar{\Pi}$ defined therein is the* unique *$\mathcal{P}$-solution, the* maximal $\mathcal{P}$-subsolution*, and the* minimal $\mathcal{P}$-supersolution *of the related problem $(\mathcal{VI})$ on $\mathcal{D} = \mathcal{D}(\bar{T}, \bar{S})$. Let $(\bar{\Pi}_h)_{h>0}$ denote a stable, monotone and consistent approximation scheme for the function $\bar{\Pi}$. Then $\bar{\Pi}_h \to \bar{\Pi}$ locally uniformly on $\mathcal{D}$ as $h \to 0^+$, provided (in case $\bar{S} < +\infty$) $\bar{\Pi}_h \to \bar{\Pi}(= \widehat{\Pi})$ at $\bar{S}$.*

Moreover these uniqueness, extremality and convergence results still hold true independently of the structure condition on ℓ in assumption 4.3, relative to arbitrary $\mathcal{P}$-solutions $\widehat{\Pi}$, resp. $\bar{\Pi}$, assumed to exist, to the associated problems $(\mathcal{VI})$.

Proof. Note, in particular, that under our assumptions:
• the functions $(r(t) - q(t) + \eta\gamma(t,S))S$ and $\sigma(t,S)S$ are locally Lipschitz continuous;
• the function f admits a *modulus of continuity* in S, in the sense that for every constant $c > 0$ there exists a continuous function $\eta_c : \mathbb{R}_+ \to \mathbb{R}_+$ with $\eta_c(0) = 0$ and such that

$$|f(t,S,x) - f(t,S',x)| \leq \eta_c(|S - S'|)$$

for any $t \in [0,T]$ and $S, S', x \in \mathbb{R}$ with $|S| \vee |S'| \vee |x| \leq c$.
The assertions are then consequences of the results of Crépey [13]. □

Remark 5.1 We refer, in particular, the reader to the last section of Crépey [13] in regard to the fact that the potential discontinuities of f at the T_is (which represent a non-standard feature from the point of view of the classic theory of viscosity solutions as presented, for instance, in Crandall et al. [12]) are not a real issue in the previous results, provided one works with the suitable Definition B.1 of viscosity solutions to our problems.

6 Applications to Convertible Bonds

As was already pointed out, a convertible bond is a special case of a convertible security. To describe the covenants of a typical convertible bond (CB), we introduce the following additional notation (for a detailed description and discussion of typical covenants of a CB, see [4]):

$\bar{N}$: the par (nominal) value,

η: the fractional loss on the underlying equity upon default,

$\bar{R}_t$: the recovery process on the CB upon default of the issuer at time t, given by $\bar{R}_t = \bar{R}(t, S_{t-})$ for a continuous bounded function $\bar{R}$,

κ : the conversion factor,

$R^{cb}_t = R^{cb}(t, S_{t-}) = (1-\eta)\kappa S_{t-} \vee \bar{R}_t$: the effective recovery process,

$\xi^{cb} = \bar{N} \vee \kappa S_T + A_T$: the effective payoff at maturity,

$\bar{P} \leq \bar{C}$: the put and call nominal payments, respectively, such that $\bar{P} \leq \bar{N} \leq \bar{C}$,

$\delta \geq 0$: the length of the call notice period (see below),

$t^{\delta} = (t + \delta) \wedge T$: the end date of the call notice period started at t.

Note that *putting* a convertible bond at τ_p effectively means either putting or converting the bond at τ_p, whichever is best for the bondholder. This implies that, accounting for the accrued interest, the effective payment to the bondholder who decides to put at time t is

$$P_t^e := \bar{P} \vee \kappa S_t + A_t. \tag{30}$$

As for *calling*, convertible bonds typically stipulate a positive *call notice period* δ clause, so that if the bond issuer makes a call at time τ_c, then the bondholder has the right to either redeem the bond for $\bar{C}$ or convert it into κ shares of stock at any time $t \in [\tau_c, \tau_c^{\delta}]$, where $\tau_c^{\delta} = (\tau_c + \delta) \wedge T$. This implies that, accounting for the accrued interest, the effective payment to the bondholder in case of exercise at time $t \in [\tau_c, \tau_c^{\delta}]$ is

$$C_t^e := \bar{C} \vee \kappa S_t + A_t. \tag{31}$$

6.1 Reduced Convertible Bonds

A CB with a positive call notice period is rather hard to price directly. To handle this difficulty, we proposed in [4] a two-step valuation method for a CB with a positive call notice period. In the first step, we search for the value of a CB upon call, by considering a suitable family of puttable bonds indexed by the time variable t (see Proposition 6.7 and 6.8). In the second step, we use the price process obtained in the first step as the payoff at call time of a CB with no call notice period, that is, with $\delta = 0$. To formalize this procedure, we find it convenient to introduce the concept of a *reduced convertible bond*, i.e., a particular convertible bond with no call notice period. Essentially, a reduced convertible bond associated with a given convertible bond with a positive call notice period is an 'equivalent' convertible bond with no call notice period, but with the payoff process at call adjusted upwards in order to account for the additional value due to the option-like feature of the positive call period for the bondholder.

Definition 6.1 (see [4]) A *reduced convertible bond* (RB) is a convertible security with coupon process C, recovery process R^{cb} and terminal payoffs L^{cb}, U^{cb}, ξ^{cb} such that (cf. (30)–(31))

$$R_t^{cb} = (1 - \eta)\kappa S_{t-} \vee \bar{R}_t, \quad L_t^{cb} = \bar{P} \vee \kappa S_t + A_t = P_t^e, \quad \xi^{cb} = \bar{N} \vee \kappa S_T + A_T,$$

and, for every $t \in [0, T]$,

$$U_t^{cb} = \mathbb{1}_{\{t < \tau_d\}} \widetilde{U}^{cb}(t, S_t) + \mathbb{1}_{\{t \geq \tau_d\}} C_t^e, \tag{32}$$

for a function $\widetilde{U}^{cb}(t, S)$ jointly continuous in time and space variables, except for negative left jumps of $-c^i$ at the T_is, and such that $\widetilde{U}^{cb}(t, S_t) \geq C_t^e$ on the event $\{t < \tau_d\}$ (so $U_t^{cb} \geq C_t^e$ for every $t \in [0, T]$).

The discounted dividend process of an RB is thus given by, for every $t \in [0, T]$,

$$\int_{[0,t]} \beta_u \, dD_u^{cb} = \int_0^{t \wedge \tau_d} \beta_u c(u) \, du + \sum_{0 \leq T_i \leq t, \, T_i < \tau_d} \beta_{T_i} c^i + \mathbb{1}_{\{0 \leq \tau_d \leq t\}} \beta_{\tau_d} R_{\tau_d}^{cb}. \tag{33}$$

Clearly, a CB with no notice period (i.e., with $\delta = 0$) is an RB with the function $\widetilde{U}^{cb}(t, S)$ given by the formula $\widetilde{U}^{cb}(t, S) = \bar{C} \vee \kappa S + A_t$. More generally, the financial interpretation of the process $\widetilde{U}^{cb}$ in an RB is that $\widetilde{U}^{cb}$ represents the value of the RB upon a call at time t. In Section 6.5, we shall formally prove that, under mild regularity assumptions in our model, any CB (no matter whether the call period is positive or not) can be interpreted and priced as an RB prior to call.

6.2 Decomposition of a Reduced Convertible Bond

In order to perform a deeper analysis of the bond and option features of a reduced convertible bond, it is useful to decompose an RB into the straight bond component, referred to as the *embedded bond*, and the option component, called the *embedded game exchange option*.

6.2.1 Embedded Bond

For an RB with the dividend process D^{cb} given by (33), we consider an elementary security with the same coupon process as the RB and with the quantities R^b and ξ^b given as follows:

$$R_t^b = \bar{R}_t, \quad \xi^b = \bar{N} + A_T, \tag{34}$$

so that

$$R_t^{cb} - R_t^b = \big((1-\eta)\kappa S_t - \bar{R}_t\big)^+ \geq 0, \quad \xi^{cb} - \xi^b = (\kappa S_T - \bar{N})^+ \geq 0.$$

This elementary security corresponds to the defaultable bond with discounted cash flows given by the expression

$$\begin{aligned}\beta_t \phi(t) &= \int_t^T \beta_u \, dD_u^b + \mathbb{1}_{\{\tau_d > T\}} \, \beta_T \xi^b \qquad (35)\\ &:= \int_t^{T\wedge\tau_d} \beta_u c(u)\, du + \sum_{t<T_i\leq T, T_i<\tau_d} \beta_{T_i} c^i + \mathbb{1}_{\{t<\tau_d\leq T\}} \, \beta_{\tau_d} R_{\tau_d}^b + \mathbb{1}_{\{\tau_d>T\}} \, \beta_T \xi^b\end{aligned}$$

and the associated functions (cf. (22)–(23))

$$f(t,S,x) = \gamma(t,S)\bar{R}(t,S) + \Gamma(t,S) - \mu(t,S)x, \quad g(S) = \bar{N}.$$

Definition 6.2 The RB with discounted cash flows given by (34)–(35) is called the *bond embedded into the RB*, or simply the *embedded bond*. It can be seen as the 'straight bond' component of the RB, that is, the RB stripped of its optional clauses.

In the sequel, in addition to the assumptions made so far, we work under the following reinforcement of Assumption 5.1.

Assumption 6.1 The functions $r(t)$, $q(t)$, $\gamma(t,S)S$, $\sigma(t,S)S$, $\gamma(t,S)\bar{R}(t,S)$ and $c(t)$ are continuously differentiable in time variable, and thrice continuously differentiable in space variable, with bounded related spatial partial derivatives.

Note that these assumptions cover typical financial applications. In particular, they are satisfied when $\bar{R}$ is constant and for well-chosen parameterizations of σ and γ, which can be enforced at the time of the calibration of the model.

Proposition 6.1 **(i)** *In the case of an RB, the elementary BSDE ($\mathcal{E}$) (cf. Definition 4.1(iii)) associated with the embedded bond admits a unique solution $(\widehat{\Phi}, Z, K = 0)$. Denoting $\widetilde{\Phi} = \widehat{\Phi} + A$, the embedded bond admits the unique arbitrage price*

$$\Phi_t = \mathbb{1}_{\{t<\tau_d\}} \widetilde{\Phi}_t, \quad t \in [0,T]. \tag{36}$$

(ii) *Moreover, we have that $\widehat{\Phi}_t = \widehat{\Phi}(t, \widetilde{S}_t)$ where the function $\widehat{\Phi}(t,S)$ is bounded, jointly continuous in time and space variables, twice continuously differentiable in space variable, and of class $\mathcal{C}^{1,2}$ on every time interval $[T_{i-1}, T_i)$ (or $[T_{I-1}, T)$, in case $i = I$). The process $\widehat{\Phi}(t, \widetilde{S}_t)$ is an Itô process with true martingale component; specifically, we have*

$$d\widehat{\Phi}_t = \big(\mu_t \widehat{\Phi}_t - (\gamma_t R_t^b + \Gamma_t)\big)\, dt + \sigma(t, \widetilde{S}_t)\widetilde{S}_t \partial_S \widehat{\Phi}_t \, dW_t = u_t \, dt + v_t \, dW_t, \tag{37}$$

where the process v belongs to $\mathcal{H}^2$.

Proof. **(i)** By standard results (see, e.g., [20, 22]), the elementary BSDE $(\mathcal{E})$ with data $(\gamma\bar{R}+\Gamma-\mu\Theta,\bar{N})$ admits a unique solution $(\widehat{\Phi},Z,K=0)$. Hence, by Proposition 4.1 (specified to the particular case of an elementary security), we obtain that the embedded bond admits a unique arbitrage price given by (36).
(ii) The elementary BSDE, yields, for every $t\in[0,T]$,

$$\widehat{\Phi}_t=\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T\big(\gamma_u R^b_u+\Gamma_u-\mu_u\widehat{\Phi}_u\big)\,du+\big(\xi^b-A_T\big)\,\Big|\,\mathcal{F}_t\Big)$$

or, equivalently (see [6]),

$$\alpha_t\widehat{\Phi}_t=\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T\alpha_u\big(\gamma_u R^b_u+\Gamma_u\big)\,du+\alpha_T\big(\xi^b-A_T\big)\,\Big|\,\mathcal{F}_t\Big).\tag{38}$$

Note that we have (cf. (24) and (21) with, by convention $A_{0-}=0$):

$$\alpha_T A_T=\int_{[0,T]}d(\alpha A)_u=\int_0^T\alpha_u\big(a(u)-\mu_u A_u\big)\,du-\sum_{0\le T_i\le T}\alpha_{T_i}c^i.$$

Plugging this into (38) yields

$$\alpha_t\widetilde{\Phi}_t=\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T\alpha_u\big(\gamma_u R^b_u\,du+c(u)\big)\,du+\sum_{t<T_i\le T}\alpha_{T_i}c^i+\alpha_T\xi^b\,\Big|\,\mathcal{F}_t\Big).$$

Let us set

$$\alpha_t\widehat{\Phi}^0_t=\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T\alpha_u\big(\gamma_u R^b_u+c(u)\big)\,du+\alpha_T(\bar{N}+A_T)\,\Big|\,\mathcal{F}_t\Big),\quad t\le T,\tag{39}$$

$$\alpha_t\widehat{\Phi}^i_t=\mathbb{E}_{\mathbb{Q}}\big(\alpha_{T_i}c^i\,\big|\,\mathcal{F}_t\big),\quad t\le T_i.\tag{40}$$

We have $\widetilde{\Phi}_T=\widehat{\Phi}^0_T$ and $\widetilde{\Phi}_t=\widehat{\Phi}^0_t+\sum_{j;T_i\le T_j\le T}\widehat{\Phi}^j_t$ on $[T_{i-1},T_i)$ (or on $[T_{I-1},T)$ in case $i=I$). Let us denote generically T or T^i by $\mathcal{T}$, and $\widehat{\Phi}^0$ or $\widehat{\Phi}^i$ by $\widehat{\Theta}$, as appropriate according to the problem at hand. Note that $\widehat{\Theta}$ is bounded. In addition, given our regularity assumptions, we have $\widehat{\Theta}_t=\widehat{\Theta}(t,\widetilde{S}_t)$, where $\widehat{\Theta}$ belongs to $\mathcal{C}^{1,2}([0,\mathcal{T})\times\mathbb{R})\cap\mathcal{C}^0([0,\mathcal{T}]\times\mathbb{R})$ (see [34, 22]). Therefore, $\widehat{\Phi}_t=\widetilde{\Phi}_t-A_t$ is given by $\widehat{\Phi}(t,\widetilde{S}_t)$ for a function $\widehat{\Phi}(t,S)$, which is jointly continuous in (t,S) on $[0,T]\times\mathbb{R}$ and twice continuously differentiable in S on $[0,T)\times\mathbb{R}$. Moreover, given (39), (40) and the above $\mathcal{C}^{1,2}$ regularity results, we have

$$\begin{aligned}d\widehat{\Phi}^0_t&=\Big(\mu_t\widehat{\Phi}^0_t-\big(\gamma_t R^b_t+c(t)\big)\Big)\,dt+\sigma(t,\widetilde{S}_t)\widetilde{S}_t\partial_S\widehat{\Phi}^0(t,\widetilde{S}_t)\,dW_t,\quad t<T,\\ d\widehat{\Phi}^i_t&=\mu_t\widehat{\Phi}^i_t\,dt+\sigma(t,\widetilde{S}_t)\widetilde{S}_t\partial_S\widehat{\Phi}^i(t,\widetilde{S}_t)\,dW_t,\quad t<T_i\wedge T,\ \text{for } i=1,2,\dots,I,\\ dA_t&=\rho(t)\,dt,\quad t\notin\{T_i\}_{i=0,1,\dots,I}\ .\end{aligned}$$

This yields

$$d\widehat{\Phi}(t,\widetilde{S}_t)=\Big(\mu_t\widetilde{\Phi}_t-\big(\gamma_t R^b_t+c(t)+\rho(t)\big)\Big)\,dt+\sigma(t,\widetilde{S}_t)\widetilde{S}_t\partial_S\widetilde{\Phi}(t,\widetilde{S}_t)\,dW_t=u_t\,dt+v_t\,dW_t.$$

Moreover, since $\widehat{\Phi}$ and u in (37) are bounded, we conclude that $v\in\mathcal{H}^2$. □

6.2.2 Embedded Game Exchange Option

The option component of an RB is formally defined as an RB with the dividend process $D^{cb}-D^b$, payment at maturity $\xi^{cb}-\xi^b$, put payment $L^{cb}_t-\Phi_t$, call payment $U^{cb}_t-\Phi_t$ and call protection lifting time $\bar{\tau}$, where Φ is the embedded bond price in (36). This can be formalized by means of the following definition.

Definition 6.3 The *embedded game exchange option* is a zero-coupon CS with discounted cash flows, for any $t \in [0,T]$ and $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$:

$$\beta_t \psi(t;\tau_p,\tau_c) = \mathbb{1}_{\{t<\tau_d\leq\tau\}}\,\beta_{\tau_d}(R^{cb}_{\tau_d} - R^b_{\tau_d}) \\ + \mathbb{1}_{\{\tau_d>\tau\}}\,\beta_\tau\Big(\mathbb{1}_{\{\tau=\tau_p<T\}}\big(L^{cb}_{\tau_p} - \Phi_{\tau_p}\big) + \mathbb{1}_{\{\tau=\tau_c<\tau_p\}}\big(U^{cb}_{\tau_c} - \Phi_{\tau_c}\big) + \mathbb{1}_{\{\tau=T\}}(\xi^{cb} - \xi^b)\Big). \tag{41}$$

Note that from the point of view of the financial interpretation (see [4] for more comments), the game exchange option corresponds to an option to exchange the embedded bond for either L^{cb}, U^{cb} or ξ^{cb} (as seen from the perspective of the holder), according to which player decides first to stop this game prior to or at T.

Also note that in the case of the game exchange option, there are no coupons involved and thus the clean price and the price coincide.

6.2.3 Solutions of the Doubly Reflected BSDEs

The following auxiliary result can be proved by inspection.

Lemma 6.1 *Given an RB, the associated functions $f(t,S,x)$, $g = g(S)$, $\ell = \ell(t,S)$ and $h = h(t,S)$ are:*
- $f = \gamma R^{cb} + \Gamma - \mu x$, $g = \bar{N} \vee \kappa S$, $\ell = \bar{P} \vee \kappa S$ *and* $h = \widetilde{U}^{cb} - A$ *for the RB;*
- $f = \gamma(R^{cb} - R^b) - \mu x$, $g = (\kappa S - \bar{N})^+$, $\ell = \bar{P} \vee \kappa S - \widehat{\Phi}$ *and* $h = \widetilde{U}^{cb} - A - \widehat{\Phi}$ *for the embedded game exchange option.* □

We will now show how our results can be applied to both an RB and an embedded game exchange option.

Proposition 6.2 **(i)** *The data f, g, ℓ, h (and $\bar{\tau}$ given, as usual, by (26)) associated to an RB satisfy all the general assumptions of Propositions 5.1–5.2*
(ii) *The BSDEs $(\mathcal{E})$ related to an RB or to the embedded game exchange option have unique solutions.*

Proof. **(i)** This can be verified directly by inspection of the related data in Lemma 6.1 (we are in fact in the situation of Example 4.1).
(ii) Given part (i), the BSDE $(\mathcal{E})$ related to an RB has a unique solution $(\widehat{\Pi}, V, K)$, by a direct application of Proposition 4.2. Now, $(\widehat{\Phi}, Z, 0)$ denoting the solution to the elementary BSDE $(\mathcal{E})$ exhibited in Proposition 6.1(i), it is immediate to check that $(\widehat{\Psi}, Y, K)$ solves the game exchange option-related problem $(\mathcal{E})$ iff $(\widehat{\Phi} + \widehat{\Psi}, Z + Y, K)$ solves the RB-related problem $(\mathcal{E})$. Hence the result for the game exchange option follows from that for the RB. □

Given an RB and the embedded game exchange option, we denote by $\widehat{\Pi}$ and $\widehat{\Psi}$ the state-processes (first components) of the solutions to the related BSDEs. The following result summarizes the valuation of an RB and the embedded game exchange option.

Proposition 6.3 **(i)** *The process Ψ_t defined as $\mathbb{1}_{\{t<\tau_d\}}\widehat{\Psi}_t$ is the unique arbitrage price of the embedded game exchange option and $(\Psi_t, \zeta^*, \tau_c^*)$ (resp. $(-\Psi_t, -\zeta^*, \tau_p^*)$) as defined in Proposition 4.1 is an issuer hedge with initial value Ψ_t (resp. holder hedge with initial value $-\Psi_t$) starting from time t for the embedded game exchange option.*
(ii) *The process Π_t defined as $\mathbb{1}_{\{t<\tau_d\}}\widetilde{\Pi}_t$, with $\widetilde{\Pi} := \widehat{\Pi} + A$, is the unique arbitrage price of the RB, and $(\Pi_t, \zeta^*, \tau_c^*)$ (resp. $(-\Pi_t, -\zeta^*, \tau_p^*)$) as defined in Proposition 4.1 is an issuer hedge with initial value Π_t (resp. holder hedge with initial value $-\Pi_t$) starting from time t for the RB.*
(iii) *With $\widehat{\Phi}$ and Φ defined as in Proposition 6.1, we have that $\Pi = \Phi + \Psi$ and $\widehat{\Pi} = \widehat{\Phi} + \widehat{\Psi}$.*

Proof. Given Proposition 6.2, statements (i) and (ii) follow by an application of Proposition 4.1., Part (iii) then follows from the general results of [4]. □

In the sequel we denote generically by $\widehat{\Theta}$ the state-process (i.e., the first component) of the solution to the BSDE related to an RB, the embedded game exchange option or the embedded bond, as suitable in the context at hand.

6.3 Variational Inequalities for Post-Protection Prices

We consider the following problems $(\mathcal{VI})$ on $\mathcal{D} = [0, T] \times \mathbb{R}$:
• for a defaultable bond

$$\begin{aligned} -\mathcal{L}\widehat{\Phi} + \mu\widehat{\Phi} - (\gamma R^b + \Gamma) = 0, \quad t < T, \\ \widehat{\Phi}(T, S) = \bar{N}, \end{aligned} \tag{42}$$

• for a game exchange option

$$\begin{aligned} \max\Big(\min\Big(-\mathcal{L}\widehat{\Psi} + \mu\widehat{\Psi} - \gamma\big(R^{cb} - R^b\big),\, \widehat{\Psi} - \big(\bar{P} \vee \kappa S - \widehat{\Phi}\big)\Big)\,,\, \widehat{\Psi} - \big(\widetilde{U}^{cb} - A - \widehat{\Phi}\big)\Big) = 0, \quad t < T, \\ \widehat{\Psi}(T, S) = (\kappa S - \bar{N})^+, \end{aligned} \tag{43}$$

• for an RB

$$\begin{aligned} \max\Big(\min\Big(-\mathcal{L}\widehat{\Pi} + \mu\widehat{\Pi} - \big(\gamma R^{cb} + \Gamma\big),\, \widehat{\Pi} - \bar{P} \vee \kappa S\Big)\,,\, \widehat{\Pi} - \big(\widetilde{U}^{cb} - A\big)\Big) = 0, \quad t < T, \\ \widehat{\Pi}(T, S) = \bar{N} \vee \kappa S. \end{aligned} \tag{44}$$

Proposition 6.4 (Post-Protection Prices) *For any of problems (42)-(44) there exists a $\mathcal{P}$-solution on $\mathcal{D}$, denoted generically as $\widehat{\Theta}(t, S)$, which determines the corresponding* post-protection price, *in the sense that*

$$\widehat{\Theta}_t = \widehat{\Theta}(t, \widetilde{S}_t), \quad t \in [\bar{\tau}, T]. \tag{45}$$

Moreover, we have uniqueness of the $\mathcal{P}$-solution and any stable, monotone and consistent approximation scheme for $\widehat{\Theta}$ converges locally uniformly to $\widehat{\Theta}$ on $\mathcal{D}$ as $h \to 0^+$.

In the case of the RB and the embedded game exchange option, the post-protection put/conversion region and the post-protection call/conversion region are given as

$$\begin{aligned} \mathcal{E}_p = \big\{(u, S) \in [0, T] \times \mathbb{R}\, ;\, \widehat{\Pi}(u, S) = \bar{P} \vee \kappa S\big\}, \\ \mathcal{E}_c = \big\{(u, S) \in [0, T] \times \mathbb{R}\, ;\, \widehat{\Pi}(u, S) = \widetilde{U}^{cb}(u, S) - A_u\big\}. \end{aligned}$$

Proof. In the case of the RB or of the embedded bond, the results follow by direct application of Propositions 6.2, 5.1(i), 5.2(i) and Corollary 5.1(i). Now, given that $\widehat{\Pi}$ and $\widehat{\Phi}$ are $\mathcal{P}$-solutions to (44) and (42), respectively, and in view of the regularity properties of $\widehat{\Phi}$ stated in Proposition 6.1(ii), therefore $\widehat{\Psi} := \widehat{\Pi} - \widehat{\Phi}$ is a $\mathcal{P}$-solution to (43). Since $\widehat{\Pi}$ and $\widehat{\Phi}$ satisfy the related identities (45), then so does $\widehat{\Psi}$, in view of Proposition 6.3(iii). Finally, given the last statement in Proposition 5.2, the game exchange option also satisfies the claimed uniqueness and convergence results. □

6.4 Variational Inequalities for Protection Prices

We now consider the following problems $(\mathcal{VI})$ on $\mathcal{D} = \mathcal{D}(\bar{T}, \bar{S})$, where the functions $\widehat{\Phi}, \widehat{\Psi}, \widehat{\Pi}$ are those of Proposition 6.4:
• for a game exchange option

$$\begin{aligned} \min\Big(-\mathcal{L}\bar{\Psi} + \mu\bar{\Psi} - \gamma\big(R^{cb} - R^b\big),\, \bar{\Psi} - \big(\bar{P} \vee \kappa S - \widehat{\Phi}\big)\Big) = 0 \text{ on } \mathrm{Int}_p\mathcal{D} \\ \bar{\Psi} = \widehat{\Psi} \text{ on } \partial_p\mathcal{D} \end{aligned} \tag{46}$$

• for an RB

$$\begin{aligned} \min\Big(-\mathcal{L}\bar{\Pi} + \mu\bar{\Pi} - \big(\gamma R^{cb} + \Gamma\big),\, \bar{\Pi} - \bar{P} \vee \kappa S\Big) = 0 \text{ on } \mathrm{Int}_p\mathcal{D} \\ \bar{\Pi} = \widehat{\Pi} \text{ on } \partial_p\mathcal{D}\ . \end{aligned} \tag{47}$$

Proposition 6.5 (Protection Prices) *For any of the problems (46)-(47) there exists a $\mathcal{P}$-solution on $\mathcal{D}$, denoted generically as $\bar{\Theta}$, that determines the corresponding protection price, in the sense that*

$$\widehat{\Theta}_t = \bar{\Theta}(t, \widetilde{S}_t), \quad t \in [0, \bar{\tau}].$$

Moreover, we have uniqueness of the $\mathcal{P}$-solution and any stable, monotone and consistent approximation scheme for $\bar{\Theta}$ converges locally uniformly to $\bar{\Theta}$ on $\mathcal{D}$ as $h \to 0^+$, provided (in case $\bar{S} < +\infty$) it converges to $\bar{\Theta}(=\widehat{\Theta})$ at $\bar{S}$.

In the case of the RB and the embedded game exchange option, the protection put/conversion region is given as

$$\bar{\mathcal{E}}_p = \big\{(u, S) \in [0, T] \times \mathbb{R}\,;\, \bar{\Pi}(u, S) = \bar{P} \vee \kappa S\big\}$$

Proof. In the case of the RB, the results follow by direct application of Propositions 6.2, 5.1(ii), 5.2(ii) and Corollary 5.1(ii). In the case of the game exchange option, we proceed by difference as in the proof of Proposition 6.4 ($\widehat{\Phi}$ denoting the same function as before). □

6.5 Convertible Bonds with Positive Call Notice Period

We now consider the case of a convertible bond with positive call notice period.

Note that between the call time t and the end of the notice period $t^\delta = (t+\delta) \wedge T$, a CB actually becomes a CB with no call clause (or *puttable bond*) over the time interval $[t, t^\delta]$, which is a special case of a puttable security (cf. Definition 3.2; formally, we set $\bar{\tau} = t^\delta$ in the related BSDE). For a fixed t, this puttable bond, denoted henceforth as the *t-PB*, has effective put payment equal to the effective call payment C^e_u, $u \in [t, t^\delta]$ of the original CB (cf. (31)) and the effective payment at maturity equal to $C^e_{t^\delta}$ (see [4]).

Lemma 6.2 *In the case of the t-PB, the associated functions $f(u, S, x)$, $g = g(S)$ and $\ell = \ell(u, S)$ are ($h = +\infty$ in all three cases below):*
- ***embedded bond** (called the **t-bond**, in the sequel): $f(u, S, x) = \gamma(u, S)R^b(u, S) + \Gamma(u, S) - \mu(u, S)x$, $g(S) = \bar{C}$ and $\ell(u, S) = -\infty$;*
- ***embedded game exchange option** (called the **t-game exchange option**, in the sequel): $f(u, S, x) = \gamma(u, S)(R^{cb} - R^b)(u, S) - \mu(u, S)x$, $g(S) = \bar{C} \vee \kappa S - \widehat{\Phi}^t(t^\delta, S)$ and $\ell(u, S) = \bar{C} \vee \kappa S - \widehat{\Phi}^t(u, S)$, where $\widehat{\Phi}^t$ is the pricing function of the t-bond (obtained by an application of Proposition 6.4, see also (48) below);*
- ***t-PB**: $f(u, S, x) = \gamma(u, S)R^{cb}(u, S) + \Gamma(u, S) - \mu(u, S)x$, $g(S) = \bar{C} \vee \kappa S$ and $\ell(u, S) = \bar{C} \vee \kappa S$.*

Note that in view of the proof of Proposition 6.7 below, it is convenient to define the related pricing problems on $[0, t^\delta] \times \mathbb{R}$, rather than merely on $[t, t^\delta] \times \mathbb{R}$. Specifically, given $t \in [0, T]$, we define the following problems $(\mathcal{VI})$ on $[0, t^\delta] \times \mathbb{R}$:
- for the t-bond

$$\begin{aligned} -\mathcal{L}\widehat{\Phi}^t + \mu\widehat{\Phi}^t - \big(\gamma R^b + \Gamma\big) = 0, \quad u < t^\delta, \\ \widehat{\Phi}^t(t^\delta, S) = \bar{C}, \end{aligned} \tag{48}$$

- for the t-game exchange option

$$\begin{aligned} \min\Big(-\mathcal{L}\widehat{\Psi}^t + \mu\widehat{\Psi}^t - \gamma\big(R^{cb} - R^b\big),\, \widehat{\Psi}^t - \big(\bar{C} \vee \kappa S - \widehat{\Phi}^t\big)\Big) = 0, \quad u < t^\delta, \\ \widehat{\Psi}^t(t^\delta, S) = \bar{C} \vee \kappa S - \widehat{\Phi}^t(t^\delta, S), \end{aligned} \tag{49}$$

- for the t-PB

$$\begin{aligned} \min\Big(-\mathcal{L}\widehat{\Pi}^t + \mu\widehat{\Pi}^t - \big(\gamma R^{cb} + \Gamma\big),\, \widehat{\Pi}^t - \bar{C} \vee \kappa S\Big) = 0, \quad u < t^\delta, \\ \widehat{\Pi}^t(t^\delta, S) = \bar{C} \vee \kappa S. \end{aligned} \tag{50}$$

Proposition 6.6 *For any of problems (48)-(50), the corresponding BSDE ($\mathcal{E}$) has a solution, and the related t-price process $\widehat{\Theta}_u^t$ can be written as $\widehat{\Theta}^t(u, \widetilde{S}_u)$, where the function $\widehat{\Theta}^t$ is a $\mathcal{P}$-solution of the related problem ($\mathcal{VI}$) on $[0, t^\delta] \times \mathbb{R}$. Moreover, the uniqueness of the $\mathcal{P}$-solution holds and any stable, monotone and consistent approximation scheme for $\widehat{\Theta}^t$ converges locally uniformly to $\widehat{\Theta}^t$ on $[0, t^\delta] \times \mathbb{R}$ as $h \to 0^+$.*

In the case of the t-PB and the t-game exchange option, the protection put/conversion region is given as

$$\mathcal{E}_p^t = \{(u, S) \in [t, t^\delta] \times \mathbb{R}\,;\, \widehat{\Pi}^t(u, S) = \bar{C} \vee \kappa S\}$$

Proof. In view of Lemma 6.2, the assertion follows by an application of Proposition 6.4. □

Proposition 6.7 (Continuous Aggregation Property) *The function $\widehat{U}(t, S) := \widehat{\Pi}^t(t, S)$ is jointly continuous in time and space variables. Hence the function $\widetilde{U}(t, S) = \widehat{U}(t, S) + A_t$ is also continuous with respect to (t, S), except for left jumps of size $-c^i$ at the T_is.*

Proof. Let $(t_n, S_n) \to (t, S)$ as $n \to \infty$. We decompose

$$\widehat{\Pi}^{t_n}(t_n, S_n) = \widehat{\Pi}^t(t_n, S_n) + (\widehat{\Pi}^{t_n}(t_n, S_n) - \widehat{\Pi}^t(t_n, S_n)).$$

By Proposition 6.6, $\widehat{\Pi}^t(t_n, S_n) \to \widehat{\Pi}^t(t, S)$ as $n \to \infty$. Moreover, denoting $\widehat{C}_t = \bar{C} \vee \kappa \widetilde{S}_t$, $F = \gamma R^{cb} + \Gamma$, we have

$$\alpha_u \widehat{\Pi}_u^t = \operatorname{esssup}_{\tau_p \in \mathcal{F}_{t^\delta}^u} \mathbb{E}_{\mathbb{Q}}\Big(\int_u^{\tau_p} \alpha_v F_v\, dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} \,\Big|\, \mathcal{F}_u\Big), \quad u \le t^\delta.$$

So, assuming t_n sufficiently close to the left of t, and in view of the Markov property of the process $\widetilde{S}$, we obtain, on the event $\{\widetilde{S}_{t_n} = S_n\}$,

$$\begin{aligned}\alpha_{t_n} \widehat{\Pi}^{t_n}(t_n, S_n) &= \operatorname{esssup}_{\tau_p \in \mathcal{F}_{t_n^\delta}^{t_n}} \mathbb{E}_{\mathbb{Q}}\Big(\int_{t_n}^{\tau_p} \alpha_v F_v\, dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} \,\Big|\, \mathcal{F}_{t_n}\Big) \\ &\le \operatorname{esssup}_{\tau_p \in \mathcal{F}_{t^\delta}^{t_n}} \mathbb{E}_{\mathbb{Q}}\Big(\int_{t_n}^{\tau_p} \alpha_v F_v\, dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} \,\Big|\, \mathcal{F}_{t_n}\Big) = \alpha_{t_n} \widehat{\Pi}^t(t_n, S_n).\end{aligned}$$

Conversely, for any $\tau_p \in \mathcal{F}_{t^\delta}^{t_n}$, we have $\tau_p^\delta := \tau_p \wedge t_n^\delta \in \mathcal{F}_{t_n^\delta}^{t_n}$, $0 \le \tau_p - \tau_p^\delta \le t - t_n$ and

$$\begin{aligned}&\Big|\int_{t_n}^{\tau_p} \alpha_v F_v\, dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} - \int_{t_n}^{\tau_p^\delta} \alpha_v F_v\, dv - \alpha_{\tau_p^\delta} \widehat{C}_{\tau_p^\delta}\Big| \\ &\qquad \le \int_{\tau_p^\delta}^{\tau_p} \alpha_v |F_v|\, dv + \big|\alpha_{\tau_p} \widehat{C}_{\tau_p} - \alpha_{\tau_p^\delta} \widehat{C}_{\tau_p^\delta}\big|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Big|\mathbb{E}_{\mathbb{Q}}\Big(\int_{t_n}^{\tau_p} \alpha_v F_v\, dv + \alpha_{\tau_p} \widehat{C}_{\tau_p} \,\Big|\, \mathcal{F}_{t_n}\Big) - \mathbb{E}_{\mathbb{Q}}\Big(\int_{t_n}^{\tau_p^\delta} \alpha_v F_v\, dv + \alpha_{\tau_p^\delta} \widehat{C}_{\tau_p^\delta} \,\Big|\, \mathcal{F}_{t_n}\Big)\Big| \\ &\quad \le \mathbb{E}_{\mathbb{Q}}\Big(\int_{\tau_p^\delta}^{\tau_p} \alpha_v |F_v|\, dv \,\Big|\, \mathcal{F}_{t_n}\Big) + \mathbb{E}_{\mathbb{Q}}\Big(\big|\alpha_{\tau_p} \widehat{C}_{\tau_p} - \alpha_{\tau_p^\delta} \widehat{C}_{\tau_p^\delta}\big| \,\Big|\, \mathcal{F}_{t_n}\Big) \\ &\quad \le c\sqrt{t - t_n}\, \|F\|_{\mathcal{H}^2} + \mathbb{E}_{\mathbb{Q}}\Big(\big|\alpha_{\tau_p} \widehat{C}_{\tau_p} - \alpha_{\tau_p^\delta} \widehat{C}_{\tau_p^\delta}\big| \,\Big|\, \mathcal{F}_{t_n}\Big)\end{aligned}$$

for some constant c. We conclude that $\widehat{\Pi}^{t_n}(t_n, S_n) - \widehat{\Pi}^t(t_n, S_n) \to 0$ as $t_n \to t^-$. But this is also true, with the same proof, as $t_n \to t^+$. Hence $\widehat{\Pi}^{t_n}(t_n, S_n) - \widehat{\Pi}^t(t_n, S_n) \to 0$ as $t_n \to t$. Finally, $\widehat{\Pi}^{t_n}(t_n, S_n) \to \widehat{\Pi}^t(t, S)$ as $t_n \to t$, as desired. □

The next result shows that a CB can be formally reduced to the corresponding RB.

Proposition 6.8 *A CB with positive notice period $\delta > 0$ can be interpreted as an RB with $\widetilde{U}^{cb}(t,S) = \widetilde{U}(t,S)$, where $\widetilde{U}(t,S)$ is the function defined in Proposition 6.7, so that (cf. (32))*

$$U_t^{cb} = \mathbb{1}_{\{\tau_d > t\}} \widetilde{U}(t, S_t) + \mathbb{1}_{\{\tau_d \le t\}} (\bar{C} \vee \kappa S_t + A_t). \tag{51}$$

Proof. The t-PB related reflected BSDE ($\mathcal{E}$) has a solution, and thus, by Proposition 4.1, the t-PB has a unique arbitrage price process $\Pi_u^t = \mathbb{1}_{\{u<\tau_d\}} \widetilde{\Pi}_u^t$ with $\widetilde{\Pi}_u^t = \widehat{\Pi}_u^t + A_u$. Hence the arbitrage price of the CB upon call time t (assuming the CB still alive at time t) is well defined, as $\Pi_t^t = \widetilde{\Pi}_t^t = \widetilde{U}(t, \widetilde{S}_t)$ (cf. Proposition 6.7).

Moreover, by Proposition 6.7, the function $\widetilde{U}(t,S)$ is jointly continuous in time and space, except for negative left jumps of $-c^i$ at the T_is, and we also have that $\Pi_t^t \ge \bar{C} \vee \kappa S_t + A_t$ on the event $\{\tau_d > t\}$. Hence U^{cb} defined as (51) satisfies all the requirements in (32). □

An important conclusion from Proposition 6.8 is that all the results of Section 6 are applicable to a CB (also in the case of a positive call notice period), since, in the simple model of the present paper, a CB may always be interpreted as an RB.

7 Numerical Issues

Let us first comment on numerical issues related to the valuation of a CS. Assume that $\bar{\tau} = 0$ (no call protection) and that we have already specified all the parameters for one of the problems (42), (43) or (44), including, in the case of (43) or (44), the function $\widetilde{U}^{cb}$. Then one can solve the problem numerically (see e.g. [2, 29]) and it is known that, under mild conditions (cf. Proposition 5.2 and the results of Section 6), suitable approximation schemes will converge towards the $\mathcal{P}$-solution of the problem as the discretization step goes to 0. Solving the PDEs related to the embedded bond is standard and thus we shall not comment on this issue.

To have a fully endogenous specification of the problem, one can take $\widetilde{U}^{cb}(t,S) = \widetilde{U}(t,S)$ as defined in Proposition 6.7 in (43) or (44), where $\widetilde{U}(t,S)$ is numerically computed by solving the related obstacle problems, using Proposition 6.6. We provide below a practical algorithm for solving, say (44), with $\widetilde{U}^{cb}(t,S) = \widetilde{U}(t,S)$, using, for example, a fully implicit finite difference scheme (see, for instance, [33]) to discretize $\mathcal{L}$:

1. Localize problems (50) for the embedded t-PBs and problem (44) for the CB. A natural choice, for the t-PBs and the CB, is to localize the problems on the spatial domain $(-\infty, \frac{\bar{C}}{\kappa}]$, with a Dirichlet boundary condition equal to κS (or a Neumann boundary condition equal to κ) at level $\frac{\bar{C}}{\kappa}$;

2. Discretize the localized domain $\mathcal{D}^{loc} = [0,T] \times (-\infty, \frac{\bar{C}}{\kappa}]$, using, say, one time step per day between 0 and T;

3. Discretize problems (50) for the embedded t-PBs on the subdomain $[t, t^\delta]$ of $\mathcal{D}^{loc}$ for t in the time grid *(one problem per value of t in the time grid)*;

4. Solve for $\widehat{\Pi}^t$ the discretized problems (50) corresponding to the embedded t-PBs for t in the time grid *(one problem per value of t in the time grid)*;

5. Discretize problem (44) for the CB on $\mathcal{D}^{loc}$ and solve the discretized problem, using the numerical approximation of $\widetilde{U}(t,S) := \widehat{\Pi}^t(t,S) + A_t$ as an input for $\widetilde{U}^{cb}(t,S)$ in (44).

The problem for the t-PB only has to be solved on the time-strip $[t, t^\delta]$ of $\mathcal{D}^{loc}$. Hence the overall computational cost for solving a CB problem (44) with positive call notice period is roughly the same as that required for solving one CB problem without call notice period, plus the cost of solving n PB problems that would be defined on the whole grid, where n is the number of time mesh points in the call notice period. For instance, for a call notice period $\delta = 1$ month and a time step of one day, we have $n = 30$. Finally, if a call protection is in force then we proceed along essentially the same lines, using the results of Section 6.4.

On Figure 1,[2] we plot the prices of the convertible bond, the embedded bond and the embedded game exchange option obtained in this way as a function of the stock level S at time 0, in the simple case where $\delta = 0$, no call protection is in force, there are no dividends (neither coupons nor recovery), and for the remaining parameters as given in Table 1. In each case, we plot the curves corresponding to default intensities of the form $\gamma(t, S) = \gamma_0(\frac{S_0}{S})^{\gamma_1}$ where $\gamma_0 = 0.02$ and γ_1 equals either 1.2 or zero. The corresponding two curves are labeled `local` and `implied` respectively.

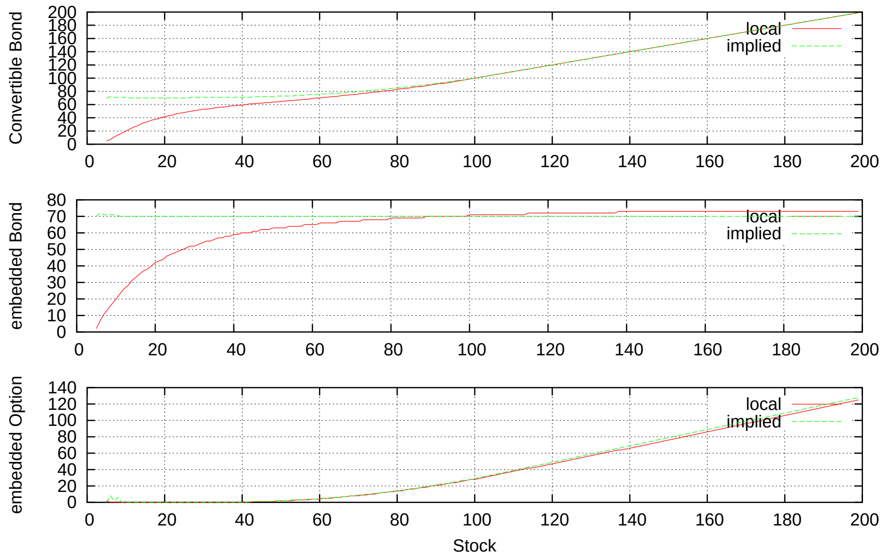

Figure 1: *The Ski-Jump Diagram and its Decomposition*

r	q	η	σ	S_0	T	$\bar{P}$	$\bar{N}$	$\bar{C}$	κ
5%	0	0	20%	100	5y	0	100	130	1

Table 1: *Parameter values*

Note that in case $\alpha = 1.2$, consistently with typical market data, the price of the CB as a function of S exhibits the so-called *ski–jump behavior*, namely, it is convex for high values of S and collapsing at the low values. This collapse at low levels of S comes from the collapse of the embedded bond component of a CB ('collapse of the bond floor', see, e.g., [4]).

An alternative for pricing would be to use numerical methods for reflected BSDEs [32, 9, 10]. The interest of these methods is to provide numerical approximations not only for the state-process $\widehat{\Pi}$ (i.e., the price of a CS) in the solution $(\widehat{\Pi}, Z, K)$ to $(\mathcal{E})$, but also for Z (i.e., the 'delta' of a CS, cf. (25)).

[2] We thank Abdallah Rahal from the Mathematics Departments at University of Evry, France, and Lebanese University, Lebanon, for numerical implementation of the model and, in particular, for generating the picture.

In the present set-up, such methods reduce to simple extensions to game problems of simulation methods for American options [30, 35, 31]. Note that these methods are not much used in the industry at this stage. Beyond the fact that they are computationally intensive, another reason is that they do not give a confidence interval, unlike standard Monte Carlo methods for European options. Yet, in order to take into account non standard call protection clauses or, more generally, to cope with highly path-dependent features, it may be necessary to resort to such simulation methods.

A further numerical issue is the *calibration* of the model, which consists in fitting some specific parameters of the model, such as the local volatility σ and the local intensity γ in our case, to market quotes of related liquidly traded assets. A larger variety of input instruments can be used in this calibration process, including traded options on the underlying equity and/or CDSs related to bond issues of the reference name (see, e.g., [1]). As it can be seen on Figure 1, the price of the embedded game exchange option enjoys much better properties than the price of the CB in terms of convexity with respect to the stock price, and thus in turn (see [4]) in terms of monotonicity with respect to the volatility. Simple numerical experiments support also the intuitive guess that the embedded bond concentrates most of the interest rate and credit risks of a convertible bond, whereas the value of the embedded game exchange option explains most of the volatility risk (note in this respect that the embedded game exchange option always has a null coupon process). These features suggest that it could be advantageous to use prices of (synthetic) embedded game exchange options, rather than prices of CBs, for the purpose of calibration (see also the discussion in the last section of [4]).

A Rolling CDS

In this appendix, we derive the dynamics of the rolling CDS, introduced in Section 2.3.1, in the context of the Markovian defaultable diffusion model of this paper. The interested reader is referred to [7] for the dynamics of the rolling CDS in a more general set-up. Since the derivation takes a simple form in the present Markovian situation, we provide a direct and self-contained proof.

It was shown in [7] that the cumulative price process $\widehat{B}$ of a rolling CDS satisfies (using the set-up of present paper)

$$d(\beta_t \widehat{B}_t) = (1 - H_t)\beta_t \alpha_t^{-1}\big(dp_t - \bar{\nu}(t, \widetilde{S}_t)\, df_t\big) + \beta_t \nu(t)\, dM_t^d,$$

where dp and df denote the stochastic differentials of the following processes, with a *fixed value* $\theta = \Theta(t)$ *of the parameter* θ therein (that is, stochastic differentials with respect to t in $\mathcal{F}_t$, but not with respect to $\theta = \Theta(t)$):

$$p_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^\theta \nu(u)\alpha_u \gamma(u, \widetilde{S}_u)\, du \,\Big|\, \mathcal{F}_t \Big)$$

and

$$f_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^\theta \alpha_u du \,\Big|\, \mathcal{F}_t \Big).$$

It is also rather straightforward to verify that the (local) martingale P, given as

$$P_t = \int_0^t \alpha_u^{-1}\, dp_u,$$

is equal to the martingale part of the process $\widehat{p}$ defined as

$$\widehat{p}_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^\theta \nu(u)\alpha_t^{-1}\alpha_u \gamma(u, \widetilde{S}_u)\, du \,\Big|\, \mathcal{F}_t \Big).$$

In particular, in the present Markovian set-up, the process P_t can also be represented as

$$P_t = \int_0^t \sigma(u, \widetilde{S}_u)\widetilde{S}_u \partial_S \widetilde{P}(u, \widetilde{S}_u)\, dW_u,$$

where $\widetilde{P}$ is the pre-default pricing function of a protection rate payment $\nu(u)$ with horizon θ.

Likewise, it is straightforward to verify that the (local) martingale F given as

$$F_t = \int_0^t \alpha_u^{-1}\, df_u,$$

is equal to the martingale part of the process $\widehat{f}$ defined as

$$\widehat{f}_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^\theta \alpha_t^{-1} \alpha_u \, du \,\Big|\, \mathcal{F}_t \Big).$$

Thus, process F_t can also be written as

$$F_t = \int_0^t \sigma(u, \widetilde{S}_u) \widetilde{S}_u \partial_S \widetilde{F}(u, \widetilde{S}_u)\, dW_u$$

where the function $\widetilde{F}$ is the pre-default pricing function of a unit rate fee payment with horizon θ. This demonstrates the validity of (12).

B Viscosity Solutions of Double Obstacle Variational Inequalities

In this appendix, we comment briefly on the definition of a viscosity solution, which is required, in the case of our obstacles problem $(\mathcal{VI})$, to cope in particular with the potential discontinuities in time of f at the T_is (in case there are discrete coupons, cf. (23)). We refer the interested reader to Crépey [13] for more details. Given a closed domain $\mathcal{D} \subseteq [0, T] \times \mathbb{R}$, we set, for $i = 1, 2, \ldots, I$,

$$\mathcal{D}^i = \mathcal{D} \cap \{T_{i-1} \le t \le T_i\}, \ \ \mathrm{Int}_p\mathcal{D}^i = \mathrm{Int}_p\mathcal{D} \cap \{T_{i-1} \le t < T_i\}. \tag{52}$$

Note that the sets $\mathrm{Int}_p\mathcal{D}^i$ partition $\mathrm{Int}_p\mathcal{D}$.

Definition B.1 **(i)** A locally bounded upper semicontinuous function Π on $\mathcal{D}$ is called a *viscosity subsolution* of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ if and only if $\Pi \le h$, and $\Pi(t,S) > \ell(t,S)$ implies

$$-\mathcal{L}\varphi(t,S) - f(t,S,\Pi(t,S)) \le 0,$$

for any $(t,S) \in \mathrm{Int}_p\mathcal{D}^i$ and $\varphi \in \mathcal{C}^{1,2}(\mathcal{D}^i)$ such that $\Pi - \varphi$ is maximal on $\mathcal{D}^i$ at (t,S), for some $i \in 1, 2, \ldots, I$.

(ii) A locally bounded lower semicontinuous function Π on $\mathcal{D}$ is called a *viscosity supersolution* of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ if and only if $\Pi \ge \ell$, and $\Pi(t,S) < h(t,S)$ implies

$$-\mathcal{L}\varphi(t,S) - f(t,S,\Pi(t,S)) \ge 0,$$

for any $(t,S) \in \mathrm{Int}_p\mathcal{D}^i$ and $\varphi \in \mathcal{C}^{1,2}(\mathcal{D}^i)$ such that $\Pi - \varphi$ is minimal on $\mathcal{D}^i$ at (t,S), for some $i \in 1, 2, \ldots, I$.

(iii) A function Π is called a *viscosity solution* of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ if and only if it is both a viscosity subsolution and a viscosity supersolution of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ (in which case, Π is a continuous function).

Remark B.1 **(i)** In case of a CS with no discrete coupons (like, for instance, the game exchange option component of a CB, which is a zero-coupon CS), the previous definitions reduce to the standard definitions of viscosity (semi-)solutions for obstacles problems (see, for instance, [23, 12]).

(ii) A classical solution of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ (if any) is necessarily a viscosity solution of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$.

(iii) A viscosity subsolution (resp. supersolution) Π of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ does not need to verify $\Pi \ge \ell$ (resp. $\Pi \le h$) on $\mathrm{Int}_p\mathcal{D}$. A viscosity solution (in particular, a classical solution, if any) Π of $(\mathcal{VI})$ on $\mathrm{Int}_p\mathcal{D}$ necessarily satisfies $\ell \le \Pi \le h$ on $\mathrm{Int}_p\mathcal{D}$.

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