A multiple-curve HJM model of interbank risk

Stéphane Crépey, Zorana Grbac and Hai-Nam Nguyen *

Laboratoire Analyse et probabilités
Université d'Évry Val d'Essonne
91037 Évry Cedex, France

January 9, 2013

Abstract

In the aftermath of the 2007-2009 financial crisis, a variety of spreads have developed between quantities that had been essentially the same until then, notably LIBOR-OIS spreads, LIBOR-OIS swap spreads, and basis swap spreads. By the end of 2011, with the sovereign credit crisis, these spreads were again significant. In this paper we study the valuation of LIBOR interest rate derivatives in a multiple-curve setup, which accounts for the spreads between a risk-free discount curve and LIBOR curves. Towards this end we resort to a defaultable HJM methodology, in which these spreads are explained by an implied default intensity of the LIBOR contributing banks, possibly in conjunction with an additional liquidity factor. Markovian short rate specifications are given in the form of an extended CIR and a Lévy Hull–White model for a risk-free short rate and a LIBOR short spread. The use of Lévy drivers leads to the more parsimonious specification. Numerical values of the FRA spreads and the basis swap spreads computed with the latter largely cover the ranges of values observed even at the peak of the 2007-2009 crisis.

Keywords: Interest Rate Derivatives, LIBOR, HJM, Multiple Curve, Interbank Risk, Lévy Processes.

MSC: 91G30, 91G20, 60G51
JEL classification: G12, E43

1 Introduction

The recent financial crisis caused a number of anomalies that were not previously observed in the fixed income markets. The interest rates whose dynamics were very closely following each other have started to diverge substantially. In particular, the spreads between the LIBOR rates and the OIS rates (the swap rates of interest rate swaps whose floating-rate payments are indexed to a compounded overnight rate) of the same maturity have been far from negligible, as well as the spreads between the swap rates of the LIBOR-indexed interest

*The research of the authors benefited from the support of the 'Chaire Risque de crédit', Fédération Bancaire Française, and of the DGE. The authors thank Jeroen Kerkhof, from Jefferies, London, for the graphs of Figure 1, and Alexander Herbertsson, from University of Gothenburg, Sweden, for his help in detailing the computations about swaptions.

rate swaps and the OIS rates. The former type of spreads is known as the LIBOR-OIS spread and the latter as the LIBOR-OIS swap spread. We refer to Section 2.1 for precise definitions of various interest rates and to Sections 4.2 and 4.3 for definitions of interest rate swaps and spreads. In Figure 1 (left) the historical EURIBOR-EONIAswap spreads in the period 2005-2010 are plotted for maturities ranging from 1 month to 12 months. Before the crisis these spreads were practically negligible, whereas at the peak of the crisis they were greater than 200 basis points for some maturities. The EURIBOR rate and the EONIAswap rate are analogs of the LIBOR rate and the OIS rate in the EUR fixed income market (cf. Section 2.1 for details). Furthermore, since the financial crisis the LIBOR rates of different maturities have exhibited notably diverse behavior, which is reflected in the so-called basis swap spreads appearing when basis swaps are priced. In a basis swap two streams of floating-rate payments linked to LIBOR rates on different tenors are exchanged (cf. Section 4.4 for details). This is why practitioners nowadays tend to produce different discount curves for different tenors; see Figure 1 (right), which displays discount functions related to the EONIAswap rates, the 3-month and the 6-month EURIBOR rates. All these phenomena are described in Filipović and Trolle (2011) as the advent of a so-called interbank risk.

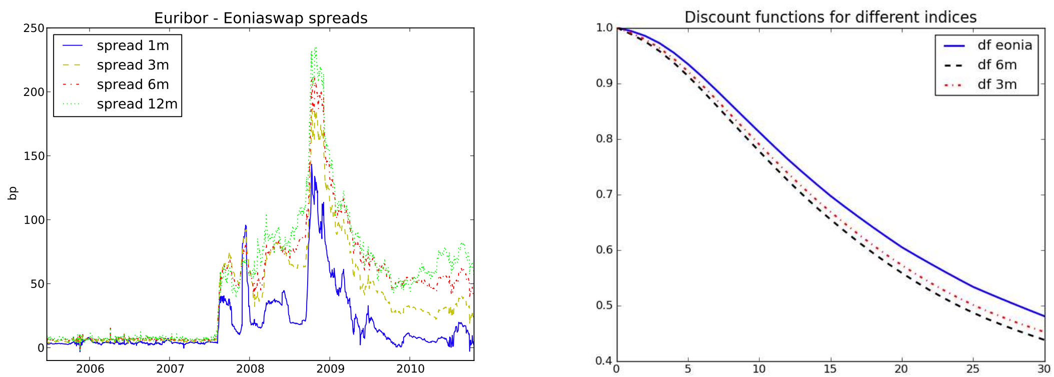

Figure 1: *Left*: Historical EURIBOR-EONIAswap spreads 2005-10. *Right*: Discount curves bootstrapped on September 2, 2010.

In addition, when valuing and hedging interest rate derivatives, the interbank risk issue comes in combination with the counterparty risk issue, which is the risk of a party defaulting in an OTC derivative contract. In this context, the questions such as which curve should be used as discounting curve, to which extent the choice of a given curve should be put in relation with counterparty risk, or possibly hidden relations between bilateral counterparty risk (accounting for the default risk of both parties) and funding costs (of funding a position in a contract in a multiple-curve environment), have become the subject of endless debates between market practitioners.

In this paper we propose a model of interbank risk for the pricing of LIBOR interest rate derivatives in a multiple-curve setup. Note that this is a model of "clean" valuation in the sense of Crépey (2012), meaning clean of counterparty risk and excess funding costs above the risk-free rate (in practice: the OIS rate). However, a counterparty risk and excess funding costs correction (CVA for Credit Valuation Adjustment in the counterparty risk terminology) can then be obtained as the value of an option on this clean price process; see for instance Crépey (2012). Actually, the main motivation for the present work is to devise a model of clean valuation of interest rate derivatives with interbank risk, tractable in itself, but also from the perspective of serving as an underlying model for CVA computations. This integration of the present clean model into a counterparty risky environment will be considered in a follow-up paper.

Resorting to the usual distinction between short rate, HJM and BGM or LIBOR market models, one can classify the interbank risk (multiple-curve in this regard, yet "clean" in the above sense) valuation literature as follows. Kijima, Tanaka, and Wong (2009) or Kenyon (2010) propose short rate approaches. Henrard (2007, 2010) derives corrected Gaussian HJM formulas under the assumption of deterministic spreads between the curves. Bianchetti (2010) resolves a two-curve issue in a cross-currency mathematical framework, deriving "quanto convexity corrections" to the usual BGM market model valuation formulas. Here the main tool is that of a change of measure/numéraire. The LIBOR market model approach is also extended in Mercurio (2010b, 2010a) and Fujii, Shimada and Takahashi (2011, 2010) in such a way that basis spreads for different tenors are modeled as different processes. A hybrid HJM-LIBOR market model is proposed in Moreni and Pallavicini (2010), where the HJM framework is employed to obtain a parsimonious model for multiple curves, using a single family of Markov driving processes. Finally, a credit risk approach is tentative in Morini (2009). However, Morini concludes on page 43 that in his model "the credit risk alone does not explain the market patterns".

We recall that LIBOR stands for London Interbank Offered Rate and is produced for 10 currencies with 15 maturities, ranging from overnight to 12 months, thus producing 150 rates each business day. The contributing banks are selected for each currency panel with the aim of reflecting the balance of the market for a given currency based upon three guiding principles: scale of market activity, credit rating and perceived expertise in the currency concerned. Each panel, ranging from 7 to 18 contributors, is chosen by the independent Foreign Exchange and Money Markets Committee (FX&MM Committee) to give the best representation of activity within the London money market for a particular currency. Twice a year the FX&MM Committee undertakes an assessment of each LIBOR panel, evaluating the contributing banks and updating the selection if necessary. This rolling construction of the LIBOR contributing group is intended to ensure that, in principle, actual defaults cannot occur within the group. However, the deterioration of the credit quality of the LIBOR contributors during the length of a LIBOR loan is greater with longer tenors, resulting in a default spread between the LIBOR markets of different tenors (OIS market in the limiting case of an overnight tenor). Moreover, the economic fundamentals of interbank risk are not only credit risk, but also liquidity risk, among other factors such as "strategic" game considerations (see Michaud and Upper (2008, page 48)), which might from time to time incite a bank to declare as LIBOR contribution a number slightly different from its internal conviction regarding "The rate at which an individual Contributor Panel bank could borrow funds, were it to do so by asking for and then accepting interbank offers in reasonable market size, just prior to 11.00 London time" (the theoretical definition of the LIBOR rate). For these interpretations and the related econometric aspects we refer the reader to the quantitative analysis of the term structure of interbank risk which was recently conducted by Filipović and Trolle (2011). Based on a data set covering the period from August 2007 until January 2011, their results show that the default component is overall the main dominant driver of interbank risk, except for short-term contracts in the first half of the sample (see Figures 3 and 4 in their paper). The second main driver is interpreted as liquidity risk, which is consistent with the claims in Morini (2009).

Here we make both the credit and the liquidity interbank risk components explicit, in the mathematical framework of a defaultable Heath–Jarrow–Morton methodology; see the seminal paper by Heath, Jarrow, and Morton (1992) and the defaultable extensions by Bielecki and Rutkowski (2000) and Eberlein and Özkan (2003). Our motivation for modeling the continuously compounded forward rates using an HJM approach, instead of dealing directly with discretely compounded LIBOR rates in a BGM framework, is twofold. On the one side, it allows one to consider simultaneously the LIBOR rates for all possible tenors (recall that one of the post-crisis spreads studied in this work is related to the LIBOR rates of various tenors). The HJM framework is capable of producing a multi-curve model with as many stochastic factors as LIBOR rates of different tenors by increasing the dimension of the driving process, while still retaining the tractability of the pricing formulas for any arbitrary correlation of stochastic factors. On the other side, this is a unified approach for a very general class of time-inhomogeneous Lévy driving processes. It is also important to mention that various short rate models can be accommodated in this setup as special cases (see Section 3 for the extended CIR and the extended Hull-White model). As will be illustrated in a follow-up work, this direct link to the short rate process r is useful in the context of counterparty risk applications, where the model of this paper can be used as an underlying model for CVA computations. Numerical issues related to our model will be mainly considered in a follow-up paper. However, the last section of this paper already makes clear that, in contrast to the conclusion of Morini (2009) in his first tentative credit risk approach, even a pure appropriate credit risk model (with liquidity component set to zero) is in fact able to explain spreads very much in line with the orders of magnitude that were observed in the market even at the peak of the crisis.

The rest of the paper is organized as follows. In Section 2, we apply an adaptation of the defaultable HJM approach to model the term structure of multiple interest rate curves. Section 3 presents a tractable pricing model within this framework which we obtain by choosing the class of nonnegative multidimensional Lévy processes as driving processes combined with deterministic volatility structures. In Section 4 the basic interest rate derivatives tied to the LIBOR rate are described and explicit valuation formulas are derived. Section 5 presents numerical results illustrating the flexibility of the model in producing a wide range of FRA spreads and basis swap spreads.

In our view the main contributions of this work are: a consistent and tractable multiple-curve HJM term structure model of interbank risk; low-dimensional extended CIR or Lévy Hull–White short rate specifications of the multiple-curve HJM setup, opening the door to the use of this model as an underlying model for interest rate derivative CVA computations; numerical evidence that an appropriately chosen credit risk setup is enough to account for even the most extreme interbank spreads observed in the market.

2 Multiple-curve HJM setup

2.1 Notation

In this subsection we introduce the main notions and notation we are going to work with. The main reference rate for a variety of interest rate derivatives is the LIBOR in the USD fixed income market and the EURIBOR in the EUR fixed income market. LIBOR (resp. EURIBOR) is computed daily as an average of the rates at which designated banks belonging to the LIBOR (resp. EURIBOR) panel believe unsecured funding for periods of length up to one year can be obtained by them (resp. by a prime bank). From now on we shall use the term LIBOR meaning any of these two rates. Another important reference rate in fixed income markets is a so-called OIS (Overnight Indexed Swap) rate, which is the swap rate of a swap whose floating rate is obtained by compounding an overnight rate, i.e. a rate at which overnight unsecured loans can be obtained in the interbank market. In the USD fixed income market this rate is the FF (Federal Funds) rate and in the EUR market it is the EONIA rate (where EONIA stands for Euro Overnight Index Average). From now on we shall use the generic term OIS rate for both fixed income markets. The OIS rate is considered by practitioners to be the best available market proxy for the risk-free rate since the risk in an overnight loan can be deemed almost negligible. On the other hand, the LIBOR rate depends on the term structure of interbank risk, which is reflected in the observed LIBOR-OIS and LIBOR-OIS swap spreads (see the left panel in Figure 1).

In this paper we introduce a default time τ^* associated with the LIBOR reference curve via a given default intensity $\gamma^*(t)$. We emphasize that τ^* is not meant to represent an actual default time of any specific entity (recall that the LIBOR panel is constantly being updated). It is merely used as an implied model of default risk for the reference curve, to quantify the credit spread component of interbank risk on a mathematically tractable "default intensity scale".

We shall work with instantaneous continuously compounded forward rates, specifying the dynamics of the term structure of the risk-free (OIS) forward interest rates $f_t(T)$ and of the forward spreads $g_t(T)$ corresponding to the risky (LIBOR) rates of the reference curve. We denote by $f^*_t(T)$ the instantaneous continuously compounded risky forward rates, so for every $0 \le t \le T$,

$$g_t(T) = f^*_t(T) - f_t(T), \tag{1}$$

where $T \in [0, \bar{T}]$ and $\bar{T}$ is a finite time horizon. The corresponding short rates r and r^* are given, for every $t \in [0, \bar{T}]$, by

$$r_t = f_t(t) \quad \text{and} \quad r^*_t = f^*_t(t). \tag{2}$$

We also define the short term spread λ by

$$\lambda_t = g_t(t) = r^*_t - r_t.$$

The discount factors associated with our two yield curves are denoted by $B_t(T)$ and $\bar{B}^*_t(T)$, respectively. These are time-t (cumulative) prices and pre-default prices of risk-free and risky zero coupon bonds with maturity T, with $B_T(T) = 1$ and $\bar{B}^*_T(T) = 1$. The bond prices are related to the forward rates via the following formulas, for $t \le T$,

$$B_t(T) = \exp\left(-\int_t^T f_t(u)du\right) \quad \text{and} \quad \bar{B}^*_t(T) = \exp\left(-\int_t^T f^*_t(u)du\right). \tag{3}$$

The T-spot LIBOR rate $L_T(T, T+\delta)$ is a simply compounded interest rate fixed at time T for the time interval $[T, T+\delta]$, which will be defined in our setup as

$$L_T(T, T+\delta) = \frac{1}{\delta}\Big(\frac{1}{\bar{B}^*_T(T+\delta)} - 1\Big). \tag{4}$$

We thus use in this definition the risky bond prices $\bar{B}^*$, where the reference entity of the risky bond is to be interpreted as consisting of (a stylized representative of) the LIBOR contributing banks.

2.2 Driving process

We consider a complete probability space $(\Omega, \mathcal{F}_{\bar{T}}, \mathbb{P})$, where $\bar{T}$ is the finite time horizon. Let $\mathcal{E} = (\mathcal{E}_t)_{t\in[0,\bar{T}]}$ denote a filtration on this space satisfying the usual conditions. The driving process $Y = (Y_t)_{0\le t\le \bar{T}}$ is assumed to be a process with independent increments and absolutely continuous characteristics (PIIAC) in the sense of Eberlein, Jacod, and Raible (2005), also called a time-inhomogeneous Lévy process in Eberlein and Kluge (2006a), or an additive process in the sense of Definition 1.6 in Sato (1999). The process Y is taken as an $\mathcal{E}$-adapted, càdlàg, $\mathbb{R}^n$-valued process, starting from zero. The law of Y_t, $t \in [0, \bar{T}]$, is described by the characteristic function, in which u denotes a row-vector in $\mathbb{R}^n$:

$$\begin{aligned} \mathbb{E}[e^{\mathrm{i}uY_t}] \quad &= \quad \exp \int_0^t \Big(\mathrm{i}ub_s - \frac{1}{2} uc_s u^\top \\ &\qquad + \int_{\mathbb{R}^n} \left(e^{\mathrm{i}ux} - 1 - \mathrm{i}uh(x) \right) F_s(dx) \Big) ds, \end{aligned} \tag{5}$$

where $b_s \in \mathbb{R}^n$, c_s is a symmetric, nonnegative definite real-valued n-dimensional matrix and F_s is a Lévy measure on $\mathbb{R}^n$, i.e. $F_s(\{0\}) = 0$ and $\int_{\mathbb{R}^n}(|x|^2 \wedge 1) F_s(dx) < \infty$, for all $s \in [0, \bar{T}]$. The function $h : \mathbb{R}^n \to \mathbb{R}^n$ is a truncation function (for example $h(x) = x\mathbf{1}_{\{|x|\le 1\}}$).

Let $\|\cdot\|$ denote the norm on the space of real n-dimensional matrices, induced by the Euclidean norm $|\cdot|$ on $\mathbb{R}^n$. The following standing assumption is satisfied:

Assumption 2.1 **(i)** The triplet $(b_t, c_t, F_t)_{0\le t\le \bar{T}}$ satisfies

$$\int_0^{\bar{T}} \left(|b_t| + \|c_t\| + \int_{\mathbb{R}^n} (1 \wedge |x|^2) F_t(dx) \right) dt < \infty;$$

(ii) There exist constants $\mathcal{K}, \varepsilon > 0$ such that

$$\int_0^{\bar{T}} \int_{|x|>1} \exp(ux) F_t(dx) dt < \infty, \tag{6}$$

for every $u \in [-(1+\varepsilon)\mathcal{K}, (1+\varepsilon)\mathcal{K}]^n$.

Condition (6) ensures the existence of exponential moments of the process Y. More precisely, (6) holds if and only if $\mathbb{E}[\exp(uY_t)] < \infty$, for all $0 \le t \le \bar{T}$ and $u \in [-(1+\varepsilon)\mathcal{K}, (1+\varepsilon)\mathcal{K}]^n$ (cf. Lemma 2.6 and Corollary 2.7 in Papapantoleon (2007)). Moreover, Y is then a special semimartingale, with the following canonical decomposition (cf. Jacod and Shiryaev (2003, II.2.38), and Eberlein, Jacod, and Raible (2005))

$$Y_t = \int_0^t b_s ds + \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}^n} x(\mu - \nu)(ds, dx), \qquad t \in [0, \bar{T}], \tag{7}$$

where μ is the random measure of jumps of Y, ν is the $\mathbb{P}$-compensator of μ, $\sqrt{c_s}$ is a measurable version of a square-root of the symmetric, nonnegative definite matrix c_s, and W is a $\mathbb{P}$-standard Brownian motion. The triplet of predictable semimartingale characteristics of Y with respect to the measure $\mathbb{P}$, denoted by $(B_t, C_t, \nu_t)_{0\le t\le \bar{T}}$, is

$$B_t = \int_0^t b_s ds, \qquad C_t = \int_0^t c_s ds, \qquad \nu([0,t] \times A) = \int_{[0,t]} \int_A F_s(dx) ds, \tag{8}$$

for every Borel set $A \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$. The triplet $(b_t, c_t, F_t)_{0\le t\le \bar{T}}$ represents the local characteristics of Y. Any of these triplets determines the distribution of Y, as the Lévy–Khintchine formula (5) obviously dictates (with $h(x) = x$, which is a valid choice for the truncation function due to (6)).

We denote by κ_s the cumulant generating function associated with the infinitely divisible distribution characterized by the Lévy triplet (b_s, c_s, F_s). One can extend κ_s to row-vectors of complex numbers $z \in \mathbb{C}^n$ such that $\Re z \in [-(1+\varepsilon)\mathcal{K}, (1+\varepsilon)\mathcal{K}]^n$. We have, for $s \in [0, \bar{T}]$,

$$\kappa_s(z) = zb_s + \frac{1}{2} zc_s z^\top + \int_{\mathbb{R}^n} (e^{zx} - 1 - zx)\, F_s(dx). \tag{9}$$

Note that (5) can be written in terms of κ:

$$\mathbb{E}[e^{\mathrm{i}uY_t}] = \exp \int_0^t \kappa_s(\mathrm{i}u) ds. \tag{10}$$

If Y is a Lévy process, in other words if Y is time-homogeneous, then (b_s, c_s, F_s), and thus also κ_s, do not depend on s. In that case, κ boils down to the log-moment generating function of Y_1. For details we refer to Papapantoleon (2007, Lemma 2.8, Remark 2.9 and Remark 2.16).

Remark 2.2 The motivation for the choice of time-inhomogeneous Lévy processes as driving processes in our model is twofold. On the one side, these processes are analytically tractable, and on the other side, they posses a high degree of flexibility, which allows for an adequate fit of the term structure of volatility smiles, i.e. of the change in the smile across maturities; see Eberlein and Kluge (2006a, 2006b) and Eberlein and Koval (2006) for applications of time-inhomogeneous Lévy processes in interest rate modeling.

2.3 Term structure of interest rates

In this subsection, we model the term structures of the risk-free and the risky interest rates.

We shall be concerned with two filtrations on the standing risk-neutral probability space $(\Omega, \mathcal{F}_{\bar{T}}, \mathbb{P})$ of this paper: the background filtration $\mathcal{E} = (\mathcal{E}_t)_{0\le t\le \bar{T}}$, and the full filtration $\mathcal{F} = (\mathcal{F}_t)_{0\le t\le \bar{T}}$ containing $\mathcal{E}$ and the information about the default time τ^*. The bond price processes $(B_t(T))_{0\le t\le T}$ and $(\bar{B}^*_t(T))_{0\le t\le T}$, and also the corresponding forward rate processes $(f_t(T))_{0\le t\le T}$ and $(f^*_t(T))_{0\le t\le T}$, for any $T \in [0, \bar{T}]$, are all $\mathcal{E}$-adapted. It is assumed that τ^* is not an $\mathcal{E}$-stopping time, but it is an $\mathcal{F}$-stopping time. Moreover, we assume that immersion holds between $\mathcal{E}$ and $\mathcal{F}$, i.e. every $\mathcal{E}$-local martingale is an $\mathcal{F}$-local martingale. We assume that τ^* possesses an $\mathcal{E}$-hazard intensity γ^*. Thus, its Azéma supermartingale is given by

$$\mathbb{P}(\tau^* > t | \mathcal{E}_t) = e^{-\int_0^t \gamma^*_s ds}, \tag{11}$$

where γ^* is an $\mathcal{E}$-adapted, nonnegative and integrable process.

Let us now specify the instantaneous continuously compounded forward rates $f_t(T)$ and the instantaneous forward spreads $g_t(T)$, which in turn provide the bond prices $B_t(T)$ and $\bar{B}^*_t(T)$ via (3). We are going to make use of the results from Eberlein and Raible (1999) and Eberlein and Kluge (2006b), where HJM models driven by time-inhomogeneous Lévy processes were developed, and the results from Bielecki and Rutkowski (2000) and Eberlein and Özkan (2003), where defaultable extensions of the HJM framework were introduced.

Contrary to the latter two papers, we choose here to model directly the forward spreads $g_t(T)$ instead of the risky forward rates $f_t^*(T)$, which is clearly equivalent due to (1). However, one should have $\bar{B}_t^*(T) \leq B_t(T)$, i.e. the risky bonds are cheaper than the risk-free bonds with the same maturity. This implies by (3) that $f_t^*(T) \geq f_t(T)$, or equivalently, $g_t(T) \geq 0$. Hence, we decide to model the forward spreads directly and study their nonnegativity in some special cases. In the next subsection two tractable nonnegative examples are provided. Let us also mention here a paper by Chiarella, Maina, and Nikitopoulos (2010), where a class of stochastic volatility HJM models admitting finite dimensional Markovian structures is proposed. They model the default-free forward rates and the forward spreads, whose dynamics are driven by correlated Brownian motions. One of the examples in the sequel, the stochastic volatility CIR model of Section 3.1, can be fit into this modeling framework.

2.3.1 Risk-free rates

The dynamics of the risk-free forward rates $f_t(T)$, for $T \in [0, \bar{T}]$, is given by

$$f_t(T) = f_0(T) + \int_0^t a_s(T) ds + \int_0^t \sigma_s(T) dY_s, \tag{12}$$

where the initial values $f_0(T)$ are deterministic, bounded and Borel measurable in T. Moreover, σ and a are stochastic processes defined on $\Omega \times [0, \bar{T}] \times [0, \bar{T}]$ taking values in $\mathbb{R}^n$ and $\mathbb{R}$, respectively. Let $\mathcal{P}$ and $\mathcal{O}$ respectively denote the predictable and the optional σ-field on $\Omega \times [0, \bar{T}]$. We recall that the predictable σ-field is the σ-field on $\Omega \times [0, \bar{T}]$ generated by all càg adapted processes and the optional σ-field is generated by all càdlàg adapted processes (considered as mappings on $\Omega \times [0, \bar{T}]$). The mappings $(\omega; s, T) \mapsto a_s(\omega; T)$ and $(\omega; s, T) \mapsto \sigma_s(\omega; T)$ are measurable with respect to $\mathcal{P} \otimes \mathcal{B}([0, \bar{T}])$. For $s > T$ we have $a_s(\omega; T) = 0$ and $\sigma_s(\omega; T) = 0$, as well as $\sup_{t, T \leq \bar{T}} (|a_t(\omega; T)| + |\sigma_t(\omega; T)|) < \infty$. These conditions ensure that we can find a "joint-version" of all $f_t(T)$ such that $(\omega; t, T) \mapsto f_t(\omega; T) \mathbf{1}_{\{t \leq T\}}$ is $\mathcal{O} \otimes \mathcal{B}([0, \bar{T}])$-measurable (see Eberlein, Jacod, and Raible (2005)). Then it follows (cf. equation (2.4) in Eberlein and Kluge (2006b)), for $t \in [0, T]$, that

$$B_t(T) = B_0(T) \exp\left(\int_0^t (r_s - A_s(T)) ds - \int_0^t \Sigma_s(T) dY_s \right), \tag{13}$$

where we set

$$A_s(T) := \int_s^T a_s(u) du\,, \quad \Sigma_s(T) := \int_s^T \sigma_s(u) du. \tag{14}$$

Inserting $T = t$ into (13), the risk-free discount factor process $\beta = (\beta_t)_{0 \leq t \leq \bar{T}}$, defined by $\beta_t = \exp\left(-\int_0^t r_s ds \right)$, can be written as

$$\beta_t = B_0(t) \exp\left(-\int_0^t A_s(t) ds - \int_0^t \Sigma_s(t) dY_s \right). \tag{15}$$

Combining this with (13) we obtain the following useful representation for the bond price process

$$B_t(T) = \frac{B_0(T)}{B_0(t)} \exp\left(\int_0^t (A_s(t) - A_s(T)) ds + \int_0^t (\Sigma_s(t) - \Sigma_s(T)) dY_s \right). \tag{16}$$

We make a standing assumption that the volatility structure is bounded in the sense that one has $0 \le \Sigma_s^i(T) \le \frac{\mathcal{K}}{2}$ for every $0 \le s \le T \le \bar{T}$ and $i \in \{1, 2, \dots, n\}$, where $\mathcal{K}$ is the constant from Assumption 2.1(ii). Note that if Y is a Brownian motion, this assumption holds with $\mathcal{K} = \infty$. In other words, the volatility structure in the Brownian case does not have to be bounded.

As is well-known, the model is free of arbitrage if the bond prices discounted at the risk-free rate, $(\beta_t B_t(T))_{0\le t\le T}$, are $\mathcal{F}$-martingales with respect to a risk-neutral measure $\mathbb{P}$. Due to the immersion property it suffices that they are $\mathcal{E}$-martingales. This is guaranteed by the following drift condition, which is assumed henceforth:

$$A_s(T) = \kappa_s(-\Sigma_s(T)), \quad s \in [0, T], \tag{17}$$

where κ_s is the cumulant of Y defined in (9). This condition can be found in Eberlein and Kluge (2006b), see equation (2.3) therein and comments thereafter. For more detailed computations, see Proposition 2.2 of Kluge (2005) in the case of deterministic volatility, and Theorem 7.9 and Corollary 7.10 of Raible (2000) for a stochastic volatility combined with a (time-homogeneous) Lévy driving process. If Y is a standard Brownian motion, condition (17) simplifies to $A_s(T) = \frac{1}{2}|\Sigma_s(T)|^2$, which is the classical HJM no-arbitrage condition.

2.3.2 Risky rates

The dynamics of the forward spreads $g_t(T)$, $t \in [0, T]$, is given by

$$g_t(T) = g_0(T) + \int_0^t a_s^*(T)ds + \int_0^t \sigma_s^*(T)dY_s, \tag{18}$$

where the initial values $g_0(T)$ are deterministic, bounded and Borel measurable in T. Moreover, $a^*(T)$ and $\sigma^*(T)$ satisfy the same measurability and boundedness conditions as $a(T)$ and $\sigma(T)$. The risky forward rates are then given by

$$f_t^*(T) = f_0^*(T) + \int_0^t \bar{a}_s^*(T)ds + \int_0^t \bar{\sigma}_s^*(T)dY_s, \tag{19}$$

where we set

$$f_0^*(T) = f_0(T) + g_0(T)\,, \;\; \bar{a}_s^*(T) = a_s(T) + a_s^*(T)\,, \;\; \bar{\sigma}_s^*(T) = \sigma_s(T) + \sigma_s^*(T).$$

The dynamics of the bond prices $(\bar{B}_t^*(T))_{0\le t\le T}$ can be obtained exactly in the same way as the dynamics of $(B_t(T))_{0\le t\le T}$ in equation (13). Therefore, for $t \in [0, T]$,

$$\bar{B}_t^*(T) = \bar{B}_0^*(T) \exp\left(\int_0^t (r_s^* - \bar{A}_s^*(T))ds - \int_0^t \bar{\Sigma}_s^*(T)dY_s\right), \tag{20}$$

where

$$\bar{A}_s^*(T) := \int_s^T \bar{a}_s^*(u)du \qquad \text{and} \qquad \bar{\Sigma}_s^*(T) := \int_s^T \bar{\sigma}_s^*(u)du. \tag{21}$$

Setting

$$A_s^*(T) := \int_s^T a_s^*(u)du \qquad \text{and} \qquad \Sigma_s^*(T) := \int_s^T \sigma_s^*(u)du,$$

we have

$$\bar{A}_s^*(T) = A_s(T) + A_s^*(T) \qquad \text{and} \qquad \bar{\Sigma}_s^*(T) = \Sigma_s(T) + \Sigma_s^*(T). \tag{22}$$

Recall from (2) that the short rate r^*_s is given by $r_s + \lambda_s$. Similarly to (16), we can rewrite the bond price $\bar{B}^*_t(T)$ as

$$\bar{B}^*_t(T) = \frac{\bar{B}^*_0(T)}{\bar{B}^*_0(t)} \exp\left(\int_0^t (\bar{A}^*_s(t) - \bar{A}^*_s(T))ds + \int_0^t (\bar{\Sigma}^*_s(t) - \bar{\Sigma}^*_s(T))dY_s \right). \tag{23}$$

In defaultable HJM models no-arbitrage requirements yield a drift condition relating the drift term $\bar{A}^*_t(T)$ and the volatility term $\bar{\Sigma}^*_s(T)$. To see it let us temporarily assume that our risky bond prices $\bar{B}^*(T)$ could be interpreted as the pre-default prices of defaultable LIBOR zero-coupon bonds; let us then study the constraints that would correspond to precluding arbitrage opportunities related to dealing with these bonds, were such bonds traded in the market. Note that such LIBOR bonds are actually not traded; not even synthetically as averages of the defaultable bonds of LIBOR contributors, since the LIBOR rates reflected in $\bar{B}^*(T)$ are only reference numbers and not transaction quotes; see the definition of the LIBOR rate in the introduction.

The defaultable bonds are assumed to pay a certain recovery upon default. We adopt the fractional recovery of a market value scheme, which specifies that in case of default of the bond issuer, the fraction of the pre-default value of the bond is paid at default time. The value at maturity of such a bond is given by

$$B^*_T(T) = \mathbf{1}_{\{\tau^* > T\}} + \mathbf{1}_{\{\tau^* \le T\}} R^* \bar{B}^*_{\tau^*-}(T) B^{-1}_{\tau^*}(T),$$

where $R^* \in [0,1]$ is the recovery and $\bar{B}^*_t(T)$ is the pre-default bond price defined in (3), for every $t \in [0,T]$. Note that receiving the amount $\mathbf{1}_{\{\tau^* \le T\}} R^* \bar{B}^*_{\tau^*-}(T)$ at τ^* is equivalent to receiving $\mathbf{1}_{\{\tau^* \le T\}} R^* \bar{B}^*_{\tau^*-}(T) B^{-1}_{\tau^*}(T)$ at T. The time-t bond price can be written as

$$B^*_t(T) = \mathbf{1}_{\{\tau^* > t\}} \bar{B}^*_t(T) + \mathbf{1}_{\{\tau^* \le t\}} R^* \bar{B}^*_{\tau^*-}(T) B^{-1}_{\tau^*}(T) B_t(T). \tag{24}$$

The immersion property implies that $\bar{B}^*_{\tau^*-}(T) = \bar{B}^*_{\tau^*}(T)$. Moreover, note that $\mathbf{1}_{\{\tau^* > t\}} \bar{B}^*_t(T) = \mathbf{1}_{\{\tau^* > t\}} B^*_t(T)$, for every $t \in [0,T]$.

Let us now study the conditions which ensure the absence of arbitrage, i.e. let us find the conditions such that $B^*(T)$ discounted at the risk-free rate, $(\beta_t B^*_t(T))_{0 \le t \le T}$, are $(\mathcal{F}, \mathbb{P})$-martingales, for all $T \in [0, \bar{T}]$. For each T the martingale condition is satisfied if

$$(\bar{B}^*_t(T) - R^*_t B_t(T))\gamma^*_t = \bar{B}^*_t(T)\xi_t(T), \qquad t \in [0,T], \tag{25}$$

where

$$\xi_t(T) := \lambda_t - \bar{A}^*_t(T) + \kappa_t(-\bar{\Sigma}^*_t(T))$$

and $(R^*_t)_{t \ge 0}$ is the terminal recovery process in the sense of Condition (HJM.8) in Section 13.1.9 of Bielecki and Rutkowski (2002). The proof of the above statement is similar to the derivation of Condition (13.24) in Bielecki and Rutkowski (2002, Section 13.1.9) in the Gaussian case. For similar conditions in (time-inhomogeneous) Lévy driven models, we refer to Eberlein and Özkan (2003) or Grbac (2010, Section 3.7).

Under the recovery scheme assumed above (i.e. the fractional recovery of a market value), one gets a particularly convenient form of the martingale condition (25). The recovery process R^* takes the following form (cf. (24))

$$R^*_t := R^* \bar{B}^*_t(T) B^{-1}_t(T),$$

which inserted into (25) yields

$$(1-R^*)\gamma^*_t = \xi_t(T), \qquad t\in[0,T]. \tag{26}$$

Since condition (26) must be true for all $T\in[0,\bar{T}]$, it is actually equivalent to the following two conditions:

$$(1-R^*)\gamma^*_t = \lambda_t \tag{27}$$

and

$$\bar{A}^*_t(T) = \kappa_t(-\bar{\Sigma}^*_t(T)). \tag{28}$$

Indeed, conditions (27) and (28) obviously imply (26). To see the converse, one has to insert $T=t$ into (26) and note that $\bar{A}^*_t(t)=0$ and $\bar{\Sigma}^*_t(t)=0$ by (14). Moreover, $\kappa_t(0)=0$ by (9), which yields (27). Condition (28) now follows from (26) by inserting $T\neq t$.

Now, in our model, the risky bonds $\bar{B}^*(T)$ are mathematical concepts which represent the interbank risk of the LIBOR group and they are neither defaultable in the classical sense, nor they are traded assets. Moreover, interbank risk does not need to consist only of credit risk, it can also have a liquidity component. We thus relax the defaultable HJM drift condition (28) into a less stringent condition

$$\bar{A}^*_t(T) = \kappa_t(-\bar{\Sigma}^*_t(T)) + \alpha_t(T), \tag{29}$$

where $\alpha(T)$ satisfies the same measurability and boundedness conditions as $A(T)$ and $A^*(T)$. Recall that the LIBOR rates are defined by (4), where $\alpha(T)$ then appears via (29) through (23). Since $\kappa_t(-\bar{\Sigma}^*_t(T))$ is related to the pure credit risk interpretation of the risky bonds, as shown above, we shall refer to it as the credit risk component of interbank risk. The remaining contribution $\alpha(T)$ will be referred to as the liquidity component of interbank risk, in reference to the econometrically demonstrated explanation of interbank risk as a mixture of credit and liquidity risk of the LIBOR contributing banks (cf. Filipović and Trolle (2011)).

However, in the remainder of this section, as well as in Sections 3 and 5, for simplicity we shall work without the liquidity component $\alpha_t(T)$ being explicitly present. In other words, we shall work under the more specific “credit” assumption (28). We emphasize that Section 4 does not rely on this assumption and all the results therein are still valid under condition (29), provided that the liquidity component $\alpha_t(T)$ is deterministic in which case we write $\alpha_t(T)=\alpha(t,T)$.

Proposition 2.3 **(i)** *The forward rate $f_t(T)$ is given by*

$$f_t(T) = f_0(T) + \int_0^t \frac{\partial}{\partial T}\kappa_s(-\Sigma_s(T))ds + \int_0^t \sigma_s(T)dY_s, \tag{30}$$

and the short rate r_t by

$$r_t = f_0(t) + \int_0^t \frac{\partial}{\partial t}\kappa_s(-\Sigma_s(t))ds + \int_0^t \sigma_s(t)dY_s. \tag{31}$$

(ii) *The forward spread $g_t(T)$ is given by*

$$\begin{aligned} g_t(T) &= g_0(T) + \int_0^t \left(\frac{\partial}{\partial T}\kappa_s(-\Sigma^*_s(T)-\Sigma_s(T)) - \frac{\partial}{\partial T}\kappa_s(-\Sigma_s(T))\right)ds \\ &\quad + \int_0^t \sigma^*_s(T)dY_s, \end{aligned} \tag{32}$$

and the short term spread λ_t by

$$\begin{aligned}\lambda_t &= g_0(t) + \int_0^t \left(\frac{\partial}{\partial t}\kappa_s(-\Sigma_s^*(t) - \Sigma_s(t)) - \frac{\partial}{\partial t}\kappa_s(-\Sigma_s(t)) \right) ds \\ &\quad + \int_0^t \sigma_s^*(t) dY_s. \end{aligned} \tag{33}$$

(iii) *The $\mathcal{E}$-intensity γ^* of the default time τ^* is given by*

$$\gamma_t^* = \frac{1}{1-R^*}\Big(g_0(t) + \int_0^t \left(\frac{\partial}{\partial t}\kappa_s(-\Sigma_s^*(t) - \Sigma_s(t)) - \frac{\partial}{\partial t}\kappa_s(-\Sigma_s(t)) \right) ds + \int_0^t \sigma_s^*(t) dY_s \Big).$$

Proof. To prove (i), note that from condition (17) it follows that

$$a_s(T) = \frac{\partial}{\partial T}\kappa_s(-\Sigma_s(T)).$$

This immediately yields (30) and (31). Similarly, to prove (ii), we make use of (28) and obtain

$$\begin{aligned} a_s^*(T) &= \bar{a}_s^*(T) - a_s(T) \\ &= \frac{\partial}{\partial T}\kappa_s(-\bar{\Sigma}_s^*(T)) - \frac{\partial}{\partial T}\kappa_s(-\Sigma_s(T)) \\ &= \frac{\partial}{\partial T}\kappa_s(-\Sigma_s^*(T) - \Sigma_s(T)) - \frac{\partial}{\partial T}\kappa_s(-\Sigma_s(T)). \end{aligned}$$

Hence, (32) and (33) follow. Finally, to prove (iii) we combine (27) and (33). □

3 The model

In this section we focus our attention on time-homogeneous Lévy processes Y. The cumulant generating function associated with Y is then given by

$$\kappa(z) := zb + \frac{1}{2}zcz^\top + \int_{\mathbb{R}^n} (e^{zx} - 1 - zx)\, F(dx),$$

where (b, c, F) is the Lévy triplet of Y_1 (compare (9)). We study conditions that ensure the nonnegativity of the risk-free interest rates and the spreads, considering in particular two cases: a pure-jump Lévy process with nonnegative components (subordinators) combined with deterministic bond price volatility structures, and a two-dimensional Brownian motion in combination with stochastic volatility structures. We shall focus in particular on the first case, which turns out to be very tractable for valuation purposes. Note that the general HJM model, as well as many short rate models, does not necessarily produce nonnegative interest rates. The standard argument is that the probability of negative interest rates is sufficiently small, and therefore this undesirable feature is still tolerable. However, when interest rates are small as in the recent years, the nonnegativity of interest rates produced by a model becomes a practically relevant issue.

3.1 Stochastic volatility CIR

Assume that the driving process $Y = (Y^1, Y^2)$ is a two-dimensional Brownian motion with correlation ϱ. The canonical decomposition (7) of Y is given by

$$Y_t = \sqrt{c}(W_t^1, W_t^2)^\top,$$

where (W^1, W^2) is a two-dimensional standard Brownian motion and the covariance matrix $c = [c_{i,j}]_{i,j=1,2}$ is such that $c_{1,1} = c_{2,2} = 1$ and $c_{1,2} = c_{2,1} = \varrho$. The cumulant generating function of Y is given by $\kappa(z) = \frac{1}{2} z c z^\top$, $z \in \mathbb{R}^2$. In order to produce nonnegative short rates and short term spreads with this driving process, the volatilities in the HJM model cannot be deterministic. We make use of the volatility specifications that produce the CIR short rate and the CIR short term spread within the HJM framework, as shown in Chiarella and Kwon (2001). Thus, we impose the following assumptions on the volatilities $\sigma_s(t)$ and $\sigma_s^*(t)$:

$$\sigma_s(t) = \left(\zeta(s)\sqrt{r_s} e^{-\int_s^t k(u)du}, 0\right), \quad \sigma_s^*(t) = \left(0, \zeta^*(s)\sqrt{\lambda_s} e^{-\int_s^t k^*(u)du}\right),$$

where ζ, ζ^*, k and k^* are deterministic functions (cf. equation (6.2) in Chiarella and Kwon (2001)). Note that the two-dimensional volatility structure above is chosen in such a way that the risk-free rates are driven only by the first Brownian motion $Y^1 =: W^r$ and the forward spreads are driven solely by $Y^2 =: W^\lambda$. Hence, we can apply directly the results from Chiarella and Kwon (2001, equation (6.3)) and obtain the following SDE for the short rate r

$$dr_t = (\rho_t - k(t) r_t)dt + \zeta(t)\sqrt{r_t} dW_t^r, \tag{34}$$

where

$$\rho_t = \frac{\partial}{\partial t} f_0(t) + k(t) f_0(t) + \int_0^t \sigma_s^2(t) ds.$$

This is a one-dimensional extended CIR short rate model. We emphasize, however, that ρ_t is non-deterministic since it depends on the non-deterministic $\sigma_s(t)$. An additional, auxiliary factor

$$\imath_t = \int_0^t \sigma_s^2(t) ds\,, \; d\imath_t = \big((\zeta(t))^2 r_t - 2k(t)\imath_t\big)dt$$

is needed to make the model Markovian in $(r_t, \imath_t)$. The forward rate volatility specification that yields the extended CIR short rate model in which k and ζ do not depend on time, was studied in Heath, Jarrow, and Morton (1992, Section 8), but in this case ρ in (34) is not available in explicit form.

Reasoning along the same lines as above yields the following SDE for the short term spread λ

$$d\lambda_t = (\rho_t^* - \kappa^*(t)\lambda_t)dt + \zeta^*(t)\sqrt{\lambda_t} dW_t^\lambda,$$

where ρ_t^* is defined accordingly. Similarly, we also define

$$\jmath_t = \int_0^t (\sigma_s^*(t))^2 ds\,, d\jmath_t = \big((\zeta^*(t))^2 \lambda_t - 2k^*(t)\jmath_t\big)dt.$$

In Theorem 2.1 of Chiarella and Kwon (2001) it was shown that the risk-free extended CIR model possesses an affine term structure with two stochastic factors. More precisely, the bond prices can be written as exponential-affine functions of the current level of the short rate r and the process $\imath$:

$$B_t(T) = \frac{B_0(T)}{B_0(t)} \exp\left(\gamma(t,T) f_0(t) - \gamma(t,T) r_t - \frac{1}{2}\gamma^2(t,T)\imath_t\right), \tag{35}$$

where

$$\gamma(t,T) = \int_t^T e^{-\int_t^u k(v)dv} du$$

is a deterministic function (combine Theorem 2.1 with (2.4) and (1.2) in Chiarella and Kwon (2001)). For risky bonds $\bar{B}^*_t(T)$ a similar expression involving in addition λ_t and $\jmath_t$ can be obtained by exactly the same reasoning and making use of the representation

$$\bar{B}^*_t(T) = B_t(T) \exp\left(-\int_t^T g_t(u)du\right), \tag{36}$$

which follows from (1) and (3).

3.2 Jumps and deterministic volatility

In CVA applications (see Crépey (2012)), Markovian specifications are required. The previous Brownian specification of the general multiple-curve HJM setup, yields a four-dimensional Markov factor model in terms of the process $(X_t = (r_t, \lambda_t, \imath_t, \jmath_t))_{0 \le t \le \bar{T}}$. In the quest of a more parsimonious Markovian specification, we now assume that the driving process Y is an n-dimensional Lévy process, whose components are subordinators, and that the volatilities are deterministic. We derive conditions that ensure the nonnegativity of the interest rates and the spreads in this setting. It is worthwhile mentioning that when Y is two-dimensional as in the previous example, this yields a two-dimensional Markov factor model in terms of $X = (r, \lambda)$, which makes this specification preferable for applications.

Let Y be an n-dimensional nonnegative Lévy process, such that its Lévy measure satisfies Assumption 2.1. Its cumulant generating function is given by

$$\kappa(z) = zb + \int_{\mathbb{R}^n_+} (e^{zx} - 1) F(dx) \tag{37}$$

for $z \in \mathbb{R}^n$ such that $z \in [-(1+\varepsilon)\mathcal{K}, (1+\varepsilon)\mathcal{K}]^n$, where $b \ge 0$ denotes the drift term and the Lévy measure F has its support in $\mathbb{R}^n_+$. We refer to Theorem 21.5 and Remark 21.6 in Sato (1999) for one-dimensional subordinators; for multi-dimensional nonnegative Lévy processes see (3.15) in Barndorff-Nielsen and Shephard (2001). Note that subordinators do not have a diffusion component and their jumps can be only positive. Examples of these processes include a compound Poisson process with positive jumps, Gamma process, inverse Gaussian (IG) process, and generalized inverse Gaussian (GIG) processes.

In the remainder of the paper we impose the following standing assumptions on the volatilities Σ and Σ^*:

Assumption 3.1 The volatilities Σ and Σ^* are nonnegative, deterministic and stationary functions. More precisely, they are given as

$$\Sigma_s(t) = \left(S^i(t-s)\right)_{1 \le i \le n} \qquad \text{and} \qquad \Sigma^*_s(t) = \left(S^{*,i}(t-s)\right)_{1 \le i \le n},$$

for every s, t such that $0 \le s \le t \le \bar{T}$, where $S^i : [0, \bar{T}] \to \mathbb{R}_+$ and $S^{*,i} : [0, \bar{T}] \to \mathbb{R}_+$, $i = 1, \dots, n$, are deterministic functions bounded by $\frac{\mathcal{K}}{2}$, where $\mathcal{K}$ is the constant from (6).

Proposition 3.2 **(i)** *The dynamics of the forward rate $f_t(T)$ and the short rate r_t are given by*

$$f_t(T) = f_0(T) - \kappa(-\Sigma_t(T)) + \kappa(-\Sigma_0(T)) + \int_0^t \sigma_s(T)dY_s \tag{38}$$

and

$$r_t = f_0(t) + \kappa(-\Sigma_0(t)) + \int_0^t \sigma_s(t)dY_s. \tag{39}$$

(ii) *The dynamics of the forward spread $g_t(T)$ and the short spread λ_t are given by*

$$\begin{aligned} g_t(T) &= g_0(T) - \kappa(-\Sigma_t^*(T) - \Sigma_t(T)) + \kappa(-\Sigma_0^*(T) - \Sigma_0(T)) \\ &\quad +\kappa(-\Sigma_t(T)) - \kappa(-\Sigma_0(T)) + \int_0^t \sigma_s^*(T)dY_s \end{aligned} \tag{40}$$

and

$$\lambda_t = g_0(t) + \kappa(-\Sigma_0^*(t) - \Sigma_0(t)) - \kappa(-\Sigma_0(t)) + \int_0^t \sigma_s^*(t)dY_s. \tag{41}$$

Proof. We begin by noting that

$$\frac{\partial}{\partial T}S^i(T-s) = -\frac{\partial}{\partial s}S^i(T-s) \text{ and } \frac{\partial}{\partial T}S^{*,i}(T-s) = -\frac{\partial}{\partial s}S^{*,i}(T-s), \tag{42}$$

for $i = 1, \dots, n$. Hence, Assumption 3.1 implies

$$\frac{\partial}{\partial T}\kappa(-\Sigma_s(T)) = -\frac{\partial}{\partial s}\kappa(-\Sigma_s(T))$$

and

$$\frac{\partial}{\partial T}\kappa(-\Sigma_s^*(T) - \Sigma_s(T)) = -\frac{\partial}{\partial s}\kappa(-\Sigma_s^*(T) - \Sigma_s(T)),$$

which follows from (42) by differentiation. Therefore, we obtain

$$\begin{aligned} \int_0^t \frac{\partial}{\partial T}\kappa(-\Sigma_s(T))ds &= -\int_0^t \frac{\partial}{\partial s}\kappa(-\Sigma_s(T))ds \\ &= -\left(\kappa(-\Sigma_t(T)) - \kappa(-\Sigma_0(T))\right), \end{aligned}$$

and similarly,

$$\int_0^t \frac{\partial}{\partial T}\kappa(-\Sigma_s^*(T) - \Sigma_s(T))ds = -\left(\kappa(-\Sigma_t^*(T) - \Sigma_t(T)) - \kappa(-\Sigma_0^*(T) - \Sigma_0(T))\right).$$

Inserting these expressions into (30) and (32) yields (38) and (40), respectively. To show (39) and (41) we note that

$$\int_0^t \frac{\partial}{\partial t}\kappa(-\Sigma_s(t))ds = \kappa(-\Sigma_0(t))$$

and

$$\int_0^t \frac{\partial}{\partial t}\kappa(-\Sigma_s^*(t) - \Sigma_s(t))ds = \kappa(-\Sigma_0^*(t) - \Sigma_0(t)),$$

due to $\kappa(-\Sigma_t(t)) = \kappa(0) = 0$ and $\kappa(-\Sigma_t^*(t) - \Sigma_t(t)) = 0$, which follows by (14) and (21) combined with (37). □

In the next two propositions we give necessary and sufficient deterministic conditions for the nonnegativity of the interest rates and the spreads. Let us denote

$$\begin{aligned} \mu(t,T) &:= f_t(T) - \int_0^t \sigma_s(T)dY_s = f_0(T) - \kappa(-\Sigma_t(T)) + \kappa(-\Sigma_0(T)) \\ \mu(t) &:= r_t - \int_0^t \sigma_s(t)dY_s = f_0(t) + \kappa(-\Sigma_0(t)), \end{aligned}$$

where the second equality in each line follows by (38) and (39), respectively. Note that $\mu(t,T)$ and $\mu(t)$ are thus deterministic.

Proposition 3.3 **(i)** *The short rate r_t is nonnegative if $\mu(t) \geq 0$, for $t \in [0,T]$.*
(ii) *Assume that the distribution of the random vector Y_1 has $[0,\infty)^n$ as its support. Then the converse of (i) is also true, i.e. if $r_t \geq 0$, then $\mu(t) \geq 0$, for every $t \in [0,T]$. Moreover, if $r_t \geq 0$, for every $t \in [0,T]$, then $f_t(T) \geq 0$, for every $T \in [0,\bar{T}]$. In words, the nonnegativity of the short rate implies the nonnegativity of the forward rate.*

Proof. Since Y has nonnegative components and the volatility σ is nonnegative by assumption, it is obvious that $\mu(t) \geq 0$ implies $r_t \geq 0$, for every t. This proves (i).

In case when the support of Y_1 is $[0,\infty)^n$, we show the converse statement by noting that

$$0 \leq \left(\int_0^t \sigma_s(t)dY_s\right)(\omega) \leq \frac{\mathcal{K}}{2}\left(\sum_{i=1}^n \int_0^t dY_s^i(\omega)\right) = \frac{\mathcal{K}}{2}\sum_{i=1}^n Y_t^i(\omega), \tag{43}$$

for every $\omega \in \Omega$. Note that since Y^i, $i = 1,\dots,n$, are increasing process, here the stochastic integrals coincide with the Stieltjes integrals, and hence we are able to do the integration pathwise. Moreover, since Y_1 has the support $[0,\infty)^n$, so does Y_t. This implies that $\mathbb{P}\left(\omega \in \Omega : \sum_{i=1}^n Y_t^i(\omega) < \varepsilon\right) > 0$, for every $\varepsilon > 0$. This combined with (43) yields that

$$\mathbb{P}\left(\omega \in \Omega : \left(\int_0^t \sigma_s(t)dY_s\right)(\omega) < \varepsilon\right) > 0,$$

for every $\varepsilon > 0$. Since $\mu(t)$ is deterministic, it follows that

$$r_t = \mu(t) + \int_0^t \sigma_s(t)dY_s \geq 0$$

only if $\mu(t) \geq 0$. Thus, we have proved the first claim in (ii). To show the second one, namely that the nonnegativity of the short rate r_t for all $t \in [0,T]$, implies the nonnegativity of the forward rate $f_t(T)$, note that

$$\mu(t,T) = \mu(T) - \kappa(-\Sigma_t(T)).$$

Since we have just proved that $r_T \geq 0$ implies $\mu(T) \geq 0$, it suffices to show that $-\kappa(-\Sigma_t(T)) \geq 0$ to deduce that $\mu(t,T) \geq 0$. But this follows easily from $\Sigma_t(T) \geq 0$ combined with (37). Thus, we have $\mu(t,T) \geq 0$, which implies $f_t(T) \geq 0$ by definition of $\mu(t,T)$. □

Completely analogously, we can derive conditions for the nonnegativity of the forward spread $g_t(T)$ and the short term spread λ_t. Let us denote

$$\begin{aligned}\mu^*(t,T) &:= g_t(T) - \int_0^t \sigma_s^*(T)dY_s \\ &= g_0(T) - \kappa(-\Sigma_t(T) - \Sigma_t^*(T)) + \kappa(-\Sigma_0(T) - \Sigma_0^*(T)) \\ &\quad + \kappa(-\Sigma_t(T)) - \kappa(-\Sigma_0(T)) \\ \mu^*(t) &:= \lambda_t - \int_0^t \sigma_s^*(t)dY_s = g_0(t) + \kappa(-\Sigma_0(t) - \Sigma_0^*(t)) - \kappa(-\Sigma_0(t)),\end{aligned}$$

which follows by (40) -(41).

Proposition 3.4 **(i)** *The short term spread λ_t is nonnegative if $\mu^*(t) \geq 0$, for every $t \in [0, T]$.*
(ii) *Assume that the distribution of Y_1 has $[0, \infty)^n$ as its support. Then the converse of (i) is also true, i.e. if $\lambda_t \geq 0$, then $\mu^*(t) \geq 0$, for every t. Moreover, if $\lambda_t \geq 0$, for every $t \in [0, T]$, then $g_t(T) \geq 0$, for every $T \in [0, \bar{T}]$, i.e. the nonnegativity of the short term spread implies the nonnegativity of the forward spread.*

Let us now assume that $Y = (Y_1, Y_2)$ is a two-dimensional nonnegative Lévy process. We shall study in more detail the dependence between its components. But before doing so, let us give an example of the volatility structures that satisfy the conditions of this section and produce nonnegative rates and spreads.

Example 3.5 (Vasicek volatility structure) Assume that the volatility of the forward rates $f_\cdot(T)$ and the volatilities of the forward spreads $g_\cdot(T)$, for $T \in [0, \bar{T}]$, are of Vasicek type, so for every $0 \leq s \leq T \leq \bar{T}$,

$$\sigma_s(T) = \left(\sigma e^{-a(T-s)}, 0\right) \ , \ \ \sigma_s^*(T) = \left(0, \sigma^* e^{-a^*(T-s)}\right), \tag{44}$$

where $\sigma, \sigma^* > 0$ and $a, a^* \neq 0$ are real constants such that μ and μ^* from Propositions 3.3(i) and 3.4(i) are nonnegative. Then

$$\Sigma_t(T) \ = \ \int_t^T \sigma_t(u)du = \left(\frac{\sigma}{a}\left(1 - e^{-a(T-t)}\right), 0\right) \ , \ \ \Sigma_t^*(T) = \left(0, \frac{\sigma^*}{a^*}\left(1 - e^{-a^*(T-t)}\right)\right)$$

and by (22)

$$\bar{\Sigma}_t^*(T) = \Sigma_t(T) + \Sigma_t^*(T) = \left(\frac{\sigma}{a}\left(1 - e^{-a(T-t)}\right), \frac{\sigma^*}{a^*}\left(1 - e^{-a^*(T-t)}\right)\right).$$

The volatilities Σ and Σ^* satisfy the standing Assumption 3.1. Moreover, inserting them into Proposition 3.2, we note that the forward rates $f_\cdot(T)$ and the short rate r are driven solely by the first subordinator Y^1, whereas the forward spreads $g_\cdot(T)$ and the short spread λ are driven by the second subordinator Y^2.

With this volatility specification, one obtains the Lévy Hull–White extended Vasicek model for the short rate r (cf. Corollary 4.5 and equation (4.11) in the risk-free setup of Eberlein and Raible (1999))

$$dr_t = a(\rho(t) - r_t)dt + \sigma dY_t^1.$$

By similar reasoning, one can obtain the Lévy Hull–White extended Vasicek model for the short term spread λ

$$d\lambda_t = a^*(\rho^*(t) - \lambda_t)dt + \sigma^* dY_t^2.$$

The functions ρ and ρ^* are deterministic functions of time which are chosen is such a way that the models fit the initial term structures $f_0(T)$ and $g_0(T)$ observed in the market. Inserting the Vasicek volatilities into equation (39) for r and equation (41) for λ, and differentiating with respect to time, one obtains ρ and ρ^*. We have

$$\rho(t) = f_0(t) + \frac{1}{a}\frac{\partial}{\partial t} f_0(t) + \kappa^1\left(\frac{\sigma}{a}\left(e^{-at}-1\right)\right) - (\kappa^1)'\left(\frac{\sigma}{a}\left(e^{-at}-1\right)\right)\frac{\sigma}{a}e^{-at},$$

and

$$\begin{aligned}
\rho^*(t) \quad = \quad & g_0(t) + \frac{1}{a^*}\frac{\partial}{\partial t} g_0(t) - \kappa^1\left(\frac{\sigma}{a}\left(e^{-at}-1\right)\right) + (\kappa^1)'\left(\frac{\sigma}{a}\left(e^{-at}-1\right)\right)\frac{\sigma}{a}e^{-at} \\
& + \kappa\left(\left(\frac{\sigma}{a}\left(e^{-at}-1\right), \frac{\sigma^*}{a^*}\left(e^{-a^*t}-1\right)\right)\right) \\
& + \frac{1}{a^*}\frac{\partial}{\partial t}\kappa\left(\left(\frac{\sigma}{a}\left(e^{-at}-1\right), \frac{\sigma^*}{a^*}\left(e^{-a^*t}-1\right)\right)\right),
\end{aligned}$$

where κ^1 is the cumulant function of Y^1.

Moreover, this model possesses an affine term structure. It means that the risk-free bond prices can be written as exponential-affine functions of the current level of the short rate r, and the risky bond prices as exponential-affine functions of the short rate r and the short term spread λ. We have

$$B_t(T) = \exp(m(t,T) + n(t,T)r_t), \tag{45}$$

where

$$\begin{aligned}
m(t,T) \quad = \quad & \log\left(\frac{B_0(T)}{B_0(t)}\right) - n(t,T)\left[f_0(t) + \kappa^1\left(\frac{\sigma}{a}\left(e^{-at}-1\right)\right)\right] \\
& - \int_0^t \left[\kappa^1\left(\frac{\sigma}{a}\left(e^{-a(T-s)}-1\right)\right) - \kappa^1\left(\frac{\sigma}{a}\left(e^{-a(t-s)}-1\right)\right)\right] ds
\end{aligned}$$

and

$$n(t,T) = -e^{at}\int_t^T e^{-au}du = \frac{1}{a}\left(e^{-a(T-t)}-1\right).$$

This result is proved in Raible (2000, Theorem 4.8). For $\bar{B}_t^*(T)$ it follows, by exactly the same reasoning and using representation (36), that

$$\bar{B}_t^*(T) = \exp(m(t,T) + n(t,T)r_t + m^*(t,T) + n^*(t,T)\lambda_t), \tag{46}$$

$$\begin{aligned}
m^*(t,T) \quad = \quad & \log\left(-\int_t^T g_0(u)du\right) - n^*(t,T)\left[g_0(t) - \kappa^1\left(\frac{\sigma}{a}\left(e^{-at}-1\right)\right)\right. \\
& \left. + \kappa\left(\frac{\sigma}{a}\left(e^{-at}-1\right), \frac{\sigma^*}{a^*}\left(e^{-a^*t}-1\right)\right)\right] \\
& - \int_0^t \left[\kappa^2\left(\frac{\sigma^*}{a^*}\left(e^{-a^*(T-s)}-1\right)\right) - \kappa^2\left(\frac{\sigma^*}{a^*}\left(e^{-a^*(t-s)}-1\right)\right)\right] ds
\end{aligned}$$

and

$$n^*(t,T) = -e^{a^*t}\int_t^T e^{-a^*u}du = \frac{1}{a^*}\left(e^{-a^*(T-t)} - 1\right),$$

where κ^2 is the cumulant function of Y^2.

Example 3.6 (Dependent drivers) In order to specify the dependence between components Y^1 and Y^2 of the driving process Y, we present here a common factor model. Possible tractable alternatives to create dependence would be to subordinate two independent Lévy processes with an independent common subordinator, or to use a Lévy copula, see Cont and Tankov (2003).

Let us assume that Y^1 and Y^2 are given as

$$Y^1 = Z^1 + Z^3 \quad \text{and} \quad Y^2 = Z^2 + Z^3,$$

where Z^i, $i = 1, 2, 3$, are mutually independent subordinators with drifts b^{Z^i} and Lévy measures F^{Z^i}. Then Y^1 and Y^2 are again subordinators (this follows by Proposition 11.10 and Theorem 21.5 in Sato (1999)) and they are obviously dependent. The Lévy measures and the cumulant functions for subordinators Y^1 and Y^2, as well as for the two-dimensional process $Y = (Y^1, Y^2)$, can be calculated explicitly, as shown below.

Consider a three-dimensional Lévy process $Z = (Z^1, Z^2, Z^3)$, consisting of mutually independent subordinators Z^i, as above. Applying Sato (1999, Exercise 12.10, page 67), the independence of Z^1, Z^2 and Z^3 implies that the Lévy measure F^Z of Z is given by

$$F^Z(A) = \sum_{i=1}^{3} F^{Z^i}(A_i), \qquad A \in \mathcal{B}(\mathbb{R}^3 \setminus \{0\}), \tag{47}$$

where for every i, $A_i = \{x \in \mathbb{R} : x\mathbf{e}_i \in A\}$ with $\mathbf{e}_i$ a unit vector in $\mathbb{R}^3$ with 1 in the i-th position and other entries zero.

Now we simply have to write Y, Y^1 and Y^2 as linear transformations of Z and apply Proposition 11.10 in Sato (1999). For example, we have $Y = UZ$, where

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Hence, $b^Y = Ub^Z$ and the Lévy measure F^Y is given, for $B \in \mathcal{B}(\mathbb{R}^2 \setminus \{0\})$, by

$$F^Y(B) = F^Z(x \in \mathbb{R}^3 : Ux \in B) = F^Z(x \in \mathbb{R}^3 : (x_1 + x_3, x_2 + x_3)^\top \in B),$$

which combined with (47) yields

$$F^Y(B) = F^{Z^1}(x \in \mathbb{R} : (x,0) \in B) + F^{Z^2}(x \in \mathbb{R} : (0,x) \in B) + F^{Z^3}(x \in \mathbb{R} : (x,x) \in B).$$

The cumulant function κ^Y of Y is given, for $z \in \mathbb{R}^2$ such that κ^{Z^i}, $i = 1, 2, 3$, below are well-defined, by

$$\kappa^Y(z) = \kappa^{Z^1}(z_1) + \kappa^{Z^2}(z_2) + \kappa^{Z^3}(z_1 + z_2).$$

This can be derived directly recalling that $\kappa^Y(z) = \log \mathbb{E}[e^{zY}]$ and using independence between Z^1, Z^2 and Z^3.

Similarly, writing each Y^i, $i = 1, 2$, as a linear transformation of Z, we obtain its Lévy measure F^{Y^i}, for $C \in \mathcal{B}(\mathbb{R} \setminus \{0\})$,

$$F^{Y^i}(C) = F^Z(x \in \mathbb{R}^3 : x_i + x_3 \in C) = F^{Z^i}(x \in \mathbb{R} : x \in C) + F^{Z^3}(x \in \mathbb{R} : x \in C)$$

and the drift $b^{Y^i} = b^{Z^i} + b^{Z^3}$, which shows that Y^i is indeed a subordinator (recall Theorem 21.5 in Sato (1999)). The cumulant function κ^{Y^i} of Y^i is given, for $z \in \mathbb{R}$ such that κ^{Z^i} and κ^{Z^3} below are well-defined, by

$$\kappa^{Y^i}(z) = \kappa^{Z^i}(z) + \kappa^{Z^3}(z).$$

To conclude this section, we describe two well-known subordinators: an inverse Gaussian (IG) process and a Gamma process. In addition, we recall an example of a subordinator belonging to the $CGMY$ Lévy family, which was introduced by Carr, Geman, Madan, and Yor (2002). Note that these processes have infinite activity, which makes them suitable drivers for the term structure of interest rates in our model.

Example 3.7 (IG process) According to Kyprianou (2006, Section 1.2.5), a process $Z = (Z_t)_{t\geq 0}$ obtained from a standard Brownian motion W by setting

$$Z_t = \inf\{s > 0 : W_s + bs > t\},$$

where $b > 0$, is an inverse Gaussian (IG) process and has the Lévy measure given by

$$F(dx) = \frac{1}{\sqrt{2\pi x^3}} e^{-\frac{b^2 x}{2}} \mathbf{1}_{\{x>0\}}\, dx.$$

The distribution of Z_t is $IG(\frac{t}{b}, t^2)$. The Lévy measure F satisfies condition (6) for any two constants $\mathcal{K}, \varepsilon > 0$ such that $(1+\varepsilon)\mathcal{K} < \frac{b^2}{2}$. Hence, the cumulant function κ exists for all $z \in (-\frac{b^2}{2}, \frac{b^2}{2})$ (actually for all $z \in (-\infty, \frac{b^2}{2})$ since F is concentrated on $(0,\infty)$) and is given by

$$\kappa(z) = b\left(1 - \sqrt{1 - 2\frac{z}{b^2}}\right). \tag{48}$$

Example 3.8 (Gamma process) The Gamma process $Z = (Z_t)_{t\geq 0}$ with parameters $\alpha, \beta > 0$ is a subordinator with Lévy measure given by

$$F(dx) = \beta x^{-1} e^{-\alpha x} \mathbf{1}_{\{x>0\}}\, dx,$$

see Kyprianou (2006, Section1.2.4). The distribution of Z_t is $\Gamma(t\beta, \alpha)$. The Lévy measure F satisfies condition (6) for any two constants $\mathcal{K}, \varepsilon > 0$ such that $(1+\varepsilon)\mathcal{K} < \alpha$. Hence, the cumulant function κ is well-defined for all $z \in (-\infty, \alpha)$ and is given by

$$\kappa(z) = -\beta \log\left(1 - \frac{z}{\alpha}\right).$$

Example 3.9 (CGMY subordinator) The CGMY Lévy process $Z = (Z_t)_{t\geq 0}$ with parameters $G = \infty$ and $Y < 1$ is a subordinator by Theorem 21.5 in Sato (1999). Its Lévy measure is given by

$$F(dx) = C\frac{\exp(-M|x|)}{|x|^{1+Y}} \mathbf{1}_{\{x>0\}}\, dx,$$

where $C, M > 0$ and $Y < 1$; see Raible (2000, A.3.2). For an overview of the main properties of the class of CGMY Lévy processes we refer to Carr, Geman, Madan, and Yor (2002) or Raible (2000, A.3.2). Note that the cumulant function κ is known in closed form for $Y < 0$ and given by

$$\kappa(z) = C\Gamma(-Y)\left((M-z)^Y - M^Y\right),$$

for all $z \in (-\infty, M)$; see Carr, Geman, Madan, and Yor (2002, Theorem 1) and Raible (2000, A.3.2).

4 Valuation of interest rate derivatives

Here we give an overview of the basic interest rate derivatives where the underlying rate is the LIBOR and calculate their value in our setup. We work with general time-inhomogeneous Lévy processes and under the assumptions of deterministic volatilities and drift terms in equations (16) and (23), as well as under the assumption (17).

Before proceeding with the valuation of interest rate derivatives, let us recall that the forward martingale measure $\mathbb{P}^T$ associated with the date $0 < T \leq \bar{T}$ is a probability measure defined on $(\Omega, \mathcal{F}_T)$ and equivalent to $\mathbb{P}$. It is characterized by the following density process

$$\frac{d\mathbb{P}^T}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{\beta_t B_t(T)}{B_0(T)},$$

where $0 \leq t \leq T$. In our setup this density process is given by (cf. (13))

$$\frac{d\mathbb{P}^T}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(-\int_0^t A_s(T)ds - \int_0^t \Sigma_s(T)dY_s \right). \tag{49}$$

Note that the density process is $\mathcal{E}$-adapted. The payoffs of the derivatives that we are going to study in the sequel are typically some combinations of deterministic functions of the LIBOR rates $L_T(T, T+\delta)$, which are $\mathcal{E}_T$-measurable random variables, for any $T \in [0, \bar{T} - \delta]$. Then we have

$$\mathbb{E}[f(L_T(T, T+\delta))|\mathcal{F}_t] = \mathbb{E}[f(L_T(T, T+\delta))|\mathcal{E}_t],$$

for any deterministic, Borel measurable function $f : \mathbb{R} \to \mathbb{R}$. This property is due to the immersion property between $\mathcal{E}$ and $\mathcal{F}$ (see Bielecki and Rutkowski (2002, Section 6.1.1)), which by assumption holds in our model. Moreover, the property holds true under any forward measure $\mathbb{P}_T$ as well, since the density process in (49) is $\mathcal{E}$-adapted. Henceforth in all computations we shall automatically replace $\mathcal{F}_t$ by $\mathcal{E}_t$.

Finally, note that in a multiple-curve setup the process $\left(\frac{\bar{B}^*_t(T)}{\bar{B}^*_t(T+\delta)}\right)_{0 \leq t \leq T}$ is not a martingale under the forward measure $\mathbb{P}^{T+\delta}$. Consequently, the forward LIBOR rate, if defined as $L_t(T, T+\delta) = \frac{1}{\delta}\left(\frac{\bar{B}^*_t(T)}{\bar{B}^*_t(T+\delta)} - 1\right)$, would be different from a forward rate implied by a forward rate agreement for the future time interval $[T, T+\delta]$, as we shall see below. In the one-curve setup, the forward LIBOR rate defined as $L_t(T, T+\delta) = \frac{1}{\delta}\left(\frac{B_t(T)}{B_t(T+\delta)} - 1\right)$ is precisely the FRA rate for $[T, T+\delta]$.

4.1 Forward rate agreements

The simplest interest rate derivative is a forward rate agreement (FRA) with inception date T and maturity $T+\delta$, where $0 \leq T \leq \bar{T} - \delta$. Let us denote the fixed rate by K and the notional amount by N. The payoff of such an agreement at maturity $T+\delta$ is equal to

$$P^{FRA}(T+\delta; T, T+\delta, K, N) = N\delta(L_T(T, T+\delta) - K),$$

where $L_T(T, T+\delta)$ is the T-spot LIBOR rate. Thus, the value of the FRA at time $t \leq T$ is calculated as the conditional expectation with respect to the forward measure $\mathbb{P}^{T+\delta}$ associated with the date $T+\delta$ and is given by

$$P^{FRA}(t; T, T+\delta, K, N) \quad = \quad N\delta B_t(T+\delta)\mathbb{E}^{\mathbb{P}^{T+\delta}}[L_T(T, T+\delta) - K|\mathcal{E}_t].$$

We emphasize again that the forward rate implied by this FRA, that is the rate K_t such that $P^{FRA}(t; T, T+\delta, K_t, N) = 0$, differs in the multiple-curve setup from the classical forward LIBOR rate.

Let us derive the value of the FRA and calculate the forward rate K_t in the model. Using definition (4) of the LIBOR rate $L_T(T, T+\delta)$ we have

$$P^{FRA}(t; T, T+\delta, K, N) = NB_t(T+\delta)\mathbb{E}^{\mathbb{P}^{T+\delta}}\left[\frac{1}{\bar{B}^*_T(T+\delta)} - \bar{K}|\mathcal{E}_t\right], \tag{50}$$

where $\bar{K} = 1 + \delta K$. The key issue is thus to compute conditional expectations of the form

$$v_t^{T,S} := \mathbb{E}^{\mathbb{P}^S}\left[\frac{1}{\bar{B}^*_T(S)}\,\Big|\,\mathcal{E}_t\right], \qquad 0 \leq t \leq T \leq S. \tag{51}$$

Inserting $S = T + \delta$ and (23) into (51) we obtain

$$\begin{aligned} v_t^{T,T+\delta} &= \frac{\bar{B}^*_0(T)}{\bar{B}^*_0(T+\delta)} \exp\left(\int_0^T (\bar{A}^*_s(T+\delta) - \bar{A}^*_s(T))ds\right) \\ &\quad \times \mathbb{E}^{\mathbb{P}^{T+\delta}}\left[\exp\left(\int_0^T (\bar{\Sigma}^*_s(T+\delta) - \bar{\Sigma}^*_s(T))dY_s\right)\,\Big|\,\mathcal{E}_t\right] \\ &= c^{T,T+\delta} \exp\left(\int_0^t (\bar{\Sigma}^*_s(T+\delta) - \bar{\Sigma}^*_s(T))dY_s\right) \\ &\quad \times \mathbb{E}^{\mathbb{P}^{T+\delta}}\left[\exp\left(\int_t^T (\bar{\Sigma}^*_s(T+\delta) - \bar{\Sigma}^*_s(T))dY_s\right)\right], \end{aligned} \tag{52}$$

where

$$c^{T,T+\delta} = \frac{\bar{B}^*_0(T)}{\bar{B}^*_0(T+\delta)} \exp\left(\int_0^T (\bar{A}^*_s(T+\delta) - \bar{A}^*_s(T))ds\right).$$

For the second equality in (52) we use the fact that $\int_0^t (\bar{\Sigma}^*_s(T+\delta) - \bar{\Sigma}^*_s(T))dY_s$ is $\mathcal{E}_t$-measurable. Moreover, since Y is a time-inhomogeneous Lévy process under the measure $\mathbb{P}^{T+\delta}$, its increments are independent (cf. Proposition 2.3 and Lemma 2.5 in Kluge (2005)). This combined with the deterministic volatility structure which is integrated with respect to Y yields the equality.

The remaining expectation can be calculated making use of Proposition 3.1 in Eberlein and Kluge (2006b), which yields

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}^{T+\delta}}\left[\exp\left(\int_t^T (\bar{\Sigma}^*_s(T+\delta) - \bar{\Sigma}^*_s(T))dY_s\right)\right] \\ &\quad = \exp\left(\int_t^T \kappa_s^{\mathbb{P}^{T+\delta}}(\bar{\Sigma}^*_s(T+\delta) - \bar{\Sigma}^*_s(T))ds\right), \end{aligned} \tag{53}$$

where $\kappa_s^{\mathbb{P}^{T+\delta}}$ denotes the cumulant function of Y under the measure $\mathbb{P}^{T+\delta}$. However, to obtain the expression for this expectation using the cumulant function κ_s of Y under the measure $\mathbb{P}$, we have the following sequence of equalities

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{T+\delta}}\left[\exp\left(\int_t^T(\bar{\Sigma}_s^*(T+\delta)-\bar{\Sigma}_s^*(T))dY_s\right)\right] \qquad (54)\\
&= \exp\left(-\int_0^T A_s(T+\delta)ds\right)\\
&\quad\times\mathbb{E}\left[\exp\left(\int_t^T(\bar{\Sigma}_s^*(T+\delta)-\bar{\Sigma}_s^*(T))dY_s-\int_0^T\Sigma_s(T+\delta)dY_s\right)\right]\\
&= \exp\left(-\int_0^T\kappa_s(-\Sigma_s(T+\delta))ds\right)\mathbb{E}\left[\exp\left(\int_0^t(-\Sigma_s(T+\delta))dY_s\right)\right]\\
&\quad\times\mathbb{E}\left[\exp\left(\int_t^T(\bar{\Sigma}_s^*(T+\delta)-\bar{\Sigma}_s^*(T)-\Sigma_s(T+\delta))dY_s\right)\right]\\
&= \exp\left(\int_t^T\left(\kappa_s(\bar{\Sigma}_s^*(T+\delta)-\bar{\Sigma}_s^*(T)-\Sigma_s(T+\delta))-\kappa_s(-\Sigma_s(T+\delta))\right)ds\right),
\end{aligned}$$

where we have used equation (49) for the first equality, and the drift condition (17) plus the independence of increments of Y for the second one. The third equality follows by Eberlein and Kluge (2006b, Proposition 3.1)). Finally, we obtain

$$\begin{aligned}
v_t^{T,T+\delta} &= c^{T,T+\delta}\exp\left(\int_0^t(\bar{\Sigma}_s^*(T+\delta)-\bar{\Sigma}_s^*(T))dY_s\right) \qquad (55)\\
&\quad\times\exp\left(\int_t^T\left(\kappa_s(\bar{\Sigma}_s^*(T+\delta)-\bar{\Sigma}_s^*(T)-\Sigma_s(T+\delta))-\kappa_s(-\Sigma_s(T+\delta))\right)ds\right).
\end{aligned}$$

Note that along the same lines one obtains a formula for the cumulant function $\kappa_s^{\mathbb{P}^{T+\delta}}$

$$\kappa_s^{\mathbb{P}^{T+\delta}}(z)=\kappa_s(z-\Sigma_s(T+\delta))-\kappa_s(-\Sigma_s(T+\delta)), \qquad (56)$$

for $z\in\mathbb{R}^n$ such that $\kappa_s(z-\Sigma_s(T+\delta))$ is well-defined. This follows by combining (53) and (54), for every $t\in[0,T]$ and for $\bar{\Sigma}_s^*(T+\delta)-\bar{\Sigma}_s^*(T)$ replaced with z.

In particular we proved the following

Proposition 4.1 *The value of the FRA at time $t=0$ is given by*

$$P^{FRA}(0;T,T+\delta,K,N) = NB_0(T+\delta)\left[v_0^{T,T+\delta}-\bar{K}\right],$$

where

$$\begin{aligned}
v_0^{T,T+\delta} &= \frac{\bar{B}_0^*(T)}{\bar{B}_0^*(T+\delta)}\exp\left(\int_0^T\left(\bar{A}_s^*(T+\delta)-\bar{A}_s^*(T)-\kappa_s(-\Sigma_s(T+\delta))\right)ds\right)\\
&\quad\times\exp\left(\int_0^T\kappa_s(\bar{\Sigma}_s^*(T+\delta)-\bar{\Sigma}_s^*(T)-\Sigma_s(T+\delta))ds\right).
\end{aligned}$$

The forward rate K_0 implied by this FRA is given by

$$K_0=\frac{1}{\delta}\left[v_0^{T,T+\delta}-1\right]. \qquad (57)$$

The spread above the one-curve forward rate given by $\frac{1}{\delta}\left(\frac{B_0(T)}{B_0(T+\delta)}-1\right)$, is equal to

$$Spread_0^{FRA} = \frac{1}{\delta}\left[v_0^{T,T+\delta}-\frac{B_0(T)}{B_0(T+\delta)}\right]. \qquad (58)$$

As soon as the driving process Y and the parameters of the model are specified, all these values can be easily computed. We provide an example in Section 5.

4.2 Interest rate swaps

An interest rate swap is a financial contract between two parties to exchange one stream of future interest payments for another, based on a specified notional amount N. Here we consider a fixed-for-floating swap, where a fixed payment is exchanged for a floating payment linked to the LIBOR rate. We assume, as is typical, that the LIBOR rate is set in advance and the payments are made in arrears. The swap is initiated at time $T_0 \geq 0$. Denote by $T_1 < \cdots < T_n$, where $T_1 > T_0$, a collection of the payment dates and by S the fixed rate. Then the time-t value of the swap for the receiver of the floating rate is given by, for $t \leq T_0$,

$$\begin{aligned} P^{Sw}(t; T_1, T_n) &= N \sum_{k=1}^{n} \delta_{k-1} B_t(T_k) \mathbb{E}^{\mathbf{P}^{T_k}} \left[L_{T_{k-1}}(T_{k-1}, T_k) - S | \mathcal{E}_t \right] \\ &= N \sum_{k=1}^{n} P^{FRA}(t; T_{k-1}, T_k, S, 1) \\ &= N \sum_{k=1}^{n} B_t(T_k) \left(v_t^{T_{k-1}, T_k} - \bar{S}_{k-1} \right), \end{aligned} \tag{59}$$

where $\delta_{k-1} = T_k - T_{k-1}$, $\bar{S}_{k-1} = 1 + \delta_{k-1} S$, and $v_t^{T_{k-1}, T_k}$ is given by (55), for every $k = 1, \ldots, n$. This formula follows directly from (50) and (51).

The swap rate $S(t; T_1, T_n)$ is the rate that makes the time-t value $P^{Sw}(t; T_1, T_n)$ of the swap equal to zero. Therefore,

Proposition 4.2 *The swap rate $S(t; T_1, T_n)$, for $t \leq T_0$, is given by*

$$S(t; T_1, T_n) = \frac{\sum_{k=1}^{n} B_t(T_k)(v_t^{T_{k-1}, T_k} - 1)}{\sum_{k=1}^{n} \delta_{k-1} B_t(T_k)}. \tag{60}$$

4.3 Overnight indexed swaps (OIS)

In an overnight indexed swap (OIS) the counterparties exchange a stream of fixed-rate payments for a stream of floating-rate payments linked to a compounded overnight rate. Let us assume the same tenor structure as in the previous subsection is given and denote again the fixed rate by S. Similarly to Filipović and Trolle (2011, Section 2.5, equation (11)), the time-t value of the swap for the receiver of the floating rate is given by, for $t \leq T_0$,

$$P^{OIS}(t; T_1, T_n) = N \left(B_t(T_0) - B_t(T_n) - S \sum_{k=1}^{n} \delta_{k-1} B_t(T_k) \right).$$

The OIS rate $OIS(t; T_1, T_n)$, for $t \leq T_0$, is given by

$$OIS(t; T_1, T_n) = \frac{B_t(T_0) - B_t(T_n)}{\sum_{k=1}^{n} \delta_{k-1} B_t(T_k)}. \tag{61}$$

The LIBOR-OIS spread at time T, for the interval $[T, T+\delta]$, where $0 \leq T \leq \bar{T} - \delta$, is thus obtained as a difference of (4) and (61) (for a single payment date) as

$$L_T(T, T+\delta) - OIS(T; T+\delta, T+\delta) = \frac{1}{\delta} \left(\frac{1}{\bar{B}_T^*(T+\delta)} - \frac{1}{B_T(T+\delta)} \right). \tag{62}$$

Note that this spread is nonnegative as soon as the forward spreads are nonnegative (cf. (1) and (3)).

The LIBOR-OIS swap spread at time $0 \le t \le T_0$ is by definition the difference between the swap rate (60) of the LIBOR-indexed interest rate swap and the OIS rate (61) and is given by

$$S(t;T_1,T_n) - OIS(t;T_1,T_n) = \frac{\sum_{k=1}^{n} B_t(T_k)(v_t^{T_{k-1},T_k} - 1) - B_t(T_0) + B_t(T_n)}{\sum_{k=1}^{n} \delta_{k-1} B_t(T_k)}. \tag{63}$$

4.4 Basis swaps

A basis swap is an interest rate swap, where two floating payments linked to the LIBOR rates of different tenors are exchanged. For example, a buyer of such a swap receives semiannually a 6m-LIBOR and pays quarterly a 3m-LIBOR, both set in advance and paid in arrears. Note that there also exist other conventions regarding the payments on the two legs of a basis swap. A more detailed account on basis swaps can be found in Mercurio (2010a, Section 5.2) and Filipović and Trolle (2011, Section 2.4 and Appendix F). Let us consider a basis swap with the two tenor structures denoted by $\mathcal{T}^1 = \{T_0^1 < \ldots < T_{n_1}^1\}$ and $\mathcal{T}^2 = \{T_0^2 < \ldots < T_{n_2}^2\}$, where $T_0^1 = T_0^2 \ge 0$, $T_{n_1}^1 = T_{n_2}^2 = \widehat{T}$, and $\mathcal{T}^1 \subset \mathcal{T}^2$. The notional amount is denoted by N and the swap is initiated at time T_0^1, so that the first payments are due at T_1^1 and T_1^2. The time-t value of such an agreement is given by, for $t \le T_0^1$,

$$\begin{aligned} P^{BSw}(t;\widehat{T},N) \quad &= \quad N\left(\sum_{i=1}^{n_1} \delta_{i-1}^1 B_t(T_i^1) \mathbb{E}^{\mathbb{P}^{T_i^1}}[L_{T_{i-1}^1}(T_{i-1}^1,T_i^1)|\mathcal{E}_t]\right. \\ &\quad \left. - \sum_{j=1}^{n_2} \delta_{j-1}^2 B_t(T_j^2) \mathbb{E}^{\mathbb{P}^{T_j^2}}[L_{T_{j-1}^2}(T_{j-1}^2,T_j^2)|\mathcal{E}_t]\right). \end{aligned} \tag{64}$$

Making use of (50) and (51) we obtain

Proposition 4.3 *The value of the basis swap at time $t \le T_0^1 = T_0^2$ is given by*

$$P^{BSw}(t;\widehat{T},N) = N\left(\sum_{i=1}^{n_1} B_t(T_i^1)\left(v_t^{T_{i-1}^1,T_i^1} - 1\right) - \sum_{j=1}^{n_2} B_t(T_j^2)\left(v_t^{T_{j-1}^2,T_j^2} - 1\right)\right), \tag{65}$$

where $v_t^{T_{k-1}^x,T_k^x}$ is given by (55), for each tenor structure $\mathcal{T}^x$, $x = 1,2$, $k = 1,\ldots,n_x$.

Note that in the classical one-curve setup the time-t value of such a swap is zero. Since the crisis, markets quote positive basis swap spreads that have to be added to the smaller tenor leg, which is consistently accounted for in our setup; see Section 5 for a numerical example. More precisely, on the smaller tenor leg the floating interest rate $L_{T_{j-1}^2}(T_{j-1}^2,T_j^2)$ at T_j^2 is replaced by $L_{T_{j-1}^2}(T_{j-1}^2,T_j^2) + S^{BSw}(t;\widehat{T})$, for every $j = 1,\ldots,n_2$, where $S^{BSw}(t;\widehat{T})$ is the basis swap spread calculated below.

Proposition 4.4 *The basis swap spread $S^{BSw}(t;\widehat{T})$ at time t is given by*

$$S^{BSw}(t;\widehat{T}) = \frac{\sum_{i=1}^{n_1} B_t(T_i^1)\left(v_t^{T_{i-1}^1,T_i^1} - 1\right) - \sum_{j=1}^{n_2} B_t(T_j^2)\left(v_t^{T_{j-1}^2,T_j^2} - 1\right)}{\sum_{j=1}^{n_2} \delta_{j-1}^2 B_t(T_j^2)}. \tag{66}$$

Proof. The result follows from (64), where $L_{T^2_{j-1}}(T^2_{j-1},T^2_j)$ is replaced by $L_{T^2_{j-1}}(T^2_{j-1},T^2_j)+S$, for every j, where S denotes the spread. The basis swap spread $S^{BSw}(t;\widehat{T})$ is then obtained by solving $P^{BSw}(t;\widehat{T},N)=0$, i.e. it is the spread S that makes the value of the basis swap zero at time t. $\square$

Let us check that the value of the basis swap in a pre-crisis one-curve setup is indeed zero. We recall that in this setup the forward LIBOR rates, which were defined using the risk-free zero coupon bonds as $\left(L_t(T,T+\delta)=\frac{1}{\delta}\left(\frac{B_t(T)}{B_t(T+\delta)}-1\right)\right)_{0\le t\le T}$, are martingales under the corresponding forward measures. We thus have

$$\begin{aligned}
P^{BSw}(t;\widehat{T},N) &= N\left(\sum_{i=1}^{n_1}\delta^1_{i-1}B_t(T^1_i)\mathbb{E}^{\mathbf{P}^{T^1_i}}[L_{T^1_{i-1}}(T^1_{i-1},T^1_i)|\mathcal{E}_t]\right.\\
&\qquad\left.-\sum_{j=1}^{n_2}\delta^2_{j-1}B_t(T^2_j)\mathbb{E}^{\mathbf{P}^{T^2_j}}[L_{T^2_{j-1}}(T^2_{j-1},T^2_j)|\mathcal{E}_t]\right)\\
&= N\left(\sum_{i=1}^{n_1}\delta^1_{i-1}B_t(T^1_i)L_t(T^1_{i-1},T^1_i)-\sum_{j=1}^{n_2}\delta^2_{j-1}B_t(T^2_j)L_t(T^2_{j-1},T^2_j)\right)\\
&= N\Big((B_t(T^1_0)-B_t(T^1_{n_1}))-(B_t(T^2_0)-B_t(T^2_{n_2}))\Big)=0,
\end{aligned}$$

by the initial assumptions $T^1_0=T^2_0$ and $T^1_{n_1}=T^2_{n_2}$.

In the multiple-curve setup we cannot use the same calculation, since now the LIBOR rates are not martingales under the classical forward measures. Hence, one ends up with formula (65), which in general yields a non-zero value of the basis swap and produces a positive basis swap spread (66) (cf. Table 2 in Section 5).

Remark 4.5 Note that in practice the floating-rate payments and the fixed-rate payments of the swaps defined in Sections 4.2 and 4.3 typically do not occur with the same frequency, as we have assumed to simplify the notation. In that case one has to work with two different tenor structures (as in the case of basis swaps) and modify the formulas accordingly. Then the price of a basis swap $P^{BSw}(t;\mathcal{T}^1,\mathcal{T}^2)$ with tenor structures $\mathcal{T}^1$ and $\mathcal{T}^2$ can be expressed as a difference of prices of two interest rate swaps which have the same tenor structure for the fixed-rate payments (e.g. $\mathcal{T}^1$) and the same fixed rate S, and the floating-rate payments are done on the two tenor structures $\mathcal{T}^1$ and $\mathcal{T}^2$ of the basis swap. More precisely, we have

$$P^{BSw}(t;\mathcal{T}^1,\mathcal{T}^2)=P^{Sw}(t;\mathcal{T}^1,\mathcal{T}^1)-P^{Sw}(t;\mathcal{T}^1,\mathcal{T}^2), \tag{67}$$

where

$$P^{Sw}(t;\mathcal{T}^1,\mathcal{T}^x)=N\left(\sum_{i=1}^{n_x}B_t(T^x_i)\left(v_t^{T^x_{i-1},T^x_i}-1\right)-\sum_{j=1}^{n_1}\delta^1_{j-1}SB_t(T^1_j)\right)$$

is the time-t value of an interest rate swap, whose fixed-rate payments are done on the tenor structure $\mathcal{T}^1$ and the floating-rate payments linked to the LIBOR on $\mathcal{T}^x$, $x=1,2$.

4.5 Caps and floors

Recall that an interest rate cap (respectively floor) is a financial contract in which the buyer receives payments at the end of each period in which the interest rate exceeds (respectively falls below) a mutually agreed strike level. The payment that the seller has to make covers exactly the difference (whenever positive) between the strike K and the interest rate at the end of each period. Every cap (respectively floor) is a series of caplets (respectively floorlets). The time-t price of a caplet with strike K and maturity $T \geq t$, which is settled in arrears, is given by

$$\begin{aligned} P^{Cpl}(t;T,K) &= \delta\, B_t(T+\delta)\mathbb{E}^{\mathbf{P}^{T+\delta}}\left[(L_T(T,T+\delta)-K)^+ \,\Big|\, \mathcal{E}_t\right] \\ &= B_t(T+\delta)\mathbb{E}^{\mathbf{P}^{T+\delta}}\left[\left(\frac{1}{\bar{B}^*_T(T+\delta)} - \bar{K}\right)^+ \,\Big|\, \mathcal{E}_t\right], \end{aligned}$$

where $\bar{K} = 1 + \delta K$.

It is worthwhile mentioning that the classical transformation of a caplet into a put option on a bond does not work in the multiple-curve setup. More precisely, the fact that the payoff $\left((1+\delta L_T(T,T+\delta)) - \bar{K}\right)^+$ settled at time $T+\delta$ is equivalent to the payoff $B_T(T+\delta)\left((1+\delta L_T(T,T+\delta)) - \bar{K}\right)^+$ settled at time T is still valid, since the OIS discounting is used. However, this will not yield the desired cancelation of discount factors. Since the LIBOR rate depends on the $\bar{B}^*(T)$ bonds and the risk-free $B(T)$ bonds are used for discounting, we have

$$B_T(T+\delta)\left((1+\delta L_T(T,T+\delta)) - \bar{K}\right)^+ = B_T(T+\delta)\left(\frac{1}{\bar{B}^*_T(T+\delta)} - \bar{K}\right)^+,$$

which cannot be simplified further as in the one-curve case.

Let us now calculate the value of the caplet at time $t = 0$ using the Fourier transform method. We have

$$\begin{aligned} P^{Cpl}(0;T,K) &= B_0(T+\delta)\mathbb{E}^{\mathbf{P}^{T+\delta}}\left[\left(\frac{1}{\bar{B}^*_T(T+\delta)} - \bar{K}\right)^+\right] \\ &= B_0(T+\delta)\mathbb{E}^{\mathbf{P}^{T+\delta}}\left[\left(e^X - \bar{K}\right)^+\right], \end{aligned}$$

where X is a random variable given by (see (23))

$$X := \log\frac{\bar{B}^*_0(T)}{\bar{B}^*_0(T+\delta)} + \int_0^T (\bar{A}^*_s(T+\delta) - \bar{A}^*_s(T))ds + \int_0^T (\bar{\Sigma}^*_s(T+\delta) - \bar{\Sigma}^*_s(T))dY_s.$$

Let us denote by $M_X^{T+\delta}$ the moment generating function of X under the measure $\mathbf{P}^{T+\delta}$, i.e.

$$M_X^{T+\delta}(z) = \mathbb{E}^{\mathbf{P}^{T+\delta}}\left[e^{zX}\right],$$

for $z \in \mathbb{R}$ such that the above expectation is finite. We have

$$\begin{aligned} M_X^{T+\delta}(z) &= \exp\left(-\int_0^T \kappa_s(-\Sigma_s(T+\delta))ds\right) \qquad (68) \\ &\quad \times \exp\left(z\left(\log\frac{\bar{B}^*_0(T)}{\bar{B}^*_0(T+\delta)} + \int_0^T \left(\bar{A}^*_s(T+\delta) - \bar{A}^*_s(T)\right)ds\right)\right) \\ &\quad \times \exp\left(\int_0^T \kappa_s\left(z\left(\bar{\Sigma}^*_s(T+\delta) - \bar{\Sigma}^*_s(T)\right) - \Sigma_s(T+\delta)\right)ds\right), \end{aligned}$$

where κ_s is the cumulant function of Y under the measure $\mathbb{P}$. The derivation of this formula follows along similar lines as the computations in Section 4.1. In particular, we have used equations (49) and (17) and Proposition 3.1 in Eberlein and Kluge (2006b).

Let us impose some conditions on the boundedness of the volatility structures Σ and Σ^* for the sake of the next result. We assume that there exists a positive constant $\widetilde{\mathcal{K}} < \frac{\mathcal{K}}{3}$ such that $\Sigma_s(T) \leq \widetilde{\mathcal{K}}$ and $\Sigma_s^*(T) \leq \widetilde{\mathcal{K}}$ componentwise and for all $s, T \in [0, \bar{T}]$ (note that this is a slightly stronger boundedness condition than the one in Assumption 3.1).

Now, applying Theorem 2.2 and Example 5.1 in Eberlein, Glau, and Papapantoleon (2010) we obtain

Proposition 4.6 *The price at time $t = 0$ of a caplet with strike K and maturity T is given by*

$$P^{Cpl}(0; T, K) = \frac{B_0(T+\delta)}{2\pi} \int_{\mathbb{R}} \frac{\bar{K}^{1+\mathrm{i}v-R} M_X^{T+\delta}(R-\mathrm{i}v)}{(\mathrm{i}v-R)(1+\mathrm{i}v-R)} dv, \tag{69}$$

for any $R \in \left(1, \frac{\mathcal{K}-\widetilde{\mathcal{K}}}{2\widetilde{\mathcal{K}}}\right)$.

Proof. One has to apply Theorem 2.2 in Eberlein, Glau, and Papapantoleon (2010) with the Fourier transform of the caplet payoff function derived in Example 5.1 of the same paper, where other prerequisites for Theorem 2.2 related to the payoff function are also checked. This Fourier transform is well-defined for any $R \in (1, +\infty)$. To ensure that $M_X^{T+\delta}(R - \mathrm{i}v)$ is finite, it suffices to take any $R \in \left(1, \frac{\mathcal{K}-\widetilde{\mathcal{K}}}{2\widetilde{\mathcal{K}}}\right)$. More precisely, for every $i = 1, \ldots, n$,

$$\begin{aligned} \left|R\left(\bar{\Sigma}_s^{i,*}(T+\delta) - \bar{\Sigma}_s^{i,*}(T)\right) - \Sigma_s^i(T+\delta)\right| &\leq R\left|\bar{\Sigma}_s^{i,*}(T+\delta) - \bar{\Sigma}_s^{i,*}(T)\right| + \left|\Sigma_s^i(T+\delta)\right| \\ &\leq R2\widetilde{\mathcal{K}} + \widetilde{\mathcal{K}} \\ &\leq \frac{\mathcal{K}-\widetilde{\mathcal{K}}}{2\widetilde{\mathcal{K}}} 2\widetilde{\mathcal{K}} + \widetilde{\mathcal{K}} < \mathcal{K}, \end{aligned}$$

and thus $M_X^{T+\delta}(R - \mathrm{i}v)$ is finite (compare (68) and recall that κ_s is well-defined for all $z \in \mathbb{C}^n$ such that $\Re z \in [-(1+\varepsilon)\mathcal{K}, (1+\varepsilon)\mathcal{K}]^n$). □

4.6 Swaptions

A swaption is an option to enter an interest rate swap with swap rate S and maturity T_n at a pre-specified date $T = T_0$. Let us consider the swap from Section 4.2. Recall that a swaption can be seen as a sequence of fixed payments $\delta_{j-1}\left(S(T; T_1, T_n) - S\right)^+$, $j = 1, \ldots, n$, that are received at payment dates $T_1, \ldots, T_n$, where $S(T; T_1, T_n)$ is the swap rate of the underlying swap at time T. Hence, the value at time t of the swaption is given by

$$P^{Swn}(t; T, T_n, S) = B_t(T) \sum_{j=1}^{n} \delta_{j-1} \mathbb{E}^{\mathbf{P}^T}\left[B_T(T_j)\left(S(T; T_1, T_n) - S\right)^+ |\mathcal{E}_t\right];$$

see Musiela and Rutkowski (2005, Section 13.1.2, p.482). At time $t = 0$ we have

$$\begin{aligned} P^{Swn}(0; T, T_n, S) &= B_0(T) \mathbb{E}^{\mathbf{P}^T}\left[\sum_{j=1}^{n} \delta_{j-1} B_T(T_j)\left(S(T; T_1, T_n) - S\right)^+\right] \\ &= B_0(T) \mathbb{E}^{\mathbf{P}^T}\left[\left(\sum_{j=1}^{n} B_T(T_j) v_T^{T_{j-1}, T_j} - \sum_{j=1}^{n} B_T(T_j) \bar{S}_{j-1}\right)^+\right], \end{aligned}$$

which follows by inserting (60). Recall that $\bar{S}_{j-1} = 1 + \delta_{j-1} S$ and $v_T^{T_{j-1},T_j}$ is given by (52).

To proceed we assume in addition that the conditions of Example 3.5 are satisfied, i.e. the driving process Y is two-dimensional nonnegative Lévy process and we assume the Vasicek volatility structures (44). Recall that for each j, $B_T(T_j)$ is given by (cf. equation (16))

$$B_T(T_j) = \frac{B_0(T_j)}{B_0(T)} \exp\left(\int_0^T (A_s(T) - A_s(T_j))ds + \int_0^T (\Sigma_s(T) - \Sigma_s(T_j))dY_s \right)$$

and $v_T^{T_{j-1},T_j}$ is given by (cf. equations (52) and (53))

$$\begin{aligned} v_T^{T_{j-1},T_j} &= c^{T_{j-1},T_j} \exp\left(\int_0^T (\bar{\Sigma}_s^*(T_j) - \bar{\Sigma}_s^*(T_{j-1}))dY_s \right) \\ &\quad \times \exp\left(\int_T^{T_{j-1}} \kappa_s^{\mathbf{P}^{T_j}} (\bar{\Sigma}_s^*(T_j) - \bar{\Sigma}_s^*(T_{j-1}))ds \right), \end{aligned}$$

where $\kappa_s^{\mathbf{P}^{T_j}}$ is given by (56) (with $T + \delta$ replaced by T_j). The volatilities appearing above can be written as

$$\begin{aligned} \Sigma_s(T) - \Sigma_s(T_j) &= \left(\frac{\sigma}{a} e^{as} \left(e^{-aT_j} - e^{-aT} \right), 0 \right) \\ \bar{\Sigma}_s^*(T_j) - \bar{\Sigma}_s^*(T_{j-1}) &= \left(\frac{\sigma}{a} e^{as} \left(e^{-aT_{j-1}} - e^{-aT_j} \right), \frac{\sigma^*}{a^*} e^{a^* s} \left(e^{-a^* T_{j-1}} - e^{-a^* T_j} \right) \right), \end{aligned}$$

which motivates us to introduce the following $\mathcal{E}_T$-measurable random vector

$$X_T = \left(\int_0^T e^{as} dY_s^1, \int_0^T e^{a^* s} dY_s^2 \right).$$

Consequently, for each j we can rewrite $B_T(T_j)$ and $v_T^{T_{j-1},T_j}$ as

$$B_T(T_j) = c^{j,0} e^{c^{j,1} X_T^1} \quad \text{and} \quad v_T^{T_{j-1},T_j} = \bar{c}^{j,0} e^{\bar{c}^j X_T},$$

where

$$\begin{aligned} c^{j,0} &= \frac{B_0(T_j)}{B_0(T)} \exp\left(\int_0^T (A_s(T) - A_s(T_j))ds \right), \\ c^{j,1} &= \frac{\sigma}{a} \left(e^{-aT_j} - e^{-aT} \right), \\ \bar{c}^{j,0} &= c^{T_{j-1},T_j} \exp\left(\int_T^{T_{j-1}} \kappa_s^{\mathbf{P}^{T_j}} (\bar{\Sigma}_s^*(T_j) - \bar{\Sigma}_s^*(T_{j-1}))ds \right), \\ \bar{c}^j &= \left(\frac{\sigma}{a} \left(e^{-aT_{j-1}} - e^{-aT_j} \right), \frac{\sigma^*}{a^*} \left(e^{-a^* T_{j-1}} - e^{-a^* T_j} \right) \right) \end{aligned}$$

are deterministic constants. Hence, the value at time $t = 0$ of the swaption depends only on the distribution of the random vector X_T under the measure $\mathbb{P}^T$:

$$\begin{aligned} P^{Swn}(0; T, T_n, S) &= B_0(T) \mathbb{E}^{\mathbf{P}^T} \left[\left(\sum_{j=1}^n c^{j,0} e^{c^{j,1} X_T^1} \bar{c}^{j,0} e^{\bar{c}^j X_T} - \sum_{j=1}^n \bar{S}_{j-1} c^{j,0} e^{c^{j,1} X_T^1} \right)^+ \right] \\ &= B_0(T) \mathbb{E}^{\mathbf{P}^T} \left[\left(\sum_{j=1}^n a^{j,0} e^{a^{j,1} X_T^1 + a^{j,2} X_T^2} - \sum_{j=1}^n b^{j,0} e^{b^{j,1} X_T^1} \right)^+ \right], \end{aligned} \tag{70}$$

where $a^{j,0} = c^{j,0}\bar{c}^{j,0}$, $a^{j,1} = c^{j,1} + \bar{c}^{j,1}$, $a^{j,2} = \bar{c}^{j,2}$, $b^{j,0} = \bar{S}_{j-1}c^{j,0}$ and $b^{j,1} = c^{j,1}$. To calculate this expectation we shall use the moment generating function $M^T_{X_T}$ of X_T under the measure $\mathbb{P}^T$, which is given explicitly in terms of the characteristics of Y by

$$\begin{aligned} M^T_{X_T}(z) &= \mathbb{E}^{\mathbb{P}^T}\left[e^{z_1 X^1_T + z_2 X^2_T}\right] \\ &= \mathbb{E}^{\mathbb{P}^T}\left[e^{\int_0^T z_1 e^{as} dY^1_s + \int_0^T z_2 e^{a^* s} dY^2_s}\right] \\ &= \exp\left(-\int_0^T \kappa\left(\left(\frac{\sigma}{a}(e^{-a(T-s)} - 1), 0\right)\right) ds\right) \\ &\quad \times \exp\left(\int_0^T \kappa\left(z_1 e^{as} - \frac{\sigma}{a}(1 - e^{-a(T-s)}), z_2 e^{a^* s}\right) ds\right), \end{aligned} \tag{71}$$

for any $z \in \mathbb{R}^2$ such that the expectation above is finite. This follows along the same lines as in (54), for a deterministic function $U(s) = \left(z_1 e^{as}, z_2 e^{a^* s}\right)$, the forward measure $\mathbb{P}_T$ and inserting the Vasicek volatility specifications.

Next, to compute the expectation in (70), one has to use a two-dimensional version of the Jamshidian decomposition and apply the Fourier transform method, similarly to Section 4.5. More precisely, let us introduce deterministic functions $\widetilde{f}, f : \mathbb{R}^2_+ \to \mathbb{R}$

$$\begin{aligned} \widetilde{f}(x_1, x_2) &= \sum_{j=1}^n a^{j,0} e^{a^{j,1} x_1 + a^{j,2} x_2} - \sum_{j=1}^n b^{j,0} e^{b^{j,1} x_1} \\ f(x_1, x_2) &= \widetilde{f}(x_1, x_2)^+. \end{aligned}$$

Then

$$P^{Swn}(0; T, T_n, S) = B_0(T)\mathbb{E}^{\mathbb{P}^T}\left[f(X^1_T, X^2_T)\right],$$

and making use of Theorem 3.2 in Eberlein, Glau, and Papapantoleon (2010), one obtains

Proposition 4.7 *The time-0 price of a swaption with swap rate S, exercise date T and maturity T_n is given by the following semi-closed formula:*

$$P^{Swn}(0; T, T_n, S) = \frac{B_0(T)}{(2\pi)^2}\int_{\mathbb{R}^2} M^T_{X_T}(R + \mathrm{i}u)\widehat{f}(\mathrm{i}R - u)du, \tag{72}$$

where $R \in \mathbb{R}^2$ is such that $M^T_{X_T}(R+\mathrm{i}u)$ is well-defined (since $M^T_{X_T}(R)$ given by (71) exists) and the function $g(x) := e^{-Rx}f(x)$ satisfies the prerequisites of Theorem 3.2 in Eberlein, Glau, and Papapantoleon (2010).

A closed analytic expression for the Fourier transform $\widehat{f}$ is not available in this case. However, it could be computed numerically as follows. Note that for each fixed $x_1 \in \mathbb{R}_+$, the function $x_2 \mapsto \widetilde{f}(x_1, x_2)$ is continuous, increasing in x_2 and $\lim_{x_2\to\infty} \widetilde{f}(x_1, x_2) = +\infty$, since $a^{j,0}, a^{j,2} > 0$. Hence, let us define

$$q(x_1) = \inf\{x_2 \in \mathbb{R}_+ : \widetilde{f}(x_1, x_2) \geq 0\}.$$

Note that $\widetilde{f}(x_1, \cdot)$ has at most one zero since it is an increasing function for every fixed x_1. Consequently,

$$f(x_1, x_2) = \widetilde{f}(x_1, x_2)^+ = \widetilde{f}(x_1, x_2)\mathbf{1}_{\{x_2 \geq q(x_1)\}}.$$

The Fourier transform of f is therefore given, for suitable $z \in \mathbb{C}^2$, by

$$
\begin{aligned}
\widehat{f}(z) &= \int_{\mathbb{R}^2} e^{\mathrm{i}zx} f(x) dx \\
&= \int_{\mathbb{R}^2} e^{\mathrm{i}zx} \widetilde{f}(x_1, x_2) \mathbf{1}_{\{x_2 \geq q(x_1)\}} dx \\
&= \int_0^\infty \int_{q(x_1)}^\infty e^{\mathrm{i}zx} \left(\sum_{j=1}^n a^{j,0} e^{a^{j,1}x_1 + a^{j,2}x_2} - \sum_{j=1}^n b^{j,0} e^{b^{j,1}x_1} \right) dx_2 dx_1. \qquad (73)
\end{aligned}
$$

Since $q(x_1)$ is obtained by numerically solving $\widetilde{f}(x_1, x_2) = 0$, we shall not obtain a closed formula for $\widehat{f}$. However, based on (73), $\widehat{f}$ can be valued numerically. We refer the reader to a follow-up numerical paper for more details about this, as well as for discussion of the prerequisites of Theorem 3.2 in Eberlein, Glau, and Papapantoleon (2010) regarding function g in Proposition 4.7.

5 Numerical example

To give a flavor of the practical behavior of the model, we consider in this section a toy example, which illustrates the ability of the model to produce a wide range of FRA spreads (reflecting a segmentation between OIS and LIBOR markets) and basis swap spreads (reflecting a segmentation between LIBOR markets of different tenors). Implementation and numerical issues will be dealt with in detail in a follow-up paper.

We work with a one-dimensional driving process Y, which is an IG process with parameter b and cumulant function κ given by (48) (see Example 3.7). Let us consider a one-dimensional Vasicek volatility structure given by

$$
\Sigma_t(T) = \frac{\sigma}{a} \left(1 - e^{-a(T-t)}\right) \quad \text{and} \quad \Sigma^*_t(T) = \frac{\sigma^*}{a^*} \left(1 - e^{-a^*(T-t)}\right).
$$

Then

$$
\bar{\Sigma}^*_t(T) = \Sigma_t(T) + \Sigma^*_t(T) = \frac{\sigma}{a} \left(1 - e^{-a(T-t)}\right) + \frac{\sigma^*}{a^*} \left(1 - e^{-a^*(T-t)}\right)
$$

by (22). The initial bond term structure is assumed to be given by

$$
B_0(T) = e^{-\bar{r}T} \quad \text{and} \quad B^*_0(T) = e^{-(\bar{r}+\bar{\lambda})T},
$$

where $\bar{r} > 0$ and $\bar{\lambda} > 0$ are some given constants.

5.1 FRAs

First let us consider an FRA and calculate the spread (58) ("FRA spread" henceforth) between the forward rate in our model and a classical, one-curve forward rate. Figure 1 in Morini (2009) shows 2007-2009 market data for FRA spreads that surged to more than 170bps at the peak of the crisis. As for the LIBOR-OIS spreads at that time, Filipović and Trolle (2011) report that the spread between the 3m-LIBOR and the 3m-OIS rate attained "366 basis points on Oct 10, 2008" (see also their Figure 1).

Table 1 displays the FRA spread in the model with $\widehat{T} = 2$, $\delta = 0.5$, $N = 1$, for three values of $\bar{\lambda}$ and for b going over a range of values from 3 to 100. The classical, one-curve FRA rate for the value $\bar{r} = 2\%$ used in the table amounts to 2.01%. The results in Table 1 finely cover the ranges of FRA rates observed in the crisis.

b	$\bar{\lambda} = 10$	$\bar{\lambda} = 50$	$\bar{\lambda} = 300$
3	413.42	454.69	714.51
4	200.33	241.18	498.31
5	113.20	153.87	409.91
6	71.75	112.33	367.84
7	49.72	90.26	345.49
8	37.00	77.52	332.59
10	24.10	64.59	319.50
12	18.28	58.75	313.59
15	14.32	54.79	309.58
20	11.89	52.36	307.11
30	10.64	51.10	305.84
50	10.22	50.68	305.42
100	10.12	50.58	305.31

Table 1: bp-FRA spreads for $\bar{r} = 2\%, a = 0.025, a^* = 0.02, \sigma = \sigma^* = 0.5$ ($\bar{\lambda}$ in bps).

5.2 Basis swap spreads

Now let us consider a 6m-LIBOR versus 12m-LIBOR basis swap with maturity $\widehat{T} = 10$ years. One can read on page 8 of Morini (2009) (see also Figure 3 therein): "From August 2008 to April 2009, the basis swap spread to exchange 6m-LIBOR with 12m-LIBOR over 1 year was strongly positive and averaged 40bps." We calculate the model spread $S^{BSw}(0; 10)$ ("basis swap spread" henceforth) using formula (66), for the same sets of model parameters as in the previous subsection. The results are displayed in Table 2 and they again cover the ranges of spreads observed in the 2007-2009 crisis.

b	$\bar{\lambda} = 10$	$\bar{\lambda} = 50$	$\bar{\lambda} = 300$
3	201.623	205.386	230.437
4	79.036	80.996	94.554
5	38.090	39.265	47.824
6	21.154	21.937	27.995
7	12.978	13.544	18.222
8	8.556	8.994	12.852
10	4.318	4.622	7.624
12	2.493	2.735	5.341
15	1.286	1.485	3.816
20	0.560	0.732	2.891
30	0.188	0.345	2.414
50	0.065	0.217	2.256
100	0.035	0.187	2.218

Table 2: bp-6m/12m-basis swap spreads for $\bar{r} = 2\%, a = 0.025, a^* = 0.02, \sigma = \sigma^* = 0.5$ ($\bar{\lambda}$ in bps).

Conclusion

In this article we develop a multiple-curve HJM model of interbank risk. The interbank risk means the spread risk between the LIBOR markets of different tenors (and also the OIS market in the limiting case of an overnight tenor), which is a significant risk since the crisis. To account for the multiple curves we resort to a general defaultable HJM mathematical formalism with tractable Markovian short-term specifications. If a defaultable HJM drift condition is satisfied, the model spreads have a pure credit risk interpretation. Otherwise, another contribution to the spreads shows up that we call the liquidity component of interbank risk, in reference to the econometrically demonstrated explanation of interbank risk as a mixture of credit and liquidity risk of the LIBOR contributing banks. Preliminary numerical results reveal that even a pure credit risk specification of the model is able to produce FRA spreads and basis swap spreads of the orders of magnitudes that were observed at the peaks of the last financial crises. A follow-up paper will deal further with the numerical issues, as well as with the integration of the clean valuation model of this paper (which is a valuation model of a contract fully collateralized at an OIS funding rate) into a CVA setup.

References

Barndorff-Nielsen, O. E. and N. Shephard (2001). Modelling by Lévy processes for financial econometrics. In O. E. Barndorff-Nielsen, T. Mikosch, and S. Resnick (Eds.), *Lévy Processes: Theory and Applications*, pp. 283–318. Birkhäuser.

Bianchetti, M. (2010). Two curves, one price. *Risk Magazine, August*, 74–80.

Bielecki, T. R. and M. Rutkowski (2000). Multiple ratings model of defaultable term structure. *Mathematical Finance 10*, 125–139.

Bielecki, T. R. and M. Rutkowski (2002). *Credit Risk: Modeling, Valuation and Hedging.* Springer.

Carr, P., H. Geman, D. Madan, and M. Yor (2002). The fine structure of asset returns: An empirical investigation. *Journal of Business 75*, 305–332.

Chiarella, C. and O. K. Kwon (2001). Classes of interest rate models under the HJM framework. *Asia-Pacific Financial Markets 8*(1), 1–22.

Chiarella, C., S. C. Maina, and C. S. Nikitopoulos (2010). Markovian defaultable HJM term structure models with unspanned stochastic volatility. *Research Paper Series 283, Quantitative Finance Research Centre, University of Technology, Sydney*.

Cont, R. and P. Tankov (2003). *Financial Modelling with Jump Processes.* Chapman and Hall/CRC Press.

Crépey, S. (2012). Bilateral counterparty risk under funding constraints – Part II: CVA. *Mathematical Finance*. (forthcoming).

Eberlein, E., K. Glau, and A. Papapantoleon (2010). Analysis of Fourier transform valuation formulas and applications. *Applied Mathematical Finance 17(3)*, 211–240.

Eberlein, E., J. Jacod, and S. Raible (2005). Lévy term structure models: no-arbitrage and completeness. *Finance and Stochastics 9*, 67–88.

Eberlein, E. and W. Kluge (2006a). Exact pricing formulae for caps and swaptions in a Lévy term structure model. *Journal of Computational Finance 9*(2), 99–125.

Eberlein, E. and W. Kluge (2006b). Valuation of floating range notes in Lévy term structure models. *Mathematical Finance 16*, 237–254.

Eberlein, E. and N. Koval (2006). A cross-currency Lévy market model. *Quantitative Finance 6*(6), 465–480.

Eberlein, E. and F. Özkan (2003). The defaultable Lévy term structure: ratings and restructuring. *Mathematical Finance 13*, 277–300.

Eberlein, E. and S. Raible (1999). Term structure models driven by general Lévy processes. *Mathematical Finance 9*, 31–53.

Filipović, D. and A. B. Trolle (2011). The term structure of interbank risk. SSRN eLibrary.

Fujii, M., Y. Shimada, and A. Takahashi (2010). A note on construction of multiple swap curves with and without collateral. *FSA Research Review 6*, 139–157.

Fujii, M., Y. Shimada, and A. Takahashi (2011). A market model of interest rates with dynamic basis spreads in the presence of collateral and multiple currencies. *Wilmott Magazine 54*, 61–73.

Grbac, Z. (2010). *Credit Risk in Lévy Libor Modeling: Rating Based Approach*. Ph. D. thesis, University of Freiburg.

Heath, D., R. Jarrow, and A. Morton (1992). Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica 60*, 77–105.

Henrard, M. (2007). The irony in the derivatives discounting. *Wilmott Magazine 2*, 92–98.

Henrard, M. (2010). The irony in the derivatives discounting part II: the crisis. *Wilmott Journal 2*, 301–316.

Jacod, J. and A. N. Shiryaev (2003). *Limit Theorems for Stochastic Processes* (2nd ed.). Springer.

Kenyon, C. (2010). Short-rate pricing after the liquidity and credit shocks: including the basis. SSRN eLibrary.

Kijima, M., K. Tanaka, and T. Wong (2009). A multi-quality model of interest rates. *Quantitative Finance 9*(2), 133–145.

Kluge, W. (2005). *Time-Inhomogeneous Lévy Processes in Interest Rate and Credit Risk Models*. Ph. D. thesis, University of Freiburg.

Kyprianou, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer.

Mercurio, F. (2010a). A LIBOR market model with a stochastic basis. *Risk Magazine, December*, 84–89.

Mercurio, F. (2010b). Interest rates and the credit crunch: new formulas and market models. *Bloomberg Portfolio Research Paper No. 2010-01-FRONTIERS*.

Michaud, F.-L. and C. Upper (2008). What drives interbank rates? Evidence from the Libor panel. *BIS Quarterly Review (March 2008)*, 47–58.

Moreni, N. and A. Pallavicini (2010). Parsimonious HJM modelling for multiple yield-curve dynamics. Preprint, arXiv:1011.0828v1.

Morini, M. (2009). Solving the puzzle in the interest rate market. SSRN eLibrary.

Musiela, M. and M. Rutkowski (2005). *Martingale Methods in Financial Modelling* (2nd ed.). Springer.

Papapantoleon, A. (2007). *Applications of Semimartingales and Lévy Processes in Finance: Duality and Valuation*. Ph. D. thesis, University of Freiburg.

Raible, S. (2000). *Lévy processes in finance: theory, numerics, and empirical facts*. Ph. D. thesis, University of Freiburg.

Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.