Delta-hedging Vega Risk?

Stéphane Crépey[*]

August 24, 2004

Abstract

In this article we compare the Profit and Loss arising from the delta-neutral dynamic hedging of options, using two possible values for the delta of the option. The first one is the Black–Scholes implied delta, while the second one is the *local* delta, namely the delta of the option in a generalized Black–Scholes model with a local volatility, recalibrated to the market smile every day. We explain why in negatively skewed markets the local delta should provide a better hedge than the implied delta during slow rallies or fast sell-offs, and a worse hedge, though to a lesser extent, during fast rallies or slow sell-offs. Since slow rallies and fast sell-offs are more likely to occur than fast rallies or slow sell-offs in negatively skewed markets (provided we have physical as well as implied negative skewness), we conclude that on average the local delta provides a better hedge than the implied delta in negatively skewed markets. We obtain the same conclusion in the case of positively skewed markets. We illustrate these results by using both simulated and real time-series of equity-index data, that have had a large negative implied skew since the stock market crash of October 1987. Moreover we check numerically that the conclusions we draw are true when transaction costs are taken into account. In the last section we discuss the case of barrier options.

Key words. Option, Black–Scholes model, implied volatility, smile, skew, local volatility, stochastic volatility, volatility regimes, model calibration, delta-hedging, transaction costs, barrier options.

1 Introduction

In option markets, using the basic Black–Scholes model for hedging may prove to be very misleading. For example, it is not unusual for traders that are delta-neutral and gamma positive in the Black–Scholes model to see their position damaged when the market moves. This is a consequence of the misspecification of the Black–Scholes model. This misspecification is most clearly visible in the *volatility smile*, namely the fact that the options' implied volatility depends in practice upon the options' moneyness and time-to-maturity. Recall that the Black–Scholes implied volatility means the constant value of the volatility parameter that, injected into the Black–Scholes formula for that option price, makes the Black–Scholes price of the option equal to its market value. Therefore if the market prices behaved as in the Black–Scholes world then the options' smile would be flat and constant. It is neither flat nor constant. Typical implied volatility patterns at fixed time-to-maturity include (i) the *skew*, i.e. an upwards slope or a downwards slope, as persistently seen on equities and equity-indices since the October 1987 stock market crash, and (ii) the *smile*, that is convex in the option strike, as it is often the case on foreign exchange markets. The term *smile* is also used for denoting the whole of the implied volatility *surface* of options with variable moneyness and time-to-maturity.

[*]Université d'Évry, Laboratoire d'Analyse et Probabilité, Département de Mathématiques, Bd François Mitterrand, 91025 Évry Cedex, France (`Stephane.Crepey@univ-evry.fr`).

In order to take the smile into account, traders often use the Black–Scholes model with the implied volatility of the option as volatility parameter. However, this is a purely *ad hoc* procedure to find a better and more elaborate model consistent with the market smile. On equity, index or currency markets, there are at least three classes of models:

a. The standard Black–Scholes model [12, 49];

b. The Dupire or fitted Black–Scholes model, that is a generalized Black–Scholes model with a local volatility, and is often referred to as the *implied tree model* [29] or the *local volatility model* [34];

c. Stochastic volatility models, that further subdivide into *spot stochastic volatility models* [40, 42, 37], *stochastic implied trees* [30] and *market models of implied volatility* [52, 17, 16, 21]. These can be regarded as the counterparts in the equity-world of the short-rate models [56, 22], term structure models [39] and market models [15, 50] on fixed-incomes.

Other improvements include *jumps* or *stochastic interest rates* in order to better handle very short-dated or long-dated options [4, 33]. The classes of models mentioned above are successive steps to better manage the *volatility risk* that arises from the fact that volatility is not constant, but is actually randomly time-varying. In fact, the only class of models in which this volatility risk can be consistently taken into account is class c (see above), that consists in *dynamic* models of volatility. Complexity and incompleteness are drawbacks.

In class b, the volatility risk is inconsistently taken into account by daily recalibration of the model to the market smile. We refer the reader to Hull & Suo [41] for a description of the way the model is used by traders. On the one hand, the model is popular among traders because it contains a sufficient number of degrees of freedom to provide, using appropriate calibration methods, a perfect fit to the market smile. This ensures that the model does not generate any arbitrage, at least when the date is set. Moreover the model is complete. Traders on the other hand dislike local volatility models for the bad dynamics they often predict for the volatility smile, notably their unrealistic forecast of rapidly flattening forward smiles. As a matter of fact it is well known that one-dimensional diffusions are misspecified. So call option prices are increasing in the underlyer in one-dimensional diffusion models [9, 23], which is often violated in real markets [5]. Recalibrating the model to the market smile every day is the way in which traders overcome the misspecification of the model.

Other studies have compared the performances of the standard Black–Scholes model with those of spot stochastic volatility models [4, 6, 46, 53], and those of models including jumps and/or stochastic interest rates [4]. So, working on 1988–1991 S&P 500 options data, Bakshi *et al* [4] found that stochastic volatility modelling improved the single-instrument delta-hedging performances over the standard Black–Scholes model, especially for out-of-the-money (OTM) calls. As for market models of implied volatility, the problem of specifying the *risk-neutral* dynamics of the implied volatility in these models is still open. Consequently, they are mainly used for statistical and risk management purposes (VaR calculations) rather than for hedging options.

In this paper we shall concentrate upon the comparison between the standard and the fitted Black–Scholes models. Since the Dupire or fitted Black–Scholes model is used by traders in the inconsistent dynamic mode consisting in recalibrating the model to the market smile every day, it is important to evaluate the performances of this use of the model for hedging vanillas or dealing with more exotic kinds of options. Relying upon analytical as well as empirical considerations, we shall compare the performances of delta-hedging strategies that use two possible values for the delta of the option. The first one is the Black–Scholes implied delta, while the second one is the so-called *local* delta, i.e. the delta of the option in a generalized Black–Scholes model with a local volatility, recalibrated to the market smile every day.

2   Review of other articles and overview of the paper

Empirical comparisons have already been made before by Dumas *et al* [32], Coleman *et al* [20, 19] and Vähämaa [55]. Their results are controversial. Dumas *et al* [32] concluded that the implied delta provided a better hedge than the local delta, while Coleman *et al* [20, 19] and Vähämaa [55] found that the reverse was true. Moreover, following the theoretical considerations in Derman [26], Vähämaa [55] found that the superiority of the local delta was more significant in jumpy market regimes, such as the jumpy market regime that followed the terrorist attacks of September 2001.

2.1   Hedging horizon

Dumas *et al* [32] concluded that the implied delta outperformed the local delta as far as hedging is concerned. However, what Dumas *et al* [32] actually put to the test, under the name *hedging performances*, consisted in fact of *prices prediction performances*. More precisely, using June 1988–December 1993 S&P 500 European option prices as input data, they calibrated local volatility models with the data of fixed days. Then they used the calibrated models for computing new option prices one week later. Alternatively, they expressed the option prices of fixed days in terms of Black–Scholes implied volatilities that they used for computing new option prices one week later. Comparing either set of predicted prices with the actual market prices one week later, they found first that the predictions of both models undervalued the true market prices, and secondly that the local volatility model undervalued them to a greater extent.

Coleman *et al* [20, 19] performed actual dynamic hedging experiments such as those we shall present in this article, based on 1993 European S&P 500 index options data and on 1997–1998 American S&P 500 futures options data. Analyzing their results in terms of the standard deviations of the final P&L (Profit and Loss), they found that the local delta outperformed the implied delta, provided the *hedging horizon* (time interval between the start and the end of the hedge) was long enough, and longer than one or two weeks.

2.2   Regimes of volatility

According to other research, the crucial point would lie not in the hedging horizon, but in the so-called *regime of volatility*. Introduced by Derman [26] and further considered by Alexander [1, §§2.3 and 6.3], this concept is at the root of the approach that led to the market models of implied volatility [52, 17, 16, 21]. Derman [26] draws a distinction between three types of market conditions, or regimes of volatility: stable, trending and jumpy.

In the stable regime, the options' implied volatilities are essentially stable. Considered in isolation, each option behaves as if it lived in its own Black–Scholes world, with the Black–Scholes implied volatility of the option as volatility parameter. The market is referred to as being *sticky strike*. In this regime it is expected that the implied delta provides a reasonably good hedge.

In a trending market it would be reasonable to expect that the smile tends to follow the underlyer. The smile would then be referred to as being *sticky delta*, i.e. time-invariant as a function of the options' *moneyness*, rather than time-invariant as a function of the options' strike. However, it seems that the market keeps being *sticky strike* in trending regimes (applying corrections after a while), so that once again the use of the implied delta seems relevant. Thus, as far as the dynamics of the smile is concerned, the stable and trending regimes are very similar. From now on we shall refer to both of them as being *slow* markets.

In contrast, in the jumpy regime, that we shall call *fast* regime, the underlyer as well as the options' smile exhibit a high level of volatility. In such fast markets with high levels of *volatility of volatility*, hedging with the Black–Scholes implied delta of the option is insufficient: one should also control the Vega Risk of the option, i.e. the risk arising from the volatility sensitivity of the option. In equity and equity-index markets, changes in implied volatilities (as well as in historical volatility, though

to a lesser extent) happen to be strongly negatively correlated with the returns in the underlyer. This is a quite general phenomenon, especially in jumpy markets [1]. It has received numerous explanations in the economico-financial literature, such as the *leverage* effect (see, for instance, Jackwerth–Rubinstein [43]). So Figure 1 displays the FTSE 100 index and the corresponding at-the-money (ATM) 3 months rolling implied volatility, as well as the implied volatility of a fixed FTSE 100 option, between October 1 1999 and March 1 2000 (yet this was a slow market, as we shall assess by a definite backtesting criterion in section 4). Given this correlation, we shall see in
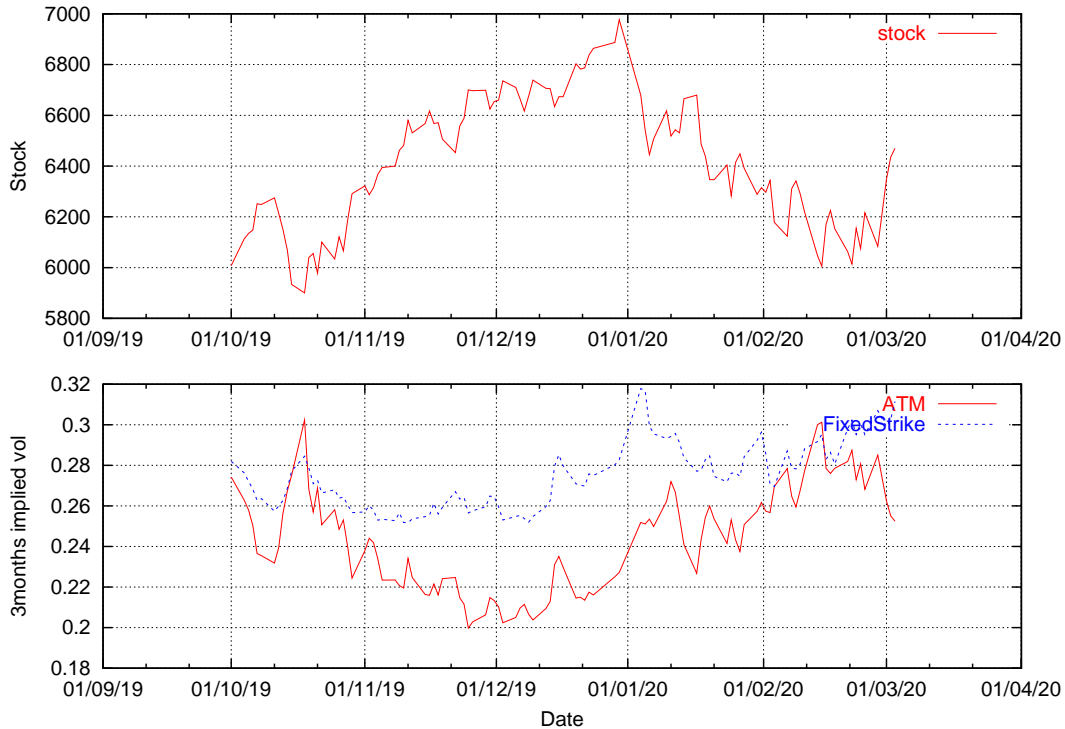

Figure 1: *Inverted mirror picture on the FTSE 100, October 1 1999–March 1 2000.*

section 3 that a natural way of managing the Vega Risk in jumpy equity markets consists in adding a correction term to the implied delta, which is tantamount to using the local delta instead of the implied delta for hedging the option.

This theory of the regimes of volatility implies that, in negatively skewed markets, the implied delta should not be worse or could even be better *on average conditionally on the fact that the market is in a slow regime*, whereas the local delta should be better *on average conditionally on the fact that the market is in a fast regime*. Using a suitable proxy for the local delta (see §3.3), Vähämaa [55] studied the P&L of delta hedged option positions on 2001 European FTSE options data, with an emphasis on the months of May (trending regime) and September (jumpy regime following the World Trade Center attacks). He found first that the standard deviation of the P&L increments over all the options in the sample at his disposal was smaller with the local delta than it was with the implied delta in both months, and secondly that the difference between the standard deviations was more significant in September[1]. Thirdly he found that the local delta outperformed the implied delta during the year 2001 as a whole.

---

[1] Vähämaa [55] assesses the significance of the differences by a bootstrapping method with 1000 resamplings.

2.3 Overview

The previous results may seem rather puzzling, sometimes even contradictory. In this article we try to understand how the delta-hedging strategies in either delta compare in different regimes and in general. In section 3 we propose a theoretical analysis. The main results of the paper are propositions 3.5 to 3.7: due to the discreteness of the hedge and to the reversion of implied volatilities towards realized volatilities, the better delta of the two depends not only on the market regime, slow or fast, but also on the *direction* of the market, upwards or downwards. So, in the case of negatively skewed markets, the local delta should provide a better hedge than the implied delta during slow rallies or fast sell-offs, and a worse hedge (though to a lesser extent) during fast rallies or slow sell-offs. Since slow rallies and fast sell-offs are more likely to occur than fast rallies or slow sell-offs in negatively skewed markets (provided we have *physical* as well as implied negative skewness), we conclude that the local delta is better *on average*, as well as *on average conditionally on the fact that the market is in a fast regime* or *on average conditionally on the fact that the market is in a slow regime* [2]. We draw the same conclusions in the case of positively skewed markets. In section 4 we present numerical results, using simulated as well as real time-series of equity-index financial data. Equity-index markets have had a large negative implied skew since the stock market crash of October 1987. We shall see that the numerical results are fully consistent with the theoretical analysis of section 3, and also, that the transaction costs do not blur the results. In the last section we consider the case of barrier options. Appendices A and B recall the mathematical definition of the Dupire model and detail the pricing and calibration procedures used in the body of the article.

3 Definition and analysis of the delta-hedging strategies

Given a stock, index or currency S, a European vanilla *call* (or, respectively, *put*) option with maturity date T and strike K, on the underlying asset S, corresponds to a right to buy (or, respectively, sell), at price K, a unit of S at time T. In the body of this article, we shall assume that the riskless interest rate r in the economy and the dividend yield q on S are both equal to zero, in order to simplify the notation. We refer the reader to Appendix A for the mathematical definition of the Dupire model [34, 1994], with arbitrary constant r and q.

3.1 Implied delta versus local delta

Consider an agent who is short of one European vanilla option struck at K and expiring at T. Delta-hedging the option consists in rebalancing a complementary position in the underlyer every time step τ[3], in order to minimize the overall exposure to small moves of the underlyer. In this article, we shall be primarily concerned with the comparison between two delta-hedging strategies, with positions Δ in the underlyer at date t given by (see Appendix A):

- the *implied delta* of the option, that is

$$\Delta \equiv \Delta^{BS} = \partial_S \Pi^{BS}_{T,K}(t, S; \Sigma_{T,K})$$

  where $\Pi^{BS}_{T,K}$ and $\Sigma_{T,K}$ denote the Black–Scholes price and implied volatility of the option;

- or, alternatively, the *local delta* of the option, that is

$$\Delta \equiv \Delta^{loc} = \partial_S \Pi_{T,K}(t, S; \sigma)$$

  where σ represents the local volatility function calibrated with the implied volatility surface observed at date t and $\Pi_{T,K}$ means the price of the option in the Dupire model thus calibrated.

---

[2]Note the difference between these conclusions and the implications of §2.2 (see also the discussion in §3.5).

[3]We shall take τ as being equal to one market day in the numerical experiments of sections 4 and 5.

We compare the P&L trajectories that we obtain by adding up the following increments comprised from the setting up of the hedge until its closure:

$$\delta P\&L = -\delta\Pi + \Delta\,\delta S \tag{1}$$

where Π is the market price of the option and $\Delta \equiv \Delta^{BS}$ or Δ^{loc}. More precisely, we aim at determining which $\Delta \equiv \Delta^{BS}$ or Δ^{loc} maintains the P&L trajectory closest to 0 throughout the hedging period. Unless otherwise stated, the hedge will be closed the first time (if any before T) the option's moneyness K/S leaves the range used for calibrating the local volatility every day (namely 0.8 to 1.2 for European calibration problems and 0.9 to 1.1 for American calibration problems, see Appendices A and B). Outside this range, the calibrated local volatility model price of the option is no longer marked-to-market; moreover the option has a low vega (sensitivity to volatility), so that the implied and the local delta get closer and closer to each other and the choice between them is no longer relevant.

Let us add that such elementary hedging schemes, using either the implied delta or the local delta, could be diversified and enhanced in several ways. So we could incorporate additional instruments like liquid options in the hedging portfolio in order to statically reduce the variance of the P&L and limit the volatility risk (see [54, 18, 27, 35]). As we shall see in section 5, this is particularly important for hedging exotic options.

3.2 Analysis in the standard Black–Scholes model

Let us recall well-known facts in an idealized Black–Scholes world with constant volatility $\sigma \equiv \Sigma$ [44, 6, 10, 48, 13, 14, 36, 38]. By using a Taylor expansion for $\delta\Pi$ and by applying the Black–Scholes equation (12) with constant volatility $\sigma \equiv \Sigma$ to Π, we obtain:

$$\delta P\&L = -\delta\Pi + \Delta\delta S = \frac{1}{2}S^2\Gamma\left(\Sigma^2\tau - (\frac{\delta S}{S})^2\right) + o(\tau)$$

where Π, $\Delta \equiv \partial_S\Pi$ and $\Gamma \equiv \partial^2_{S^2}\Pi$ represent the option's *price*, *delta* and *gamma* in the Black–Scholes model. Hence, the distribution of the Profit and Loss

$$P\&L = \sum\delta P\&L$$

(where the sum is extended over the lifetime of the option) is asymptotically symmetric and centered as τ tends to 0. Moreover, P&L converges in probability to 0 as τ tends to 0. The convergence rate depends on the regularity of the option's payoff. In the case of a European vanilla call or put option, the standard deviation of P&L is dominated by $\sqrt{\tau}$.

3.3 Analysis in a local volatility model

Let us now operate in the framework of a fixed local volatility model (see Appendix A). By using a Taylor expansion for $\delta\Pi$ and by applying the Black–Scholes equation (12) with local volatility $\sigma \equiv \sigma(t, S)$ to Π, we obtain:

$$\delta P\&L^{loc} = -\delta\Pi + \Delta^{loc}\delta S = \frac{1}{2}S^2\Gamma^{loc}\left(\sigma(t, S)^2\tau - (\frac{\delta S}{S})^2\right) + o(\tau) \tag{2}$$

where Π, Δ^{loc} and Γ^{loc} are the option's price and its Greeks in the model. Since the local volatility model price of a vanilla call or put option is convex in the spot price of the underlyer (see, for instance, Crépey [23, theorem 4.3]), then Γ^{loc} is positive. Therefore:

Proposition 3.1 *In a local volatility model, whether $\delta P\&L^{loc}$ is negative or positive depends, up to the order $o(\tau)$, on whether $(\frac{\delta S}{S})^2$ is larger or smaller than $\sigma(t, S)^2\tau$ – or, equivalently, on the relative position of the so-called* realized volatility $\frac{|\delta S|}{S\sqrt{\tau}}$ *with respect to the local volatility $\sigma(t, S)$.*

Up to the order $o(\tau)$, the (physical as well as risk-neutral) expectation of the square of the realized volatility (the so-called *realized variance* [25]) is equal to the square of the local variance $\sigma(t, S)^2$. Consequently:

Proposition 3.2 *In a local volatility model, the distribution of $\delta P\&L^{loc}$ is asymptotically centered as $\tau \to 0$.*

Since
$$\delta P\&L^{BS} - \delta P\&L^{loc} \;=\; (\Delta^{BS} - \Delta^{loc})\, \delta S \,, \tag{3}$$
we deduce that $\delta P\&L^{BS}$ is *directional*, unlike $\delta P\&L^{loc}$. This means that $\delta P\&L^{loc}$ is driven by a term in δS. And so:

Proposition 3.3 *In a local volatility model, asymptotically as $\tau \to 0$:*

  a. *$\delta P\&L^{loc}$ is driven by terms in τ and $(\delta S)^2$;*

  b. *$\delta P\&L^{BS}$ remains directional;*

  c. *Consequently the fluctuations (such as measured by the standard deviation) of $\delta P\&L^{BS}$ are one order of magnitude greater than those of $\delta P\&L^{loc}$.*

Moreover, we have:
$$\Pi_{T,K}(t, S; \sigma) = \Pi_{T,K}^{BS}(t, S; \Sigma_{T,K}) \,.$$

Denoting by ν^{BS} the Black–Scholes implied vega of the option (that is *positive*, for a European vanilla option), this implies that
$$\Delta^{loc} = \Delta^{BS} + \nu^{BS}\partial_S\Sigma \,. \tag{4}$$

In local volatility models, it is well known that the implied volatility can be interpreted as an average of the local volatilities on the most likely paths between (t, S) and (T, K) (see Derman *et al* [31, Appendix] and Berestycki *et al* [11]). Therefore, in the case of a monotonic implied skew, one may expect the local volatility to be skewed in the same direction. Assuming that the dividend and interest rates are sufficiently close to each other, Coleman *et al* [19, p. 9] proved that $\partial_S\Sigma$ has the same sign as the skew, by using the call-put parity relation. Moreover, in the case where the value of local volatility is independent of time and varies linearly with underlyer level, Derman *et al* [31, Appendix] showed that the implied skew gives an approximation for $\partial_S\Sigma$, i.e.

$$\partial_S\Sigma \approx \partial_K\Sigma \,. \tag{5}$$

For example, the previous linearity assumption is legitimate in the case of equity-index market skews of options of fixed maturity (see Derman *et al* [28]). Real skews also exhibit a significant term structure. Omitting this aspect and applying rule (5) maturity by maturity in (4), we obtain the following proxy for the local delta:

$$\Delta^{loc} \approx \Delta^{BS} + \nu^{BS}\partial_K\Sigma \,. \tag{6}$$

The implied skew $\partial_K\Sigma$ can be read straightaway in the market. Therefore this proxy allows some people to estimate the local delta without having to calibrate the model. It is used, for instance, in Vähämaa [55] (see §2.2).

Assuming furthermore that σ is *negatively skewed*, then (6) implies that $\Delta^{loc} \leq \Delta^{BS}$, hence the following statement.

Proposition 3.4 *In a negatively skewed local volatility model, given a fixed rebalancing time interval τ:*

  a. *$\Delta^{loc} \leq \Delta^{BS}$;*

    b. $\delta P\&L^{BS}$ *is larger or smaller than* $\delta P\&L^{loc}$ *according to whether* δS *is positive or negative.*

Table 1 illustrates the results of propositions 3.1 and 3.4 in a negatively skewed local volatility model. During slow rallies and fast sell-offs (highlighted in red in Table 1), the local delta provides a better hedge ($|\delta P\&L^{loc}| \leq |\delta P\&L^{BS}|$). In contrast, during fast rallies or slow sell-offs (in blue) the implied delta might provide a better hedge ($|\delta P\&L^{BS}| \leq |\delta P\&L^{loc}|$), though to a lesser extent. By "to a lesser extent" we mean that as a whole the situation seems more favourable to the local delta since the red cases in Table 1 are necessarily in its favour while the blue ones are indeterminate in general. Moreover there is no reason to think that the cases that are favourable to the implied delta would be more favourable to the implied delta than the cases favourable to the local delta are favourable to the local delta. So the difference between the two δP&Ls is always equal to $(\Delta^{BS} - \Delta^{loc})\,\delta S$, an a priori unbiased quantity from this point of view. In practice let us try to see which delta is better in the blue cases. It is natural to expect that this depends on the rebalancing time interval τ. For moderately small τ (such as *one day*, as we shall see numerically in sections 4 and 5), the dispersion of the realized volatility along the sampled trajectories is the dominant factor. This dispersion is caused by the discreteness of the hedge, enhanced by the local nature of the volatility in the model. Therefore the implied delta is likely to be better in the blue cases, for moderately small τ. But keep in mind that we are currently operating under the assumption of a local volatility model, in which the local delta would become a perfect hedge along any trajectory of the underlyer if the rebalancing frequency went to infinity (see proposition 3.3).

	Slow	**Fast**
Rally	$0 \leq \delta P\&L^{loc} \leq \delta P\&L^{BS}$	$\delta P\&L^{loc} \leq -(\delta P\&L^{BS})^{-}$
Sell-Off	$(\delta P\&L^{BS})^{+} \leq \delta P\&L^{loc}$	$\delta P\&L^{BS} \leq \delta P\&L^{loc} \leq 0$

Table 1: *Market regimes in a negatively skewed local volatility model.*

In a negatively skewed model, slow rallies and fast sell-offs are more likely to occur than fast rallies or slow sell-offs (slow rallies and fast sell-offs are the *dominant regimes* under negative skewness). Therefore, from the above considerations, it is reasonable to expect that the local delta is better *on average*, as well as *on average conditionally on the fact that the market is in a fast regime* or *on average conditionally on the fact that the market is in a slow regime* — This seems reasonable, except for one point, and that might be an important practical issue. The increments δS in the δP&Ls are issued from the *physical* stock process (9), which can be very different from the *risk-neutral* stock process (11). As shown in Bakshi *et al* [7], negatively skewed risk-neutral distributions are possible even when the physical returns' distribution is symmetric. This leads us to the following statement.

Proposition 3.5 *In a negatively skewed local volatility model, given a moderately small rebalancing time interval τ (such as one day):*

    a. the local delta provides a better hedge in a slow rally or a fast sell-off, while the implied delta may provide a better hedge, though to a lesser extent, in a fast rally or a slow sell-off;

    b. provided we have physical *as well as implied negative skewness, the local delta is better on average, as well as on average conditionally on the fact that the market is in a fast regime, or on average conditionally on the fact that the market is in a slow regime.*

In a *positively* skewed local volatility model (Table 2), we must reverse the order of the lines in Table 1, and the dominant regimes are also reversed (provided we have *physical* as well as implied positive skewness), hence the following statement.

Proposition 3.6 *In a positively skewed local volatility model, given a moderately small rebalancing time interval τ (such as one day):*

	Slow	Fast
Rally	$(\delta P\&L^{BS})^+ \le \delta P\&L^{loc}$	$\delta P\&L^{BS} \le \delta P\&L^{loc} \le 0$
Sell-Off	$0 \le \delta P\&L^{loc} \le \delta P\&L^{BS}$	$\delta P\&L^{loc} \le -(\delta P\&L^{BS})^-$

Table 2: *Market regimes in a positively skewed local volatility model.*

a. *the local delta provides a better hedge in a fast rally or a slow sell-off, while the implied delta may provide a better hedge, though to a lesser extent, in a slow rally or a fast sell-off;*

b. *provided we have* physical *as well as implied positive skewness, the local delta is better on average, as well as on average conditionally on the fact that the market is in a fast regime, or on average conditionally on the fact that the market is in a slow regime.*

3.4 Analysis in real markets

In real markets, we can decompose the P&L increments in the following way:

$$\begin{cases} \delta P\&L^{loc} &= (-\delta\Pi^{loc} + \Delta^{loc}\delta S) + (\delta\Pi^{loc} - \delta\Pi) \\ \delta P\&L^{BS} &= (-\delta\Pi^{loc} + \Delta^{BS}\delta S) + (\delta\Pi^{loc} - \delta\Pi) \end{cases} \tag{7}$$

where $\delta\Pi$ denotes the increment of the market price of the option between the dates t and $t + \tau$ while $\delta\Pi^{loc}$ represents the price increment predicted by the local volatility model calibrated at date t, given the new value of the underlyer at date $t + \tau$. In the right-hand side of (7), the first terms behave as in the analysis of §3.3, while the second terms are due to the misspecification at date $t + \tau$ of the local volatility model calibrated at date t. This misspecification arises from the fact that the market-makers have revised their anticipations between date t and date $t + \tau$, according to the new value of the underlyer observed at date $t + \tau$ (and also, from time to time, according to more punctual economico-political macro news or events). It seems reasonable to expect that (i) at fast market regimes with high levels of realized volatility, the market-makers will have a tendency to push the options' implied volatility upwards compared to those predicted by the model calibrated at date t, whereas (ii) at slow market regimes, the market-makers will have a tendency to push the options' implied volatility downwards compared to those predicted by the model calibrated at date t. Provided we deal with vanilla options, that are vega positive, this implies that (i) $\delta\Pi^{loc} \le \delta\Pi$ at fast market regimes and (ii) $\delta\Pi \le \delta\Pi^{loc}$ at slow market regimes. Let us assume that the market is negatively skewed. By comparison with the situation in a negatively skewed local volatility model, $\delta P\&L^{BS}$ and $\delta P\&L^{loc}$ are pushed away from 0 by the same amount in Table 1. So the situation depicted in Table 1 still holds true in the real market. In the dominant market regimes in particular (in red in Table 1), the revision of their anticipations by the market-makers merely worsens the two $\delta P\&L$s by the same amount. It is easy to see that the conclusions are the same in a positively skewed market. Thus:

Proposition 3.7 *Assuming that the main explanation of the movement of the market smile between t and $t + \tau$, beyond the movement predictable in the date t-calibrated model knowing the new value of the underlyer at date $t + \tau$, is a reversion towards the most recent realized volatilities (including the volatility realized between t and $t + \tau$), then propositions 3.5 and 3.6 apply not only in local volatility models, but also in real markets.*

3.5 Overall recommendation

If the assumption in proposition 3.7 holds true and provided we have physical as well as implied negative skewness, then the local delta provides a better hedge on average, as well as on average conditionally on the fact that the market is in a fast regime, or on average conditionally on the

fact that the market is in a slow regime. Then our overall recommendation is to use the local delta rather than the implied delta. We make the same recommendation in a persistently positively skewed market.

By comparison, Derman [26] implies that, in negatively skewed markets, the implied delta should not be worse or could even be better on average conditionally on the fact that the market is in a slow regime, while the local delta should be better on average conditionally on the fact that the market is in a fast regime. In this analysis one needs to know what is or will be the actual market regime, fast or slow, for making one's choice of a delta. The question of knowing which delta is better on average is left unanswered.

However it is rather natural to think that the physical market skewness may be correlated with the market regime (there should be more physical negative skewness at fast regimes), conditionally on the assumption of an implied negative skew. In this case the outperformance of the local delta *on average conditionally on the fact that the market is in a fast regime* would be larger than its outperformance *on average conditionally on the fact that the market is in a slow regime* or *on average in general*. In this respect, the statistical studies about the hedging performances of either delta are of particular interest. You should remember that if we exclude the first paper by Dumas *et al* [32], most subsequent papers in the field, such as Coleman *et al* [20, 19] or Vähämaa [55], conclude that on average the local delta provides a better hedge than the implied delta, especially in jumpy markets.

4 Report on numerical experiments

We are going to put the theory of section 3 to the test in order to assess which delta of the two provides a better hedge in practice. First we shall examine numerical results obtained by simulation within a fixed calibrated local volatility model. Then we shall present real-life hedging experiments using only the trajectory effectively followed by the underlyer in the data set.

4.1 Description of the data

Local volatility models are applicable to any equity, index or currency asset. However, since the calibration of such models requires a liquid market of vanilla options quoted on a sufficiently broad range of strikes distributed on several maturities, we shall concentrate on major equity-indices. Equity-index markets have been exhibiting a large negative implied skew since the stock market crash of October 1987. A common interpretation of the skew is that final investors tend to overprice OTM puts with respect to OTM calls because of a portfolio insurance premium on OTM puts. As a consequence (see proposition 3.4.*a*), the gap between the implied and the local delta of at-the-money vanilla options usually lies between 5% and 20%. However note that the *physical* negative skewness on these markets is much smaller, when there is any, than the implicit skewness. Thus, it is questionable whether or not we are in the preconditions of proposition 3.5.*b* (or the similar proposition in real markets, see proposition 3.7).

Since similar experiments involving several indices (S&P 500, DAX, FTSE 100, SMI and Dow Jones) always led to the same qualitative conclusions, we present the results obtained with two of them, namely the FTSE 100 and the DAX. On the FTSE index both European and American vanilla options are available. We used real settlement option prices quoted on the LIFFE[4] and on the DTB[5] during 1999–2000 for the FTSE 100 and during 2001 for the DAX. These data sets as well as an Excel spreadsheet with further results on these and other indices (S&P 500, SMI and Dow Jones) are available on request. The historical skewness of the returns over the mentioned years are -1.04 for the DAX and -0.06 for FTSE 100, which amounts to much less negative skewness than in

---

[4]International Financial Futures and Options Exchange, http://www.liffe-data.com.

[5]Deutsche Terminbörse, http://deutsche-boerse.com/INTERNET/EXCHANGE/index_e.htm.

the risk-neutral distributions of these indices over the same periods.

4.2 Numerical experiments in a calibrated local volatility model

First we present hedging experiments in the framework of a fixed local volatility model. Yet in order to keep as close as possible to the market we do not use an exogenously and arbitrarily fixed local volatility model, but we resort to a model calibrated with one of our real option prices data sets. Thus we calibrated a local volatility function, using the European vanilla option prices observed on the DAX index on 24 August 2001 and the associated zero-coupon curve as input instruments (see Appendices A and B). Then we simulated 1000 trajectories of the underlyer between August 24 and September 10 2001, in the physical local volatility model defined by the local volatility thus calibrated and by a physical drift ρ arbitrarily taken as zero in (9). Along each simulated trajectory of the index, we delta-hedged the European vanilla call option with maturity $T = 0.556$ and strike $K = 5400$, by using either the local delta or the implied delta for rebalancing the position every market day. We chose this particular option as nearest-to-the-money among those traded on the DAX index on August 24, 2001.

The average absolute final P&L was found to be 3.65% of the initial option premium with the implied delta, versus 1.36% with the local delta. Figure 2 displays the histograms of (i) the increments δP&L aggregated over the 1000 simulations and (ii) the final P&L on September 10 2001, using the implied delta or the local delta for hedging. Further descriptive statistics are given in Table 3. The local final P&L exhibits much less standard deviation (yet slightly more negative skewness and excess kurtosis) than the implied final P&L. By using either a variance equality F-test or a shuffling procedure with 5000 resamplings, the difference between these standard deviations was found to be significant at any confidence level. The correlation that appears in Table 3 is the empirical correlation between the final P&Ls and the realized volatility (annualized historical volatility) along the simulated trajectories. In accordance with the analysis in §3.3, this correlation is highly negative and significant in the case of the local delta, and much less so in the case of the implied delta. Thus, the fluctuations of the realized volatility account for 72.85% of the variance of the local final P&L, versus 36.73% of the variance of the implied final P&L.

We also computed statistics concerning the relative value of the absolute areas enclosed by the P&L trajectories. These absolute areas are natural indicators of how well either delta managed in its task of keeping the P&L trajectory as close as possible to zero over the hedging period. The absolute final P&L would be another indicator of the same kind, but the enclosed absolute area indicator is a smoother one. Concretely, for every simulated trajectory of the index, we computed the log-difference between the local and the implied *enclosed absolute area* (the sum of the absolute values of the P&L over the hedging period). The average value for this log-difference was -0.68 and the extrema were -2.98 and 1.92. In other words on average the local enclosed absolute area was 1.97 times smaller than the implied enclosed absolute area. However, in many cases, the local enclosed absolute area was greater than (and up to 6.84 times greater than) the implied enclosed absolute area. In such cases the implied delta provided a better hedge than the local delta, though this concerns a local volatility model, in which the local delta would become a perfect hedge along any underlying trajectory if the rebalancing frequency went to infinity. As explained in the previous section, this is due to the dispersion of the realized volatility along the sampled trajectories. This dispersion is caused by the discreteness of the hedge and enhanced by the local nature of the volatility in the model. For a similar phenomenon in the standard Black–Scholes framework, the reader is referred to Kamal [44].

4.3 Real-life hedging experiments on market data

Now we are going to consider real-life hedging experiments on market data, using only the trajectory effectively followed by the underlyer in the data set. Our analysis relies upon individual P&L trajectories and basic statistics for the P&L increments. For further statistical evidence, the reader
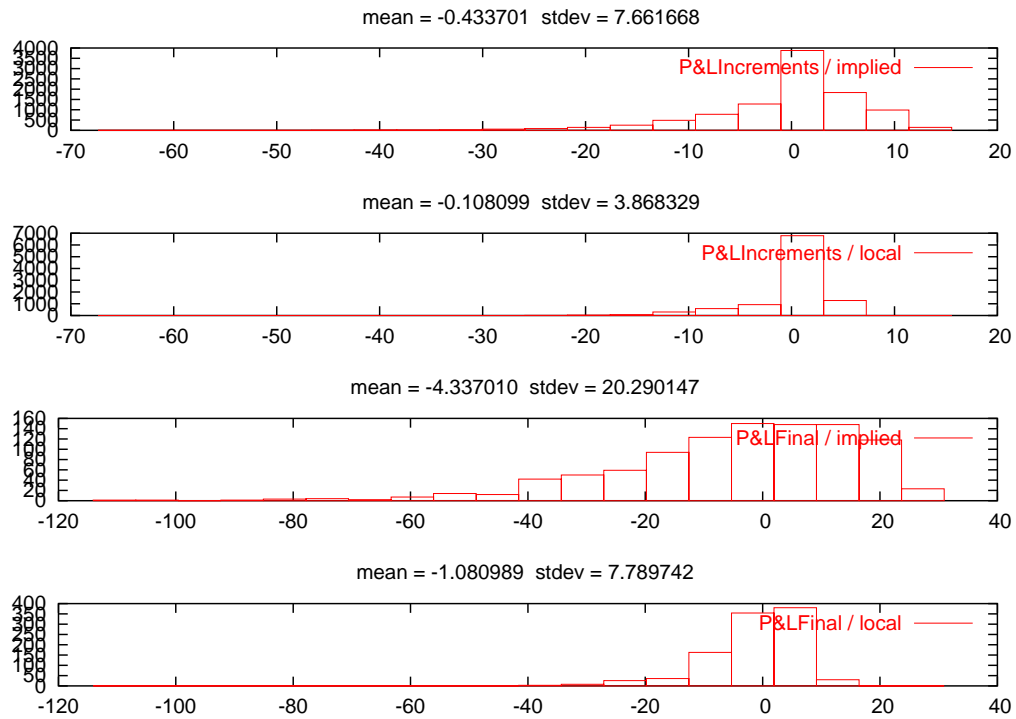
Figure 2: *Delta-hedge of a European vanilla call option in a local volatility model calibrated with real option prices, using the implied delta or the local delta for hedging; P&L histograms (aggregated over 1000 simulations).*

	Average	Stdev	Skewn.	Exc. Kurt.	Min	Max	Correl	$\mathbf{R}^2$
final P&L implied	-4.33	20.29	-1.10	1.87	-110.45	27.30	-60.60%	36.73%
final P&L local	-1.08	7.78	-1.27	2.28	-35.71	12.34	-85.35%	72.85%
Realized volatility	22.81%	7.33%	0.71	0.88	6.19%	54.72%	100%	100%

Table 3: *Descriptive statistics (experiments on simulated trajectories, 1000 simulations).*

is referred to Coleman *et al* [20, 19] or Vähämaa [55].

Figure 3 shows the results of the daily delta-hedging of a European vanilla call option with maturity $T = 3/4$ (nine months) on the DAX index. The option hedged was nearest-to-the-money among those that were traded on the index at the beginning of the hedging period on May 2, 2001. The curves labelled `tree/implied` and `tree/local` illustrate respectively the results of the hedge with the implied delta and the local delta. The initial values of the implied and local deltas lie around 60% and 50%. The difference between the two deltas narrows as time goes on, as the option is further and further from the money.

On the upper graph in Figure 3, the curves labelled `current` and `forward`, respectively, represent the trajectories of the index S_t and of the lagged index S_{t-1}. So the relative position of the two curves controls the sign of δS. Provided that the dynamics of the underlyer is locally approximated by a local volatility model, it also controls the relative position of δP&L^{BS} and δP&L^{loc}, by proposition 3.4.*b*. The fact that the relative position of S_t and S_{t-1} controls the relative position of δP&L^{BS} and δP&L^{loc} can be checked directly in Figure 3.

The curves labelled `Gauss/local` illustrate the results of the hedge of the option, using the real trajectory of the index and the daily price of the option in the local volatility model calibrated on the *first day* of the hedging period[6]. The curves labelled `Gauss/implied` illustrate the results of the hedge of the option by using the Black–Scholes implied prices and Greeks extracted from the model prices derived in the `Gauss/local` experiment. So the option prices used in the `Gauss` experiments are not the market prices of the option, but the prices of the option in a fixed local volatility model. The `Gauss` curves allow us to put the theory of §3.3 to the test and to quantify the discretization error, i.e. the part of the P&Ls due to the discreteness of the hedge in the `tree` curves. The residual part of the P&Ls is due to the misspecification of the model.

According to Coleman *et al*, the results of Dumas *et al* arose from the fact that the latter hedged their options over too short a period, namely one week (see §2.1). Yet in this experiment we see that the implied delta outperforms the local delta by far, not only during the first days or weeks of hedging, but also during the whole period. Moreover, this occurs not only in the marked-to-market framework (`tree` experiments), but also in the fixed local volatility model framework (`Gauss` experiments). We have already encountered the fact that such a phenomenon is possible within a fixed local volatility model, both theoretically in §3.3 and numerically in §4.2.

We now consider other experiments where the local delta outperforms the implied delta. Figure 4 illustrates the delta-hedging of a European vanilla call option with maturity $T = 0.556$ and strike $K = 5400$ on the DAX index, starting on August 24, 2001. For clarity, Figure 4 only shows the curves in the marked-to-market framework (curves labelled `tree` in the previous experiment), and no longer those corresponding to a fixed local volatility model (curves previously labelled `Gauss`). The local delta (curves labelled `local` in Figure 4) outperforms the implied delta by far (curves labelled `implied`).

Figure 5 illustrates the delta-hedging of the European vanilla call option with maturity $T = 0.46$ and strike $K = 6025$ on the FTSE index, starting on October 1, 1999 (this option is the one whose implied volatility trajectory was displayed in Figure 1, labelled `FixedStrike` therein). European vanilla option prices on the index were used for calibrating the model every day. The local delta provides a better hedge than the implied delta.

Figure 6 illustrates the delta-hedging of the European vanilla call option with maturity $T = 0.309$ and strike $K = 5950$ on the FTSE index, starting on October 1, 1999. *American* vanilla option prices were used for calibrating the model every day (see Appendices A and B). Once again the local delta outperforms the implied delta.

As we shall see below, the last two experiments do not unfold in jumpy market conditions. So the

---

[6]As explained in Appendix B, these prices are computed by an implicit finite difference scheme which is exactly solved by the Gauss algorithm, hence the label "Gauss" for these curves.
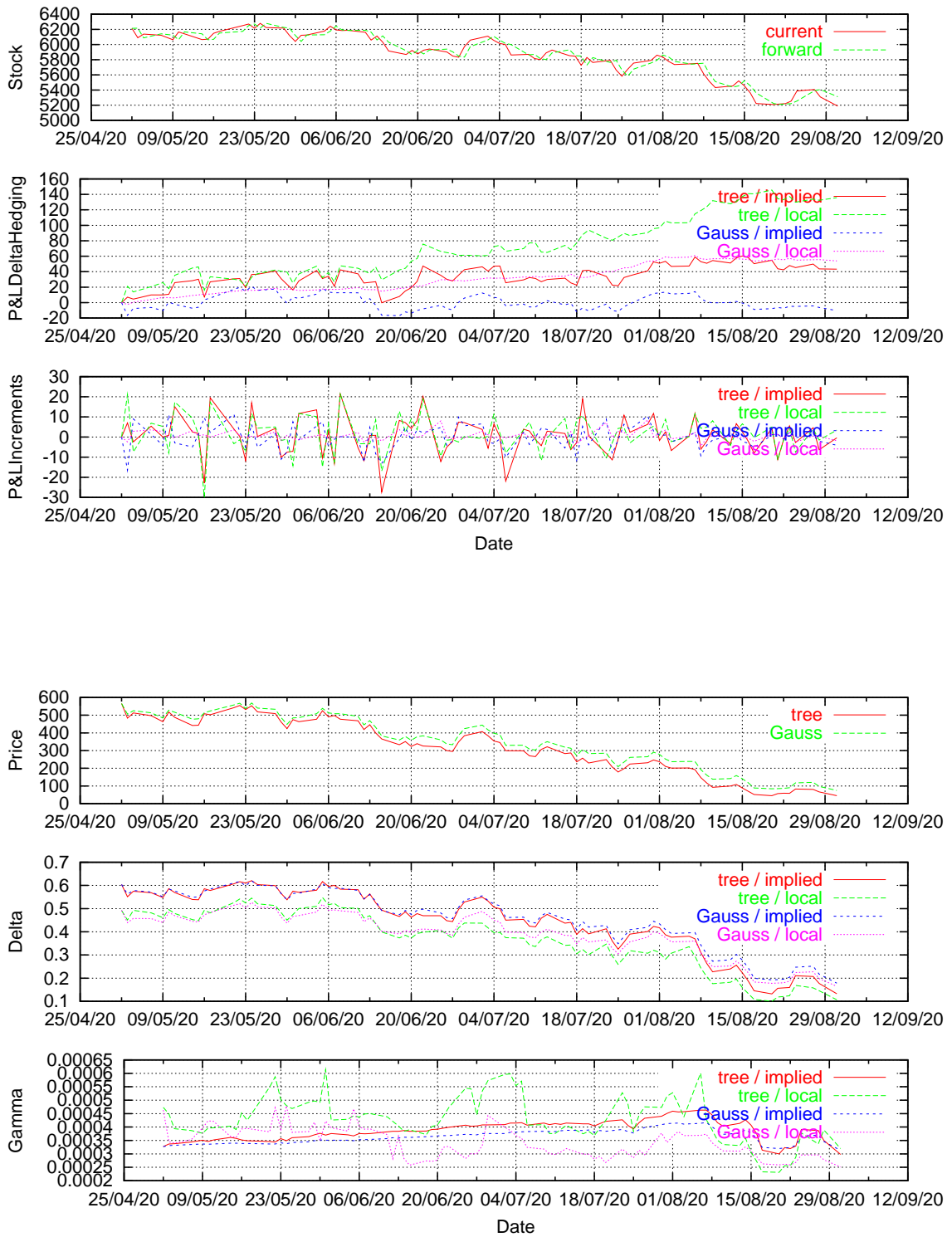
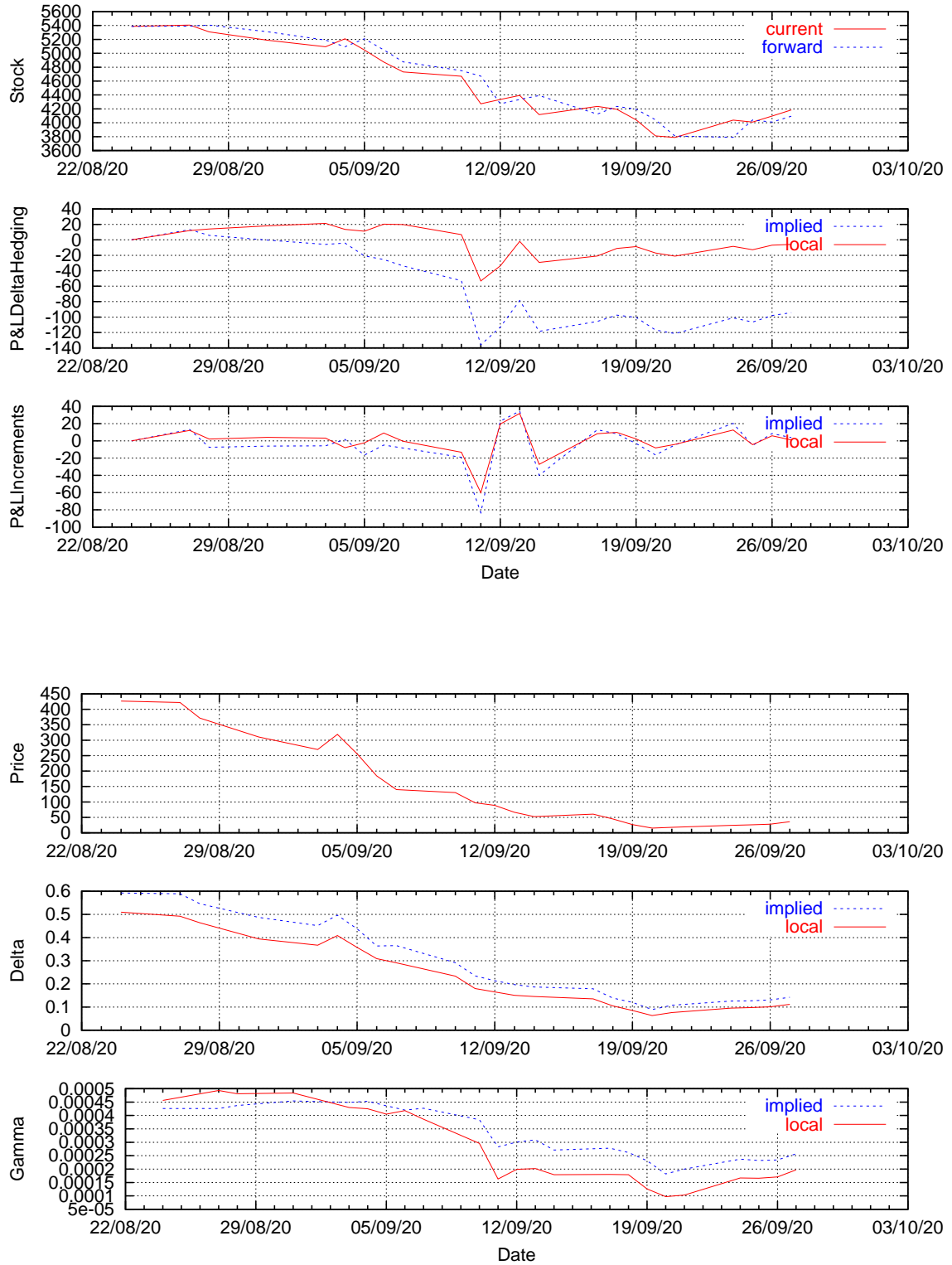Figure 3: *Delta-hedge on the DAX index, 02/05/01–31/08/01. (Top) P&L (Bottom) Greeks.*

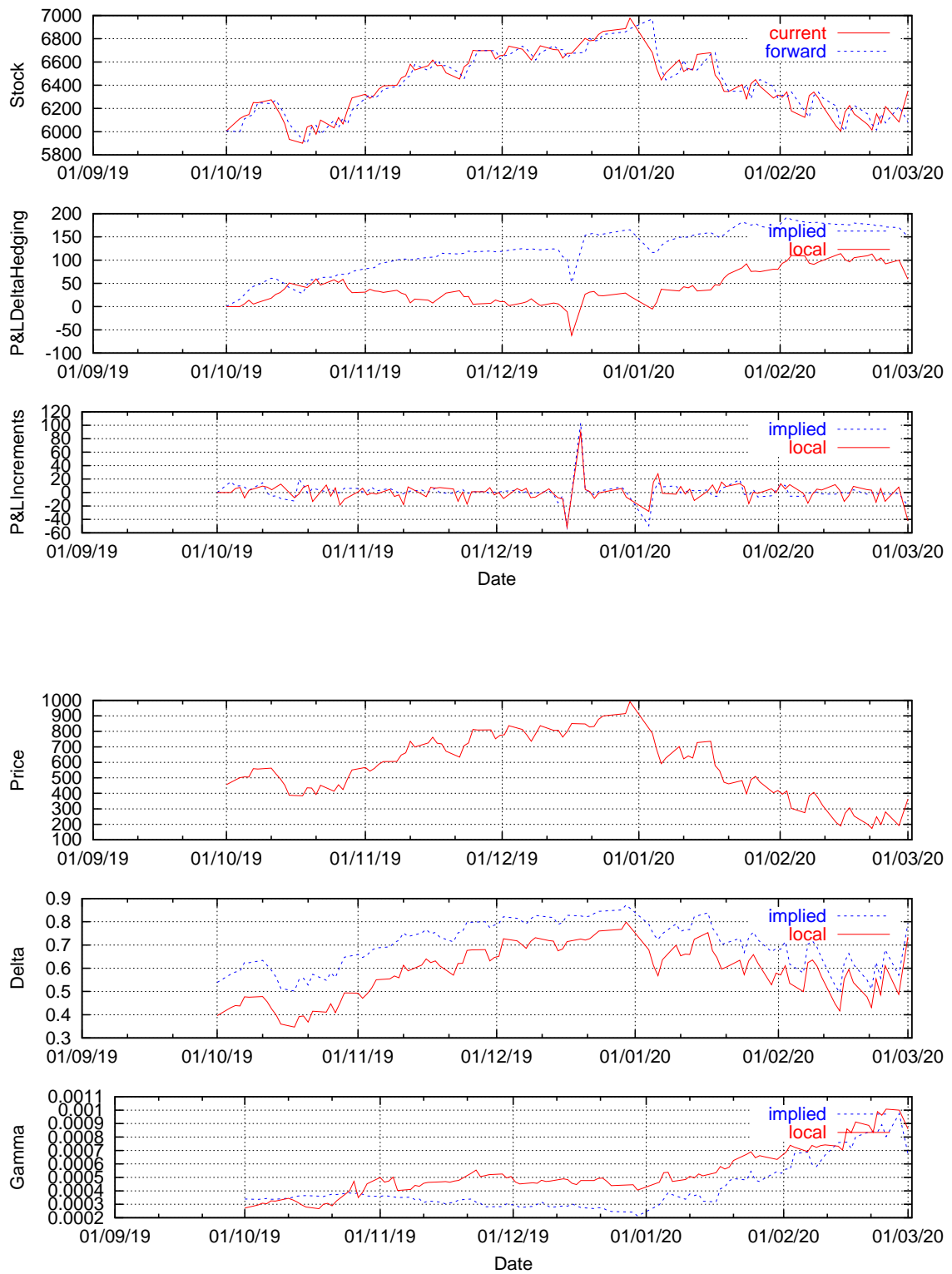Figure 4: *Delta-hedge on the DAX index, 24/08/01–27/09/01. (Top) P&L (Bottom) Greeks.*

Figure 5: *Delta-hedge on the FTSE index using European option prices for the calibration of the model, 01/10/99–01/03/00. (Top) P&L (Bottom) Greeks.*

corresponding results do not tally with the analysis of Derman [26], according to which the implied delta should work as well or better than the local delta, in stable or trending market regimes.

Table 4 sums up descriptive statistics about the previous experiments[7]. The results are divided into three horizontal parts. The first part gives performance measures of the hedges. The second part gives elements of volatility analysis. In the third part we draw conclusions as to the market regime, the better delta of the two and the consistency with the theory of section 3, in each of the considered cases.

In order to identify the market regime in a given sequence of data, we resort to the following backtesting criterion: the market is said to be *fast* or *slow* over a given hedging period according to whether the (annualized daily historical) realized volatility is larger or smaller than the initial implied volatility of the option hedged. Among the previous experiments, only one corresponds to a fast market according to the criterion mentioned above, namely the experiment on the DAX index 24/08/01–27/09/01 (fast regime following the WTC attacks of September 11 2001). As we can see in Table 4, it is the experiment with the largest realized volatility (by far, 53.99%) as well as the highest volatility of implied volatility (by far, 66.33%). If we omit the `Gauss` experiment that occurs in the framework of a fixed local volatility model, it is also the experiment with the largest negative correlation between the returns and the option's implied volatility changes. Yet we think that this is less significant: our experience on more sequences and indices shows that fast markets according to the criterion mentioned above exhibit more realized volatility and more volatility of volatility than slow markets, but not necessarily more negative correlation (results available on request, see also §2.2). Pairing the market regime "fast" or "slow" with the sign of the average return of the index over each hedging period, we finally obtain the market regimes such as they are displayed in Table 4.

As a backtesting criterion to assess which delta provides a better hedge in a given experiment, we resort to the sign of the log-difference between the absolute areas enclosed by the P&L trajectories (see §4.2). The correspondence between the market regimes and the better delta of the two is exactly as it is predicted by the theory of section 3 (see Table 1). Note that during the slow sell-off on the DAX index 02/05/01–31/08/01, which corresponds to one of the a priori indeterminate blue cases in Table 1, the implied delta provides a better hedge, either in the `Gauss` framework of a fixed local volatility model, or in the marked-to-market `tree` framework. This conforms to the discussion in §3.3. We also remark that the absolute log-difference between the local and the implied P&L enclosed absolute areas is essentially of the same order of magnitude in the cases that are favourable to the implied delta as in the cases that are favourable to the local delta. Consistently with the conclusions of proposition 3.7, all our numerical results on market data conform to the statements in proposition 3.5 (point *a*, at least; we dot not have enough data to validate *b*). In Figure 3 in particular, observe how the `tree` trajectories look like the `Gauss` trajectories, but with more momentum (the `tree` trajectories are pushed away further from the time axis). Except for the case of the `Gauss` experiment, also note that (i) the initial implied volatilities of the options are larger than the average implied volatilities over the corresponding hedging periods in the slow markets, whereas (ii) in the fast market the initial implied volatility of the option is smaller than the corresponding average implied volatility. So the implied volatilities of the options tend to increase in fast markets and decrease in slow markets. This is quite in line with the reversion of implied volatilities towards realized volatilities that is postulated in proposition 3.7. By contrast, in the case of the `Gauss` experiment that occurs in the framework of a fixed local volatility model, the implied volatility of the option tends to *rise* over the hedging period, under the single effect of the negative correlation with the returns in the index.

As an alternative hedging performance criterion, we could have resorted to the standard deviation of the δP&Ls. These are also displayed in Table 4. The standard deviations seem to display an overall bias in favour of the local delta. But first, the standard deviation of the δP&Ls is a less straightforward hedging performance measure than the absolute area enclosed by the P&L

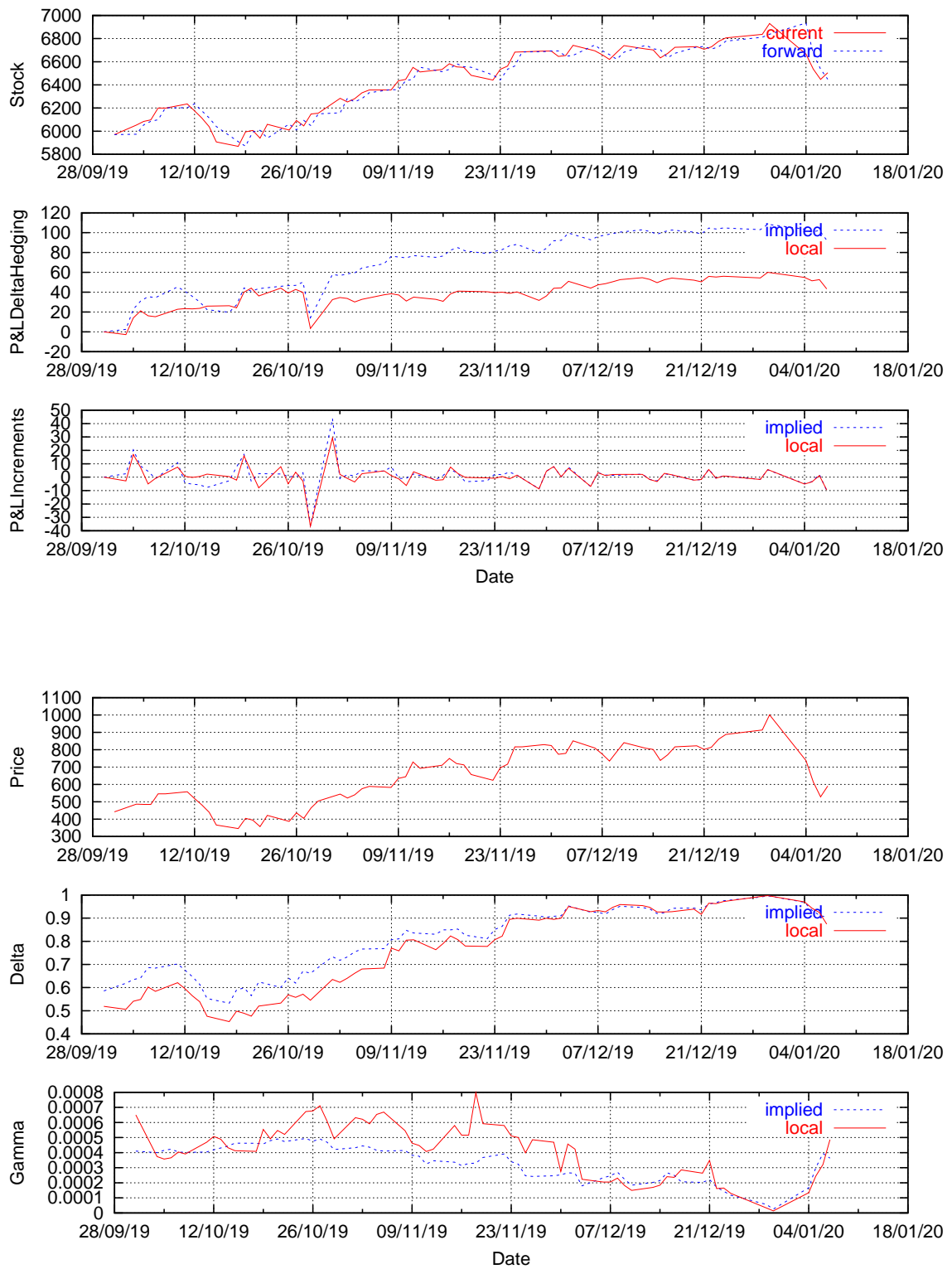[7]The last column in Table 4 illustrates results concerning a barrier option that will be considered in section 5.

Figure 6: *Delta-hedge on the FTSE index using American option prices for the calibration of the model, 01/10/99–07/01/00. (Top) P&L (Bottom) Greeks.*

	DAX 02/05 31/08 Gauss	DAX 02/05 31/08 tree	FTSE 01/10 01/03 Europ.	FTSE 01/10 07/01 Amer.	DAX 24/08 27/09	DAX 24/08 27/09 Barrier
final $|P\&L^{BS}|$ (% initial option premium)	1.82%	7.63%	33.02%	20.89%	22.09%	90.53%
final $|P\&L^{loc}|$ (% initial option premium)	9.52%	23.99%	12.97%	9.80%	1.38%	36.64%
Enclosed abs. area (log-difference local−implied)	1.38	0.73	-1.02	-0.62	-1.44	-0.93
stdev $\delta\mathbf{P\&L}^{BS}$	5.76	9.06	13.67	8.63	23.26	87.89
stdev $\delta\mathbf{P\&L}^{loc}$	2.13	8.68	13.95	7.48	17.46	46.88
relative difference	-63.04%	-4.12%	2.09%	-13.34%	-24.95%	-46.66%
Significance of the difference (F-Test p-values)	100.00%	64.87%	58.29%	87.62%	90.11%	99.77%
Significance of the difference (Shuffling p-values)	100.00%	60.86%	53.66%	65.98%	70.60%	95.92%
Trans. volumes implied	2.51	2.66	4.70	2.83	1.39	2.35
Trans. volumes local	2.27	2.48	5.08	2.93	1.20	2.24
Trans. costs implied (% initial option premium)	4.13%	4.38%	9.28%	5.74%	2.62%	7.45%
Trans. costs local (% initial option premium)	3.75%	4.09%	10.05%	5.94%	2.27%	7.12%
Realized volatility (annualized)	18.96%		22.00%	17.31%	53.99%	
Initial implied volatility	21.82%		28.19%	28.65%	22.67%	
Average implied volatility	23.11%	21.24%	27.44%	26.22%	28.64%	
Volatility of implied volatility	13.82%	10.11%	25.83%	31.21%	66.33%	
Correl implied vol changes / index returns	-88.52%.	-63.52%	-57.23%	-44.43%	-85.11%	
Average daily return	-0.22%		0.07%	0.13%	-1.15%	
Market regime	Slow sell-off		Slow rally	Slow rally	Fast sell-off	
better delta of the two	Implied	Implied	Local	Local	Local	Local
Consistency with the analysis of §3	Yes	Yes	Yes	Yes	Yes	Yes

Table 4: Descriptive statistics (real-life hedging experiments on market data).

trajectories. Secondly, the confidence levels as to the significance of the differences between the implied and the local standard deviations are quite low (except for the `Gauss` case, which corresponds to computations in a local volatility model). In Table 4, these confidence levels are assessed by p-values computed in two ways: either by variance equality F-tests, or by a shuffling procedure with 5000 resamplings, which is more relevant on non-Gaussian data (see the histograms and descriptive statistics in Figure 2 and Table 3).

4.4 Analysis of the transaction costs

The next question we wish to consider is whether the transaction costs caused by the dynamic rebalancing of the positions are biased in favour of either delta. In Table 4, the lines `Trans. volumes` indicate the number of units in the underlyer sold or purchased during each hedging experiment, including the setting up and the closure of the position (since the latter occurs before the maturity of the option, see §3.1). Multiplying these volumes by 15 basis points (bps) × the initial value of the underlyer gives a rough estimate of the transaction costs incurred in each case. These are also given in Table 4, expressed as a percentage of the initial premium of the option. The 15 bps correspond to 10 bps of commission costs for the execution of the orders plus 5 bps of half bid-ask spreads on each sale or purchase (on major equity-index markets the bid-ask spread does not amount to more than 10 bps).

The transaction costs are quite close in either delta, and are one order of magnitude closer than the final absolute P&Ls in either delta (also expressed in Table 4 as a percentage of the initial premia of the options). Moreover these transaction costs would be lower if the position was rebalanced whenever it is required by the gamma of the option, instead of rebalancing it every market day.

5 Barrier options

Up to this point we have only considered the hedging of vanilla options. In practice, people are also interested in more exotic products, like barrier options, that become active (barriers *in*) or inactive (barriers *out*) if the underlyer reaches pre-established levels. For instance, down-and-out puts are used by directional traders in order to get low cost exposure to a market move where the trader has a view on the size of the expected move within a particular time frame.

5.1 Pricing

For dealing with barrier options, a common practice consists in using the Black–Scholes implied volatility of the *vanilla part* of the option, i.e. the structure without the barrier, to compute the corresponding *Black–Scholes price and Greeks* of the barrier option. Thus, Figures 7 and 8 display the prices and Greeks of two barrier options, computed in two alternative ways:

- (Curves labelled `implied`) As the Black–Scholes price and Greeks of the barrier options, where the implied volatility that is used corresponds to the price of the vanilla part of the structure computed with a trinomial tree;

- (Curves labelled `local`) With a trinomial tree truncated at the barrier's level (see Appendix B).

In both cases the trinomial tree that is used has a local volatility calibrated with the DAX option prices of May 2 2001. The barriers are set at level `(1+barriersMoneyness)*underlyer` (barriers up) or `(1-barriersMoneyness)*underlyer` (barriers down). The options are also endowed with a rebate equal to 5% of the initial value of the underlyer. In the case of Figure 8 corresponding to an American put option endowed with a down-and-in barrier, we resorted to a trinomial tree with 200

time steps to compute the implied volatility of the American vanilla part of the structure and the associated prices and Greeks of the barrier option. Notice the substantial discrepancy between the *implied* prices and Greeks of the barrier options and their *local* prices and Greeks, especially in the case of the American option in Figure 8. These results illustrate the model risk inherent to barrier options [41].
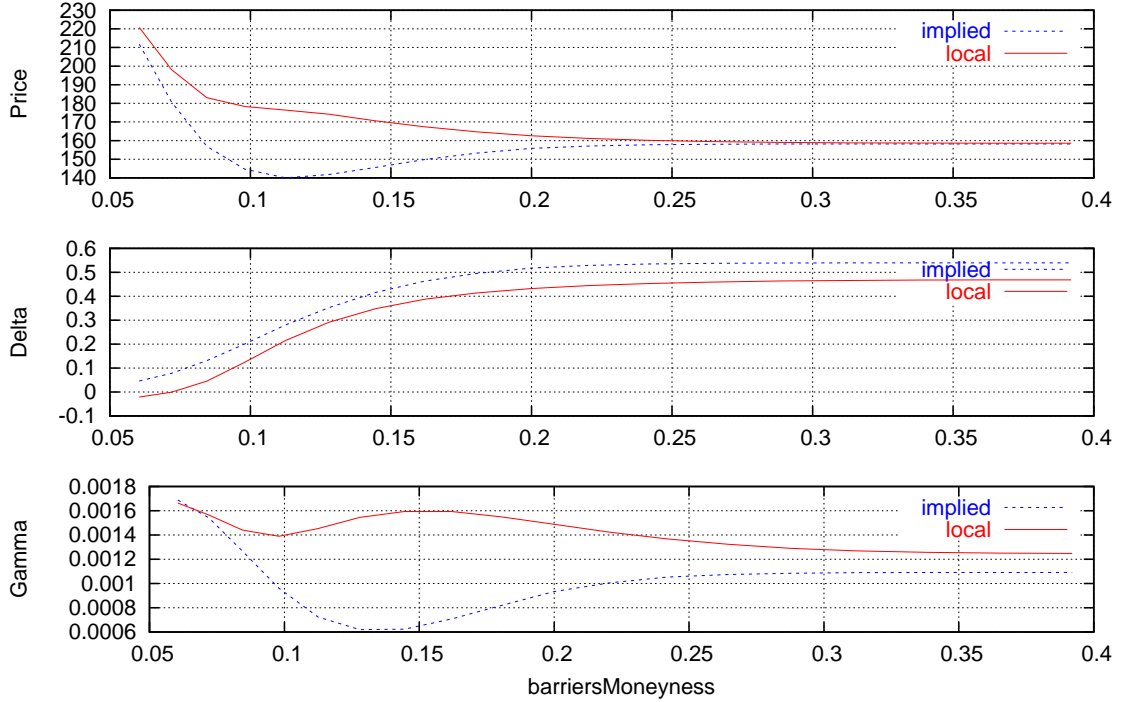

Figure 7: *ATM European double-knock-out call with 5%-rebate and one month time-to-maturity.*

5.2 Hedging

It is customary to make a distinction between *regular* barriers that are triggered when the option is out-of-the-money and *reverse* barriers that are triggered when the option is in-the-money. Regular barrier options are not much harder to hedge than vanilla options. In contrast, reverse barriers may be very dangerous because of a mixed Gamma/Vega exposure, i.e. the fact that the Greeks of the option may change of sign in the neighbourhood of the barrier.

Let us redo the analysis of section 3 in the case where the gamma and the vega of the option are negative. We first operate in the framework of a negatively skewed local volatility model as in §3.3. By applying (6) with a negative vega, we expect the implied delta to be *smaller* than the local delta. Then, by using (2) with a negative gamma, we obtain the results displayed in Table 5. The dominant market regimes (the red cases, provided we have *physical* as well as implied negative skewness) are still favourable to the local delta. Hence the local delta provides a better hedge than the implied delta, on average as well as on average conditionally on the fact that the market is in a fast regime or on average conditionally on the fact that the market is in a slow regime. Now we are going to extend the analysis to the case of real markets as in §3.4. Since the option is vega *negative*, the reversion of implied volatilities towards realized volatilities implies that (i) $\delta\Pi^{loc} \leq \delta\Pi$ at slow market regimes and (ii) $\delta\Pi \leq \delta\Pi^{loc}$ at fast market regimes. Let us assume that the market is negatively skewed. By
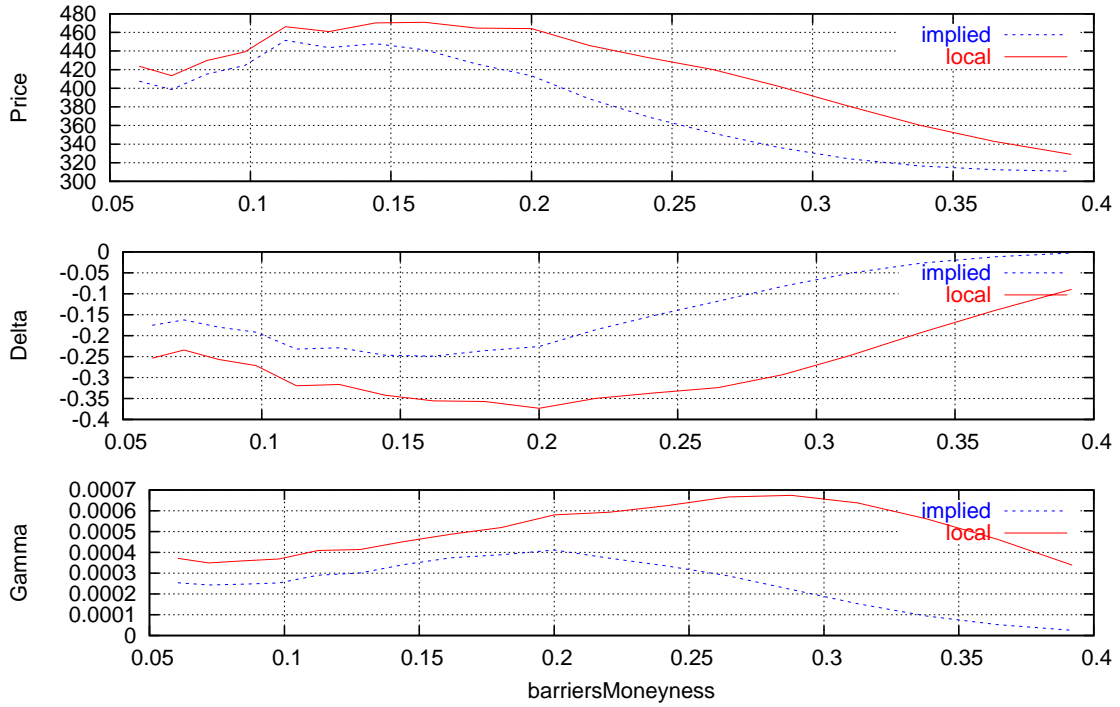
Figure 8: *ATM American down-and-in put with 5%-rebate and half-year time-to-maturity.*

	Slow	**Fast**
Rally	$\delta P\&L^{BS} \le \delta P\&L^{loc} \le 0$	$(\delta P\&L^{BS})^+ \le \delta P\&L^{loc}$
Sell-Off	$\delta P\&L^{loc} \le -(\delta P\&L^{BS})^-$	$0 \le \delta P\&L^{loc} \le \delta P\&L^{BS}$

Table 5: *Negative Gamma/Vega exposure in a negatively skewed local volatility model.*

comparison with the case of a local volatility model, δP&L^{BS} and δP&L^{loc} are pushed away from 0 by the same amount in Table 5. So the situation depicted in Table 5 still holds true in the real market. Moreover, we can draw similar conclusions in the case of a positively skewed market. We conclude that propositions 3.5 to 3.7 are valid and applicable not only to vanilla options, but also to options with a mixed Gamma/Vega exposure.

However, we should have some reservations. First, the previous analysis simplifies the problem by assuming that the option is always either in a positive gamma/vega phase, or in a negative gamma/vega phase. Secondly, this analysis cannot be automatically extended to all situations and to all types of barriers; every new case must be considered separately. Thirdly, the fact that the local delta provides a hedge *better* than the implied delta does not guarantee that it provides a *good* hedge. When one deals with exotic options, hedging the spot exposure is generally not enough. As we explained in §3.1, multi-instrument hedging schemes should be used.

Figure 9 illustrates the delta-hedging of a European put option on the DAX index with maturity $T = 0.556$ and strike $K = 5400$, endowed with a down-and-out reverse barrier at level $H = 3500$ (no rebate). The hedge starts on August 24, 2001. We have already considered the hedge of the corresponding vanilla call option, illustrated in Figure 4. Since we do not know market prices for this barrier option, we use the implied price or the local price of the barrier option (see §5.1) according to whether the implied delta or the local delta is used for the hedge. Notice that in Figure 9 the option gamma is negative in the time frame of the hedge. Accordingly, as expected, the local delta is larger than the implied delta. Descriptive statistics concerning the results are displayed in the last column of Table 4. Since European vanilla call and put options having the same characteristics have the same implied volatility, we have already done the volatility analysis of the vanilla part of the structure when we considered the vanilla call option in §4.3. The market is in a fast sell-off and the P&L trajectories behave as it is predicted by the theory (see cell in the lower right-hand corner of Table 5). However, regarding the absolute hedging performances, you should note in Table 4 that the final absolute P&Ls, as well as the standard deviations of the δP&Ls, either implied or local, are one order of magnitude greater than they were in the vanilla case. This may be partly due to the high *leverage* of barrier options, namely the low cost of barrier options in comparison with vanilla options, but it more generally emphasizes the need to resort to more elaborate multi-instrument hedging schemes for dealing with exotic options.

6   Conclusion

There is analytical as well as empirical support for the view that the local delta of an option, namely its delta in a generalized Black–Scholes model with a local volatility function recalibrated to the market smile every day, should be preferred to the Black–Scholes implied delta for rebalancing an option's delta-hedge in negatively skewed markets, provided that the physical underlying process as well as the risk-neutral process are negatively skewed. This view is supported by numerical tests on equity-index market data and is also in line with the statistical evidence in Coleman *et al* [20, 19] or Vähämaa [55]. The fact that fast markets may exhibit more physical negative skewness than slow markets might be an explanation for the results in Vähämaa [55] according to which the outperformance of the local delta compared to the implied delta is more significant in fast markets. So Derman's intuition may be right [26]. Moreover, we draw the same conclusions in the case of positively skewed markets, and we show that our conclusions are true when transaction costs are taken into account. When barrier options are considered we find that the local delta outperforms the implied delta but there is a need to resort to more elaborate multi-instrument hedging schemes in order to obtain acceptable absolute hedging performances. This last point will be dealt with in the future.
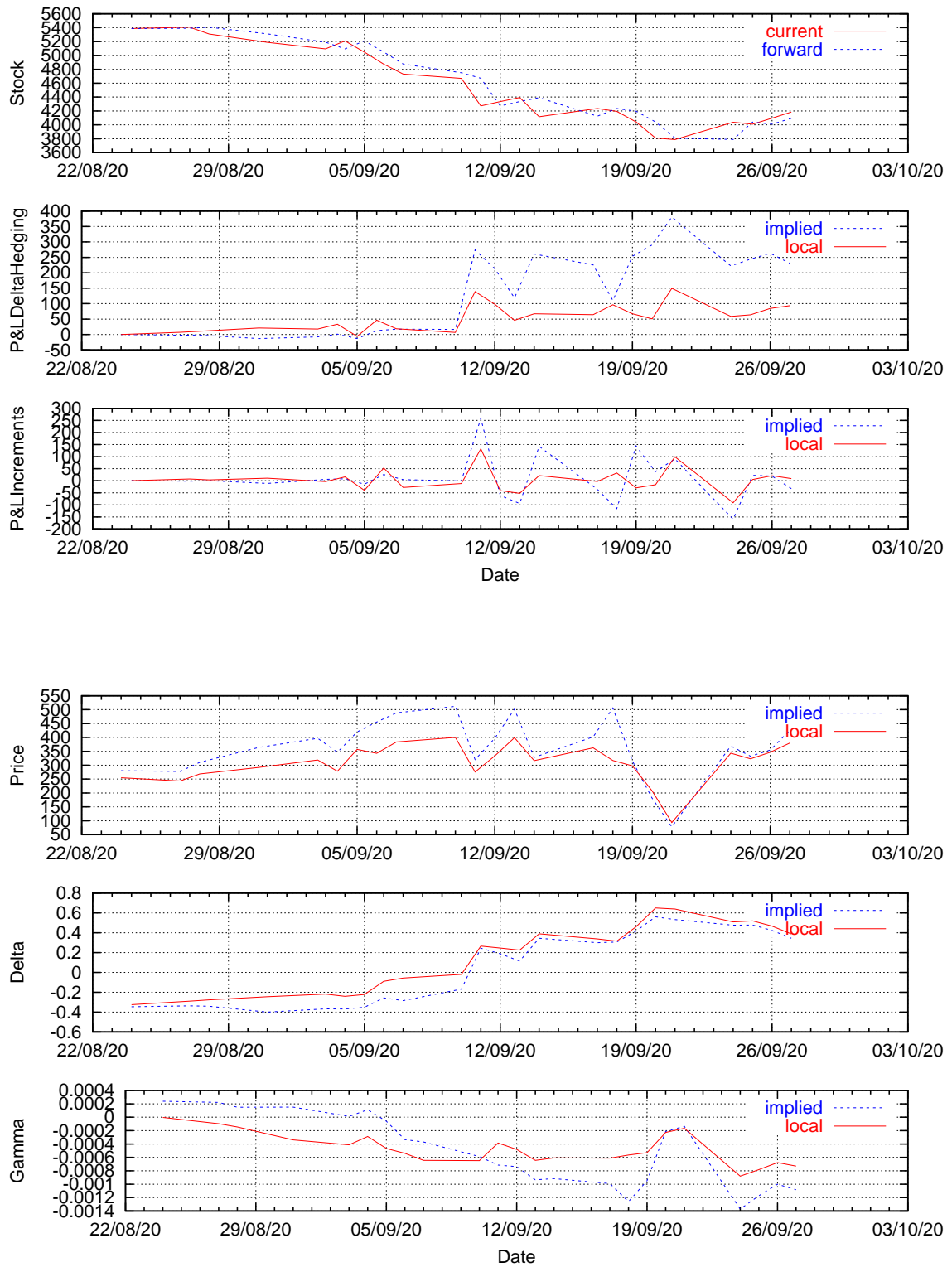
Figure 9: *Delta-hedge of a European down-and-out put option on the DAX index, 24/08/01–27/09/01. (Top) P&L (Bottom) Greeks.*

Acknowledgements

A   The Dupire model

Since the treatment for put options is entirely similar to that for call options, we shall focus on call options in the following paragraphs. Let us assume that the riskless interest rate r in the economy and the dividend yield q on S are both constant. Then, given a European vanilla call option on S with maturity date T and strike K, we have the following *arbitrage bounds* on the market price $\pi_{T,K}$ of the call at date t:

$$(Se^{-q(T-t)} - Ke^{-r(T-t)})^+ \leq \pi_{T,K} \leq Se^{-q(T-t)} \ . \tag{8}$$

The *Dupire* or *generalized Black–Scholes* model [34, 1994] assumes that the spot price of the underlying follows a diffusion of the type

$$dS_t = S_t(\rho(t, S_t)dt + \sigma(t, S_t)dW_t) \ , \quad t > t_0 \ ; \quad S_{t_0} = S_0 \tag{9}$$

where W denotes a standard Brownian motion W under the physical probability. The local volatility, $\sigma \equiv \sigma(t, S)$, is a definite deterministic (though not directly observable) time-space function. In the particular case where the function σ actually is a constant $\sigma \equiv \Sigma$, we obtain the *standard Black–Scholes model* [12, 1973] with volatility parameter Σ. The physical drift, $\rho \equiv \rho(t, S)$, may itself be a function of time and space. We shall see below that this drift does not impact the price of the options in the model. Yet it plays a part in the discussion on hedging performances in §3.3.

Let us suppose that the market is liquid, non arbitrable and perfect. Then one can show that European vanilla call options on S have a theoretical fair price in the Dupire model, which we shall denote by $\Pi_{T,K}(t_0, S_0; \sigma)$, where

$$\Pi_{T,K}(t_0, S_0; \sigma) = e^{-r(T-t_0)}\mathrm{E}_P^{t_0, S_0}(S_T - K)^+ \ . \tag{10}$$

The symbol P represents the so-called *risk-neutral probability*, under which

$$dS_t = S_t((r - q)dt + \sigma(t, S_t)dW_t) \ , \quad t > t_0 \ ; \quad S_{t_0} = S_0 \ . \tag{11}$$

So the fair value of the option in the model does not depend on the physical drift ρ, though ρ may be co-responsible, with the local volatility σ, for fat tails or skewness in the *physical* stock returns. This is because the option's risk is entirely market diversifiable in this model. The Dupire model is *complete*. Alternatively to the probabilistic representation (10) for the option model price Π, this price can be given as the solution to the *Black–Scholes backward parabolic equation* in the variables (t_0, S_0), namely

$$\begin{cases} -\partial_t\Pi - (r - q)S\partial_S\Pi - \dfrac{1}{2}\sigma(t, S)^2 S^2 \partial_{S^2}^2\Pi + r\Pi \ = \ 0, \quad t < T \\ \Pi|_T \ \equiv \ (S - K)^+ \ . \end{cases} \tag{12}$$

Thus (10) can be seen as the Feynman–Kac representation for the solution of (12). The fact that (12) is a well-posed characterization of the option price in a suitable Sobolev space, for any measurable and positively bounded local volatility σ, was established in Crépey [23, theorem 4.3].

In the standard Black–Scholes model where the local volatility σ actually is a constant $\sigma \equiv \Sigma$, the call option price and delta are given by the well-known *Black–Scholes formulas*:

$$\Pi_{T,K}^{BS}(t, S; \Sigma) = Se^{-q(T-t)}\mathcal{N}(d_+) - Ke^{-r(T-t)}\mathcal{N}(d_-) \tag{13}$$

$$\Delta_{T,K}^{BS}(t, S; \Sigma) \equiv \partial_S\Pi_{T,K}^{BS}(t, S; \Sigma) = e^{-q(T-t)}\mathcal{N}(d_+)$$

where $\mathcal{N}$ is the standard Gaussian distribution and

$$d_\pm \equiv \frac{\ln(\frac{S}{K}) + (r - q)(T - t)}{\Sigma\sqrt{T - t}} \pm \frac{1}{2}\Sigma\sqrt{T - t} \ .$$

As Σ grows from 0 to infinity, the Black–Scholes price (13) strictly increases from one to the other arbitrage bound in (8). So we define the *Black–Scholes implied volatility* $\Sigma_{T,K}$ of the call as the unique constant volatility for which the market price $\pi_{T,K}$ of the call equals its price in the associated standard Black–Scholes model. The *implied volatility surface* is the map of the implied volatilities for all $T \geq t$ and $K > 0$, or (T, K) in a suitable discretization.

In the Dupire model, the local volatility, σ, is an unknown function of time and stock. The *calibration problem* is the inverse problem that consists in inferring this function from the market-quoted prices of liquid options, typically European vanilla call and put options with various strikes and maturities. Subsequently, the local volatility function thus calibrated can be used to price *exotic* (non vanilla) options or value hedge-ratios consistently with the market.

Since the local volatility has an infinity of degrees of freedom while the set of input instruments is finite, the calibration problem is under-determined and ill-posed. To overcome this ill-posedness, several stabilizing procedures can be used [34, 29, 3, 45, 24, 11] (see Appendix B).

A variant of the calibration problem, also considered in this article and documented in Appendix B, consists in the calibration of a local volatility with *American* option prices. American options may be exercised at any date by (and not only at) the maturity T. The arbitrage bounds on the American call option with market price $\pi_{T,K}$ are

$$\max_{\tau \in [t,T]} (Se^{-q(T-\tau)} - Ke^{-q(T-\tau)})^+ \leq \pi_{T,K} \leq S \ . \tag{14}$$

Even in the standard Black–Scholes model, there are no closed pricing formulas for American options: only numerical procedures are available. Yet it is known that the standard Black–Scholes price of an American vanilla call option is non-decreasing from one to the other arbitrage bound in (14) as the volatility parameter Σ grows from 0 to infinity. We thus define the *Black–Scholes implied volatility* $\Sigma_{T,K}$ of the American call as the *largest* constant volatility for which the market price $\pi_{T,K}$ of the call equals its price in the associated standard Black–Scholes model, computed thanks to an accurate numerical scheme. The *American calibration problem* consists in inferring a local volatility function from the implied volatility surface of all quoted American vanilla option prices.

B   Pricing and calibration procedures

The solutions of the generalized Black–Scholes equation with local volatility (12) are computed numerically by using trinomial trees (that is, explicit finite difference schemes) or fully implicit finite difference schemes. The latter are solved by the Gauss algorithm, that is an exact method of resolution of tridiagonal linear systems in linear time. In the case of American options, we resort to the so-called splitting algorithm consisting in having each step of backward diffusion (explicit or implicit) followed by the application of a threshold at the level of the option's payoff. The reader is referred to Crépey [24, Appendix] for further details. To deal with barriers we use the same schemes except that (i) the domain of resolution is truncated at the barriers' level and (ii) the scheme is modified at the barriers in order to take into account their exact nature and location. These modifications are standard and described, for instance, in Morton & Mayers [51].

Regarding the calibration methods, Coleman *et al* [19] have already noted, and it has been confirmed by our experience, that the dynamic hedging performances of the calibrated models are quite robust with respect to the reasonable procedure that is used. By "reasonable" procedure, we mean a

procedure suitably stabilized for overcoming the ill-posedness of the calibration problem [34, 29, 3, 45, 24, 11]. In this article, we use either the entropic regularization method of Avellaneda *et al* [3] (an implementation using weighted least squares as described in Avellaneda *et al* [2, section 4] or Martini *et al* [47]), or the variant in Crépey [24] of the Tikhonov variational regularization method of Lagnado & Osher [45], according to whether European or American options are considered as input instruments. The point is that the entropic regularization method is faster when it is available, and sufficiently accurate and stable for our purposes, but it only works with European options as input instruments. Both algorithms calibrate a local volatility function in a trinomial tree setting. The previous references provide a full explanation and numerical benchmarks. A state-of-the-art implementation of these algorithms, developed by Claude MARTINI, Rama CONT, Pierre COHORT, Steven FARCY, José DA FONSECA and Stéphane CRÉPEY, is embedded in the `Calibration Engine` (Artabel SA, http://www.artabel.net).

As a preprocessing stage of all the European calibration procedures, we discard the options with prices outside the arbitrage bounds (8), as well as the less liquid options with moneyness K/S_0 smaller than 0.8 or larger than 1.2. In fact, in the implementation, we do not use constant interest rates and dividend yields, but rather deterministic term structures of interest rates and dividend yields. The term structures of the interest rates are extracted from the relevant zero-coupon curves. Moreover, to match the issue of asynchronous data, we resort to artificial term structures of dividend yields. These are computed in order to minimize the sum of the squares of the differences between the two terms of the theoretical call-put parity relation. These squares are summed up over all the available pairs of call and put options with the same strike and maturity in the input data.

As a preprocessing stage of the American calibration procedures, we discard the options with prices outside the American arbitrage bounds (14), as well as the less liquid options with moneyness smaller than 0.9 or larger than 1.1. Since we lack a suitable call-put parity relation that would allow us to gauge dividends, we do not use any dividends in the models calibrated with American option prices.

References

[1] C. ALEXANDER. *Market Models, A Guide to Financial Data Analysis*, Wiley, 2001.

[2] M. AVELLANEDA, R. BUFF, C. FRIEDMAN, N. GRANDCHAMP, L. KRUK, J. NEWMAN. Weighted Monte Carlo: A new Technique for Calibrating Asset-Pricing Models, *International J. of Theoretical and Applied Finance*, 4:1 (2000), pp. 1–29.

[3] M. AVELLANEDA, C. FRIEDMAN, R. HOLMES AND D. SAMPERI. Calibrating volatility surfaces via relative-entropy minimization, *Applied Math. Finance*, 41 (1997), pp. 37–64.

[4] G. BAKSHI, C. CAO AND Z. CHEN. Empirical Performance of Alternative Option Pricing Models, *Journal of Finance*, Vol. 52, No. 5 (1997), pp. 2003–2049.

[5] G. BAKSHI, C. CAO AND Z. CHEN. Do Call Prices and the Underlying Stock Always Move in the Same Direction?, *Review of Financial Studies*, 13 (Fall) (2000), pp. 549–584.

[6] G. BAKSHI AND N. KAPADIA. Delta Hedged Gains and the Pricing of Volatility Risk, *Review of Financial Studies*, Vol. 16 (2) (2003), pp. 527–566.

[7] G. BAKSHI, N. KAPADIA AND D. MADAN. Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options, *Review of Financial Studies*, 16 1 (2003), pp. 101–143.

[8] D. BATES. Post-'87 Crash Fears in the S&P 500 Futures Option Market, *Journal of Econometrics*, 94:1/2 (2000), pp. 181–238.

[9] Y. BERGMAN, B. GRUNDY AND Z. WIENER. General properties of option prices, *Journal of Finance*, Vol. 51, No. 5 (1996), pp. 1573–1610.

[10] D. BERTSIMAS, L. KOGAN, AND A. LO. When is Time Continuous, *Journal of Financial Economics*, 55 (2000), pp. 173–204.

[11] H. BERESTYCKI, J. BUSCA AND I. FLORENT. Asymptotics and calibration of local volatility models. *Quantitative Finance*, 2 (2002), pp. 61–69.

[12] F. BLACK AND M. SCHOLES. The pricing of options and corporate liabilities, *J. Pol. Econ.*, 81 (1973), pp. 637–659.

[13] J.-P. BOUCHAUD, G. IORI, D. SORNETTE. Real-world options: smile and residual risk, *Risk Magazine*, 9 (3) (1996), pp. 61–65.

[14] J.-P. BOUCHAUD, D. SORNETTE. The Black–Scholes option pricing problem in mathematical finance: generalization and extensions for a large class of stochastic processes, *Journal de Physique I France*, 4 (1994), pp. 863–861.

[15] A. BRACE, D. GATAREK AND M. MUSIELA. The market model of interest rate dynamics, *Mathematical Finance*, 7(2) (1997), pp. 127–154.

[16] A. BRACE, B. GOLDYS, J. VAN DER HOEK, AND R. WOMERSLEY. Market models in the stochastic implied volatility framework, *Working Paper* S02-11, Department of Statistics, University of New South Wales (2002).

[17] A. BRACE, B. GOLDYS, F. KLEBANER, AND R. WOMERSLEY. Market model of stochastic implied volatility with application to the BGM model, *Working Paper* S01-1, Department of Statistics, University of New South Wales (2001).

[18] P. CARR, K. ELLIS AND V. GUPTA. Static Hedging of Exotic Options, *Journal of Finance*, 53(3) (1987) pp. 1165–91.

[19] T.F. COLEMAN, Y. KIM, Y. LI AND A. VERMA. Dynamic Hedging With a Deterministic Local Volatility Function Model, *CTC Manhattan Finance Publications* (2000).

[20] T. COLEMAN, Y. LI AND A. VERMA. Reconstructing the unknown volatility function. *Journal of Computational Finance*, 2 (1999), 3, pp. 77–102.

[21] R. CONT, J. DA FONSECA AND V. DURRLEMAN. Stochastic models of implied volatility surfaces. *Economic Notes*, 31 2 (2002).

[22] J. COX, J. INGERSOLL AND S. ROSS. A Theory of the Term Structure of Interest Rates. *Econometrica*, 53 (2) (1985), pp. 385–407.

[23] S. CRÉPEY. Calibration of the local volatility in a generalized Black–Scholes model using Tikhonov regularization. *SIAM Journal on Mathematical Analysis*, Vol. 34 No 5 (2003), pp. 1183–1206.

[24] S. CRÉPEY. Calibration of the local volatility in a trinomial tree using Tikhonov regularization. *Inverse Problems*, 19 (2003), pp. 91–127.

[25] K. DEMETERFI, E. DERMAN, M. KAMAL AND J. ZOU. More than you ever wanted to know about volatility swaps, *Quantitative Strategies Research Notes*, March 1999. pp. 78–95.

[26] E. DERMAN. Regimes of Volatility. *Risk magazine*, April 1999.

[27] E. DERMAN, D. ERGENER AND I. KANI. Static Options Replication, *Journal of Derivatives*, 2 (1995), pp. 78–95.

[28] E. DERMAN, M. KAMAL AND I. KANI. Trading and Hedging Local Volatility. *Goldman Sachs Selected Quantitative Strategies Reports*, October 1996.

[29] E. DERMAN AND I. KANI. Riding on a smile, *Risk*, 7 (1994), pp. 32–39.

[30] E. DERMAN AND I. KANI. Stochastic Implied Trees: Arbitrage Pricing with Stochastic Term and Strike Structure of Volatility. *International Journal of Theoretical and Applied Finance*,1 (1998), 61–110.

[31] E. DERMAN, I. KANI AND J. ZOU. The Local Volatility Surface — Unlocking the Information in Index Option Prices. *Goldman Sachs Selected Quantitative Strategies Reports*, December 1995.

[32] B. DUMAS, J. FLEMING AND R. WHALEY. Implied volatility functions: empirical tests, *J. of Finance*, 53 (1998), 6, pp. 2059–2106.

[33] D. DUFFIE, J. PAN AND K. SINGLETON. Transform Analysis and Asset Pricing for Affine Jump-Diffusions, *Econometrica*, 68 (2000), 6, pp. 1343–1376.

[34] B. DUPIRE. Pricing with a smile, *Risk*, 7 (1994), pp. 18–20.

[35] D. DUPONT. Hedging barrier options: current methods and alternatives, *Working Paper* (2001).

[36] N. EL KAROUI, M. JEANBLANC AND S. SHREVE. Robustness of the Black and Scholes Formula, *Mathematical finance*, 8 (1998), pp. 93–126.

[37] J.-P. FOUQUE, G. PAPANICOLAOU AND K. SIRCAR. *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, 2000.

[38] E. GOBET AND E. TEMAM. Discrete time hedging errors for options with irregular payoffs, *Finance and Stochastics*, 5(3) (2001), pp. 357-367.

[39] D. HEATH, R. JARROW, A. MORTON. Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation, *Econometrica*, 60(1) (1992), pp. 77–105.

[40] S. HESTON. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies*, 6(2) (1993), pp. 327–43.

[41] J. HULL AND W. SUO. A Methodology for Assessing Model risk and its Application to the Implied Volatility Function Model. *Journal of Financial and Quantitative Analysis*, Vol. 37 No. 2 (2002), pp.297–318.

[42] J. HULL AND A. WHITE. The Pricing of Options on Assets with Stochastic Volatilities, *Journal of Finance*, 52 (1987), pp. 281-300.

[43] J. JACKWERTH AND M. RUBINSTEIN. Recovering Probabilities and Risk Aversion from Option Prices and Realized Returns, in: *Essays in Honor of Fisher Black*, editor: Bruce Lehmann, Oxford University Press, Oxford, forthcoming.

[44] M. KAMAL. When You Cannot Hedge Continuously: The Corrections to Black–Scholes. *Goldman Sachs Equity Derivatives Research* (1998).

[45] R. LAGNADO AND S. OSHER. A Technique for Calibrating Derivative Security Pricing Models: Numerical Solution of an Inverse Problem, *Journal of Computational Finance*, 1 (1997), pp. 13–25.

[46] R. LEE. Implied and Local Volatilities under Stochastic Volatility, *International Journal of Theoretical and Applied Finance*, Vol 4 No 1 (2001), pp. 45–89.

[47] C. MARTINI, P. COHORT AND S. FARCY. Minimum relative-entropy calibration: theory, algorithms and numerical experiments, *Mathematical Finance Satellite meeting to AMAM 03*, February 14–15, 2003, Nice, France.

[48] C. MARTINI AND C. PATRY. Variance Optimal Hedging in the Black–Scholes Model for a given Number of Transactions, *RR INRIA 3767*, September 1999.

[49] R.C. MERTON. The theory of rational option pricing, *Bell J. Econ. Man. Sc.*, 4 (1973), pp. 141–183.

[50] K. MILTERSEN, K. SANDMANN AND D. SONDERMANN. Closed-Form Solutions for Term Structure Derivatives with Log- Normal Interest Rates, *J. of Finance*, 52(1) (1997), pp. 409–430.

[51] K. MORTON AND D. MAYERS. *Numerical Solution of Partial Differential Equations* (1994), Cambridge university press.

[52] P. SCHÖNBUCHER. A market model for stochastic implied volatility, *Phil. Trans. R. Soc.*, A 357 (1999), pp. 2071-92.

[53] R. SIRCAR. Hedging under Stochastic Volatility, *Quantitative Analysis in Financial Markets*, Vol. 2 (2000), M. Avellaneda ed., World Scientific.

[54] N. TALEB. *Dynamic Hedging: Managing Vanilla and Exotic Options* (1997), Wiley.

[55] S. VÄHÄMAA. Delta hedging with the smile, *Proceedings of the 2003 Multinational Finance Society Conference* (2003).

[56] O. VASICEK. An equilibrium characterization of the term structure, *Journal of Financial Economics*, 5 (1977), pp. 177–188.