Reflected and Doubly Reflected BSDEs with Jumps: A Priori Estimates and Comparison

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September 18, 2008

Abstract. It is now established that under quite general circumstances, including in models with jumps, existence of a solution to a reflected BSDE is guaranteed under mild conditions, whereas existence of a solution to a doubly reflected BSDE is essentially equivalent to the so-called Mokobodski condition. As for uniqueness of solutions, it holds under mild integrability conditions. However, for practical purposes, existence and uniqueness is not enough. In order to give further developments to these results in Markovian set-ups, one also need a (simply or doubly) reflected BSDE to be well-posed, in the sense that the solution satisfies suitable *bound and error estimates*, and one further needs a suitable *comparison theorem*. In this paper we derive such estimates and comparison results. In the last section applicability of the results is illustrated on a pricing problem in finance.

Key words: Reflected BSDEs, Jumps, A priori Estimates, Comparison Theorem, Markovian BSDEs, Finance, Convertible Bonds.

1 Introduction

It is now established that under quite general circumstances, including in models with jumps, existence of a solution to a (simply) reflected BSDE (RBSDE for short in the sequel) is guaranteed under mild conditions, whereas existence of a solution to a *doubly reflected* BSDE (R2BSDE) is equivalent to the so-called *Mokobodski condition*. This condition essentially postulates the existence of a quasimartingale between the barriers (see in particular Hamadène–Hassani [22, Theorem 4.1] and previous works in this direction [12, 23, 27, 28, 20, 18]). As for uniqueness of solutions, it is guaranteed under mild integrability conditions (see e.g. Hamadène–Hassani [22, Remark 4.1]).

However, for practical purposes, existence and uniqueness is not enough. Let us for instance consider the application of R2BSDEs to convertible bonds in finance (see Section 6 and [5, 6, 8]). In this case

*The research of S. Crépey benefited from the support of the Europlace Institute of Finance and of Ito33.

the state-process (first component) Y of a solution to a related R2BSDE may be interpreted in terms of an arbitrage price process for the bond. As demonstrated in [7], the mere existence of a solution to the related R2BSDE is a result with important theoretical consequences in terms of pricing and hedging the bond. Yet, in order to give further developments to these results in Markovian set-ups, we also need the R2BSDE to be *well-posed*, in the sense that the solution satisfies suitable *bound and error estimates*, and we also need a suitable *comparison theorem*.

Now, as opposed to the situation prevailing for RBSDEs (see, e.g., El Karoui et al. [16]), universal a priori estimates cannot be obtained for R2BSDEs. In order to get estimates for R2BSDEs, one needs to specialize the problem a little bit. Likewise, universal comparison theorems do not hold in models with jumps (see [2] for a counter-example in the simple case of a BSDE, without barriers).

Section 2 presents an abstract set-up in which our results are derived, as well as the BSDEs under consideration (Subsection 2.1). In Sections 3 and 4 we establish the a priori bound and error estimates (Theorem 3.2) and our comparison theorem (Theorem 4.2). The a priori error estimates immediately imply uniqueness of a solution to our problems (Subsection 5.1). Assuming an additional martingale representation property and the quasi-left continuity of the barriers, we then give existence results (Subsection 5.2). In Section 6 we show that all the required assumptions are satisfied in the case of the convertible bonds related R2BSDEs, in a rather generic Markovian specification of our abstract set-up. These R2BSDEs thus admit (unique) solutions.

These results can be used to develop a related variational inequality approach in the Markovian case (see [10, 11]).

2 Set-Up

In all the paper we work with a finite time horizon $T > 0$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F}_T = \mathcal{F}$, satisfying the usual conditions of right-continuity and completeness. By default we declare that a *random variable* is $\mathcal{F}$-measurable, and that a *process* is defined on the time interval $[0, T]$ and $\mathbb{F}$-adapted. We may and do assume that all semimartingales are càdlàg, without loss of generality.

Let $B = (B_t)_{t\in[0,T]}$ be a d-dimensional standard Brownian motion. Given an auxiliary measured space $(E, \mathcal{B}_E, \rho)$, where ρ is a non-negative σ-finite measure on $(E, \mathcal{B}_E)$, let $\mu = (\mu(dt, de))_{t\in[0,T],e\in E}$ be an *integer valued random measure* on $\big([0, T] \times E, \mathcal{B}([0, T]) \otimes \mathcal{B}_E\big)$. Denoting $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}_E$ where $\mathcal{P}$ is the predictable sigma field on $\Omega \times [0, T]$, recall that an integer valued random measure μ on $\big([0, T] \times E, \mathcal{B}([0, T]) \otimes \mathcal{B}_E\big)$ is an optional and $\widetilde{\mathcal{P}}$ – sigma finite, $\mathbb{N} \cup \{+\infty\}$ – valued random measure such that $\mu(\omega, \{t\} \times E) \le 1$, identically (Jacod–Shiryaev [25, Definition II.1.13 page 68]; see also [1, 30]).

We assume that the compensator of μ is defined by $\zeta_t(\omega, e)\rho(de)dt$, for a $\widetilde{\mathcal{P}}$-measurable non-negative uniformly bounded (random) function ζ. The motivation for the introduction of the random density ζ is to account for dependence between factors in applications, for instance in the context of financial modeling (see section 6.2 and [10, 11, 3, 9]). We refer the reader to the literature [1, 25, 30] regarding the definition of the integral process of $\widetilde{\mathcal{P}}$-measurable integrands with respect to random measures such as $\mu(dt, de)$, its compensator $dt \otimes \zeta d\rho := \zeta_t(\omega, e)\rho(de)dt$, or its compensatrix (compensated measure) $\widetilde{\mu}(dt, de) = \mu(dt, de) - \zeta_t(\omega, e)\rho(de)dt$.

By default in the sequel, all (in)equalities between random quantities are to be understood $d\mathbb{P}$ – almost surely, $d\mathbb{P} \otimes dt$ – almost everywhere or $d\mathbb{P} \otimes dt \otimes \zeta d\rho$ – almost everywhere, as suitable in the situation at hand. For simplicity we omit all dependences in ω of any process or random function in the notation.

We denote by:
- $|X|$, the (d-dimensional) Euclidean norm of a vector or row vector X in $\mathbb{R}^d$ or $\mathbb{R}^{1\otimes d}$;
- $\mathcal{M}_\rho = \mathcal{M}(E, \mathcal{B}_E, \rho; \mathbb{R})$, the set of measurable functions from $(E, \mathcal{B}_E, \rho)$ to $\mathbb{R}$ endowed with the topology of convergence in measure;

• for $v \in \mathcal{M}_\rho$ and $t \in [0,T]$:

$$|v|_t = \Big[\int_E v(e)^2 \zeta_t(e)\rho(de)\Big]^{\frac{1}{2}} \in \mathbb{R}_+ \cup \{+\infty\}\ ; \tag{1}$$

• $\mathcal{B}(\mathcal{O})$, the Borel sigma field on $\mathcal{O}$, for any topological space $\mathcal{O}$.

Let us now introduce some Banach (or Hilbert, in case of $\mathcal{L}^2$, $\mathcal{H}^2_d$ or $\mathcal{H}^2_\mu$) spaces of processes or random functions:
• $\mathcal{L}^2$, the space of square integrable real-valued ($\mathcal{F}_T$-measurable) random variables ξ such that

$$\|\xi\|_2 := \Big(\mathbb{E}\Big[\xi^2\Big]\Big)^{\frac{1}{2}} < +\infty\,;$$

• $\mathcal{S}^p_d$, for any real $p \geq 2$ (or $\mathcal{S}^p$, in case $d=1$), the space of $\mathbb{R}^d$-valued càdlàg processes X such that

$$\|X\|_{\mathcal{S}^p_d} := \Big(\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t|^p\Big]\Big)^{\frac{1}{p}} < +\infty\,;$$

• $\mathcal{H}^2_d$ (or $\mathcal{H}^2$, in case $d=1$), the space of $\mathbb{R}^{1\otimes d}$-valued predictable processes Z such that

$$\|Z\|_{\mathcal{H}^2_d} := \Big(\mathbb{E}\Big[\int_0^T |Z_t|^2\,dt\Big]\Big)^{\frac{1}{2}} < +\infty\ ;$$

• $\mathcal{H}^2_\mu$, the space of $\widetilde{\mathcal{P}}$-measurable functions $V : \Omega \times [0,T] \times E \to \mathbb{R}$ such that (cf. (1))

$$\|V\|_{\mathcal{H}^2_\mu} := \Big(\mathbb{E}\Big[\int_0^T |V_t|^2_t dt\Big]\Big)^{\frac{1}{2}} = \Big(\mathbb{E}\Big[\int_0^T \int_E V_t(e)^2 \zeta_t(e)\rho(de)dt\Big]\Big)^{\frac{1}{2}} < +\infty\ ;$$

• $\mathcal{A}^2$, the space of finite variation continuous processes K with (continuous and non decreasing) Jordan components $K^{\pm} \in \mathcal{S}^2$ null at time 0;
• $\mathcal{A}^2_i$, the space of non-decreasing processes in $\mathcal{A}^2$.

Remark 2.1 By a slight abuse of notation, we shall also write $\|X\|_{\mathcal{H}^2}$ for $\Big(\mathbb{E}\Big[\int_0^T |X_t|^2\,dt\Big]\Big)^{\frac{1}{2}}$ in the case of a progressively measurable (not necessarily predictable) real-valued process X.

Observe that in particular:
• $\int_0^{\cdot} Z_t dB_t$ and $\int_0^{\cdot}\int_E V_t(e)\tilde{\mu}(dt,de)$ are (true) martingales, for any $Z \in \mathcal{H}^2_d$ and $V \in \mathcal{H}^2_\mu$;
• $K = K^+ - K^-$, and $K^{\pm}$ define mutually singular measures on $\mathbb{R}^+$, for any $K \in \mathcal{A}^2$;
• $K = K^+$, for any $K \in \mathcal{A}^2_i$.

It is worth noting that our results admit a straightforward extension to the case where the Brownian motion B is replaced by a more general continuous local martingale. In this case, the space $\mathcal{H}^2_d$ is defined as the space of $\mathbb{R}^{1\otimes d}$-valued predictable processes Z such that

$$\|Z\|_{\mathcal{H}^2_d} := \Big(\mathbb{E}\Big[\int_0^T |Z_t|^2\,d\langle B\rangle_t\Big]\Big)^{\frac{1}{2}} < +\infty$$

($\|X\|_{\mathcal{H}^2}$ being still defined as $\|X\|_{\mathcal{H}^2} = \Big(\mathbb{E}\Big[\int_0^T X_t^2\,dt\Big]\Big)^{\frac{1}{2}}$, in the case of a progressively measurable real-valued process X).

2.1 Reflected and Doubly Reflected BSDEs

2.1.1 Basic Problems

Let us be given a real-valued random variable *(terminal condition)* ξ, and a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^{1\otimes d})\otimes\mathcal{B}(\mathcal{M}_\rho)$-measurable *driver coefficient* $g:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^{1\otimes d}\times\mathcal{M}_\rho\to\mathbb{R}$. In all the paper, we work under the following Standing Assumptions:
(**H.0**) $\xi\in\mathcal{L}^2$;
(**H.1.i**) $g_\cdot(y,z,v)$ is a progressively measurable process, for any $y\in\mathbb{R}$, $z\in\mathbb{R}^{1\otimes d}$, $v\in\mathcal{M}_\rho$;
(**H.1.ii**) $\|g_\cdot(0,0,0)\|_{\mathcal{H}^2}<+\infty$;
(**H.1.iii**) g is uniformly Λ – Lipschitz continuous with respect to (y,z,v), in the sense that Λ is a constant such that for any $t\in[0,T]$ and $(y,z,v),(y',z',v')\in\mathbb{R}\times\mathbb{R}^{1\otimes d}\times\mathcal{M}_\rho$, identically:

$$|g_t(y,z,v)-g_t(y',z',v')|\le\Lambda(|y-y'|+|z-z'|+|v-v'|_t)\ .$$

We also introduce the *barriers* (or *obstacles*) L and U, such that:
(**H.2.i**) L and U are càdlàg processes in $\mathcal{S}^2$;
(**H.2.ii**) $L_t\le U_t$, $t\in[0,T)$ and $L_T\le\xi\le U_T$, $\mathbb{P}$-a.s.

Definition 2.2 A *solution to the R2BSDE with data* (g,ξ,L,U) is a quadruple (Y,Z,V,K) such that:

$$\left.\begin{array}{l}
(i)\ Y\in\mathcal{S}^2, Z\in\mathcal{H}^2_d, V\in\mathcal{H}^2_\mu, K\in\mathcal{A}^2\\
(ii)\ Y_t=\xi+\displaystyle\int_t^T g_s(Y_s,Z_s,V_s)ds+K_T-K_t\\
\qquad\qquad -\displaystyle\int_t^T Z_s dB_s-\int_t^T\int_E V_s(e)\widetilde{\mu}(ds,de)\ \text{ for any }\ t\in[0,T],\ \mathbb{P}\text{-a.s.}\\
(iii)\ L_t\le Y_t\le U_t\ \text{ for any }\ t\in[0,T],\ \mathbb{P}\text{-a.s.,}\\
\ \text{and}\ \displaystyle\int_0^T(Y_t-L_t)dK_t^+=\int_0^T(U_t-Y_t)dK_t^-=0,\ \mathbb{P}\text{-a.s.}
\end{array}\right\}(\mathcal{E})$$

The inequalities and the integral conditions in $(\mathcal{E})$(iii) are called the *barrier constraints* and the *minimality conditions*, respectively.

Let us now consider the case when there is only one barrier, say, for instance, a lower barrier L. A *solution to the RBSDE with data* (g,ξ,L) is a quadruple (Y,Z,V,K) such that:

$$\left.\begin{array}{l}
(i)\ Y\in\mathcal{S}^2, Z\in\mathcal{H}^2_d, V\in\mathcal{H}^2_\mu, K\in\mathcal{A}^2_i\\
(ii)\ Y_t=\xi+\displaystyle\int_t^T g_s(Y_s,Z_s,V_s)ds+K_T-K_t\\
\qquad\qquad -\displaystyle\int_t^T Z_s dB_s-\int_t^T\int_E V_s(e)\widetilde{\mu}(ds,de)\ \text{ for any }\ t\in[0,T],\ \mathbb{P}\text{-a.s.}\\
(iii)\ L_t\le Y_t\ \text{ for any }\ t\in[0,T],\ \mathbb{P}\text{-a.s. and}\ \displaystyle\int_0^T(Y_t-L_t)dK_t=0,\ \mathbb{P}\text{-a.s.}
\end{array}\right\}(\mathcal{E}')$$

When there is no barrier, we define likewise *solutions to the BSDE with data* (g,ξ).

Remark 2.3 (i) All these definitions (as well as the ones introduced in section 2.1.2 below) admit obvious extensions to problems in which the driving term contains a further finite variation process A (not necessarily absolutely continuous).
(ii) Since the integrands are càdlàg and the integrators lie in $\mathcal{A}^2$ in the minimality conditions, these are equivalent to

$$\int_0^T(Y_{t-}-L_{t-})dK_t^+=0\ ,\ \int_0^T(U_{t-}-Y_{t-})dK_t^-=0\ .$$

2.1.2 Extensions with stopping time

Motivated by applications (see [5, 7, 8]), we now consider two generalizations of the above problems involving a further stopping time $\tau \in \mathcal{T}$.

Reflected BSDE with random terminal time A *solution to a BSDE, resp. RBSDE, resp. R2BSDE with random terminal time* τ is defined as in Definition 2.2, with the only difference being that T is replaced by τ therein (including in the definition of the involved spaces of random variables, processes and random functions; so in particular we assume here that ξ is $\mathcal{F}_\tau$-measurable). A solution to a BSDE with random terminal τ is thus defined over the random time interval $[0,\tau] \subseteq [0,T]$.

In particular we denote in the sequel by $(\bar{\mathcal{E}}')$ the RBSDE with random terminal time τ and data (g,ξ,L) on $[0,\tau]$ (assuming in this case that ξ is $\mathcal{F}_\tau$-measurable). Note that in the special case $\tau = T$, $(\bar{\mathcal{E}}')$ reduces to $(\mathcal{E}')$. So $(\bar{\mathcal{E}}')$ is a first possible generalization of $(\mathcal{E}')$.

Remark 2.4 (i) Given a solution (Y,Z,V,K) to $(\bar{\mathcal{E}}')$ on $[0,\tau]$, let us prolongate (Y,Z,V,K) to the whole interval $[0,T]$ so that on $(\tau,T]$ the prolongated processes and random functions Y, K, Z and V satisfy $Y = Y_\tau$, $K = K_\tau, Z = V = 0$. One thus gets a solution to the RBSDE $(\mathcal{E}')$ with data $(\mathbb{1}_{\cdot\leq\tau} g, \xi, L_{\cdot\wedge\tau})$. Note that the data $(\mathbb{1}_{\cdot\leq\tau} g, \xi, L_{\cdot\wedge\tau})$ satisfy (H.0), (H.1) and (the Assumptions regarding L in) (H.2) on $[0,T]$, provided (g,ξ,L) satisfy (H.0), (H.1) and (H.2) with τ instead of T therein. Given these observations, the estimates and comparison results derived in this paper for solutions to RBSDEs (on $[0,T]$) will thus in effect be applicable to solutions to $(\bar{\mathcal{E}}')$.
(ii) BSDEs with random terminal time were introduced in Darling and Pardoux [13] (without barriers and in a context of Brownian filtrations). In [13], the random terminal time is a priori unbounded, whereas in this paper $0 \leq \tau \leq T$. In this respect, the situation that we consider here is rather elementary.

Upper barrier with delayed activation We shall also consider τ*-R2BSDEs*, namely the generalization of the R2BSDE $(\mathcal{E})$ on $[0,T]$ in which the upper barrier U is inactive before τ. Formally, we replace U by $\bar{U}_t := \mathbb{1}_{\{t<\tau\}}\infty + \mathbb{1}_{\{t\geq\tau\}}U_t$ in $(\mathcal{E})$(iii), with the convention that $0 \times \pm\infty = 0$. The resulting problem is denoted by $(\bar{\mathcal{E}})$. Note that in the special case $\tau = 0$, resp. $\tau = T$, $(\bar{\mathcal{E}})$ reduces to $(\mathcal{E})$, resp. $(\mathcal{E}')$. Thus $(\bar{\mathcal{E}})$ is a generalization of both $(\mathcal{E})$ and $(\mathcal{E}')$.

3 A Priori Bound and Error Estimates

A (càdlàg) *quasimartingale* X can be defined as a difference of two non-negative supermartingales (see sections VI.38 to VI.42 and Appendix 2 of Dellacherie and Meyer [14]; see also Protter [31, Chapter III, section 4]). Among the various decompositions $X = X^1 - X^2$ of a quasimartingale X as a difference of two non-negative supermartingales X^1 and X^2, there exists a (unique) decomposition $X = \bar{X}^1 - \bar{X}^2$, referred to as the *Rao decomposition* of X in the sequel, which is *minimal* in the sense that $X^1 \geq \bar{X}^1$, $X^2 \geq \bar{X}^2$, for any such decomposition $X = X^1 - X^2$ ([14, section VI.40]). Also note that any quasimartingale X *belonging to* $\mathcal{S}^2$ is a *special semimartingale* with canonical decomposition $X = X_0 + M + A$ such that M is a *uniformly integrable martingale* and A is a *predictable finite variation process of integrable variation* ([14, Appendix 2.4]).

We shall now see that when L (resp. U) is a quasimartingale in $\mathcal{S}^2$, we have an explicit representation for the process K^+ (resp. K^-) of a solution to $(\mathcal{E})$ (Lemma 3.1). This will enable us to derive related a priori bound and error estimates in Theorem 3.2.

The results of this section thus extend to R2BSDEs with jumps the results of El Karoui et al. [16] (see also [15] for a survey) regarding RBSDEs in a continuous set-up: representation of K^+ (cf. [16, Proposition 4.2]) and a priori bound and error estimates (cf. [16, Propositions 3.5 and 3.6]).
Note that in El Karoui et al. [16], the representation of K^+ is incidental and the estimates are

universal, whereas in our case, the representation of K^+ or K^- is actually used in the derivation of the estimates, assuming that one of the barriers is a quasimartingale in $\mathcal{S}^2$ (or a suitable limit in $\mathcal{S}^2$ of quasimartingales).
We only state and prove the results regarding L. The results for U follow by considering the problem with data $(-g, -\xi, -L, -U)$.

Lemma 3.1 (i) *Let (Y, Z, V, K) be a solution to $(\mathcal{E})$, in case when L is a quasimartingale in $\mathcal{S}^2$ with canonical decomposition*

$$L_t = L_0 + M_t + A_t \ , \ t \in [0, T] \tag{2}$$

for a uniformly integrable martingale M and a predictable process of integrable variation A. Then

$$dK_t^+ \le \mathbb{1}_{\{Y_t = L_t\}} \big(g_t^-(Y_t, Z_t, V_t)dt + dA_t^- \big) \ , \tag{3}$$

where $A = A^+ - A^-$ is the Jordan decomposition of A.
(ii) *If, in addition,*

$$dA_t^- \le \alpha_t dt \tag{4}$$

for a progressively measurable time-integrable process α, then K^+ is an Lebesgue-absolutely continuous process with density k^+ such that

$$k_t^+ \le \mathbb{1}_{\{Y_t = L_t\}} \big(g_t^-(Y_t, Z_t, V_t) + \alpha_t \big) \ , \ t \in [0, T] \ . \tag{5}$$

Proof Note that (3) immediately implies (5), under condition (4). Therefore it only remains to prove (i). By $(\mathcal{E})$, we have:

$$\begin{aligned} d(Y_t - L_t) = \quad & -g_t(Y_t, Z_t, V_t)dt - d(K_t^+ - K_t^-) - dA_t \\ & + Z_t \, dB_t + \int_E V_t(e) \widetilde{\mu}(dt, de) - dM_t \ . \end{aligned} \tag{6}$$

Besides, we have by application of the Meyer–Tanaka formula to the semimartingale $Y - L$, denoting by Θ the *local time* of $Y - L$ at 0 (see e.g. [31, page 214]):

$$\begin{aligned} & d(Y_t - L_t)^+ = -\mathbb{1}_{\{Y_t > L_t\}} g_t(Y_t, Z_t, V_t)dt \\ & - \mathbb{1}_{\{Y_t > L_t\}} dK_t^+ + \mathbb{1}_{\{Y_t > L_t\}} dK_t^- - \mathbb{1}_{\{Y_t > L_t\}} dA_t \\ & + \mathbb{1}_{\{Y_t > L_t\}} Z_t \, dB_t + \int_E \mathbb{1}_{\{Y_{t-} > L_{t-}\}} V_t(e) \, \widetilde{\mu}(dt, de) - \mathbb{1}_{\{Y_{t-} > L_{t-}\}} dM_t \\ & + \mathbb{1}_{\{Y_{t-} > L_{t-}\}} (Y_t - L_t)^- + \mathbb{1}_{\{Y_{t-} \le L_{t-}\}} (Y_t - L_t)^+ + \frac{1}{2} d\Theta_t \ . \end{aligned} \tag{7}$$

By the lower barrier constraint on Y, we have that

$$(Y - L)^- = 0 \ , \ (Y - L)^+ = Y - L \ , \ \mathbb{1}_{\{Y_{t-} = L_{t-}\}} dK_t^+ = dK_t^+ \ .$$

Whence by identification of (6) and (7):

$$\begin{aligned} & \mathbb{1}_{\{Y_{t-} = L_{t-}\}} \big(Z_t \, dB_t + \int_E V_t(e) \widetilde{\mu}(dt, de) - dM_t \big) = \\ & \qquad \mathbb{1}_{\{Y_t = L_t\}} \big(g_t^+(Y_t, Z_t, V_t)dt + dA_t^+ \big) + \frac{1}{2} d\Theta_t + \mathbb{1}_{\{Y_{t-} = L_{t-}\}} \Delta(Y - L)_t \\ & \qquad + dK_t^+ - \mathbb{1}_{\{Y_t = L_t\}} \big(g_t^-(Y_t, Z_t, V_t)dt + dA_t^- + dK_t^- \big) \ . \end{aligned} \tag{8}$$

Since M is integrable, the second line of (8) defines a non-decreasing integrable process. Denoting its compensator by R and its compensatrix by $\widetilde{R}$, it comes:

$$\begin{aligned} & \mathbb{1}_{\{Y_{t-} = L_{t-}\}} \big(Z_t \, dB_t + \int_E V_t(e) \widetilde{\mu}(dt, de) - dM_t \big) - d\widetilde{R}_t = \\ & \qquad dR_t - \mathbb{1}_{\{Y_t = L_t\}} \big(g_t^-(Y_t, Z_t, V_t)dt + dA_t^- + dK_t^- \big) + dK_t^+ \ . \end{aligned} \tag{9}$$

Note that A^- is predictable, like A (see Dellacherie and Meyer [14, page 129]). Since K^+ is continuous, all terms are predictable in the second line of (9), whence equality *to zero* in (9). In particular:

$$dK_t^+ + dR_t = \mathbb{1}_{\{Y_t = L_t\}}\big(\, g_t^-(Y_t, Z_t, V_t)dt \, + dA_t^- \, + dK_t^- \big) \, , \tag{10}$$

whence

$$dK_t^+ \leq \mathbb{1}_{\{Y_t = L_t\}}\big(\, g_t^-(Y_t, Z_t, V_t)dt \, + dA_t^- \, + dK_t^- \big) \, . \tag{11}$$

Inequality (3) follows by mutual singularity of K^+ and K^-. □

The proof of the following Theorem (a priori bound and error estimates) is deferred to Appendix A.

Theorem 3.2 *We consider a sequence of R2BSDEs of the form considered in Lemma 3.1(i), with data and solutions indexed by n, the data being bounded in the sense that the driver coefficients g^n are Λ – equi-Lipschitz continuous, and for some constant c_1 :*

$$\|\xi^n\|_2^2 + \|g^n_{\cdot}(0,0,0)\|_{\mathcal{H}^2}^2 + \|L^n\|_{\mathcal{S}^2}^2 + \|U^n\|_{\mathcal{S}^2}^2 + \|A^{n,-}\|_{\mathcal{S}^2}^2 \; \leq \; c_1 \; . \tag{12}$$

Then we have for some constant $c(\Lambda)$:

$$\|Y^n\|_{\mathcal{S}^2}^2 + \|Z^n\|_{\mathcal{H}_d^2}^2 + \|V^n\|_{\mathcal{H}_\mu^2}^2 + \|K^{n,+}\|_{\mathcal{S}^2}^2 + \|K^{n,-}\|_{\mathcal{S}^2}^2 \leq c(\Lambda)c_1 \; . \tag{13}$$

Indexing by ^{n,p} the differences $\cdot^n - \cdot^p$, we also have:

$$\begin{aligned} &\|Y^{n,p}\|_{\mathcal{S}^2}^2 + \|Z^{n,p}\|_{\mathcal{H}_d^2}^2 + \|V^{n,p}\|_{\mathcal{H}_\mu^2}^2 + \|K^{n,p}\|_{\mathcal{S}^2}^2 \leq \\ &\quad c(\Lambda)c_1 \Big(\|\xi^{n,p}\|_2^2 + \|g^{n,p}_{\cdot}(Y^n_{\cdot}, Z^n_{\cdot}, V^n_{\cdot})\|_{\mathcal{H}^2}^2 + \|L^{n,p}\|_{\mathcal{S}^2} + \|U^{n,p}\|_{\mathcal{S}^2} \Big) \; . \end{aligned} \tag{14}$$

Assume further that the barriers L^n satisfy the assumptions of Lemma 3.1(ii), so $dA^{n,-} \leq \alpha_t^n dt$ for some progressively measurable processes α^n with $\|\alpha^n\|_{\mathcal{H}^2}$ finite for every $n \in \mathbb{N}$. Then we may replace $\|L^n\|_{\mathcal{S}^2}^2$ and $\|L^{n,p}\|_{\mathcal{S}^2}$ by $\|L^n\|_{\mathcal{H}^2}^2$ and $\|L^{n,p}\|_{\mathcal{H}^2}$ in (12) and (14).

Suppose additionally that $\|\alpha^n\|_{\mathcal{H}^2}$ is bounded over $\mathbb{N}$ and that when $n \to \infty$:
- *$g^n_{\cdot}(Y_{\cdot}, Z_{\cdot}, V_{\cdot})$ $\mathcal{H}^2$-converges to $g_{\cdot}(Y_{\cdot}, Z_{\cdot}, V_{\cdot})$ locally uniformly w.r.t. $(Y, Z, V) \in \mathcal{S}^2 \times \mathcal{H}_d^2 \times \mathcal{H}_\mu^2$, and*
- *(ξ^n, L^n, U^n) $\mathcal{L}^2 \times \mathcal{H}^2 \times \mathcal{S}^2$-converges to (ξ, L, U).*

Then (Y^n, Z^n, V^n, K^n) $\mathcal{S}^2 \times \mathcal{H}_d^2 \times \mathcal{H}_\mu^2 \times \mathcal{S}^2$-converges to a solution (Y, Z, V, K) of $(\mathcal{E})$. Moreover, (Y, Z, V, K) also satisfies (13)–(14) (with "$n = \infty$" therein).

Remark 3.1 (i) By symmetry, analog results are valid when the U^n are quasimartingales in $\mathcal{S}^2$ (with $dA^{n,+} \leq \alpha_t^n dt$ for some progressively measurable processes α^n such that $\|\alpha^n\|_{\mathcal{H}^2}$ is bounded over $n \in \mathbb{N}$, for the last part of the theorem).
(ii) The reader can check by inspection of the proofs in Appendix A that Theorem 3.2 is in fact valid for more general sequences of τ-R2BSDEs (see section 2.1.2), given a further stopping time $\tau \in \mathcal{T}$ (the same for every n).

In the case of RBSDEs like $(\mathcal{E}')$, the following results can be proven along the same lines as Theorem 3.2.

Theorem 3.3 *Let us consider a sequence of RBSDEs, the data being bounded in the sense that the driver coefficients g^n are Λ – equi-Lipschitz continuous, and for some constant c_1 :*

$$\|\xi^n\|_2^2 + \|g^n_{\cdot}(0,0,0)\|_{\mathcal{H}^2}^2 + \|L^n\|_{\mathcal{S}^2}^2 \; \leq \; c_1 \; . \tag{15}$$

Then we have for some constant $c(\Lambda)$:

$$\|Y^n\|_{\mathcal{S}^2}^2 + \|Z^n\|_{\mathcal{H}_d^2}^2 + \|V^n\|_{\mathcal{H}_\mu^2}^2 + \|K^n\|_{\mathcal{S}^2}^2 \leq c(\Lambda)c_1 \; . \tag{16}$$

Indexing by ^{n,p} *the differences* $\cdot^n - \cdot^p$, *we also have:*

$$\|Y^{n,p}\|^2_{\mathcal{S}^2} + \|Z^{n,p}\|^2_{\mathcal{H}^2_d} + \|V^{n,p}\|^2_{\mathcal{H}^2_\mu} + \|K^{n,p}\|^2_{\mathcal{S}^2} \leq \tag{17}$$
$$c(\Lambda)c_1\Big(\|\xi^{n,p}\|^2_2 + \|g^{n,p}_\cdot(Y^n_\cdot, Z^n_\cdot, V^n_\cdot)\|^2_{\mathcal{H}^2} + \|L^{n,p}\|_{\mathcal{S}^2}\Big)\ .$$

If, moreover, the barriers L^n *satisfy the assumptions of Lemma 3.1(ii), then we may replace* $\|L^n\|^2_{\mathcal{S}^2}$ *and* $\|L^{n,p}\|_{\mathcal{S}^2}$ *by* $\|L^n\|^2_{\mathcal{H}^2}$ *and* $\|L^{n,p}\|_{\mathcal{H}^2}$ *in (15) and (17).*

Suppose that when $n \to \infty$:
• $g^n_\cdot(Y_\cdot, Z_\cdot, V_\cdot)$ $\mathcal{H}^2$*-converges to* $g_\cdot(Y_\cdot, Z_\cdot, V_\cdot)$ *locally uniformly w.r.t.* $(Y, Z, V) \in \mathcal{S}^2 \times \mathcal{H}^2_d \times \mathcal{H}^2_\mu$, *and*
• (ξ^n, L^n) $\mathcal{L}^2 \times \mathcal{S}^2$*-converges to* (ξ, L) *(or merely* (ξ^n, L^n) $\mathcal{L}^2 \times \mathcal{H}^2$*-converges to* (ξ, L)*, in case when the barriers* L^n *are as in Lemma 3.1(ii)).*
Then (Y^n, Z^n, V^n, K^n) $\mathcal{S}^2 \times \mathcal{H}^2_d \times \mathcal{H}^2_\mu \times \mathcal{S}^2$*-converges to a solution* (Y, Z, V, K) *of* $(\mathcal{E}')$*. Moreover,* (Y, Z, V, K) *also satisfies (16)–(17) (with "*$n = \infty$*" therein).* □

4 Comparison

In this section we specialize (H.1) to the case where

$$g_t(y, z, v) = \widetilde{g}_t\Big(y, z, \int_E v(e)\eta_t(e)\zeta_t(e)\,\rho(de)\Big)\ , \tag{18}$$

for a $\widetilde{\mathcal{P}}$-measurable non-negative function $\eta_t(e)$ with $|\eta_t|_t$ uniformly bounded, and a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^{1\otimes d}) \otimes \mathcal{B}(\mathbb{R})$-measurable function $\widetilde{g} : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1\otimes d} \times \mathbb{R} \to \mathbb{R}$ such that:
(H.1.i)' $\widetilde{g}_\cdot(y, z, r)$ is a progressively measurable process, for any $y \in \mathbb{R}$, $z \in \mathbb{R}^{1\otimes d}$, $r \in \mathbb{R}$;
(H.1.ii)' $\|\widetilde{g}_\cdot(0, 0, 0)\|_{\mathcal{H}^2} < +\infty$;
(H.1.iii)' $|\widetilde{g}_t(y, z, r) - \widetilde{g}_t(y', z', r')| \leq \Lambda\big(|y - y'| + |z - z'| + |r - r'|\big)$, for any $t \in [0, T]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^{1\otimes d}$ and $r, r' \in \mathbb{R}$;
(H.1.iv)' $r \mapsto \widetilde{g}_t(y, z, r)$ is non-decreasing, for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1\otimes d}$.

Using in particular the fact that

$$\left|\int_E (v(e) - v'(e))\eta_t(e)\zeta_t(e)\,\rho(de)\right| \leq |v - v'|_t|\eta_t|$$

with $|\eta_t|_t$ uniformly bounded, so g defined by (18) satisfies (H.1).

Our next goal is to prove a comparison result for $(\mathcal{E})$ (or $(\mathcal{E}')$, see Remark 4.1(ii)) in this case, thus extending to RBSDEs and R2BSDEs the comparison theorem of Barles et al. [2, Proposition 2.6 page 63] (see also Royer [32]) for classic BSDEs (without barriers). We refer the reader to Barles et al. [2, Remark 2.7 page 64] for a counter-example in the general case, not assuming (H.1.iv)'.

To this end we shall first prove the following Lemma relative to a linear BSDE (without barriers). This BSDE is slightly non-standard inasmuch as its driving term contains a finite variation non absolutely continuous process. This poses no special problem, however (see Remark 2.3(i)).

Lemma 4.1 (Linear BSDE) *Let us be given* $\xi \in \mathcal{L}^2$*, a process* $A \in \mathcal{A}^2$ *and*

$$\widetilde{g}_t(y, z, r) = \beta_t y + z\pi_t^{\mathsf{T}} + \kappa_t r$$

for uniformly bounded predictable real-valued, resp. $\mathbb{R}^{1\otimes d}$*-valued, processes* β *and* κ*, resp.* π*, with* $\kappa\eta > -1$*. Let* (Y, Z, V) *solve the BSDE with terminal condition* ξ *at* T *and driving term defined by, for* $t \in [0, T]$:

$$A_t + \int_0^t \widetilde{g}_s\Big(y, z, \int_E v(e)\eta_s(e)\zeta_s(e)\rho(de)\Big)ds\ .$$

Then, for any $\tau \in \mathcal{T}$:

$$\Gamma_0 Y_0 = \mathbb{E}\Big[\Gamma_\tau Y_\tau + \int_0^\tau \Gamma_s dA_s \,\Big|\mathcal{F}_0\Big] \ , \ \mathbb{P}\text{-}a.s. \tag{19}$$

where the càdlàg adjoint process Γ *is the solution of the following linear (forward) SDE:*

$$d\Gamma_t = \Gamma_{t-}\big(\beta_t dt + \pi_t dB_t + \kappa_t \int_E \eta_t(e)\widetilde{\mu}(dt,de)\big) \ , \ t \in [0,T] \tag{20}$$

with initial condition $\Gamma_0 = 1$. *In particular,* $\Gamma > 0$ *on* $[0,T]$.

Proof. Using (20), the integration by parts formula gives, for $\tau \in \mathcal{T}$:

$$\begin{aligned}
\Gamma_0 Y_0 = \Gamma_\tau Y_\tau &+ \int_0^\tau \Gamma_{s-}\Big[dA_s + \Big(\beta_s Y_s + Z_s \pi_s^{\mathsf{T}} + \kappa_s \int_E V_s(e)\eta_s(e)\zeta_s(e)\rho(de)\Big)\, ds\Big] \\
&- \int_0^\tau \Gamma_{s-} Z_s dB_s - \int_0^\tau \int_E \Gamma_{s-} V_s(e) \widetilde{\mu}(ds,de) \\
&- \int_0^\tau Y_{s-}\Gamma_{s-}\Big(\beta_s ds + \pi_s dB_s + \kappa_s \int_E \eta_s(e)\widetilde{\mu}(ds,de)\Big) \\
&- \int_0^\tau \Gamma_s Z_s \pi_s^{\mathsf{T}}\, ds - \int_0^\tau \int_E \Gamma_{s-} V_s(e)\kappa_s\eta_s(e)\mu(ds,de) \\
= \Gamma_\tau Y_\tau &+ \int_0^\tau \Gamma_s dA_s - \int_0^\tau \Gamma_s \big(Z_s + Y_s \pi_s\big)\, dB_s \\
&- \int_0^\tau \int_E \Gamma_{s-}\big[(1+\kappa_s\eta_s(e))V_s(e) + \kappa_s\eta_s(e)Y_{s-}\big]\widetilde{\mu}(ds,de) \ .
\end{aligned}$$

In particular $\Gamma Y + \int_0^\cdot \Gamma_s dA_s$ is a local martingale. Moreover, $\sup_{[0,T]}|Y|$ belongs to $\mathcal{L}^2$, and so does (by Burkholder's inequality) $\sup_{[0,T]}|\Gamma|$, hence their product is integrable. Thus the local martingale $\Gamma Y + \int_0^\cdot \Gamma_s dA_s$ is a uniformly integrable martingale, whose value at time 0 is the $\mathcal{F}_0$-conditional expectation of its value at the stopping time $\tau \in \mathcal{T}$. This yields (19). Finally, we recognize in Γ the *stochastic exponential* of

$$\Theta := \int_0^\cdot \beta_s ds + \int_0^\cdot \pi_s dB_s + \int_0^\cdot \int_E \kappa_s \eta_s(e)\widetilde{\mu}(ds,de) \ ,$$

which is explicitely given in terms of Θ by

$$\Gamma_t = e^{\Theta_t - \frac{1}{2}\langle \Theta^c\rangle_t} \prod_{0<s\le t} \big(1+\Delta\Theta_s\big)e^{-\Delta\Theta_s} \ , \ t\in[0,T] \ . \tag{21}$$

Therefore $\Gamma > 0$, since $\kappa\eta > -1$. □

Theorem 4.2 *Let* (Y,Z,V,K) *and* (Y',Z',V',K') *be solutions to the R2BSDEs with data* (g,ξ,L,U) *and* (g',ξ',L',U') *satisfying assumptions (H.0)–(H.1)–(H.2). We assume further that* g *satisfies (H.1)'. Then* $Y \le Y'$, $d\mathbb{P}\otimes dt$ *– almost everywhere, whenever:*
(i) $\xi \le \xi'$, $\mathbb{P}$ *– almost surely,*
(ii) $g.(Y'_.,Z'_.,V'_.) \le g'_.(Y'_.,Z'_.,V'_.)$, $d\mathbb{P}\otimes dt$ *– almost everywhere,*
(iii) $L \le L'$ *and* $U \le U'$, $d\mathbb{P}\otimes dt$ *– almost everywhere.*

Proof. We write the proof in case $d = 1$, for notational simplicity. Let us denote $\overline{\xi} = \xi - \xi'$, and for

$t \in [0,T]$:

$$\delta_t = g_t(Y'_t, Z'_t, V'_t) - g'_t(Y'_t, Z'_t, V'_t)$$

$$\beta_t = \begin{cases} \left(Y_t - Y'_t\right)^{-1}\left(g_t(Y_t, Z_t, V_t) - g_t(Y'_t, Z_t, V_t)\right) & \text{if} \quad Y_t \neq Y'_t \\ 0 & \text{if} \quad Y_t = Y'_t, \end{cases}$$

$$\pi_t = \begin{cases} \left(Z_t - Z'_t\right)^{-1}\left(g_t(Y'_t, Z_t, V_t) - g_t(Y'_t, Z'_t, V_t)\right) & \text{if} \quad Z_t \neq Z'_t \\ 0 & \text{if} \quad Z_t = Z'_t \end{cases}$$

$$\kappa_t = \begin{cases} \dfrac{g_t(Y'_t, Z'_t, V_t) - g_t(Y'_t, Z'_t, V'_t)}{\int_E (V_t(e) - V'_t(e))\eta_t(e)\zeta_t(e)\rho(de)} & \text{if} \quad \int_E \left(V_t(e) - V'_t(e)\right)\eta_t(e)\zeta_t(e)\rho(de) \neq 0 \\ 0 & \text{if} \quad \int_E \left(V_t(e) - V'_t(e)\right)\eta_t(e)\zeta_t(e)\rho(de) = 0\,. \end{cases}$$

By assumption (H.1)' on g, we have:

$$g_t(Y'_t, Z'_t, V_t) - g_t(Y'_t, Z'_t, V'_t) = \\ \widetilde{g}_t\Big(Y'_t, Z'_t, \int_E V_t(e)\eta_t(e)\zeta_t(e)\,\rho(de)\Big) - \widetilde{g}_t\Big(Y'_t, Z'_t, \int_E V'_t(e)\eta_t(e)\zeta_t(e)\,\rho(de)\Big)\,.$$

The Lipschitz continuity property of $\widetilde{g}$ with respect to (y, z, r) implies that β, π, κ are real-valued uniformly bounded progressively measurable processes. Moreover $\|\delta\|_{\mathcal{H}^2}$ is finite. Furthermore $\kappa \geq 0$ on $[0,T]$, by assumption (H.1.iv)' on g.

Now, by linearity, $\left(\overline{Y}, \overline{Z}, \overline{V}\right) := \left(Y - Y',\, Z - Z', V - V'\right)$ solves the following linear BSDE with terminal condition $\bar{\xi} = \xi - \xi'$ at T, in which $A_t := K_t - K'_t + \int_0^t \delta_s ds$ (see Remark 2.3(i)):

$$\overline{Y}_t = \bar{\xi} + A_T - A_t + \int_t^T \Big(\overline{Y}_s\beta_s + \overline{Z}_s\pi_s + \kappa_s \int_E \overline{V}_s(e)\eta_s(e)\zeta_s(e)\rho(de)\Big)\,ds \\ - \int_t^T \overline{Z}_s\,dB_s - \int_t^T \int_E \overline{V}_s(e)\widetilde{\mu}(ds, de)\,,\ t \in [0,T]\,.$$

Lemma 4.1 then yields, for any $\tau \in \mathcal{T}$:

$$\Gamma_0 \overline{Y}_0 = \mathbb{E}\Big[\Gamma_\tau \overline{Y}_\tau + \int_0^\tau \Gamma_s\delta_s ds + \int_0^\tau \Gamma_s d(K^+_s + K'^-_s) - \int_0^\tau \Gamma_s d(K'^+_s + K^-_s)\Big|\mathcal{F}_0\Big]\,. \tag{22}$$

Now:
- $\kappa \geq 0$, hence $\Gamma > 0$, by Lemma 4.1;
- $\delta \leq 0$ and dK'^+, $dK^- \geq 0$.

Therefore choosing

$$\tau = \inf\Big\{\,s \in [0,T]\,;\, Y_s = L_s\Big\} \wedge \inf\Big\{\,s \in [0,T]\,;\, Y'_s = U'_s\Big\} \wedge T$$

then $\overline{Y}_\tau \leq 0$ and $K^+ = K'^- = 0$ on $[0,\tau]$, yielding $\overline{Y}_0 \leq 0$, $\mathbb{P}$ – almost surely, by (22). Since time 0 plays no special role in the problem, we have in fact $Y_t \leq Y'_t$, $\mathbb{P}$ – almost surely, for any $t \in [0,T]$. As Y and Y' are càdlàg processes, we conclude that $Y_t \leq Y'_t$ for any $t \in [0,T]$, $\mathbb{P}$ – almost surely. □

Remark 4.1 (i) By inspection of the above proof, it appears that one may relax assumptions (H.1.ii) and (H.1.iii) on g' into $\|g'_\cdot(Y'_\cdot, Z'_\cdot, V'_\cdot)\|_{\mathcal{H}^2} < \infty$ in Theorem 4.2.
(ii) This comparison theorem admits obvious specifications to RBSDEs and BSDEs. We thus recover Barles et al. [2, Proposition 2.6 page 63] (see also Royer [32]).

5 Existence and Uniqueness Results

Recall that $(\bar{\mathcal{E}}')$ is more general than $(\mathcal{E}')$, whereas $(\bar{\mathcal{E}})$ can be considered as a generalization of either $(\mathcal{E})$ or $(\mathcal{E}')$ (see section 2.1.2). So some of the statements are in a sense redundant in Propositions 5.1 and 5.2 below. However we find it convenient to state them explicitly, for more clarity.

5.1 Uniqueness

Proposition 5.1 *Under assumptions (H.0)–(H.1)–(H.2):*
(i) *Uniqueness holds for $(\mathcal{E})$ and $(\mathcal{E}')$;*
(ii) *Given a further stopping time $\tau \in \mathcal{T}$, uniqueness holds for the RBSDE with random terminal time $(\bar{\mathcal{E}}')$ (assuming ξ $\mathcal{F}_\tau$-measurable) and for the τ-R2BSDE $(\bar{\mathcal{E}})$.*

Proof. (i) Uniqueness for $(\mathcal{E}')$ results directly from the error estimate (17). As for $(\mathcal{E})$, careful examination of the proof of estimate (14) in section A.2 shows that in the special case $L^{n,p} = U^{n,p} = 0$, estimate (14) can be *strengthened under weaker Assumptions*, namely we have

$$\|Y^{n,p}\|^2_{\mathcal{S}^2} + \|Z^{n,p}\|^2_{\mathcal{H}^2_d} + \|V^{n,p}\|^2_{\mathcal{H}^2_\mu} + \|K^{n,p}\|^2_{\mathcal{S}^2} \le \\ c(\Lambda)c_1\Big(\|\xi^{n,p}\|^2_{\mathcal{L}^2} + \|g^{n,p}_\cdot(Y^n_\cdot, Z^n_\cdot, V^n_\cdot)\|^2_{\mathcal{H}^2}\Big) \tag{23}$$

for any sequence of R2BSDEs with common barriers L and U and such that

$$\|\xi^n\|^2_2 + \|g^n_\cdot(0,0,0)\|^2_{\mathcal{H}^2} \le c_1$$

(without any of the Assumptions specific to Lemma 3.1). Uniqueness for $(\mathcal{E})$ then directly follows from (23).
(ii) Given Remark 2.4(i), uniqueness for $(\bar{\mathcal{E}}')$ follows from the uniqueness, by part (i), for the RBSDE with data $(\mathbb{1}_{\cdot\le\tau}g, \xi, L_{\cdot\wedge\tau})$. Finally, uniqueness for $(\bar{\mathcal{E}})$ can be established as that for $(\mathcal{E})$ above, given Remark 3.1(ii). □

5.2 Existence

In this section we work under the following *square integrable martingale predictable representation* assumption:
(**H**) Every square integrable martingale M admits a representation

$$M_t = M_0 + \int_0^t Z_s\,dB_s + \int_0^t \int_E V_s(e)\widetilde{\mu}(ds,de)\,,\quad t\in[0,T] \tag{24}$$

for some $Z \in \mathcal{H}^2_d$ and $V \in \mathcal{H}^2_\mu$.

We also strengthen Assumption (H.2.i) into:
(**H.2.i**)′ L and U are càdlàg *quasi-left continuous* processes in $\mathcal{S}^2$.
Recall that for a càdlàg process X, quasi-left continuity is equivalent to the existence of sequence of totally inaccessible stopping times which exhausts the jumps of X, whence ${}^pX = X_{\cdot-}$ (Jacod–Shiryaev [25, Propositions I.2.26 page 22 and I.2.35 page 25]). We thus work in this section under assumptions (H)–(H.0)–(H.1)–(H.2)', where (H.2)' denotes (H.2) with (H.2.i) replaced by (H.2.i)'.

The proof of the following proposition, which is essentially contained in earlier results by Hamadène and Ouknine [21] and Hamadène [22], is given in Appendix B. By *the Mokobodski condition* in this proposition, we mean the existence of a quasimartingale X with Rao components in $\mathcal{S}^2$ and such that $L \le X \le U$ over $[0,T]$. This is of course tantamount to the existence of non-negative supermartingales X^1, X^2 belonging to $\mathcal{S}^2$ and such that $L \le X^1 - X^2 \le U$ over $[0,T]$ (cf. first paragraph of section 3). X is then obviously a quasimartingale in $\mathcal{S}^2$. Note that the question whether any quasimartingale in $\mathcal{S}^2$ has Rao components in $\mathcal{S}^2$ is unsolved, to the best of our knowledge.

Proposition 5.2 *Assuming (H)–(H.0)–(H.1)–(H.2)':*
(i) *Existence holds for $(\mathcal{E}')$ and (assuming that ξ is $\mathcal{F}_\tau$-measurable, here) $(\bar{\mathcal{E}}')$;*
(ii) *Existence of a solution to $(\mathcal{E})$ is equivalent to the Mokobodski condition, which also implies existence of a solution to $(\bar{\mathcal{E}})$. In particular, existence holds for $(\mathcal{E})$, whence $(\bar{\mathcal{E}})$, when L or U is a quasimartingale with Rao components in $\mathcal{S}^2$ (in which case, L or U is obviously a quasimartingale in $\mathcal{S}^2$ as postulated in Lemma 3.1(i)).*

The complete characterization of existence for $(\bar{\mathcal{E}})$ depends of course on the specification of the stopping time τ. Recall that in the special case $\tau = T$, $(\bar{\mathcal{E}})$ reduces to $(\mathcal{E}')$ (whence always a solution to $(\bar{\mathcal{E}})$ in this case), whereas in the special case $\tau = 0$ $(\bar{\mathcal{E}})$ reduces to $(\mathcal{E})$ (whence in this case equivalence between existence of a solution to $(\bar{\mathcal{E}})$ and the Mokobodski condition).

6 An Application in Finance

In the case of the convertible bonds related R2BSDEs in finance (see section 1), the lower barrier L is given by a call payoff functional of the underlying stock price process S, the latter being typically modeled as a jump-diffusion (with possibly random coefficients). This motivates the following developments.

6.1 Abstract Set-Up

Proposition 6.1 *Let S be given as an Itô-Lévy process with square integrable special semimartingale decomposition components, so*

$$S_t = S_0 + \int_0^t a_s ds + \int_0^t z_s dB_s + \int_0^t \int_E v_s(e)\widetilde{\mu}(ds, de)\ ,\ t \in [0, T] \tag{25}$$

for some $z \in \mathcal{H}^2_d$, $v \in \mathcal{H}^2_\mu$, and a progressively measurable time-integrable process a such that $\|a\|_{\mathcal{H}^2} < +\infty$. Let in turn L be given as $L = S \vee c$, for some constant $c \in \mathbb{R} \cup \{-\infty\}$.
Then L is a (càdlàg) quasi-left continuous quasimartingale with Rao components in $\mathcal{S}^2$. Moreover L satisfies all the conditions in Lemma 3.1 (including the hypotheses on L in (H.2)), with in particular a^-, the negative part of a in (25), for α in (4)–(5).

Proof. We have by the Meyer–Tanaka (or simply Itô-Lévy, in case $c = -\infty$) formula, much like in the proof of Lemma 3.1:

$$\begin{aligned} dL_t = {} & \mathbb{1}_{\{S_t>c\}} z_t\, dB_t + \int_E \mathbb{1}_{\{S_{t-}>c\}} v_t(e)\, \widetilde{\mu}(dt, de) - \mathbb{1}_{\{S_t>c\}} a^-_t\, dt \\ & + \mathbb{1}_{\{S_{t-}>c\}} (S_t - c)^- + \mathbb{1}_{\{S_{t-}\le c\}} (S_t - c)^+ + \frac{1}{2} d\Theta_t + \mathbb{1}_{\{S_t>c\}} a^+_t\, dt \end{aligned} \tag{26}$$

where Θ is the local time of S at c (or 0, in case $c = -\infty$). We thus have for $t \in [0,T]$:

$$\begin{aligned} L_t = \mathbb{E}\Bigg[& L_T - \int_t^T \mathbb{1}_{\{S_u>c\}} a_u du - \frac{1}{2}(\Theta_T - \Theta_t) \\ & - \sum_{t<u\le T} \mathbb{1}_{\{S_{u-}>c\}} (S_u - c)^- + \mathbb{1}_{\{S_{u-}\le c\}} (S_u - c)^+ \big| \mathcal{F}_t \Bigg] = L^1_t - L^2_t \end{aligned} \tag{27}$$

where we set, for $t \in [0,T]$:

$$\begin{aligned} L^1_t &= \mathbb{E}\big[L^+_T + \int_t^T \mathbb{1}_{\{S_u>c\}} a^-_u du \big| \mathcal{F}_t\big] \\ L^2_t &= \mathbb{E}\big[L^-_T + \int_t^T \mathbb{1}_{\{S_u>c\}} a^+_u du + \frac{1}{2}(\Theta_T - \Theta_t) + \\ & \qquad \sum_{t<u\le T} \mathbb{1}_{\{S_{u-}>c\}} (S_u - c)^- + \mathbb{1}_{\{S_{u-}\le c\}} (S_u - c)^+ \big| \mathcal{F}_t\big] \end{aligned}$$

Here L^1 and L^2 are non-negative supermartingales, as optional projections of non-increasing processes. Moreover, L and L^1, and thus, in turn, L^2, belong to $\mathcal{S}^2$. L is therefore a quasimartingale with Rao components in $\mathcal{S}^2$.

Observe further that the second line of (26) defines a non-decreasing integrable process. Denoting by R and $\widetilde{R}$ its compensator and its compensatrix, we get:

$$dL_t = \mathbb{1}_{\{S_t>c\}} z_t \, dB_t + \int_E \mathbb{1}_{\{S_{t-}>c\}} v_t(e) \, \widetilde{\mu}(dt, de) - d\widetilde{R}_t \\ + dR_t - \mathbb{1}_{\{S_t>c\}} a_t^- \, dt \, . \tag{28}$$

So the predictable finite variation component A of L is given by $A = R - \int_0^\cdot \mathbb{1}_{\{S_t>c\}} a_t^- \, dt$, where R and $\int_0^\cdot \mathbb{1}_{\{S_t>c\}} a_t^- \, dt$ are non-decreasing processes, thus the Jordan component A^- of A satisfies $dA_t^- \le \mathbb{1}_{\{S_t>c\}} a_t^- \, dt$. □

6.2 Jump–Diffusion Setting with Regimes

Motivated by applications (see [9, 7, 10, 11, 4]), we now present a rather generic specification for a Markovian model F (which in the context of financial applications will correspond to a Markovian *factor process* underlying a financial derivative), and we show how it fits into the abstract set-up of the present paper.

6.2.1 Specification of the Model

Given integers d and k, we define the following linear operator $\mathcal{G}$ acting on regular functions $u = u^i(t,x)$ for $(t,x,i) \in [0,T] \times \mathbb{R}^d \times I$, where $I = \{1, \cdots, k\}$:

$$\mathcal{G}u^i(t,x) = \partial_t u^i(t,x) + \tfrac{1}{2} \sum_{l,q=1}^d a^i_{l,q}(t,x) \partial^2_{x_l x_q} u^i(t,x) \\ + \sum_{l=1}^d \Big(b^i_l(t,x) - \int_{\mathbb{R}^d} \delta^i_l(t,x,y) f^i(t,x,y) m(dy) \Big) \partial_{x_l} u^i(t,x) \\ + \int_{\mathbb{R}^d} \big(u^i(t, x + \delta^i(t,x,y)) - u^i(t,x) \big) f^i(t,x,y) m(dy) \\ + \sum_{j \in I} \lambda_{i,j}(t,x) (u^j(t,x) - u^i(t,x)) \, . \tag{29}$$

In this equation, $m(dy)$ is a *finite jump measure* on $\mathbb{R}^d$, and all the coefficients are Borel-measurable functions such that:
• the $a^i(t,x)$ are d-dimensional *covariance* matrices, with $a^i(t,x) = \sigma^i(t,x)\sigma^i(t,x)^\top$ for some d-dimensional *dispersion* matrices $\sigma^i(t,x)$:
• the $b^i(t,x)$ are d-dimensional *drift* vector coefficients;
• the *intensity* functions $f^i(t,x,y)$ are bounded, and the *jump size* functions $\delta^i(t,x,y)$ are absolutely integrable with respect to $m(dy)$;
• the $[\lambda_{i,j}(t,x)]_{i,j \in I}$ are *intensity* matrices such that the $\lambda_{i,j}(t,x)$ are non-negative and bounded for $i \neq j$, and $\lambda_{i,i}(t,x) = -\sum_{j \in I \setminus \{i\}} \lambda_{i,j}(t,x)$.
We shall often find convenient to denote $v(t,x,i,\cdots)$ rather than $v^i(t,x,\cdots)$ for a function v of $(t,x,i,\cdots)$, and $\lambda(t,x,i,j)$, for $\lambda_{i,j}(t,x)$. For instance, the notation $f(t,X_t,N_t,y)$ (or even $f(t,F_t,y)$, with $F_t = (X_t,N_t)$ below) will typically be used rather than $f^{N_t}(t,X_t,y)$. Also note that a function u on $[0,T] \times \mathbb{R}^d \times I$ may equivalently be referred to as a system $u = (u^i)_{i \in I}$ of functions $u^i = u^i(t,x)$ on $[0,T] \times \mathbb{R}^d$.

The construction of a model corresponding to the previous data is a non-trivial issue treated in detail in [10] (see also [11], or see Theorems 4.1 and 5.4 in Chapter 4 of Ethier and Kurtz [19] for

abstract conditions regarding the existence and uniqueness of a solution to the martingale problem with generator $\mathcal{G}$). We will thus be rather formal at this point of the present paper, referring the reader to [10, 11] for the complete statement of "suitable conditions" below.
So "under suitable conditions" (see [10, 11]), there exists a stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$ on $[0, T]$ endowed with a d-dimensional Brownian motion B, an *integer-valued random measure* χ and an $(\Omega, \mathbb{F}, \mathbb{P})$-Markov càdlàg process $F = (X, N)$ on $[0, T]$ with initial condition (x, i) at time 0, such that:
• defining ν as the integer-valued random measure on I which counts the transitions $\nu_t(j)$ of N to state j between time 0 and time t, the $\mathbb{P}$-compensatrix $\widetilde{\nu}$ of ν is given by

$$d\widetilde{\nu}_t(j) = d\nu_t(j) - \mathbb{1}_{\{N_t \neq j\}} \lambda(t, F_t, j)\, dt \tag{30}$$

(with $\lambda(s, F_t, j) = \lambda_{N_t, j}(s, X_t)$), whence the following canonical special semimartingale representation for N :

$$dN_t = \sum_{j \in I} \lambda(t, F_t, j)(j - N_t)\, dt + \sum_{j \in I} (j - N_{t-})\, d\widetilde{\nu}_t(j)\ ,\ t \in [0, T]\ ; \tag{31}$$

• the $\mathbb{P}$-compensatrix $\widetilde{\chi}$ of χ is given by

$$\widetilde{\chi}(dt, dy) = \chi(dt, dy) - f(t, F_t, y) m(dy) dt,$$

and the $\mathbb{R}^d$-valued process X satisfies, for $t \in [0, T]$:

$$dX_t = b(t, F_t)\, dt + \sigma(t, F_t)\, dB_t + \int_{\mathbb{R}^d} \delta(t, F_{t-}, y)\, \widetilde{\chi}(dy, dt)\ . \tag{32}$$

Besides the following estimates are available, for any $p \in [2, +\infty)$:

$$\|X\|^p_{\mathcal{S}^p_d} \le C_p \big(1 + |x|^p\big)\ . \tag{33}$$

We then have the following variant of the Itô formula (see, e.g., Jacod [24, Theorem 3.89 page 109]), where ∂u denotes the row-gradient of $u = u^i(t, x)$ with respect to x :

$$\begin{aligned} du(t, F_t) &= \mathcal{G}u(t, F_t) dt + \partial u(t, F_t) \sigma(t, F_t) dB_t \\ &\quad + \int_{\mathbb{R}^d} \big(u(t, X_{t-} + \delta(t, F_{t-}, y), N_{t-}) - u(t, F_{t-})\big) \widetilde{\chi}(dy, dt) \\ &\quad + \sum_{j \in I} (u(t, X_{t-}, j) - u(t, F_{t-})) d\widetilde{\nu}_t(j)\ ,\ t \ge 0 \end{aligned} \tag{34}$$

for any system $u = (u^i)_{i \in I}$ of functions $u^i = u^i(t, x)$ of class $\mathcal{C}^{1,2}$ on $[0, T] \times \mathbb{R}^d$. In particular $(\Omega, \mathbb{F}, \mathbb{P}, F)$ is a solution to the *time-dependent local martingale problem* with generator $\mathcal{G}$ and initial condition (t, x, i) (see Ethier and Kurtz [19, sections 7.A and 7.B]).

Finally, still "under suitable conditions" (see [11]), every $(\Omega, \mathbb{F}, \mathbb{P})$-square integrable martingale M in this model admits a representation

$$M_t = M_0 + \int_0^t Z_s\, dB_s + \int_0^t \int_{\mathbb{R}^d} \widetilde{V}_s(y) \widetilde{\chi}(dy, ds) + \sum_{j \in I} \int_0^t \widetilde{W}_s(j) d\widetilde{\nu}_s(j)\ ,\quad t \in [0, T] \tag{35}$$

for some $Z \in \mathcal{H}^2_d$, $\widetilde{V} \in \mathcal{H}^2_\chi$ and $\widetilde{W} \in \mathcal{H}^2_\nu$.

6.2.2 Mapping with the Abstract Set-Up

Let 0_d stand for the null in $\mathbb{R}^d$. The model $F = (X, N)$ is thus a rather generic Markovian specification of our abstract set-up, with (cf. section 2):

• E, the subset $\big(\mathbb{R}^d \times \{0\}\big) \cup \big(\{0_d\} \times I\big)$ of $\mathbb{R}^{d+1}$;

• $\mathcal{B}_E$, the sigma field generated by $\mathcal{B}(\mathbb{R}^d)\times\{0\}$ and $\{0_d\}\times\mathcal{I}$ on E, where $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{I}$ stand for the Borel sigma field on $\mathbb{R}^d$ and the sigma field of all parts of I, respectively;

• $\rho(de)$ and $\zeta_t(e)$ respectively given by, for any $e=(y,j)\in E$:

$$\rho(de)=\begin{cases} m(dy) & \text{if} \quad j=0\\ 1 & \text{if} \quad y=0_d\end{cases}, \quad \zeta_t(e)=\begin{cases} f(t,F_t,y) & \text{if} \quad j=0\\ \mathbb{1}_{\{N_t\neq j\}}\lambda(t,F_t,j) & \text{if} \quad y=0_d\end{cases}$$

• μ, the integer-valued random measure on $\big([0,T]\times E,\mathcal{B}([0,T])\otimes\mathcal{B}_E\big)$ counting the jumps of X of size $y\in A$ and the jumps of N to state j between 0 and t, for any $t\geq 0$, $A\in\mathcal{B}(\mathbb{R}^d), j\in I$.

We denote for short:

$$(E,\mathcal{B}_E,\rho)=(\mathbb{R}^d\oplus I,\mathcal{B}(\mathbb{R}^d)\oplus\mathcal{I},m(dy)\oplus\mathbb{1})\ .$$

So, in the present context:

$$\mathcal{M}_\rho\equiv\mathcal{M}(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d),m(dy);\mathbb{R})\times\mathbb{R}^k \tag{36}$$

and the compensator of μ is given by, for any $t\geq 0, A\in\mathcal{B}(\mathbb{R}^d), j\in I$, with $A\oplus\{j\}:=\big(A\times\{0\}\big)\cup\big(\{0_d\}\times\{j\}\big)$:

$$\int_0^t\int_{A\oplus\{j\}}\zeta_s(e)\rho(de)ds=\int_0^t\int_A f(s,F_s,y)m(dy)ds+\int_0^t\mathbb{1}_{\{N_s\neq j\}}\lambda(s,F_s,j)\,ds\ .$$

Note finally that (35) is a martingale representation of the form (24), with for $e=(y,j)$:

$$V_s(de)=\begin{cases}\widetilde{V}_s(y)\text{ if } j=0\\ \widetilde{W}_s(j)\text{ if } y=0_d\ .\end{cases}$$

Hence the model F has the martingale representation property (H).

6.3 Markovian BSDEs

We consider in this model the BSDE naturally connected with the Itô formula (34), namely for $t\geq 0$:

$$-\,dY_t=g(t,F_t,Y_t,Z_t,V_t)dt-Z_tdB_t-\int_{\mathbb{R}^d}\widetilde{V}_t(y)\widetilde{\chi}(dy,dt)-\sum_{j\in I}\widetilde{W}_t(j)d\widetilde{\nu}_t(j)$$

with $V=(\widetilde{V},\widetilde{W})$, possibly supplemented by suitable barrier and minimality conditions, and for a suitable driver coefficient $g(t,F_t,y,z,v)$ where $v=(\widetilde{v},\widetilde{w})\in\mathcal{M}(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d),m(dy);\mathbb{R})\times\mathbb{R}^k$ (cf. (36)).

Let P denote the class of functions u on $[0,T]\times\mathbb{R}^d\times I$ such that u^i is Borel-measurable with polynomial growth in x for any $i\in I$. Let us further be given real-valued continuous *running cost functions* $\widetilde{g}_i(t,x,u,z,r)$ (where $(u,z,r)\in\mathbb{R}^k\times\mathbb{R}^{1\otimes d}\times\mathbb{R}$), *terminal cost functions* $\Psi^i(x)$, and *lower and upper obstacle functions* $\ell^i(t,x)$ and $h^i(t,x)$, such that:
(**M.0**) Ψ lies in P;
(**M.1.i**) $(t,x,i)\mapsto\widetilde{g}_i(t,x,0,0,0)$ lies in P;
(**M.1.ii**) $\widetilde{g}$ is uniformly $\Lambda-$Lipschitz continuous with respect to (u,z,r), in the sense that Λ is a constant such that for every $(t,x,i)\in[0,T]\times\mathbb{R}^d\times I$ and $(u,z,r),(u',z',r')\in\mathbb{R}^k\times\mathbb{R}^{1\otimes d}\times\mathbb{R}$:

$$|\widetilde{g}_i(t,x,u,z,r)-\widetilde{g}_i(t,x,u',z',r')|\leq\Lambda\left(|u-u'|+|z-z'|+|r-r'|\right)\ ;$$

(**M.1.iii**) $\widetilde{g}$ is non-decreasing with respect to r;
(**M.2.i**) ℓ and h lie in P;
(**M.2.ii**) $\ell\leq h$, $\ell(T,\cdot)\leq\Psi\leq h(T,\cdot)$;

We define for any $(t,y,z,v)\in[t,T]\times\mathbb{R}\times\mathbb{R}^{1\otimes d}\times\mathcal{M}_\rho$, with $v=(\widetilde{v},\widetilde{w})\in\mathcal{M}(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d),m(dy);\mathbb{R})\times\mathbb{R}^k$:

$$g(t,F_t,y,z,v)=\widetilde{g}(t,F_t,\widetilde{u}_t,z,\widetilde{r}_t)-\sum_{j\in I\setminus\{N_t\}}w_j\lambda(t,F_t,j)\ , \tag{37}$$

where $\widetilde{u}_t=\widetilde{u}_t(y,\widetilde{w})$ and $\widetilde{r}_t=\widetilde{r}_t(\widetilde{v})$ are defined as

$$(\widetilde{u}_t)^j=\begin{cases} y, & j=N_t \\ y+\widetilde{w}_j, & j\neq N_t\end{cases}\ ,\ \widetilde{r}_t=\int_{\mathbb{R}^d}\widetilde{v}(y)f(t,F_t,y)m(dy)\ . \tag{38}$$

We then consider the data

$$g_t(\omega,y,z,v)=g(t,F_t,y,z,v)\ ,\ \xi=\Psi(F_T)\ ,\ L_t=\ell(t,F_t)\ ,\ U_t=h(t,F_t)\ . \tag{39}$$

Remark 6.1 The connection between the Markovian R2BSDEs with data of the form (39) and the Markovian R2BSDEs which appear in risk-neutral pricing problems in finance (see [7]) is established in [10] (see also [11]).

Proposition 6.2 *The data (39) satisfy assumptions (H.0)–(H.1)–(H.2)'.*

Proof. Given (M.0)–(M.1)–(M.2) and the estimate (33) on X, the verification of (H.0)–(H.1)–(H.2)' is straightforward (see [10] for every detail). □

Within model F we are able to specify a concrete class of processes S which satisfy the conditions of Proposition 6.1. We thus have the following

Lemma 6.3 *Let $\phi=(\phi^i)_{i\in I}$ be a system of real-valued functions $\phi^i=\phi^i(t,x)$ of class $\mathcal{C}^{1,2}$ on $[0,T]\times\mathbb{R}^d$ such that*

$$\phi,\,\mathcal{G}\phi,\,\partial\phi\sigma,\,(t,x,i)\mapsto\int_{\mathbb{R}^d}|\phi^i(t,x+\delta^i(t,x,y))|m(dy)\ \in P\ . \tag{40}$$

Then the process S defined by, for $t\in[0,T]$:

$$S_t=\phi(t,F_t)\ ,$$

is an Itô-Lévy process with square integrable special semimartingale decomposition components, with related process a in (25) given as $a_t=\mathcal{G}\phi(t,F_t)$, for $t\in[0,T]$.

Proof. Under our polynomial growth assumptions and given the estimates (33) on X, the result follows by application of the Itô formula (34) to $\phi(t,F_t)$. □

Example 6.2 The standing example we have in mind for S in Proposition 6.1 is $S=X^1$, the first component of X of our model $F=(X,N)$ (assuming $d\geq1$ therein). This corresponds to the case where $\phi^i(t,x)=x_1$ in Lemma 6.3. Note that in this case:

$$\mathcal{G}\phi=b_1,\ \partial\phi\sigma=\sigma_1,\ \int_{\mathbb{R}^d}|\phi^i(t,x+\delta^i(t,x,y))|m(dy)=\int_{\mathbb{R}^d}|x_1+\delta^i_1(t,x,y)|m(dy)\ ,$$

so that (40) reduces to

$$b_1,\,\sigma_1,\,(t,x,i)\mapsto\int_{\mathbb{R}^d}|\delta^i_1(t,x,y)|m(dy)\in P\ . \tag{41}$$

Theorem 6.4 *Given the data (39) with ℓ specified as $\phi \vee c$ where ϕ satisfies (40) (e.g., $\phi = x_1$, assuming (41)) and for some constant $c \in \mathbb{R} \cup \{-\infty\}$, then the related R2BSDE $(\mathcal{E})$ admits a unique solution (Y, Z, V, K). Moreover K^+ is an Lebesgue-absolutely continuous process with density k^+ satisfying (5). The RBSDE $(\mathcal{E}')$ also admits a unique solution. Finally, given a further stopping time $\tau \in \mathcal{T}$, the RBSDE with random terminal time $(\bar{\mathcal{E}}')$ (assuming ξ $\mathcal{F}_\tau$-measurable, here) and the τ-R2BSDE $(\bar{\mathcal{E}})$ also have unique solutions.*

Proof. First, our model F has the martingale representation property (H) (see end of section 6.2.2). Moreover assumptions (H.0)–(H.1)–(H.2)' are satisfied, by Proposition 6.2. Finally, L is a quasimartingale with Rao components in $\mathcal{S}^2$, by application of Proposition 6.1 and Lemma 6.3 (see also Example 6.2 in case $\phi = x_1$). Therefore $(\mathcal{E})$ admits a unique solution (Y, Z, V, K), by Proposition 5.2(i). Moreover all the conditions of Lemma 3.1(ii) are fulfilled, by Proposition 6.1. Consequently K^+ is an Lebesgue-absolutely continuous process with density k^+ satisfying (5). The remaining results follow likewise by application of Proposition 5.2. □

A Proof of Theorem 3.2

In this appendix, c denotes a "large" constant which may change from line to line. We do not track the dependency of the constants line after line, letting the reader check in the end that the overall dependency is indeed like stated in Theorem 3.2.

A.1 Proof of the bound estimate

We have to show that there exists a constant c with the required dependencies such that, for any $t \in [0, T]$ and $n \in \mathbb{N}$:

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t^n|^2 + \int_0^T |Z_s^n|^2 ds + \int_0^T \int_E |V_s^n(e)|^2 \zeta_s(e)\rho(de)ds + (K_T^{n,+})^2 + (K_T^{n,-})^2\Big] \le c \,. \tag{42}$$

We omit indices ^{n} in the rest of this section, to alleviate the notation. Standard computations based on Itô's formula and Gronwall's Lemma yield:

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T Y_s^2 ds + \int_0^T |Z_s|^2 ds + \int_0^T \int_E |V_s(e)|^2 \zeta_s(e)\rho(de)ds\Big] \\ &\qquad \le c\,\mathbb{E}\Big[\xi^2 + \int_0^T g_s^2(0,0,0)ds + \int_0^T |L_s| dK_s^+ + \int_0^T |U_s| dK_s^-\Big] \,. \end{aligned} \tag{43}$$

Besides, using (3) and the Lipschitz continuity property of g, we have:

$$\begin{aligned} \mathbb{E}\Big[(K_T^+)^2\Big] &\le \mathbb{E}\Big[(A_T^-)^2 + \int_0^T g_s^2(0,0,0)ds + \int_0^T |Y_s|^2 ds + \int_0^T |Z_s|^2 ds \\ &\qquad + \int_0^T \int_E |V_s(e)|^2 \nu(ds, de)\Big] \\ &\le \mathbb{E}(A_T^-)^2 + c\mathbb{E}\Big[\xi^2 + \int_0^T g_s^2(0,0,0)ds + \int_0^T |L_s| dK_s^+ + \int_0^T |U_s| dK_s^-\Big] \,, \end{aligned} \tag{44}$$

by (43). Moreover, we have likewise by the related R2BSDE:

$$\mathbb{E}\Big(K_T^+ - K_T^-\Big)^2 \le c\mathbb{E}\Big[\xi^2 + \int_0^T g_s^2(0,0,0)ds + \int_0^T |L_s| dK_s^+ + \int_0^T |U_s| dK_s^-\Big] \,. \tag{45}$$

So, combining (44) and (45):

$$\mathbb{E}\Big[(K_T^+)^2+(K_T^-)^2\Big] \le c\mathbb{E}\Big[\xi^2+(A_T^-)^2+\int_0^T g_s^2(0,0,0)ds+\sup_{0\le s\le T}L_s^2+\sup_{0\le s\le T}U_s^2\Big] \tag{46}$$

and finally

$$\mathbb{E}\Big[|Y_t|^2+\int_0^T|Z_s|^2ds+\int_0^T\int_E|V_s(e)|^2\zeta_s(e)\rho(de)ds+(K_T^+)^2+(K_T^-)^2\Big] \le c\mathbb{E}\Big[\xi^2+(A_T^-)^2+\int_0^T g_s^2(0,0,0)ds+\sup_{0\le s\le T}L_s^2+\sup_{0\le s\le T}U_s^2\Big]\,. \tag{47}$$

Applying Itô's formula to Y^2 again, and taking first suprema in time, then expectations, we deduce (42) by the Burkholder inequality.

Moreover, in the case $dA^{n,-}\le\alpha_t^n dt$ for some progressively measurable processes α^n with $\|\alpha^n\|_{\mathcal{H}^2}$ finite, we have by application of Lemma 3.1(ii):

$$dK^{n,+}=k_t^{+,n}dt \text{ with } k_t^{+,n}\le \mathbb{1}_{\{Y_t^n=L_t^n\}}\big(g_t^n(Y_t^n,Z_t^n,V_t^n)^-+\alpha_t^n\big)\,.$$

In particular $\|k^{n,+}\|_{\mathcal{H}^2}$ is finite, by the previous results. One may then replace $\sup_{0\le s\le T}L_s^2$ by $\int_0^T L_s^2ds$ in (46) and (47), and then in turn $\|L^n\|_{\mathcal{S}^2}^2$ by $\|L^n\|_{\mathcal{H}^2}^2$ in (12).

A.2 Proof of the error estimate (14)

Expliciting indices ^{n} and ^{p} again, we get by the Itô formula and the Lipschitz continuity property of g, with "$\dot{\le}$" standing for "$\le$ *up to a martingale term*":

$$\begin{aligned}&(Y_t^n-Y_t^p)^2+\int_t^T|Z_s^n-Z_s^p|^2ds+\int_t^T\int_E|V_s^n(e)-V_s^p(e)|^2\zeta_s(e)\rho(de)ds\ \dot{\le}\\ &|\xi^n-\xi^p|^2+2\int_t^T|g_s^n(Y_s^n,Z_s^n,V_s^n)-g_s^p(Y_s^n,Z_s^n,V_s^n)|^2\,ds\\ &+c\int_t^T|Y_s^n-Y_s^p|^2\,ds+\frac12\int_t^T|Z_s^n-Z_s^p|^2\,ds\\ &+\frac12\int_t^T\int_E|V_s^n(e)-V_s^p(e)|^2\zeta_s(e)\rho(de)ds+2\int_t^T(Y_s^n-Y_s^p)(dK_s^n-dK_s^p)\,.\end{aligned}$$

Now, by the barriers conditions:

$$\int_t^T(Y_s^n-Y_s^p)(dK_s^n-dK_s^p)\le \int_t^T(L_s^n-L_s^p)(dK_s^{n,+}-dK_s^{p,+})-(U_s^n-U_s^p)(dK_s^{n,-}-dK_s^{p,-})\,. \tag{48}$$

Thus

$$\begin{aligned}&\mathbb{E}\Big[|Y_t^n-Y_t^p|^2+\frac12\int_t^T|Z_s^n-Z_s^p|^2ds+\frac12\int_t^T\int_E|V_s^n(e)-V_s^p(e)|^2\zeta_s(e)\rho(de)ds\Big]\le\\ &c\,\mathbb{E}\Big[|\xi^n-\xi^p|^2+\int_t^T|Y_s^n-Y_s^p|^2\,ds\\ &+\int_t^T|g_s^n(Y_s^n,Z_s^n,V_s^n)-g_s^p(Y_s^n,Z_s^n,V_s^n)|^2\,ds\\ &+\sup_{0\le s\le T}|L_s^n-L_s^p|(K_T^{n,+}+K_T^{p,+})+\sup_{0\le s\le T}|U_s^n-U_s^p|(K_T^{n,-}+K_T^{p,-})\Big]\,.\end{aligned} \tag{49}$$

Using arguments already used in the previous section, we get the required control over $\|Y^{n,p}\|^2_{\mathcal{S}^2} + \|Z^{n,p}\|^2_{\mathcal{H}^2_d} + \|V^{n,p}\|^2_{\mathcal{H}^2_\mu}$ by Gronwall's Lemma, estimate (13) and Burkholder inequality. The control over $\|K^{n,p}\|^2_{\mathcal{S}^2}$ follows using the equation for $K^{n,p}$ deduced of the related R2BSDEs.

Moreover, in the case where $dA^{n,-} \le \alpha^n_t dt$ for some progressively measurable processes α^n with $\|\alpha^n\|_{\mathcal{H}^2}$ finite (see end of section A.1), then the barriers conditions (48) write:

$$\int_t^T (Y^n_s - Y^p_s)(dK^n_s - dK^p_s) \le \int_t^T (L^n_s - L^p_s)(k^{n,+}_s - k^{p,+}_s)ds - \int_t^T (U^n_s - U^p_s)(dK^{n,-}_s - dK^{p,-}_s) \,.$$

We thus have (49) with $\int_0^T |L^n_s - L^p_s|(k^{n,+}_s + k^{p,+}_s)ds$ instead of $\sup_{0\le s\le T}|L^n_s - L^p_s|(K^{n,+}_T + K^{p,+}_T)$ therein, which in turn implies (14) with $\|L^{n,p}\|_{\mathcal{H}^2}$ instead of $\|L^{n,p}\|_{\mathcal{S}^2}$ therein.

A.3 Convergence proof

We now turn to the situation considered in the last part of the Theorem. In this case, we are for each n in the situation of Lemma 3.1(ii), whence

$$dK^{n,+} = k^{+,n}_t dt \text{ with } k^{+,n}_t \le \mathbb{1}_{\{Y^n_t = L^n_t\}}\big(g^n_t(Y^n_t, Z^n_t, V^n_t)^- + \alpha^n_t\big) \,.$$

So $\|k^{n,+}\|_{\mathcal{H}^2}$ is bounded, by the results of the previous section (assuming $\|\alpha^n\|_{\mathcal{H}^2}$ bounded).

(Y^n, Z^n, V^n) is bounded in $\mathcal{S}^2 \times \mathcal{H}^2_d \times \mathcal{H}^2_\mu$, by (13). Hence (Y^n, Z^n, V^n, K^n) is a Cauchy sequence in $\mathcal{S}^2 \times \mathcal{H}^2_d \times \mathcal{H}^2_\mu \times \mathcal{S}^2$, by (14). Therefore (Y^n, Z^n, V^n, K^n) $\mathcal{S}^2 \times \mathcal{H}^2_d \times \mathcal{H}^2_\mu \times \mathcal{S}^2$-converges to some limiting process (Y, Z, V, K). Let us show that (Y, Z, V, K) solves $(\mathcal{E})$.
By the bound estimate (13), we have that $\mathbb{E}\big[(K^{n,+}_T)^2\big] \le c$, so the $K^{n,+}$ are bounded in $\mathcal{H}^2$, as are the K^n, whence the $K^{n,-}$. Besides, $\|k^{n,+}\|^2_{\mathcal{H}^2}$ is bounded, as noticed above. Thus by application of the Banach–Mazur Lemma (see Cvitanic–Karatzas [12, page 2046] and references therein), there exist, for every $n \in \mathbb{N}$, an integer $N(n) \ge n$ and weights $w^n_j \ge 0$ with $\sum_{j=n}^{N(n)} w^n_j = 1$ such that:

$$\widetilde{K}^{n,\pm} = \sum_{j=n}^{N(n)} w^n_j K^{j,\pm} \to \widetilde{K}^{\pm} \text{ and } \widetilde{k}^{n,+} = \sum_{j=n}^{N(n)} w^n_j k^{j,+} \to \widetilde{k}^{+} \text{ in } \mathcal{H}^2 \text{ as } n \to \infty \,.$$

This implies in particular that $\widetilde{K}^+ = \int_0^\cdot \widetilde{k}^+_u du$ (cf. Cvitanic–Karatzas [12, page 2047]). Moreover, since

$$K^{n,+} - K^{n,-} = K^n \text{ with } K^{n,\pm} \in \mathcal{A}^2_i \,,$$

thus

$$\widetilde{K}^+ - \widetilde{K}^- = K \text{ with } d\widetilde{K}^{\pm} \ge 0$$

(and $\widetilde{K}^{\pm}_0 = 0$), by passage to the limit in $\mathcal{H}^2$. So finally $\widetilde{K}^{\pm} \in \mathcal{A}^2_i$, using also the continuity of K. In addition, by passage to the limit, estimate (13) holds for $(Y, Z, V, \widetilde{K}^+, \widetilde{K}^-)$, and the process (Y, Z, V, K), with $K = \widetilde{K}^+ - \widetilde{K}^-$, satisfies the limiting equation (ii) in $(\mathcal{E})$. We also have $L \le Y \le U$.
Finally there comes, using the fact that $\int_0^T (U^n_t - Y^n_t)dK^{n,-}_t = 0$ in the second line:

$$\begin{aligned} 0 \le \int_0^T (U_t - Y_t)d\widetilde{K}^-_t &= \int_0^T (U_t - Y_t)(d\widetilde{K}^-_t - dK^{n,-}_t) + \int_0^T (U_t - Y_t)dK^{n,-}_t \\ &= \int_0^T (U_t - Y_t)(d\widetilde{K}^-_t - dK^{n,-}_t) + \int_0^T (U_t - U^n_t + Y^n_t - Y_t)dK^{n,-}_t \,. \end{aligned}$$

Now, $\int_0^T (U_t - U_t^n + Y_t^n - Y_t) dK_t^{n,-}$ converges to 0 in expectation, by $(\mathcal{S}^2)^2$-convergence of (Y^n, U^n) to (Y, U) and bound estimate (13) on the $K^{n,-}$. Besides, we have convergence in $\mathcal{H}^2$, hence in measure, of $\widetilde{K}^- - \widetilde{K}^{n,-}$ to 0 (at least, along a suitable subsequence). Moreover, by Proposition 1.5(d) in Mémin–Slominski [29] (see also Prigent [30, Theorem 1.4.2(4) page 102]), the sequence $(\widetilde{K}^- - \widetilde{K}^{n,-})_n$ is *predictably uniformly tight* (see Jacod–Shiryaev [25, VI.6a page 377]), as converging in law (to 0) with $(\widetilde{K}_t^- - \widetilde{K}_t^{n,-})_n$ bounded in $\mathcal{L}^2$ for every $t \in [0, T]$. Therefore $\int_0^T (U_t - Y_t)(d\widetilde{K}_t^- - d\widetilde{K}_t^{n,-})$ converges in measure (for the Skorokhod topology) to 0 (Jacod–Shiryaev [25, Theorem VI.6.22(c) page 383], see also Prigent [30, §1.4]), so that finally $\int_0^T (U_t - Y_t) d\widetilde{K}_t^- = 0$. Likewise, $\int_0^T (Y_t - L_t) d\widetilde{K}_t^+ = 0$.

Since $K = \widetilde{K}^+ - \widetilde{K}^+$ with $\widetilde{K}^\pm \in \mathcal{A}_i^2$, so the Jordan components $K^\pm$ of K are also in $\mathcal{A}_i^2$ and such that $K^\pm \leq \widetilde{K}^\pm$. Thus $\int_0^T (U_t - Y_t) dK_t^- = \int_0^T (Y_t - L_t) dK_t^+ = 0$. □

B Proof of Proposition 5.2

B.1 Basic Problems

With the exception of Becherer [3], previous works on BSDEs with jumps (see e.g. [33, 2, 21, 18, 22]) deal more specifically with the case where the integer-valued random measure μ is a Poisson random measure. Becherer [3] treats the case of a classic BSDE (no barriers) in the present set-up, thus extending to the case of a random density $\zeta_t(e)$ the results of [33, 2].

We leave to the reader the routine task to check that all the results in [21, 18, 22] can be immediately extended to the abstract set-up of the present paper. So our RBSDE $(\mathcal{E}')$ admits a (unique) solution (see Hamadène and Ouknine [21]). As for $(\mathcal{E})$, we know by Hamadène–Hassani [22, Theorem 4.1 and Remark 4.2] that the existence of a solution to $(\mathcal{E})$ is equivalent to the Mokobodski condition. In particular, existence holds for $(\mathcal{E})$ when L or U is a quasimartingale with Rao components in $\mathcal{S}^2$.

Remark B.1 By application of Theorem 3.3(ii) and in view of Remark 3.1(i), existence for $(\mathcal{E})$ also holds when L (or U) is a limit in $\mathcal{S}^2$ of quasimartingales L^n (resp. U^n) with Rao components in $\mathcal{S}^2$, provided the predictable finite variation components $A^{n,-}$ of L^n (resp. $A^{n,+}$ of U^n) have densities α^n with $\|\alpha^n\|_{\mathcal{H}^2}$ bounded over $n \in \mathbb{N}$.

B.2 Extensions with stopping time

Given a further stopping time $\tau \in \mathcal{T}$, we now consider the variants of the above problems introduced in section 2.1.2.

B.2.1 Reflected BSDE with random terminal time

By inspection of the arguments of Hamadène and Ouknine [21], it appears that the existence result for $(\mathcal{E}')$ admits an immediate extension to the case of a reflected BSDE with random terminal time τ (in the sense of Darling and Pardoux [13], but in the rather elementary situation where our stopping time τ is bounded here, cf. Remark 2.4(ii)). So, assuming that ξ is $\mathcal{F}_\tau$-measurable, existence of a solution to the RBSDE $(\bar{\mathcal{E}}')$ also holds true.

B.2.2 Upper barrier with delayed activation

We finally consider the τ-R2BSDE $(\bar{\mathcal{E}})$. Note that in applications (see [5, 7, 8]), τ is typically given as a predictable stopping time. In this case, the upper barrier $\bar{U}$ has a jump at a predictable stopping time, and (H.2.i)' (or an immediate adaptation to the case of an $\mathbb{R} \cup \{+\infty\}$-valued upper barrier) is not satisfied by $\bar{U}$. This is why the τ-R2BSDE deserves a separate treatment.

In order to show that the τ-R2BSDE $(\bar{\mathcal{E}})$ with data (g, ξ, L, U, τ) has a solution under the Mokobodski condition, let then $(\widehat{Y}, \widehat{Z}, \widehat{V}, \widehat{K})$ denote the solution to $(\mathcal{E})$. This solution is indeed known to exist (and be unique) under the Mokobodski condition, by the results reviewed in section B.1. Let likewise $(\bar{Y}, \bar{Z}, \bar{V}, \bar{K})$ denote the solution, known to exist by the result of section B.2.1, to the RBSDE with random terminal time τ and data $(\widehat{Y}_\tau, g, L)$ on $[0, \tau]$. Now, defining (Y, Z, V, K) by

$$
\begin{gathered}
Y := \bar{Y} \mathbb{1}_{t<\tau} + \widehat{Y} \mathbb{1}_{t \geq \tau} \\
K^+ := \bar{K} \mathbb{1}_{t<\tau} + [\widehat{K}^+ + (\bar{K}_\tau - \widehat{K}^+_\tau)] \mathbb{1}_{t \geq \tau} \ , \ K^- := (\widehat{K}^- - \widehat{K}^-_\tau) \mathbb{1}_{t \geq \tau} \\
Z := \bar{Z} \mathbb{1}_{t \leq \tau} + \widehat{Z} \mathbb{1}_{t>\tau} \ , \ V := \bar{V} \mathbb{1}_{t \leq \tau} + \widehat{V} \mathbb{1}_{t>\tau} \ ,
\end{gathered}
$$

then by construction (Y, Z, V, K) is a solution to the τ-R2BSDE $(\bar{\mathcal{E}})$ on $[0, T]$.

Acknowledgements. It is our pleasure to thank Monique Jeanblanc for kind advice and useful discussions throughout the work.

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