A BSDE Approach to Counterparty Risk under Funding Constraints

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June 11, 2011

Abstract

This paper deals with the valuation and hedging of counterparty risk on OTC derivatives. Our study is done in a multiple-curve setup reflecting the various funding constraints (or costs) involved, allowing one to investigate the question of interaction between counterparty risk and funding.

The correction in value of a contract due to counterparty risk under funding constraints is represented as the value of an option on the value of the contract clean of counterparty risk and excess funding costs. We develop a reduced-form backward stochastic differential equations (BSDE) approach to the problem of pricing and hedging this correction, the so-called Credit Valuation Adjustment (CVA for short). In the Markov setup, explicit CVA pricing and hedging schemes are formulated in terms of semilinear CVA PDEs.

The take-away message of the paper is twofold. Firstly, for properly valuing and hedging a counterparty risky contract under funding constraints, it is necessary to focus on a party of interest (rather than on the contract in itself) and to consider explicitly the three pillars of its position consisting of the contract, its hedging portfolio and its funding portfolio. Secondly, the counterparty risk two stages valuation and hedging methodology (counterparty risky price obtained as clean price minus CVA) which is currently emerging for practical reasons in banks, is also useful in the mathematical analysis of the problem.

Keywords: Counterparty Risk, Funding Constraints, Pricing and Hedging, Backward Stochastic Differential Equation (BSDE), Credit Valuation Adjustment (CVA).

*The research of the author benefited from the support of the 'Chaire Risque de crédit', Fédération Bancaire Française, and of the DGE. The author thanks Zorana Grbac, Marek Rutkowski, Tom Bielecki, Giovanni Cesari, Jeroen Kerkhof, Jean-Paul Laurent and Monique Jeanblanc for their involvement at various levels throughout the preparation of this work.

1 Introduction

Counterparty risk is the risk of either party defaulting in an OTC derivative contract (or portfolio of contracts). This is the native form of credit risk, which affects any OTC transaction between two parties, as opposed to reference credit risk which is present in the cash-flows of credit derivatives. An early treatment of counterparty risk can be found for instance in Chapter 14 of Bielecki and Rutkowski (2002). The interest in counterparty risk, along with counterparty risk itself, has exploded since the crisis, when it was realized that the resilience of a bank to a major financial turmoil, is largely determined by its ability to properly value and hedge this risk. The reader is referred to Cesari et al. (2010) or Gregory (2009) for practically oriented presentations.

In this paper we deal with valuation and hedging of a generic contract, to be understood in practice as a portfolio of OTC derivatives, between two defaultable counterparties. These two parties, which will be referred to as "the bank" and "the investor", are tied by a legal agreement, the Credit Support Annex (CSA), prescribing the collateralization scheme and the close-out cash-flow in case of default of either party. The aim of such an agreement is to mitigate counterparty risk. Collateral means cash or various possible eligible securities posted through margin calls as default guarantee by the two parties. The CSA close-out cash-flow is the terminal cash-flow, including the accumulated collateral at that time, to occur in case of default of either party.

A counterparty risk related issue, especially when dealing with bilateral counterparty risk, is a proper accounting for the costs and benefits of funding one's position into the contract. From the perspective of say the bank (and symmetrically so for the investor), this lets a third party enter the scene, namely the funder of the position of the bank. This also gives rise to another close-out cash-flow in case the bank is indebted toward its funder at its time of default. Interaction between the pricing, the hedging and the funding problems, has recently become a major topic of concern for practitioners, reflected for instance in Piterbarg (2010), Morini and Prampolini (2010), Burgard and Kjaer (2010) or Burgard and Kjaer (2011).

A particular trading desk has only a precise view on its own activity. It therefore lacks the global view, and specifically the aggregated data, needed to properly value the CSA cash-flows. Therefore in major investment banks today the trend is to have a central CVA desk in charge of collecting the global information and of valuing and hedging counterparty risk. Here CVA stands for Credit Value Adjustment. The value-and-hedge of the contract is then obtained as the difference between the "clean" value-and-hedge provided by the trading desk (clean of counterparty risk and excess funding costs), and a value-and-hedge adjustment computed by the CVA desk.

This allocation of tasks between the various industry trading desks of an investment bank, and the central CVA desk, motivates the present mathematical CVA approach to the problem of valuing and hedging counterparty risk. Moreover this is done in a multiple-curve setup accounting for the various funding constraints (or costs) involved, allowing one to investigate the question of interaction between counterparty risk and funding.

We develop in this paper a reduced-form CVA backward stochastic differential equations (BSDE) approach to these problems, where the reduction of filtration is with respect to the default times of the bank and the investor.

1.1 Outline of the Paper

In Section 2, we revise the cash-flows at hand and characterize the hedging error arising from a given pricing and hedging scheme, accounting in particular for the funding cash-flows.

Given potential non-linearities in the funding cash-flows, it is not possible to get rid of funding costs through discounting as in a classical one-curve setup. In Section 3, the cash-flows are thus priced instead under an "additive, flat" extension of the classical "multiplicative, discounted" risk-neutral assumption. We also derive the dynamic hedging interpretation of our "additive risk-neutral" price.

Since the pioneering works of Damiano Brigo and his coauthors (see for instance Brigo and Capponi (2010) in a context of bilateral counterparty risk), it is well understood that the CVA can be viewed as an option, the so-called Contingent Credit Default Swap (CCDS), on the clean value of the contract. Section 4 extends to a non linear multiple-curve setup the representation of the CVA as the price of a CCDS. Our CVA accounts not only for counterparty risk, but also for funding costs. The CCDS is then a dividend-paying option, where the dividends correspond to these costs. Note that due to different funding conditions, the (even bilateral) CVAs are not the same to the two parties.

We then develop in Section 5 a practical reduced-form CVA backward stochastic differential equations (BSDE) approach, to the problem of pricing and hedging counterparty risk under funding constraints. Counterparty risk and funding corrections to the clean price-and-hedge of the contract are represented as the solution to a pre-default CVA BSDE stated with respect to a reference filtration, in which defaultability of the two parties only shows up through their default intensities.

In the Markovian setup of Section 6, explicit CVA pricing and hedging schemes are formulated in terms of semilinear pre-default CVA PDEs.

The **main contributions** of this paper consist of:

- An additive risk-neutral pricing approach to the funding issue, shown to be consistent with pricing by replication in the case of complete markets,
- A reduced-form CVA BSDE modeling, valuing and hedging methodology.

The problem of valuing and hedging counterparty risk under funding constraints, is thus reduced to solving Markovian pre-default CVA BSDEs, or (if the space-dimension allows) equivalent semilinear PDEs. Our **main results** in this direction are Proposition 6.6 and Corollary 6.7, which yield concrete recipes for risk-managing the contract as a whole or its CVA component, according to the following objective of the bank: minimizing the (risk-neutral) variance of the cost process (which is essentially the hedging error) of the contract or of its CVA component, whilst achieving a perfect hedge of the jump-to-default exposure. As an aside, this paper also contributes to shed some light on the debate about unilateral versus bilateral counterparty risk.

Note that the mathematical BSDE modeling approach of this paper is consistent with the American Monte Carlo technology which is advocated for practical computations in Cesari et al. (2010).

The **take-away message** of the paper is twofold. Firstly, for properly valuing and hedging counterparty risk in a multiple-curve setup reflecting the presence of various funding costs, it is necessary to focus on a party of interest, say the bank, and to consider the "system" consisting of the bank, the investor and the funder of the bank. One must also have a clear view of the three equally important pillars of this party's position consisting of the contract itself, its hedging portfolio and its funding portfolio (as opposed to getting rid of the funding component of the position by discounting at the risk-free rate in a classical one-curve setup).

Secondly, the counterparty risk two stages valuation and hedging methodology (counterparty risky price obtained as clean price minus CVA) which is currently emerging for practical reasons in banks, is also useful in the mathematical analysis of the problem. This makes the CVA not only a very important and legitimate financial object, but also a valuable mathematical tool.

1.2 Modeling Issues

1.2.1 Unilateral or Bilateral Counterparty Risk?

In principle the possibility of one's own default should be accounted for by a suitable correction, actually standing as a benefit (the so-called DVA for Debt Valuation Adjustment), to the value of the contract. There is a debate among practitioners however regarding the relevance of accounting for one's own credit risk as a benefit through bilateral counterparty risk valuation. The point is that since selling protection on oneself is typically illegal or hardly doable in practice, it is not really possible to hedge one's own credit risk. The principle of risk-neutral valuation of bilateral counterparty risk is thus questionable.

But the practical justification for using a model of bilateral counterparty risk is that unilateral valuation of counterparty risk induces a significant, unreconcilable gap between the CVAs computed by the two parties. This implies that a CSA cannot be agreed on the basis of unilateral counterparty risk valuations.

We will come back on this issue in the last Subsection of the paper.

If in the end one does not want to account for bilateral counterparty risk, one simply considers a model of unilateral counterparty risk, which corresponds in our formalism to letting $\theta = +\infty$ everywhere below (for unilateral counterparty risk from the perspective of the bank).

1.2.2 Immersion Hypothesis and the Case of Credit Derivatives

We believe that the reduced-form approach of this paper is appropriate to deal with counterparty risk on all kinds of derivatives, except for credit derivatives. Indeed a reduced-form approach draws its computational power from, essentially, an immersion hypothesis between the reference filtration "ignoring" the default times of the two parties, and the filtration progressively enlarged by the latter. This immersion hypothesis implies a kind of weak or indirect dependence between the reference contract and the default times of the two parties (see Jeanblanc and Le Cam (2008) for a detailed discussion). This is fine for non-credit derivatives, but it is not consistent with the strong credit dependence effects that may hold between the two parties and the underlying names of a credit portfolio.

Moreover, in the case of credit derivatives, the reduced-form approach of this paper, besides losing in relevance from the point of view of financial modeling, also loses from its computational appeal. With credit derivatives the discontinuous and high-dimensional nature of the problem is such that the gain in tractability resulting from the above reduction of filtration, is not so tangible.

Remark 1.1 We refer the reader to Assefa, Bielecki, Crépey, and Jeanblanc (2011) and Bielecki and Crépey (2011) regarding possible approaches to appropriately deal with CVA on credit derivatives. Note that ideally counterparty risk should not be considered at the level of a specific class of assets, but at the level of all the contracts between two counterparties under a given CSA, and in fact ultimately, after summation, at the aggregated level of all the CSAs of a bank. The construction of a global model and methodology for valuing and hedging a CSA hybrid book of derivatives, including credit derivatives, will be dealt with in future research.

1.2.3 Funding Constraints

By funding assets we mean riskless, finite variation assets which are used for funding a position. In the classical one-curve setup with risk-free rate r_t, there is only one funding asset, the so-called savings account, growing at rate r_t. The savings account is thus the inverse of the risk-free discount factor $\beta_t = e^{-\int_0^t r_s ds}$. In this paper, we do not postulate the existence of the savings account. The risk-free rate r_t simply corresponds to the time-value of money, and one can only think of β_t^{-1} as a "fictitious" savings account. What we shall have instead is the coexistence of various funding assets with different growth rates in the economy.

This immediately raises the question of arbitrage that might result from trading between these rates. Of course these can simply reflect different levels of credit-riskiness, so that a related arbitrage opportunity is only a pre-default view, disregarding losses-upon-defaults. However, even without credit risk, different funding rates may consistently coexist in an economy, reflecting trading constraints, or, in other words, liquidity funding costs. The rationale here is that a given funding rate is only accessible for a definite notional and for a specific purpose, so that funding arbitrage strategies are either not possible, or not sought for by the parties. A good example in the context of counterparty risk is that of the collateral, in which the two parties must have a contractually defined amount ($\Gamma_t^{\pm}$ below) at any point in time.

2 Cash-Flows and Strategies

In this Section, we revise the cash-flows and characterize the hedging error arising from a given pricing and hedging scheme, detailing in particular funding cash-flows.

2.1 Contract

We consider a contract, to be understood as a generic CSA portfolio of OTC derivatives, between a bank and an investor. We denote by θ and $\overline{\theta}$ the default times of the bank and of the investor, in the sense of the times at which promised dividends and margin calls, cease to be paid by a distressed party.

Let $T \in \mathbb{R}_+$ denote the time horizon of the contract, and let $(\Omega, \mathcal{G}_T, \mathcal{G})$, where $\mathcal{G} = (\mathcal{G}_t)_{t\in[0,T]}$, stand for a filtered space which is used throughout the paper for describing the evolution of a financial market model. The filtration $\mathcal{G}$ as well as any other filtration in the paper, are assumed to satisfy the usual conditions. All random times are $[0, T] \cup \{+\infty\}$-valued. The default times θ and $\overline{\theta}$ are $\mathcal{G}$-stopping times. All random variables are $\mathcal{G}_T$-measurable. All processes are defined over $[0, T]$ and $\mathcal{G}$-adapted. We endow the measurable space $(\Omega, \mathcal{G}_T)$ with a probability measure $\mathbb{P}$, which is fixed throughout the paper, and will later be interpreted as a specific martingale pricing measure. We assume in particular that $\mathbb{P}$ is equivalent to the historical probability measure $\widehat{\mathbb{P}}$ over $(\Omega, \mathcal{G}_T)$. We denote by $\mathbb{E}_t$ the conditional expectation given $\mathcal{G}_t$. All cash-flows that appear in the paper are assumed to be $\mathbb{P}$-integrable. By default, all price and value processes (including the collateral margin amount Γ_t) are assumed to be semimartingales, and all semimartingales (including finite variation processes) are taken in a càdlàg version; all inequalities between random quantities are to be understood $d\mathbb{P}$-almost surely or $dt \otimes d\mathbb{P}$-almost everywhere, as suitable.

We assume that default times cannot occur at fixed times, which is for instance satisfied in all intensity models of credit risk. We denote

$$\tau = \theta \wedge \overline{\theta}\,,\ \ \bar{\tau} = \tau \wedge T,$$

where $\bar{\tau}$ represents the effective time horizon of our problem, since there will be no cash-flows after it. We let D represent the clean or promised cumulative dividend process of the contract, assumed to be of finite variation. A promised dividend dD_t is only effectively paid if none of the parties defaulted by time t, resulting in the effective dividend process C such that $dC_t = \mathbb{1}_{t<\tau} dD_t$.

In order to mitigate counterparty risk, the contract is collateralized. Collateral consists of cash or various possible eligible securities posted through CSA regulated margin calls as default guarantee by the two parties. We model collateral in this paper by means of an algebraic margin amount Γ_τ passing from the bank to the investor at time $\tau < T$. So, before τ, a positive Γ_t represents an amount "lent" by the bank to the investor (and remunerated as such by the investor), but devoted to become the property of the investor in case of default of either party at time τ (if $< T$). Symmetrically, before τ, a positive $(-\Gamma_t)$ represents an amount "lent" by the investor to the bank (and remunerated as such by the bank), but devoted to become the property of the bank in case of default of either party at time $\tau < T$.

There is also a CSA close-out cash-flow $\mathbb{1}_{\tau<T} R^i$ from the bank to the investor at time of default $\tau < T$, in which R^i is a $\mathcal{G}_\tau$-measurable random variable which will be specified in Subsection 4.3.

We shall focus henceforth on the bank shortening the contract to the investor under the rules of a given CSA, and setting up a related hedge. By the bank shortening the contract to the investor we mean that all the cash-flows of the contract are paid by the bank. This is conventional however since promised cash-flows are algebraic. For instance $\Delta D_t = \pm 1$ means a bullet cash-flow of ± 1 "paid" by the bank to the investor at time t.

We also call external funder (or funder for short) a generic third-party[1] insuring funding of the position of the bank. External here stands in opposition to the internal source of funding provided to the bank by the investor via the remuneration of the margin amount. For simplicity we assume the external funder to be default-free.

In the context of this paper where the focus is on counterparty risk, recoveries upon default are more conveniently excluded from dividends and accounted for separately as boundary conditions. We shall thus distinguish two categories of related cash-flows:

- Dividends, in the sense of all pre-default cash-flows involving the bank, decomposing into:
  - Counterparty clean or promised contract dividends;
  - Gains on the hedging instruments before time τ;
  - The dt-cost/benefit of funding the position/investing into it;

[1] Possibly composed in practice of several entities and/or devices.

    * This includes in particular the remuneration of the collateral;

- Close-out cash-flows, meaning cash-flows at the default time τ (if $< T$), consisting of:
  - The CSA close-out cash-flow, or recovery on the contract paid by the bank to the investor upon default of either party;
    * This includes in particular the delivery of the collateral;
  - A close-out funding cash flow from the funder to the bank in case of a default of the bank.

Apart from the promised dividends of the contract and the remuneration of the collateral, which are exchanged between the two parties, all other cash-flows differ between them. This induces an asymmetry between the parties, to the consequence that the value of the contract is not the same from their perspectives. This is why we say above that we focus on the bank. Of course symmetrical considerations apply to the investor, but with non-symmetrical hedging positions and funding conditions.

2.2 Hedging Assets

Let $\mathcal{P}$ denote the $\mathbb{R}^d$-valued semimartingale price process of a family of hedging (risky, infinite variation) assets, and let $\mathcal{C}$ stand for the corresponding $\mathbb{R}^d$-valued cumulative effective dividend process. The finite variation dividend process $\mathcal{C}$ represents all the primary promised cash-flows that are granted to a holder of the primary risky assets before time τ.

An hedging asset can be traded either in swapped form, at no upfront payment, or (at least for a physical asset as opposed to a natively swapped primary asset, see below) directly on a primary market. Hedging assets traded in swapped form include (counterparty risk clean) CDS-s on the two parties which are typically used for hedging the counterparty jump-to-default exposure of the contract. Note that a fixed CDS (of a given contractual spread in particular) cannot be traded dynamically in the market. Indeed, only freshly emitted CDS-s can be entered into, at no cost and at the related fair market spread, at a given time. What is used in practice for hedging corresponds to the concept of a rolling CDS, formally introduced in Bielecki, Jeanblanc, and Rutkowski (2008), which is essentially a self-financing trading strategy in market CDS-s. So, much like as in futures contracts, the value of a rolling CDS is null at any point in time, yet due to the trading gains of the strategy, the related cumulative value process is not zero. The case of hedging assets traded in swapped form also covers the situation of a physical (as opposed to natively swapped) hedging asset traded via a repo market.

We assume in this paper that every hedging asset can be traded in swapped form, either as a natively swapped instrument rolled over time, or, for a physical asset, via a corresponding repo market. In mathematical terms, trading the hedging asset with price $\mathcal{P}^i_t$ in swapped form effectively means than one uses, instead of the original (physical or fixed swap) asset, a synthetic asset with price process $\mathcal{S}^i_t = 0$ and gain process given by

$$d\mathcal{P}^i_t - \big(r^i_t \mathcal{P}^i_t + c^i_t\big)dt + d\mathcal{C}^i_t, \tag{1}$$

where:

- In case of a physical primary asset traded via a repo market, the basis c^i corresponds to the so-called repo basis; the meaning of all other terms in (1) is clear;

- In case of a natively swapped asset rolled over time, the different terms in (1) are to be understood as[2]

$$d\mathcal{P}_t^i = d\bar{\mathcal{P}}_t^{i,t_0} \mid_{t_0=t} \,, \;\; \mathcal{P}_t^i = \bar{\mathcal{P}}_t^{i,t} = \mathcal{S}_t^i = 0 \,, \;\; c^i = 0 \,, \;\; d\mathcal{C}_t^i = d\bar{\mathcal{C}}_t^{i,t_0} \mid_{t_0=t} \tag{2}$$

  where $(\bar{\mathcal{P}}_t^{i,t_0})_{t\geq t_0}$ is the price process at time t of the corresponding fixed (as opposed to rolled) swap emitted at time $t_0 \leq t$, with dividend process $(\bar{\mathcal{C}}_t^{i,t_0})_{t\geq t_0}$.

Note that all the dt-funding costs in this paper are expressed in terms of a basis, like c^i in (1), to the risk-free cost at rate r_t.

2.3 Funding Assets

This Subsection provides a comprehensive specification of funding cash-flows. The corresponding notion of a self-financing trading strategy will be derived in Subsection 2.4. A general formulation of the pricing and hedging problem under abstract funding constraints will then be given in Subsection 2.5.

Regarding funding of the hedging instruments, we suppose that the hedging position in a primary risky asset is either entirely swapped, or funded in totality by the external lender, and that this choice is given and fixed once for all at time 0 for every hedging instrument. We let a superscript ^{s} refer to the subset of the hedging instruments traded in swapped form, and $^{\bar{s}}$ refer to the subset, complement of ^{s}, of (physical) hedging instruments which are traded directly on a primary market (and are therefore funded together with the contract by the external funder).

Regarding the remuneration of the margin amount, we restrict ourselves for simplicity here to collateral posted as cash. We follow the most common CSA covenant under which the party getting the collateral can use it in its trading, as opposed to a covenant where collateral is segregated by a third party in order to avoid the so-called re-hypothecation risk (see Bielecki and Crépey (2011)). Specific CSA rates $r_t + b_t$ and $r_t + \bar{b}_t$, where b and $\bar{b}$ stand for related bases, are then typically used to remunerate the collateral owned by either party. This results in a dt-remuneration of the margin amount which is worth

$$(r_t + b_t)\Gamma_t^+ dt - (r_t + \bar{b}_t)\Gamma_t^- dt = r_t\Gamma_t dt + \big(b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^-\big)dt$$

to the bank, and the opposite to the investor. We assume further that the bank can lend money to (respectively borrow money from) its external funder at an excess cost over the risk-free rate r_t determined by a funding credit and/or liquidity basis λ (respectively $\bar{\lambda}$).

Note that even though the two parties are defaultable, the mechanism of collateralization makes them practically default-free[3] as far as aspects related to the collateral are concerned. Since the external funder is assumed to be default-free, thus regarding funding cash-flows of the bank, default risk is purely on the bank's side, and confined to external funding cash-flows. Namely, in case the bank is indebted to its (default-free) funder at time $\tau = \theta < T$, then the bank could not be in a position to reimburse its external debt, which results as we shall see below in a close-out funding cash-flow from the external funder to the bank. This cash-flow corresponds to the the funding side of "the bank benefiting from its own default".

[2] See Bielecki, Jeanblanc, and Rutkowski (2008) and Bielecki, Crépey, Jeanblanc, and Rutkowski (2010) for more details.

[3] Neglecting re-hypothecation issues, see Bielecki and Crépey (2011).

In order to account for the above funding specifications in a classical formalism of self-financing trading strategies, let us introduce the following funding assets on $[0,\bar{\tau}]$ (with all initial conditions set to one):

- Two collateral funding assets, B^0 and $\bar{B}^0$, evolving as

$$dB_t^0 = (r_t + b_t)B_t^0 dt\,,\ \ d\bar{B}_t^0 = (r_t + \bar{b}_t)\bar{B}_t^0 dt, \tag{3}$$

  dedicated to the funding of the positive and the negative part of the margin account,

- Two external funding assets, B^f and $\bar{B}^f$, evolving as

$$dB_t^f = (r_t + \lambda_t)B_t^f dt\,,\ \ d\bar{B}_t^f = (r_t + \bar{\lambda}_t)\bar{B}_t^f dt - (1-\mathfrak{r})\bar{B}_{t-}^f \delta_\theta(dt) \tag{4}$$

  where the symbol δ denotes a Dirac measure; these are the investing and funding assets of the bank by its external lender.

The $\mathcal{G}_\theta$-measurable random variable $\mathfrak{r}$ in (4) represents the recovery rate of the bank towards its external funder. The case $\mathfrak{r} = 1$ can be seen as a model of partial default in which at time θ the bank only defaults on its contractual commitments with regard to the investor, but not on its funding debt with respect to its funder. This case can also be used for modeling the situation of a bank in a global net lender position, so that it actually does not need any external lender. In case cash is needed for funding its position, the bank simply uses its own cash. The case $\mathfrak{r} < 1$ can be seen as a model of total default time θ of the bank, which defaults at time θ not only on its commitments in the contract with regard to the investor, but also on its related funding debt.

2.4 Trading Strategies

The valuation and hedging task for the bank shortening the contract to the investor, consists in devising a price and a dynamic hedging portfolio for the contract sold to the investor, whilst getting funded by its external lender.

A hedge process is defined as a predictable and locally bounded, $\mathbb{R}^d$-valued row-vector process ζ over $[0,\bar{\tau}]$, representing the number of units of the primary risky assets which are held in the hedging portfolio. By price-and-hedge of the contract for the bank shortening it to the investor, we mean any pair-process $(\bar{\Pi},\zeta)$ over $[0,\bar{\tau}]$, where $\bar{\Pi}$ is an $\mathbb{R}$-valued semimartingale such that $\bar{\Pi}_{\bar{\tau}} = \mathbb{1}_{\tau<T}R^i$ (the CSA close-out cash-flow), and ζ is a hedge process. By hedging error process of the price-and-hedge $(\bar{\Pi},\zeta)$, we mean $\varrho = \bar{\Pi} - \bar{\mathcal{W}}$, where $\bar{\mathcal{W}}$ is the value process of the collateralization, hedging and funding portfolio, the strategy being funded as described in Subsection 2.3. So, for $t \in [0,\bar{\tau}]$,

$$\bar{\mathcal{W}}_t = \Big(\Gamma_t^+ - \Gamma_t^-\Big) + \Big(\zeta_t^s \mathcal{S}_t^s + \zeta_t^{\bar{s}}\mathcal{P}_t^{\bar{s}}\Big) + \Big((\bar{\mathcal{W}}_t - \Gamma_t - \zeta_t^{\bar{s}}\mathcal{P}_t^{\bar{s}})^+ - (\bar{\mathcal{W}}_t - \Gamma_t - \zeta_t^{\bar{s}}\mathcal{P}_t^{\bar{s}})^-\Big) \tag{5}$$

where the three terms in the right-hand side correspond to the amounts respectively invested as collateral, into the hedging risky assets (swapped and non swapped components ζ^s and $\zeta^{\bar{s}}$, see the explanations surrounding Equation (1)) and into the external funding assets. Note $\mathcal{S}_t = 0$, so a hedging instrument traded in swapped form does not contribute to the value $\bar{\mathcal{W}}_t$ directly, however it will contribute below to its dynamics, via the related gain process in (1). Equivalently to (5), let us put in a more formal notation

$$\bar{\mathcal{W}}_t = \eta_t^0 B_t^0 + \bar{\eta}_t^0 \bar{B}_t^0 + \zeta_t^s \mathcal{S}_t^s + \zeta_t^{\bar{s}}\mathcal{P}_t^{\bar{s}} + \eta_t^f B_t^f + \bar{\eta}_t^f \bar{B}_t^f \tag{6}$$

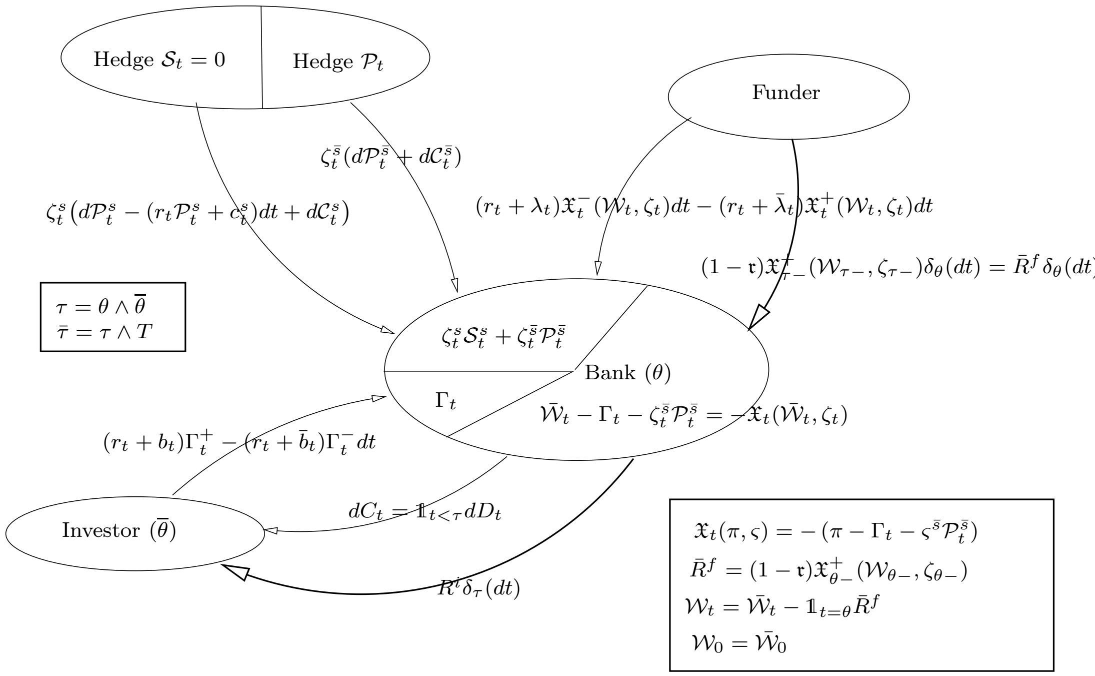

Figure 1: Cash-flows of the bank over $[0, \bar{\tau}]$.

with

$$\eta_t^0 = \frac{\Gamma_t^+}{B_t^0}\,,\ \ \bar{\eta}_t^0 = -\frac{\Gamma_t^-}{\bar{B}_t^0}\,,\ \ \eta_t^f = \frac{(\bar{\mathcal{W}}_t - \Gamma_t - \zeta_t^{\bar{s}}\mathcal{P}_t^{\bar{s}})^+}{B_t^f}\,,\ \ \bar{\eta}_t^f = -\frac{(\bar{\mathcal{W}}_t - \Gamma_t - \zeta_t^{\bar{s}}\mathcal{P}_t^{\bar{s}})^-}{\bar{B}_t^f}$$

and $\eta_t^i = -\frac{\zeta_t^i P_t^i}{B_t^i}$, for $i = 1, \ldots, d$. Following a standard terminology, we then say in view of (6) that the strategy $(\bar{\Pi}, \zeta)$ of the bank is self-financing if and only if $\bar{\mathcal{W}}_0 = \bar{\Pi}_0$ and for $t \in [0, \bar{\tau}]$

$$\begin{aligned} d\bar{\mathcal{W}}_t = &-dC_t + \eta_t^0 dB_t^0 + \bar{\eta}_t^0 d\bar{B}_t^0 + \\ &\zeta_t^s\big(d\mathcal{P}_t^s - (r_t\mathcal{P}_t^s + c_t^s)dt + d\mathcal{C}_t^s\big) + \zeta_t^{\bar{s}}(d\mathcal{P}_t^{\bar{s}} + d\mathcal{C}_t^{\bar{s}}) + \eta_t^f dB_t^f + \bar{\eta}_{t-}^f d\bar{B}_t^f \end{aligned} \tag{7}$$

where the "minus" in $\bar{\eta}_{t-}^f$ is needed[4] because $\bar{B}_t^f$ jumps at time θ (and process $\bar{\eta}^f$ is not predictable). Figure 1 (see also the proof of Proposition 2.1 below) displays a graphical representation of all the related cash-flows over $[0, \bar{\tau}]$. We shall now derive the dynamics of the hedging error process $\varrho = \bar{\Pi} - \bar{\mathcal{W}}$ of a self-financing strategy. We denote for every real number π and $\mathbb{R}^d$-valued row-vector ς

$$\begin{aligned} f_t(\pi, \varsigma) &= b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^- + \lambda_t\big(\pi - \Gamma_t - \varsigma^{\bar{s}}\mathcal{P}_t^{\bar{s}}\big)^+ - \bar{\lambda}_t\big(\pi - \Gamma_t - \varsigma^{\bar{s}}\mathcal{P}_t^{\bar{s}}\big)^- - \varsigma^s c_t^s \\ \mathfrak{X}_t(\pi, \varsigma) &= -\big(\pi - \Gamma_t - \varsigma^{\bar{s}}\mathcal{P}_t^{\bar{s}}\big) \end{aligned} \tag{8}$$

where $f_t(\bar{\mathcal{W}}_t, \zeta_t)$ will be interpreted as the dt-excess-funding-benefit of the bank, and $\mathfrak{X}_{t-}(\bar{\mathcal{W}}_{t-}, \zeta_{t-})$ as the (algebraic) debt of the bank towards its funder at time t. Let finally for $t \in [0, \bar{\tau}]$

$$\Pi_t^* = \bar{\Pi}_t - \mathbb{1}_{t=\theta}\bar{R}^f\,,\ \ \mathcal{W}_t = \bar{\mathcal{W}}_t - \mathbb{1}_{t=\theta}\bar{R}^f, \tag{9}$$

[4] We thank Marek Rutkowski for pointing this out as well as for a significant contribution in a reorganization and clarification of this part of the paper.

where $\bar{R}^f := (1-\mathfrak{r})\mathfrak{X}^+_{\theta-}(\mathcal{W}_{\theta-},\zeta_{\theta-})$ will appear below as the close-out cash-flow from the external funder to the bank at time $\tau=\theta<T$.

Proposition 2.1 *Under the funding specifications of Subsection 2.3, a price-and-hedge* $(\bar{\Pi},\zeta)$ *is self-financing if and only if* $\mathcal{W}_0=\Pi^*_0(=\bar{\Pi}_0)$ *and for* $t\in[0,\bar{\tau}]$

$$d\mathcal{W}_t = -dC_t + \big(r_t\mathcal{W}_t + f_t(\mathcal{W}_t,\zeta_t)\big)dt + \zeta_t(d\mathcal{P}_t - r_t\mathcal{P}_t dt + d\mathcal{C}_t). \tag{10}$$

Proof. Plugging (3)-(4) into (7) and using also the current specification of the funding policy regarding hedging assets, yields that the strategy is self-financing if and only if for $t\in[0,\bar{\tau}]$

$$\begin{aligned}
d\bar{\mathcal{W}}_t =& - dC_t + (r_t+b_t)\Gamma^+_t dt - (r_t+\bar{b}_t)\Gamma^-_t dt + \zeta_t(d\mathcal{P}_t + d\mathcal{C}_t) - \zeta^s\left(r_t\mathcal{P}^s_t + c^s_t\right)dt\\
&+ (r_t+\lambda_t)\big(\bar{\mathcal{W}}_t - \Gamma_t - \zeta^{\bar{s}}_t\mathcal{P}^{\bar{s}}_t\big)^+dt - (r_t+\bar{\lambda}_t)\big(\bar{\mathcal{W}}_t - \Gamma_t - \zeta^{\bar{s}}_t\mathcal{P}^{\bar{s}}_t\big)^-dt\\
&- \bar{\eta}^f_{\tau-}(1-\mathfrak{r})\bar{B}^f_{\tau-}\delta_\theta(dt)\\
=& - dC_t + r_t(\bar{\mathcal{W}}_t - \zeta_t\mathcal{P}_t)dt + \zeta_t(d\mathcal{P}_t + d\mathcal{C}_t) + b_t\Gamma^+_t dt - \bar{b}_t\Gamma^-_t dt - \zeta^s c^s_t dt\\
&+ \lambda_t\big(\bar{\mathcal{W}}_t - \Gamma_t - \zeta^{\bar{s}}_t\mathcal{P}^{\bar{s}}_t\big)^+dt - \bar{\lambda}_t\big(\bar{\mathcal{W}}_t - \Gamma_t - \zeta^{\bar{s}}_t\mathcal{P}^{\bar{s}}_t\big)^-dt\\
&+ (1-\mathfrak{r})\left(\bar{\mathcal{W}}_{\tau-} - \Gamma_{\tau-} - \zeta^{\bar{s}}_{\tau-}\mathcal{P}^{\bar{s}}_{\tau-}\right)^-\delta_\theta(dt)\\
=& - dC_t + r_t\bar{\mathcal{W}}_t dt + \zeta_t(d\mathcal{P}_t - r_t\mathcal{P}_t dt + d\mathcal{C}_t) + f_t(\bar{\mathcal{W}}_t,\zeta_t)dt\\
&+ (1-\mathfrak{r})\mathfrak{X}^+_{\tau-}(\bar{\mathcal{W}}_{\tau-},\zeta_{\tau-})\delta_\theta(dt).
\end{aligned}$$

□

2.5 General Price-and-Hedge

As illustrated in Subsection 2.3, the exact nature of the funding cash-flows depends on the specification of a funding policy defined in terms of related funding riskless assets. For the sake of clarity one shall work henceforth with the following abstract, formal definition of a general (self-financing) price-and-hedge, in which the funding component of the hedging portfolio only shows up through the dt-excess-benefit-funding coefficient f, and through the funding close-out cash-flow R^f, without explicit reference to specific funding assets. We shall thus consider as given an abstract dt-excess-benefit-funding coefficient $f_t(\pi,\varsigma)$, as well as an abstract external debt function $\mathfrak{X}_t(\pi,\varsigma)$, where $\pi\in\mathbb{R}$ and $\varsigma\in\mathbb{R}^d$. A $\mathcal{G}_\theta$-measurable random variable $\mathfrak{r}$ represents as before the recovery rate of the bank towards its external funder.

The following definition is put in the form of a Forward-Backward Stochastic Differential Equation (FBSDE, see Ma and Yong (2007)) in $(\mathcal{W},\Pi^*,\zeta)$. What solving the FBSDE would mean is solving the related control problem, that is finding a general price-and-hedge $(\bar{\Pi},\zeta)$ such that the corresponding hedging error process ϱ in the second line of (11) has "nice" properties in terms of arbitrage (typically: ϱ being a martingale under some probability measure, like $\mathbb{P}$, equivalent to the historical measure $\widehat{\mathbb{P}}$) and replication (typically: ϱ being small in some appropriate norm). This would be a fairly non-standard FBSDE however, and we shall not try to solve it in this form, rather introducing soon a more tractable BSDE.

Definition 2.2 (General Price-and-Hedge) Given a hedge process ζ, let $(\mathcal{W},\Pi^*,\zeta,\varrho)$ satisfy the initial conditions $\mathcal{W}_0=\Pi^*_0$, $\varrho_0=0$ and for $t\in[0,\bar{\tau}]$

$$\begin{aligned}
d\mathcal{W}_t &= -dC_t + \big(r_t\mathcal{W}_t + f_t(\mathcal{W}_t,\zeta_t)\big)dt + \zeta_t(d\mathcal{P}_t - r_t\mathcal{P}_t dt + d\mathcal{C}_t)\\
d\Pi^*_t &= -dC_t + \big(r_t\mathcal{W}_t + f_t(\mathcal{W}_t,\zeta_t)\big)dt + \zeta_t(d\mathcal{P}_t - r_t\mathcal{P}_t dt + d\mathcal{C}_t) + d\varrho_t
\end{aligned} \tag{11}$$

along with a terminal condition $\Pi^*_{\bar\tau} = \mathbb{1}_{\tau<T}\bar R$ where

$$\bar R = R^i - \mathbb{1}_{\tau=\theta}\bar R^f \tag{12}$$

in which $\bar R^f := (1-\mathfrak{r})\mathfrak{X}^+_{\tau-}(\mathcal{W}_{\tau-},\zeta_{\tau-})$.

One then calls general price-and-hedge with hedging error ϱ, the pair-process $(\bar\Pi,\zeta)$ where for $t\in[0,\bar\tau]$

$$\bar\Pi_t := \Pi^*_t + \mathbb{1}_{t=\theta}\bar R^f.$$

We say that $(\bar\Pi,\zeta)$ is a replicating strategy if $\varrho_{\bar\tau}=0$ almost surely.

Observe that $\bar R$ represents the total close-out cash-flow delivered by the bank at time $\tau<T$ (CSA close-out cash flow R^i paid to the investor minus close-out funding cash-flow $\mathbb{1}_{\tau=\theta}\bar R^f$ got from the external funder). Also note that under the funding specifications of Subsection 2.3 this definition is consistent with the developments of Subsection 2.4. In the abstract Definition 2.2 we focus on processes Π^* and $\mathcal{W}$ rather on $\bar\Pi$ and $\bar{\mathcal{W}}$ that concurrently showed-up in the specific setup of Subsections 2.3-2.4, because Π^* (actually, ultimately Π to be introduced in Definition 3.4 below) and $\mathcal{W}$ will be more convenient mathematically. By a slight abuse of terminology we call $\mathcal{W}$ the value of the hedging portfolio. To be precise, it is actually process $\bar{\mathcal{W}}$ which corresponds to what should be called exactly the value of the collateralization, hedging and funding portfolio.

Also observe that in case $f=0$ and $\mathfrak{r}=1$ (classical one-curve setup without excess funding costs), one recovers the usual notion of a self-financing hedging strategy with related wealth process $\mathcal{W}$, so that the funding base f and the funding close-out cash-flow $\bar R^f$ can be interpreted as our corrections to a classical one-curve setup.

3 Martingale Pricing

In this Section we deal with the pricing of the contract shortened by the bank to the investor, under the funding conditions of the bank defined by the coefficients f and c. Note that given possible non-linearities in the excess funding benefit coefficient f, it will not be possible to get rid of the funding costs in the pricing through discount factors as in a linear one-curve setup (unless one resorts to an endogenous discount factor depending on the value of the contract). Cash-flows will be priced instead in this Section under an "additive, flat" extension of the classical "multiplicative, discounted" risk-neutral assumption. We also derive the dynamic hedging interpretation of such an additive risk-neutral price.

Recall the expression $d\mathcal{P}^i_t - \big(r_t\mathcal{P}^i_t + c^i_t\big)dt + d\mathcal{C}^i_t$ in (1) for the gain process of a buy-and-hold position into an hedging asset traded in swapped form. Let $\mathcal{M}$ denote the gain process of all hedging instruments traded in swapped form, so $\mathcal{M}_0=0$ and for $t\in[0,\bar\tau]$

$$d\mathcal{M}_t = d\mathcal{P}_t - \big(r_t\mathcal{P}_t + c_t\big)dt + d\mathcal{C}_t. \tag{13}$$

Our standing probability measure $\mathbb{P}$ is henceforth interpreted as a risk-neutral pricing measure on the primary market of hedging instruments traded in swapped form, in the sense that

Assumption 3.1 The primary risky gain process $\mathcal{M}$ is an $\mathbb{R}^d$-valued $(\mathcal{G},\mathbb{P})$-martingale.

By arbitrage, let us mean a self-financing strategy with a related gain at time $\bar{\tau}$ which is almost surely non-negative, and which is positive with a positive probability (under the historical or any equivalent probability measure). Since the historical probability measure $\widehat{\mathbb{P}}$ is equivalent to $\mathbb{P}$, Assumption 3.1 precludes arbitrage opportunities that might result from pure primary trading strategies only involving primary hedging assets traded in swapped form. It can be considered as an "additive" version of the "multiplicative" risk-neutral assumption which is more commonly used through the language of discounting at the risk-free rate r_t in the one-curve literature. Under this assumption it is convenient to rewrite (10) in martingale form as

$$d\mathcal{W}_t = \big(r_t\mathcal{W}_t + g_t(\mathcal{W}_t, \zeta_t)\big)dt - dC_t + \zeta_t d\mathcal{M}_t \tag{14}$$

where for $\pi \in \mathbb{R}$ and $\varsigma \in \mathbb{R}^d$

$$g_t(\pi, \varsigma) = f_t(\pi, \varsigma) + \varsigma c_t. \tag{15}$$

Example 3.2 Under the specification of Subsection 2.3, one gets

$$\begin{aligned} g_t(\pi, \varsigma) &= b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^- + \lambda_t\big(\pi - \Gamma_t - \varsigma^{\bar{s}}\mathcal{P}_t^{\bar{s}}\big)^+ - \bar{\lambda}_t\big(\pi - \Gamma_t - \varsigma^{\bar{s}}\mathcal{P}_t^{\bar{s}}\big)^- - \varsigma^s c_t^s + \varsigma c_t \\ &= b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^- + \lambda_t\big(\pi - \Gamma_t - \varsigma^{\bar{s}}\mathcal{P}_{t-}^{\bar{s}}\big)^+ - \bar{\lambda}_t\big(\pi - \Gamma_t - \varsigma^{\bar{s}}\mathcal{P}_{t-}^{\bar{s}}\big)^- + \varsigma^{\bar{s}} c_t^{\bar{s}} \end{aligned} \tag{16}$$

which only depends on ς through $\varsigma^{\bar{s}}$, the position in the hedging assets funded together with the contract by the external lender.

Remark 3.3 In the case of a physical (as opposed to a natively swapped) primary asset, the coefficient c^i corresponds to the related repo basis, and one might wonder why in Example 3.2 the repo rates eventually present in g are actually those of the hedging instruments which are not traded in swapped form. An interpretation is that in case of a hedging instrument traded in swapped form, the opportunity of getting it funded at the excess cost c^i is exploited, whereas for a hedging instrument not traded in swapped form this opportunity is not, creating an (algebraic) "loss of income" which should be reflected in the final "pricing formula", and therefore in the coefficient g of the corresponding pricing equation to be introduced in Definition 3.4 below.

3.1 $\mathbb{P}$-Price-and-Hedge BSDE

The class of general price-and-hedges introduced in Definition 2.2 is too large for practical purposes. This leads us to introduce the following more restrictive definition. Given a hedge ζ and a semimartingale Π, we denote $R = R^i - \mathbb{1}_{\tau=\theta}R^f$, in which

$$R^f := (1 - \mathfrak{r})\mathfrak{X}_{\tau-}^+(\Pi_{\tau-}, \zeta_{\tau-}). \tag{17}$$

Let us stress that R implicitly depends on $(\Pi_{\tau-}, \zeta_{\tau-})$ in this notation.

Definition 3.4 ($\mathbb{P}$-price-and-hedge) Let a pair (Π, ζ) made of a $\mathcal{G}$-semimartingale Π and a hedge ζ satisfy the following BSDE on $[0, \bar{\tau}]$:

$$\begin{aligned} &\Pi_{\bar{\tau}} = \mathbb{1}_{\tau<T}R \text{ and for } t \in [0, \bar{\tau}] : \\ &d\Pi_t + dC_t - \big(r_t\Pi_t + g_t(\Pi_t, \zeta_t)\big)dt = d\nu_t \end{aligned} \tag{18}$$

for some $\mathcal{G}$-martingale ν null at time 0. Letting for $t \in [0, \bar{\tau}]$

$$\bar{\Pi}_t := \Pi_t + \mathbb{1}_{t=\theta} R^f,$$

process $(\bar{\Pi}, \zeta)$ is then said to be a $\mathbb{P}$-price-and-hedge. The related cost process is the $\mathcal{G}$-martingale ε defined by $\varepsilon_0 = 0$ and for $t \in [0, \bar{\tau}]$

$$d\varepsilon_t = d\nu_t - \zeta_t d\mathcal{M}_t \tag{19}$$

where ν is the $\mathcal{G}$-martingale component of Π in (18), and $\mathcal{M}$ is the $\mathcal{G}$-martingale component (13) of the primary risky price process $\mathcal{P}$.

Equivalently to the BSDE (18) in differential form, one can write in integral form, for $t \in [0, \bar{\tau}]$ (recall $\beta_t = e^{-\int_0^t r_s ds}$)

$$\beta_t \Pi_t = \mathbb{E}_t \Big(\int_t^{\bar{\tau}} \beta_s dC_s \; - \int_t^{\bar{\tau}} \beta_s g_s(\Pi_s, \zeta_s) ds + \beta_{\bar{\tau}} \mathbb{1}_{\tau < T} R \Big). \tag{20}$$

The reader is referred to El Karoui, Peng, and Quenez (1997) for a general reference about BSDEs in finance, and to Example 1.1 therein as a basic example of use of BSDEs in connection with valuation and hedging under funding constraints (different borrowing and lending rates). The $\mathbb{P}$-price-and-hedge BSDE (18) is made non-standard by the random terminal time $\bar{\tau}$, the dependence of the terminal condition R in $(\Pi_{\tau-}, \zeta_{\tau-})$, the contract effective dividend term dC_t, and finally the fact that it is not driven by an explicit set of fundamental martingales like Brownian motions and/or compensated jump measures. In this last regard, the representation (19) rather suggests that this BSDE will be solved with respect to the "market martingale" $\mathcal{M}$, up to a (typically orthogonal) martingale ε. The issue of well-posedness of the $\mathbb{P}$-price-and-hedge BSDE is postponed to the next sections, where it will be more conveniently discussed in terms of a corresponding CVA BSDE.

By construction, a $\mathbb{P}$-price-and-hedge $(\bar{\Pi}, \zeta)$ is a general price-and-hedge in the sense of Definition 2.2. In Subsection 3.2 we shall comment upon a $\mathbb{P}$-price-and-hedge from the points of view of arbitrage, hedging and computational tractability. But let us first derive the equations for the wealth of the corresponding hedging portfolio $\mathcal{W}$ and for the corresponding hedging error ϱ. One thus has the following

Lemma 3.5 *Given a $\mathbb{P}$-price-and-hedge $(\bar{\Pi}, \zeta)$ and the related process Π, let a process $\mathcal{W}$ be defined by the first line in (11), starting from the initial condition $\mathcal{W}_0 = \Pi_0$; let then a process Π^* be defined by, for $t \in [0, \bar{\tau}]$*

$$\Pi^*_t = \bar{\Pi}_t - \mathbb{1}_{\tau=\theta} \bar{R}^f$$

where $\bar{R}^f := (1 - \mathfrak{r}) \mathfrak{X}^+_{\tau-}(\mathcal{W}_{\tau-}, \zeta_{\tau-})$. Let finally $\varrho = \Pi^* - \mathcal{W}$ on $[0, \bar{\tau}]$. Then $(\bar{\Pi}, \zeta)$ is a general price-and-hedge with wealth $\mathcal{W}$ of the hedging portfolio such that for $t \in [0, \bar{\tau}]$

$$\Big(\beta_t \Pi_t - \int_0^t \beta_s g_s(\Pi_s, \zeta_s) ds \Big) - \Big(\beta_t \mathcal{W}_t - \int_0^t \beta_s g_s(\mathcal{W}_s, \zeta_s) ds \Big) = \int_0^t \beta_s d\varepsilon_s \tag{21}$$

and hedging error ϱ such that for $t \in [0, \bar{\tau}]$

$$\begin{aligned} d\varrho_t &= d\varepsilon_t + \big(r_t \varrho_t + g(\Pi_t, \zeta_t) - g(\mathcal{W}_t, \zeta_t) \big) dt \; - \; \mathbb{1}_{\tau=\theta} (1 - \mathfrak{r}) (\bar{R}^f - R^f) \delta_\tau(dt) \\ &= d\varepsilon_t + \big(r_t \varrho_t + g(\Pi_t, \zeta_t) - g(\mathcal{W}_t, \zeta_t) \big) dt \\ &\qquad - \; \mathbb{1}_{\tau=\theta} (1 - \mathfrak{r}) \Big(\mathfrak{X}^+_{\tau-}(\mathcal{W}_{\tau-}, \zeta_{\tau-}) - \mathfrak{X}^+_{\tau-}(\Pi_{\tau-}, \zeta_{\tau-}) \Big) \delta_\tau(dt) \end{aligned} \tag{22}$$

(and $\varrho_0 = \varepsilon_0 = 0$).

Proof. Identity (21) immediately follows from (14), (18) and (19) (plus the fact that $\mathcal{W}_0 = \Pi_0$). Rewritten in term of the hedging error $\varrho = \Pi^* - \mathcal{W}$, Equation (14) for the value $\mathcal{W}$ of the hedging portfolio of $(\bar{\Pi}, \zeta)$ yields that for $t \in [0, \bar{\tau}]$

$$d\Pi_t^* = \big(r_t \mathcal{W}_t + g_t(\mathcal{W}_t, \zeta_t)\big)dt - dC_t + \zeta_t d\mathcal{M}_t + d\varrho_t. \tag{23}$$

Besides, the equation part (second line) in the $\mathbb{P}$-price-and-hedge BSDE (18) can be written in terms of the cost $d\varepsilon_t = d\nu_t - \zeta_t d\mathcal{M}_t$ in (19) as

$$d\Pi_t = \big(r_t \Pi_t + g_t(\Pi_t, \zeta_t)\big)dt - dC_t + \zeta_t d\mathcal{M}_t + d\varepsilon_t. \tag{24}$$

Since $\Pi_t - \Pi_t^* = \mathbb{1}_{t=\theta}(\bar{R}^f - R^f)$, substracting (24) from (23) yields (22). □

3.2 Arbitrage, Replication and Computational Issues

Assume first that it is possible to find a $\mathbb{P}$-price-and-hedge process $(\bar{\Pi}, \zeta)$ with a vanishing cost process $\varepsilon = 0$, and second that for this $(\bar{\Pi}, \zeta)$ and the related process Π, uniqueness holds for the following forward SDE in Y: $Y_0 = \Pi_0$ and for $t \in [0, \bar{\tau}]$:

$$d(\beta_t Y_t) - \beta_t g_t(Y_t, \zeta_t)dt = d(\beta_t \Pi_t) - \beta_t g_t(\Pi_t, \zeta_t)dt.$$

Via the BSDE machinery (see, e.g., El Karoui, Peng, and Quenez (1997)), the first assumption is typically met by application of a predictable representation property of $\mathcal{G}$-martingales (whenever available), whereas the second assumption is a technical requirement guaranteeing that Π and $\mathcal{W}$ coincide if they solve the same forward SDE. Under these assumptions, one gets by (21) with $\varepsilon = 0$ therein that $\mathcal{W} = \Pi$. It follows that $R^f = \bar{R}^f$, and therefore by (22) that $\varrho = \varepsilon = 0$. In this case the $\mathbb{P}$-price-and-hedge process $(\bar{\Pi}, \zeta)$ is thus a replicating strategy.

We refer the reader to Burgard and Kjaer (2010), Burgard and Kjaer (2011) and to the corresponding development in Bielecki, Crépey, and Rutkowski (2011), for practical examples of replication. Since replication being possible or not ultimately relies on a predictable representation property of $\mathcal{G}$-martingales, replicability typically holds not only for a particular contract, but for any financial derivative with $\mathcal{G}$-adapted and integrable cash-flows. We shall thus refer to this case henceforth as the "complete market" case.

In a more general, "incomplete" market, the cost ε of a $\mathbb{P}$-price-and-hedge $(\bar{\Pi}, \zeta)$, and in turn its hedging error ϱ, can only be reduced up to a level "proportional" to the "degree of incompleteness" of the primary market. The bank shortening the contract to the investor can thus only partially hedge its position, ending-up with a non-vanishing hedging error $\varrho_{\bar{\tau}}$.

Remark 3.6 (Arbitrage) In the complete market case or if $\mathfrak{r} = 1$, then the Dirac-driven term vanishes in (22). Under suitable conditions, one can then change the measure $\mathbb{P}$ into an equivalent measure $\mathbb{Q}$ such that the hedging error ϱ is a $\mathbb{Q}$-martingale. This excludes that $\varrho_{\bar{\tau}}$ could be non-negative almost surely and positive with positive probability. In conclusion a $\mathbb{P}$-price-and-hedge $(\bar{\Pi}, \zeta)$ cannot be an arbitrage in this case. On the opposite, in an incomplete market with moreover $\mathfrak{r} < 1$, a $\mathbb{P}$-price-and-hedge $(\bar{\Pi}, \zeta)$ is, in principle, arbitrable.

A non-arbitrable strategy would be a general price-and-hedge $(\bar{\Pi}, \zeta)$ such that the triplet $(\mathcal{W}, \bar{\Pi}, \zeta)$ in Definition 2.2 solves the related FBSDE, in the sense in particular that the hedging error ϱ would be a martingale under some probability measure $\mathbb{Q}$ equivalent to the historical measure $\widehat{\mathbb{P}}$. However in an incomplete market and with moreover $\mathfrak{r} < 1$

this FBSDE seems intractable. Our $\mathbb{P}$-price-and-hedge BSDE can be viewed as a simplified version of this theoretical FBSDE. The price to pay for this simplification is that it opens the door to an arbitrage (unless the market is complete or $\mathfrak{r} = 1$, in which case the $\mathbb{P}$-price-and-hedge BSDE and the above FBSDE are essentially equivalent). However we believe that this arbitrage is quite theoretical (the corresponding "free lunch" seems quite difficult to lock in).

In view of the above arbitrage, hedging and computational considerations, we restrict ourselves to $\mathbb{P}$-price-and-hedges in the sequel. For brevity we write henceforth "a price-and-hedge (Π, ζ)" when the related pair-process $(\bar{\Pi}, \zeta)$ is a $\mathbb{P}$-price-and-hedge. By price related to a hedge process ζ, we mean any process Π such that (Π, ζ) is a price-and-hedge (solves the BSDE (18)). Also in the sequel we simply call (18) the price BSDE, as opposed to CVA BSDEs to appear later in the paper.

These appellations are slightly abusive since given a $\mathbb{P}$-price-and-hedge $(\bar{\Pi}, \zeta)$, the actual price of the contract at time $\tau = \theta < T$ is $\bar{\Pi}_\theta = R^i$ and not $\Pi_\theta = R$. However this is immaterial since nobody cares about the price of the contract at time θ (which is given by the CSA close-out cash-flow R^i). What matters is the price for $t < \tau$, in which case $\Pi_t = \bar{\Pi}_t = \Pi^*_t$. Also, no confusion may arise between the "new" ($\mathbb{P}$-)price-and-hedge (Π, ζ) terminology and the "old" price-and-hedge $(\bar{\Pi}, \zeta)$ terminology of Subsection 2.4, because we now switched to the abstract setup of Subsection 2.5 and will not come back in the sequel to anything specifically related to Subsections 2.3-2.4.

From the BSDE point of view, a particularly simple situation will be the one where

$$g_t(\pi, \varsigma) = g_t(\pi)\,, \;\; \mathfrak{X}^+_t(\pi, \varsigma) = \mathfrak{X}^+_t(\pi). \tag{25}$$

We call it the fully swapped hedge case in reference to its financial interpretation under the funding specifications of Subsection 2.3.

Remark 3.7 (Symmetries) Similarly to the funding benefit coefficient g and the external funding recovery rate $\mathfrak{r}$ of the bank, one can introduce a funding cost coefficient $\overline{g}$ and an external funding recovery rate $\bar{\mathfrak{r}}$ for the investor. Note that in case where $\mathfrak{r} = \bar{\mathfrak{r}} = 1$ and $g = \overline{g}$ with $g = g(\pi)$ and $\overline{g} = \overline{g}(\pi)$, all cash-flows are symmetric from the point of view of the two parties. It is only in this case that the seller price of the bank will agree with the buyer price of the investor. Otherwise (and as soon in particular as the g-coefficients do depend on ς), funding induces an asymmetry between the two parties, resulting in a short bank price of the contract, different from its long investor price (and in turn different CVAs later in the paper). An example of symmetric funding costs is the setup of Fujii and Takahashi (2011), where excess funding costs reduce to collateral bases b and $\bar{b}$ as in our Subsection 2.3, and there are no funding close-out cash-flows involved (so $\mathfrak{r} = \bar{\mathfrak{r}} = 1$). Since collateral remuneration cash-flows are between the two parties of the contract (they do not involve external entities), collateral bases does not break the symmetry in our sense.[5]

Another notable specification, corresponding to the setup of Piterbarg (2010), is the linear case where $g = g(\pi)$ is affine and $\mathfrak{r}$ is equal to one. The bank has then a common buyer and seller price. Under the funding specifications of Subsection 2.3, the linear case corresponds to $b = \bar{b}$ and $\lambda = \bar{\lambda}$.

[5]Fujii and Takahashi (2011) consider in their paper a different notion of symmetry, which may be broken even in their setup.

Finally the one-curve setup corresponds to the case where $\mathfrak{r} = 1$ and all the bases are equal to 0, so $g = c = 0$. The only funding rate[6] in the economy is then the risk-free interest rate r.

These various specifications are discussed in detail in Bielecki, Crépey, and Rutkowski (2011).

4 CVA

Having identified the price BSDE (18) as key in the modeling of counterparty risk under funding constraints, we devote the sequel of this paper to the study of this BSDE. Again, this BSDE is made non-standard by the random terminal time $\bar{\tau}$, the dependence of the terminal condition R in $(\Pi_{\tau-}, \zeta_{\tau-})$, the dividend term dC_t, and the fact that it is not driven by an explicit set of fundamental martingales like Brownian motions and/or compensated jump measures.

Interestingly enough, the notion of CVA, which recently emerged for practical reasons in banks, will appear as a useful mathematical device to cope with these technicalities. Since the pioneering works of Damiano Brigo and his coauthors (see for instance Brigo and Capponi (2010) in a context of bilateral counterparty credit risk), it is well understood that the CVA can be viewed as an option, the so-called Contingent Credit Default Swap (CCDS), on the clean value of the contract. This Section extends to a non linear multiple-curve setup the notion of CVA and its representation as the price of a CCDS. In our setup the CVA actually accounts not only for counterparty risk, but also for excess funding costs. The CCDS is then a dividend-paying option, where the dividends correspond to these costs.

4.1 Bilateral Reduced Form Setup

We assume henceforth that the model filtration $\mathcal{G}$ can be decomposed into $\mathcal{G} = \mathcal{F} \vee \mathcal{H}^{\theta} \vee \mathcal{H}^{\overline{\theta}}$, where $\mathcal{F}$ is some reference filtration and $\mathcal{H}^{\theta}$ and $\mathcal{H}^{\overline{\theta}}$ stand for the natural filtrations of θ and $\overline{\theta}$. Let also $\bar{\mathcal{G}} = \mathcal{F} \vee \mathcal{H}$, where $\mathcal{H}$ is the natural filtration of $\bar{\tau}$ (or, equivalently, of τ). We refer the reader to Bielecki and Rutkowski (2002) for the standard material regarding the reduced-form approach in credit risk modeling. The Azéma supermartingale associated with τ is the process G defined by, for $t \in [0, T]$,

$$G_t = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t). \tag{26}$$

We assume that G is a positive, continuous and non-increasing process. This is a classical, slight relaxation of the so-called immersion or $(\mathcal{H})$-hypothesis of $\mathcal{F}$ into $\bar{\mathcal{G}}$, see Jeanblanc and Le Cam (2008) for a detailed discussion.

Lemma 4.1 **(i)** *An $\mathcal{F}$-local martingale stopped at τ is a $\bar{\mathcal{G}}$-local martingale, and a $\bar{\mathcal{G}}$-local martingale stopped at τ is a $\mathcal{G}$-local martingale.*
(ii) *An $\mathcal{F}$-adapted càdlàg process cannot jump at τ. One thus has that $\Delta X_\tau = 0$ almost surely, for every $\mathcal{F}$-adapted càdlàg process X.*

Proof. **(i)** Since τ has a positive, continuous and non-increasing Azéma supermartingale, it is known from Elliot, Jeanblanc, and Yor (2000) that an $\mathcal{F}$-local martingale stopped at τ, is a $\bar{\mathcal{G}}$-local martingale. Besides, two successive applications of the Dellacherie-Meyer

[6] Assuming the existence of a riskless asset with growth rate r.

Key Lemma (see for instance Bielecki and Rutkowski (2002)) yield that for every $\bar{\mathcal{G}}$-adapted integrable process M, one has for every $0 \leq s \leq t \leq T$

$$\begin{aligned}\mathbb{E}\left(M_{t\wedge\tau}\,|\,\mathcal{G}_s\right) &= \mathbb{1}_{s\geq\tau}M_\tau \;+\; \mathbb{1}_{s<\tau}\frac{\mathbb{E}\left(M_{t\wedge\tau}\mathbb{1}_{s<\tau}\,|\,\mathcal{F}_s\right)}{\mathbb{P}\left(s<\tau\,|\,\mathcal{F}_s\right)}\\ &= \mathbb{1}_{s\geq\tau}M_{s\wedge\tau} \;+\; \mathbb{1}_{s<\tau}\mathbb{E}\left(M_{t\wedge\tau}\,|\,\bar{\mathcal{G}}_s\right),\end{aligned}$$

which, in case M is a $\bar{\mathcal{G}}$-martingale, boils down to $M_{s\wedge\tau}$. A $\bar{\mathcal{G}}$-martingale stopped at τ is thus a $\mathcal{G}$-martingale. A standard localization argument then yields that a $\bar{\mathcal{G}}$-local martingale stopped at τ is a $\mathcal{G}$-local martingale.
(ii) As G is continuous, τ avoids $\mathcal{F}$-stopping times in the sense that $\mathbb{P}(\tau=\sigma)=0$ for any $\mathcal{F}$-stopping time σ (see, e.g., Coculescu and Nikeghbali (2011)). Besides, by Theorem 4.1 page 120 in He, Wang, and Yan (1992), there exists a sequence of $\mathcal{F}$-stopping times exhausting the jump times of an $\mathcal{F}$-adapted càdlàg process. This proves part (ii). □

4.2 Clean Price

In the sequel, the risk-free discount factor process β, or equivalently the risk-free short rate process r, and the clean dividend process D, are assumed to be $\mathcal{F}$-adapted. In order to define the related CVA process Θ, we now introduce the clean price process P of the contract. The clean price process is a fictitious, instrumental value process, which corresponds to the price of the contract without counterparty risk nor excess funding costs. In the present bilateral reduced-form setup, the clean price process P of the contract is naturally defined by, for $t \in [0,T]$,

$$\beta_t P_t = \mathbb{E}\left(\int_t^T \beta_s dD_s \,\Big|\, \mathcal{F}_t\right). \tag{27}$$

The discounted cumulative clean price,

$$\beta\widehat{P} := \beta P + \int_{[0,\cdot]} \beta_t dD_t,$$

is thus an $\mathcal{F}$-martingale. The corresponding clean $\mathcal{F}$-martingale M on $[0,T]$, to be compared with the $\mathcal{G}$-martingale component ν of Π in the price BSDE (18), is defined by, for $t \in [0,T]$,

$$dM_t = dP_t + dD_t - r_t P_t dt, \tag{28}$$

along with the terminal condition $P_T = 0$.

Lemma 4.2 **(i)** *The clean price process P satisfies for $t \in [0,\bar{\tau}]$,*

$$\beta_t P_t = \mathbb{E}_t\big[\int_t^{\bar{\tau}} \beta_s dD_s + \beta_{\bar{\tau}} P_{\bar{\tau}}\big]. \tag{29}$$

(ii) *There can be no promised dividend of the contract nor jump of the clean price process at the default time τ, so $\Delta D_\tau = \Delta P_\tau = 0$ almost surely.*

Proof. **(i)** Since the discounted cumulative clean price $\beta\widehat{P}$ is an $\mathcal{F}$-martingale, so by Lemma 4.1(i), process $\beta\widehat{P}$ stopped at τ is a $\mathcal{G}$-martingale, integrable by standing assumption in this paper, thus (29) follows.
(ii) Since all our semimartingales are taken in a càdlàg version, then by Lemma 4.1(ii) the $\mathcal{F}$-semimartingales D and P cannot jump at τ. □

4.3 CSA Close-Out Cash-Flow

Before moving to CVA we now need to specify R^i in the CSA close-out cash-flow $\mathbb{1}_{\tau<T}R^i$. Toward this end we define a $\mathcal{G}_\tau$-measurable random variable χ as

$$\chi = Q_\tau - \Gamma_\tau \tag{30}$$

where Q denotes the so-called CSA fair value process of the contract, expectation of future cash-flows or so, in a sense defined by the CSA. From the point of view of financial interpretation, χ represents the (algebraic) debt of the bank to the investor at time τ, given as the CSA fair value Q_τ less the margin amount Γ_τ (since the latter is 'instantaneously transferred' to the investor at time τ). We then set

$$R^i = \Gamma_\tau + \mathbb{1}_{\tau=\theta}\big(\rho\chi^+ - \chi^-\big) - \mathbb{1}_{\tau=\bar{\theta}}\big(\bar{\rho}\chi^- - \chi^+\big) - \mathbb{1}_{\theta=\bar{\theta}}\chi \tag{31}$$

in which the $[0,1]$-valued $\mathcal{G}_\theta$- and $\mathcal{G}_{\bar{\theta}}$-measurable random variables ρ and $\bar{\rho}$ denote the recovery rates of the bank and the investor to each other. So:

- If the investor defaults at time $\overline{\theta} < \theta \wedge T$, then $R^i = \Gamma_\tau - \big(\bar{\rho}\chi^- - \chi^+\big)$,
- If the bank defaults at time $\theta < \overline{\theta} \wedge T$, then $R^i = \Gamma_\tau + \rho\chi^+ - \chi^-$,
- If the bank and the investor default simultaneously at time $\theta = \overline{\theta} < T$, then $R^i = \Gamma_\tau + \rho\chi^+ - \bar{\rho}\chi^-$.

Note that the margin amount Γ typically depends on Q, often in a rather path dependent way. We refer the reader to Bielecki and Crépey (2011) regarding this and other, theoretically minor, yet practically important issues, like haircut, re-hypothecation risk and segregation, or the cure period. All these can also be accommodated in our setup.

4.4 CVA Representation

With R^i thus specified in $R = R^i - \mathbb{1}_{\tau=\theta}R^f$, we are now ready to introduce the CVA process Θ of the bank. Recall from Definition 3.4 that unless $\mathfrak{r} = 1$, the terminal condition $R = R^i - \mathbb{1}_{\tau=\theta}R^f$ in a solution (Π, ζ) to the price BSDE (18), implicitly depends on $(\Pi_{\tau-}, \zeta_{\tau-})$, via $R^f = (1-\mathfrak{r})\widehat{\mathfrak{X}}^+_{\theta-}$, where $\widehat{\mathfrak{X}}_t$ is used as a shorthand for $\mathfrak{X}_t(\Pi_t, \zeta_t)$. Also note that

$$\begin{aligned}
&P_\tau - R \\
&\quad = P_\tau - Q_\tau + \chi - \mathbb{1}_{\tau=\theta}\big(\rho\chi^+ - \chi^-\big) + \mathbb{1}_{\tau=\bar{\theta}}\big(\bar{\rho}\chi^- - \chi^+\big) + \mathbb{1}_{\theta=\bar{\theta}}\chi + \mathbb{1}_{\tau=\theta}(1-\mathfrak{r})\widehat{\mathfrak{X}}^+_{\theta-} \\
&\quad = P_\tau - Q_\tau + \mathbb{1}_{\tau=\theta}\Big((1-\rho)\chi^+ + (1-\mathfrak{r})\widehat{\mathfrak{X}}^+_{\tau-}\Big) - \mathbb{1}_{\tau=\bar{\theta}}(1-\bar{\rho})\chi^-.
\end{aligned} \tag{32}$$

One can then state the following

Definition 4.3 *Given a solution* (Π, ζ) *to the price BSDE* (18), *the corresponding CVA process* Θ *is defined by* $\Theta = P - \Pi$ *on* $[0, \bar{\tau}]$. *In particular,* $\Theta_{\bar{\tau}} = \mathbb{1}_{\tau<T}\xi$, *where*

$$\begin{aligned}
\xi &:= P_\tau - R \\
&= P_\tau - Q_\tau + \mathbb{1}_{\tau=\theta}\Big((1-\rho)\chi^+ + (1-\mathfrak{r})\widehat{\mathfrak{X}}^+_{\tau-}\Big) - \mathbb{1}_{\tau=\bar{\theta}}(1-\bar{\rho})\chi^-.
\end{aligned} \tag{33}$$

Remark 4.4 The clean contract is assumed to be funded at the risk-free rate r_t. The clean price P is thus not only clean of counterparty risk, but also of excess funding costs. Our Credit Valuation Adjustment (CVA) should thus rather be called Credit and Funding Value Adjustment. We stick to the name Credit Valuation Adjustment (CVA) for simplicity.

The following result extends to the multiple-curve setup, the one-curve CVA representation results of Brigo and Capponi (2010) or Assefa et al. (2011). Note that in a multiple-curve setup this representation, in the form of Equation (34) below, is implicit. Namely, the right-hand side of (34) involves Θ and ζ, via R in ξ and via g in the integral term. This is at least the case unless $\mathfrak{r}=1$ and a funding coefficient $g(\pi,\varsigma)=g(\pi)$ is linear in π, so that one can get rid of these dependencies by a suitable adjustment of the discount factor (see Example 5.4).

Proposition 4.5 *Let us be given a hedge ζ and $\mathcal{G}$-semimartingales Π and Θ such that $\Theta = P - \Pi$ on $[0,\bar\tau]$. The pair-process (Π,ζ) is a solution to the price BSDE (18) if and only if Θ satisfies for $t\in[0,\bar\tau]$*

$$\beta_t\Theta_t = \mathbb{E}_t\Big[\beta_{\bar\tau}\mathbb{1}_{\tau<T}\xi + \int_t^{\bar\tau}\beta_s g_s(P_s-\Theta_s,\zeta_s)ds\Big]. \tag{34}$$

Proof. Recall $P_T=0$ and $\Delta D_\tau=0$, so $P_{\bar\tau}=\mathbb{1}_{\tau<T}P_\tau$ and $\mathbb{1}_{\tau<T}\Delta D_{\bar\tau}=0$. Taking the difference between (29) and (20), one thus gets for $t\in[0,\bar\tau]$

$$\begin{aligned}\beta_t\left(P_t-\Pi_t\right) &= \mathbb{E}_t\Big[\beta_{\bar\tau}\mathbb{1}_{t<\bar\tau}\big(\Delta D_{\bar\tau}-\mathbb{1}_{\bar\tau<\tau}\Delta D_{\bar\tau}\big)+\int_t^{\bar\tau}\beta_s g_s(\Pi_s,\zeta_s)ds+\beta_{\bar\tau}\mathbb{1}_{\tau<T}\left(P_\tau-R\right)\Big]\\ &= \mathbb{E}_t\Big[\int_t^{\bar\tau}\beta_s g_s(\Pi_s,\zeta_s)ds+\beta_{\bar\tau}\mathbb{1}_{\tau<T}\xi\Big]\end{aligned}$$

which is Equation (34) in Θ. □

One thus recovers in the multiple-curve setup, the general interpretation of the CVA as the price of the so-called contingent credit default swap (CCDS), which is an option on the debt χ (sitting via R in ξ) of the bank to the investor at time τ. However, in a multiple-curve setup, this is a dividend-paying option, paying not only the amount ξ at time $\tau<T$, but also dt-dividends at rate $g_t(P_t-\Theta_t,\zeta_t)-r_t\Theta_t$ between times 0 and $\bar\tau$.

Example 4.6 Under the funding specifications of Subsection 2.3 with the coefficient g given by (16), and in the fully swapped hedge case with $\bar{s}=\emptyset$, the CVA representation of Equation (34) writes

$$\begin{aligned}\beta_t\Theta_t &= \mathbb{E}_t\big[\mathbb{1}_{\tau<T}\beta_{\bar\tau}\left(P_\tau-Q_\tau\right)\big]\\ &\quad+\mathbb{E}_t\Big[\mathbb{1}_{\tau=\theta<T}\beta_{\bar\tau}(1-\rho)(Q_\tau-\Gamma_\tau)^+\Big]\\ &\quad-\mathbb{E}_t\Big[\mathbb{1}_{\tau=\bar\theta<T}\beta_{\bar\tau}(1-\bar\rho)(Q_\tau-\Gamma_\tau)^-\Big]\\ &\quad+\mathbb{E}_t\Big[\mathbb{1}_{\tau=\theta<T}\beta_{\bar\tau}(1-\mathfrak{r})(P_{\tau-}-\Theta_{\tau-}-\Gamma_{\tau-})^-+\int_t^{\bar\tau}\beta_s\Big(b_t\Gamma_t^+ + \lambda_s\left(P_s-\Gamma_s-\Theta_s\right)^+\Big)ds\Big]\\ &\quad-\mathbb{E}_t\Big[\int_t^{\bar\tau}\beta_s\Big(\bar b_t\Gamma_t^- + \bar\lambda_s\left(P_s-\Gamma_s-\Theta_s\right)^-\Big)ds\Big].\end{aligned}$$

From the perspective of the bank, the five lines in this decomposition of the (net) CVA Θ, can respectively be interpreted as a replacement cost, a positive debt value adjustment, a negative (non-algebraic, strict) credit value adjustment, a positive excess funding benefit and a negative excess funding cost.

4.4.1 CCDS Static Hedging Interpretation

If the clean contract with price process P and the CCDS were traded assets, a static replication scheme of the bank shortening the contract to the investor and funding it by its external funder would consist in, given a price process Π solving the price BSDE (18) for $\zeta = 0$:

- At time 0, using the proceeds Π_0 from the shortening of the contract and $\Theta_0 = P_0 - \Pi_0$ from the shortening of a CCDS to buy the clean contract at price P_0,
- On the time interval $(0, \bar{\tau})$, holding P and $(-\Theta)$, transferring to the investor all the dividends dD_t which are perceived by the bank through its owning of P, and incurring dt-costs at rate $r_t P_t + g(\Pi_t, 0) - r_t \Theta_t = g_t(\Pi_t, 0) + r_t \Pi_t$. These costs exactly match the dt-funding benefits from the short naked (non dynamically hedged) position in the contract.

Thus, at time $\bar{\tau}$:

- If $\bar{\tau} = \tau < T$, the bank is left with an amount $P_\tau - \Theta_\tau = P_\tau - \xi = R$, which is exactly the close-out cash-flow it must deliver to the investor and to its funder,
- If $\bar{\tau} = T$, there are no cash-flows at $\bar{\tau}$.

In both cases the bank is left break-even at $\bar{\tau}$. But of course this static buy-and-hold replication strategy is not practical, since neither the clean contract nor the CCDS are traded assets. One is thus led to dynamic hedging. Here a question arises whether one should try to hedge the contract globally, or[7] to hedge the clean contract P separately from the CVA component Θ of Π. In order to address these issues one needs to dig further into the analysis of the cost process $d\varepsilon = d\nu - \zeta d\mathcal{M}$ of a price-and-hedge (Π, ζ).

5 Pre-Default BSDE Modeling

We develop in this Section a practical reduced-form CVA BSDE approach to the problem of pricing and hedging counterparty risk under funding constraints. Counterparty risk and funding corrections to the clean price-and-hedge of the portfolio are obtained as the solution to a pre-default BSDE stated with respect to the reference filtration, in which defaultability of the two parties only shows up through their default intensities.

5.1 Reduction of Filtration

Let us call the CVA BSDE of the bank, the $\mathcal{G}$-BSDE on the random time interval $[0, \bar{\tau}]$, with terminal condition $\mathbb{1}_{\tau<T}\xi$ at $\bar{\tau}$, and driver coefficient $g_t(P_t - \vartheta, \varsigma) - r_t\vartheta$, $\vartheta \in \mathbb{R}, \varsigma \in \mathbb{R}^d$. The following Lemma rephrases Proposition 4.5 in BSDE terms.

Lemma 5.1 *Given $\mathcal{G}$-semimartingales Π and Θ summing-up to P and a hedge ζ, (Π, ζ) solving the price BSDE is equivalent to (Θ, ζ) solving the corresponding CVA BSDE.*

Passing from the price BSDE in (Π, ζ) to the CVA BSDE in (Θ, ζ), allows one to get rid of the dC_t-term (promised dividend of the contract) in (18). This makes the CVA BSDE more convenient than the price BSDE (18). We assume in the sequel that:

[7] If any freedom in this is left by the internal organization of the bank.

- The $\mathcal{G}$-semimartingale (margin amount) Γ_t is $\mathcal{F}$-adapted. This makes financial sense since securities eligible as collateral are only cash or very basic securities which should not be affected by the default of either party. By Lemma 4.1(ii), one then almost surely has that $\Delta\Gamma_\tau = 0$;

- The CSA fair value process Q is given as a left-limit process, and is thus $\mathcal{G}$-predictable. This makes financial sense since what Q_τ is really meant to be is a notion of fair value of the contract right before the default time τ of either party;

- The recovery rates ρ, $\bar{\rho}$ and $\mathfrak{r}$ can be represented as ρ_θ, $\bar{\rho}_{\overline{\theta}}$ and $\mathfrak{r}_\theta$, for some $\mathcal{G}$-predictable processes ρ_t, $\bar{\rho}_t$ and $\mathfrak{r}_t$.

By Theorem 67.b in Dellacherie and Meyer (1975), the $\mathcal{G}_{\tau-}$-measurable random variables $\mathbb{P}(\tau = \theta \,|\, \mathcal{G}_{\tau-})$ and $\mathbb{P}(\tau = \overline{\theta} \,|\, \mathcal{G}_{\tau-})$ can be represented as p_τ and $\overline{p}_\tau$, for some $\mathcal{G}$-predictable process p and $\overline{p}$. Since $\Delta\Gamma_\tau = 0$, there exists in virtue of the same theorem an $\mathcal{F}$-predictable process with the same value as Γ at τ; in other words one can thus henceforth assume that process Γ is in fact $\mathcal{F}$-predictable.

The debt χ of the bank to the investor, and, given a price-and-hedge (Π, ζ), the terminal payoff ξ of a CCDS, are then the values at time τ of the $\mathcal{G}$-predictable process χ_t and of the $\mathcal{G}$-progressively measurable process ξ_t such that for $t \in [0, T]$

$$\begin{aligned} \chi_t &= Q_t - \Gamma_t \\ \xi_t &= (P_t - Q_t) + \mathbb{1}_{t \geq \theta}\Big((1-\rho_t)\chi_t^+ + (1-\mathfrak{r}_t)\widehat{\mathfrak{X}}_{t-}^+\Big) - \mathbb{1}_{t \geq \overline{\theta}}(1-\bar{\rho}_t)\chi_t^- \end{aligned} \tag{35}$$

where $\widehat{\mathfrak{X}}_t$ is used as a shorthand for $\mathfrak{X}_t(\Pi_t, \zeta_t)$. Let further for $t \in [0, T]$, $\pi \in \mathbb{R}$ and $\varsigma \in \mathbb{R}^d$

$$\bar{\xi}_t(\pi, \varsigma) = (P_t - Q_t) + p_t\Big((1-\rho_t)\chi_t^+ + (1-\mathfrak{r}_t)\mathfrak{X}_{t-}^+(\pi, \varsigma)\Big) - \overline{p}_t(1-\bar{\rho}_t)\chi_t^- . \tag{36}$$

Let also J denote the non-default indicator process such that $J_t = \mathbb{1}_{t<\tau}$ for $t \in [0, \bar{\tau}]$. Observe that given a hedge ζ, Θ solving Equation (34) over $[0, \bar{\tau}]$ is equivalent to $\Theta = J\bar{\Theta} + (1 - J)\mathbb{1}_{\tau<T}\xi$, for a process $\bar{\Theta}$ such that for $t \in [0, \bar{\tau}]$

$$\beta_t\bar{\Theta}_t = \mathbb{E}_t\Big[\beta_{\bar{\tau}}\mathbb{1}_{\tau<T}\bar{\xi}_\tau(P_{\tau-} - \bar{\Theta}_{\tau-}, \zeta_{\tau-}) + \int_t^{\bar{\tau}} \beta_s g_s(P_s - \bar{\Theta}_s, \zeta_s)ds\Big]. \tag{37}$$

To simplify the problem further, we now introduce an equivalent pre-default CVA BSDE over $[0, T]$, relative to the pre-default filtration $\mathcal{F}$. The following result is classical, see for instance Bielecki, Crépey, Jeanblanc, and Rutkowski (2009) for precise references. We denote by Y_- the left-limiting process (whenever well-defined) of a process Y.

Lemma 5.2 *For any $\mathcal{G}$-adapted, respectively $\mathcal{G}$-predictable process X over $[0, T]$, there exists a unique $\mathcal{F}$-adapted, respectively $\mathcal{F}$-predictable, process $\widetilde{X}$ over $[0, T]$, called the pre-default value process of X, such that $JX = J\widetilde{X}$, respectively $J_-X = J_-\widetilde{X}$ over $[0, T]$.*

Given the structure of the data, we may therefore assume without loss of generality that process $g_t(P_t - \vartheta, \varsigma)$ is $\mathcal{F}$-progressively measurable for every $\vartheta \in \mathbb{R}$, $\varsigma \in \mathbb{R}^d$, and that all the processes (including for instance p and $\overline{p}$) which appear as building blocks in $\bar{\xi}$, are $\mathcal{F}$-predictable. We assume further that the Azéma supermartingale G of τ is time-differentiable. This allows one to define the hazard intensity $\gamma_t = -\frac{d \ln G_t}{dt}$ of τ, so $G_t = e^{-\int_0^t \gamma_s ds}$. We then define the credit-risk-adjusted-interest-rate $\widetilde{r}$ and the credit-risk-adjusted-discount-factor $\widetilde{\beta}$ as, for $t \in [0,T]$,

$$\widetilde{r}_t = r_t + \gamma_t \,,\ \widetilde{\beta}_t = \beta_t G_t = \beta_t \exp(-\int_0^t \gamma_s ds) = \exp(-\int_0^t \widetilde{r}_s ds).$$

One can then state the following

Definition 5.3 The pre-default CVA BSDE of the bank is the $\mathcal{F}$-BSDE in $(\widetilde{\Theta}, \zeta)$ on $[0,T]$ with a null terminal condition at T, and with driver coefficient

$$\widetilde{g}_t(P_t - \vartheta, \varsigma) = g_t(P_t - \vartheta, \varsigma) + \gamma_t \widetilde{\xi}_t(P_t - \vartheta, \varsigma) - \widetilde{r}_t \vartheta \tag{38}$$

where $\widetilde{\xi}_t(\pi, \varsigma)$ denotes for every $\pi \in \mathbb{R}$ and $\varsigma \in \mathbb{R}^d$ the $\mathcal{F}$-progressively measurable process defined by, for $t \in [0,T]$

$$\widetilde{\xi}_t(\pi, \varsigma) = (P_t - Q_t) + p_t\Big((1-\rho_t)\chi_t^+ + (1-\mathfrak{r}_t)\mathfrak{X}_t^+(\pi,\varsigma)\Big) - \overline{p}_t(1-\bar{\rho}_t)\chi_t^- . \tag{39}$$

An $\mathcal{F}$-special semimartingale $\widetilde{\Theta}$ and a hedge ζ to the contract, thus solve the pre-default CVA BSDE if and only if

$$\begin{cases} \widetilde{\Theta}_T = 0, \text{ and for } t \in [0,T]: \\ -d\widetilde{\Theta}_t = \widetilde{g}_t(P_t - \widetilde{\Theta}_t, \zeta_t)dt - d\widetilde{\mu}_t \end{cases} \tag{40}$$

where $\widetilde{\mu}$ is the $\mathcal{F}$-local martingale component of $\widetilde{\Theta}$. Or equivalently to the second line in (40): For $t \in [0,T]$,

$$-d(\widetilde{\beta}_t \widetilde{\Theta}_t) = \widetilde{\beta}_t\Big(g_t(P_t - \widetilde{\Theta}_t, \zeta_t) + \gamma_t \widetilde{\xi}_t(P_t - \widetilde{\Theta}_t, \zeta_t)\Big)dt - \widetilde{\beta}_t d\widetilde{\mu}_t. \tag{41}$$

Remark 5.4 (Linear Case) In the linear case with $\mathfrak{r} = 1$ and $g_t(P - \vartheta, \varsigma) = g_t^*(P) - \lambda_t^* \vartheta$, the CVA equations (34) and (41) respectively boil down to the explicit representations

$$\beta_t^* \Theta_t = \mathbb{E}_t\Big[\beta_{\bar\tau}^* \mathbb{1}_{\tau<T}\xi + \textstyle\int_t^{\bar\tau} \beta_s^* g_s^*(P_s)ds\Big] \tag{42}$$

$$\widetilde{\beta}_t^* \widetilde{\Theta}_t = \mathbb{E}\Big[\textstyle\int_t^T \widetilde{\beta}_s^*\Big(g_s^*(P_s^*) + \gamma_s \widetilde{\xi}_s(P_s - \widetilde{\Theta}_s, \zeta_s)\Big)ds \,\Big|\, \mathcal{F}_t\Big] \tag{43}$$

for the funding-cost-adjusted-discount-factors

$$\beta_t^* = \exp(-\int_0^t (r_s + \lambda_s^*)ds)\,,\ \ \widetilde{\beta}_t^* = \exp(-\int_0^t (\widetilde{r}_s + \lambda_s^*)ds).$$

Remark 5.5 (CSA Fair Valuation and Collateralization Schemes) It is implicitly understood above that the CSA fair value process Q, present in ξ via $\chi = Q - \Gamma$ in (35), is an exogenous process, as in the standard clean CSA fair value scheme $Q = P_-$. An a priori unusual situation from this point of view, yet one which is sometimes considered in the counterparty risk literature, at least in the classical one-curve setup, is the so-called pre-default CSA fair value scheme $Q = \Pi_-$. In a multiple-curve setup with counterparty risky prices which differ from the perspectives of the two parties, this scheme seems hardly workable in practice. Yet from a BSDE point of view this dependence in Π of Q can be accounted for at no harm, by letting $Q = P_{-} - \widetilde{\Theta}_{-}$ everywhere in the coefficient $\widetilde{g}_t$ of the pre-default CVA BSDE (40).

Note however that in order to meet ISDA requirements, a real-life collateralization scheme Γ is typically path dependent in Q (see Section 3.2 of Bielecki and Crépey (2011)). Under the pre-default CSA fair valuation scheme, and in case of a path dependent collateralization, one ends-up with a time-delayed BSDE with a coefficient depending on the past of $\widetilde{\Theta}$. This raises a mathematical difficulty of the pre-default CSA fair valuation scheme since even for a Lipschitz coeffficient, a time-delayed BSDE may only have a solution for T small enough, depending on the Lipschitz constant of the coefficient (see Delong and Imkeller (2010)).

5.2 Modeling Assumption

From now on, our approach to deal with the price BSDE (18) will consist in modeling the counterparty risky price process Π via the corresponding pre-default CVA process $\widetilde{\Theta}$. In this Section we work under the following

Assumption 5.6 The pre-default CVA BSDE (40) admits a solution $(\widetilde{\Theta}, \zeta)$.

We shall now examine the consequences of this assumption regarding existence of a solution (Π, ζ) to the price BSDE (18) in this Subsection, and analysis of the cost process ε of (Π, ζ) in Subsection 5.3.

By standard arguments, the compensated jump-to-default process $H_t = (1 - J_t) - \int_0^t J_s \gamma_s ds$, is a $\bar{\mathcal{G}}$-martingale over $[0, T]$. Since it is stopped at τ, Lemma 4.1(i) implies that it is also a $\mathcal{G}$-martingale. Let $\widehat{\xi}_t$ be a shorthand for $\widetilde{\xi}_t(P_t - \widetilde{\Theta}_t, \zeta_t)$. The following results (first line of (46) in particular) are key in the sequel.

Proposition 5.7 *Under Assumption 5.6:*
(i) *The pair* (Θ, ζ) *with* Θ *defined over* $[0, \bar{\tau}]$ *as*

$$\Theta := J\widetilde{\Theta} + (1 - J)\mathbb{1}_{\tau<T}\xi \tag{44}$$

solves the CVA BSDE (34) *over* $[0, \bar{\tau}]$. *Therefore, the pair* (Π, ζ) *with*

$$\Pi := P - \Theta = J(P - \widetilde{\Theta}) + (1 - J)\mathbb{1}_{\tau<T}R \tag{45}$$

solves the price BSDE (18) *over* $[0, \bar{\tau}]$*;*
(ii) *The* $\mathcal{G}$*-martingale component* ν *of the counterparty risky price* $\Pi = P - \Theta$ *and the* $\mathcal{G}$*-martingale component* $\mu = M - \nu$ *of* Θ,[8] *satisfy for* $t \in [0, \bar{\tau}]$*:*

$$\begin{aligned} d\mu_t &= d\widetilde{\mu}_t - \Big((\xi_t - \widetilde{\Theta}_t)dJ_t + \gamma_t(\widehat{\xi}_t - \widetilde{\Theta}_t)dt\Big) \\ d\nu_t &= d\widetilde{\nu}_t - \Big((R_t - \widetilde{\Pi}_t)dJ_t + \gamma_t(\widetilde{R}_t - \widetilde{\Pi}_t)dt\Big)\,. \end{aligned} \tag{46}$$

Here $\widetilde{\Pi} := P - \widetilde{\Theta}$ *is the pre-default value process of* Π, $\widetilde{\nu} := M - \widetilde{\mu}$ *is an* $\mathcal{F}$*-local martingale component of* $\widetilde{\Pi}$, *and the* $\mathcal{G}$*-progressively measurable process* R_t *and the* $\mathcal{F}$*-progressively measurable process* $\widetilde{R}_t$ *are defined by, for* $t \in [0, T]$,

$$\begin{aligned} R_t &= \Gamma_t + \mathbb{1}_{t\geq\theta}\Big((\rho_t\chi_t^+ - \chi_t^-) - (1 - \mathfrak{r}_t)\widehat{\mathfrak{X}}_{t-}^+\Big) - \mathbb{1}_{t\geq\bar{\theta}}\big(\bar{\rho}_t\chi_t^- - \chi_t^+\big) - \mathbb{1}_{t\geq\theta=\bar{\theta}}\chi_t \\ \widetilde{R}_t &= \Gamma_t + p_t\Big((\rho_t\chi_t^+ - \chi_t^-) - (1 - \mathfrak{r}_t)\widehat{\mathfrak{X}}_t^+\Big) - \bar{p}_t\big(\bar{\rho}_t\chi_t^- - \chi_t^+\big) - q_t\chi_t \end{aligned} \tag{47}$$

[8]Recall (18) and (28) for the definition of ν and M.

in which $\widehat{\mathfrak{X}}_t$ *stands as a shorthand for* $\mathfrak{X}_t(\widetilde{\Pi}_t, \zeta_t)$, *and* q *in* $\widetilde{R}$ *is an* $\mathcal{F}$*-predictable process such that* $q_\tau = \mathbb{P}(\theta = \overline{\theta} \,|\, \mathcal{G}_{\tau-})$.

Proof. **(i)** Standard reduction-of-filtration computations[9] exploiting the pre-default CVA BSDE (40) which is solved by $(\widetilde{\Theta}, \zeta)$ over $[0, T]$, show via (37) that (Θ, ζ) solves the CVA BSDE (34) over $[0, \bar{\tau}]$. By Lemma 5.1, the pair (Π, ζ), where $\Pi := P - \Theta$, thus solves the price BSDE (18). Also recall $P_T = 0$, which justifies the right-hand side identity in (45).
(ii) Let us introduce the Doléans-Dade $\mathcal{G}$-martingale $\mathcal{E}$ such that for $t \in [0, \bar{\tau}]$

$$\mathcal{E}_t = J_t G_t^{-1} = 1 - \int_0^t \mathcal{E}_{u-} \, dH_u.$$

In the present intensity setup with $G_t = e^{-\int_0^t \gamma_s ds}$, one has for $t \in [0, \bar{\tau}]$

$$\widetilde{\beta}_t \mathcal{E}_t = \beta_t J_t \, , \;\; \widetilde{\beta}_t \mathcal{E}_{t-} = \beta_t J_{t-}$$

and therefore

$$d(\beta_t \Theta_t) = d(\mathcal{E}_t \widetilde{\beta}_t \widetilde{\Theta}_t) + \beta_t \xi \delta_\tau(dt). \tag{48}$$

It may happen that the $\mathcal{F}$-semimartingale $(\widetilde{\beta}\widetilde{\Theta})$ fails to be also a $\mathcal{G}$-semimartingale, so a direct application of the $\mathcal{G}$-integration by parts formula to $(\mathcal{E}\widetilde{\beta}\widetilde{\Theta})$ is not possible. However, by Lemma 4.1, the process $\widetilde{\beta}\widetilde{\Theta}$ stopped at τ is a $\mathcal{G}$-semimartingale. It is also clear that $\mathcal{E}\widetilde{\beta}\widetilde{\Theta} = \mathcal{E}\widetilde{\beta}_{\cdot\wedge\tau}\widetilde{\Theta}_{\cdot\wedge\tau}$. Hence by applying the integration by parts formula to $(\mathcal{E}\widetilde{\beta}_{\cdot\wedge\tau}\widetilde{\Theta}_{\cdot\wedge\tau})$, we obtain since the $\mathcal{F}$-semimartingale $\widetilde{\beta}\widetilde{\Theta}$ cannot jump at τ

$$d(\mathcal{E}_t \widetilde{\beta}_{t\wedge\tau} \widetilde{\Theta}_{t\wedge\tau}) = \mathcal{E}_{t-} \Big(d\big(\widetilde{\beta}_{t\wedge\tau} \widetilde{\Theta}_{t\wedge\tau}\big) - \widetilde{\beta}_t \widetilde{\Theta}_t \, dH_t \Big).$$

Plugging this into (48) and using the pre-default CVA BSDE in discounted form (41), yields for $t \in [0, \bar{\tau}]$

$$\begin{aligned} d(\beta_t \Theta_t) &= \mathcal{E}_{t-} \Big(d\big(\widetilde{\beta}_{t\wedge\tau} \widetilde{\Theta}_{t\wedge\tau}\big) - \widetilde{\beta}_t \widetilde{\Theta}_t \, dH_t \Big) + \beta_t \xi \delta_\tau(dt) \\ &= \beta_t \big(- g_t(P_t - \widetilde{\Theta}_t, \zeta_t) dt - \gamma_t \widehat{\xi}_t dt + d\widetilde{\mu}_t - \widetilde{\Theta}_t \, dH_t \big) + \beta_t \xi \delta_\tau(dt) \end{aligned}$$

so

$$d\mu_t = d\widetilde{\mu}_t - (\xi_t - \widetilde{\Theta}_t) dJ_t - \gamma_t (\widehat{\xi}_t - \widetilde{\Theta}_t) dt.$$

This proves the first line in (46). One then has for $t \in [0, \bar{\tau}]$

$$\begin{aligned} d\nu_t = dM_t - d\mu_t &= (dM_t - d\widetilde{\mu}_t) + \Big((\xi_t - \widetilde{\Theta}_t) dJ_t + \gamma_t (\widehat{\xi}_t - \widetilde{\Theta}_t) dt \Big) \\ &= d\widetilde{\nu}_t - \Big((R_t - \widetilde{\Pi}_t) dJ_t + \gamma_t (\widetilde{R}_t - \widetilde{\Pi}_t) dt \Big) \end{aligned}$$

where the last equality follows by algebraic manipulations similar to (32). This proves the second line in (46). □

Remark 5.8 The jump-to-default exposure corresponding to the dJ-term in either line of (46) can be seen as a marked process, where the mark corresponds to the default being a default of the investor alone, of the bank alone, or a joint default. Consistently with this interpretation, the compensator of either dJ-term in (46) corresponds to the "average jump size" given by the dt-term in the same line, where the average is taken with respect to the probabilities of the marks, conditionally on the fact that a jump occurs at time τ.

[9] See, e.g., Bielecki, Crépey, Jeanblanc, and Rutkowski (2009).

5.3 Cost Processes Analysis

Let us now assume for the $\mathcal{G}$-martingale component $\mathcal{M}$ of the primary risky assets price process $\mathcal{P}$, a structure analogous to the one derived in the second line of (46) for the $\mathcal{G}$-martingale component ν of Π. One thus assumes that $\mathcal{M}$ is given by $\mathcal{M}_0 = 0$ and for $t \in [0, \bar{\tau}]$

$$d\mathcal{M}_t = d\widetilde{\mathcal{M}}_t - \Big(\big(\mathcal{R}_t - \widetilde{\mathcal{P}}_t\big) dJ_t + \gamma_t (\widetilde{\mathcal{R}}_t - \widetilde{\mathcal{P}}_t) dt \Big) \tag{49}$$

for an $\mathcal{F}$-local martingale $\widetilde{\mathcal{M}}$, a $\mathcal{G}$-progressively measurable primary recovery process $\mathcal{R}_t$, and an $\mathcal{F}$-progressively measurable process $\widetilde{\mathcal{R}}_t$ such that $\gamma_t(\widetilde{\mathcal{R}}_t - \widetilde{\mathcal{P}}_t)dt$ compensates $\big(\mathcal{R}_t - \widetilde{\mathcal{P}}_t\big)dJ_t$ over $[0, \bar{\tau}]$.

For every hedges ϕ and ζ, to be understood as hedges of the contract clean price P and price Π, let $\eta = \phi - \zeta$ denote the corresponding hedge of the CVA component Θ of Π. Let then the cost processes $\varepsilon^{P,\phi}$, $\varepsilon^{\Theta,\eta}$ and $\varepsilon^{\Pi,\zeta}$ be defined by $\varepsilon_0^{P,\phi} = \varepsilon_0^{\Theta,\eta} = \varepsilon_0^{\Pi,\zeta} = 0$, and for $t \in [0, \bar{\tau}]$

$$d\varepsilon_t^{P,\phi} = dM_t - \phi_t d\mathcal{M}_t\,,\ \ d\varepsilon_t^{\Theta,\eta} = d\mu_t - \eta_t d\mathcal{M}_t\,,\ \ d\varepsilon_t^{\Pi,\zeta} = d\varepsilon_t^{P,\phi} - d\varepsilon_t^{\Theta,\eta} = d\nu_t - \zeta_t d\mathcal{M}_t. \tag{50}$$

One retrieves in particular $\varepsilon^{\Pi,\zeta} = \varepsilon$, the cost process of (Π, ζ) formerly introduced in (19), related to the corresponding hedging error ϱ by Equation (22). An immediate application of (46) and (49) yields,

Proposition 5.9 *For $t \in [0, \bar{\tau}]$,*

$$\begin{aligned}
d\varepsilon_t^{P,\phi} &= \Big(dM_t - \phi_t d\widetilde{\mathcal{M}}_t\Big) + \phi_t\big(\mathcal{R}_t - \widetilde{\mathcal{P}}_t\big)dJ_t + \gamma_t\phi_t\big(\widetilde{\mathcal{R}}_t - \widetilde{\mathcal{P}}_t\big)dt && (51)\\
d\varepsilon_t^{\Theta,\eta} &= \Big(d\widetilde{\mu}_t - \eta_t d\widetilde{\mathcal{M}}_t\Big) - \Big(\big(\xi_t - \widetilde{\Theta}_t\big) - \eta_t\big(\mathcal{R}_t - \widetilde{\mathcal{P}}_t\big)\Big)dJ_t && (52)\\
&\quad -\gamma_t\Big(\big(\widehat{\xi}_t - \widetilde{\Theta}_t\big) - \eta_t\big(\widetilde{\mathcal{R}}_t - \widetilde{\mathcal{P}}_t\big)\Big)dt \\
d\varepsilon_t^{\Pi,\zeta} &= \Big(d\widetilde{\nu}_t - \zeta_t d\widetilde{\mathcal{M}}_t\Big) - \Big(\big(R_t - \widetilde{\Pi}_t\big) - \zeta_t\big(\mathcal{R}_t - \widetilde{\mathcal{P}}_t\big)\Big)dJ_t && (53)\\
&\quad -\gamma_t\Big(\big(\widetilde{R}_t - \widetilde{\Pi}_t\big) - \zeta_t\big(\widetilde{\mathcal{R}}_t - \widetilde{\mathcal{P}}_t\big)\Big)dt.
\end{aligned}$$

We thus get decompositions of the related cost processes as $\mathcal{F}$-local martingales stopped at τ, hence $\mathcal{G}$-local martingales, plus $\mathcal{G}$-compensated jump-to-default exposures. These decompositions can then be used for devising specific pricing and hedging schemes, such as pricing at the cost of hedging by replication (if possible), or of hedging only pre-default risk, or of hedging only the jump-to-default risk (dJ-terms), or of min-variance hedging, etc. This will now be made practical in a Markovian setup.

6 Markovian Case

In a Markovian setup to be specified in Subsection 6.1, explicit CVA pricing and hedging schemes will now be formulated in terms of semilinear pre-default CVA PDEs. More precisely, we shall relate suitable notions of orthogonal solutions to the pre-default CVA BSDE to:

- From a financial point of view, corresponding min-variance hedging strategies of the bank, based on the cost processes analysis of Subsection 5.3;
- From a mathematical point of view, classical Markovian BSDEs driven by an explicit set of fundamental martingales given in the form of a multi-variate Brownian motion and a compensated jump measure.

These Markovian BSDEs will be well posed under mild conditions, yielding related orthogonal solutions to the pre-default CVA BSDE, and providing in turn the corresponding min-variance hedges to the bank. This approach will be developed for three different min-variance hedging objectives, respectively considered in Subsections 6.2, 6.3 and 6.4. In the end the preferred criterion (we mainly see the analysis of Subsection 6.2 as preparatory to those of Subsections 6.3 and 6.4) can be optimized by solving (numerically if need be) the related Markovian BSDE, or (if found more efficient) by solving an equivalent semilinear parabolic PDE. Also we shall see that this methodology can be applied to either the risk-management of the overall contract, or of its CVA component in isolation. But in all cases the pre-default CVA BSDE will be the key in the mathematical analysis of the problem.

Our main results in this Section are Proposition 6.6 and Corollary 6.7, which yield concrete recipes for risk-managing the contract as a whole or its CVA component, according to the following objective of the bank: minimizing the variance[10] of the cost process of the contract or of its CVA component, whilst achieving a perfect hedge of the jump-to-default exposure.

As explained in the introduction, a clean price-and-hedge (P, ϕ) is typically determined by the industry trading desks of the bank. The central CVA desk is then left with the task of devising a CVA price-and-hedge (Θ, η). Consistently with this logic, given a clean price-and-hedge (P, ϕ), a solution $(\widetilde{\Theta}, \zeta)$ to the pre-default CVA BSDE will be sought henceforth in the form $(\widetilde{\Theta}, \phi - \eta)$, where an $\mathcal{F}$-adapted triplet $(\widetilde{\Theta}, \eta, \epsilon)$ solves

$$\begin{cases} \widetilde{\Theta}_T = 0, \text{ and for } t \in [0, T] : \\ -d\widetilde{\Theta}_t = \widetilde{g}_t(P_t - \widetilde{\Theta}_t, \phi_t - \eta_t)dt - \big(\eta_t d\widetilde{\mathcal{M}}_t + d\epsilon_t\big) \end{cases} \quad (54)$$

for an $\mathcal{F}$-predictable integrand η and an $(\mathcal{F}, \mathbb{P})$-local martingale ϵ. The pre-default CVA BSDE in form (54) is indeed equivalent to the original pre-default CVA BSDE (40), provided one lets $\eta = \phi - \zeta$, and ϵ in (54) is defined in turn through

$$d\widetilde{\mu}_t = d\widetilde{\Theta}_t - \widetilde{g}_t(P_t - \widetilde{\Theta}_t, \phi_t - \eta_t)dt = \eta_t d\widetilde{\mathcal{M}}_t + d\epsilon_t$$

(and $\epsilon_0 = 0$). We call henceforth CVA price-and hedge, any pair-process (Θ, η) such that $(\widetilde{\Theta}, \eta, \epsilon)$, with $\widetilde{\Theta} = J\Theta$ and ϵ defined through $(\widetilde{\Theta}, \eta)$ by the second line of (54) (and $\epsilon_0 = 0$), solves (54), meaning that ϵ thus defined is an $(\mathcal{F}, \mathbb{P})$-local martingale.

6.1 Factor Process

We assume further that the pre-default CVA BSDE thus redefined in terms of η rather than ζ is Markovian, in the sense that any of its adapted (respectively predictable) input data of the form $\mathcal{D}_t$ is given as a Borel-measurable[11] function $\mathcal{D}(t, X_t)$ (respectively $\mathcal{D}(t, X_{t-})$) of an

[10] Risk-neutral variance, under $\mathbb{P}$, for computational tractability.

[11] Typically continuous.

$\mathcal{F}$-Markov factor process X. So in particular $(P_t, \phi_t) = (P(t, X_t), \phi(t, X_t))$. Consequently, one has for an obviously defined deterministic function $\widetilde{g}(t, x, \pi, \varsigma)$:

$$\widetilde{g}_t(P_t - \widetilde{\Theta}_t, \phi_t - \eta_t)dt = \widetilde{g}\big(t, X_t, P(t, X_t) - \widetilde{\Theta}_t, \phi(t, X_t) - \eta_t\big)dt.$$

We shall use as drivers of the pre-default factor process X an $\mathbb{R}^q$-valued $\mathcal{F}$-Brownian motion W and an $\mathcal{F}$-compensated integer-valued random measure N on $[0, T] \times \mathbb{R}^q$, for some integer q. Given coefficients $b(t, x)$, $\sigma(t, x)$, $\delta(t, x, y)$ and $F(t, x, dy)$ to be specified depending on the application at hand (see for instance Crépey and Grbac (2011)), we assume that the pre-default factor process X satisfies the following Markovian $\mathcal{F}$-forward SDE in $\mathbb{R}^q$: an initial condition $X_0 = x$ of X given as an observable or calibratable constant, and for $t \in [0, T]$

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t + \delta(t, X_{t-}) \cdot dN_t, \tag{55}$$

with an intensity measure of N of the form $F(t, X_t, dx)dt$. In (55) and below we denote for every matrix-valued function $f = f(t, x, y)$ on $[0, T] \times \mathbb{R}^q \times \mathbb{R}^q$ (like $f = \delta$ in (55))

$$f(t, X_{t-}) \cdot dN_t = \int_{\mathbb{R}^q} f(t, X_{t-}, x) N(dt, dx)\,, \quad \big(f \cdot F\big)(t, x) \quad = \int_{\mathbb{R}^q} f(t, x, y) F(t, x, dy).$$

The matrix-integrals are performed entry by entry of f, so that one ends up in both cases with matrices of the same dimensions as f. Note that a factor process X with an intensity of N depending on X can be classically constructed by change of probability measure, see, e.g., Part II of Crépey (2011).

Given a vector-valued function $u = u(t, x)$ on $[0, T] \times \mathbb{R}^q$, let $\nabla u(t, x)$ denote the Jacobian matrix of u with respect to x at (t, x), and let δu be the function on $[0, T] \times \mathbb{R}^q \times \mathbb{R}^q$ such that for every $(t, x, y) \in [0, T] \times \mathbb{R}^q \times \mathbb{R}^q$

$$\delta u(t, x, y) = u(t, x + \delta(t, x, y)) - u(t, x).$$

One assumes further that $\widetilde{\mathcal{P}}_t = \widetilde{\mathcal{P}}(t, X_t)$, for some pre-default primary risky assets pricing function $\widetilde{\mathcal{P}}$, so that the $\mathcal{F}$-local martingale component $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ in (49) writes

$$d\widetilde{\mathcal{M}}_t = (\nabla \widetilde{\mathcal{P}} \sigma)(t, X_t) dW_t + \delta \widetilde{\mathcal{P}}(t, X_{t-}) \cdot dN_t.$$

Further analysis of the cost processes ε-s in (51)-(53) depends on a hedging criterion of the bank. In the following Subsections we shall propose three tractable approaches, all of them involving, to some extent, min-variance hedging. In case the primary market is rich enough to allow for replication, min-variance hedging of course reduces to hedging by replication. Moreover we shall consider the two issues of hedging the contract globally, or to only hedge its CVA. In all cases the mathematical analysis will ultimately rely on the pre-default CVA BSDE (54).

Remark 6.1 We refer the reader to Schweizer (2001) for a survey about various quadratic error minimization approaches which can be used in incomplete markets, and the corresponding issues regarding the choice of a pricing measure. A first class of so-called mean-self-financing approaches resorts to non-self-financing trading strategies and introduces a related notion of cost of a strategy, analyzed in terms of the so-called minimal martingale measure. A second class of so-called mean-variance hedging approaches sticks to self-financing trading strategies and aims at minimizing a quadratic hedging error. In this second approach a central role is played by the so-called variance-optimal martingale measure.

Since one only works with self-financing strategies in this paper and one shall ultimately aim at minimizing some quadratic cost criterion, our methodology is closer to a mean-variance hedging approach. Note in this regard that our "cost process" $\varepsilon^{\Pi,\zeta} = \varepsilon$ of a price-and-hedge (Π, ζ) is not the cost of a strategy in a mean-self-financing sense, but corresponds rather to a hedging error in a mean-variance hedging sense (the close relation between ε and the corresponding hedging error properly said, ϱ, being provided by Equation (22)).

However for the sake of tractability we only consider in this work minimization under the martingale pricing measure $\mathbb{P}$, whereas most of the theoretical difficulty with mean-variance hedging (and also with mean-self-financing approaches) comes from the fact that one aims at minimizing the hedging error under the historical probability measure. To emphasize this difference we write in our case min-variance hedging instead of mean-variance hedging. Also note that our min-variance hedging will only be done with respect to the reference filtration $\mathcal{F}$ on the top of a given choice of a hedging strategy regarding the jump-to-default exposure of the bank (no hedge in Subsection 6.2, perfect hedge in Subsection 6.3 and hedge of an isolated default of the investor in Subsection 6.4), as opposed to directly minimizing the variance relative to the "big" filtration $\mathcal{G}$.

Given vector-valued functions $u = u(t,x)$ and $v = v(t,x)$ on $[0,T] \times \mathbb{R}^q$, we denote

$$(u,v)(t,x) = (\nabla u\sigma)(\nabla v\sigma)^{\mathsf{T}}(t,x) \,+\, \big((\delta u \delta v^{\mathsf{T}}) \cdot F\big)(t,x), \tag{56}$$

in which $^{\mathsf{T}}$ stands for "transposed". So, if u and v are n- and m-dimensional vector-functions of (t,x), one ends-up with an $\mathbb{R}^{n\times m}$-valued matrix-function (u,v) of (t,x).

6.2 Min-Variance Hedging of Market Risk

Our first objective will be to min-variance hedge the market risk corresponding to the term $d\widetilde{\mu}_t - \eta_t d\widetilde{\mathcal{M}}_t$ in the CVA cost process $\varepsilon^{\Theta,\eta}$ in (52), or $d\widetilde{\nu}_t - \zeta_t d\widetilde{\mathcal{M}}_t$ in the overall contract cost process $\varepsilon^{\Pi,\zeta} = \varepsilon$ in (53).

Regarding (52), this is tantamount to seeking for a solution $(\widetilde{\Theta}, \eta, \epsilon)$ to the pre-default CVA BSDE (54) in which ϵ is $\mathcal{F}$-orthogonal to $\widetilde{\mathcal{M}}$ (cf. Proposition 5.2 in El Karoui, Peng, and Quenez (1997) Given such an orthogonal solution $(\widetilde{\Theta}, \eta, \epsilon)$ to (54), and if moreover $\widetilde{\Theta}_t = \widetilde{\Theta}(t, X_t)$, one then has by a standard min-variance oblique bracket formula,[12] in the $(\cdot,\cdot)$ notation of (56):

$$\eta_t = \tfrac{d<\widetilde{\mu},\widetilde{\mathcal{M}}>_t}{dt} \left(\tfrac{d<\widetilde{\mathcal{M}}>_t}{dt}\right)^{-1} = \left(\left(\widetilde{\Theta}, \widetilde{\mathcal{P}}\right)\Lambda\right)(t, X_{t-}) =: \eta(t, X_{t-}) \tag{57}$$

where we let $\Lambda = \left(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}\right)^{-1}$. Here invertibility of the $\mathcal{F}_t$-conditional covariance matrix $\frac{d<\widetilde{\mathcal{M}}>_t}{dt}$ is assumed.

This leads to the following Markovian BSDE in $(\widetilde{\Theta}(t,X_t), (\nabla\widetilde{\Theta}\sigma)(t,X_t), \delta\widetilde{\Theta}(t,X_{t-},\cdot))$ over $[0,T]$:

$$\begin{cases} \widetilde{\Theta}(T, X_T) = 0, \text{ and for } t \in [0,T] : \\ -\, d\widetilde{\Theta}(t,X_t) = \widehat{g}\left(t, X_t, \widetilde{\Theta}(t,X_t), (\nabla\widetilde{\Theta}\sigma)(t,X_t), \left((\delta\widetilde{\Theta}\delta\widetilde{\mathcal{P}}^{\mathsf{T}}) \cdot F\right)(t,X_t)\right) dt \\ \qquad -\, (\nabla\widetilde{\Theta}\sigma)(t,X_t)dW_t - \delta\widetilde{\Theta}(t,X_{t-}) \cdot dN_t, \end{cases} \tag{58}$$

[12] See, e.g., Part I of Crépey (2011).

with for every $(t,x,\vartheta,z,w) \in [0,T] \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^d$ (for row-vectors z, w)

$$\widehat{g}\,(t,x,\vartheta,z,w) = \widetilde{g}\,(t,x,P(t,x)-\vartheta,\phi(t,x)-\widehat{\eta}(t,x,\vartheta,z,w))$$

where we let

$$\widehat{\eta}(t,x,\vartheta,z,w) = \Big(z(\nabla\widetilde{\mathcal{P}}\sigma)^{\mathsf{T}}(t,x)+w\Big)\,\Lambda(t,x).$$

Indeed one then has in view of (56) and (57):

$$\widehat{\eta}(t,X_t,\widetilde{\Theta}(t,X_t),(\nabla\widetilde{\Theta}\sigma)(t,X_t),\Big((\delta\widetilde{\Theta}\delta\widetilde{\mathcal{P}}^{\mathsf{T}})\cdot F\Big)(t,X_t)) = \eta(t,X_t)$$

$$\widehat{g}\,\Big(t,X_t,\widetilde{\Theta}(t,X_t),(\nabla\widetilde{\Theta}\sigma)(t,X_t),\Big((\delta\widetilde{\Theta}\delta\widetilde{\mathcal{P}}^{\mathsf{T}})\cdot F\Big)(t,X_t)\Big)\,dt =$$
$$\widetilde{g}\,\Big(t,X_t,P(t,x)-\widetilde{\Theta}(t,X_t),\phi(t,X_t)-\eta(t,X_t)\Big)\,dt.$$

We refer the reader to the literature (see, e.g., Parts II and III of Crépey (2011)) regarding the fact that under mild Lipschitz-continuity and square-integrability conditions on the coefficient $\widehat{g}$, the Markovian BSDE (58) has a unique square-integrable solution; moreover, the pre-default CVA function $\widetilde{\Theta} = \widetilde{\Theta}(t,x)$ in this solution can be characterized as the unique solution in suitable spaces to the following semi-linear partial integro-differential equation (PDE for short):

$$\begin{cases} \widetilde{\Theta}(T,x) = 0\,,\ x \in \mathbb{R}^q \\ (\partial_t + \mathcal{X})\,\widetilde{\Theta}(t,x) + \widehat{g}(t,x,\widetilde{\Theta}(t,x),(\nabla\widetilde{\Theta}\sigma)(t,x),\big((\delta\widetilde{\Theta}\delta\widetilde{\mathcal{P}}^{\mathsf{T}})\cdot F\big)(t,x)) = 0 \text{ on } [0,T)\times\mathbb{R}^q, \end{cases} \tag{59}$$

where $\mathcal{X}$ stands for the infinitesimal generator of X.

Remark 6.2 In the classical BSDE-with-jumps literature (see for instance Royer (2006) or Crépey and Matoussi (2008)), it is typically postulated that the driver coefficient, $\widehat{g}$ in the case of the Markovian BSDE (58), only depends on $\delta\widetilde{\Theta}(t,X_{t-},\cdot)$ through one average of $\delta\widetilde{\Theta}(t,X_{t-},\cdot)$ against some jump measure, rather than through d such averages in (58) (note the last argument w of $\widehat{g}$ is a row-vector in $\mathbb{R}^d$). By inspection of the proof in Crépey and Matoussi (2008), the comparison principle which is established there, and which is key in the connection between a BSDE and a PDE approach to a semilinear parabolic equation (see for instance Part III of Crépey (2011), can be elevated from the scalar case to the case of any finite number of averages. However this comparison is, as already in the scalar case, subject to a monotonicity condition of $\widehat{g}$ with respect to w, so

$$\widehat{g}\,(t,x,\vartheta,z,w) \le \widehat{g}\,\big(t,x,\vartheta,z,w'\big) \text{ if } w_i \le w_i',\, i=1,\ldots,d.$$

Of course these technicalities vanish in case of a fully swapped hedge satisfying (25) so that $\widetilde{g}\big(t,x,\pi,\varsigma\big) = \widetilde{g}\big(t,x,\pi\big)$ (see Remark 6.8 for the corresponding Markovian BSDEs and semi-linear PDEs).

Proposition 6.3 *Assuming invertibility of the primary-risky-assets-covariance-matrix* $\Big(\widetilde{\mathcal{P}},\widetilde{\mathcal{P}}\Big)$, *the solution* $\widetilde{\Theta} = \widetilde{\Theta}(t,x)$ *to* (59) *yields, via* (57) *for* η *and in turn* (54) *for* ϵ, *an orthogonal solution* $(\widetilde{\Theta},\eta,\epsilon)$ *to the pre-default CVA BSDE* (54).

The CVA-market-risk-min-variance hedge is thus given by Formula (57) *as*

$$\eta_t = \eta(t,X_{t-}) = \left(\Big(\widetilde{\Theta},\widetilde{\mathcal{P}}\Big)\Big(\widetilde{\mathcal{P}},\widetilde{\mathcal{P}}\Big)^{-1}\right)(t,X_{t-}).$$

Process ϵ *in the solution to* (54) *is the residual CVA market risk under the CVA hedge* η.

We now consider hedging of the market risk $d\widetilde{\nu}_t - \zeta_t d\widetilde{\mathcal{M}}_t$ of the overall contract cost process $\varepsilon^{\Pi,\zeta} = \varepsilon$ in (53). Let the clean hedge ϕ be specifically given here as the coefficient of regression in an $\mathcal{F}$-orthogonal decomposition $dM = \phi d\widetilde{\mathcal{M}} + de$. By the min-variance oblique bracket formula, one thus has for $t \in [0, \bar{\tau}]$

$$\phi_t = \frac{d\langle M, \widetilde{\mathcal{M}}\rangle_t}{dt}\left(\frac{d\langle \widetilde{\mathcal{M}}\rangle_t}{dt}\right)^{-1} = \left(\left(\widetilde{P}, \widetilde{\mathcal{P}}\right)\Lambda\right)(t, X_{t-}) =: \phi(t, X_{t-}). \tag{60}$$

Besides, in view of (51)-(53), it holds that

$$d\widetilde{\nu}_t - \zeta_t d\widetilde{\mathcal{M}}_t = \big(dM_t - \phi_t d\widetilde{\mathcal{M}}_t\big) - \big(d\widetilde{\mu}_t - \eta_t d\widetilde{\mathcal{M}}_t\big).$$

Therefore, $dM - \phi d\widetilde{\mathcal{M}}$ and $d\widetilde{\mu} - \eta d\widetilde{\mathcal{M}}$ being $\mathcal{F}$-orthogonal to $d\widetilde{\mathcal{M}}$, implies the same property for $d\widetilde{\nu} - \zeta d\widetilde{\mathcal{M}}$. In other words, Proposition 6.3 admits the following

Corollary 6.4 *For ϕ given as the regression coefficient of M against $\widetilde{\mathcal{M}}$, the strategy $\zeta_t := (\phi - \eta)(t, X_{t-})$ is a min-variance hedge of the market risk component $d\widetilde{\nu} - \zeta d\widetilde{\mathcal{M}}$ of the contract cost process $\varepsilon^{\Pi,\zeta} = \varepsilon$. The residual market risk of the contract hedged in this way is given by $e - \epsilon$.*

6.3 Min-Variance Hedging Constrained to Perfect Hedging of Jump-to-Default Risk

The previous approach disregards the jump-to-default risk corresponding to the dJ-terms in (52) or (53). We now wish to min-variance hedge the market risk corresponding to the term $d\widetilde{\mu}_t - \eta_t d\widetilde{\mathcal{M}}_t$ in the CVA cost process $\varepsilon^{\Theta,\eta}$ in (52) (respectively $d\widetilde{\nu}_t - \zeta_t d\widetilde{\mathcal{M}}_t$ in the overall contract cost process $\varepsilon^{\Pi,\zeta} = \varepsilon$ in (53)), under the constraint that one perfectly hedges the jump-to-default risk corresponding to the dJ-term in (52) (respectively (53)). Note that in view of the marked point process interpretation provided in Remark 5.8, cancelation of the dJ-term in any of Equation (51) to (53), implies cancelation of the dt-driven process which compensates it in the same equation. We are thus equivalently minimizing the variance of the cost processes $\varepsilon^{\Theta,\eta}$ or $\varepsilon^{\Pi,\zeta} = \varepsilon$ under the constraint of perfectly hedging the jump-to-default exposure.

Let us re-order if need be the primary risky assets so that the first ones (if any) cannot jump at time τ, and the last ones (if any) can jump at time τ. We then let a superscript ^{0} refer to the subset of the hedging instruments with price processes which cannot jump at time τ, so $\mathcal{R}^0 = \widetilde{\mathcal{R}}^0 = \widetilde{\mathcal{P}}^0$, and we let ^{1} refer to the subset, complement of ^{0}, of the hedging instruments which can jump at time τ.[13] The CVA cost equation (52) can thus be rewritten as, for $t \in [0, \bar{\tau}]$:

$$\begin{aligned} d\varepsilon_t^{\Theta,\eta} = &\Big(d\widetilde{\mu}_t - \eta_t^0 d\widetilde{\mathcal{M}}_t^0 - \eta_t^1 d\widetilde{\mathcal{M}}_t^1\Big) - \Big(\big(\xi_t - \widetilde{\Theta}_t\big) - \eta_t^1\big(\mathcal{R}_t^1 - \widetilde{\mathcal{P}}_t^1\big)\Big) dJ_t \\ &- \gamma_t\Big(\big(\widehat{\xi}_t - \widetilde{\Theta}_t\big) - \eta_t^1\big(\widetilde{\mathcal{R}}_t^1 - \widetilde{\mathcal{P}}_t^1\big)\Big) dt. \end{aligned} \tag{61}$$

The condition that a CVA price-and-hedge (Θ, η) perfectly hedges the dJ-term in (61) writes:

$$\xi_t - \widetilde{\Theta}_{t-} = \eta_t^1\big(\mathcal{R}_t^1 - \widetilde{\mathcal{P}}_t^1\big),\ t \in [0, \bar{\tau}] \tag{62}$$

[13] This is to an harmless abuse of notation that Y^0 and Y^1 do not represent anymore the "coordinates ^{0} and ^{1}" of an $\mathbb{R}^d$-valued vector Y, or these are now "group-coordinates".

where it should be noted in view of (35) that ξ_t is, via $(1-\mathfrak{r}_t)\widehat{\mathfrak{X}}^+_{t-}$, a random function of $\widetilde{\Theta}_{t-}$ and $\zeta_{t-}=\phi_{t-}-\eta_{t-}$. Viewed as an equation to be solved in η^1_t, condition (62) is thus very implicit unless one is in the special case where (in the present Markov setup)

$$(1-\mathfrak{r}(t,X_t))\mathfrak{X}^+(t,X_t,\pi,\varsigma)=(1-\mathfrak{r}(t,X_t))\mathfrak{X}^+(t,X_t,\pi) \tag{63}$$

does not depend on ς, so that ξ_t does not depend on η_{t-}. In this case, in view of the expression of ξ_t in (35), depending on whether one considers a model of unilateral counterparty risk ($\theta=\infty$), of bilateral counterparty risk without joint default of the bank and of the investor ($\theta,\overline{\theta}<\infty$ with $\theta\neq\overline{\theta}$ almost surely), or of bilateral counterparty risk with a possible joint default of the bank and of the investor, then Equation (62) respectively boils down to a system of one, two or three linear equations to be satisfied by η^1_t.

We work in this Subsection under the assumption that Equation (62) does have a solution of the form

$$\eta^1_t=\eta^1_t(\widetilde{\Theta}_{t-})=\eta^1(t,X_{t-},\widetilde{\Theta}_{t-}). \tag{64}$$

Under condition (63), this is satisfied under a mild non-redundancy condition on the hedging instruments in group ^{1}, with η^1 typically uni-variate in case $\theta=\infty$, bi-variate in case $\theta,\overline{\theta}<\infty$ with $\theta\neq\overline{\theta}$, and tri-variate otherwise. For a case of existence of a solution of the form (64) to Equation (62) without condition (63), we refer the reader to Bielecki, Crépey, and Rutkowski (2011), extending Burgard and Kjaer (2010) and Burgard and Kjaer (2011

Remark 6.5 (Discussion of Condition (63)**)** Condition (63) holds in the case of a fully swapped hedge as well as in the partial default case where $\mathfrak{r}=1$. Again in specific cases a solution of the form $\eta^1_t(\widetilde{\Theta}_{t-})$ to Equation (62) may be found without condition (63), see Bielecki, Crépey, and Rutkowski (2011). In the case without (63), a possible idea to recover (if need be) (63) however is to forget about the close-out funding cash-flow $R^f=(1-\mathfrak{r}_\theta)\widehat{\mathfrak{X}}^+_{\theta-}$ in R, thus working everywhere as if $\mathfrak{r}$ was equal to one, whilst using a dt-funding-excess-benefit coefficient $f_t(\pi,\varsigma)$ adjusted to

$$\widetilde{f}_t(\pi,\varsigma)=f_t(\pi,\varsigma)-\gamma_t p_t(1-\mathfrak{r}_t)\mathfrak{X}^+_t(\pi,\varsigma). \tag{65}$$

The problem thus modified then satisfies (63). The adjusted funding benefit coefficient $\widetilde{f}_t(\pi,\varsigma)$ represents a pure liquidity (as opposed to credit risk) funding benefit coefficient. Using this approach thus also allows one to nicely decouple the credit risk ingredients in the model, represented by θ and $\overline{\theta}$, from the liquidity funding ingredients, represented by the adjusted excess funding benefit coefficient $\widetilde{f}$ and the repo basis c.

Note that simply ignoring the close-out funding cash-flow R^f (as in Burgard and Kjaer (2010) and Burgard and Kjaer (2011)) without adjusting f, would induce a valuation and hedging bias. In contrast, accordingly adjusting f as in (65) makes it at least correct from the valuation point of view, for every fixed ζ. But this correctness in value is only for a given hedge process ζ. Since a central point in all this is precisely on how to choose ζ, we believe this adjustment approach is fallacious.

Now, for any CVA hedge η with components η^1 of η in group ^{1} given as a solution $\eta^1_t(\widetilde{\Theta}_{t-})$ to (62), the CVA cost process (61) reduces to

$$d\varepsilon^{\Theta,\eta}_t=d\widetilde{\mu}_t-\eta^0_t d\widetilde{\mathcal{M}}^0_t-\eta^1_t d\widetilde{\mathcal{M}}^1_t. \tag{66}$$

This leads us to seek for a solution (Θ, η) to the problem of min-variance hedging of the CVA constrained to perfect hedging of CVA jump-to-default risk, with η_t of the form

$$\eta_t = \big(\eta^0_t, \eta^1_t(\widetilde{\Theta}_{t-})\big), \tag{67}$$

and with $(\widetilde{\Theta}, \eta, \epsilon)$ solving the pre-default CVA BSDE (54), where ϵ is defined through $\widetilde{\Theta}$ and η by the second line of (54). Note in view of the pre-default CVA BSDE (54) that $d\epsilon_t$ then boils down to $d\varepsilon^{\Theta,\eta}_t$ in (66), the variance of which one would like to minimize. Now, in order to minimize the variance of $d\varepsilon^{\Theta,\eta}_t = d\epsilon_t$ among all solutions $(\widetilde{\Theta}, \eta, \epsilon)$ of (54) such that $\eta^1_t = \eta^1_t(\widetilde{\Theta}_{t-})$, one must choose η^0 as the coefficient of regression of $d\bar{\mu}_t := d\widetilde{\mu}_t - \eta^1_t(\widetilde{\Theta}_{t-})d\widetilde{\mathcal{M}}^1_t$ against $d\widetilde{\mathcal{M}}^0_t$. In other words we are now looking for a solution $(\widetilde{\Theta}, \eta, \epsilon)$ to the pre-default CVA BSDE (54), with $\eta^1_t = \eta^1_t(\widetilde{\Theta}_{t-})$ and with $d\widetilde{\mu}_t - \eta^1_t(\widetilde{\Theta}_{t-})d\widetilde{\mathcal{M}}^1_t - \eta^0_t d\widetilde{\mathcal{M}}^0_t$ orthogonal to $d\widetilde{\mathcal{M}}^0_t$. In such a solution, assuming further a deterministic $\widetilde{\Theta}_t = \widetilde{\Theta}(t, X_t)$, it comes by the min-variance oblique bracket formula:

$$\begin{aligned}
\eta^0_t &= \frac{d < \bar{\mu}, \widetilde{\mathcal{M}}^0 >_t}{dt}\left(\frac{d < \widetilde{\mathcal{M}}^0 >_t}{dt}\right)^{-1} \\
&= \Big(\Big(\widetilde{\Theta}, \widetilde{\mathcal{P}}^0\Big)\,\Lambda^0\Big)(t, X_{t-}) - \eta^1(t, X_{t-}, \widetilde{\Theta}(t, X_{t-}))\Big(\Big(\widetilde{\mathcal{P}}^1, \widetilde{\mathcal{P}}^0\Big)\,\Lambda^0\Big)(t, X_{t-}) \\
&=: \eta^0(t, X_{t-})
\end{aligned} \tag{68}$$

where we let $\Lambda^0 = \Big(\widetilde{\mathcal{P}}^0, \widetilde{\mathcal{P}}^0\Big)^{-1}$, assumed to exist. This leads us to the following Markovian BSDE in $(\widetilde{\Theta}(t, X_t), (\nabla\widetilde{\Theta}\sigma)(t, X_t), \delta\widetilde{\Theta}(t, X_{t-}, \cdot))$ over $[0, T]$:

$$\begin{cases}
\widetilde{\Theta}(T, X_T) = 0, \text{ and for } t \in [0, T]: \\
-\,d\widetilde{\Theta}(t, X_t) = \bar{g}\Big(t, X_t, \widetilde{\Theta}(t, X_t), (\nabla\widetilde{\Theta}\sigma)(t, X_t), \big((\delta\widetilde{\Theta}\delta(\widetilde{\mathcal{P}}^0)^\mathsf{T}) \cdot F\big)(t, X_t)\Big)\,dt \\
\qquad -\,(\nabla\widetilde{\Theta}\sigma)(t, X_t)dW_t - \delta\widetilde{\Theta}(t, X_{t-}) \cdot dN_t,
\end{cases} \tag{69}$$

with for every $(t, x, \vartheta, z, w) \in [0, T] \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^{d_0}$, in which d_0 is the number of assets in group ^{0}:

$$\bar{g}\,(t, x, \vartheta, z, w) = \widetilde{g}\,\big(t, x, P(t, x) - \vartheta, \phi(t, x) - \big(\bar{\eta}^0(t, x, \vartheta, z, w), \eta^1(t, x, \vartheta)\big)\big)$$

where we let

$$\bar{\eta}^0(t, x, \vartheta, z, w) = \Big(z(\nabla\widetilde{\mathcal{P}}^0\sigma)^\mathsf{T}(t, x) + w\Big)\,\Lambda^0(t, x) - \eta^1(t, x, \vartheta)\,\Big(\Big(\widetilde{\mathcal{P}}^1, \widetilde{\mathcal{P}}^0\Big)\,\Lambda^0\Big)\,(t, x).$$

Indeed one then has in view of (56) and (68)

$$\begin{aligned}
&\bar{\eta}^0\,\Big(t, X_t, \widetilde{\Theta}(t, X_t), (\nabla\widetilde{\Theta}\sigma)(t, X_t), \big((\delta\widetilde{\Theta}\delta(\widetilde{\mathcal{P}}^0)^\mathsf{T}) \cdot F\big)(t, X_t)\Big) = \eta^0(t, X_t) \\
&\bar{g}\,\Big(t, X_t, \widetilde{\Theta}(t, X_t), (\nabla\widetilde{\Theta}\sigma)(t, X_t), \big((\delta\widetilde{\Theta}\delta(\widetilde{\mathcal{P}}^0)^\mathsf{T}) \cdot F\big)(t, X_t)\Big)\,dt \\
&\qquad = \widetilde{g}\,\Big(t, X_t, P(t, X_t)\widetilde{\Theta}(t, X_t), \phi(t, X_t) - \big(\eta^0(t, X_t), \eta^1(t, X_t, \widetilde{\Theta}(t, X_t))\big)\Big)\,dt.
\end{aligned}$$

Now, under mild technical conditions, the Markovian BSDE (69) has a unique solution, and[14] the pre-default CVA function $\widetilde{\Theta} = \widetilde{\Theta}(t, x)$ in this solution can be characterized as the

[14]Up to the monotonicity condition of Remark 6.2, applying here to $\bar{g}$.

unique solution to the following semilinear PDE:

$$\begin{cases} \widetilde{\Theta}(T,x) = 0\,,\ x \in \mathbb{R}^q \\ (\partial_t + \mathcal{X})\,\widetilde{\Theta}(t,x) + \bar{g}(t,x,\widetilde{\Theta}(t,x),(\nabla\widetilde{\Theta}\sigma)(t,x),\big((\delta\widetilde{\Theta}\delta(\widetilde{\mathcal{P}}^0)^{\mathsf{T}})\cdot F\big)(t,x)) = 0 \text{ on } [0,T)\times\mathbb{R}^q. \end{cases} \tag{70}$$

One then has by virtue of the above analysis,

Proposition 6.6 *Assume existence of a solution $\eta^1_t = \eta^1(\widetilde{\Theta}_{t-})$ to Equation* (62) *and invertibility of the group ^{0}-primary-risky-assets-covariance-matrix $\left(\widetilde{\mathcal{P}}^0,\widetilde{\mathcal{P}}^0\right)$. Then the solution $\widetilde{\Theta} = \widetilde{\Theta}(t,x)$ to* (70) *yields, via* (67)-(68) *for η and* (54) *for ϵ, a solution $(\widetilde{\Theta},\eta,\epsilon)$ to the pre-default CVA BSDE* (54)*, such that $\eta^1_t = \eta^1_t(\widetilde{\Theta}_{t-})$ and $d\widetilde{\mu} - \eta^1_t(\widetilde{\Theta}_{t-})d\widetilde{\mathcal{M}}^1_t - \eta^0_t d\widetilde{\mathcal{M}}^0_t$ is orthogonal to $d\widetilde{\mathcal{M}}^0_t$.*

The min-variance hedge of the CVA (market risk) constrained to perfect hedge of the CVA jump-to-default risk, is thus given as $\big(\eta^0_t,\eta^1_t(\widetilde{\Theta}_{t-})\big)$, where $\eta^1_t(\widetilde{\Theta}_{t-})$ is the assumed solution to (62)*, and where $\eta^0_t = \eta^0(t,X_{t-})$ is in turn given by Formula* (68)*:*

$$\eta^0_t = \left(\left(\widetilde{\Theta},\widetilde{\mathcal{P}}^0\right)\left(\widetilde{\mathcal{P}}^0,\widetilde{\mathcal{P}}^0\right)^{-1}\right)(t,X_{t-}) - \eta^1_t(\widetilde{\Theta}_{t-})\left(\left(\widetilde{\mathcal{P}}^1,\widetilde{\mathcal{P}}^0\right)\left(\widetilde{\mathcal{P}}^0,\widetilde{\mathcal{P}}^0\right)^{-1}\right)(t,X_{t-}).$$

Process $\epsilon = \varepsilon^{\Theta,\eta}$ represents the residual CVA (market) risk under this CVA hedge η.

We now consider the constrained min-variance hedging problem of the contract as a whole, rather than simply of its CVA component. We assume further that the hedge ϕ of the contract clean price P, only involves the primary assets in group ^{0}, and that ϕ^0 is given as the coefficient of regression in an $\mathcal{F}$-orthogonal decomposition $dM = \phi^0 d\widetilde{\mathcal{M}}^0 + d\bar{e}$, so

$$\phi^0_t = \frac{d{<}M,\widetilde{\mathcal{M}}^0{>}_t}{dt}\left(\frac{d{<}\widetilde{\mathcal{M}}^0{>}_t}{dt}\right)^{-1} = \left(\left(\widetilde{P},\widetilde{\mathcal{P}}^0\right)\Lambda^0\right)(t,X_{t-}) =: \phi^0(t,X_{t-}).$$

For $(\widetilde{\Theta},\eta,\epsilon)$ as in Proposition 6.6 and for $\zeta := \phi - \eta$, the cost equations (51)-(53) boil down to

$$\begin{aligned} d\varepsilon^{P,\phi}_t &= dM_t - \phi^0_t d\widetilde{\mathcal{M}}^0_t = d\bar{e}_t \\ d\varepsilon^{\Theta,\eta}_t &= d\widetilde{\mu}_t - \eta^0_t d\widetilde{\mathcal{M}}^0_t - \eta^1_t(\widetilde{\Theta}_{t-})d\widetilde{\mathcal{M}}^1_t = d\epsilon_t \\ d\varepsilon^{\Pi,\zeta}_t &= d\varepsilon_t = d\varepsilon^{P,\phi}_t - d\varepsilon^{\Theta,\eta}_t \\ &= d\widetilde{\nu}_t - \zeta^0_t d\widetilde{\mathcal{M}}^0_t + \eta^1_t(\widetilde{\Theta}_{t-})d\widetilde{\mathcal{M}}^1_t. \end{aligned}$$

Therefore $dM_t - \phi^0_t d\widetilde{\mathcal{M}}^0_t$ and $d\widetilde{\mu}_t - \eta^0_t d\widetilde{\mathcal{M}}^0_t - \eta^1_t(\widetilde{\Theta}_{t-})d\widetilde{\mathcal{M}}^1_t$ being $\mathcal{F}$-orthogonal to $d\widetilde{\mathcal{M}}^0_t$, implies the same property for $d\widetilde{\nu}_t - \zeta^0_t d\widetilde{\mathcal{M}}^0_t + \eta^1_t(\widetilde{\Theta}_{t-})d\widetilde{\mathcal{M}}^1_t$. Proposition 6.6 thus admits the following

Corollary 6.7 *For ϕ^0 given as the regression coefficient of M against $\widetilde{\mathcal{M}}^0$, the strategy $\zeta_t = \big(\phi^0(t,X_{t-}) - \eta^0(t,X_{t-}), -\eta^1_t(\widetilde{\Theta}_{t-})\big)$, is a min-variance hedge of the contract (market risk), under the contract jump-to-default perfect hedge constraint that $\zeta^1_t = -\eta^1_t(\widetilde{\Theta}_{t-})$. The residual (market) risk of the contract hedged in this way is given by $\varepsilon^{\Pi,\zeta} = \varepsilon = \bar{e} - \epsilon$.*

Remark 6.8 Under the fully swapped hedge condition (25), which in the current Markov setup implies (63) through a more specific $\widetilde{g}\big(t,x,\pi,\varsigma\big)=\widetilde{g}\big(t,x,\pi\big)$, the Markovian BSDEs (58) and (69) equally boil down to:

$$\begin{cases}\widetilde{\Theta}(T,X_T)=0,\ \text{and for } t\in[0,T]:\\ -\,d\widetilde{\Theta}(t,X_t)=\widetilde{g}\left(t,X_t,P(t,X_t)-\widetilde{\Theta}(t,X_t)\right)dt-(\nabla\widetilde{\Theta}\sigma)(t,X_t)dW_t-\delta\widetilde{\Theta}(t,X_{t-})\cdot dN_t,\end{cases}\tag{71}$$

with a related semilinear PDE given as

$$\begin{cases}\widetilde{\Theta}(T,x)=0\,,\ x\in\mathbb{R}^q\\ (\partial_t+\mathcal{X})\,\widetilde{\Theta}(t,x)+\widetilde{g}(t,x,\widetilde{\Theta}(t,x))=0 \text{ on } [0,T)\times\mathbb{R}^q.\end{cases}\tag{72}$$

The strategies of Proposition 6.3-Corollary 6.4 and Proposition 6.6-Corollary 6.7 differ however.

6.4 Unilateral or Bilateral in the End?

The practical importance of hedging counterparty risk in terms not only of market risk, but also of jump-to-default exposure, was revealed in the last financial crisis. But, since selling one's own CDS is illegal, whether it is practically possible to hedge one's own default is rather dubious, due to lacking of suitable hedging instruments.

In very specific cases the bank can hedge its own default by repurchasing its own bond, see Burgard and Kjaer (2010), Burgard and Kjaer (2011) and the related development in Bielecki, Crépey, and Rutkowski (2011). Otherwise the bank can resort to a variant of the approach of Subsection 6.3 consisting in min-variance hedging of market risk constrained to perfect hedging of the investor's jump-to-default risk, whilst not hedging its own default. Only hedging the investor's jump-to-default risk means hedging the dJ-term in (61) on the random set $\{\overline{\theta}<\theta\wedge T\}$. In view of the CVA cost equation (61) and given the specification (35) of ξ_t, this boils down to the following univariate explicit linear equation to be satisfied by a scalar process $\eta^1_t=\eta^1_t(\widetilde{\Theta}_{t-})$, for $t\in[0,T]$:

$$P_t-Q_t-(1-\bar{\rho}_t)\chi^-_t-\widetilde{\Theta}_{t-}=\eta^1_t\big(\mathcal{R}^1_t-\widetilde{\mathcal{P}}^1_t\big).\tag{73}$$

Min-variance hedging the CVA market risk of the contract (or of the contract as a whole) subject to perfect hedge of the investor's isolated jump-to-default, thus boils down to min-variance hedging the CVA market risk of the contract (or of the contract as a whole) subject to η^1_t being defined by (73), which can be done along similar lines as in Subsection 6.3, yielding easily derived analogs of Proposition 6.6 and Corollary 6.7. Note this involves no technical condition like (63).

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