Forward Monte-Carlo Scheme for PDEs: Multi-Type Marked Branching Diffusions

Pierre Henry-Labordère[1]

[1]Global markets Quantitative Research, SOCIÉTÉ GÉNÉRALE

Outline

Contents

- Stochastic representation of semi-linear PDEs: Counterparty risk (and American options).
- Review of Numerical Methods:
  - Brute-force “Monte-Carlo of Monte-Carlo" method (with nested simulations).
  - BSDEs.
  - Gradient representation.
  - Branching diffusions.
- Marked branching diffusions.
- Numerical results.
- Multi-type marked branching diffusions: Extensions to fully non-linear PDEs [joint work with X. Tan, N. Touzi].

Semi-linear PDEs: CVA examples

- Two types of PDEs:

$$\partial_t u + \mathcal{L}u + ru + r_1 u^+ = 0 \,,\; u(T,x) = \psi(x) \;:\; \text{PDE1}$$

$$\partial_t u + \mathcal{L}u + ru + r_1 M + r_2 M^+ + r_3 u^+ = 0 \;:\; \text{PDE2}$$
$$\partial_t M + \mathcal{L}M + r_4 M = 0 \,,\; M(T,x) = \psi(x)$$

- Toy example:

$$\partial_t u + \mathcal{L}u - \beta u^+ = 0 \,,\; u(T,x) = \psi(x)$$

A brut-force algorithm

- Feynman-Kac's formula:

$$u(t,x) = \mathbb{E}_t^{\mathbb{P}}[\psi(X_T)] - \int_t^T \beta \mathbb{E}_t^{\mathbb{P}}[u^+(s,X_s)]ds$$

- Approximation (β is small)[1]:

$$u(t,x) \simeq \mathbb{E}_t^{\mathbb{P}}[\psi(X_T)] - \sum_{i=1}^{n} \beta \mathbb{E}_t^{\mathbb{P}}[\left(\mathbb{E}_{t_i}^{\mathbb{P}}[\psi(X_T)]\right)^+]\Delta t_i$$

- Leads to “Monte-Carlo of Monte-Carlo" approach (with nested simulations). Complexity: $O(N^2)$.
- Can we design an algorithm with complexity $O(N)$?

[1] exact for PDE2.

1-BSDE [Pardoux-Peng]

- 1-BSDE:

$$
\begin{aligned}
dX_t &= b(t, X_t)dt + \sigma(t, X_t).dW_t \\
dY_t &= \beta Y_t^+ dt + Z_t \sigma(t, X_t).dW_t \\
Y_T &= \psi(X_T)
\end{aligned}
$$

  where (Y, Z) adapted processes.
- Unique solution: $(Y_t = u(t, X_t), Z_t = \sigma(t, X_t)\partial_x u(t, X_t))$.
- Discretization scheme ($Y_{t_{i-1}}$ is forced to be $\mathcal{F}_{t_{i-1}}$-adapted):

$$
Y_{t_{i-1}} = \mathbb{E}^{\mathbb{P}}_{t_{i-1}}[Y_{t_i}] \left(1_{\mathbb{E}^{\mathbb{P}}_{t_{i-1}}[Y_{t_i}]>0} \frac{1-(1-\theta)\beta\Delta t_i}{1+\theta\beta\Delta t_i} + 1_{\mathbb{E}^{\mathbb{P}}_{t_{i-1}}[Y_{t_i}]<0} \right)
$$

- Needs the computation of $\mathbb{E}^{\mathbb{P}}_{t_{i-1}}[Y_{t_i}]$ by regression methods. Quite difficult and time-consuming, specially for multi-asset portfolios.

Gradient representation [Talay-al], [Jourdain]

- Let u be the solution of

$$\partial_t u + \frac{1}{2}\sigma^2(t,x)\partial_x^2 u + f(u) = 0 \; u(T,x) = \psi(x)$$

- By differentiating w.r.t. x :

$$\partial_t \Delta + \left(\left(\sigma \partial_x \sigma\right) \partial_x + \frac{1}{2}\sigma^2(t,x)\partial_x^2 \right) \Delta + f'\left(u\right) \Delta = 0$$

- Interpreted as a Fokker-Planck PDE:

$$u(t,x) = -\int_{\mathbb{R}^+} \psi'(a) da \, \mathbb{E}_t^{\mathbb{P}}[1(X_T^a - S) e^{\int_t^T f'(u(T+t-s, X_s^a))ds}]$$
$$dX_s^a = \sigma(T+t-s, X_s^a) dB_s + \left(\sigma \partial_2 \sigma\right)\left(T+t-s, X_s^a\right) ds$$

Branching Diffusions [MCKean]

- Branching diffusions first introduced by McKean for KPP type PDE:

$$\partial_t u(t,x) + \mathcal{L}u + \beta\left(\sum_{k=1}^{\infty} p_k u^k - u\right) = 0 \text{ in } \mathbb{R}^d \times \mathbb{R}_+$$
$$u(T,x) = \psi(x) \text{ in } \mathbb{R}^d$$

- Restrictive algebraic non-linearity:

$$f(u) \equiv \sum_{k=0}^{\infty} p_k u^k \ , \ \sum_{k=0}^{\infty} p_k = 1 \ , \ 0 \le p_k \le 1$$

- Feynman-Kac's formula:

$$u(t,x) \quad = \quad \mathbb{E}_t[1_{\tau > T}\psi(X_T)] + \sum_{k=0}^{\infty} p_k \mathbb{E}_t[u^k(\tau, X_\tau) 1_{\tau < T}]$$

Probability interpretation

- Let a single particle starts at the origin, performs an Itô diffusion motion on $\mathbb{R}^d$, after a mean β exponential time dies and produces k descendants with probability p_k. Then, the descendants perform independent Itô diffusion motions on $\mathbb{R}^d$ from their birth locations, die and produce descendants after a mean $\beta(\cdot)$ exponential times, etc. This process is called a d-dimensional branching diffusion with a branching rate $\beta > 0$.
- Stochastic representation [strong Markov property]:

$$u(t, x) = \mathbb{E}_t[\prod_{i=1}^{N_T} \psi(z_T^i)]$$

Marked branching diffusions [PHL]

- Algebraic semi-linear PDE:

$$\partial_t u + \mathcal{L}u + \Phi(u) = 0$$

with $\Phi(u) = \beta(F(u) - u)$ and $F(u) = \sum_{k=0}^{M} a_k u^k$.

- From Feynman-Kac's formula:

$$u(t,x) = \mathbb{E}_t[1_{\tau > T}\psi(X_T)] + \mathbb{E}_t[F(u_\tau)1_{\tau < T}]$$

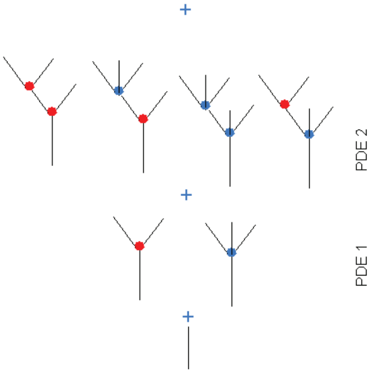
- Recursively solved in terms of multiple exp. random times τ_i:

$$\begin{aligned} u(t,x) &= \mathbb{E}_t[1_{\tau_0 > T}\psi(X_T)] \\ &+ \mathbb{E}_t[F\left(\mathbb{E}_\tau[1_{\tau_0 > T}\psi(X_T)] + \mathbb{E}_\tau[F(u_{\tau_2})1_{\tau_2 < T}]\right) 1_{\tau < T}] \end{aligned}$$

Marked branching diffusions (2)

- Stochastic representation:

$$u(t,x) = \mathbb{E}_t[\prod_{i=1}^{N_T} \psi(z_T^i) \prod_{k=0}^{M} \left(\frac{a_k}{p_k}\right)^{\omega_k}]$$



Marked branching Brownian motion (2)

- Algebraic PDE type 2:

$$\partial_t u(t,x) + \mathcal{L}u(t,x) + \beta(F(\mathbb{E}_t[\psi(X_T)]) - u(t,x)) = 0$$

- Feynman-Kac's formula:

$$u(t,x) = \mathbb{E}_t[1_{\tau > T}\psi(X_T)] + \mathbb{E}_t[F(\mathbb{E}_\tau[\psi(X_T)])\,1_{\tau < T}]$$

- As compared to the previous section, we have the term $F(\mathbb{E}_\tau[\psi(X_T)])\,1_{\tau < T}$. This term can be computed using the previous algorithm by imposing that the particle can default only once. This corresponds to the first three diagrams in Fig. (1).

Convergence

Proposition 1

Let us assume that $\psi \in \mathrm{L}^\infty(\mathbb{R}^d)$. Set $q(s) := \sum_{k=0}^M |a_k| ||\psi||_\infty^{k-1} s^k$.

1. Case $q(1) > 1$: We have $u \in \mathrm{L}^\infty([0, T] \times \mathbb{R}^d)$ if there exists $X \in \mathbb{R}_+^*$ such that

$$\int_1^X \frac{ds}{q(s) - s} = \beta T$$

   In the particular case of one branching type k, the sufficient condition for convergence reads as

$$|a_k| ||\psi||_\infty^{k-1} \left(1 - e^{-\beta T(k-1)}\right) < 1$$

2. Case $q(1) \leq 1$: $u \in \mathrm{L}^\infty([0, T] \times \mathbb{R}^d)$ for all T.

Optimal probabilities

By assuming that $\psi \in \mathrm{L}^\infty(\mathbb{R}^d)$, the expectation in (1) can then be bounded by

$$|\hat{u}(0,x)| \quad \leq \quad \mathbb{E}_{0,x}[\prod_{k=0}^{M} \left(\frac{|a_k|}{p_k}\right)^{\omega_k} ||\psi||_\infty^{N(\omega)}] = ||\psi||_\infty \hat{\mathbb{P}}\left(T, -\ln\frac{|a_k|}{p_k} - \ln||\psi||_\infty^{k-1}\right)$$

$$p_k = \frac{|a_k|||\psi||_\infty^k}{\sum_{i=0}^{M} |a_i|||\psi||_\infty^i}$$

Bias

Proposition 2

Let us assume that $\underline{F}(v)$ and $\overline{F}(v)$ are two polynomials satisfying (**Comp**), the sufficient condition in Prop. 1 for a maturity T and

$$\underline{F}(x) \leq x^+ \leq \overline{F}(x)$$

We denote $\underline{v}$ and $\overline{v}$ the corresponding solutions of (PDE($\underline{F}$, $\overline{F}$)) and v the solution of (PDE(v^+)). Then

$$\underline{v} \leq v \leq \overline{v}$$

Numerical Experiments

- We have implemented our algorithm for the two PDE types

$$\partial_t u + \mathcal{L}u + \beta(F(u) - u) = 0 \; , \; u(T,x) = 1_{x>1} \; : \; \mathrm{PDE1}$$

and

$$\partial_t u + \mathcal{L}u + \beta(F(\mathbb{E}_t[1_{X_T>1}]) - u) = 0 \; , \; u(T,x) = 1_{x>1} \; : \; \mathrm{PDE2}$$

- $\mathcal{L}$ is the Itô generator of a geometric Brownian motion with a volatility $\sigma_{\mathrm{BS}} = 0.2$ and the Poisson intensity is $\beta = 0.05$.

Numerical Experiment 1

N	Fair(PDE2)	Stdev(PDE2)	Fair(PDE1)	Stdev(PDE1)
12	20.78	0.78	21.31	0.79
14	22.25	0.39	21.37	0.39
16	21.97	0.19	21.76	0.20
18	21.90	0.10	21.51	0.10
20	21.86	0.05	21.48	0.05
22	**21.81**	0.02	**21.50**	0.02

Table: MC price quoted in percent as a function of the number of MC paths 2^N. PDE pricer(PDE1) = **21.82**. PDE pricer(PDE2) = **21.50**. Non-linearity $F(u) = \frac{1}{2}\left(u^3 - u^2\right)$.

Numerical Experiment 2

N	Fair(PDE2)	Stdev(PDE2)	Fair(PDE1)	Stdev(PDE1)
12	21.14	0.78	20.00	0.78
14	21.56	0.38	19.90	0.39
16	21.62	0.19	20.25	0.20
18	21.31	0.10	20.39	0.10
20	21.38	0.05	20.36	0.05
22	**21.36**	0.02	**20.40**	0.02

Table: MC price quoted in percent as a function of the number of MC paths 2^N. PDE pricer(PDE1) = **21.37**. PDE pricer(PDE2) = **20.39**. Non-linearity $F(u) = \frac{1}{3}\left(u^3 - u^2 - u^4\right)$.

Numerical Experiment 3

- The semi-linear PDE in $\mathbb{R}^d$

$$\partial_t u + \mathcal{L}u + u^2 = 0$$

blows up in finite-time if and only if $d \leq 2$ for any bounded positive payoff [Sugitani].

Maturity(Year)	BBM alg.(Stdev)	PDE
0.5	71.66(0.09)	71.50
1	157.35(0.49)	157.17
1.1	$\infty(\infty)$	∞

Table: MC price quoted in percent as a function of the maturity for the non-linearity $F(u) = u^2 + u$. $\psi(x) \equiv 1_{x>1}$.

Polynomial approximation

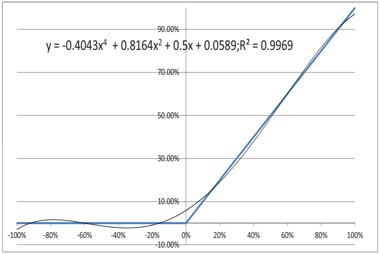


Figure: u^+ versus its polynomial approximation.

Algorithm: Final recipe

1. Simulate the assets and the Poisson default time[2].
2. At each default time, produce k descendants with probability p_k. For PDE type 2, the particles are not allowed to die anymore.
3. Evaluate for each particle alive the payoff

$$\prod_{i=1}^{N_T} \psi(z_T^i) \prod_{k=0}^{M} \left(\frac{a_k}{p_k}\right)^{\omega_k}$$

where ω_k denotes the number of branching type k.

[2] The intensity β can stochastic (Cox process).

Two PDE types

- We have implemented our algorithm for the two PDE types

$$\partial_t u + \frac{1}{2} x^2 \sigma_{\mathrm{BS}}^2 \partial_x^2 u - \beta u^+ = 0 \,,\; u(T,x) = 2.1_{x>1} - 1 \;:\; \mathrm{PDE1}$$

and

$$\partial_t u + \frac{1}{2} x^2 \sigma_{\mathrm{BS}}^2 \partial_x^2 u - \beta \mathbb{E}_t[2.1_{x>1} - 1]^+ = 0 \;:\; \mathrm{PDE2}$$

with Poisson intensities $\beta = 1\%$ and $\beta = 3\%$. $\sigma_{\mathrm{BS}} = 20\%$.

Numerical example 1

Maturity(Year)	PDE with poly.	BBM alg.	PDE
2	11.62	11.63(0.00)	11.62
4	16.54	16.53(0.00)	16.55
6	20.28	20.27(0.00)	20.30
8	23.39	23.38(0.00)	23.41
10	26.11	26.09(0.00)	26.14

Table: MC price quoted in percent as a function of the maturity for PDE 1 with $\beta = 1\%$.

Maturity(Year)	PDE with poly.	BBM alg.(Stdev)	PDE
2	11.62	11.64(0.00)	11.63
4	16.56	16.55(0.02)	16.57
6	20.32	20.30(0.00)	20.34
8	23.45	23.45(0.00)	23.48
10	26.20	26.18(0.00)	26.24

Table: MC price quoted in percent as a function of the maturity for PDE 2 with $\beta = 1\%$.

Numerical example 2

Maturity(Year)	PDE with poly.	BBM alg.	PDE
2	12.34	12.35(0.00)	12.35
4	17.72	17.71(0.00)	17.75
6	21.77	21.76(0.00)	21.82
8	25.07	25.06(0.00)	25.14
10	27.89	27.88(0.00)	27.98

Table: MC price quoted in percent as a function of the maturity for PDE 1 with $\beta = 3\%$.

Maturity(Year)	PDE with poly.	BBM alg.(Stdev)	PDE
2	12.38	12.39(0.00)	12.39
4	17.88	17.86(0.00)	17.91
6	22.08	22.07(0.01)	22.14
8	25.58	25.57(0.01)	25.66
10	28.62	28.60(0.01)	28.74

Table: MC price quoted in percent as a function of the maturity for PDE 2 with $\beta = 3\%$.

Multi-type Marked branching diffusions

Joint work with X. Tan and N. Touzi.

- Semi-linear PDE system with polynomial non-linearities:

$$\partial_t u_i(t,x) + \mathcal{L} u_i + \beta_i (F_i(u_0,\ldots,u_N) - u_i) = 0, \quad u_i(T,x) = \psi_i(x) \ , \ \forall \ i = 0 \cdots, N$$

where

$$F_i(u_1,\ldots,u_N) = \sum_{j=0}^{\infty} M_{ij} \prod_{p=1}^{N} u_p^{\mu^i_p(j)}$$

- **Formula**:

$$\hat{u}_i(t,x) = \mathbb{E}\Big[\prod_{j=0}^{N} \prod_{i=1}^{N^j_T} \psi_j(z^i_T) \prod_{j=0}^{N} \prod_{k=1}^{\infty} M_{jk}^{\omega_j(k)} | z^i_t = x, N^j_t = \delta_{ji} \Big]$$

Fully non-linear PDE - toy example

Burgers:

$$\partial_t u + \frac{\sigma^2}{2}\partial_x^2 u + \frac{\beta}{2}(\partial_x u)^2 = 0\ ,\ \ u(T,x) = \psi(x) \in C^\infty(\mathbb{R})$$

Solution: $u(t,x) = \frac{\sigma^2}{\beta}\ln \mathbb{E}_{t,x}[e^{\frac{\beta}{\sigma^2}\psi(X_T)}]$

Bootstrapping method (set $u_0 = u$ and $u_i = \partial_x^i u$):

$$\partial_t u_0 + \frac{\sigma^2}{2}\partial_x^2 u_0 + \frac{\beta}{2}u_1^2 = 0\ ,\ \ u_0(T,x) = \psi(x)$$

$$\partial_t u_1 + \frac{\sigma^2}{2}\partial_x^2 u_1 + \beta u_1 u_2 = 0\ ,\ \ u_1(T,x) = \partial_x \psi(x)$$

$$\partial_t u_2 + \frac{\sigma^2}{2}\partial_x^2 u_2 + \beta\left(u_2^2 + u_1 u_3\right) = 0\ ,\ \ u_2(T,x) = \partial_x^2 \psi(x)$$

$$\cdots$$

$$\partial_t u_K + \frac{1}{2}\partial_x^2 u_K = 0\ ,\ \ u_K(T,x) = \partial_x^K \psi(x)$$

$\rightarrow$ Semi-linear PDE system with polynomial non-linearities!

Numerical example

3 species:

N	Fair	Stdev
12	2.01	0.09
14	2.40	0.28
16	2.14	0.09
18	2.19	0.03
20	**2.20**	0.02

Table: MC price quoted in percent as a function of the number of MC paths 2^N. $T = 1$ year. Exact price $\left(-\frac{\sigma^2}{2} \ln\left(1 - \frac{2}{3}T\right)\right)$ = **2.20**. Non-linearity $\beta = 1$, $\sigma = 0.2$, $\psi(x) = x^2/3$. Blow-up for $T \geq 1.5$ as expected.

Fully non-linear PDE - toy example

- One-dimensional UVM:

$$\partial_t u + \frac{1}{2}\underline{\sigma}^2 \partial_x^2 u + \frac{1}{2}\left(\overline{\sigma}^2 - \underline{\sigma}^2\right)\left(\partial_x^2 u\right)^+ = 0 \ , \ u(T,x) = \psi(x)$$

- Set $u = e^{\beta(T-t)} v$ with $\beta = \frac{1}{2}\left(\overline{\sigma}^2 - \underline{\sigma}^2\right)$:

$$\partial_t v + \frac{1}{2}\underline{\sigma}^2 \partial_x^2 v + \frac{1}{2}\left(\overline{\sigma}^2 - \underline{\sigma}^2\right)\left(\left(\partial_x^2 v\right)^+ - v\right) = 0 \ , \ v(T,x) = \psi(x)$$

- We approximate Γ^+ by a polynomial $P(\Gamma)$[3]:

$$\partial_t v + \frac{1}{2}\underline{\sigma}^2 \partial_x^2 v + \frac{1}{2}\left(\overline{\sigma}^2 - \underline{\sigma}^2\right)\left(P\left(\partial_x^2 v\right) - v\right) = 0$$

[3] This is not really an approximation. In practise, rather than taking $\sigma = \overline{\sigma}\theta(\Gamma) + \underline{\sigma}(1 - \theta(\Gamma))$, we can use some smoother functions of Γ, for example requiring more comfortable break-even levels as the gamma notional increases.

Bootstrap+ truncation

$$\partial_t v_0 + \frac{1}{2}\underline{\sigma}^2 \partial_x^2 v_0 + \frac{1}{2}\left(\overline{\sigma}^2 - \underline{\sigma}^2\right)\left(P\left(v_2\right) - v_0\right) = 0, \quad v_0(T,x) = \psi(x)$$

$$\partial_t v_1 + \frac{1}{2}\underline{\sigma}^2 \partial_x^2 v_1 + \frac{1}{2}\left(\overline{\sigma}^2 - \underline{\sigma}^2\right)\left(P'\left(v_2\right) v_3 - v_1\right) = 0, \quad v_1(T,x) = \psi'(x)$$

$$\partial_t v_2 + \frac{1}{2}\underline{\sigma}^2 \partial_x^2 v_2 + \frac{1}{2}\left(\overline{\sigma}^2 - \underline{\sigma}^2\right)\left(P^{(2)}\left(v_2\right) v_3^2 + P'\left(v_2\right) v_4 - v_2\right) = 0, \quad v_2(T,x) = \psi^{(2)}(x)$$

$$\partial_t v_3 + \frac{1}{2}\underline{\sigma}^2 \partial_x^2 v_3 + \frac{1}{2}\left(\overline{\sigma}^2 - \underline{\sigma}^2\right)\left(P^{(3)}(v_2) v_3^3 + +3P^{(2)}(v_2) v_3 v_4 + P'(v_2) v_5 - v_3\right) = 0,$$

$$\ldots$$

$$\partial_t v_K + \frac{1}{2}\underline{\sigma}^2 \partial_x^2 v_K = 0, \quad v_K(T,x) = \psi^{(K)}(x)$$

In practise, $\frac{1}{2}\left(\overline{\sigma}^2 - \underline{\sigma}^2\right) \preccurlyeq 1$ (i.e. small perturbation).

Numerical example

5 species:

N	Fair	Stdev
12	20.18	0.51
14	20.13	0.26
16	19.94	0.13
18	19.94	0.06
20	19.96	0.03

Table: MC price quoted in percent as a function of the number of MC paths 2^N. $T = 10$ year. Exact price = **20**. “Non-linearity" $P(\Gamma) = \Gamma$, $\sigma = 0.2$, $\psi(x) = x^2/2$.

N	Fair	Stdev
12	12.21	0.25
14	12.14	0.13
16	11.99	0.06
18	11.92	0.03
20	11.95	0.02

Table: MC price quoted in percent as a function of the number of MC paths 2^N. $T = 10$ year. Exact price = **11.96**. Non-linearity $P(\Gamma) = \Gamma^2/2$, $\sigma = 0.2$, $\psi(x) = x^2/2$.

Conclusions

1. Forward MC scheme for fully non-linear parabolic PDEs.
2. Applicable in higher dimensions (no grid space).
3. No regressions and finite elements required.
4. Algorithm fully parallelizable (independent particles - no interaction).

Some references

- PHL: *Counterparty Risk Valuation: A Marked Branching Diffusion Approach*, ssrn(2012), submitted.
- PHL, Tan, X., Touzi, N. : *A numerical algorithm for a class of BSDEs via branching processes*, in preparation.