Pricing Convertible Bonds with Call Protection

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Abstract

In this paper we deal with the issue of pricing numerically by simulation convertible bonds. A convertible bond can be seen as a coupon-paying and callable American option. Moreover call times are typically subject to constraints, called call protections, preventing the issuer from calling the bond at certain sub-periods of time. The nature of the call protection may be very path-dependent, like a path dependence based on a 'large' number d of Boolean random variables, leading to high-dimensional pricing problems. Deterministic pricing schemes are then ruled out by the curse of dimensionality, and simulation methods appear to be the only viable alternative.

We consider in this paper various possible clauses of call protection. We propose in each case a reference, but heavy, if practical, deterministic pricing scheme, as well as a more efficient (as soon as d exceeds a few units) and practical Monte Carlo simulation/regression pricing scheme.

In each case we derive the pricing equation, study the convergence of the Monte Carlo simulation/regression scheme and illustrate our results by reports on numerical experiments. One thus gets a practical and mathematically justified approach to the problem of pricing by simulation convertible bonds with highly path-dependent call protection.

More generally, this paper is an illustration of the real abilities of simulation/regression numerical schemes for high to very high-dimensional pricing problems, like systems of $2^{d=30}$ scalar coupled partial differential equations that arise in the context of the application at hand in this paper.

1 Introduction

We consider the issue of pricing numerically by simulation convertible bonds on an underlying stock S. A convertible bond pays coupons from time 0 onwards, until a *terminal*

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payoff

$$\mathbf{1}_{\zeta=\tau<T}\ell(\tau,S_\tau)+\mathbf{1}_{\theta<\tau}h(\theta,S_\theta)+\mathbf{1}_{\zeta=T}g(S_T) \tag{1}$$

occurs at the minimum $\zeta=\tau\wedge\theta$ of two $[0,T]$-valued stopping times θ and τ. Here the *put time* τ and the *call time* θ are $[0,T]$-valued stopping times under the control of the holder and the issuer of the bond, respectively, and:

- $g(S_T)$ corresponds to a *terminal payoff* that is paid by the issuer to the holder at time T if the contract was not exercised before the maturity time T;
- $\ell(\tau,S_\tau)$, respectively $h(\theta,S_\theta)$, corresponds to an *early put payoff*, respectively *early call payoff*, that is paid by the issuer to the holder of the claim in the event of early termination of the contract at the initiative of the holder, respectively issuer.

Note that *early put payoff* and *early call payoff* are in practice to be understood as *put or conversion payoff* and *call or conversion payoff*, against a cash amount of money or a number[1] of stocks S, since in any of the events of an early 'put' or 'call', the holder has in fact the choice between being granted a pre-specified cash amount of money, or enforcing conversion of the convertible bond against a number of stocks S, the latter possibility explaining the name of the product.
Convertible bonds are thus products with early exercise clauses both by the holder (put clauses, like with American options) and the issuer (call clauses) of the claim, and represent as such the main practical example of a *game option* [22].

Moreover, call times θ are typically subject to constraints, called *call protections*, preventing the issuer from calling the bond on certain random time intervals. In addition, the nature of the call protection may be very path-dependent. For instance, given a trigger level $\bar{S}$ and non-negative integers $l\leq d$, the 'l out of the last d' call protection clause means that call is allowed at a given time t if and only if S has been $\geq\bar{S}$ at $\geq l$ among the last d daily closing prices preceding the current time t. This clause, which can be found in actual convertible bond contracts with d of the order of thirty (for thirty days, that is one month), leads, after extension of the state space to ensure Markovianity of the set-up, to systems of 2^d coupled pricing partial differential equations, corresponding to a path dependence based on the d Boolean parameters '$S\geq\bar{S}$ or $<\bar{S}$' at every of the last d call protection monitoring times. For d greater than, say, ten, deterministic pricing schemes are ruled out by the curse of dimensionality (see Table 10 in section 6.3.1), and simulation methods appear to be the only viable alternative.

From the mathematical point of view, the study of game options with call protection leads to doubly reflected backward stochastic differential equations with an upper barrier which is only active on random time intervals. These equations will be called henceforth 'doubly reflected BSDEs with an *intermittent* upper barrier', or RIBSDE for short , where the 'I' in RIBSDE stands for 'intermittent'. Such RIBSDEs and, in the Markovian case, the related variational inequalities (VI for short henceforth), were first introduced in Crépey [16]. In [14], we subsequently established a convergence rate for a discrete time approximation scheme by simulation to an RIBSDE. The purpose of this paper is to assess the practical value of this approach, on the benchmark problem of pricing by simulation convertible bonds with call protection.

[1] Taken equal to one for notational simplicity in this paper.

1.1 Outline of the paper

Section 2 lays out the set-up. Section 3 presents the convertible bond pricing equations and their approximation. We then review in Section 4 basic results and methodologies in the simplest cases of no call (American options) or no call protection (game options). Section 5 deals with stylized cases of 'continuously monitored' call protection. More realistic forms of discretely monitored call protection are handled in Section 6. Conclusions are drawn in Section 7.

2 Set-Up

2.1 Primary Market Model

Given a finite horizon $T > 0$, the evolution of a financial market model is modeled throughout in terms of stochastic processes defined on a continuous time stochastic basis $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$, where $\mathbb{Q}$ denotes a risk-neutral pricing measure. We denote by $\mathcal{T}_t$ (or simply $\mathcal{T}$, in case $t = 0$) the set of $[t, T]$-valued stopping times, and by $\mathbb{E}_t$ (or simply $\mathbb{E}$, in case $t = 0$) the $\mathbb{Q}$ − conditional expectation given $\mathcal{F}_t$ operator (which in case $t = 0$ reduces to $\mathbb{Q}$-expectation).

Specifically, in this paper, we shall work all along with the following *local drift and volatility model* of an $\mathbb{R}_+$-valued underlying stock-process S :

$$dS_t = S_t \left(b(t, S_t)dt + \sigma(t, S_t)dW_t\right) \ , \ S_0 = x \ , \tag{2}$$

where:
• W is a univariate $\mathbb{Q}$ − Brownian motion,
• $b(t, S)$ is a local drift coefficient, to be interpreted as a *risk-neutral drift coefficient*, possibly accounting for riskless interest-rate, dividend yields on S, and/or credit-risk adjustment on S, and
• $\sigma(t, S)$ is a local volatility function.
Equivalently, S follows a one-dimensional diffusion with generator of (t, S) given by, denoting $\partial_t u = \frac{\partial u}{\partial t}$, $\partial u = \frac{\partial u}{\partial S}$, $\partial^2 u = \frac{\partial^2 u}{\partial S^2}$ for every function $u = u(t, S)$:

$$\mathcal{G}u \equiv \partial_t u + bS\partial u + \frac{1}{2}\sigma^2 S^2 \partial^2 u \ . \tag{3}$$

We postulate the standard Lipschitz and growth assumptions on the coefficients ensuring that the SDE (2) admits a unique positive (strong, say) solution S, where T stands for the maturity of a generic contingent claim on S.

More precisely, we shall work with a credit-risk adjusted risk-neutral drift coefficient b of the form

$$b(t, S) = r(t) - q(t) + \eta\gamma(t, S) \ , \tag{4}$$

where:
• the *riskless short interest rate* $r(t)$, the *equity dividend yield* $q(t)$ on S, and the *local default intensity* $\gamma(t, S) \geq 0$ of the firm issuing the bond are bounded, Borel-measurable functions, and
• $\eta \leq 1$ is a real constant, to be interpreted as the *fractional loss* on S in case of a default

of the firm issuing the bond.
We refer the reader to [5] for every details and background, and for the credit-risk interpretation of this parameterization. To be even more specific, we shall let

$$\gamma(t,S) = \gamma_0(S_0/S)^\alpha\ ,\ \sigma(t,S) = \sigma\ , \tag{5}$$

for non-negative default intensity parameters γ_0 and α, and for a constant volatility parameter σ.

By a suitable choice of the model parameters, this simple equity-to-credit framework allows one to account for rather different situations. For instance:
• The 'total default' case with $\alpha > 0$ and $\eta = 1$ can be used in the situation where S represents the value of the equity of the reference entity (firm issuing the convertible bond), so S jumps to zero in case of default of the reference entity, assuming no recovery upon the stock of the reference entity upon default;
• The 'partial default' case with $\alpha = 0$ and $\eta = 0$ can be used in the situation where S represents the value of the equity of a firm different from that issuing the bond (case of the so-called *exchangeable bonds*).

We finally denote by $\beta_t = e^{-\int_0^t \mu(s,S_s)ds}$ the risk-neutral *credit-risk adjusted discount factor*, where $\mu(t,S) = r(t) + \gamma(t,S)$ is the *credit-risk adjusted interest rate.* Let also $B(t,S) = b(t,S)S$, $\Sigma(t,S) = \sigma(t,S)S$. Thus

$$\begin{aligned} &\partial\mu(t,S) = \partial\gamma(t,S) = -\alpha\gamma_0 S_0^\alpha/S^{1+\alpha} \\ &B(t,S) = (r(t) - q(t))S + \eta\gamma_0 S_0^\alpha S^{1-\alpha}\ ,\ \Sigma(t,S) = \sigma S \\ &\partial B(t,S) = r(t) - q(t) + (1-\alpha)\eta\gamma_0(S_0/S)^\alpha\ ,\ \partial\Sigma(t,S) = \sigma\ . \end{aligned} \tag{6}$$

2.1.1 First-Variation Process

We assume that B and Σ are of class $\mathcal{C}_b^1$ with Lipschitz first derivatives. It is then well-known that the so-called *first variation* or *flow* process ∇ of S, formally given by, writing explicitly the initial condition x of S as a superscript in this equation (cf. (2)),

$$\nabla_t = \lim_{\varepsilon\to 0+} (2\varepsilon)^{-1}(S_t^{x+\varepsilon} - S_t^{x-\varepsilon})\ , \tag{7}$$

is a well-defined process, which can be characterized as the unique solution to the following SDE:

$$d\nabla_t = \nabla_t\Big(\partial B(t,S_t)dt + \partial\Sigma(t,S_t)dW_t\Big)\ ,\ \ \nabla_0 = 1. \tag{8}$$

2.1.2 Time-Discretization

For numerical purposes, one needs to approximate (2) and (8) on a discrete time-grid $\mathfrak{t} = \{0 = t_0 < t_1 < \ldots t_i < \ldots < t_n = T\}$. In this context we will often find convenient to denote the time by i rather than t_i, so:
• X_i and $u_i(X_i)$ will be used as short-hands for X_{t_i} and $u(t_i, X_{t_i})$, given a function $u = u(t,x)$ and a continuous-time process X,
• $\mathbb{E}_i$ will refer to the conditional expectation with respect to the discrete information flow

$\mathcal{F}_i$ until time i, and
• $\mathcal{T}_i$ will stand for the set of stopping times ν taking their values in $\{i, \ldots, n\}$.

We shall thus consider the Euler schemes for S and ∇ defined by $S_0 = x$, $\nabla_0 = 1$, and for every $i = 0, \ldots, n-1$:

$$\begin{aligned} S_{i+1} &= S_i\Big(1 + b_i(S_i)(t_{i+1} - t_i) + \sigma_i(S_i)(W_{t_{i+1}} - W_{t_i})\Big) \\ \nabla_{i+1} &= \nabla_i\Big(1 + \partial B_i(S_i)(t_{i+1} - t_i) + \partial\Sigma_i(S_i)(W_{t_{i+1}} - W_{t_i})\Big) \; . \end{aligned} \tag{9}$$

2.2 Convertible Bond

Given a non-decreasing sequence of stopping times ϑ, which will represent times of *switching of call protection* in the financial interpretation, let $\mathcal{T}_t^{\vartheta}$ (or simply $\mathcal{T}^{\vartheta}$, in case $t = 0$) denote the set of all the $\cup_{l>0}[\vartheta_{2l-1}, \vartheta_{2l}) \cup \{T\}$ – valued stopping times. We consider a convertible bond continuously paying coupons $c(t, S_t)dt$ from time 0 onwards, until a terminal payoff (1) is paid at time $\zeta = \tau \wedge \theta$, where $(\tau, \theta) \in \mathcal{T} \times \mathcal{T}^{\vartheta}$, with, specifically,

$$\ell(t, S_t) = \ell(S_t) = \bar{P} \vee S_t \, , \; h(t, S_t) = h(S_t) = \bar{C} \vee S_t \, , \; g(S_T) = \xi = \bar{N} \vee S_T \tag{10}$$

for non-negative constants $\bar{P} \leq \bar{N} \leq \bar{C}$. The fact that θ has to be chosen by the bond's issuer in the constrained set $\mathcal{T}_t^{\vartheta}$ means that one effectively deals with an intermittent call protection, forbidding issuer calls on the 'even' time intervals $[\vartheta_{2l}, \vartheta_{2l+1})$ (see [16, 14]). In view of this restriction on call times, in addition to the nominal call payoff process $h(S_t)$, one introduces the *effective call payoff process* accounting for the call protection, defined by, for $t \in [0, T]$,

$$U_t = \sum_{l \geq 0} \mathbf{1}_{[\vartheta_{2l}, \vartheta_{2l+1})} \infty + \sum_{l>0} \mathbf{1}_{[\vartheta_{2l-1}, \vartheta_{2l})} h(S_t) \; , \tag{11}$$

and one sets likewise $L_t = \ell(S_t)$.

Accounting for credit risk and recovery on the bond upon default (see [5]), one assumes the following form of the coupon rate function c:

$$c(t, S) = \bar{c}(t) + \gamma(t, S)\left((1 - \eta)S \vee \bar{R}\right) \; , \tag{12}$$

where $\bar{c}$ denotes a *nominal coupon rate* function, and $\bar{R}$ stands for a *nominal recovery on the bond upon default.* We also denote $f(t, S, y) = c(t, S) - \mu(t, S)y$.

So

$$\begin{aligned} &\partial\ell(S) = \mathbf{1}_{S \geq \bar{P}} \, , \; \partial h(S) = \mathbf{1}_{S \geq \bar{C}} \, , \; \partial g(S) = \mathbf{1}_{S \geq \bar{N}} \\ &\partial c(t, S) = \partial\gamma(t, S)\left((1 - \eta)S \vee \bar{R}\right) + (1 - \eta)\gamma(t, S)\mathbf{1}_{(1-\eta)S \geq \bar{R}} \; . \end{aligned} \tag{13}$$

Remark 2.1 In practice coupons are discrete rather than continuously paid, which results in a discrete stream of nominal coupons instead of the continuously paid nominal coupon rate $\bar{c}$ in (12). In the theoretical description of the model and algorithms in this paper we consider continuously paid coupons for simplicity of presentation. However, following the guidelines of [3, 5, 16], the results and methods of this paper can be readily extended to the case of discrete coupons. In the numerical experiments we actually work with discrete coupons, using the methodology of [3, 5, 16] in this regard.

3 Pricing Equations and Their Approximation

3.1 Stochastic Pricing Equation

By application of [16, 14], our convertible bond pricing reflected BSDE $(\mathcal{E})$ writes, denoting $c_t = c(t, S_t)$, $\mu_t = \mu(t, S_t)$, and using the convention that $0 \times \infty = 0$ in the last line:

$$\Pi_T = \xi, \text{ and for } t \in [0, T),$$
$$\begin{cases} -d\Pi_t = (c_t - \mu_t \Pi_t)dt + dA_t - \Delta_t \sigma(t, S_t) S_t dW_t \\ L_t \le \Pi_t \le U_t\,,\ (\Pi_t - L_t)dA_t^+ = (U_{t-} - \Pi_{t-})dA_t^- = 0 \end{cases}$$

to be solved in (Π, Δ, A), where in particular A stands for a finite variation process with square integrable *Jordan components*[2] $A^{\pm}$, continuous but for possible jumps of A^- at the 'even' times ϑ_{2l}s in $(0, T)$. Allowing for discontinuities of A^- at the ϑ_{2l}s is indeed necessary to ensure existence of a solution to $(\mathcal{E})$ satisfying the constraint $\ell(S_t) \le \Pi_t \le h(S_t)$ on the 'odd' intervals $[\vartheta_{2l-1}, \vartheta_{2l})$, whereas on the 'even' intervals $[\vartheta_{2l}, \vartheta_{2l+1})$, only the lower constraint $\ell(S_t) \le \Pi_t$ is in force. Uniqueness of a solution holds under the minimality conditions $(\Pi_t - L_t)dA_t^+ = (U_{t-} - \Pi_{t-})dA_t^- = 0$ (see [16]).

The following (rather standard, see [4] or [16] for a proof) *verification principle* establishes the connection between a solution (Π, Δ, A), assumed to exist, to $(\mathcal{E})$, and the problem of pricing the convertible bond under the risk-neutral measure $\mathbb{Q}$.

Proposition 3.1 **(i)** Π *is the conditional* $\mathbb{Q}$*-value process of the Dynkin game with cost criterion* $\mathbb{E}_t \pi^t(\tau, \theta)$ *on* $\mathcal{T}_t \times \mathcal{T}_t^{\vartheta}$*, where* $\pi^t(\tau, \theta)$ *is the* $\mathcal{F}_{\zeta = \tau \wedge \theta}$*-measurable random variable defined by*

$$\beta_t \pi^t(\tau, \theta) = \int_t^{\zeta} \beta_s c_s ds + \beta_{\zeta} \big(\mathbf{1}_{\{\zeta = \tau < T\}} L_{\tau} + \mathbf{1}_{\{\zeta = \theta < \tau\}} U_{\theta} + \mathbf{1}_{\{\zeta = T\}} \xi \big)\ .$$

One thus has $\mathbb{Q}$ *– almost surely, for every* $t \in [0, T]$,

$$\operatorname{esssup}_{\tau \in \mathcal{T}_t} \operatorname{essinf}_{\theta \in \mathcal{T}_t^{\vartheta}} \mathbb{E}_t \pi^t(\tau, \theta) = \Pi_t = \operatorname{essinf}_{\theta \in \mathcal{T}_t^{\vartheta}} \operatorname{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}_t \pi^t(\tau, \theta)\ . \tag{14}$$

More precisely, for any $t \in [0, T]$ *and for any* $\varepsilon > 0$*, the pair of stopping times* $(\tau^{\varepsilon}, \theta^{\varepsilon}) \in \mathcal{T}_t \times \mathcal{T}_t^{\vartheta}$ *given by*

$$\begin{aligned} \tau^{\varepsilon} &= \inf \Big\{ s \in [t, T]\,;\, \Pi_s \le L_s + \varepsilon \Big\} \wedge T \\ \theta^{\varepsilon} &= \inf \Big\{ s \in \cup_{l > 0} [\vartheta_{2l-1} \vee t, \vartheta_{2l} \vee t)\,;\, \Pi_s \ge U_s - \varepsilon \Big\} \wedge T, \end{aligned} \tag{15}$$

is an ε *– saddle-point for this Dynkin game at time* t*, in the sense that one has, for any* $(\tau, \theta) \in \mathcal{T}_t \times \mathcal{T}_t^{\vartheta}$*,*

$$\mathbb{E}_t \pi^t(\tau, \theta^{\varepsilon}) - \varepsilon \le \Pi_t \le \mathbb{E}_t \pi^t(\tau^{\varepsilon}, \theta) + \varepsilon. \tag{16}$$

(ii) *If the reflecting process* A *in the solution of* $(\mathcal{E})$ *is continuous, then the pair of stopping times* $(\tau^{\star}, \theta^{\star}) \in \mathcal{T}_t \times \mathcal{T}_t^{\vartheta}$ *obtained by setting* $\varepsilon = 0$ *in (15), is a saddle-point of the game. One thus has, for any* $(\tau, \theta) \in \mathcal{T}_t \times \mathcal{T}_t^{\vartheta}$*,*

$$\mathbb{E}_t \pi^t(\tau, \theta^{\star}) \le \Pi_t \le \mathbb{E}_t \pi^t(\tau^{\star}, \theta).$$

[2] Terms $A^{\pm}$ of the unique decomposition $A = A^+ - A^-$ as the difference of two non-decreasing processes null at 0 and defining mutually singular random measures on $[0, T]$.

As a consequence, Π is an arbitrage price process for the bond (see [4, 16, 5]). Given a suitable set of hedging instruments, the time-0 price Π_0 is also a *bilateral super-hedging price*, in the sense that there exists a self-financing super-hedging strategy for the issuer of the bond from any issuer initial wealth greater than Π_0, and a self-financing super-hedging strategy for the holder of the bond from any holder initial wealth greater than $(-\Pi_0)$. The price Π_0 is also the infimum of the initial wealths of all the issuer's self-financing super-hedging strategies, and this infimum is attained as a minimum in case the reflecting process A is continuous in the solution of $(\mathcal{E})$.

3.2 Markovian Case

We assume henceforth that the ϑ_ls are given as the successive times of exit from and entrance into a domain K, for an auxiliary finite-valued, pure jump, *call protection process* H. So $\vartheta_0 = 0$, and for every $l > 0$,

$$\vartheta_{2l-1} = \inf\{t > \vartheta_{2l-2}\,;\, H_t \notin K\} \wedge T\,,\ \vartheta_{2l} = \inf\{t > \vartheta_{2l-1}\,;\, H_t \in K\} \wedge T\,. \tag{17}$$

The effective call payoff process (11) can thus be rewritten as

$$U_t = \mathbf{1}_{H_t\in K}\infty + \mathbf{1}_{H_t\notin K} h(S_t) =: \bar{h}(X_t)\,, \tag{18}$$

with $X_t = (S_t, H_t)$.

One denotes henceforth $u(t, x, k)$ or $u_k(t, x)$ depending on what is more convenient in the context at hand, functions u of three arguments t, x, k where the third argument k takes its values in a discrete set, so that k can be thought of as referring to the index of a component of a vector or system of functions of (t, x).

In case the pair-process X is Markov, it is expected in view of (18) and established in later sections under suitable technical conditions that one should have, for $t \in [0, T]$:[3]

$$\Pi_t = u(t, X_t)\,,\ \Delta_t = \partial_S u(t, X_t)\,, \tag{19}$$

for a pricing function $u = u_k(t, S)$, where the mute variable k corresponds to H_t in the probabilistic interpretation.

3.3 Generic Simulation Pricing Schemes

Regarding *simulations*, indexed by $j = 1, \ldots, m$, we denote further by, given the continuous-time process $(X_t)_{t\in[0,T]}$ and its Euler scheme $(X_i)_{0\le i\le n}$:

- X_i^j, the value of X_i on the j^{th} simulated trajectory,
- $\mathbb{E}_i^j$, the conditional expectation given $X_i = X_i^j$.

On a stochastically generated mesh $(X_i^j)_{0\le i\le n}^{1\le j\le m}$, the generic simulation pricing scheme (u_i^j, δ_i^j) of [14] for estimating $u(t_i, X_i^j)_{0\le i\le n}^{1\le j\le m}$ and $(\partial_S u(t_i, X_i^j))_{0\le i\le n-1}^{1\le j\le m}$ writes: $u_n = g$, and then for $i = n-1 \ldots 0$, for $j = 1 \ldots m$

$$\begin{aligned} u_i^j &= \min\Big(\bar{h}(X_i^j)\,,\ \max\Big(\ell(S_i^j),\, e^{-\mu_i^j}\mathbb{E}_i^j\big(u_{i+1} + \mathrm{h}c_{i+1}\big)\Big)\Big) \\ \delta_i^j &= \frac{\mathbb{E}_i^j\left\{u_{i+1}(W_{i+1} - W_i)\right\}}{\sigma_i(S_i^j)S_i^j\mathrm{h}}\,. \end{aligned} \tag{20}$$

[3] Provided the function u is sufficiently regular, regarding the delta.

Here ℓ and $\bar{h}$ are the put and effective call payoff functions. Of course the 'min' in the first line plays no role on the simulated trajectories of X for which $H_i^j \in K$, since in this case $\bar{h}(X_i^j) = +\infty$.

The simulation pricing scheme thus ultimately relies on the computation of the conditional expectations sitting in (20) (for $i \geq 1$, since for $i = 0$ the conditional expectations reduces to expectations). For computing at time step $i \geq 1$ the conditional expectations $\mathbb{E}_i^j$ in (20) (conditional expectation given $t = t_i, S_i = S_i^j, H_i = H_i^j$), one performs, for every k, *non-linear regressions*[4] of $(u_{i+1}^j + \mathrm{hc}_{i+1}^j)^{j \in \Omega_i^k}$ and $(u_{i+1}^j (W_{i+1}^j - W_i^j))^{j \in \Omega_i^k}$ against $(S_i^j)^{j \in \Omega_i^k}$, where Ω_i^k denotes the subset of the indices j of the trajectories such that $H_i^j = k$. For the indices k's such that the set Ω_i^k is empty or too small for the non-linear regressions over Ω_i^k to be doable or significant, the 'missing regression functions' are set to zero. Denoting by $u_i^\dagger(\cdot, k)$'s and $\delta_i^\dagger(\cdot, k)$'s the regression functions obtained in this way, one then approximates in (20), for every $j = 1, \ldots, m$

$$\mathbb{E}_i^j \big(u_{i+1} + \mathrm{hc}_{i+1} \big) \approx u_i^\dagger (S_i^j, H_i^j) \, , \; \mathbb{E}_i^j \big(u_{i+1}^j (W_{i+1}^j - W_i^j) \big) \approx \delta_i^\dagger (S_i^j, H_i^j) \; .$$

Note that this procedure for computing the conditional expectations in (20) can be interpreted as using a *method of cells*[4] in the direction of the k variable and whatever method of choice in the direction of the S variable, a method of cells again being a simple and robust alternative, for estimating the pricing function $u = u_k(t, S)$.

One can also recover from the pricing function estimated by (20) the following estimates of the call/put regions and of the optimal call/put policies:

$$\begin{aligned} \mathcal{E}_p &= \big\{ (i, X_i^j) \, ; \, u_i^j = \ell(X_i^j) \big\} \, , \; \tau^j = \inf\{0 \leq i \leq n \, ; \, X_i^j \in \mathcal{E}_p\} \wedge n \\ \mathcal{E}_c &= \big\{ (i, X_i^j) \, ; \, H_i^j \notin K \text{ and } u_i^j = h_i(S_i^j) \big\} \, , \; \theta^j = \inf\{0 \leq i \leq n \, ; \, H_i^j \notin K \text{ and } X_i^j \in \mathcal{E}_c\} \wedge n \; . \end{aligned} \tag{21}$$

Comments 3.1 **(i)** As opposed to the case of straight 'European' Monte Carlo pricing schemes, confidence intervals are not available in the case of time-iterative simulation/regression pricing schemes. However Beveridge and Joshi [1] recently showed how to derive bounds on the prices by resorting to a suitable extension to game options of the dual American Monte Carlo approach of Rogers [30]. Moreover it is of course possible to derive a confidence interval of the *estimation method* (if not a confidence interval on the true values), by running the simulation/regression scheme for, say, fifty seeds of the random generator, and computing the standard deviation of the estimated prices and deltas.
(ii) Regarding alternative methods for computing conditional expectations in the context of pricing by simulation: *Malliavin Calculus* methods, *quantization* methods, etc., we refer the reader to, among others, Lions and Régnier [25], Bouchard et al. [7], or Pagès and Bally [28]. Note however that Malliavin Calculus methods are typically harder to implement than non-linear regression methods, and that quantization methods suffer significantly the curse of dimensionality. This is why we only resort in this paper to non-linear regression methods.

3.3.1 Price and Delta at Time 0

The previous scheme suffers from the accumulation of errors that occur through the iterated computation of the conditional expectations in (20). To limit the impact of these errors,

[4]See the Appendix.

an often preferred alternative, as far as the prices and delta at *at time 0* are concerned, consists in only retaining from (20) the estimated optimal stopping policies τ^js and θ^js in (21). These can then be used for computing estimates $\widetilde{u}_0$ and $\widetilde{\delta}_0$ of the option price and delta at time 0 alternative to u_0 and δ_0 in (20). We thus have the following *MC forward estimates* for the option price and delta at time 0, in which $\zeta^j = \tau^j \wedge \theta^j$:

$$\widetilde{u}_0 = \frac{1}{m}\sum_{j=1}^{m}\Big\{\mathrm{h}\sum_{i=1}^{\zeta^j}\beta_i^j c_i(S_i^j) + \beta_{\zeta^j}^j\Big(\mathbf{1}_{\{\zeta^j=\tau^j<n\}}\ell(S_{\tau^j}^j) + \mathbf{1}_{\{\theta^j<\tau^j\}}h(S_{\theta^j}^j) + \mathbf{1}_{\{\zeta^j=n\}}g(S_n^j)\Big)\Big\}$$

$$\begin{aligned}\widetilde{\delta}_0 = \frac{1}{m}\sum_{j=1}^{m}\Big\{&\mathrm{h}\sum_{i=1}^{\zeta^j}\beta_i^j\left(\beta_i^j\partial c_i(S_i^j)\nabla_i^j + \varepsilon_i^j c_i(S_i^j)\right)\\ &+\beta_{\zeta^j}^j\left(\mathbf{1}_{\{\zeta^j=\tau^j<n\}}\partial\ell(S_{\tau^j}^j) + \mathbf{1}_{\{\theta^j<\tau^j\}}\partial h(S_{\theta^j}^j) + \mathbf{1}_{\{\tau^j\wedge\theta^j=n\}}\partial g(S_n^j)\right)\nabla_{\zeta^j}^j\\ &+\varepsilon_{\zeta^j}^j\left(\mathbf{1}_{\{\zeta^j=\tau^j<n\}}\ell(S_{\tau^j}^j) + \mathbf{1}_{\{\theta^j<\tau^j\}}h(S_{\theta^j}^j) + \mathbf{1}_{\{\tau^j\wedge\theta^j=n\}}g(S_n^j)\right)\Big\},\end{aligned} \tag{22}$$

where we set $\beta_l^j = e^{-\mathrm{h}\sum_{k=0}^{l-1}\mu_k^j}$, and where $\varepsilon_i^j = -\mathrm{h}\beta_i^j\sum_{l=1}^{i}\partial\mu_l(S_l^j)\nabla_l^j$ is a discretization of the first-variation process

$$-\beta_t\int_0^t\partial\mu(s,S_s)\nabla_s ds$$

of $\beta_t = e^{-\int_0^t\mu(s,S_s)ds}$. See also (6) and (13) for the expression of the partial derivatives involved in the expression of $\widetilde{\delta}_0$.

Remark 3.2 The simulation/regression price simulation estimates u_0 and $\widetilde{u}_0$ are essentially the ones that were developed in the 'no call' American case, by Tsitsiklis and VanRoy using iteration on the values in [31] ('MC backward estimate' u_0 in (20)), and by Longstaff and Schwartz using iteration on the policies in [26] ('MC forward estimate' $\widetilde{u}_0$ in (22)). See Chapter 6 of Glasserman [19] for a survey about American Monte Carlo pricing methods. As for the deltas, the delta MC backward estimate in (20) is standard in the numerical BSDE literature, and the delta MC forward estimate in (22) can be found for instance in Theorem 2.3 of Gobet [20].

3.4 Convergence Results

In this section we discuss the theoretical convergence of the price and delta estimates (20) and (22).

3.4.1 Time-Discretization

Let us first review convergence results regarding the time-discretization of $(\mathcal{E})$ which is implicit in (20). Fully accurate statements would entail the introduction of a sub-grid (*reflection grid*) of $\mathfrak{t} = (t_i)_{0\le i\le n}$ such that the reflections in the first line of (20) only operate at times of the reflection grid, rather than at every $i = 0\dots n$. Since we found no significant difference in the numerical results by using the full time-discretization grid $\mathfrak{t}$ as reflection grid, this is omitted for the sake of simplicity of presentation. Denoting by $(\widehat{\Pi}_{t_i})_{0\le i\le n}$ the discrete time approximation of $(\Pi_t)_{t\in[0,T]}$ which is implicit in (20) and by $|\mathfrak{t}| = \max_{1\le i\le n} t_i - t_{i-1}$, one thus has,

Proposition 3.2 **(i) (Bouchard and Chassagneux [6], Chassagneux [13])** *In the 'no call' American or 'no protection' game cases to be considered in section 4 below, one has,*

$$\max_{i\leq n-1}\sup_{t\in[t_i,t_{i+1})}\mathbb{E}\Big[|\Pi_t-\widehat{\Pi}_{t_i}|^2\Big]\leq C|\mathfrak{t}|^{\frac{1}{2}}\ ;$$

(ii) (Chassagneux and Crépey [14]) *In the cases of discretely monitored call protection to be considered in section 6 below, one has, for every* $\varepsilon>0$*,*

$$\max_{i\leq n-1}\sup_{t\in[t_i,t_{i+1})}\mathbb{E}\Big[|\Pi_t-\widehat{\Pi}_{t_i}|^2\Big]\leq C_\varepsilon|\mathfrak{t}|^{\frac{1}{2}-\varepsilon}\ .$$

Comments 3.3 **(i)** These convergence rates exploid the semi-convex regularity of the payoff functions ℓ and h in (10). In case of less regular (only Lipschitz) payoff functions, downgraded convergence rates are available, see [14].
(ii) In the 'no call' or 'no protection' cases, convergence rates are also available for the deltas, see Theorem 6.1 in [13] and Theorem 4.1 in [6]. The other, non-trivial, protection cases of this paper, are currently under research in this regard.
(iii) Convergence in the cases of continuously monitored call protection considered in section 5, will be discussed in that section.

Monte Carlo Backward versus Forward Estimates In the 'no call' American case of section 4.1, we refer the reader to Remark 5.5 of Bouchard and Chassagneux [6] for the derivation of representations similar to (22) relatively to the *discretely reflected BSDE* associated to the continuous Euler scheme of an underlying diffusion. In fact, in the no call case, and in the no protection case as well, working as they do but using a discrete time Euler scheme of the underlying diffusion instead of the continuous Euler scheme in their case, would give rise to space-continuous analogs of the 'forward' representations (22) for the quantities denoted in [14] by $\widetilde{Y}_0$ and $\widehat{Z}_0$. These quantities represent the values of discrete time approximations, convergent with the rates stated in Proposition 3.2, to the option's price and delta at time 0.[5] The MC forward estimates (22) are thus in fact based on the same time-discretisation of $(\mathcal{E})$ as the MC backward estimates (20), the one with the rates recalled in Proposition 3.2, so that (20) and (22) only differ by space-discretization effects, to be commented upon in section 3.4.2 below.

Remark 3.4 In cases of other than trivial (American or game) call protection, the forward representation of the price at time 0 is still valid, but the forward representation of the delta at time 0 is not established. So, in all non-trivial cases of sections 5 and 6, the MC forward estimate $\widetilde{\delta}_0$ is currently unsupported in terms of mathematical convergence results, even from the mere point of view of the time-discretisation which is implicit in (22), not even mentioning space-discretisation issues.

3.4.2 Space-Discretization

Building on the above time-discretization results, a complete time-space convergence analysis can then be conducted by proceeding along the lines of Lemor, Gobet and Warin [24] and

[5]To be precise, $\widehat{Z}$ is the integrand of the Brownian motion in the stochastic integral represention of the discrete time approximation, denoted by $\bar{Z}$ in [14], of Z.

Lemor [23], who show how to control the cumulative regression approximation error resulting from the non-linear regressions in space repeatedly performed over the discrete time grid. Since we have nothing to add in this regard, we directly refer the interested reader to the above references.

Note that in practice (see, e.g., section 4.3 below), the MC forward estimates (22) are typically found to work better than the MC backward estimates (20). A commonly admitted interpretation of this better practical behavior is that, as mentioned above, (22) does not directly suffer from the accumulation of errors in (20). Note however that this interpretation is a bit flawed since (22) actually relies on (20), which is used in a first stage for deriving the τ^j/θ^js in (21). Some people argue that the pricing function u is typically not very sensitive to the optimal stopping policy, which in their view would explain why, altogether, (22) is more accurate than (20). It is our opinion however that a serious theoretical study of this phenomenon still remains to be done.

4 American and Game Options

In this section we review basic results and methodologies in the simplest cases of no call (American options) and no call protection (game options). More and more realistic protection cases will be considered in later sections.

4.1 No Call

We first consider the simplest 'no call' case in which $\vartheta_1 = T$, so $\cup_{l>0}[\vartheta_{2l-1}, \vartheta_{2l}) \cup \{T\} = \{T\}$, and the set $\mathcal{T}^{\vartheta}$ of the admissible call times is reduced to the singleton $\{T\}$. This corresponds to the case of dividend-paying American options, which, if not for dividends, has been studied in depth in the literature (see Remark 3.2).

Given an American claim with coupon rate c, early payoff process L and payoff at maturity ξ, let us introduce the *cumulative discounted payoff process* defined by, for $t \in [0, T]$,

$$\int_0^t \beta_s c_s ds + \beta_t \big(\mathbf{1}_{t<T} L_t + \mathbf{1}_{t=T}\xi\big) \ . \tag{23}$$

4.1.1 Pricing Equations

Let us first recover the pricing equations in this case. As is well known, in the risk-neutral model (2), the discounted price process $\beta\Pi$ of the above claim is given by the *Snell envelope* of the cumulative discounted payoff process (23), which yields the following simple form of (14), for $t \in [0, T]$,

$$\beta_t \Pi_t = \text{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}_t \left\{ \int_t^{\tau} \beta_s c_s ds + \beta_\tau \big(\mathbf{1}_{\{\tau<T\}} L_\tau + \mathbf{1}_{\{\tau=T\}}\xi\big) \right\} \ .$$

In this case the stochastic pricing equation $(\mathcal{E})$ reduces to

$$\begin{array}{l} \Pi_T = \xi, \ \text{and for } t \in [0,T], \\ \left\{ \begin{array}{l} -d\Pi_t = (c_t - \mu_t \Pi_t)dt + dA_t - \Delta_t \sigma(t, S_t) S_t dW_t \\ L_t \le \Pi_t \, , \ (\Pi_t - L_t) dA_t = 0 \end{array} \right. \end{array} \tag{24}$$

for a *non-decreasing* reflecting process $A = A^+$, *continuous* since there are no ϑ_{2l}s in $(0,T)$ involved. So the price process Π is continuous as well.

One then has by application of the results of [18] that the reflected BSDE (24) is well-posed, and that the expected representation (19) holds,[6] where $X = S$ (no auxiliary process H is needed here), and where the pricing function $u = u(t,S)$ can be characterized as the unique continuous viscosity solution [15] with polynomial growth in S to the deterministic pricing equation $(\mathcal{VI})$, which in this case reduces to the following scalar variational inequality:

$$\begin{cases} u = g \text{ at } T \\ \max\Big(\mathcal{G}u + c - \mu u, \ell - u\Big) = 0 \text{ on } [0,T) \times (0,+\infty) \,. \end{cases} \tag{25}$$

4.1.2 Deterministic Numerical Scheme

Let us briefly recall the generic multinomial recombining tree algorithm for solving (25), on a tensorized time-space grid (t_i, S^j) discretizing the time-state space, where i and j index the time and space step in the algorithm: $u_n(j) = g(S^j)$ for $j = 1,\dots,m$, and then for $i = n-1\dots 0$, for $j = 1\dots m$,

$$u_i^j = \max\left(\ell\left(S^j\right), e^{-\mu_i^j \mathrm{h}} \sum_l p_i^{j,l}(u_{i+1}^{j+l} + \mathrm{h}c_{i+1}^{j+l})\right) \,. \tag{26}$$

Here the $p_i^{j,l}$s are suitable weights which can be interpreted as the $(i, i+1)$ – conditional transition probabilities of a time-inhomogenous Markov chain $(S_i)_{0\le i\le n}$ approximating $(S_t)_{t\in[0,T]}$, so, for every l,

$$p_i^{j,l} = \mathbb{Q}(S_{i+1} = S^{j+l} \,|\, S_i = S^j) \,.$$

The $p_i^{j,l}$s are typically obtained by substitution of Taylor expansions for u and its derivatives in (25), followed by the numerical solution of a linear system in the case of implicit schemes. Such deterministic approximation schemes are convergent if $p \ge 0$, see, for instance, Morton and Mayers [27], or Duffy [17]. An approximation $(\delta_i^j)_{0\le i\le n}^{2\le j\le m-1}$ for the *delta function* $\Delta(t,S) = \partial u(t,S)$ at the interior points of the time-space grid can then be obtained by the formula, for every $i = 0\dots n$ and $j = 2\dots m-1$,

$$\delta_i^j = \frac{u_i^{j+1} - u_i^{j-1}}{S_i^{j+1} - S_i^{j-1}} \,. \tag{27}$$

4.1.3 Simulation Pricing Schemes

For solving (25) by simulation, a possible procedure consists in writing a dynamic programming equation as of (26), but on a stochastically generated, non tensorized mesh $(S_i^j)_{0\le i\le n}^{1\le j\le m}$, using an appropriate discretization scheme S_i for the underlying diffusion S (like for instance the Euler scheme of section 2.1.2). We thus get the following 'stochastic version' of (26), in which we recognize the generic simulation scheme (20) particularized to the present case: $u_n^j = g(S_n^j)$ for $j = 1\dots m$, and then for $i = n-1\dots 0$, for $j = 1\dots m$,

$$u_i^j = \max\Big(\ell(S_i^j), e^{-\mu_i^j \mathrm{h}} \mathbb{E}_i^j\big(u_{i+1} + \mathrm{h}c_{i+1}\big)\Big) \,, \tag{28}$$

[6] Provided the function u is sufficiently regular, regarding the delta.

where $\mathbb{E}_i^j$ stands for the conditional expectation given $S_i = S_i^j$. A numerical approximation $\delta = (\delta_i^j)$ of the delta function on the grid could be deduced from (u_i^j) by the formula (27), however convergence of this estimate is not established, and it typically exhibits a very high variance in practice. To approximate the delta, better, convergent alternatives, are the regression estimates of (20) or (22) (see Comment 3.3(ii) and the last paragraph of section 3.4.1).

At step $i \geq 1$, the conditional expectations in these schemes can be computed by non-linear regression of $(u_{i+1}^j + \mathrm{h}c_{i+1}^j)^{1\leq j\leq m}$ and $(u_{i+1}^j(W_{i+1}^j - W_i^j))^{1\leq j\leq m}$ against $(S_i^j)^{1\leq j\leq m}$.

Also, alternatively to the above estimates u_0 and δ_0 for the price and delta at time 0, one may use the MC forward estimates $\widetilde{u}_0$ and $\widetilde{\delta}_0$.

4.2 No Protection

We now discuss the case of standard game options without call protection, so $\vartheta_1 = 0, \vartheta_2 = T$, and $\mathcal{T}_t^\vartheta = \mathcal{T}_t$.

4.2.1 Pricing Equations

In this case the stochastic pricing equation $(\mathcal{E})$ writes:

$$\begin{array}{l} \Pi_T = \xi, \text{ and for } t \in [0,T], \\ \begin{cases} -d\Pi_t = (c_t - \mu_t \Pi_t)dt + dA_t - \Delta_t \sigma(t, S_t) S_t dW_t \\ L_t \leq \Pi_t \leq U_t \,,\; (\Pi_t - L_t) dA_t^+ = (U_t - \Pi_t) dA_t^- = 0 \end{cases} \end{array} \tag{29}$$

to be solved in (Π, Δ, A), where again A and Π are sought for as continuous processes. Moreover, one has $\Pi_t = v(t, S_t)$, where the pricing function v is the solution of the following pricing equation:

$$\begin{cases} v = g \text{ at } T \\ \min\left(\max\left(\mathcal{G}v + c - \mu v, \ell - v\right), h - v\right) = 0 \text{ on } [0,T) \times (0, +\infty) \,. \end{cases} \tag{30}$$

The ‘no call’ deterministic pricing algorithm (26) simply needs to be amended as: $v_n(j) = g(S^j)$ for $j = 1 \ldots m$, and then for $i = n-1 \ldots 0$, for $j = 1 \ldots m$,

$$v_i^j = \min\left(h_i\left(S^j\right), \max\left(\ell\left(S^j\right), e^{-\mu_i^j \mathrm{h}} \sum_l p_i^{j,l}(v_{i+1}^{j+l} + \mathrm{h}c_{i+1}^{j+l})\right)\right) \,. \tag{31}$$

A delta estimate $(\delta_i^j)_{0\leq i\leq n}^{2\leq j\leq m-1}$ can then be deduced from $(v_i^j)_{0\leq i\leq n}^{1\leq j\leq m}$ by the formula (27) with v instead of u therein.

Moreover it is possible to conduct a convergence analysis of (31) similar to that of (26) in the ‘no call’ case, see the comments following (26) in section 4.1.2.

4.2.2 Simulation Schemes

Likewise, the stochastic pricing scheme (28) simply needs to be amended as follows (cf. (20)): $v_n^j = g(S_n^j)$ for $j = 1 \ldots m$, and then for $i = n-1 \ldots 0$, for $j = 1 \ldots m$,

$$v_i^j = \min\left(h_i(S_i^j), \max\left(\ell(S_i^j), e^{-\mu_i^j \mathrm{h}} \mathbb{E}_i^j\left(v_{i+1} + \mathrm{h}c_{i+1}\right)\right)\right) \,. \tag{32}$$

One also has the delta estimate δ_i^j in (20), with v_i^j instead of u_i^j therein. Again, for $i \geq 1$, the conditional expectations at time step i in the algorithm can be computed by non-linear regression against $(S_i^j)^{1 \leq j \leq m}$, and alternatively to the estimates v_0 and δ_0 for the price and delta at time 0, one may use the related MC forward estimates as of (22).

4.3 Numerical Experiments

Since the case of American options is very well documented in the literature, we do not provide any numerical results in this regard, directly considering game options. The general data of Table 1 will be used by default throughout the paper.

$\bar{P}$	$\bar{N}$	$\bar{C}$	$\bar{R}$	η	σ	r	q	γ_0	α	m
0	100	103	0	1	0.2	0.05	0	0.02	1.2	10^4

Table 1: *General Data.*

In all the numerical experiments of this paper, we use a constant time-step $t_{i+1} - t_i = \text{h}$, where the time step index varies from 0 to n, with:
• h = six hours (four time steps per day) in the case of the simulation pricing schemes for solving $(\mathcal{E})$, and
• h = one day in the case of deterministic schemes for solving $(\mathcal{VI})$.

The space-steps in the S variable are, the superscript ^{j} referring to a generic space step index:[7]
• $S^{j+1} - S^j = 0.5$ in the case of deterministic schemes, and
• Cells of diameter one (segments of $\mathbb{R}_+$ of length one) in the case of simulation/regression methods involving a *method of cells* in the direction of the S variable (see the Appendix).

Regarding the deterministic numerical schemes, fully implicit finite differences schemes are used throughout the paper.

For a maturity $T = 125$ days and a nominal coupon rate $\bar{c} = 0$ in (12), Table 2 shows the standard deviation of fifty estimates of the game option price and delta obtained by changing the seed of the random generator, using:
• on one hand, the MC backward estimate,
• on the other hand, the MC forward estimate.
In both cases a global parametric regression basis $1, S, S^2$ is used for estimating the conditional expectations (see the Appendix).

The MC forward estimates have a much lower deviation, as expected.

	Value VI	Dev MC Bd	Dev MC Fd
Price	102.049	0.821	0.010
Delta	0.416	0.071	0.019

Table 2: *VI Values and MC Backward and Forward Standard Deviations over 50 trials for the option prices and deltas ($S_0 = 100.55$).*

[7] *Space step* in the sense of a *trajectory's index* varying between 1 and m, in the case of simulation pricing schemes.

Table 3 shows the option prices and deltas computed for various S_0s by the MC forward estimates, or by the deterministic scheme (31) for the VI (30). The errors (%Err) in this and the following Tables are the unsigned *percentage relative errors* of the MC prices with respect to the VI prices. So an error of '1' in the table means a relative error of $\pm 1\%$ of the MC price with respect to the VI price.

S_0	VI Price	%Err MC Bd	%Err MC Fd	VI delta	%Err MC Bd	%Err MC Fd
98.55	101.246	1.90	0.04	0.376	1.07	0.07
99.55	101.637	1.92	0.01	0.396	0.95	0.50
100.55	102.049	1.99	0.01	0.416	2.77	0.67
101.55	102.479	1.65	0.07	0.435	3.97	3.47

Table 3: *MC versus VI prices and deltas.*

The MC forward estimates are more accurate than the MC backward estimates. ***MC forward estimates are used by default henceforth.*** *MC estimate* or *simulation pricing scheme* are thus to be understood in the sequel as MC *forward* estimate or *forward* simulation pricing scheme. ***Moreover in view of Remark 3.4 we shall in all non-trivial protection cases (i.e., other than the American and game cases of section 4) focus on prices, forgetting about deltas for which convergence of the MC forward estimate $\widetilde{\delta}_0$ is not established mathematically.***

An important issue in this paper, especially with the path-dependent call protection clauses to be considered in section 6, is of course that of computation times. To make the computation time information below less contingent on the implementation, it will be given in terms of multiples of the average time needed to solve the scalar variational inequalities of this section 4.3, scaled to one second. ***So a computation time of one second henceforth is to be understood not in the sense of a 'physical' second, but of the average time needed to solve the scalar variational inequalities of this section 4.3***, which is actually less than one second on nowadays computers in a compiled programming language like C++.

Table 4 thus compares in terms of accuracy and computation time, the results for $S_0 = 100.55$ (so VI price = 102.049, VI delta = 0.416, third line of Table 3) corresponding to various possible bases of regression in S (see the Appendix) :
• either global regression schemes against powers of S, via one empirical covariance matrix inversion per time step in the simulation,
• or a simple method of cells, for which no matrix inversion is required.

For instance, the %Err MC Price is 2.149 with the regression basis $(1, S)$, and 0.018 with the regression basis $(1, S, S^2, S^3)$, versus 0.014 (truncated to 0.01 in Table 3) with the regression basis $(1, S, S^2)$. Note that regarding the global regression schemes the computation times significantly increase with the size of the basis which is used, due to the increasing complexity of the matrix inversion involved. The results of Table 4 thus illustrate the sensitivity of the results to the regression basis, both in terms of accuracy and of computation time. As far as global regression schemes are concerned, the choice of $(1, S, S^2)$ appears to be a good compromise, given the number $m = 10^4$ simulations and the time step h = 6 hrs for the Euler scheme which are used, among other model parameters.

	%Err MC Price	%Err MC Delta	CPU Time
$(1, S)$	2.14	5.03	0.5s
$(1, S, S^2)$	0.014	0.67	0.7s
$(1, S, S^3)$	0.018	0.69	1.5s
Method of Cells	0.049	0.89	0.5s

Table 4: *MC errors and CPU times depending on the regression basis which is used for S.*

5 Continuously Monitored Call Protection

In this section we study the two following forms of continuously monitored call protection, given a *trigger level* $\bar{S} > S_0$:
• call forbidden until the first time ϑ_1 that $S \geq \bar{S}$, and call possible for $t \geq \vartheta_1$, which corresponds to the sequence ϑ given by

$$\vartheta_1 = \inf\{t \geq 0\,;\, S_t \geq \bar{S}\} \wedge T\ ,$$

and $\vartheta_l = T$ for $l \geq 2$,
• the 'intermittent' analog of the previous clause, so, roughly, call possible *whenever* $S \geq \bar{S}$, and forbidden otherwise. But as such this clause is ill-posed mathematically (see Remark 5.1), so that the following modified clause will be considered instead, given a second trigger level $\underline{S} < \bar{S}$: call forbidden until the first time ϑ_1 that $S \geq \bar{S}$, possible hereafter until the first time $\vartheta_2 > \vartheta_1$ that $S < \underline{S}$, etc., which corresponds to the non-decreasing sequence ϑ of stopping times given by, for every $l > 0$:

$$\vartheta_{2l-1} = \inf\{t > \vartheta_{2l-2}\,;\, S_t \geq \bar{S}\} \wedge T\ ,\ \vartheta_{2l} = \inf\{t > \vartheta_{2l-1}\,;\, S_t \leq \underline{S}\} \wedge T\ . \tag{33}$$

Remark 5.1 For $\underline{S} = \bar{S}$, (33) would typically not define an increasing sequence of stopping times. This is why we need to set $\underline{S} < \bar{S}$.

5.1 Vanilla protection

We first consider the call protection *until a stopping time* ϑ_1, so $\vartheta_2 = T$, and the effective call payoff process U_t accounting for the call protection in $(\mathcal{E})$ reduces to

$$U_t = \mathbf{1}_{\{t<\vartheta_1\}}\infty + \mathbf{1}_{\{t\geq\vartheta_1\}}h(S_t)\ . \tag{34}$$

Since $\vartheta_2 = T$, the processes $A^{\pm}$ and Π are again continuous in this case.

It is intuitively clear and established in [5] that the $\mathbb{Q}$-price process Π of a convertible bond with call protection before ϑ_1 coincides on $[\vartheta_1, T]$ with the no protection price process of section 4.2. The only remaining issue is thus the 'protection pricing problem', in the sense of the characterization and computation of the price process on the protection time interval $[0, \vartheta_1)$.

The most basic kind of lifting time ϑ_1 of a call protection would of course consist in a constant $\vartheta_1 = T_1 \in [0, T)$. In this case the protection pricing problem reduces to a no call pricing problem as of section 4.1.

In this section we shall consider the case of a continuously-monitored call protection, we call it 'vanilla protection', corresponding to

$$\vartheta_1 = \inf\{t \in \mathbb{R}_+ \,;\, S_t \geq \bar{S}\} \wedge T \;, \tag{35}$$

for some trigger level $\bar{S} > S_0$. Note that this case falls outside the scope of [14], since ϑ_1 in (35) is not discrete-valued. But this case is studied in details in [16, 5].

5.1.1 Deterministic Pricing Equation

The following equation for the related protection pricing function $u = u(t, S)$ on the domain $[0, T] \times (0, \bar{S}]$ was established in [5]:

$$\left\{ \begin{array}{l} u = v \text{ on } (\{T\} \times (0, \bar{S}]) \cup ([0, T] \times \{\bar{S}\}) \\ \max\left(\mathcal{G}u + c - \mu u, \ell - u\right) = 0 \text{ on } [0, T) \times (0, \bar{S}) \end{array} \right. \tag{36}$$

where v is the no call protection pricing function of section 4.2.1, solution of the deterministic pricing equation (30). Moreover, provided that the functions u and v are sufficiently regular for Itô formulas to be applicable, one has, for $t \in [0, T]$:

$$\Delta_t = \mathbf{1}_{\{t \leq \vartheta_1\}} \partial_S u(t, S_t) + \mathbf{1}_{\{t > \vartheta_1\}} \partial_S v(t, S_t) \;. \tag{37}$$

Knowing an approximation (v_i^j) of v, computed for instance by the deterministic scheme (31) of section 4.2.1, the Cauchy-Dirichlet problem (36) can be solved by standard finite differences deterministic numerical schemes, like

$$u_i^j = \max\left(\ell\left(S^j\right), \, e^{-\mu_i^j \mathrm{h}} \sum_l p_i^{j,l} (u_{i+1}^{j+l} + \mathrm{h} c_{i+1}^{j+l})\right) \tag{38}$$

at the grid points interior to the domain $[0, T] \times (0, \bar{S}]$, with a Dirichlet boundary condition $u = v$ at the grid points on the time-space boundary '$T \cup \bar{S}$'. Convergence results for this scheme can be found in Crépey [16].

Remark 5.2 In fact the convergence results of [16] can only be considered as partial results, since one only gets the convergence of the scheme for u on $[0, T] \times (0, \bar{S}]$ *conditionally on its convergence* on $[0, T) \times \{\bar{S}\}$, for which no explicit condition is given. Moreover the convergence analysis of [16] is conducted under the working assumption that the true value for v is plugged on $[0, T) \times \{\bar{S}\}$ in the approximation scheme for u, whereas in practice one has to use an approximation (v_i^j) of v. Finally this analysis only yields convergence, not convergence rates.

5.1.2 Simulation Scheme

Given a stochastically generated mesh $(S_i^j)_{0 \leq i \leq n}^{1 \leq j \leq m}$, and setting for every $j = 1 \ldots m$ (cf. (35))

$$\vartheta_1^j = \inf\{0 \leq i \leq n \,;\, S_i^j \geq \bar{S}\} \wedge n \;,$$

a simulation algorithm for estimating the $u(t_i, S_i^j)$'s writes: $u_n = v_n = g$, and then for $i = n-1 \ldots 0$, for $j = 1 \ldots m$:

- $v_i^j = \min\Big(h_i(S_i^j)\,,\ \max\Big(\ell(S_i^j),\, e^{-\mu_i^j \mathrm{h}}\mathbb{E}_i^j\big(v_{i+1} + \mathrm{h}c_{i+1}\big)\Big)\Big)$,
- If $i \geq \vartheta_1^j$, $u_i^j = v_i^j$, else

$$u_i^j = \max\Big(\ell\left(S_i^j\right),\, e^{-\mu_i^j \mathrm{h}}\mathbb{E}_i^j\big(u_{i+1} + \mathrm{h}c_{i+1}\big)\Big)\ . \tag{39}$$

So (v_i^j) here is but the no call protection price of section 4.2.2, estimated by simulation as in (32). For $i \geq 1$ the conditional expectations in (39) may be computed by non-linear regression of $(u_{i+1}^j + \mathrm{h}c_{i+1}^j)^{j\in\Omega_i}$ against $(S_i^j)^{j\in\Omega_i}$, where Ω_i denotes the *subset of the trajectories* j *on which* $\{i < \vartheta_1^j\}$.

Convergence Analysis This case falls outside the scope of of [14], where the more realistic set-up of a discretely monitored call protection (to be considered in this paper in section 6 below) is favored in the theoretical convergence analysis. However, by inspection of the proofs of Chassagneux [13] and Bouchard–Menozzi [9], it is rather clear that convergence rates for the above scheme to the solution (Π, Δ, A) of the related BSDE $(\mathcal{E})$, in the form here of (29) with U given by (34)-(35) therein, could be obtained by combining general reflected BSDE convergence arguments of [13] with approximation of the call time by the technique of [9]. Note that the related convergence results regarding Π do not have the same limitations than those mentioned in Remark 5.2 regarding the deterministic approximation of u in section 5.1.1. Moreover, one also gets in this way convergence and convergence rates regarding Δ, which in view of (37) essentially corresponds to the gradient of u and v, and to the hedging strategy in the financial interpretation.

Price at Time 0 We can recover from the protection pricing function u above the following estimates of the protection put region and of the optimal protection put policy:

$$\widetilde{\mathcal{E}}_p = \big\{(i, S_i^j)\,;\, u_i^j = \ell(S_i^j)\big\}\,,\ \widetilde{\tau}^j = \inf\{0 \leq i \leq \vartheta_1^j\,;\, S_i^j \in \widetilde{\mathcal{E}}_p\} \wedge n\ . \tag{40}$$

One then has the following MC forward estimate for the protection price at time 0, with $\widetilde{\zeta}^j = \widetilde{\tau}^j \wedge \vartheta_1^j$,

$$\widetilde{u}_0 = \frac{1}{m}\sum_{j=1}^{m}\left\{\mathrm{h}\sum_{i=1}^{\widetilde{\zeta}^j}\beta_i^j c_i(S_i^j) + \beta_{\widetilde{\zeta}^j}^j\left(\mathbf{1}_{\{\widetilde{\tau}^j<\vartheta_1^j\}}\ell_{\widetilde{\tau}^j}(S_{\widetilde{\tau}^j}^j) + \mathbf{1}_{\{\vartheta_1^j\leq\widetilde{\tau}^j\}}v_{\vartheta_1^j}^j\right)\right\}\ . \tag{41}$$

Remark 5.3 It is possible to write likewise a forward estimate $\widetilde{\delta}_0$ of the time-0 protection delta $\partial_S u(0, S_0)$ (cf. (22)). However, using the notation introduced in the last paragraph of section 3.4.1, it is not clear whether the 'forward representation' of $\widehat{Z}_0$ is indeed valid in cases with call protection, so that convergence of the delta's forward estimate is not guaranteed either (see Remark 3.4).

5.2 Intermittent Vanilla protection

We now consider the 'intermittent analog' of the 'until' clause of section 5.1. Let thus the non-decreasing sequence ϑ of stopping times be defined by (33), for activating and desactivating trigger levels $\bar{S}$ and $\underline{S}$ with $\underline{S} < \bar{S}$ (see Remark 5.1).

5.2.1 Deterministic Pricing Equation

We are in the Markovian case of section 3.2 for the call protection set $K = \{0\}$ and the Boolean-valued call protection process H_t defined by $H_0 = 0$, H jumping from 0 to 1 at the ϑ_ls such that $S_{\vartheta_l} \geq \bar{S}$ and $H_{\vartheta_l -} = 0$, and H jumping from 1 to 0 at the ϑ_l's such that $S_{\vartheta_l} \leq \underline{S}$ and $H_{\vartheta_l -} = 1$. In particular one has $H_t = 1$, hence call is possible, on $\{(\omega, t); S_t \geq \bar{S}\}$, and $H_t = 0$, hence call is forbidden, on $\{(\omega, t); S_t \leq \underline{S}\}$.

The pair $X = (S, H)$ is a Markov process. It is thus expected that $\Pi_t = u(t, S_t, H_t)$, for a pricing function $u = u_k(t, S)$, with $k \in \{0, 1\}$.

Note however that the call protection ϑ is continuously monitored in time, and not discretely monitored at the dates of a finite time grid $\mathfrak{T}$. Continuously monitored intermittent call protection clauses fall outside the scope of the results of [16, 14]. Yet, in the present case, existence of a solution to the related *pricing RIBSDE* $(\mathcal{E})$ follows by application of the results of Peng and Xu [29]. More precisely, existence follows from an immediate extension of these results to the case of an $\mathbb{R} \cup \{+\infty\}$ – valued upper barrier U, noting that the results of Peng and Xu, even if stated for real-valued barriers, only use the square-integrability of the random variable $\sup_{t \in [0,T]} U_t^-$, a condition which is satisfied in the present case where $U_t^- = h(S_t)^-$.

Building upon this existence result, and uniqueness holding by standard arguments, one can then proceed much like in [16, 14] to prove that $\Pi_t = u(t, S_t, H_t)$ on $[0, T]$, for a pricing function $u = u_k(t, S)$. The formally related deterministic pricing equation $(\mathcal{VI})$ writes:

$$\begin{cases} u = g \text{ at } T \\ u_0(t, \bar{S}) = u_1(t, \bar{S}) \\ \max\left(\mathcal{G}u_0 + c - \mu u_0, \ell - u_0\right) = 0 \text{ on } [0, T) \times (0, \bar{S}) \\ u_1(t, \underline{S}) = \min(u_0(t, \underline{S}), h(t, \underline{S})) \\ \min\left(\max\left(\mathcal{G}u_1 + c - \mu u_1, \ell - u_1\right), h - u_1\right) = 0 \text{ on } [0, T) \times (\underline{S}, +\infty)\ . \end{cases} \tag{42}$$

One thus gets a system of two equations in the pair of functions (u_0, u_1), with u_0 and u_1 respectively defined on $[0, T] \times (0, \bar{S}] \times \{0, 1\}$ and $[0, T] \times [\underline{S}, +\infty) \times \{0, 1\}$. This system can practically be solved by standard finite differences deterministic numerical schemes on a fixed time-space grid, like (cf. (38)): $u_n^{j,0} = u_n^{j,1} = g(S_n^j)$ for $j = 1 \dots m$, and then for $i = n - 1 \dots 0$, for $j = 1 \dots m$:

$$\begin{cases} u_i^{j,0} = \max\left(\ell\left(S^j\right),\ e^{-\mu_i^j \mathrm{h}} \sum_l p_i^{j,l} (u_{i+1}^{j+l,0} + \mathrm{h} c_{i+1}^{j+l})\right) \\ u_i^{j,1} = \min\left(h_i\left(S^j\right), \max\left(\ell\left(S^j\right),\ e^{-\mu_i^j \mathrm{h}} \sum_l p_i^{j,l} (u_{i+1}^{j+l,1} + \mathrm{h} c_{i+1}^{j+l})\right)\right) \end{cases} \tag{43}$$

where, in the right hand side, $u_{i+1}^{j+l,0}$ is to be understood as $u_{i+1}^{j+l,1}$ for $S^{j+l} \geq \bar{S}$, and $u_{i+1}^{j+l,1}$ is to be understood as $\min(u_{i+1}^{j+l,0}, h_{i+1}^{j+l})$ for $S^{j+l} \leq \underline{S}$.

Note however that the analytic characterization of the pricing function $u = (u_0, u_1)$ as unique solution in some sense to (43) (continuous viscosity solution with growth conditions, presumably), and the related deterministic schemes convergence results, are not established yet. This is due to the absence of stability results so far, beyond existence and uniqueness, for the stochastic pricing equation $(\mathcal{E})$. More precisely, the absence of stability results makes it difficult to establish the continuity of the pricing function $u = (u_0, u_1)$, which would be a prerequisite in proving that u solves (42) in the continuous viscosity sense.

S_0	100.55	101.55	102.55	103.55
MC price	103.841	103.724	103.713	103.55
VI price	103.874	103.785	103.693	103.55
%Err	0.032	0.059	0.019	0

Table 5: *Vanilla protection ($\bar{S} = 103$).*

S_0	100.55	101.55	102.55	103.55
MC price	110.082	110.819	111.351	111.809
VI price	110.324	110.896	111.488	112.099
%Err	0.22	0.069	0.12	0.26

Table 6: *Same as Table 5 but for $\bar{S} = 120$.*

5.2.2 Simulation Pricing Scheme

The related simulation pricing algorithm is (20), specified to the set-up of section 5.2.1. Note that convergence rates for this scheme fall outside the scope of [14], since one deals here with a continuously monitored intermittent form of call protection.

5.3 Numerical Experiments

5.3.1 Vanilla protection

Using the general data of Table 1 in section 4.3 along with a maturity $T = 180$ days and a nominal coupon of $\bar{c} = 1.2$ per month (cf. (12) and Remark 2.1), Tables 5 and 6 show the vanilla protection convertible bond prices computed in two ways:
• First, by the simulation pricing scheme of section 5.1.2, where the conditional expectations are estimated by a method of cells in S,
• Second, by the deterministic numerical scheme of section 5.1.1.
The accuracy of the simulation pricing scheme is quite satisfactory.

In the last column of Table 5 one can see that both schemes compute the exact bond's value, which is equal to S_0, in case $S_0 = 103.55 > 103 = \bar{S}$ (see [5]).

5.3.2 Intermittent Vanilla protection

We now consider convertible bonds with intermittent call protection as of section 5.2, that we evaluate by the deterministic scheme of section 5.2.1, or, alternatively, by the simulation method of section 5.2.2. The results are presented in Table 7 for the same data as above with $\bar{S} = 103$, and $\underline{S} = 97$. In spite of the lack of theoretical convergence results mentioned above, the simulation results are quite accurate in practice. As already observed in Table 5, both schemes give the exact value, which is equal to the stock price S_0, for $S_0 \geq \bar{S} = 103$.

S_0	92	93	94	102	103	104
VI price	104.128	104.147	104.154	103.716	103	104
MC %Err	0.094	0.051	0.012	0.001	0	0

Table 7: *Intermittent Vanilla protection*

6 Discretely Monitored Call Protection

In the previous section we tentatively studied two forms of continuously monitored call protection. In the 'intermittent' case, some mathematical difficulties arose. However in practice call protection is monitored in discrete time, e.g., reexamined at the end of each trading day, rather than in continuous time.

As we shall now see in this section, it turns out that it is not only closer to actual contracts, but also, easier from the mathematical point of view (even if more 'complex' at first sight), to work with the actual, discretely monitored call protection. The path dependence of certain clauses can then pose computational challenges, but this is where the simulation pricing schemes reveal their interest, as compared with deterministic schemes which are ruled out by the curse of dimensionality.

Let $\mathfrak{T} = \{T_0 = 0 < T_1 \ldots < T_N = T\}$ denote an increasing sequence of fixed *monitoring times*, so that the ϑ_ls are now $\mathfrak{T}$-valued stopping times. Well-posedness of $(\mathcal{E})$ and the expected representation (19) for its solution then hold by application of [16, 14]. Moreover, the upper semi-continuous envelope $\bar{u}$ [15] of the pricing function u in (19) can be characterized analytically as the *largest viscosity sub-solution with polynomial growth in* S [14] (*solution* for short henceforth) of a system of variational inequalities indexed by k, referred to hereafter as $(\mathcal{VI})$. Discontinuities of the pricing function u may and typically do arise at the T_Is in relation with the discontinuities of the pricing process Π at the 'even' times ϑ_{2l}s in $(0, T)$. In spite of this technical difficulty the deterministic pricing equation $(\mathcal{VI})$ can, in principle, be solved by standard deterministic pricing schemes, such as finite differences θ-schemes, which are shown to be convergent in [16, 14] (simple convergence results, without convergence rates, holding under a suitable refinement of the scheme to deal with the discontinuities of the function u).

But the specific form of $(\mathcal{VI})$ depends on the very nature of the call protection process H, so that the deterministic pricing equation $(\mathcal{VI})$ will be specified later in the text in every considered case. Already note that the deterministic pricing schemes are a bit theoretical in the sense that due to the potentially very path dependent nature of the call protection (high-dimensionality of the call protection process H), one can be led to huge, impractical systems of variational inequalities.

6.1 'l last' protection

Given $\bar{S} > S_0$ and an integer l, let in this section H_t stand for the number of consecutive monitoring dates T_Is with $S_{T_I} \geq \bar{S}$ from time t backwards, capped at l. In particular, at any given time t, one has $H_t = 0$ if S was smaller than $\bar{S}$ at the last monitoring date $\leq t$. We now consider the 'l last' call protection clause corresponding to this protection process H and to the protection set $K = \{0, \ldots, l-1\}$ (see section 3.2). So call is possible at a given time t iff S was $\geq \bar{S}$ at the last l monitoring dates $\leq t$.

6.1.1 Deterministic Pricing Equation

One then has by application of [16, 14] that $\Pi_t = u(t, S_t, H_t)$ on $[0, T]$, for a pricing function $u = u_k(t, S)$ with $k = 0 \dots l$, where the restrictions of the u_ks to every set $[T_{I-1}\, T_I) \times [0, +\infty)$ are continuous, and where the limit

$$u_k(T_I-, S) = \lim_{(t,x)\to(T_I,S) \text{ with } t<T_I} u_k(t, x) \tag{44}$$

exists for every $k = 0 \dots l$, $I \geq 1$ and $S \neq \bar{S}$ (but typically not for $S = \bar{S}$). The deterministic pricing equation $(\mathcal{VI})$ now assumes the form of the following *Cauchy cascade* of variational inequalities:

For I decreasing from N to 1:
• At $t = T_I$, for $k = 0 \dots l$,

$$u_k(T_I-, S) = \begin{cases} u_{k+1}(T_I, S), \text{ or } u_k(T_I, S) \text{ if } k = l, & \text{for } \{S > \bar{S}\}\,, \\ u_0(T_I, S), \text{ or } \min(u_0(T_I, S), h(T_I, S)) \text{ if } k = l, & \text{for } \{S < \bar{S}\}\,, \end{cases} \tag{45}$$

Or, in case $I = N$, $u_k(T_I-, S) = g(S)$ for $S > 0$;
• On the time interval $[T_{I-1}, T_I)$,

$$\begin{aligned} &\max\left(\mathcal{G}u_k + c - \mu u_k\,,\ \ell - u_k\right) = 0\,,\ k = 0 \dots l - 1 \\ &\min\left(\max\left(\mathcal{G}u_l + c - \mu u_l\,,\ \ell - u_l\right), h - u_l\right) = 0\,. \end{aligned} \tag{46}$$

The (upper semi-continuous envelope $\bar{u}$ [14] of the) pricing function u can then be characterized as the largest viscosity subsolution of $(\mathcal{VI})$ with polynomial growth in S.

The Cauchy cascade $(\mathcal{VI})$ can practically be solved by standard deterministic numerical schemes, shown to be convergent in [14] under a suitable refinement of the scheme to deal with the singularities of u.

6.1.2 Simulation Scheme

Let us be given a time mesh $(t_i)_{0\leq i\leq n}$ *refining the protection monitoring grid* $\mathfrak{T}$. To solve the above problem by simulation, one first generates a stochastic grid $(S_i^j, H_i^j)_{0\leq i\leq n}^{1\leq j\leq m}$ by an Euler scheme for S, using past values of S to fill H.

Example 6.1 With say $r = q = \eta = 0$, h = six hours (four time-steps per day) and $T_{I+1} - T_I$= one day (daily monitored call protection):
• Simulate, starting from S_0 given, $S_{6\text{h}} = S_0(1 + \sigma\sqrt{\text{h}}\varepsilon_1)$, $S_{12\text{h}} = S_{6\text{h}}(1 + \sigma\sqrt{\text{h}}\varepsilon_2)$, $S_{18\text{h}} = .., S_{24\text{h}}, S_{30\text{h}}, ..., S_T$ for IID standard Gaussian random variables ε_i;
• Whenever t_i coincides with one the T_Is (i.e., one every fourth i), update the variable H, so: $H_0 = 0$, $H_{6\text{h}} = H_0$, $H_{12\text{h}} = H_0$, $H_{18\text{h}} = H_0$, $H_{1day} = 1$ if $S_{1day} \geq \bar{S}$, otherwise $H_{1day} = 0$, etc. (capped at l) until H_T;
• Redo this m times to get m trajectories $(S_i^j, H_i^j)_{0\leq i\leq n}^{1\leq j\leq m}$.

We then use the generic scheme of section 3.3 specified to the context of this section, referred to henceforth as MC_l.

6.2 'l out of the last d' protection

Given a further integer $d \geq l$, let now $H_t = (H_t^p)1_{\leq p \leq d}$ represent the vector of the indicator functions of the events $S_{T_I} \geq \bar{S}$ at the last d monitoring dates preceding time t. We now consider the 'l out of the last d' call protection clause corresponding to this new protection process H and to the protection set $K = \{k = (k_p)_{1 \leq p \leq d} \in \{0,1\}^d \,;\, |k| < l\}$ with $|k| := \sum_{1 \leq p \leq d} k_p$ (see section 3.2). So call is possible at a given time t whenever $S \geq \bar{S}$ on at least l among the last d monitoring dates $\leq t$.

6.2.1 Deterministic Pricing Equation

One then has by application of [16] that $\Pi_t = u(t, S_t, H_t)$ on $[0, T]$, for a pricing function $u = u_k(t, S)$ over $[0, T] \times (0, +\infty) \times \{0,1\}^d$, where the restrictions of the u_ks to every set $[T_{I-1}\, T_I) \times [0, +\infty)$ are continuous, and where the limit $u_k(T_I-, S)$ exists in the sense of (44) for every $k \in \{0,1\}^d$, $I \geq 1$ and $S \neq \bar{S}$, but typically fails to exist for $S = \bar{S}$. The deterministic pricing equation $(\mathcal{VI})$ assumes the form of the following Cauchy cascade of variational inequalities, with

$$k_+ = k_+(k, S) = (k_1, \ldots, k_{d-1}, \mathbf{1}_{S \geq \bar{S}}) \tag{47}$$

in the jump condition (48):

For I decreasing from N to 1:
• At $t = T_I$, for every $k \in \{0,1\}^d$,

$$\begin{array}{l} u_k(T_I-, S) = u_{k_+}(T_I, S), \quad \text{or } \min(u_{k_+}(T_I, S), h(T_I, S)) \text{ if } k \notin K \text{ and } k_+ \in K, \\ \text{for every } S \neq \bar{S}\ , \end{array} \tag{48}$$

Or, in case $I = N$, $u_k(T_I-, S) = g(S)$ for $S > 0$,
• On the time interval $[T_{I-1}, T_I)$, for every $k \in \{0,1\}^d$,

$$\begin{array}{l} \max\left(\mathcal{G}u_k + c - \mu u_k\, ,\ \ell - u_k\right) = 0\, ,\ k \in K \\ \min\left(\max\left(\mathcal{G}u_k + c - \mu u_k\, ,\ \ell - u_k\right), h - u_k\right) = 0\, ,\ k \notin K\ . \end{array}$$

Again this problem can in theory be solved by standard deterministic numerical schemes (provided appropriate care is taken of the singularities of the solution u), but this is now a system of 2^d equations, which precludes the use of this scheme for d greater than, say, ten or so.

6.2.2 Simulation Scheme

To solve this problem by simulation, given a time grid $(t_i)_{0 \leq i \leq n}$ refining the protection monitoring grid $\mathfrak{T} = (T_I)$, we generate a stochastic grid $(S_i^j, H_i^j)_{0 \leq i \leq n}^{1 \leq j \leq m}$ in the obvious way, using past values of S to fill H.

Example 6.2 In the set-up of Example 6.1 and for $d = 30$:
• Simulate $S_0, S_{6\mathrm{h}}, \ldots, S_T$ (with S_0 given) as before;
• Whenever t_i coincides with one of the T_Is (i.e., one every fourth i), update the vector

H, so: $H_0 = (0, \ldots, 0)$ is given as the null in $\{0,1\}^{30}$, $H_{6\mathrm{h}} = H_0$, $H_{12\mathrm{h}} = H_0$, $H_{18\mathrm{h}} = H_0$, $H_{1day} = (\bar{H}_0, \mathbf{1}_{S_{1day} \geq \bar{S}})$ where $\bar{H}_0$ stands for the vector made of the first 29 components (zeros) of H_0, $H_{30\mathrm{h}} = H_{1day}$, and so on until H_T;
• Redo this m times to get m trajectories $(S_i^j, H_i^j)_{0 \leq i \leq n}^{1 \leq j \leq m}$.

We then use the generic simulation scheme of section 3.3 specified to the context of this section, referred to henceforth as $\mathrm{MC}_{l,d}$. Note that H_i^j is now given as a vector of d Boolean variables, instead of an integer in $\{0, \ldots, l\}$ in the MC_l scheme of section 6.1.2. In the present setting, the m simulated trajectories of X are bucketed into 2^d cells, corresponding to the 2^d possible states of the Boolean vector k. For large (but typical, like $d = 30$) values of d, values of k for which the subset Ω_i^k of the indices j such that $H_i^j = k$ is empty or too small for the related non-linear regressions to be doable (see section 3.3), will thus be the rule rather than the exception. However this is in a sense the power of the simulation approach, which automatically selects the most likely states of the vector k *relatively to an initial condition* (S_0, H_0), as opposed to deterministic schemes, which loop over *all the possible states* of k.

6.3 Numerical Experiments

6.3.1 No protection

Note that in case $l = 0$, both the 'l last' and the 'l out of the last d' clause effectively reduce to the no protection (game) case of section 4.2.

Using the data of Section 4.3, we thus computed the prices by the simulation pricing schemes of section 6.1.2 and 6.2.2 (MC_d and $\mathrm{MC}_{0,d}$ in the tables), and we validated the results by the deterministic scheme (31). The accuracy of the results is displayed in Tables 8 and 9. Note that in the situation of this section, the deterministic scheme (31) and the ones of sections 6.1.1 or 6.2.1 produce the same numbers, but in time independent of d for (31), versus linear in d and exponential in d for the ones of sections 6.1.1 and 6.2.1.

Table 10 displays the average computation times for the MC results of Tables 8 and 9, versus those corresponding to numerical solution by the VIs cascade scheme of section 6.2.1 — at least, for $d \leq 10$, since for greater values of d, the VIs times become prohibitively long. An important practical conclusion to be drawn from Table 10 is that in the 'general' 'l out of the last d' case with $0 < l < d$ for which the only available deterministic scheme is the one of section 6.2.1, deterministic schemes are simply ruled out by the curse of dimensionality (unless d is very small), and the simulation scheme of section 6.2.2 is the only viable alternative.

S_0	$d = 1$	5	10	20	30
98.55	0.04	0.04	0.04	0.04	0.04
99.55	0.02	0.02	0.02	0.02	0.02
100.55	0.05	0.05	0.05	0.05	0.05
101.55	0.07	0.07	0.07	0.07	0.07

Table 8: *%Err MC No protection versus MC_d for various '$l = 0$ out of the last d' cases.*

S_0	$d = 1$	5	10	20	30
98.55	0.04	0.04	0.04	0.04	0.04
99.55	0.02	0.02	0.01	0.02	0.02
100.55	0.05	0.05	0.01	0.05	0.05
101.55	0.07	0.07	0.08	0.07	0.07

Table 9: *%Err MC No protection versus $MC_{0,d}$ for various '$l = 0$ out of the last d' cases.*

d	1	5	10	20	30
$\mathrm{VI}_{0,d}$	1.0s	16.1s	465.0s	—	—
MC_d	0.6s	0.6s	0.7s	0.7s	0.7s
$\mathrm{MC}_{0,d}$	0.5s	0.6s	0.9s	1.4s	1.9s

Table 10: *$VI_{0,d}$ versus MC_d Average Computation Times corresponding to Tables 8 and 9.*

6.3.2 'l last'

We now consider bonds with path-dependent call protection as of section 6.1, that we evaluate by the deterministic scheme of section 6.1.1, or, alternatively, by one of the following simulation schemes:

MC_l The 'l last' simulation scheme of section 6.1.2, using methods of cells for the computations of the conditional expectations, or

$\mathrm{MC}_{l,l}$ The 'l out of the last d' simulation scheme of section 6.2.2 with $d = l$ therein, using methods of cells again for the computation of the conditional expectations.

The results are presented in Tables 11 for $S_0 = 100$ and 12 for $S_0 = 90$, using the general data of section 5.3. Both schemes are quite accurate. In these experiments the highly path-dependent, '$l + 1$ − dimensional' scheme $\mathrm{MC}_{l,l}$ ('$l + 1$ − dimensional' in the sense of the stock S plus l Boolean variables), is in fact more accurate, even for large l's, than the path-dependent, 'bi-dimensional' scheme MC_l ('bi-dimensional' in the sense of the stock S plus one integer variable in $\{0, \ldots, l\}$). Table 13 gives the average computation times corresponding to Tables 11 and 12.

6.3.3 'l out of the last d'

We finally price bonds with highly path-dependent call protection as of section 6.2. The bonds are evaluated by the deterministic scheme of section 6.2.1, or, alternatively, by the

l	1	5	10	20	30
VI_l price	103.91	105.10	106.03	107.22	108.01
MC_l %Err	0.04	0.16	0.47	0.88	1.34
$\mathrm{MC}_{l,l}$ %Err	0.04	0.15	0.03	0.04	0.24

Table 11: *'l last' protection ($S_0 = 100$).*

l	1	5	10	20	30
VI_l price	104.07	104.50	104.81	105.17	105.37
MC_l %Err	0.04	0.20	0.32	0.40	0.60
$MC_{l,l}$ %Err	0.098	0.087	0.066	0.007	0.037

Table 12: *'l last' protection ($S_0 = 90$).*

l	1	5	10	20	30
VI_l	1.5s	4.0s	7.0s	13.1s	19.2s
MC_l	1.3s	1.4s	1.4s	1.6s	1.7s
$MC_{l,l}$	1.0s	1.5s	2.0s	3.4s	4.5s

Table 13: *Average computation times relative to Tables 11 and 12.*

'l out of the last d' Monte Carlo scheme of section 6.2.2, using three alternative methods for estimating the conditional expectations involved in the simulation scheme, based on simulated trajectories of the process $X = (S, H)$ as of section 6.2.2:

$MC_{l,d}$ Conditional expectations computed by a method of cells in (S, H) as described in section 6.2.2,

MC_d Conditional expectations computed by a method of cells in $(S, |H|)$, where $|H|$ is the number of ones in H,

$MC^\sharp_{l,d}$ Conditional expectations computed by a method of cells in $(S, |H|^\sharp)$, where $|H|^\sharp$ is defined as the *number of ones in H, starting from the $(l - |H|)^{th}$ zero in H.*

In the general 'l out of the last d' case the MC_d-algorithm is of course biased. So is the the $MC^\sharp_{l,d}$-algorithm, however the latter can be thought of as a 'good' approximate algorithm based on the 'good regressor' $|H|^\sharp$ for estimating the highly path-dependent conditional expectations. The rationale for using for 'preferring' $|H|^\sharp$ to $|H|$ as a regressor, is that in the 'l out of the last d' case, the entries of H preceding its $(l - |H|)^{th}$ zero are of little relevance to the price, since these entries will necessarily have to be superseded by new ones before the bond may become callable.

Example 6.3 Assuming for instance $d = 10$, $l = 8$:
• If $H = (1, 1, 1, 1, \mathbf{0}, 1, 1, 1, 0, 0)$, then $l - |H| = 8 - 7 = 1$ and $|H|^\sharp = 3$ (the number of ones on the right of the first zero, in bold in H),
• If $H = (1, 1, 1, 0, 1, 1, 1, \mathbf{0}, 0, 0)$, then $l - |H| = 8 - 6 = 2$ and $|H|^\sharp = 0$ (the number of ones on the right of the second zero, in bold in H).

The approximate $MC^\sharp_{l,d}$-algorithm in the highly path-dependent 'l out of the last d' case is in fact inspired by the 'exact algorithm' MC_l of section 6.1.2 in the 'l last' case. Note in particular that in case $l = d$, an 'l out of the last d' call protection reduces to an 'l last' call protection, $|H|^\sharp = |H|$, and the $MC^\sharp_{l,d}$-algorithm of this section reduces to the MC_d-algorithm.

Results for $S_0 = 100$ or 90 and $d = 5$ or 10 are presented in Tables 14 to 17. For larger values of d, Table 18 compares $MC_{l,d}$ and $MC^\sharp_{l,d}$ in terms of standard deviations over 50 trials corresponding to different seeds of the random generator, and of relative difference between the $\mathrm{MC}_{l,d}$ and the $\mathrm{MC}^\sharp_{l,d}$ prices.

l	2	3	5
$\mathrm{VI}_{l,d}$ price	104.07	104.43	105.10
MC_d %Err	0.89	1.93	2.33
$\mathrm{MC}_{l,d}$ %Err	0.21	0.15	0.15
$\mathrm{MC}^\sharp_{l,d}$ %Err	0.19	0.23	0.18

Table 14: *'l out of the last d' protection ($d = 5, S_0 = 100$).*

l	2	3	5
$\mathrm{VI}_{l,d}$ price	104.10	104.25	104.50
MC_d %Err	0.33	0.61	0.83
$\mathrm{MC}_{l,d}$ %Err	0.01	0.01	0.10
$\mathrm{MC}^\sharp_{l,d}$ %Err	0.155	0.108	0.034

Table 15: *'l out of the last d' protection ($d = 5, S_0 = 90$).*

l	2	5	10
$\mathrm{VI}_{l,d}$ price	104.27	104.87	106.03
MC_d %Err	0.91	2.43	3.01
$\mathrm{MC}_{l,d}$ %Err	0.01	0.15	0.03
$\mathrm{MC}^\sharp_{l,d}$ %Err	0.04	0.26	0.38

Table 16: *'l out of the last d' protection ($d = 10, S_0 = 100$).*

The 'approximate' $\mathrm{MC}^\sharp_{l,d}$-algorithm appears to be reasonably accurate, and significantly better than the MC_d-algorithm, too severely biased in the general 'l out of the last d' case. The interest of the $\mathrm{MC}^\sharp_{l,d}$-algorithm with respect to the $\mathrm{MC}_{l,d}$-algorithm is of course that it is faster — not 'exponentially' but by some factor, since both schemes are simulation schemes, 'only polynomially' affected by the dimension of the state space. Essentially the complexity of the $\mathrm{MC}^\sharp_{l,d}$-algorithm is the same as that of the MC_d-algorithm (see for instance Table 19 giving the computation times of Table 16, or Table 10 above), so we shall not give more comprehensive computation time reports in this regard.

These results thus illustrate another interesting feature of simulation as opposed to deterministic numerical schemes, namely the possibility to work with a 'good approximate' (even if not 'exact'), low-dimensional regressor.

7 Conclusions

The numerical results of this paper illustrate the good performances of the simulation/regression pricing scheme. Given the theoretical convergence results of [14], one thus ends up with a both practical and mathematically justified approach to the problem of pricing by simulation convertible bonds with highly path-dependent call protection.

l	2	5	10
$\mathrm{VI}_{l,d}$ price	104.24	104.41	104.82
MC_d %Err	0.48	1.23	1.43
$\mathrm{MC}_{l,d}$ %Err	0.01	0.02	0.07
$\mathrm{MC}^{\sharp}_{l,d}$ %Err	0.05	0.09	0.32

Table 17: *'l out of the last d' protection* ($d = 10, S_0 = 90$).

l	5	10	20	30
Dev $\mathrm{MC}_{l,d}$	0.056	0.061	0.086	0.152
Dev $\mathrm{MC}^{\sharp}_{l,d}$	0.060	0.069	0.092	0.175
$\%\mathrm{Err}^{\flat}$	0.09	0.24	0.72	1.06

Table 18: *Standard Deviations over 50 trials and* $\%Err^{\flat}$ ($d = 30, S_0 = 102.55$).

More generally, this paper is an illustration of the power of the simulation approach, which automatically selects and loops over **'the most likely' states** X_i of an underlying, potentially high-dimensional, but also often 'very degenerate', factor process X, **given an initial condition** $X_0 = x$. By contrast, deterministic schemes loop over *all the possible states* of X_i, regardless of their 'likelihood' (but for the choice of the computational domain, typically centered around x). Simulation schemes thus compute **'where light is'**. By contrast, deterministic schemes compute **'everywhere'**, including in 'obscure', and also in a sense useless, regions of the state space. In the context of path-dependent payoffs with a high-dimensional but very degenerate factor process, simulation schemes thus *exploit* the degeneracy of the factor process, whereas the path-dependence literally 'kills' the deterministic schemes.

A Computing Conditional Expectations by Simulation

Pricing game options by simulation ultimately reduces to the numerical computation of conditional expectations, which can simply be done by a combination of simulation and regression tools. In this Appendix we provide a brief and informal review about this, referring the interested reader to, for instance, Chapter 6 of Glasserman [19], for details and references.

Let ξ and X denote real- and $\mathbb{R}^q$-valued square integrable random variables. Under suitable conditions, the conditional expectation $\mathbb{E}(\xi|X)$ is equal to the L^2-projection of ξ over the vector space of random variables spanned by the measurable and bounded functions of X. So, in terms of a basis $(\varphi^l)_{l\in\mathbb{N}}$ of the set of the functions from $\mathbb{R}^q$ to $\mathbb{R}$,

$$\mathbb{E}(\xi|X) = \mathbb{L}(\xi|(\varphi^l(X))_{l\in\mathbb{N}}) ,$$

l	2	5	10
$\mathrm{MC}_{l,d}$	2.0s	2.1s	2.0s
MC_d	1.4s	1.4s	1.4s
$\mathrm{MC}^{\sharp}_{l,d}$	1.5s	1.6s	1.5s

Table 19: *MC Computation times corresponding to Table 16.*

where $\mathbb{L}$ stands for the L^2-projection operator. Given pairs $(X^j, \xi^j)^{1\leq j\leq m}$ simulated independently according to the law of (X, ξ), the conditional expectation $\mathbb{E}(\xi|X)$ may thus be simulated by linear regression of the ξ^js against the $(\varphi^l(X^j))^{1\leq j\leq m}_{1\leq l\leq p}$, where the ***truncation order*** p is a parameter in the method. The computational cost of this 'non-linear regression' is of the order of $\mathrm{O}(mp^2)$ to form the regression matrix, plus the computational time of solving a (typically numerically ill-conditioned) linear system of size p.

We refer the interested reader to the monograph by Györfi et al. [21] for every detail about these simulation/regression approaches for computing the *regression function*

$$x \mapsto \rho(x) = \mathbb{E}(\xi|X = x) \ .$$

In a nutshell, the (truncated) regression basis may be:
• Either *parametric*, i.e., made of functions parameterized by a few parameters, or *non-parametric*, meaning in practice that it is made of functions parameterized by a very large set, like one function by point of a discretization of the state space;
• Either *global*, that is, made of functions supported by the whole state space or with 'large' support, or *local*, to be understood as made of functions with 'small' support.

One typically deals with either a *parametric and global* regression basis, like a regression basis made of a few monomials parameterized by their coefficients, or a *non-parametric and local* basis, like a regression basis made of the indicator functions of the cells of a grid of hypercubes partitioning the state space, a method referred to as the *method of cells* in the body of this article.
Statistic theory tells us that a global basis is preferable in case of a 'regular' regression function $\rho(x)$, especially in case where a good guess is available as for the shape of ρ. This guess can then be used to define the regression basis.
Otherwise a local basis should be preferred, as simpler and often more robust in terms of implementation. Note in this respect that in the simplest case of a method of cells, the "regression" does not involve the solution of a linear system (more precisely the solution of the 'system' is straightforward, the 'regression matrix' being diagonal in this case).

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