ARBITRAGE PRICING OF DEFAULTABLE GAME OPTIONS WITH APPLICATIONS TO CONVERTIBLE BONDS

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1 Introduction

It is widely acknowledged (see, for instance, [16, 24, 28]) that a convertible bond has a natural interpretation as a defaultable bond supplemented with an option to exchange this defaultable bond for a given number κ of shares. Thus, convertible bonds are often advertised as products with upside potential and limited downside risk. However, after years of steady growth, the market of convertible bonds has suffered an unprecedented drawback in April–May 2005. Many hedge funds closed their convertible bond positions, while new convertible bond issues became more and more rare. This was largely due to persistently low credit default swap (CDS) spreads and low volatilities that limited the potential benefit of convertible bond arbitrage and to regulatory changes that made financing by means of convertible bond a less attractive alternative to straight bond financing than before. In addition, some practitioners blamed this crisis on inadequate understanding of the product, that let people think for a while that convertible bonds were a win-win mixture to both issuers and holders, up to the point where disappointment changed their mind the other way around. So, many actors in the equity-to-credit universe closed their positions after the unexpected simultaneous rise in the General Motors CDS spreads and stock price in May 2005 (see Zuckerman [29]). Accordingly, the industry realized more urgently the need to switch from Black–Scholes to more pertaining models, and to reconsider the approach and use of models in general (see Ayache [3]).

In this paper, we attempt to shed more light on the mathematical modeling of convertible bonds, thus continuing the previous research presented, for instance, in [1, 4, 14, 16, 18, 24, 26, 27, 28]. In particular, we consider the problem of the decomposition of a convertible bond into bond component and option component. This decomposition is indeed well established in the case of an 'exchange option', when the conversion can only occur at maturity (see Margrabe [24]). However, it was not yet studied in the case of a real-life convertible bond. More generally, we shall consider generic *defaultable game options* and *defaultable convertible securities*, which encompass defaultable convertible bonds (and also more standard defaultable American or European options) as special cases. Moreover, we shall examine these contracts in the framework of a fairly general market model in which prices of primary assets are assumed to follow semimartingales (see Delbaen and Schachermayer [15] or Kallsen and Kühn [18]) and a random moment of default is exogenously given.

The paper is organized as follows. In Section 2, we describe the general set-up. In the present paper, we work in a general semimartingale model, which is arbitrage-free, but possibly incomplete. In Section 3, the valuation of game options is reviewed. As a prerequisite for further developments, we provide in Proposition 3.1 a characterization of the set of ex-dividend arbitrage prices of a game option with dividends in terms of related Dynkin games. The proof of this result is based on a rather straightforward application of Theorem 2.9 in Kallsen and Kühn [18]. In Section 4, we introduce the concepts of defaultable game option and defaultable convertible security. As a consequence of Proposition 3.1, we obtain a result on arbitrage pricing of these securities. In Section 5, defaultable convertible bonds are formally defined and their basic properties are analyzed. Also, we introduce the concept of reduced convertible bond, in order to handle the case of a convertible bond with a positive call notice period. Section 6 is devoted to pertinent decompositions of arbitrage prices of game options and convertible bonds. The main result of this section is Theorem 6.1, which furnishes a rigorous decomposition of the arbitrage price of a defaultable game option as the sum of the price of a reference straight bond and an embedded game exchange option.

The present paper provides also a theoretical underpinning for a more extensive research continued in Bielecki et al. [6, 7, 8], where more specific market models are introduced and more explicit valuation and hedging results are established. In [6], we derive valuation results for a game option in the framework of a default risk model based on the hazard process and we provide a characterization of minimal hedging strategies for a game option in terms of a solution of the related doubly reflected backward stochastic differential equation. In [7, 8], we introduce Markovian pre-default models of credit risk and we show howt pricing and hedging problems for convertible bonds can solved with the use of the associated variational inequalities.

2 Primary Market

We assume throughout that the evolution of the primary market can be modeled in terms of stochastic processes defined on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$, where $\mathbb{P}$ denotes the statistical probability measure. We can and do assume that the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ satisfies the usual conditions, and that all $(\mathbb{G}, \mathbb{P})$-semimartingales are càdlàg (recall that $(\mathbb{G}, \mathbb{P})$-semimartingales are also $(\mathbb{G}, \mathbb{Q})$-semimartingales for any $\mathbb{Q} \sim \mathbb{P}$ [11, 25]). Moreover we declare that a *process* has to be ($\mathbb{G}$-)adapted, by definition.

We assume that the primary market is composed of the savings account and of d risky assets, such that, given a finite horizon date $T > 0$:
• the *discount factor* process β, that is, the inverse of the savings account, is a finite variation, continuous, positive and bounded process;
• the prices of primary risky assets are semimartingales.

The primary risky assets, with $\mathbb{R}^d$-valued price process X, pay dividends, whose cumulative value process, denoted by $\mathcal{D}$, is assumed to be a finite variation $\mathbb{R}^d$-valued process. Given the price X, we define the *cumulative price* $\widehat{X}$ of primary risky assets as

$$\widehat{X}_t := X_t + \widehat{\mathcal{D}}_t, \tag{1}$$

where

$$\widehat{\mathcal{D}}_t := \beta_t^{-1} \int_{[0,t]} \beta_u d\mathcal{D}_u$$

(by default we denote by $\int_0^t$ integrals over $(0, t]$, otherwise me mention the domain of integration as a subscript of $\int$). In the financial interpretation, $\widehat{\mathcal{D}}_t$ represents the current value at time t of all dividend payments of the assets over the period $[0, t]$, under the assumption that all dividends are immediately reinvested in the savings account.

A predictable trading strategy (ζ^0, ζ) built on the primary market has the wealth process Y given as,

$$Y_t = \zeta_t^0 \beta_t^{-1} + \zeta_t X_t \quad t \in [0, T] \tag{2}$$

(where ζ is a *row vector*). Accounting for dividends, we say that a portfolio (ζ^0, ζ) is *self-financing* whenever ζ is $\beta\widehat{X}$-integrable and if we have, for $t \in [0, T]$,

$$d(\beta_t Y_t) = \zeta_t d(\beta_t \widehat{X}_t). \tag{3}$$

Note that the related notion of stochastic integral is the generalized notion of *vector* (as opposed to *componentwise*) *stochastic integral* developed in Cherny–Shiryaev [11]. This is indeed the pertaining notion of stochastic integral to be used in relation with the Fundamental Theorems of Asset Pricing such as [15] (see [11, 6]).

In (3), we recognize the standard self-financing condition for a trading strategy (ζ^0, ζ) in non dividend paying primary risky assets, that we shall call *the equivalent non-dividend-paying synthetic assets*, with price vector $\widehat{X}$. In view of this equivalence, the following definition is natural.

Definition 2.1 We say that $(X_t)_{t\in[0,T]}$ *is an arbitrage price for our primary market with dividend-paying assets*, if and only if $(\widehat{X}_t)_{t\in[0,T]}$ is an arbitrage price for the equivalent market with non-dividend-paying synthetic assets, in the sense that $(\widehat{X}_t)_{t\in[0,T]}$ satisfies the standard *No Free Lunch with Vanishing Risk* (NFVLR) condition of Delbaen and Schachermayer [15].

Then, by application of the main theorem in [15], we have that $(X_t)_{t\in[0,T]}$ is an arbitrage price for the primary market if and only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ for which $\beta\widehat{X}$ is a sigma martingale under $\mathbb{Q}$ (see [15, 11]). In the sequel, we assume that $(X_t)_{t\in[0,T]}$ is an arbitrage price for the primary market and we denote by $\mathcal{M}$ the set of *risk-neutral measures* on the primary market, defined as the set of probability measures $\mathbb{Q} \sim \mathbb{P}$ for which $\beta\widehat{X}$ is a sigma martingale under $\mathbb{Q}$.

Note that even though the assumption of market completeness is not formally required for our results, the practical interest of some of them (those based on the converse part in Theorem 3.1) may be limited to the case of complete markets. Otherwise, integrability conditions like (7) below are typically violated (see Remark 4.3). This is not a major practical issue, however, since in practice one can often "complete the market", so that integrability conditions like (7) will be satisfied for the unique risk-neutral measure. For an illustration of this approach, we refer the reader to [8].

3 Game Options

As it is well known, a convertible bond with no call notice period can be formally seen as a special case of the so-called *game option*, introduced in Kifer [20] (see also Kallsen and Kühn [18]). For this reason, we first provide a brief overview of concepts and results related to game options.

3.1 Payoffs of a Game Option

Let 0 (respectively T) stand for the *inception date* (respectively the *maturity date*) of a game option. For any $t \in [0,T]$, we write $\mathcal{G}^t_T$ to denote the set of all stopping times with values in $[t,T]$.

Definition 3.1 A *game option* is a contract with the terminal payoff at time $\tau_p \wedge \tau_c$ given by (from the perspective of the holder)

$$\mathbb{1}_{\{\tau_p \le \tau_c\}} \mathcal{L}_{\tau_p} + \mathbb{1}_{\{\tau_p > \tau_c\}} \mathcal{U}_{\tau_c}, \tag{4}$$

where $\tau_p, \tau_c \in \mathcal{G}^0_T$ are stopping times under the control of the holder and the issuer of a game option respectively. Additionally, a game option pays *dividends*, given by a real-valued process D with finite variation. The *put payoff process* $\mathcal{L} = (\mathcal{L}_t)_{t\in[0,T]}$ and the *call payoff process* $\mathcal{U} = (\mathcal{U}_t)_{t\in[0,T]}$ are càdlàg, $\mathbb{R} \cup \{+\infty\}$-valued processes, such that $\mathcal{L} \le \mathcal{U}$ and $\mathcal{L}_T = \mathcal{U}_T$. Moreover, defining the *cumulative payoffs* of a game option with dividends as the processes $\widehat{\mathcal{L}} := \mathcal{L} + \widehat{D}$ and $\widehat{\mathcal{U}} := \mathcal{U} + \widehat{D}$, where $\widehat{D}_t := \beta_t^{-1} \int_{[0,t]} \beta_u \, dD_u$, we assume that there exists a constant c such that

$$\beta_t \widehat{\mathcal{L}}_t \ge -c, \quad t \in [0,T]. \tag{5}$$

We refer to τ_c (respectively τ_p) as the moment of *call* (respectively *put*) of a game option.

Remarks 3.1 (i) The case of dividend-paying game options is not explicitly dealt with by Kifer [20] or Kallsen and Kühn [18]. However, as we shall argue in what follows, all the results in [18] can be extended to this situation.
(ii) In [18], the payoff processes $\mathcal{L}$ and $\mathcal{U}$ are implicitly assumed to be specified in relative terms with respect to a certain numeraire. In the present work, we prefer to make explicit the presence of the discount factor β.
(iii) Kallsen and Kühn [18] postulate that the lower payoff process $\mathcal{L}$ is non-negative. However, as long as the discounted lower payoff is bounded from below (cf. (5)) all their results are applicable by a simple shift argument.
(iv) One can deduce from (4) that we impose the priority of τ_p over τ_c, meaning that the terminal payment equals $\mathcal{L}_{\tau_p}$ (rather than $\mathcal{U}_{\tau_p}$) on the event $\{\tau_p = \tau_c\}$. We thus follow here Kallsen and Kühn [18], from which we will deduce Proposition 3.1 below. Note, however, that in the general context of game options this assumption is known to be essentially immaterial, in the sense that is has typically no bearing neither on the price of a game option nor on the optimal stopping rules (cf. [20]).

3.2 Arbitrage Valuation of a Game Option

The concept of an arbitrage price of a game option can be introduced in various ways. Kallsen and Kühn [18] make the distinction between a *static* and a *dynamic* approach. The former point of view corresponds to the assumption that only a buy-and-hold strategy in the derivative asset is allowed, whereas the primary assets can be traded dynamically. In the latter approach, it is assumed that a derivative asset becomes liquid and negotiable asset, so that it can be traded together with the primary assets during the whole period $[0,T]$. Consequently, in a dynamic approach, in order to determine a price process of a derivative asset, it is postulated that the extended market, including this derivative asset, remains arbitrage-free. In this work, we shall adopt the dynamic point of view.

For the formal definition of a (dynamic) arbitrage price process of a game option, we refer the reader to Kallsen and Kühn [18, Definition 2.6]. As elaborated in [18], this definition is based on an extension to markets containing game options of the *No Free Lunch with Vanishing Risk* condition, introduced by Delbaen and Schachermayer [15, Definition 2.8], using the notion of an admissible trading strategy involving primary assets and the game option. Without entering into details, let us note that admissible strategies in this sense include, in particular, trading strategies in the primary assets only, provided that the corresponding wealth process is bounded from below. The case of dividend-paying primary assets and/or game option is not explicitly treated in [18]. However, the results of [18] can be applied to the case of dividend-paying primary assets and/or game option by resorting to the transformation of prices into cumulative prices described in Section 2 and that we already used to characterize no-arbitrage prices in our primary risky market with dividends.

As a reality check of pertinency of Kallsen and Kühn's definition of an arbitrage price of a game option and of our extension to the case of dividend-paying assets, we show in forthcoming papers [6, 7, 8] that in more specific models, in which we are able to identify well determined processes as arbitrage prices in the sense of this definition, these processes can alternatively be characterized as minimal hedging prices.

We decided not to reproduce here the full statement of Definition 2.6 in [18], since it is rather technical and will not be explicitly used in the sequel. To proceed, it will be enough for us to make use of the following characterization of an arbitrage price.

We are interested in studying a problem of time evolution of an arbitrage price of a game option. Therefore, we shall formulate the problem in a dynamic way by pricing the game option at any time $t \in [0,T]$. Given $t \in [0,T]$ and stopping times $\tau_p, \tau_c \in \mathcal{G}_T^t$, let the *ex-dividend cumulative cash flow of the game option at time* t stand for the random variable $\theta(t;\tau_p,\tau_c)$ such that

$$\beta_t\theta(t;\tau_p,\tau_c) := \beta_\tau \widehat{D}_\tau - \beta_t \widehat{D}_t + \beta_\tau \Big(\mathbb{1}_{\{\tau=\tau_p\}} \mathcal{L}_{\tau_p} + \mathbb{1}_{\{\tau<\tau_p\}} \mathcal{U}_{\tau_c} \Big)$$

with $\tau = \tau_p \wedge \tau_c$. We shall argue that $\theta(t;\tau_p,\tau_c)$ represents the terminal cash flow paid at time τ of a non-dividend paying game option equivalent to the original game option with dividends. Note that the random variable $\theta(t;\tau_p,\tau_c)$ is not $\mathcal{G}_t$-measurable for $t<T$, but it is merely $\mathcal{G}_\tau$-measurable. This is, of course, expected, since it represents payments occurring between the current date t and the exercise time τ.

The proof of the following result relies on a rather straightforward application of Theorem 2.9 in Kallsen and Kühn [18].

Proposition 3.1 *Assume that a real-valued process* $(\Theta_t)_{t\in[0,T]}$ *satisfies the following two conditions:*
(i) Θ *is a semimartingale and*
(ii) *there exists* $\mathbb{Q} \in \mathcal{M}$ *such that* Θ *is the* $\mathbb{Q}$*-value process of the Dynkin game related to the game option, in the sense that*

$$\begin{aligned} \operatorname{esssup}_{\tau_p\in\mathcal{G}_T^t} \operatorname{essinf}_{\tau_c\in\mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big) &= \Theta_t \\ &= \operatorname{essinf}_{\tau_c\in\mathcal{G}_T^t} \operatorname{esssup}_{\tau_p\in\mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big), \quad t\in[0,T]. \end{aligned} \tag{6}$$

Then the $\mathbb{R}^{d+1}$*-valued process* (X,Θ) *is an (ex-dividend) arbitrage price for the extended market composed of the primary market and the game option. Moreover, the converse holds true under the following integrability condition*

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,T]} \beta_t \widehat{\mathcal{L}}_t \,\Big|\, \mathcal{G}_0 \Big) < \infty, \quad \text{a.s.} \tag{7}$$

Recall that the fact that the Dynkin game has a *value* at time t means that we have equality between the lower value of the game, corresponding to the left-hand side of (6), and the upper value, as given by its right-hand side. It is well known that the lower value of a game is always less or equal to the upper value, but they do not need to coincide, in general. For general results on Dynkin games, see, for instance, Dynkin [17], Kifer [21], Lepeltier and Maingueneau [23].

In the situation of Proposition 3.1, we shall briefly say in the sequel that $(\Theta_t)_{t\in[0,T]}$ is an *arbitrage price for the game option*, whenever $(X_t, \Theta_t)_{t\in[0,T]}$ is an arbitrage price for the extended market consisting of the primary market and the game option.

Proof of Proposition 3.1. By the definition of arbitrage prices of dividend-paying assets, $(X_t, \Theta_t)_{t\in[0,T]}$ is an arbitrage price for the extended market with dividends, if and only if $(\widehat{X}_t, \widehat{\Theta}_t)_{t\in[0,T]}$ is an arbitrage price for the equivalent extended market without dividends, where $\widehat{\Theta}_t := \Theta_t + \widehat{D}_t$. Now, by an application of Kallsen and Kühn [18, Theorem 2.9], under condition (7) (which is actually only used for the converse part of the theorem), this is equivalent to the fact that $\beta\widehat{X}$ is a sigma martingale under some $\mathbb{P}$-equivalent probability measure $\mathbb{Q}$, and that $\widehat{\Theta}$ is a semimartingale equal to the $\mathbb{Q}$-value of the Dynkin game without dividends and with terminal payoffs $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{U}}$. Specifically, $\widehat{\Theta}$ satisfies, for $t \in [0, T]$,

$$\begin{aligned}\operatorname{esssup}_{\tau_p\in\mathcal{G}_T^t}\operatorname{essinf}_{\tau_c\in\mathcal{G}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\widehat{\theta}_t(\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big) &= \widehat{\Theta}_t \\ &= \operatorname{essinf}_{\tau_c\in\mathcal{G}_T^t}\operatorname{esssup}_{\tau_p\in\mathcal{G}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\widehat{\theta}_t(\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big)\end{aligned} \tag{8}$$

with $\widehat{\theta}_t(\tau_p, \tau_c) = \theta_t(\tau_p, \tau_c) + \widehat{D}_t$, or equivalently,

$$\beta_t\widehat{\theta}_t(\tau_p,\tau_c) = \beta_\tau\Big(\mathbb{1}_{\{\tau=\tau_p\}}\widehat{\mathcal{L}}_{\tau_p} + \mathbb{1}_{\{\tau<\tau_p\}}\widehat{\mathcal{U}}_{\tau_c}\Big).$$

It now suffices to observe that (8) is equivalent to (6). □

Proposition 3.1 essentially reduces the study of an arbitrage price of a game option to the study of the value, under a risk-neutral measure $\mathbb{Q}$, of the corresponding Dynkin game, with the issuer playing the role of the minimizer and the holder being the maximizer. It is not surprising that this general result covers in particular the case of American and European options.

Definition 3.2 An *American option* is a game option with $\mathcal{U}_t = \infty$ for $t \in [0, T)$. A *European option* is an American option such that

$$\beta_t\widehat{\mathcal{L}}_t \le \beta_T\widehat{\mathcal{L}}_T, \quad t \in [0,T]. \tag{9}$$

By applying Proposition 3.1, we deduce that the $\mathbb{Q}$-value $\widehat{\Theta}_t$ of an American option becomes the essential supremum with respect to stopping times $\tau_p \in \mathcal{G}_T^t$, specifically,

$$\widehat{\Theta}_t = \operatorname{esssup}_{\tau_p\in\mathcal{G}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\widehat{\theta}_t(\tau_p,T)\,\big|\,\mathcal{G}_t\big) = \beta_t^{-1}\operatorname{esssup}_{\tau_p\in\mathcal{G}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\beta_{\tau_p}\widehat{\mathcal{L}}_{\tau_p}\,\big|\,\mathcal{G}_t\big),$$

whereas for a European option it reduces to the following conditional expectation

$$\widehat{\Theta}_t = \mathbb{E}_{\mathbb{Q}}\big(\widehat{\theta}_t(T,T)\,\big|\,\mathcal{G}_t\big) = \beta^{-1}\,\mathbb{E}_{\mathbb{Q}}\big(\beta_T\widehat{\mathcal{L}}_T\,\big|\,\mathcal{G}_t\big).$$

4 Defaultable Game Options and Convertible Securities

In this section, we introduce fairly general subclasses of game options, namely *defaultable game option* and *defaultable convertible securities* (CS, for short), which encompass as special cases such financial instruments as convertible bonds, which will be discussed in some detail in Section 5.1, or convertible preferred stocks, as well as defaultable American or European options.

4.1 Defaultable Game Options

Let an $[0, +\infty]$-valued stopping time τ_d represent the *default time* of a reference entity. In broad terms, a *defaultable game option* is a game option with the following cash flows that are paid by the

issuer of the contract and received by the holder of the contract:
• a *dividend stream* D_t subject to rules specified in the contract,
• a *put payment* L_t made at the put time $t = \tau_p$ chosen by the holder if $\tau_p \leq \tau_c$ and $\tau_p < \tau_d \wedge T$; the rules governing the determination of the amount L_t are specified in the contract;
• a *call payment* U_t made at time $t = \tau_c$ (chosen by the issuer) provided that $\tau_c < \tau_p \wedge \tau_d \wedge T$; moreover, the call time may be subject to the constraint that $\tau_c \geq \bar{\tau}$, where $\bar{\tau}$ is the lifting time of the call protection; the rules governing the determination of the amount U_t are specified in the contract,
• a *payment at maturity* ξ made at time T provided that $T < \tau_d$ and $T \leq \tau_p \wedge \tau_c$.

Moreover, the contract is terminated at default time τ_d if $\tau_d \leq \tau_p \wedge \tau_c \wedge T$. In particular, there are no more cash flows related to this contract after τ_d. In this setting the dividend stream D additionally includes a possible recovery payment made at the default time. Of course, there is also the initial cash flow, namely, the purchasing price of the contract paid at the initiation time by the holder and received by the issuer.

The informal description of a defaultable game option is formalized through the following definition, in which H stands for the *default indicator process* $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$ of the reference entity.

Definition 4.1 A *defaultable game option* (GO) is a game option with cumulative put and call payoff processes $\widehat{\mathcal{L}} = (\widehat{\mathcal{L}}_t)_{t\in[0,T]}$ and $\widehat{\mathcal{U}} = (\widehat{\mathcal{U}}_t)_{t\in[0,T]}$ given by

$$\widehat{\mathcal{L}}_t = \widehat{D}_t + \mathbb{1}_{\{\tau_d > t\}}\big(\mathbb{1}_{\{t<T\}} L_t + \mathbb{1}_{\{t=T\}}\xi\big), \tag{10}$$

$$\widehat{\mathcal{U}}_t = \widehat{D}_t + \mathbb{1}_{\{\tau_d > t\}}\big(\mathbb{1}_{\{t<T\}} \bar{U}_t + \mathbb{1}_{\{t=T\}}\xi\big), \tag{11}$$

where:
• $\widehat{D}_t = \beta_t^{-1}\int_{[0,t]}\beta_u\,dD_u$, where the *dividend process* $D = (D_t)_{t\in[0,T]}$ equals

$$D_t = \int_{[0,t]}(1 - H_u)\,dC_u + \int_{[0,t]} R_u\,dH_u;$$

here, the *coupon process* $C = (C_t)_{t\in[0,T]}$ is a process with bounded variation and the *recovery process* $R = (R_t)_{t\in[0,T]}$ is a real-valued process;
• the *put/conversion payment* process $L = (L_t)_{t\in[0,T]}$ is a real-valued, càdlàg process;
• the process $\bar{U} = (\bar{U}_t)_{t\in[0,T]}$ equals

$$\bar{U}_t = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\geq\bar{\tau}\}}U_t,$$

where the lifting time of a *call protection* is modeled as a given stopping time $\bar{\tau} \in \mathcal{G}_T^0$ and where the *call payment* $U = (U_t)_{t\in[0,T]}$ is a real-valued, càdlàg process such that $L_t \leq \bar{U}_t$ for $t \in [0, \tau_d \wedge T)$, or equivalently

$$L_t \leq U_t \quad \text{for} \quad t \in [\tau_d \wedge \bar{\tau}, \tau_d \wedge T); \tag{12}$$

• the *payment at maturity* ξ is a $\mathcal{G}_T$-measurable real random variable.

Moreover, R, L and ξ are assumed to be bounded from below, hence (5) is satisfied.

Convention. In what follows, we shall consider various sub-classes of defaultable game options. For brevity, we shall usually omit the term *defaultable* so that we shall refer to game options, American options, convertible securities, etc., rather than defaultable game options, defaultable American options, defaultable convertible securities, etc. In particular, the general notions of game options, American options and European options introduced in Section 3, are no longer used in the sequel.

Recall that $\mathcal{G}_T^t$ denotes the set of all stopping times with values in $[t, T]$. For any $t \in [0, T]$, let also $\bar{\mathcal{G}}_T^t$ stand for $\{\tau \in \mathcal{G}_T^t\,;\, \tau \wedge \tau_d \geq \bar{\tau} \wedge \tau_d\}$, where the *lifting time of a call protection of a game option*, $\bar{\tau}$, is given in $\mathcal{G}_T^0$. Note that in the case of a game option, given the specification (10)-(11) of $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{U}}$ with $\bar{U}_t = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\geq\bar{\tau}\}}U_t$, condition (6) can be rewritten as follows

$$\begin{aligned}\text{esssup}_{\tau_p\in\mathcal{G}_T^t}\text{essinf}_{\tau_c\in\bar{\mathcal{G}}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big) &= \Pi_t \\ = \text{essinf}_{\tau_c\in\bar{\mathcal{G}}_T^t}\text{esssup}_{\tau_p\in\mathcal{G}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big), &\quad t \in [0,T],\end{aligned} \tag{13}$$

where for $t \in [0,T]$ and $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$, the *ex-dividend cumulative cash flow of a game option* is given by

$$\beta_t \pi(t; \tau_p, \tau_c) := \beta_\tau \widehat{D}_\tau - \beta_t \widehat{D}_t + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau < \tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau = T\}} \xi \Big) \tag{14}$$

with $\tau = \tau_p \wedge \tau_c$. We thus have the following theorem, as a rather straightforward consequence of Proposition 3.1.

Theorem 4.1 *If a process* Π *is a semimartingale and if there exists* $\mathbb{Q} \in \mathcal{M}$ *such that (13) is verified, then* Π *is an arbitrage price for the game option with ex-dividend cumulative cash flow* π*. Moreover, the converse holds true provided condition (7) is satisfied for* $\widehat{\mathcal{L}}$ *given by (10).*

Remarks 4.1 (i) The restriction that the issuer of game option is prevented from making a call on some random time interval $[0, \bar{\tau})$ where $\bar{\tau} \in \mathcal{G}_T^0$ (see the informal description of a game option) is implicitly enforced in Definition 4.1 by putting $\bar{U}_t = \infty$ on the random interval $[0, \bar{\tau})$.
(ii) Note that $\pi(t; \tau_p, \tau_c) = 0$ for any $t \geq \tau_d$. Therefore, the (ex-dividend) arbitrage price of a game option is necessarily equal to 0, for $t \geq \tau_d$. In what follows, an arbitrage price associated with a risk-neutral measure $\mathbb{Q}$ as in Theorem 4.1, will be called the $\mathbb{Q}$*-price* of a game option.
(iii) In view of our formulation of the problem, the put or call decisions may take place after the default time τ_d. Nevertheless, the discounted cumulative payoff processes $\beta\widehat{\mathcal{L}}$ and $\beta\widehat{\mathcal{U}}$ are constant on the set $\{t \geq \tau_d\}$ (note that the processes D and $\beta\widehat{D}$ are stopped at τ_d). Thus, effectively, the game is stopped at the default time τ_d unless the decision to stop it was already made prior to τ_d.

Recall that we also have the companion concepts of an American option and a European option, namely, a game option that is either an American or a European option in the sense of Definition 3.2. An *American option*, namely a game option with $\bar{U} = \infty$, can equivalently be seen as a non-callable game option, namely a game option with $\bar{\tau} = T$.

Definition 4.2 An American option becomes a European option provided that L is chosen to be a negatively large enough constant (depending on the other data of the American option). In the special case where R and ξ are bounded (from below and from above), such a European option will be referred to as an *elementary security* (ES).

Remarks 4.2 Consider a defaultable coupon-paying bond with (positive or negative) bounded coupons, bounded recovery payoff and a bounded face value. This bond can be formally treated as an ES, provided that we take $\bar{U} = \infty$ and we additionally introduce the constant process L which makes the inequality $\beta_t \widehat{\mathcal{L}}_t \leq \beta_T \widehat{\mathcal{L}}_T$ hold for every $t \in [0,T]$. Of course, the choice of L is somewhat arbitrary, in the sense that L will not appear explicitly in the valuation formula for the bond (see part (ii) in Theorem 4.2).

We shall now apply Theorem 4.1 in order to characterize arbitrage prices of an American option and a European option.

Theorem 4.2 (i) *If a process* $\bar{\Pi} = (\bar{\Pi}_t)_{t \in [0,T]}$ *is a semimartingale and there exists* $\mathbb{Q} \in \mathcal{M}$ *such that*

$$\bar{\Pi}_t = \operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\bar{\pi}(t; \tau_p) \,\big|\, \mathcal{G}_t\big), \quad t \in [0,T],$$

where the ex-dividend cumulative cash flow $\bar{\pi}(t; \tau_p)$ *of an American option can be represented as follows, for* $t \in [0,T]$ *and* $\tau_p \in \mathcal{G}_T^t$,

$$\beta_t \bar{\pi}_t(\tau_p) = \beta_{\tau_p} \widehat{D}_{\tau_p} - \beta_t \widehat{D}_t + \mathbb{1}_{\{\tau_d > \tau_p\}} \beta_{\tau_p} \big(\mathbb{1}_{\{\tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau_p = T\}} \xi \big),$$

then $\bar{\Pi}$ *is an arbitrage price of the related American option. Moreover, the converse holds true provided that (7) is satisfied for* $\widehat{\mathcal{L}}$ *given by (10).*
(ii) *If there exists* $\mathbb{Q} \in \mathcal{M}$ *such that*

$$\Phi_t = \mathbb{E}_{\mathbb{Q}}(\phi(t) \,|\, \mathcal{G}_t), \quad t \in [0,T],$$

where the ex-dividend cumulative cash flow $\phi(t)$ of a European option can be represented as follows, for $t \in [0,T]$,

$$\beta_t \phi(t) = \beta_T \widehat{D}_T - \beta_t \widehat{D}_t + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi, \tag{15}$$

then the process $\Phi = (\Phi_t)_{t \in [0,T]}$ is an arbitrage price of the related European option. Moreover, the converse holds true provided that

$$\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} \big(\beta_T \widehat{\mathcal{L}}_T \,\big|\, \mathcal{G}_0 \big) < \infty, \quad \text{a.s.}, \tag{16}$$

where $\widehat{\mathcal{L}}_T = \widehat{D}_T + \mathbb{1}_{\{\tau_d > T\}} \xi$.

Proof. Since an American option and a European option are special cases of a game option, their ex-dividend cumulative cash flows are given by the general formula (14). By saying that they can be represented as $\bar{\pi}(t;\tau_p)$ and $\phi(t)$, respectively, we mean that for the valuation purposes the general payoff $\pi(t;\tau_p,\tau_c)$ can be reduced to either $\bar{\pi}(t;\tau_p)$ or $\phi(t)$. Note that, consistently with the notation, an American cash flow $\bar{\pi}(t;\tau_p)$ does not depend on τ_c whereas a European cash flow $\phi(t)$ is independent of both τ_p and τ_c.

Part (i) of the theorem follows by a straightforward application of Theorem 4.1. To prove the second part, we observe that

$$\beta_t \Phi_t = \mathbb{E}_{\mathbb{Q}}(\beta_t \phi(t) \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(\beta_T \widehat{D}_T + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi \,|\, \mathcal{G}_t) - \beta_t \widehat{D}_t,$$

and thus Φ given by (15) is a semimartingale. □

4.2 Defaultable Convertible Securities

Let us now introduce the concept of a *defaultable convertible security*, referred to as a *convertible security* or briefly a CS in what follows. A CS is a financial contract that can be situated somewhere between a defaultable game option and a much more specific *defaultable convertible bond*, which will be discussed in some detail in Section 5. Let S denote one of the primary risky assets, to be interpreted as the *underlying asset* of a CS.

In broad terms, a *convertible security* (CS) with underlying S is a game option with recovery process R such that:

- the put payment L_t represents in fact a *put/conversion payment* made at the put/conversion time $t = \tau_p$; usually, the payment L_t depends on the value S_t of the underlying asset and it corresponds to the right of the holder of the CS to convert it to a predetermined number of shares of this asset – hence the name of convertible – or to receive a predetermined cash flow;
- conversion is typically still possible at default time τ_d or at maturity time T, provided that the CS is still alive.

The specific nature of CS payments motivates the following definition.

Definition 4.3 A *defaultable convertible security* (CS) with the underlying asset S is a game option such that the processes R, L and the random variable ξ satisfy the following inequalities, for some positive reals a, b, c:

$$\begin{aligned} -c &\leq R_t \leq a \vee bS_t, \quad t \in [0,T], \\ -c &\leq L_t \leq a \vee bS_t, \quad t \in [0,T], \\ -c &\leq \xi \leq a \vee bS_T. \end{aligned} \tag{17}$$

Given our assumptions, we then have for (modified) positive reals a, b :

$$\widehat{\mathcal{L}}_t \leq a \vee bS_{t \wedge \tau_d}, \quad t \in [0,T], \tag{18}$$

so that in the case of a CS, the following condition enforces (7):

$$\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} \Big(\sup_{t \in [0, T \wedge \tau_d]} S_t \,\Big|\, \mathcal{G}_0 \Big) < \infty, \quad \text{a.s.} \tag{19}$$

Remarks 4.3 In view of (19), it should be emphasized that whether condition (7) holds or does not hold crucially depends on what is chosen as primary market (S and the other primary risky assets). So in a simple jump-diffusion model for S, condition (19) typically fails to hold if the underlying market consists of the savings accounts and S alone, whereas it is satisfied in a market completed by a CDS (see [8]).

Let us finally note that an ES (see Definition 4.2) is a special case of a CS.

5 Defaultable Convertible Bonds

We shall now address the issue of arbitrage valuation of a convertible bond with real-life features, including the call notice period. As will be explained in Section 5.3, it is rather difficult to directly value a convertible bond with a positive call notice period, since it does not fulfill conditions of the definition of a game option, in general. To circumvent this difficulty, we shall introduce the concept of a *reduced convertible bond*, that is, a convertible bond whose value upon call is exogenously given as a certain stochastic process. In that case, we may assume, without loss of generality, that the bond has no call notice period. Since a reduced convertible bond is a special case of a convertible security (hence it is a game option) the valuation results in the previous sections are directly applicable to a reduced convertible bond.

5.1 Covenants of a Defaultable Convertible Bond

To describe the covenants of a typical *defaultable convertible bond* (CB), we introduce the following additional notation:

$\bar{N}$: the par (nominal) value,

c_t^{cb}: the continuous coupon rate, a bounded process,

T_i, c^i, $i = 0, 1, \ldots, K$ ($T_0 = c^0 = 0$): coupon dates and amounts; the coupon dates $T_0, \ldots, T_K$ are deterministic fixed times with $T_{K-1} < T \leq T_K$; the coupon amounts c^i are $\mathcal{F}_{T_{i-1}}$-measurable and bounded, for $i = 1, 2, \ldots, K$,

A_t: the accrued interest at time t, specifically,

$$A_t = \frac{t - T_{i_t-1}}{T_{i_t} - T_{i_t-1}} \, c^{i_t},$$

where i_t is the integer satisfying $T_{i_t-1} \leq t < T_{i_t}$; in view of our assumptions on the coupons, $(A_t)_{t\in[0,T]}$ is a càdlàg process,

$\bar{R}_t$: the recovery process on the CB upon default of the issuer at time t, a bounded process,

κ : the conversion factor,

$R_t^{cb} = \bar{R}_t \vee \kappa S_t$: the effective recovery process,

D_t^{cb} : the cumulative dividend process (to be specified below),

$\xi^{cb} = \bar{N} \vee \kappa S_T + A_T$: the payoff at maturity,

$\bar{P} \leq \bar{C}$: the put and call nominal payments, respectively,

$\delta \geq 0$: the length of the call notice period (see the detailed description below),

$t^\delta = (t + \delta) \wedge T$: the end date of the call notice period started at t.

We shall now present a detailed description of specific CB covenants. Let us consider a CB at any date $t \in [0, T]$ at which it is still alive. Then we have the following provisions:

put/conversion provision – at any time $\tau_p \in [t, \tau_c \wedge \tau_d \wedge T]$, where τ_c is a stopping time under the discretion of the issuer, the bond holder may convert a CB to κ shares of equity. In addition, at any time $\tau_p \in [t, \tau_c \wedge \tau_d \wedge T)$, and possibly also at τ_c if $\tau_c < \tau_d \wedge T$, the holder may put (return) the bond to the issuer for a nominal put payment $\bar{P}$ pre-agreed at time of issuance.

Only one of the two above decisions may be executed. Since the bond holder is also entitled to receive a relevant accrued interest payment, the *effective put/conversion payment* collected in case of put or conversion (depending on which one is more favorable to the holder) at time τ_p (if $\tau_p < T$) equals $L^{cb}_{\tau_p} = \bar{P} \vee \kappa S_{\tau_p} + A_{\tau_p}$, where κ denotes the conversion ratio. The effective put payment in case $\tau_p = T$ is considered separately (see the *promised payment* below).

call provision – the issuer has the right to call the bond at any time $\tau_c \in [t, \tau_p \wedge \tau_d \wedge T)$, where τ_p is a random time under the discretion of the holder, for a nominal call payment $\bar{C}$ pre-agreed at time of issuance. More precisely, there is a fixed call notice period $\delta \geq 0$ (typically, one month) such that if the issuer calls the bond at time τ_c, then the bond holder has either to redeem the bond for $\bar{C}$ or convert the bond into κ shares of stock, at any time u at its convenience in $[\tau_c, \tau_c^\delta]$, where $\tau_c^\delta = (\tau_c + \delta) \wedge T$. Accounting for accrued interest, the *effective call/conversion payment* to the holder at time u is $\bar{C} \vee \kappa S_u + A_u$.

call protection – typically, a CB also includes *call protections*, either *hard* or *soft*. For instance, the issuer's right to call a CB early becomes active only after a certain period of time has lapsed since the original issue date. A CB, which can't be called under any circumstances during the initial time period $[0, \bar{T})$, is subject to *hard* call protection. Alternatively, a CB that is non-callable unless the stock price reaches a certain predetermined level, say $\bar{S}$, is subject to *soft* call protection. The introduction of the stopping times $\bar{\tau}$ in $\mathcal{G}^0_T$ and of the associated class $\bar{\mathcal{G}}^t_T \subseteq \mathcal{G}^t_T$ allow one to model quite general kinds of call protections. So hard call protections correspond to $\tau_c \in \bar{\mathcal{G}}^t_T$ with $\bar{\tau} = \bar{T}$ and standard soft call protections to $\tau_c \in \bar{\mathcal{G}}^t_T$ with $\bar{\tau} = \inf\{t \in \mathbb{R}_+ \,;\, S_t \geq \bar{S}\} \wedge T$.

promised payment – the issuer agrees to pay to the bond holder, at any coupon date T_i such that $T_i < \tau_d$ and $T_i \leq \tau_p \wedge \tau_c \wedge T$, a bounded coupon amount c^i. He also agrees to pay the par value $\bar{N}$ at the maturity date T, provided that $T < \tau_d$ and $T \leq \tau_p \wedge \tau_c$. Since the bond holder may still convert at time T, we define the *effective payment at maturity* as $\xi^{cb} = \bar{N} \vee \kappa S_T + A_T$; it is collected at time T if the CB is still alive at T.

recovery structure at default – it is assumed throughout that in the case of default at time $\tau_d \leq \tau_p \wedge \tau_c \wedge T$, the *effective recovery* $R^{cb}_{\tau_d} = \bar{R}_{\tau_d} \vee \kappa S_{\tau_d}$ is recovered. Indeed, we assume that the CB can still be converted at default time τ_d.

It is typically assumed that $\bar{P} \leq \bar{N} \leq \bar{C}$, which we also suppose in the following.

Remarks 5.1 (i) As specified above, at maturity the bond holder is allowed to convert, but not to put, the bond. Some authors allow for a put decision at maturity date as well. In fact, allowing put decisions at maturity would not change anything, as long as one supposes (as we do) that $\bar{P} \leq \bar{N}$. Indeed, if $\bar{P} \leq \bar{N}$, we have $N_T = (\bar{N} \vee \kappa S_T) + A_T = (\bar{P} \vee \bar{N} \vee \kappa S_T) + A_T$.
(ii) It should be stressed that we do not consider the default decision to be a decision variable in the sense of 'optimal default' studied in corporate finance. In other words, the default time is exogenously given random time, as opposed to call and put/conversion times. It would be possible to extend our study by allowing for two possible times of default: the exogenous time τ_d^{ex} chosen by the nature and the endogenous default time τ_d^{en} which is optimally chosen by the bond issuer. Note that τ_d^{en} must not be identified with τ_c since call provisions are parts of the contract, whereas the bankruptcy provisions are not.
(iii) An important issue in the valuation of a CB is the so-called *dilution effect*. Dilution is the fact that the equity price may drop upon conversion, due to the sudden increase of the number of shares in circulation [13]. In practice, the importance of this effect depends on the number of bond holders, who decide to convert simultaneously. In our framework, we deal with a representative holder, who is supposed to make optimal decisions. Therefore, the whole issue of the convertible bond will be converted at the same time, so that a jump in the stock price upon conversion is expected. To account for dilution, one could introduce a fractional loss $0 \leq \nu \leq 1$ of the stock price at put or conversion, so that $\bar{P} \vee \kappa S_{\tau_p} = \bar{P} \vee \kappa(1-\nu) S_{\tau_p -}$. However, in the abstract framework considered in this paper, this would be immaterial.
(iv) A further possible covenant of a CB is *resettability*. *Resettability* means that to compensate

for fluctuations in S, the conversion ratio κ may depend on S_t in a particular way specified in the bond indenture. It is straightforward to check that all the results in this paper remain valid, if one assumes that $\kappa_t = \kappa(S_t)$ for some bounded Borel function κ.
(v) There exist soft call protection clauses more sophisticated than the one mentioned above, such as clauses preventing the issuer to call a CB unless the stock has been above a certain level for a given amount of time. A soft call protection always introduces a certain path-dependency to the valuation problem (cf. [2, 22]). However, we shall see that it does not complicate much the analysis from a general point of view. Naturally, it makes computationally heavier the numerical resolution of the pricing variational inequalities in a Markovian model (see [8]).
(vi) In practice, coupons of a CB are purely discrete. However, frequently, in the literature on CBs or in CB software pricing models, a simplifying assumption is made that coupons are paid continuously. Here both forms of coupons are represented.
(vii) In practice, $\bar{R}$ is generally specified as $\bar{X}\bar{Y}$, where:
– the *default claim* process $\bar{X}$ is specified in the indenture of a CB. Typically, $\bar{X}$ is simply equal to the bond par value, or the bond par value plus the accrued interest;
– the *recovery rate* process $\bar{Y}$ depends on legal specifications, such as the seniority of the related debt, etc. In practice, $\bar{Y}$ tends to be lower in periods with more defaults. However, this statistical observation holds under the real-world probability, with no obvious consequences under the market pricing measure [5]. A common recovery assumption is the so-called *face recovery* assumption, which means that $\bar{X}$ is equal to $\bar{N}$ and that $\bar{Y}$ is a given constant (typically, $\bar{Y} = 40\%$ for investment grade issues).
(viii) Upon default, the stock price process typically falls sharply. To account for this effect, one should introduce, in a model for the stock process S, a fractional loss upon default $0 \le \eta \le 1$, such that $S_{\tau_d} = (1-\eta)S_{\tau_d -}$ (see, for instance, [7]). However, in the abstract framework considered in this paper, this particular feature of the stock price is irrelevant.

Definition 5.1 In accordance with the CB covenants, the dividend process D^{cb} of a CB is given by the expression

$$D_t^{cb} = \int_0^{t\wedge\tau_d} c_u^{cb}\,du + \sum_{0\le T_i\le t,\, T_i<\tau_d} c^i + \mathbb{1}_{\{0\le\tau_d\le t\}} R_{\tau_d}^{cb}, \quad t\in[0,T]. \tag{20}$$

As in Section 4.1, we define the auxiliary process $\widehat{D}^{cb}$ representing the cumulative dividends of a CB by setting

$$\widehat{D}_t^{cb} = \beta_t^{-1}\int_{[0,t]} \beta_u\, dD_u^{cb}, \quad t\in[0,T].$$

Recall also that we write

$$R_{\tau_d}^{cb} = \bar{R}_{\tau_d} \vee \kappa S_{\tau_d}, \quad L_t^{cb} = \bar{P}\vee\kappa S_t + A_t, \quad \xi^{cb} = \bar{N}\vee\kappa S_T + A_T.$$

5.2 Convertible Bonds without Call Notice Period

Let us first assume that a convertible bond has no call notice period so that $\delta = 0$.

Definition 5.2 A *convertible bond with no call notice period* is a convertible security with the cumulative put and call payoff processes $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{U}}$ given by the expressions

$$\widehat{\mathcal{L}}_t^{cb} = \widehat{D}_t^{cb} + \mathbb{1}_{\{\tau_d>t\}}\left(\mathbb{1}_{\{t<T\}}L_t^{cb} + \mathbb{1}_{\{t=T\}}\xi^{cb}\right), \tag{21}$$

$$\widehat{\mathcal{U}}_t^{cb} = \widehat{D}_t^{cb} + \mathbb{1}_{\{\tau_d>t\}}\left(\mathbb{1}_{\{t<T\}}\bar{U}_t^{cb} + \mathbb{1}_{\{t=T\}}\xi^{cb}\right), \tag{22}$$

where we set

$$\bar{U}_t^{cb} = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\ge\bar{\tau}\}}(\bar{C}\vee\kappa S_t + A_t), \quad t\in[0,T]. \tag{23}$$

It is a routine task to check that the processes $\widehat{\mathcal{L}}^{cb}$ and $\widehat{\mathcal{U}}^{cb}$ satisfy all technical assumptions stated in Section 4.1. The arbitrage valuation of a convertible bond with no call notice period is thus covered by Theorem 4.1. A more challenging issue is the arbitrage valuation of convertible bonds with a positive call notice period.

5.3 Reduced Convertible Bonds

In Section 5.4, we shall discuss a CB with a positive call notice period and we shall propose a recursive procedure to value such a bond. In the first step, we shall value this bond upon call. In the second step, we shall use this price as the payoff at call time of a CB with no call notice period. This idea motivates us to introduce the following auxiliary concept.

Definition 5.3 A *reduced convertible bond* (RB) is a convertible security with the cumulative put payoff process $\widehat{\mathcal{L}}^{cb}$ given by (21) and the cumulative call payoff process $\widehat{\mathcal{U}}^{cb}$ given by (22) with

$$\bar{U}_t^{cb} = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\geq\bar{\tau}\}}U_t^{cb}, \quad t \in [0,T],$$

where $(U_t^{cb})_{t\in[0,T]}$ is a càdlàg process that is required to satisfy the following inequality

$$U_t^{cb} \geq \bar{C} \vee \kappa S_t + A_t, \quad t \in [0,T]. \tag{24}$$

The financial interpretation of the exogenously given process U^{cb} is that U_t^{cb} represents the value of our reduced convertible bond upon a call at time t.

Note that a CB with no call notice period is an RB (with U^{cb} defined by equality in (24)). The same remark applies to a *puttable bond* (PB), that is, a convertible bond with no call clause. In the latter case, we also set $\bar{\tau} = T$; hence a puttable bond is a special case of an American option.

Since a reduced convertible bond is a convertible security (hence a game option), in order to obtain a characterization of an arbitrage price of a reduced convertible bond it suffices to apply Theorem 4.1.

In the next section, we shall examine a method of interpreting and valuing a convertible bond with positive call notice period as a reduced convertible bond, based on an endogenous specification of the random variable U_t^{cb} as an arbitrage price of a certain puttable bond starting at time t.

5.4 Convertible Bonds with Positive Call Notice Period

A convertible bond with a positive call notice period δ is a contract involving the following decisions:
- the holder's decision to put/convert the bond at time τ_p,
- the issuer's decision to call the bond at time τ_c, and
- the holder's decision to put/convert the bond at time $\tau_p' \in [\tau_c, \tau_c + \delta]$, assuming that the bond has been called at time τ_c.

This leads to the following definition involving three stopping times.

Definition 5.4 A *convertible bond with a positive call notice period* is a contract with cumulative payoff, as seen from the perspective of the holder, given by

$$\mathbb{1}_{\{\tau_p\leq\tau_c\}}\widehat{\mathcal{L}}_{\tau_p}^{cb} + \mathbb{1}_{\{\tau_p>\tau_c\}}\widehat{\mathcal{U}}_{\tau_p'}^{c}, \tag{25}$$

where $\tau_c \in \bar{\mathcal{G}}_T^0$ is a stopping time under the control of the issuer, $\tau_p \in \mathcal{G}_T^0$ and $\tau_p' \in \mathcal{G}_{\tau_c^\delta}^{\tau_c}$ are stopping times under the control of the holder. The cumulative payoff is paid at τ_p on the event $\{\tau_p \leq \tau_c\}$ and τ_p' on the event $\{\tau_p > \tau_c\}$. Moreover, $\widehat{\mathcal{L}}^{cb}$ is given by (21) and $\widehat{\mathcal{U}}^c$ is given by the formula

$$\widehat{\mathcal{U}}_t^c = \widehat{D}_t^{cb} + \mathbb{1}_{\{\tau_d>t\}}(\bar{C} \vee \kappa S_t + A_t). \tag{26}$$

When a convertible bond with a positive call notice period is called at some date t, it becomes a puttable bond (see e.g. Kwok and Lau [22]). This particular puttable bond (puttable in a broad sense, meaning that it can be either convertible, or both convertible and puttable), which is referred to as the t-PB in what follows, is endowed with the same characteristics as a considered convertible bond, except that:

(i) the inception date of the t-PB is t, its maturity is $t^\delta = (t+\delta) \wedge T$, and its nominal is equal to the call payment $\bar{C}$,

(ii) the coupon schedule of the t-PB is the trace on $(t, t^\delta]$ of the coupon schedule of a CB,

(iii) the effective put/conversion payment of the t-PB is equal to the effective call/conversion payment $\bar{C} \vee \kappa S_u + A_u$ of a CB, at any date $u \in [t, t^\delta)$.

In (ii), we excluded t from the coupon schedule of the t-PB, because any coupon falling at call time is already paid to the bond holder via the convertible bond.

So the t-PB is the puttable bond with the inception date t, the maturity date t^δ, and the ex-dividend cumulative cash flow $\bar{\pi}^t(u;\tau_p)$, $u \in [t, t^\delta]$, given by

$$\beta_u \bar{\pi}^t(u;\tau_p) = \beta_{\tau_p} \widehat{D}^{cb}_{\tau_p} - \beta_u \widehat{D}^{cb}_u + \mathbb{1}_{\{\tau_d > \tau_p\}} \beta_{\tau_p} \big(\bar{C} \vee \kappa S_{\tau_p} + A_{\tau_p}\big),$$

where τ_p belongs to $\mathcal{G}^u_{t^\delta}$, that is, τ_p is a stopping time taking values in $[u, t^\delta]$.

Recall that a puttable bond can be seen as an example of an American option. A convertible bond with positive call notice period is thus a contract that *becomes* an American option upon call. In particular, as is apparent from Definition 5.4, a convertible bond with positive call notice period does not fit the definition of a game option. To circumvent this difficulty, it will be treated in this work as a contract that *pays* upon call the value of an American option. At least in complete markets or, more generally, in cases where the arbitrage prices of defaultable American options are unambiguously defined, this interpretation seems acceptable. Note also that for practical purposes one can often 'complete the market', in the sense that arbitrage prices of defaultable game options become uniquely defined (see, for instance, [8]).

More precisely, our guess is that when the t-PBs have unique arbitrage price processes $\bar{\Pi}^t = (\bar{\Pi}^t_u)_{u \in [t, t^\delta]}$, $t \in [0, T]$, and when the collection $(\bar{\Pi}^t_t)_{t \in [0,T]}$ of random variables can be aggregated as a càdlàg process, then any arbitrage price for the 'equivalent' RB with

$$U^{cb}_t = \mathbb{1}_{\{\tau_d > t\}} \bar{\Pi}^t_t + \mathbb{1}_{\{\tau_d \le t\}} (\bar{C} \vee \kappa S_t + A_t) \tag{27}$$

is indeed an arbitrage price for the convertible bond with positive call notice period. This is, in a sense, the ad hoc definition of an arbitrage price of a convertible bond with positive call notice period that is adopted in this paper. Whether this conjecture can be formalized in reference to an extended notion of arbitrage price generalizing the one in Kallsen and Kühn [18] and applicable to an extended notion of game option covering Definition 5.4 is left for future research.

In particular the following Proposition shows that the specification (27) of U^{cb} satisfies (24), assuming no arbitrage (note that at most one of the t-PBs may be alive in the market at the same time, so that at any given time we deal with an extended market composed of the primary market plus one t-PB).

Proposition 5.1 *Assuming (19), let us fix $t \in [0, T]$. If $(\bar{\Pi}^t_u)_{u \in [t, t^\delta]}$ is an arbitrage-free price of the t-PB then $\bar{\Pi}^t_t \ge \bar{C} \vee \kappa S_t + A_t$, on the event $\{\tau_d > t\}$.*

Proof. By part (i) in Theorem 4.2 (see also (18)), there exists $\mathbb{Q} \in \mathcal{M}$ such that

$$\bar{\Pi}^t_t = \operatorname{esssup}_{\tau_p \in \mathcal{G}^t_{t^\delta}} \mathbb{E}_{\mathbb{Q}}\big(\bar{\pi}^t(t;\tau_p) \,\big|\, \mathcal{G}_t\big). \tag{28}$$

By considering the stopping time $\tau_p = t$ in the right-hand side of (28), we obtain the inequality $\bar{\Pi}^t_t \ge \bar{C} \vee \kappa S_t + A_t$ on the event $\{\tau_d > t\}$. □

6 Decomposition of Defaultable Game Options

We now introduce in Sections 6.1 and 6.2 the pertinent decompositions of cash flows and prices of game options with respect to some reference elementary security. In Section 6.3, we provide a (non-unique) decomposition of a reduced convertible bond into a *bond component* and a *game option component*. This representation allows us to discuss the commonly used terms of 'CB spread' and 'CB implied volatility' in Section 6.4 (see, for instance, Connolly [13]). To further motivate this point, let us consider some relevant market data (data provided by courtesy of Crédit Agricole, Paris).

Table 1 provides market quotes on convertible bonds issued by the three companies of the CAC40 (French stock index) on May 10, 2005. The CB prices are Mid-Market Trading Euro Prices and *CB implied volatilities* (CB IV) are Offer-Side Implied Volatilities. In accordance with the French convention for quoting convertible bonds, the bonds nominal values in Table 1 have been scaled by a factor κ^{-1}, so that the data in Table 1 correspond to a conversion ratio κ equal to 1. For instance, the price of the scaled Alcatel CB is equal to 17.42 euros. Immediate conversion would be for one share of stock priced at 8.39 euros and the scaled nominal of the convertible bond is equal to 16.18 euros.

CB	Stock Price	Nominal	CB Price	Credit Spread	CB IV
Alcatel 4.75% Jan-11	8.39	16.18	17.42	135 bp	30.2%
Pinault 2.50% Jan-08	77.80	90.97	93.98	65 bp	21.5%
Cap Gemini 2.00 % Jun-09	25.25	39.86	41.80	65 bp	33.9%

Table 1: CB data on names of the CAC40 on May 10, 2005

For comparison, Table 2 shows market quotes on the closest listed option for each case considered in Table 1. The 'closest listed option' means the listed vanilla option with strike and maturity as close as possible to the scaled nominal and to the 'CB expected life', i.e. the most likely time of call, put, conversion or default, as forecasted by financial analysts.

CB	CB Expected Life	Option Strike and Expiry	Option IV
Alcatel 4.75% Jan-11	Oct-10	13.0 Dec-09	30.7%
Pinault 2.50% Jan-08	Nov-07	90.0 Dec-07	20.5%
Cap Gemini 2.00 % Jun-09	May-09	40.0 Dec-08	35.6%

Table 2: CBs and the closest listed options

Investors are presumed to use the information in Tables 1 and 2 to assess relative value of convertible bonds and options, and to take positions as a consequence. For instance, in some circumstances traders use to say that buying a convertible bond is 'a cheap way to buy volatility'. This means that in their view, the option component of a convertible bond is 'cheaper' (has a lower Black-Scholes implied volatility) than the corresponding listed vanilla option. It is thus a bit surprising that, to the best of our knowledge, the exact meaning of a 'CB spread' and a 'CB implied volatility' (CB IV in Table 1) is not fixed in the literature. Indeed, except for the 'exchange option' case when the conversion can only occur at maturity and there are no put or call clauses (see Margrabe [24]), a rigorous decomposition of a convertible bond into a bond and option components is not known in the general case of a defaultable convertible bond with call and/or put covenants. In accordance with this theoretical gap, the implied data displayed on the information systems available to traders are frequently insufficiently documented. Typically, such numbers are derived under the tacit assumption of some model of the stock price (possibly with jumps) in which the volatility parameter of S is well defined (for such a model, see, for instance, [8]). The value of this volatility parameter is then calibrated to the market price of the CB, which is priced by some ad hoc numerical procedures (tree or finite-differences methods). But nothing guarantees that these methods of extracting the implied volatility of the CB make sense, nor result in well-posed numerical procedures.

In particular, at the intuitive level, it seems plausible that the strike of the option embedded into a general convertible bond is a floating strike equal to the current price of a defaultable bond. So, we conjecture that the implied volatilities for convertible bonds, as given in Table 1, are not directly comparable with the corresponding implied volatilities for the closest listed options, as given in Table 2. Of course, to examine this conjecture, we need to formally define the implied volatility of a convertible bond.

6.1 Cash Flow Decomposition of a Defaultable Game Option

Let us consider a game option corresponding to the data set $(D, L, U, \bar{\tau}, \xi)$, as specified by Definition 3.1. Assume that we are given some *reference elementary security*, in the sense of Definition 4.2, with the ex-dividend cumulative cash flow given by the expression

$$\beta_t \phi(t) = \beta_T \widehat{D}^b_T - \beta_t \widehat{D}^b_t + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi^b, \tag{29}$$

where the recovery process sitting in D^b, and the payment at maturity ξ^b, are supposed to be bounded.

The first goal is to describe the cash flows of the portfolio obtained by combining the long position in the GO with the short position in the reference ES (see formula (33) below).

Remarks 6.1 Assuming that R and ξ are bounded (note however that this is not satisfied in the case of a typical CB, see Section 6.3 for a specific treatment of convertible bonds), as the reference elementary security for the GO we may take the GO contract stripped of its game features, that is, the otherwise equivalent GO in which the only admissible decision times τ_p and τ_c are $\tau_p = \tau_c = T$. This is not, of course, the only possible choice for the reference ES, but in many instances this will be the most natural choice, provided that this reference security is indeed traded.

For any probability measure $\mathbb{Q} \in \mathcal{M}$, we define the process $\Phi_t = \mathbb{E}_{\mathbb{Q}}(\phi(t) \,|\, \mathcal{G}_t)$ for $t \in [0, T]$. Note that by part (ii) in Theorem 4.2, the process Φ is actually an arbitrage price for the ES, associated with the probability measure $\mathbb{Q}$, that is, the $\mathbb{Q}$-price of the ES.

Lemma 6.1 (i) *The ex-dividend cumulative cash flow of the GO can be decomposed as follows:*

$$\pi(t; \tau_p, \tau_c) = \phi(t) + \varphi(t; \tau_p, \tau_c), \quad t \in [0, T], \tag{30}$$

where $\phi(t)$ is given by (29)*, and thus it represents the ex-dividend cumulative cash flow of the reference ES, and $\varphi(t; \tau_p, \tau_c)$ is given by the formula*

$$\begin{aligned} \beta_t \varphi(t; \tau_p, \tau_c) &= \beta_\tau (\widehat{D}_\tau - \widehat{D}^b_\tau) - \beta_t (\widehat{D}_t - \widehat{D}^b_t) \\ &\quad + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} \left(L_{\tau_p} - \phi(\tau_p) \right) + \mathbb{1}_{\{\tau < \tau_p\}} \left(U_{\tau_c} - \phi(\tau_c) \right) + \mathbb{1}_{\{\tau = T\}} (\xi - \xi^b) \Big). \end{aligned} \tag{31}$$

(ii) *Let $\mathbb{Q}$ be any probability measure from $\mathcal{M}$. Then we have*

$$\mathbb{E}_{\mathbb{Q}}(\varphi(t; \tau_p, \tau_c) \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(\psi(t; \tau_p, \tau_c) \,|\, \mathcal{G}_t), \quad t \in [0, T], \tag{32}$$

where $\psi(t; \tau_p, \tau_c)$ is defined by

$$\begin{aligned} \beta_t \psi(t; \tau_p, \tau_c) &= \beta_\tau (\widehat{D}_\tau - \widehat{D}^b_\tau) - \beta_t (\widehat{D}_t - \widehat{D}^b_t) \\ &\quad + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} \left(L_{\tau_p} - \Phi_{\tau_p} \right) + \mathbb{1}_{\{\tau < \tau_p\}} \left(U_{\tau_c} - \Phi_{\tau_c} \right) + \mathbb{1}_{\{\tau = T\}} (\xi - \xi^b) \Big). \end{aligned} \tag{33}$$

Proof. The decomposition of cash flows stated in part (i) is straightforward. For part (ii), we recall from the proof of part (ii) in Theorem 4.2 that

$$\beta_t \Phi_t = \mathbb{E}_{\mathbb{Q}}(\beta_T \widehat{D}^b_T + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi^b \,|\, \mathcal{G}_t) - \beta_t \widehat{D}^b_t,$$

and thus

$$\beta_{\tau_p} \Phi_{\tau_p} = \mathbb{E}_{\mathbb{Q}}(\beta_T \widehat{D}^b_T + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi^b \,|\, \mathcal{G}_{\tau_p}) - \beta_{\tau_p} \widehat{D}^b_{\tau_p} = \mathbb{E}_{\mathbb{Q}}(\beta_{\tau_p} \phi(\tau_p) \,|\, \mathcal{G}_{\tau_p}),$$

where the first equality follows from Doob's optional sampling theorem and the second follows from the definition of ϕ. Hence, by taking iterated conditional expectations, we obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau < \tau_d\}} \mathbb{1}_{\{\tau = \tau_p < T\}} \beta_\tau \phi(\tau_p) \,|\, \mathcal{G}_t) &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau_p < \tau_d\}} \mathbb{1}_{\{\tau = \tau_p < T\}} \beta_{\tau_p} \phi(\tau_p) \,\big|\, \mathcal{G}_{\tau_p} \big) \,\Big|\, \mathcal{G}_t \Big) \\ &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\tau_p < \tau_d\}} \mathbb{1}_{\{\tau = \tau_p < T\}} \beta_{\tau_p} \, \mathbb{E}_{\mathbb{Q}}\big(\phi(\tau_p) \,\big|\, \mathcal{G}_{\tau_p} \big) \,\Big|\, \mathcal{G}_t \Big) \\ &= \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau < \tau_d\}} \mathbb{1}_{\{\tau = \tau_p < T\}} \beta_\tau \Phi_{\tau_p} \,\big|\, \mathcal{G}_t \big) \end{aligned}$$

where we have used the fact that the random variable $\mathbb{1}_{\{\tau_p<\tau_d\}}\mathbb{1}_{\{\tau=\tau_p<T\}}\beta_{\tau_p}$ is $\mathcal{G}_{\tau_p}$-measurable. Using the same arguments, we also get $\beta_{\tau_c}\Phi_{\tau_c}=\mathbb{E}_{\mathbb{Q}}(\beta_{\tau_c}\phi(\tau_c)\,|\,\mathcal{G}_{\tau_c})$ and thus

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau<\tau_d\}}\mathbb{1}_{\{\tau<\tau_p\}}\beta_\tau\phi(\tau_c)\,|\,\mathcal{G}_t) &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\tau_c<\tau_d\}}\mathbb{1}_{\{\tau<\tau_p\}}\beta_{\tau_c}\,\mathbb{E}_{\mathbb{Q}}\big(\phi(\tau_c)\,\big|\,\mathcal{G}_{\tau_c}\big)\,\Big|\,\mathcal{G}_t\Big)\\ &= \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau<\tau_d\}}\mathbb{1}_{\{\tau<\tau_p\}}\beta_\tau\Phi_{\tau_c}\,\big|\,\mathcal{G}_t\big).\end{aligned}$$

It is now easily seen that equality (32) is valid. □

Assume that we are given a GO and we have already chosen some reference ES. Then we define the $\mathbb{Q}$-*exchange GO*, as the GO with dividend process $D-D^b$, put payment $L-\Phi$, call payment $U-\Phi$, call protection lifting time $\bar{\tau}$ and payment at maturity $\xi-\xi^b$. In other words, the $\mathbb{Q}$-exchange GO is the GO with ex-dividend cumulative cash flow $\psi(t;\tau_p,\tau_c)$ given by (33).

Remarks 6.2 (i) Note that, for a given GO, the cash flow of the $\mathbb{Q}$-exchange GO depends not only on the choice of the reference ES, but also on the choice of a probability measure $\mathbb{Q}\in\mathcal{M}$, through the definition of the $\mathbb{Q}$-price process Φ of the ES.
(ii) Since the process Φ is an arbitrage price for the reference ES, the $\mathbb{Q}$-exchange GO has the financial interpretation as the game option to exchange the reference ES for either L or U (as seen from the perspective of the holder), according to which player decides first to stop this game. This interpretation is particularly transparent when the reference ES is specified as in Remarks 6.1, since in that case (33) reduces to

$$\beta_t\psi(t;\tau_p,\tau_c)=\mathbb{1}_{\{\tau_d>\tau\}}\,\beta_\tau\Big(\mathbb{1}_{\{\tau=\tau_p<T\}}\left(L_{\tau_p}-\Phi_{\tau_p}\right)+\mathbb{1}_{\{\tau<\tau_p\}}\left(U_{\tau_c}-\Phi_{\tau_c}\right)\Big). \tag{34}$$

The contract with the cash flow given by the last formula can be seen as the zero-dividend, pure option component of the GO.
(iii) In the special case of the decomposition of a CS, the $\mathbb{Q}$-exchange GO is also a CS.

6.2 Price Decomposition of a Defaultable Game Option

We are now in the position to derive the price decomposition of a game option with respect to some reference security. We assume that we are given a GO and the reference ES, as described in the previous section. The following result follows easily from Theorem 4.1 and Lemma 6.1.

Theorem 6.1 *Assuming (7), let $\mathbb{Q}\in\mathcal{M}$ be given and let Φ be the arbitrage $\mathbb{Q}$-price of the reference ES.*
(i) *If Π is an arbitrage $\mathbb{Q}$-price for the GO then $\Psi=\Pi-\Phi$ is an arbitrage $\mathbb{Q}$-price for the $\mathbb{Q}$-exchange GO.*
(ii) *If Ψ is an arbitrage $\mathbb{Q}$-price for the $\mathbb{Q}$-exchange GO then $\Pi=\Phi+\Psi$ is an arbitrage $\mathbb{Q}$-price for the GO.*

Proof. Let us prove (i). Using (13) and (30), we obtain

$$\begin{aligned}\Psi_t=\Pi_t-\Phi_t &= \operatorname{esssup}_{\tau_p\in\mathcal{G}^t_T}\operatorname{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big)-\mathbb{E}_{\mathbb{Q}}\big(\phi(t)\,\big|\,\mathcal{G}_t\big)=\\ &\operatorname{esssup}_{\tau_p\in\mathcal{G}^t_T}\operatorname{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\varphi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big)=\operatorname{esssup}_{\tau_p\in\mathcal{G}^t_T}\operatorname{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\psi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big),\end{aligned}$$

where the last equality is a consequence of (32). Moreover, the difference $\Psi=\Pi-\Phi$ is obviously a semimartingale. Thus Ψ is an arbitrage $\mathbb{Q}$-price for the $\mathbb{Q}$-exchange GO, by the 'if' part of Theorem 4.1. The proof of part (ii) is similar to that of part (i). □

Let us stress that Ψ need not be positive and thus Π need not be greater than Φ, in general. We have, however, the following result.

Corollary 6.1 *Under the assumptions of Theorem 6.1(i) or 6.1(ii), if the process $D-D^b$ is non-decreasing, $U\geq\Phi$ on $[\tau_d\wedge\bar{\tau},\tau_d\wedge T)$ and $\xi\geq\xi^b$, then $\Psi\geq 0$ and thus $\Pi\geq\Phi$.*

Proof. Let us show that at any time $t < \tau_d \wedge T$ such that $L_t < \Phi_t$, exchanging the ES for the payoff L_t is suboptimal for the holder of the $\mathbb{Q}$-exchange GO. Towards this end, we define $\breve{\psi}(t; \tau_p, \tau_c)$ by the formula

$$\begin{aligned}\beta_t \breve{\psi}(t; \tau_p, \tau_c) &= \beta_\tau(\widehat{D}_\tau - \widehat{D}^b_\tau) - \beta_t(\widehat{D}_t - \widehat{D}^b_t) \\ &\quad + \mathbb{1}_{\{\tau_d > \tau\}}\, \beta_\tau \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} \big(L_{\tau_p} - \Phi_{\tau_p}\big)^+ + \mathbb{1}_{\{\tau < \tau_p\}} \big(U_{\tau_c} - \Phi_{\tau_c}\big) + \mathbb{1}_{\{\tau = T\}} (\xi - \xi^b) \Big)\end{aligned}$$

and we denote $\breve{\mathcal{G}}^t_T = \{\tau \in \mathcal{G}^t_T \,;\, L_\tau \geq \Phi_\tau \text{ if } \tau < T\}$. For any $\tau_p \in \mathcal{G}^t_T$, the stopping time $\breve{\tau}_p$, given by the formula

$$\breve{\tau}_p = \mathbb{1}_{\{L_{\tau_p} \geq \Phi_{\tau_p}\}} \tau_p + \mathbb{1}_{\{L_{\tau_p} < \Phi_{\tau_p}\}} T,$$

belongs to $\breve{\mathcal{G}}^t_T$. Since the process $D - D^b$ is non-decreasing we have that the process $\beta_t(\widehat{D}_t - \widehat{D}^b_t))$ is non-decreasing as well. We assumed also that $U \geq \Phi$ on $[\tau_d \wedge \bar{\tau}, \tau_d \wedge T)$ and $\xi \geq \xi^b$, so that the following inequalities hold, for any $\tau_p \in \mathcal{G}^t_T$ and $\tau_c \in \bar{\mathcal{G}}^t_T$,

$$\psi(t; \breve{\tau}_p, \tau_c) \geq \psi(t; \tau_p, \tau_c), \quad \breve{\psi}(t; \breve{\tau}_p, \tau_c) \geq \breve{\psi}(t; \tau_p, \tau_c).$$

Since obviously $\breve{\psi}(t; \tau_p, \tau_c) = \psi(t; \tau_p, \tau_c)$ for any $\tau_p \in \breve{\mathcal{G}}^t_T$, we obtain

$$\begin{aligned}&\operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^t_T} \operatorname{esssup}_{\tau_p \in \mathcal{G}^t_T} \mathbb{E}_{\mathbb{Q}}\big(\breve{\psi}(t; \tau_p, \tau_c) \,\big|\, \mathcal{G}_t\big) = \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^t_T} \operatorname{esssup}_{\tau_p \in \breve{\mathcal{G}}^t_T} \mathbb{E}_{\mathbb{Q}}\big(\breve{\psi}(t; \tau_p, \tau_c) \,\big|\, \mathcal{G}_t\big) \\ &\quad = \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^t_T} \operatorname{esssup}_{\tau_p \in \breve{\mathcal{G}}^t_T} \mathbb{E}_{\mathbb{Q}}\big(\psi(t; \tau_p, \tau_c) \,\big|\, \mathcal{G}_t\big) = \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^t_T} \operatorname{esssup}_{\tau_p \in \mathcal{G}^t_T} \mathbb{E}_{\mathbb{Q}}\big(\psi(t; \tau_p, \tau_c) \,\big|\, \mathcal{G}_t\big)\end{aligned}$$

and thus

$$\begin{aligned}\Psi_t &= \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^t_T} \operatorname{esssup}_{\tau_p \in \mathcal{G}^t_T} \mathbb{E}_{\mathbb{Q}}\big(\psi(t; \tau_p, \tau_c) \,\big|\, \mathcal{G}_t\big) \\ &= \operatorname{esssup}_{\tau_p \in \mathcal{G}^t_T} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^t_T} \mathbb{E}_{\mathbb{Q}}\big(\breve{\psi}(t; \tau_p, \tau_c) \big|\, \mathcal{G}_t\big) \geq 0.\end{aligned}$$

We conclude that $\Pi_t = \Phi_t + \Psi_t \geq \Phi_t$. □

6.3 Decomposition of a Reduced Convertible Bond

We shall now specialize our previous results to the case of a reduced convertible bond (hence, in particular, to the case of a convertible bond without call notice period). We thus postulate that the dividend process is of the form D^{cb} given by (20), that is,

$$D^{cb}_t = \int_0^{t \wedge \tau_d} c^{cb}_u \, du + \sum_{0 \leq T_i \leq t,\, T_i < \tau_d} c^i + \mathbb{1}_{\{0 \leq \tau_d \leq t\}} R^{cb}_{\tau_d}, \quad t \in [0, T].$$

In order to provide the most pertinent price decomposition of the RB, we choose as the reference instrument the ES with dividend process

$$D^b_t = \int_0^{t \wedge \tau_d} c^{cb}_u \, du + \sum_{0 \leq T_i \leq t,\, T_i < \tau_d} c^i + \mathbb{1}_{\{0 \leq \tau_d \leq t\}} R^b_{\tau_d}, \quad t \in [0, T],$$

that is, the ES with the same coupon process as the RB and with R^b and ξ^b given as follows (see Section 5.1):

$$R^b_t = \bar{R}_t, \quad \xi^b = \bar{N} + A_T. \tag{35}$$

It is thus clear that

$$R^{cb}_t - R^b_t = (\kappa S_t - \bar{R}_t)^+ \geq 0, \quad \xi^{cb} - \xi^b = (\kappa S_T - \bar{N})^+ \geq 0.$$

So, in this case the reference security is the defaultable bond with ex-dividend cumulative cash flow $\phi(t)$ given by the expression

$$\begin{aligned}\beta_t \phi(t) &= \beta_T \widehat{D}^b_T - \beta_t \widehat{D}^b_t + \mathbb{1}_{\{\tau_d > T\}}\, \beta_T \xi^b \\ &= \int_t^{T \wedge \tau_d} \beta_u c^{cb}_u \, du + \sum_{t < T_i \leq T,\, T_i < \tau_d} \beta_{T_i} c^i + \mathbb{1}_{\{t < \tau_d \leq T\}}\, \beta_{\tau_d} R^b_{\tau_d} + \mathbb{1}_{\{\tau_d > T\}}\, \beta_T \xi^b.\end{aligned}$$

It is clear that this reference bond can be interpreted as the pure bond component of the RB, that is, the RB stripped of its optional clauses. Therefore, we shall call it the *bond embedded in the RB*, or simply the *embedded bond*. Given a probability measure $\mathbb{Q} \in \mathcal{M}$, the process $\Phi_t = \mathbb{E}_{\mathbb{Q}}(\phi(t) \,|\, \mathcal{G}_t)$ is the $\mathbb{Q}$-price of the embedded bond.

Since the RB and the embedded bond have the same coupon schedule, the $\mathbb{Q}$-exchange GO is the zero-coupon GO, with the ex-dividend cumulative cash flow $\psi(t;\tau_p,\tau_c)$ given by the expression

$$\begin{aligned}\beta_t\psi(t;\tau_p,\tau_c) &= \mathbb{1}_{\{t<\tau_d\le\tau\}}\,\beta_{\tau_d}\big(R^{cb}_{\tau_d}-R^b_{\tau_d}\big) \\ &\quad + \mathbb{1}_{\{\tau_d>\tau\}}\,\beta_\tau\Big(\mathbb{1}_{\{\tau=\tau_p<T\}}\big(L^{cb}_{\tau_p}-\Phi_{\tau_p}\big)+\mathbb{1}_{\{\tau<\tau_p\}}\big(\bar U^{cb}_{\tau_c}-\Phi_{\tau_c}\big)+\mathbb{1}_{\{\tau=T\}}\big(\xi^{cb}-\xi^b\big)\Big).\end{aligned} \tag{36}$$

This particular $\mathbb{Q}$-exchange GO will be referred to as the *embedded game $\mathbb{Q}$-exchange option*. As for any $\mathbb{Q}$-exchange GO (see Section 6.1), the cash flow of the embedded game $\mathbb{Q}$-exchange option depend on the choice of the probability measure $\mathbb{Q}$ through the definition of the price process Φ of the embedded bond.

The embedded game $\mathbb{Q}$-exchange option has the natural interpretation as a game option to exchange the embedded bond for either L^{cb} or U^{cb}, according to who decides first to stop this game. In the case of default, the contract stipulates that the recovery payoffs of the RB and the embedded bond are exchanged. Finally, if the embedded game contract is still alive at maturity date T, the terminal payoffs are exchanged.

Having picked $\mathbb{Q} \in \mathcal{M}$, let Π and Ψ stand for the arbitrage $\mathbb{Q}$-price of the RB and the embedded game $\mathbb{Q}$-exchange option, respectively. Theorem 6.1 can be directly applied to this specification of an RB. Note that we are in the case of a CS, so (19) is enough for (7) (see Remark 4.3). Since an explicit expression for the price decomposition of an RB with respect to the embedded bond is easy to obtain, it is not reported here. The following result is worth stating, however.

Corollary 6.2 *Consider an RB and the embedded bond with R^b and ξ^b given by (35). Under the assumptions of Theorem 6.1(i) or 6.1(ii), in the special case of a zero-coupon RB ($c^{cb} = 0$ and $c^i = 0$ for any i), assuming that the process β is non-increasing and $\bar R \le \bar N$, we have that $\Psi \ge 0$ and thus $\Pi \ge \Phi$.*

Proof. Under the present assumptions we have by (24)

$$\Phi_t \le \bar N \le \bar C \le U^{cb}_t, \quad \tau_d \wedge \bar\tau \le t < \tau_d \wedge T,$$

so that the inequality $\Pi \ge \Phi$ is an immediate consequence of Corollary 6.1. □

Remarks 6.3 (i) When used in the "reverse-engineering" mode, Corollary 6.2 may have the practical interest to make traders realize that Ψ need not be positive, and thus Π need not be greater than Φ, in general. Indeed, the positivity of Ψ is only obtained under quite restrictive assumptions.
(ii) The possibility of the negative value of Ψ is related to the fact that we consider a callable reduced bond, but we have chosen the non-callable embedded bond as the reference security. Hence the value of the reference bond can be greater than the call price at the moment of call of the corresponding callable reduced bond.
In other words, the price of a callable and convertible bond can be either higher or lower than the price of an equivalent non-callable and non-convertible bond. It would be thus interesting to take as the reference security the callable version of the embedded bond. In that case, one would expect to have the positive value for the embedded game option, since this game option should reduce to a vulnerable American option with non-negative payoffs at default and at maturity.
(iii) Under the assumptions of Corollary 6.2, the reference zero-coupon bond is equivalent to a callable zero-coupon bond with the same nominal value $\bar N$ and call price $\bar C \ge \bar N$ (since in fact a callable zero-coupon bond with the call price $\bar N \ge \bar C$ will never be called if interest rates are non-negative). By contrast, if we deal with a coupon-paying reduced bond this argument is no longer valid, since now that assumption $\bar N \ge \bar C$ does not ensure that the callable version of the reference bond will never be called.

6.4 Spread and Implied Volatility of a Convertible Bond

We are now in the position to discuss the notion of CB spread and implied volatilities. Let us first define the spread and the Black–Scholes implied volatility of a reduced convertible bond. Note that the second part of this definition refers to a Black–Scholes model in which the related interest rate and equity dividend inputs are considered as given deterministic time-functionals (they can in fact easily be retrieved independently from the market), so that the volatility is the only parameter that is left unspecified in the underlying Black–Scholes model.

Definition 6.1 Let us consider an RB under the assumptions of Theorem 6.1(i) or (ii), with the RB and the embedded bond here as the CS and the ES there. The *RB spread* is defined as the credit spread consistent with the price Φ for the embedded bond. By an *RB Black–Scholes implied volatility* we mean any value Σ of the Black–Scholes volatility of the stock price process S that is consistent with the price Ψ for the embedded game exchange option.

In Definition 6.1, we work under a fixed risk-neutral measure $\mathbb{Q} \in \mathcal{M}$, which is given by the general assumptions of Theorem 6.1. In practical applications, it is convenient to think of $\mathbb{Q}$ as the "pricing measure" chosen by the market to price the RB and the embedded bond.

Consistency of Σ with the price Ψ for the embedded game exchange option means that the price of this option in a standard Black–Scholes model with volatility Σ is equal to Ψ. In practice, the pricing of the embedded game exchange option in a standard Black–Scholes model with a constant volatility σ can be done by solving the related double obstacle variational inequalities. These variational inequalities correspond to the sub-case of variational inequalities examined in [8] when the volatility is constant and there is no default intensity.

In the general framework considered here, the properties of the RB implied volatility can not be analyzed in detail. In particular, it is not clear whether it is possible to map every possible arbitrage price process for the game exchange option to a well-defined and unique Black-Scholes implied volatility process.

Nevertheless, it seems clear at the intuitive level that the embedded bond concentrates most of the interest rate and credit risks of the convertible bond, whereas the embedded game exchange option (which has no coupons nor nominal payment) concentrates most of the volatility risk. Thus, using the price of the embedded bond to infer credit spread, and the price of the embedded game exchange option to imply volatility, seems very natural.

Another possible benefit of our decomposition might be for the joint calibration of a 'realistic' (unlike the above standard Black–Scholes model!) convertible bond pricing model. The simplest example of such a model is the jump-diffusion model with local default intensity $\gamma(t, S)$, and possibly also local volatility $\sigma(t, S)$, see e.g. [8, 1, 4]. For consistency with market data, the functions γ and σ are typically decreasing in S. Now, for such a negatively skewed function γ, this model is not convexity-preserving. This means that the price at time $t < T$ of a European claim with payoff given by a convex function of S_T, is not necessarily convex in the stock price. As a matter of fact, the market price at time $t < T$ of a real-life CB is typically non convex in the stock price either: it exhibits the so-called *ski–jump behavior*, namely convex for high S and collapsing at low S. The collapse at low S comes from the collapse of the embedded bond component of the CB ('collapse of the bond floor'). But there is no reason why the embedded game exchange component should exhibit a similar collapse at low S (recall that the embedded game exchange pays no coupons and has no nominal either). Therefore in terms of convexity with respect to the stock price and monotonicity with respect to the volatility, the price of the embedded game exchange should enjoy much better properties than the price of the CB, both in the market and in any realistic pricing model. This is an incentive to use (synthetic) embedded game exchange prices rather CB prices to calibrate such model.

Finally, note that the embedded game exchange option of an RB can be seen as an equity option, but with a floating strike, equal at any date t to the current value Φ_t of the embedded bond. This clarifies the intuitive statements made at the beginning of this section and confirms our conjecture that the implied volatility of a CB (as given for instance by Definition 6.1) in Table 1 and the implied volatility for the closest listed option in Table 2 are in fact of a completely different nature.

7 Conclusions

The game option decomposition of a convertible bond examined in this paper is intended to provide a proper way to extract the bond's implied volatility. As argued in Section 6, this is indeed a major practical issue faced by traders. Incidentally, the related theoretical study also shows that the received opinion that a convertible bond should be worth more than its *bond floor* (the traders' name of the embedded bond) may prove incorrect under some circumstances (see Remark 6.3).

It is worth noting that some problems associated with the study of game options and convertible securities in a general semimartingale framework remain still open, however. Most notably, it is not clear whether it is possible to generalize Proposition 3.1 to the case of extended game options, such as the ones that arise naturally in the study of real-life convertible bond with positive call notice period. In the present paper and the follow-up works, we develop an alternative approach, based on a conjecture that the valuation of such a convertible bond can be done recursively. However, this conjecture remains to be justified in general, that is, under market incompleteness.

It is fair to say that the results of the present work are of a rather general nature and thus they do not furnish an explicit solution to the valuation and hedging problems for convertible bonds. At the same time, however, they have a clear advantage of being universal, in the sense that they are valid in virtually any arbitrage-free model of the security market. In subsequent papers [6, 7, 8], we continue this research in the framework of more specific models of the primary market. First, in [6], we examine the issue of pricing and hedging defaultable game options in the hazard process credit risk model through a solution to a suitable doubly reflected backward stochastic differential equation. In the subsequent papers [7, 8], we express solutions to the pricing and hedging problems for convertible securities in a jump-diffusion model of the stock price in terms of solutions of the associated variational inequalities.

References

[1] Andersen, L. and Buffum, L.: Calibration and implementation of convertible bond models. Working paper, 2003.

[2] Avellaneda, M. and Wu, L.: Pricing Parisian-style options with a lattice method. *International Journal of Theoretical and Applied Finance* 2 (1999), 1–16.

[3] Ayache, E.: The equity-to-credit problem (or the story of calibration, co-calibration and re-calibration). *The Best of Wilmott,* Vol. 1, Wiley, 2005, pp. 79–107.

[4] Ayache, E., Forsyth, P. and Vetzal, K.: Valuation of convertible bonds with credit risk. *The Journal of Derivatives*, Fall 2003.

[5] Bakshi, G., Madan, D.B. and Zhang, F.X.: Understanding the role of recovery in default risk models: Empirical comparisons and implied recovery rates. *FEDS Working Paper*, 2001.

[6] Bielecki, T.R., Crépey, S., Jeanblanc, M. and Rutkowski, M.: Valuation and hedging of defaultable game options in a hazard process model. Working paper, 2006.

[7] Bielecki, T.R., Crépey, S., Jeanblanc, M. and Rutkowski, M.: Defaultable options in a Markovian pre-default intensity model of credit risk. Work in preparation, 2006.

[8] Bielecki, T.R., Crépey, S., Jeanblanc, M. and Rutkowski, M.: Convertible bonds in a jump-diffusion model of credit risk. Work in preparation.

[9] Bielecki, T.R. and Rutkowski, M.: *Credit Risk: Modeling, Valuation and Hedging.* Springer-Verlag, Berlin, 2002.

[10] Brennan, M.J. and Schwartz, E.S.: Analyzing convertible bonds. *Journal of Financial and Quantitative Analysis* 15 (1980), 907–929.

[11] Cherny, A. and Shiryaev, A.: Vector stochastic integrals and the fundamental theorems of asset pricing. *Proceedings of the Steklov Institute of mathematics*, 237 (2002), pp. 6–49.

[12] Chesney, M., Jeanblanc, M. and Yor, M.: Brownian excursion and Parisian barrier options. *Annals of Applied Probability* 29 (1997), 165–184.

[13] Connolly, K.B.: *Pricing Convertible Bonds*, Wiley, Chichester New York, 1998.

[14] Davis, M. and Lischka, F.: Convertible bonds with market risk and credit risk. In *Applied Probability*, eds. R. Chan, Y-K. Kwok, D. Yao and Q. Zhang, Studies in Advanced Mathematics, American Mathematical Society/International Press 2002, pp. 45–58.

[15] Delbaen, F. and Schachermayer, W.: The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen* 312 (1997), 215–250.

[16] Derman, E.: Valuing convertible bonds as derivatives. Goldman Sachs Selected Quantitative Strategies Reports, November 1994.

[17] Dynkin, E.B.: Game variant of a problem on optimal stopping. *Soviet Math. Dokl.* 10 (1969), 270–274.

[18] Kallsen, J. and Kühn, C.: Convertible bonds: financial derivatives of game type. In: *Exotic Option Pricing and Advanced Lévy Models*, Edited by Kyprianou, A., Schoutens, W. and Wilmott, P., Wiley, 2005, pp. 277–288.

[19] Kallsen, J. and Kühn, C.: Pricing derivatives of American and game type in incomplete markets. *Finance and Stochastics* 8 (2004), 261–284.

[20] Kifer, Y.: Game options. *Finance and Stochastics* 4 (2000), 443–463.

[21] Kifer, Y.: Optimal stopping in games with continuous time. *Theory Probab. Appl.* 16 (1971), 545–550.

[22] Kwok, Y. and Lau, K.: Anatomy of option features in convertible bonds. *Journal of Futures Markets* 24 (2004), 513–532.

[23] Lepeltier, J.-P. and Mainguenau, M.: Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodski. *Stochastics* 13 (1984), 25–44.

[24] Margrabe, W.: The value of an option to exchange one asset for another. *The Journal of Finance* 33 (1978), 177–186.

[25] Protter, P. E., *Stochastic Integration and Differential Equations*, Second edition, Springer, 2004.

[26] Sîrbu, M., Pikovsky, I. and Shreve, S.: Perpetual convertible bonds. *SIAM J. Control Optim.* 43 (2004), 58–85.

[27] Sîrbu, M. and Shreve, S.: A two-person game for pricing convertible bonds. Working paper, 2005.

[28] Tsiveriotis, K. and Fernandes, C.: Valuing convertible bonds with credit risk. *Journal of Fixed Income* 8 (1998), 95–102.

[29] Zuckerman, G.: Hedge funds stumble even when walking – 'Conservative' wagers turn sour, leading to fears of a shakeout; A one-two punch on a GM bet, *Wall Street Journal*, May 18, 2005.