

A defaultable HJM multiple-curve term structure model

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Abstract

In the aftermath of the 2007–09 financial crisis, a variety of spreads have developed between quantities that had been essentially the same until then, notably LIBOR-OIS spreads, LIBOR-OIS swap spreads, and basis swap spreads. In this paper we study the valuation of LIBOR interest rate derivatives in a multiple-curve setup, which accounts for a discrepancy between a risk-free discount curve and a LIBOR fixing curve. Toward this end we resort to a defaultable HJM methodology, in which this discrepancy is modeled by an implied default intensity of the LIBOR contributing banks.

Keywords: Interest Rate Derivatives, LIBOR, HJM, Multiple Curve, Interbank Risk, Lévy Processes.

MSC: 91G30, 91G20, 60G51

JEL classification: G12, E43

1 Introduction

In the aftermath of the 2007–09 financial crisis, a variety of spreads have developed between quantities that had been essentially the same until then, notably LIBOR-OIS spreads, LIBOR-OIS swap spreads, and basis swap spreads (see Figure 1). This is reckoned in Filipović and Trolle (2011) as the advent of a so-called interbank risk.

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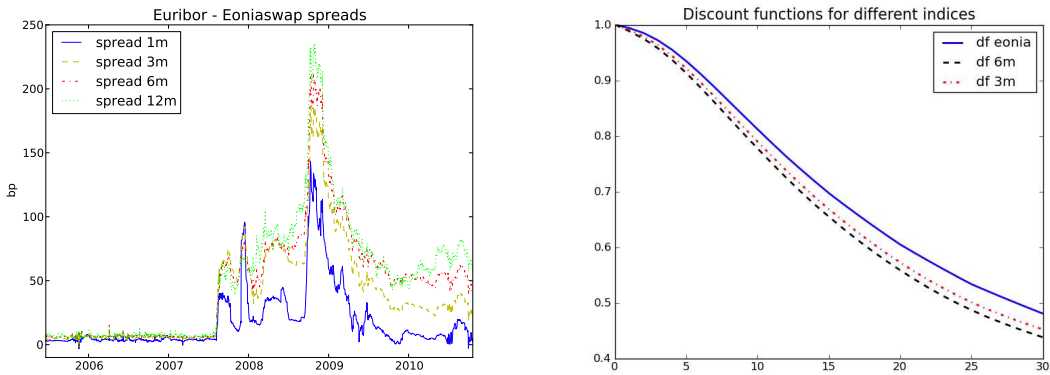


Figure 1: *Left*: Historical Euribor-Eonia swap spreads 2005-10. *Right*: Discount curves bootstrapped on September 2 2010.

In addition, when valuing and hedging interest rate derivatives, the interbank risk issue comes in combination with the counterparty risk issue, which is the risk of a party defaulting in an OTC derivative contract. In this context, which curve should be used as discounting curve, to which extent the choice of a given curve should be put in relation with counterparty risk, or possibly hidden relations between bilateral counterparty risk (accounting for the default risk of both parties) and funding costs (of funding a position into a contract in a multiple-curve environment), have become subject of endless debates between market practitioners.

In this paper we propose a model of interbank risk for the pricing of LIBOR interest rate derivatives in a multiple-curve setup. Note that this is a model of “clean” valuation in the sense of Crépey (2011), meaning clean of counterparty risk and excess funding costs over the risk-free rate. However, a counterparty risk and excess funding costs correction (CVA for Credit Valuation Adjustment in the counterparty risk terminology) can then be obtained as the value of an option on this clean price process; see for instance Crépey (2011). Actually, the initial motivation for the present work was to devise a model of clean valuation of interest rate derivatives with interbank risk, tractable in itself and also in the perspective of serving as underlying model for CVA computations. This integration of the present clean model into a counterparty risky environment will be considered in a follow-up paper.

Resorting to the usual distinction between short rate, HJM and BGM or LIBOR market models, one can classify the interbank risk (multiple-curve in this regard, yet “clean” in the above sense) valuation literature as follows. Kijima, Tanaka, and Wong (2009) or Kenyon (2010) propose short rate approaches. Henrard (2007, 2009) derives corrected Gaussian HJM formulas under the assumption of deterministic spreads between the curves. Bianchetti (2010) resolves a two-curve issue in a cross-currency mathematical framework, deriving “quanto convexity corrections” to the usual BGM market model valuation formulas. Here the main tool is that of a change of measure/numéraire. The LIBOR market model approach is also extended in Mercurio (2009, 2010) and Fujii, Shimada and Takahashi (2009, 2010) in such a way that each basis spread is modeled as a different process. A hybrid HJM-LIBOR market model is proposed in Moreni and Pallavicini (2010), where the HJM framework is employed to obtain a parsimonious model for multiple curves, using a single family of

Markov driving processes. Finally, a first credit risk approach is tentative in Morini (2009). However, Morini concludes on page 43 that in his model “the credit risk alone does not explain the market patterns”.

In this paper we also resort to a credit risk mathematical setup. Let us add that by credit risk here we do not mean counterparty risk of the parties of a contract (we actually deliberately disregard counterparty risk in a clean valuation perspective). What we mean here is simply an interpretation (or “measurement”) of LIBOR quotes (which are the main input to most interest rate derivative cash flows) in excess over the risk-free rate, in the mathematically tractable scale of an implied default intensity of the LIBOR contributing banks. Note that a rolling construction of the LIBOR group is precisely intended to the effect that, in principle, actual defaults cannot occur within the LIBOR group. We are also fully aware that the economic fundamentals of interbank risk are not only credit risk, but also liquidity risk, among other factors such as “strategic” game considerations (see Michaud and Upper (2008, page 48)), which might from time to time incite a bank to declare as LIBOR contribution a number slightly different from its intimate conviction regarding “The rate at which an individual Contributor Panel bank could borrow funds, were it to do so by asking for and then accepting interbank offers in reasonable market size, just prior to 11.00 London time” (the theoretical definition of the LIBOR).

More precisely, we shall follow a defaultable Heath–Jarrow–Morton methodology for modeling the term structure of multiple interest rates; see the seminal paper by Heath, Jarrow, and Morton (1992) and the defaultable extensions by Bielecki and Rutkowski (2000) and Eberlein and Özkan (2003). Numerical issues related to our model will be mainly considered in a follow-up paper. However, the last section of this paper already makes clear that, in counterpoint to Morini (2009) conclusions in his first tentative credit risk approach, an appropriate credit risk model is in fact able to explain spreads very much in line with the orders of magnitude that were observed in the market even at the peak of the crisis. These findings are also in line with a quantitative analysis of the term structure of interbank risk which was recently conducted by Filipović and Trolle (2011). Based on a data set covering the period from August 2007 until January 2011, their results show that the default component is overall the main dominant driver of interbank risk, except for short-term contracts in the first half of the sample (see Figures 3 and 4 in their paper). The second main driver is interpreted as liquidity risk, consistently with the claims in Morini (2009). We point out in this regard that, even though we did not see the necessity of it yet (and therefore did not do it for the sake of parsimony of the model), a simple amendment to our model allows to make explicit also a non-default component of interbank risk. For this it is enough to add one more component to the driver of our risky interest rate and the HJM-type valuation formulas can be derived in exactly the same manner as below.

Besides, our motivation for modeling the continuously compounded forward rates in a HJM fashion, instead of dealing directly with discretely compounded LIBORs in a BGM perspective, is twofold. On the one side, it allows one to consider simultaneously the LIBORs for all possible tenors (recall that one of the post-crisis spread studied in this work is between LIBORs of various tenors). The HJM framework is capable of producing a multi-curve model with as many stochastic factors as LIBORs of different tenors by increasing the dimension of the driving process, while still retaining the tractability of the pricing formulas for any arbitrary correlation of stochastic factors. On the other side, this is a unified approach for a very general class of time-inhomogeneous Lévy driving processes. It is also important to mention that various short rate models can be accommodated in this setup as special cases (see Section 3 for the extended CIR and the extended Hull-White model). As

will be illustrated in a follow-up work, this direct link to the short rate process r is useful in the context of counterparty risk applications, where the model of this paper can be used as an underlying model for CVA computations.

In our view the main contributions of this work are: a consistent and tractable defaultable HJM term structure model of interbank risk; low-dimensional extended CIR or Lévy Hull–White short rate specifications of the defaultable HJM setup, opening the door to the use of this model as underlying model to interest rate derivatives CVA computations; empirical evidence that an appropriately chosen credit risk setup is enough to account for even the most extreme interbank spreads ever observed in the market.

The rest of the paper is organized as follows. In Section 2, we apply a defaultable HJM approach to model the term structure of multiple interest rate curves. Section 3 presents a tractable pricing model within this framework which we obtain by choosing the class of non-negative multidimensional Lévy processes as driving processes combined with deterministic volatility structures. In Section 4 the basic interest rate derivatives tied to LIBOR are described and explicit valuation formulas are derived. Section 5 presents numerical results illustrating the flexibility of the model in producing a wide range of FRA and basis swap spreads.

2 Defaultable HJM setup

2.1 Notation

In this subsection we introduce the main notions and notation we are going to work with. The basic reference rate for a variety of interest rate derivatives is the LIBOR in the USD fixed income market and the EURIBOR in the EUR fixed income market. LIBOR (resp. EURIBOR) is computed daily as an average of the rates at which designated banks belonging to the LIBOR (resp. EURIBOR) panel believe unsecured funding for periods of length up to one year can be obtained by them (resp. by a prime bank). From now on we shall use the term LIBOR meaning any of these two rates. Another important reference rate in fixed income markets is a so-called OIS (Overnight Indexed Swap) rate, which is the rate at which overnight unsecured loans can be obtained in the interbank market. In the USD fixed income market it is the FF (Federal Funds) rate and in the EUR market it is the EONIA (Euro Overnight Index Average) rate. From now on we shall use the generic term OIS for any of these rates. The OIS rate is considered by practitioners to be the best available market proxy for the risk-free rate since the risk in an overnight loan can be deemed almost negligible. On the other hand, the LIBOR depends on the term structure of interbank risk, which is reflected in the observed LIBOR-OIS and LIBOR-OIS swap spreads (see the left panel in Figure 1).

In this paper we introduce a default time τ^* associated to the LIBOR reference curve via a given default intensity $\gamma^*(t)$. Again, τ^* is not meant to represent an actual default time of some specific entity (recall that the LIBOR panel is constantly being updated). It is merely used as an implied model of default risk for the reference curve, to quantify interbank risk in a mathematically tractable “default intensity scale”. This being said, our credit risk formalism is consistent however with the empirical evidence in Filipović and Trolle (2011) that default risk is a major component of the interbank risk.

We shall work with instantaneous continuously compounded forward rates, specifying the dynamics of the term structure of the risk-free forward interest rates $f_t(T)$ and of the

forward credit spreads $g_t(T)$ corresponding to the risky rates of the reference curve. We denote by $f_t^*(T)$ the instantaneous continuously compounded risky forward rates, so for every $0 \leq t \leq T$,

$$g_t(T) = f_t^*(T) - f_t(T). \quad (1)$$

The corresponding short rates r and r^* are given by

$$r_t = f_t(t) \quad \text{and} \quad r_t^* = f_t^*(t). \quad (2)$$

We also define the short term credit spread process λ by, for $t \in [0, \bar{T}]$,

$$\lambda_t = g_t(t) = r_t^* - r_t.$$

The discount factors associated with our two yield curves are denoted by $B_t(T)$ and $\bar{B}_t^*(T)$, respectively. These are time- t (cumulative) prices and pre-default prices of risk-free and risky zero coupon bonds with maturity T , with $B_T(T) = 1$ and $\bar{B}_T^*(T) = 1$. The bond prices are related to the forward rates via the following formulas, for $t \leq T$,

$$B_t(T) = \exp\left(-\int_t^T f_t(u)du\right) \quad \text{and} \quad \bar{B}_t^*(T) = \exp\left(-\int_t^T f_t^*(u)du\right). \quad (3)$$

The T -spot LIBOR $L_T(T, T + \delta)$ is a simply compounded interest rate fixed at time T for the time interval $[T, T + \delta]$, which will be defined in our setup as

$$L_T(T, T + \delta) = \frac{1}{\delta} \left(\frac{1}{\bar{B}_T^*(T + \delta)} - 1 \right). \quad (4)$$

We thus use in this definition the pre-default risky bond prices \bar{B}^* , where the reference entity of the risky bond is to be interpreted as consisting of (a stylized representative of) the LIBOR contributing banks.

2.2 Driving process

We consider a filtered probability space $(\Omega, \mathcal{F}_{\bar{T}}, \mathbb{P})$ and a finite time horizon \bar{T} . Let $\mathcal{E} = (\mathcal{E}_t)_{t \in [0, \bar{T}]}$ denote a filtration on this space satisfying the usual conditions. The driving process $Y = (Y_t)_{0 \leq t \leq \bar{T}}$ is assumed to be a process with independent increments and absolutely continuous characteristics (PIIAC) in the sense of Eberlein, Jacod, and Raible (2005), also called a time-inhomogeneous Lévy process in Eberlein and Kluge (2006a), or an additive process in the sense of Definition 1.6 in Sato (1999). Process Y is taken as an \mathcal{E} -adapted, càdlàg, \mathbb{R}^n -valued process, starting from zero. The law of Y_t , $t \in [0, \bar{T}]$, is described by the characteristic function, in which u denotes a row-vector in \mathbb{R}^n :

$$\begin{aligned} \mathbb{E}[e^{iuY_t}] &= \exp \int_0^t \left(i u b_s - \frac{1}{2} u c_s u^\top \right. \\ &\quad \left. + \int_{\mathbb{R}^n} (e^{iux} - 1 - i u h(x)) F_s(dx) \right) ds, \end{aligned} \quad (5)$$

where $b_s \in \mathbb{R}^n$, c_s is a symmetric, non-negative definite real-valued n -dimensional matrix and F_s is a Lévy measure on \mathbb{R}^n , i.e. $F_s(\{0\}) = 0$ and $\int_{\mathbb{R}^n} (|x|^2 \wedge 1) F_s(dx) < \infty$, for all $s \in [0, \bar{T}]$. The function $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a truncation function (e.g. $h(x) = x \mathbf{1}_{\{|x| \leq 1\}}$).

Let $\|\cdot\|$ denote the norm on the space of real n -dimensional matrices, induced by the Euclidean norm $|\cdot|$ on \mathbb{R}^n . The following standing assumption is satisfied:

Assumption 2.1 (i) The triplet (b_t, c_t, F_t) satisfy

$$\int_0^{\bar{T}} \left(|b_t| + \|c_t\| + \int_{\mathbb{R}^n} (1 \wedge |x|^2) F_t(dx) \right) dt < \infty;$$

(ii) There exist constants $\mathcal{K}, \varepsilon > 0$ such that

$$\int_0^{\bar{T}} \int_{|x|>1} \exp(ux) F_t(dx) dt < \infty, \quad (6)$$

for every $u \in [-(1 + \varepsilon)\mathcal{K}, (1 + \varepsilon)\mathcal{K}]^n$.

Condition (6) ensures the existence of exponential moments of the process Y . More precisely, (6) holds if and only if $\mathbb{E}[\exp uY_t] < \infty$, for all $0 \leq t \leq \bar{T}$ and $u \in [-(1 + \varepsilon)\mathcal{K}, (1 + \varepsilon)\mathcal{K}]^n$ (cf. Lemma 2.6 and Corollary 2.7 in Papapantoleon (2007)). Moreover, Y is then a special semimartingale, with the following canonical decomposition (cf. Jacod and Shiryaev (2003, II.2.38), and Eberlein, Jacod, and Raible (2005))

$$Y = \int_0^\cdot b_s ds + \int_0^\cdot \sqrt{c_s} dW_s + \int_0^\cdot \int_{\mathbb{R}^n} x(\mu - \nu)(ds, dx), \quad (7)$$

where μ is the random measure of the jumps of Y , ν is the \mathbb{P} -compensator of μ , $\sqrt{c_s}$ is a measurable version of a square-root of the symmetric, non-negative definite matrix c_s , and W is a \mathbb{P} -standard Brownian motion. The triplet of predictable semimartingale characteristics of Y with respect to the measure \mathbb{P} , denoted by (B, C, ν) , is

$$B = \int_0^\cdot b_s ds, \quad C = \int_0^\cdot c_s ds, \quad \nu([0, \cdot] \times A) = \int_{[0, \cdot]} \int_A F_s(dx) ds, \quad (8)$$

for every Borel set $A \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$. The triplet (b, c, F) represents the local characteristics of Y . Any of these triplets determines the distribution of Y , as the Lévy–Khintchine formula (5) obviously dictates (with $h(x) = x$, which is a valid choice for the truncation function due to (6)).

We denote by κ_s the cumulant generating function associated with the infinitely divisible distribution characterized by the Lévy triplet (b_s, c_s, F_s) . For a row-vector $z \in \mathbb{C}^n$ such that $\Re z \in [-(1 + \varepsilon)\mathcal{K}, (1 + \varepsilon)\mathcal{K}]^n$, we have, for $s \in [0, \bar{T}]$,

$$\kappa_s(z) = zb_s + \frac{1}{2} zc_s z^\top + \int_{\mathbb{R}^n} (e^{zx} - 1 - zx) F_s(dx). \quad (9)$$

Note that (5) can be written in terms of κ :

$$\mathbb{E}[e^{iuY_t}] = \exp \int_0^t \kappa_s(iu) ds. \quad (10)$$

If Y is a Lévy process, in other words if Y is time-homogeneous, then (b_s, c_s, F_s) , and thus also κ_s , do not depend on s . In that case, κ boils down to the log-moment generating function of Y_1 . For details we refer to Papapantoleon (2007, Lemma 2.8, Remark 2.9 and Remark 2.16).

2.3 Term structure of interest rates

In this subsection, we model the risk-free and the risky term structure of interest rates.

We shall be concerned with two filtrations on the standing risk-neutral probability space $(\Omega, \mathcal{F}_{\bar{T}}, \mathbb{P})$ of this paper: the default-free filtration $\mathcal{E} = (\mathcal{E}_t)_{0 \leq t \leq \bar{T}}$, and the full filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq \bar{T}}$ containing \mathcal{E} and the information about the default time τ^* . The default-free bond price process $B_\cdot(T)$, the pre-default bond price process $\bar{B}_\cdot(T)$, and the corresponding forward rate processes $f_\cdot(T)$ and $f^*_\cdot(T)$, for any $T \in [0, \bar{T}]$, are all \mathcal{E} -adapted. It is assumed that τ^* is not an \mathcal{E} -stopping time, but it is an \mathcal{F} -stopping time. Moreover, we assume that immersion holds between \mathcal{E} and \mathcal{F} . We assume that τ^* possesses an \mathcal{E} -hazard intensity γ^* . Thus, its Azéma supermartingale is given by

$$\mathbb{Q}(\tau^* > t | \mathcal{E}_t) = e^{-\int_0^t \gamma_s^* ds}, \quad (11)$$

where γ^* is an \mathcal{E} -adapted, non-negative and integrable process.

The risky bonds are assumed to pay a certain recovery upon default. We adopt the fractional recovery of market value scheme, which specifies that in case of default of the bond issuer, the fraction of the pre-default value of the bond is paid at the default time. The value at maturity of such a bond is given by

$$B_T^*(T) = \mathbf{1}_{\{\tau^* > T\}} + \mathbf{1}_{\{\tau^* \leq T\}} R^* \bar{B}_{\tau^*-}^*(T) B_{\tau^*}^{-1}(T),$$

where $R^* \in [0, 1]$ is the recovery and $\bar{B}_\cdot(T)$ was defined in (3). Note that receiving the amount $\mathbf{1}_{\{\tau^* \leq T\}} R^* \bar{B}_{\tau^*-}^*(T)$ at τ^* is equivalent to receiving $\mathbf{1}_{\{\tau^* \leq T\}} R^* \bar{B}_{\tau^*-}^*(T) B_{\tau^*}^{-1}(T)$ at T . The time- t price of such a bond can be written as

$$B_t^*(T) = \mathbf{1}_{\{\tau^* > t\}} \bar{B}_t^*(T) + \mathbf{1}_{\{\tau^* \leq t\}} R^* \bar{B}_{\tau^*-}^*(T) B_{\tau^*}^{-1}(T) B_t(T). \quad (12)$$

The immersion property implies that $\bar{B}_{\tau^*-}^*(T) = \bar{B}_{\tau^*}^*(T)$. Moreover, note that $\mathbf{1}_{\{\tau^* > t\}} \bar{B}_t^*(T) = \mathbf{1}_{\{\tau^* > t\}} B_t^*(T)$, for every $t \in [0, T]$.

Let us now specify the instantaneous continuously compounded forward rates $f_t(T)$ and the instantaneous forward credit spreads $g_t(T)$, which in turn provide the default-free bond prices $B_t(T)$ and the pre-default bond prices $\bar{B}_t^*(T)$ via (3). We are going to make use of the results from Eberlein and Raible (1999) and Eberlein and Kluge (2006b), where HJM models driven by time-inhomogeneous Lévy processes were developed, and the results from Bielecki and Rutkowski (2000) and Eberlein and Özkan (2003), where defaultable extensions of the HJM framework were introduced.

Contrary to the latter two papers, we choose here to model directly the forward credit spreads instead of the risky forward rates. Clearly, in order to model the pre-default term structure, it is equivalent to specify either the forward rates $f_t^*(T)$, or the forward credit spreads $g_t(T)$. However, by no-arbitrage one has $\bar{B}_t^*(T) \leq B_t(T)$, i.e. the risky bonds are cheaper than the default-free bonds with the same maturity. This implies by (3) that one should have $f_t^*(T) \geq f_t(T)$, or equivalently, $g_t(T) \geq 0$. Hence, we decide to model the forward credit spreads directly and study their non-negativity in some special cases. In the next subsection we provide two tractable non-negative examples. Let us also mention here a paper by Chiarella, Maina, and Nikipoulos (2010), where a class of stochastic volatility HJM models admitting finite dimensional Markovian structures is proposed. They model the default-free forward rates and the forward credit spreads, whose dynamics are driven by correlated Brownian motions. One of our examples in the sequel, the stochastic volatility CIR model of Section 3.1, could be fit into this modeling framework.

2.3.1 Risk-free rates

The dynamics of the risk-free forward rates $f_t(T)$, for $T \in [0, \bar{T}]$, is given by

$$f_t(T) = f_0(T) + \int_0^t a_s(T) ds + \int_0^t \sigma_s(T) dY_s, \quad (13)$$

where the initial values $f_0(T)$ are deterministic, bounded and Borel measurable in T . Moreover, σ and a are stochastic processes defined on $\Omega \times [0, \bar{T}] \times [0, \bar{T}]$ taking values in \mathbb{R}^n and \mathbb{R} , respectively. Let \mathcal{P} and \mathcal{O} respectively denote the predictable and the optional σ -field on $\Omega \times [0, \bar{T}]$. The mappings $(\omega; s, T) \mapsto a_s(\omega; T)$ and $(\omega; s, T) \mapsto \sigma_s(\omega; T)$ are measurable with respect to $\mathcal{P} \otimes \mathcal{B}([0, \bar{T}])$. For $s > T$ we have $a_s(\omega; T) = 0$ and $\sigma_s(\omega; T) = 0$, as well as $\sup_{t, T \leq \bar{T}} (|a_t(\omega; T)| + |\sigma_t(\omega; T)|) < \infty$. These conditions ensure that we can find a “joint-version” of all $f_t(T)$ such that $(\omega; t, T) \mapsto f_t(\omega; T) \mathbf{1}_{\{t \leq T\}}$ is $\mathcal{O} \otimes \mathcal{B}([0, \bar{T}])$ -measurable (see Eberlein, Jacod, and Raible (2005)). Then it follows (cf. equation (2.4) in Eberlein and Kluge (2006b)), for $t \in [0, T]$,

$$B_t(T) = B_0(T) \exp \left(\int_0^t (r_s - A_s(T)) ds - \int_0^t \Sigma_s(T) dY_s \right), \quad (14)$$

where we set

$$A_s(T) := \int_s^T a_s(u) du, \quad \Sigma_s(T) := \int_s^T \sigma_s(u) du. \quad (15)$$

Inserting $T = t$ into (14), the risk-free discount factor process $\beta = (\beta_t)_{0 \leq t \leq \bar{T}}$, defined by $\beta_t = \exp \left(- \int_0^t r_s ds \right)$, can be written as

$$\beta_t = B_0(t) \exp \left(- \int_0^t A_s(t) ds - \int_0^t \Sigma_s(t) dY_s \right). \quad (16)$$

Combining this with (14) we obtain the following useful representation for the bond price process

$$B_t(T) = \frac{B_0(T)}{B_0(t)} \exp \left(\int_0^t (A_s(t) - A_s(T)) ds + \int_0^t (\Sigma_s(t) - \Sigma_s(T)) dY_s \right). \quad (17)$$

We make a standing assumption that the volatility structure is bounded in the sense that one has $0 \leq \Sigma_s^i(T) \leq \frac{\mathcal{K}}{2}$ for every $0 \leq s \leq T \leq \bar{T}$ and $i \in \{1, 2, \dots, n\}$, where \mathcal{K} is the constant from Assumption 2.1(ii). Note that if Y is a Brownian motion, this assumption holds with $\mathcal{K} = \infty$. In other words, the volatility structure in the Brownian case does not have to be bounded.

As is well-known, the model is free of arbitrage if the bond prices discounted at the risk-free rate, $\beta B_\cdot(T)$, are \mathcal{F} -martingales with respect to a risk-neutral measure \mathbb{P} . Due to the immersion property it suffices that they are \mathcal{E} -martingales. This is guaranteed by the following drift condition:

$$A_s(T) = \kappa_s(-\Sigma_s(T)), \quad s \in [0, T], \quad (18)$$

where κ_s is the cumulant of Y defined in (9). This condition can be found in Eberlein and Kluge (2006b), see equation (2.3) therein and comments thereafter. For more detailed computations, see Proposition 2.2 of Kluge (2005) in case of deterministic volatility, and Theorem 7.9 and Corollary 7.10 of Raible (2000) for a stochastic volatility combined with a (time-homogeneous) Lévy driving process. If Y is a standard Brownian motion, condition (18) simplifies to $A_s(T) = \frac{1}{2} |\Sigma_s(T)|^2$, which is the classical HJM no-arbitrage condition.

2.3.2 Risky rates

The dynamics of the forward credit spreads $g_t(T)$, $t \in [0, T]$, is given by

$$g_t(T) = g_0(T) + \int_0^t a_s^*(T) ds + \int_0^t \sigma_s^*(T) dY_s, \quad (19)$$

where the initial values $g_0(T)$ are deterministic, bounded and Borel measurable in T . Moreover, a^* and σ^* satisfy the same measurability and boundedness conditions as a and σ . The risky forward rates are then given by

$$f_t^*(T) = f_0^*(T) + \int_0^t \bar{a}_s^*(T) ds + \int_0^t \bar{\sigma}_s^*(T) dY_s, \quad (20)$$

where we set

$$f_0^*(T) = f_0(T) + g_0(T), \quad \bar{a}_s^*(T) = a_s(T) + a_s^*(T), \quad \bar{\sigma}_s^*(T) = \sigma_s(T) + \sigma_s^*(T).$$

The dynamics of the bond prices $\bar{B}^*(T)$ can be obtained exactly in the same way as the dynamics of $B(T)$ in equation (14). Therefore, for $t \in [0, T]$,

$$\bar{B}_t^*(T) = \bar{B}_0^*(T) \exp \left(\int_0^t (r_s^* - \bar{A}_s^*(T)) ds - \int_0^t \bar{\Sigma}_s^*(T) dY_s \right), \quad (21)$$

where

$$\bar{A}_s^*(T) := \int_s^T \bar{a}_s^*(u) du \quad \text{and} \quad \bar{\Sigma}_s^*(T) := \int_s^T \bar{\sigma}_s^*(u) du. \quad (22)$$

Recall from (2) that the short rate r_s^* is given by $r_s + \lambda_s$.

Similarly to (17), we can rewrite the bond prices process $\bar{B}^*(T)$ as follows

$$\bar{B}_t^*(T) = \frac{\bar{B}_0^*(T)}{\bar{B}_0^*(t)} \exp \left(\int_0^t (\bar{A}_s^*(t) - \bar{A}_s^*(T)) ds + \int_0^t (\bar{\Sigma}_s^*(t) - \bar{\Sigma}_s^*(T)) dY_s \right). \quad (23)$$

For the sake of model parsimony we require in addition that the defaultable bond prices discounted at the risk-free rate, $\beta B^*(T)$, are $(\mathcal{F}, \mathbb{P})$ -martingales, for all $T \in [0, \bar{T}]$. Note that this constraint would correspond to precluding arbitrage opportunities related to dealing with the risky bonds $B^*(T)$, were such risky bonds traded in the market (which they are actually not, even not synthetically as averages of risky bonds of LIBOR contributors, since the LIBORs reflected in $B^*(T)$ are only reference numbers and not transaction quotes; see the definition of the LIBOR in the introduction).

For each T this additional martingale condition is satisfied if

$$(\bar{B}_t^*(T) - R_t^* B_t(T)) \gamma_t^* = \bar{B}_t^*(T) \alpha_t(T), \quad t \in [0, T], \quad (24)$$

where

$$\alpha_t(T) := \lambda_t - \bar{A}_t^*(T) + \kappa_t(-\bar{\Sigma}_t^*(T))$$

and $(R_t^*)_{t \geq 0}$ is the terminal recovery process in the sense of Condition (HJM.8) in Section 13.1.9 of Bielecki and Rutkowski (2002). The proof of the above statement is similar to the derivation of Condition (13.24) in Bielecki and Rutkowski (2002, Section 13.1.9) in the Gaussian case. For similar conditions in (time-inhomogeneous) Lévy driven models, we refer to Eberlein and Özkan (2003) or Grbac (2010, Section 3.7).

Under the fractional recovery of market value scheme which is assumed in this paper, one gets a particularly convenient form of the drift condition (24). The recovery process R takes the following form (cf. (12))

$$R_t^* := R^* \bar{B}_t^*(T) B_t^{-1}(T),$$

which inserted into (24) yields

$$(1 - R^*) \gamma_t^* = \alpha_t(T), \quad t \in [0, T]. \quad (25)$$

Since condition (25) has to hold for all $T \in [0, \bar{T}]$, it is actually equivalent to the following two conditions:

$$(1 - R^*) \gamma_t^* = \lambda_t \quad (26)$$

and

$$\bar{A}_t^*(T) = \kappa_t(-\bar{\Sigma}_t^*(T)). \quad (27)$$

Indeed, conditions (26) and (27) obviously imply (25). To see the converse, one has to insert $t = T$ into (25) and note that $\bar{A}_t^*(t) = 0$ and $\bar{\Sigma}_t^*(t) = 0$ by (15). Moreover, $\kappa_t(0) = 0$ by (9). This yields (26). Condition (27) now follows from (25) by inserting $t \neq T$.

We work henceforth under the following

Assumption 2.2 The no-arbitrage conditions (18), (26) and (27) are satisfied.

Proposition 2.3 (i) *The forward rate $f.(T)$ is given by*

$$f_t(T) = f_0(T) + \int_0^t \frac{\partial}{\partial T} \kappa_s(-\Sigma_s(T)) ds + \int_0^t \sigma_s(T) dY_s, \quad (28)$$

and the short rate r by

$$r_t = f_0(t) + \int_0^t \frac{\partial}{\partial t} \kappa_s(-\Sigma_s(t)) ds + \int_0^t \sigma_s(t) dY_s. \quad (29)$$

(ii) *The forward spread $g.(T)$ is given by*

$$\begin{aligned} g_t(T) &= g_0(T) + \int_0^t \left(\frac{\partial}{\partial T} \kappa_s(-\Sigma_s^*(T) - \Sigma_s(T)) - \frac{\partial}{\partial T} \kappa_s(-\Sigma_s(T)) \right) ds \\ &\quad + \int_0^t \sigma_s^*(T) dY_s, \end{aligned} \quad (30)$$

and the short term spread λ by

$$\begin{aligned} \lambda_t &= g_0(t) + \int_0^t \left(\frac{\partial}{\partial t} \kappa_s(-\Sigma_s^*(t) - \Sigma_s(t)) - \frac{\partial}{\partial t} \kappa_s(-\Sigma_s(t)) \right) ds \\ &\quad + \int_0^t \sigma_s^*(t) dY_s. \end{aligned} \quad (31)$$

(iii) *The \mathcal{E} -intensity γ^* of the default time τ^* is given by*

$$\gamma_t^* = \frac{1}{1 - R^*} \left(g_0(t) + \int_0^t \left(\frac{\partial}{\partial t} \kappa_s(-\Sigma_s^*(t) - \Sigma_s(t)) - \frac{\partial}{\partial t} \kappa_s(-\Sigma_s(t)) \right) ds + \int_0^t \sigma_s^*(t) dY_s \right).$$

Proof. To prove (i), note that from condition (18) it follows that

$$a_s(T) = \frac{\partial}{\partial T} \kappa_s(-\Sigma_s(T)).$$

This immediately yields (28) and (29). Similarly, to prove (ii), we make use of (27) and obtain

$$\begin{aligned} a_s^*(T) &= \bar{a}_s^*(T) - a_s(T) \\ &= \frac{\partial}{\partial T} \kappa_s(-\bar{\Sigma}_s^*(T)) - \frac{\partial}{\partial T} \kappa_s(-\Sigma_s(T)) \\ &= \frac{\partial}{\partial T} \kappa_s(-\Sigma_s^*(T) - \Sigma_s(T)) - \frac{\partial}{\partial T} \kappa_s(-\Sigma_s(T)), \end{aligned}$$

Hence, (30) and (31) follow. Finally, to prove (iii) we combine (26) and (31). \square

3 The model

In this section we focus our attention on time-homogeneous Lévy processes Y . The cumulant generating function associated with Y is then given by

$$\kappa(z) := zb + \frac{1}{2}zcz^\top + \int_{\mathbb{R}^n} (e^{zx} - 1 - zx) F(dx),$$

where (b, c, F) is the Lévy triplet of Y_1 (compare (9)). We study conditions that ensure the non-negativity of the risk-free interest rates and the credit spreads, considering in particular two cases: a pure-jump Lévy process with non-negative components (subordinators) combined with deterministic bond price volatility structures, and a two-dimensional Brownian motion combined with stochastic volatility structures. We shall focus in particular on the first case, which turns out to be very tractable for valuation purposes. Note that the general HJM model, as well as many short rate models, does not necessarily produce non-negative interest rates. The standard argument is that the probability of negative interest rates is sufficiently small, and therefore this undesirable feature is still tolerable. However, when interest rates are small as in the recent years, the non-negativity of interest rates produced by a model becomes a practically relevant issue.

3.1 Stochastic volatility CIR

Assume that the driving process $Y = (Y^1, Y^2)$ is a two-dimensional Brownian motion with correlation ρ . The canonical decomposition (7) of Y is given by

$$Y_t = \sqrt{c}(W_t^1, W_t^2)^\top,$$

where (W^1, W^2) is a two-dimensional standard Brownian motion and the covariance matrix $c = [c_{i,j}]_{i,j=1,2}$ is such that $c_{1,1} = c_{2,2} = 1$ and $c_{1,2} = c_{2,1} = \rho$. The cumulant generating function of Y is given by $\kappa(z) = \frac{1}{2}zcz^\top$, $z \in \mathbb{R}^2$. In order to produce non-negative short rates and short term spreads with this driving process, the volatilities in the HJM model cannot be chosen deterministic. We make use of the volatility specifications that produce the CIR short rate and the CIR short term spread within the HJM framework, as shown in

Chiarella and Kwon (2001). Thus, we impose the following assumptions on the volatilities $\sigma_s(t)$ and $\sigma_s^*(t)$:

$$\sigma_s(t) = \left(\zeta(s)\sqrt{r_s}e^{-\int_s^t k(u)du}, 0 \right), \quad \sigma_s^*(t) = \left(0, \zeta^*(s)\sqrt{\lambda_s}e^{-\int_s^t k^*(u)du} \right),$$

where ζ , ζ^* , k and k^* are deterministic functions (cf. equation (6.2) in Chiarella and Kwon (2001)). Note that the two-dimensional volatility structure above is chosen in such a way that the risk-free rates are driven only by the first Brownian motion $Y^1 =: W^r$ and the credit spreads are driven solely by $Y^2 =: W^\lambda$. Hence, we can apply directly the results from Chiarella and Kwon (2001, equation (6.3)) and obtain the following SDE for the short rate r

$$dr_t = (\rho_t - k(t)r_t)dt + \zeta(t)\sqrt{r_t}dW_t^r, \quad (32)$$

where

$$\rho_t = \frac{\partial}{\partial t}f_0(t) + k(t)f_0(t) + \int_0^t \sigma_s^2(t)ds.$$

This is a one-dimensional extended CIR short rate model. We emphasize, however, that ρ_t is non-deterministic since it depends on the non-deterministic $\sigma_s(t)$. An additional, auxiliary factor

$$v_t = \int_0^t \sigma_s^2(t)ds, \quad dv_t = ((\zeta(t))^2r_t - 2k(t)v_t)dt$$

is needed to make the model (r_t, v_t) Markov. The forward rate volatility specification that yields the extended CIR short rate model in which k and ζ do not depend on time, was studied in Heath, Jarrow, and Morton (1992, Section 8), but in this case ρ in (32) is not available in explicit form.

Reasoning along the same lines as above yields the following SDE for the short term spread λ

$$d\lambda_t = (\rho_t^* - \kappa^*(t)\lambda_t)dt + \zeta^*(t)\sqrt{\lambda_t}dW_t^\lambda,$$

where ρ_t^* is defined accordingly. Similarly, we also define

$$j_t = \int_0^t (\sigma_s^*(t))^2ds, \quad dj_t = ((\zeta^*(t))^2\lambda_t - 2k^*(t)j_t)dt.$$

In Theorem 2.1 of Chiarella and Kwon (2001) it was shown that the risk-free extended CIR model possesses an affine term structure with two stochastic factors. More precisely, the bond prices can be written as exponential-affine functions of the current level of the short rate r and the process v :

$$B_t(T) = \frac{B_0(T)}{B_0(t)} \exp \left(\gamma(t, T)f_0(t) - \gamma(t, T)r_t - \frac{1}{2}\gamma^2(t, T)v_t \right), \quad (33)$$

where

$$\gamma(t, T) = \int_t^T e^{-\int_t^u k(v)dv} du$$

is a deterministic function (combine Theorem 2.1 with (2.4) and (1.2) in Chiarella and Kwon (2001)). For defaultable bonds $\bar{B}_t^*(T)$ a similar expression involving in addition λ_t and j_t can be obtained by exactly the same reasoning and making use of the representation

$$\bar{B}_t^*(T) = B_t(T) \exp \left(- \int_t^T g_t(u)du \right), \quad (34)$$

which follows from (1) and (3).

3.2 Jumps and deterministic volatility

In CVA applications (see Crépey (2011)), Markovian specifications are used. The previous Brownian specification of the general HJM defaultable setup, yields a four-dimensional Markov factor process $X = (r, \lambda, \iota, j)$. In the quest of a more parsimonious Markovian specification, we now assume that the driving process Y is an n -dimensional Lévy process, whose components are subordinators, and that the volatilities are deterministic. We derive conditions that ensure the non-negativity of the interest rates and the credit spreads in this setting. It is worthwhile mentioning that when Y is two-dimensional as in the previous example, this yields a two-dimensional Markov factor process $X = (r, \lambda)$, which makes this specification preferable for applications.

Let Y be an n -dimensional non-negative Lévy process, such that its Lévy measure satisfies Assumption 2.1. Its cumulant generating function is given by

$$\kappa(z) = zb + \int_{\mathbb{R}_+^n} (e^{zx} - 1) F(dx) \quad (35)$$

for $z \in \mathbb{R}^n$ such that $z \in [-(1 + \varepsilon)\mathcal{K}, (1 + \varepsilon)\mathcal{K}]^n$, where $b \geq 0$ denotes the drift term and the Lévy measure F has its support in \mathbb{R}_+^n . We refer to Theorem 21.5 and Remark 21.6 in Sato (1999) for one-dimensional subordinators; for multi-dimensional non-negative Lévy processes see (3.19) in Barndorff-Nielsen and Shephard (2000). Note that subordinators do not have a diffusion component and their jumps can be only positive. Examples of these processes include a compound Poisson process with positive jumps, Gamma process, inverse Gaussian (IG) process, and generalized inverse Gaussian (GIG) processes.

In the remainder of the paper we impose the following standing assumptions on the bond price volatilities Σ and Σ^* :

Assumption 3.1 Volatilities Σ and Σ^* are non-negative, deterministic and stationary functions. More precisely, they are given as follows

$$\Sigma_s(t) = \left(S^i(t-s) \right)_{1 \leq i \leq n} \quad \text{and} \quad \Sigma_s^*(t) = \left(S^{*,i}(t-s) \right)_{1 \leq i \leq n},$$

for every s, t such that $0 \leq s \leq t \leq \bar{T}$, where $S^i : [0, \bar{T}] \rightarrow \mathbb{R}_+$ and $S^{*,i} : [0, \bar{T}] \rightarrow \mathbb{R}_+$, $i = 1, \dots, n$, are deterministic functions bounded by $\frac{\mathcal{K}}{2}$, where \mathcal{K} is the constant from (6).

Proposition 3.2 (i) *The dynamics of the forward rates $f_t(T)$ and r are given by*

$$f_t(T) = f_0(T) - \kappa(-\Sigma_t(T)) + \kappa(-\Sigma_0(T)) + \int_0^t \sigma_s(T) dY_s \quad (36)$$

and

$$r_t = f_0(t) + \kappa(-\Sigma_0(t)) + \int_0^t \sigma_s(t) dY_s. \quad (37)$$

(ii) *The dynamics of the credit spreads $g_t(T)$ and λ are given by*

$$\begin{aligned} g_t(T) &= g_0(T) - \kappa(-\Sigma_t^*(T) - \Sigma_t(T)) + \kappa(-\Sigma_0^*(T) - \Sigma_0(T)) \\ &\quad + \kappa(-\Sigma_t(T)) - \kappa(-\Sigma_0(T)) + \int_0^t \sigma_s^*(T) dY_s \end{aligned} \quad (38)$$

and

$$\lambda_t = g_0(t) + \kappa(-\Sigma_0^*(t) - \Sigma_0(t)) - \kappa(-\Sigma_0(t)) + \int_0^t \sigma_s^*(t) dY_s. \quad (39)$$

Proof. We begin by noting that

$$\frac{\partial}{\partial T} S^i(T-s) = -\frac{\partial}{\partial s} S^i(T-s) \text{ and } \frac{\partial}{\partial T} S^{*,i}(T-s) = -\frac{\partial}{\partial s} S^{*,i}(T-s), \quad (40)$$

for $i = 1, \dots, n$. Hence, Assumption 3.1 implies

$$\frac{\partial}{\partial T} \kappa(-\Sigma_s(T)) = -\frac{\partial}{\partial s} \kappa(-\Sigma_s(T))$$

and

$$\frac{\partial}{\partial T} \kappa(-\Sigma_s^*(T) - \Sigma_s(T)) = -\frac{\partial}{\partial s} \kappa(-\Sigma_s^*(T) - \Sigma_s(T)),$$

which follows from (40) by differentiation. Therefore, we obtain

$$\begin{aligned} \int_0^t \frac{\partial}{\partial T} \kappa(-\Sigma_s(T)) ds &= -\int_0^t \frac{\partial}{\partial s} \kappa(-\Sigma_s(T)) ds \\ &= -(\kappa(-\Sigma_t(T)) - \kappa(-\Sigma_0(T))), \end{aligned}$$

and similarly,

$$\int_0^t \frac{\partial}{\partial T} \kappa(-\Sigma_s^*(T) - \Sigma_s(T)) ds = -(\kappa(-\Sigma_t^*(T) - \Sigma_t(T)) - \kappa(-\Sigma_0^*(T) - \Sigma_0(T))).$$

Inserting these expressions into (28) and (30) yields (36) and (38), respectively. To show (37) and (39) we note that

$$\int_0^t \frac{\partial}{\partial t} \kappa(-\Sigma_s(t)) ds = \kappa(-\Sigma_0(t))$$

and

$$\int_0^t \frac{\partial}{\partial t} \kappa(-\Sigma_s^*(t) - \Sigma_s(t)) ds = \kappa(-\Sigma_0^*(t) - \Sigma_0(t)),$$

due to $\kappa(-\Sigma_t(t)) = \kappa(0) = 0$ and $\kappa(-\Sigma_t^*(t) - \Sigma_t(t)) = 0$, which follows by (15) and (22) combined with (35). \square

In the next two propositions we give necessary and sufficient deterministic conditions for the non-negativity of the interest rates and credit spreads. Note that by (36)-(37), one has that

$$\begin{aligned} f_t(T) - \int_0^t \sigma_s(T) dY_s &= f_0(T) - \kappa(-\Sigma_t(T)) + \kappa(-\Sigma_0(T)) =: \mu(t, T) \\ r_t - \int_0^t \sigma_s(t) dY_s &= f_0(t) + \kappa(-\Sigma_0(t)) =: \mu(t), \end{aligned}$$

where $\mu(t, T)$ and $\mu(t)$ are thus deterministic.

Proposition 3.3 (i) *The short rate r_t is non-negative if $\mu(t) \geq 0$, for $t \in [0, T]$.*
(ii) *Assume that the distribution of the random vector Y_1 has $[0, \infty)^n$ as its support. Then the converse of (i) is also true, i.e. if $r_t \geq 0$, then $\mu(t) \geq 0$, for every $t \in [0, T]$. Moreover, if $r_t \geq 0$, for every $t \in [0, T]$, then $f_t(T) \geq 0$, for every $T \in [0, \bar{T}]$. In words, the non-negativity of the short rate implies the non-negativity of the forward rate.*

Proof. Since Y has non-negative components and the volatility σ is non-negative by assumption, it is obvious that $\mu(t) \geq 0$ implies $r_t \geq 0$, for every t . This proves (i).

In case when the support of Y_1 is $[0, \infty)^n$, we show the converse statement by noting that

$$0 \leq \left(\int_0^t \sigma_s(t) dY_s \right) (\omega) \leq \mathcal{K} \left(\sum_{i=1}^n \int_0^t dY_s^i(\omega) \right) = \mathcal{K} \sum_{i=1}^n Y_t^i(\omega), \quad (41)$$

for every $\omega \in \Omega$. Note that since Y^i , $i = 1, \dots, n$, are increasing process, here the stochastic integrals coincide with the Stieltjes integrals, and hence we are able to do the integration pathwise. Moreover, since Y_1 has the support $[0, \infty)^n$, so does Y_t . This implies that $\mathbb{P}(\omega \in \Omega : \sum_{i=1}^n Y_t^i(\omega) < \varepsilon) > 0$, for every $\varepsilon > 0$. This combined with (41) yields that

$$\mathbb{P} \left(\omega \in \Omega : \left(\int_0^t \sigma_s(t) dY_s \right) (\omega) < \varepsilon \right) > 0,$$

for every $\varepsilon > 0$. Since $\mu(t)$ is deterministic, it follows that

$$r_t = \mu(t) + \int_0^t \sigma_s(t) dY_s \geq 0$$

only if $\mu(t) \geq 0$. Thus, we proved the first claim in (ii). To show the second one, namely that the non-negativity of the short rate r_t for all $t \in [0, T]$, implies the non-negativity of the forward rate $f_t(T)$, note that

$$\mu(t, T) = \mu(T) - \kappa(-\Sigma_t(T)).$$

Since we have just proved that $r_T \geq 0$ implies $\mu(T) \geq 0$, it suffices to show that $-\kappa(-\Sigma_t(T)) \geq 0$ to deduce that $\mu(t, T) \geq 0$. But this follows easily from $\Sigma_t(T) \geq 0$ combined with (35). Thus, we have $\mu(t, T) \geq 0$, which implies $f_t(T) \geq 0$ by definition of $\mu(t, T)$. \square

Completely analogously, we can derive conditions for the non-negativity of the forward spread $g_t(T)$ and the short term spread λ . Let us denote

$$\begin{aligned} \mu^*(t, T) &:= g_t(T) - \int_0^t \sigma_s^*(T) dY_s \\ &= g_0(T) - \kappa(-\Sigma_t(T) - \Sigma_t^*(T)) + \kappa(-\Sigma_0(T) - \Sigma_0^*(T)) \\ &\quad + \kappa(-\Sigma_t(T)) - \kappa(-\Sigma_0(T)) \\ \mu^*(t) &:= \mu^*(t, t) = \lambda_t - \int_0^t \sigma_s^*(t) dY_s = g_0(t) + \kappa(-\Sigma_0(t) - \Sigma_0^*(t)) - \kappa(-\Sigma_0(t)), \end{aligned}$$

which follows by (38) -(39).

Proposition 3.4 (i) *The short term spread λ_t is non-negative if $\mu^*(t) \geq 0$, for every $t \in [0, T]$.*

(ii) *Assume that the distribution of Y_1 has $[0, \infty)^n$ as its support. Then the converse of (i) is also true, i.e. if $\lambda_t \geq 0$, then $\mu^*(t) \geq 0$, for every t . Moreover, if $\lambda_t \geq 0$, for every $t \in [0, T]$, then $g_t(T) \geq 0$, for every $T \in [0, \bar{T}]$, i.e. the non-negativity of the short term spread implies the non-negativity of the forward spread.*

Let us now assume that Y is a two-dimensional non-negative Lévy process. We shall study in more detail the dependence between its components. But before doing so, let us give an example of the volatility structures that satisfy the conditions of this section and produce non-negative rates and spreads.

Example 3.5 (Vasicek volatility structure) Assume that the volatility of the forward rates $f.(T)$ and the volatilities of the forward spreads $g.(T)$ are of the Vasicek type, so for every $0 \leq s \leq T \leq \bar{T}$,

$$\sigma_s(T) = \left(\sigma e^{-a(T-s)}, 0 \right), \quad \sigma_s^*(T) = \left(0, \sigma^* e^{-a^*(T-s)} \right), \quad (42)$$

where $\sigma, \sigma^* > 0$ and $a, a^* \neq 0$ are real constants such that μ and μ^* from Propositions 3.3(i) and 3.4(i) are non-negative. Then

$$\Sigma_t(T) = \int_t^T \sigma_t(u) du = \left(\frac{\sigma}{a} \left(1 - e^{-a(T-t)} \right), 0 \right), \quad \Sigma_t^*(T) = \left(0, \frac{\sigma^*}{a^*} \left(1 - e^{-a^*(T-t)} \right) \right).$$

These volatilities Σ and Σ^* satisfy the standing Assumption 3.1. Moreover, inserting them into Proposition 3.2, we note that the forward rates $f.(T)$ and the short rate r are driven solely by the first subordinator Y^1 , whereas the forward spreads $g.(T)$ and the short spread λ are driven by the second subordinator Y^2 .

With this volatility specification, one obtains the Lévy Hull–White extended Vasicek model for the short rate r (cf. Corollary 4.5 and equation(4.11) in the default-free setup of Eberlein and Raible (1999))

$$dr_t = a(\rho(t) - r_t)dt + \sigma dY_t^1.$$

By similar reasoning, one can obtain the Lévy Hull–White extended Vasicek model for the short term spread λ

$$d\lambda_t = a^*(\rho^*(t) - \lambda_t)dt + \sigma^* dY_t^2.$$

The functions ρ and ρ^* are deterministic functions of time which are chosen in such a way that the models fit the initial term structures $f_0(T)$ and $g_0(T)$ observed in the market. Inserting the Vasicek volatilities into equation (37) for r and equation (39) for λ , and differentiating with respect to time, one obtains ρ and ρ^* . For ρ we have

$$\rho(t) = f_0(t) + \frac{1}{a} \frac{\partial}{\partial t} f_0(t) + \kappa^1 \left(\frac{\sigma}{a} (e^{-at} - 1) \right) - (\kappa^1)' \left(\frac{\sigma}{a} (e^{-at} - 1) \right) \frac{\sigma}{a} e^{-at},$$

where κ^1 is the cumulant function of Y^1 (compare equations (4.10) and(4.11) in Eberlein and Raible (1999)), and ρ^* is derived in a similar fashion.

Moreover, this model possesses an affine term structure. It means that the default-free bond prices can be written as exponential-affine functions of the current level of the short rate r , and the pre-default defaultable bond prices as exponential-affine functions of the short rate r and the short term spread λ :

$$B_t(T) = \exp(m(t, T) + n(t, T)r_t), \quad (43)$$

where

$$\begin{aligned} m(t, T) &= \log \left(\frac{B_0(T)}{B_0(t)} \right) - n(t, T) \left(f_0(t) + \int_0^t \frac{\partial}{\partial t} \kappa^1 \left(\frac{\sigma}{a} (e^{-a(t-s)} - 1) \right) ds \right) \\ &\quad - \int_0^t \left[\kappa^1 \left(\frac{\sigma}{a} (e^{-a(T-s)} - 1) \right) - \kappa^1 \left(\frac{\sigma}{a} (e^{-a(t-s)} - 1) \right) \right] ds \end{aligned}$$

and

$$n(t, T) = -e^{at} \int_t^T e^{-au} du = \frac{1}{a} \left(e^{-a(T-t)} - 1 \right).$$

This result for default-free zero coupon bonds $B_t(T)$ is proved in Raible (2000, Theorem 4.8). For defaultable bonds $\bar{B}_t^*(T)$ it follows, by exactly the same reasoning and using representation (34), that

$$\bar{B}_t^*(T) = \exp(m(t, T) + n(t, T)r_t + m^*(t, T) + n^*(t, T)\lambda_t), \quad (44)$$

where the deterministic functions $m^*(\cdot, T)$ and $n^*(\cdot, T)$ can be defined similarly as $m(\cdot, T)$ and $n(\cdot, T)$ above.

3.2.1 Dependent drivers

In order to specify the dependence between components Y^1 and Y^2 of the driving process Y , the simplest way is a common factor model, that we present here (an alternative would be to use a Lévy copula, see Cont and Tankov (2003)).

Let us assume that Y^1 and Y^2 are given as follows

$$Y^1 = Z^1 + Z^3 \quad \text{and} \quad Y^2 = Z^2 + Z^3,$$

where Z^i , $i = 1, 2, 3$, are mutually independent subordinators with drifts b^{Z^i} and Lévy measures F^{Z^i} . Then Y^1 and Y^2 are again subordinators (this follows by Proposition 11.10 and Theorem 21.5 in Sato (1999)) and they are obviously dependent. The Lévy measures and the cumulant functions for subordinators Y^1 and Y^2 , as well as for the two-dimensional process $Y = (Y^1, Y^2)$, can be calculated explicitly, as shown below.

Consider a three-dimensional Lévy process $Z = (Z^1, Z^2, Z^3)$, consisting of mutually independent subordinators Z^i , as above. Applying Sato (1999, Exercise 12.10, page 67), independence of Z^1, Z^2 and Z^3 , implies that the Lévy measure F^Z of Z is given by

$$F^Z(A) = \sum_{i=1}^3 F^{Z^i}(A_i), \quad A \in \mathcal{B}(\mathbb{R}^3 \setminus \{0\}), \quad (45)$$

where for every i , $A_i = \{x \in \mathbb{R} : x\mathbf{e}_i \in A\}$ with \mathbf{e}_i a unit vector in \mathbb{R}^3 with 1 in the i -th position and other entries zero.

Now we simply have to write Y , Y^1 and Y^2 as linear transformations of Z and apply Proposition 11.10 in Sato (1999). For example, we have $Y = UZ$, where

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Hence, $b^Y = Ub^Z$ and the Lévy measure F^Y is given, for $B \in \mathcal{B}(\mathbb{R}^2 \setminus \{0\})$, by

$$F^Y(B) = F^Z(x \in \mathbb{R}^3 : Ux \in B) = F^Z(x \in \mathbb{R}^3 : (x_1 + x_3, x_2 + x_3)^\top \in B),$$

which combined with (45) yields

$$F^Y(B) = F^{Z^1}(x \in \mathbb{R} : (x, 0) \in B) + F^{Z^2}(x \in \mathbb{R} : (0, x) \in B) + F^{Z^3}(x \in \mathbb{R} : (x, x) \in B).$$

The cumulant function κ^Y of Y is given, for $z \in \mathbb{R}^2$ such that κ^{Z^i} , $i = 1, 2, 3$, below are well-defined, by

$$\kappa^Y(z) = \kappa^{Z^1}(z_1) + \kappa^{Z^2}(z_2) + \kappa^{Z^3}(z_1 + z_2).$$

This can be derived directly recalling that $\kappa^Y(z) = \log \mathbb{E}[e^{zY}]$ and using independence between Z^1, Z^2 and Z^3 .

Similarly, writing each Y^i as a linear transformation of Z , we obtain its Lévy measure F^{Y^i} , for $C \in \mathcal{B}(\mathbb{R} \setminus \{0\})$,

$$F^{Y^i}(C) = F^Z(x \in \mathbb{R}^3 : x_i + x_3 \in C) = F^{Z^i}(x \in \mathbb{R} : x \in C) + F^{Z^3}(x \in \mathbb{R} : x \in C)$$

and the drift $b^{Y^i} = b^{Z^i} + b^{Z^3}$, which shows that Y^i is indeed a subordinator (recall Theorem 21.5 in Sato (1999)). The cumulant function κ^{Y^i} of Y^i is given, for $z \in \mathbb{R}$ such that κ^{Z^i} and κ^{Z^3} below are well-defined, by

$$\kappa^{Y^i}(z) = \kappa^{Z^i}(z) + \kappa^{Z^3}(z).$$

To conclude this section, we describe two well-known subordinators: an inverse Gaussian (IG) process and a Gamma process. In addition, we recall an example of a subordinator belonging to the *CGMY* Lévy family. Note that these processes have infinite activity, which makes them suitable drivers for the term structure of interest rates in our model.

Example 3.6 (IG process) According to Kyprianou (2006, Section 1.2.5), a process $Z = (Z_t)_{t \geq 0}$ obtained from a standard Brownian motion W by setting

$$Z_t = \inf\{s > 0 : W_s + bs > t\},$$

where $b > 0$, is an inverse Gaussian (IG) process and has the Lévy measure given by

$$F(dx) = \frac{1}{\sqrt{2\pi x^3}} e^{-\frac{b^2 x}{2}} \mathbf{1}_{\{x > 0\}} dx.$$

The distribution of Z_t is $IG(\frac{t}{b}, t^2)$. The Lévy measure F satisfies condition (6) for any two constants $\mathcal{K}, \varepsilon > 0$ such that $(1 + \varepsilon)\mathcal{K} < \frac{b^2}{2}$. Hence, the cumulant function κ exists for all $z \in (-\frac{b^2}{2}, \frac{b^2}{2})$ (actually for all $z \in (-\infty, \frac{b^2}{2})$ since F is concentrated on $(0, \infty)$) and is given by

$$\kappa(z) = b \left(1 - \sqrt{1 - 2\frac{z}{b^2}} \right).$$

Example 3.7 (Gamma process) The Gamma process Z with parameters $\alpha, \beta > 0$ is a subordinator with Lévy measure given by

$$F(dx) = \beta x^{-1} e^{-\alpha x} \mathbf{1}_{\{x > 0\}} dx,$$

see Kyprianou (2006, Section 1.2.4). The Lévy measure F satisfies condition (6) for any two constants $\mathcal{K}, \varepsilon > 0$ such that $(1 + \varepsilon)\mathcal{K} < \alpha$. Hence, the cumulant function κ is well-defined for all $z \in (-\infty, \alpha)$ and is given by

$$\kappa(z) = -\beta \log \left(1 - \frac{z}{\alpha} \right).$$

Example 3.8 (CGMY subordinator) The CGMY Lévy process Z with parameters $G = \infty$ and $Y \leq 0$ is a subordinator by Theorem 21.5 in Sato (1999). Its Lévy measure is given by

$$F(dx) = C \frac{\exp(-M|x|)}{|x|^{1+Y}} \mathbf{1}_{\{x > 0\}} dx,$$

where $C, M > 0$ and $Y \leq 0$; see Raible (2000, A.3.2). For an overview of the main properties of the class of CGMY Lévy processes we refer to Carr, Geman, Madan, and Yor (2002) or Raible (2000, A.3.2).

4 Valuation of interest rate derivatives

Here we give an overview of the basic interest rate derivatives where the underlying rate is the LIBOR and calculate their value in our setup. We work under the assumptions of Section 2 and Section 3.2 (jumps and deterministic volatilities). We emphasize that our setup provides a versatile multi-curve model of LIBORs of different tenors, which is relevant for pricing of multi-tenor derivatives such as basis swaps. For instance, if one wishes to have one stochastic driving factor for each tenor and for the risk-free rate, then it suffices to consider a three-dimensional process Y , where the first component drives the risk-free rates and the remaining two components are reserved for the credit spread.

Before proceeding with the interest rate derivative valuation, let us recall that a forward martingale measure \mathbb{P}^T associated with the date $0 < T \leq \bar{T}$ is a probability measure defined on (Ω, \mathcal{F}_T) and equivalent to \mathbb{P} . It is characterized by the following density process

$$\frac{d\mathbb{P}^T}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \frac{\beta_t B_t(T)}{B_0(T)}.$$

In our setup this density process is given by (cf. (14))

$$\frac{d\mathbb{P}^T}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp \left(- \int_0^t A_s(T) ds - \int_0^t \Sigma_s(T) dY_s \right). \quad (46)$$

Note that the density process is \mathcal{E} -adapted. The payoffs of the derivatives that we are going to study in the sequel are typically some combinations of deterministic functions of the LIBORs $L_T(T, T + \delta)$, which is an \mathcal{E}_T -measurable random variable, for any $T \in [0, \bar{T} - \delta]$. Then we have

$$\mathbb{E}[f(L_T(T, T + \delta)) | \mathcal{F}_t] = \mathbb{E}[f(L_T(T, T + \delta)) | \mathcal{E}_t],$$

for any deterministic, Borel measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$. This property is equivalent to the immersion property between \mathcal{E} and \mathcal{F} (see Bielecki and Rutkowski (2002, Section 6.1.1)), which by assumption holds in our model. Moreover, the property holds true under any forward measure \mathbb{P}_T as well, since the density process in (46) is \mathcal{E} -adapted. Henceforth in all computations we shall replace automatically \mathcal{F}_t by \mathcal{E}_t .

Finally, note that in a multiple-curve setup the forward price process $\left(\frac{\bar{B}_t^*(T)}{\bar{B}_t^*(T+\delta)} \right)_{0 \leq t \leq T}$ is NOT a martingale under the forward measure $\mathbb{P}^{T+\delta}$. Consequently, the forward LIBOR, which would be defined as $L_t(T, T + \delta) = \frac{1}{\delta} \left(\frac{\bar{B}_t^*(T)}{\bar{B}_t^*(T+\delta)} - 1 \right)$, is different from a forward rate implied by a forward rate agreement for the future time interval $[T, T + \delta]$, as we shall see below. In the one-curve setup, the forward LIBOR defined as $L_t(T, T + \delta) = \frac{1}{\delta} \left(\frac{B_t(T)}{B_t(T+\delta)} - 1 \right)$ is precisely the FRA rate for $[T, T + \delta]$.

4.1 Forward rate agreements

The simplest interest rate derivative is a forward rate agreement (FRA) with inception date T and maturity $T + \delta$. Let us denote the fixed rate by K and the notional amount by N . The payoff of such an agreement at maturity $T + \delta$ is equal to

$$P^{FRA}(T + \delta; T, T + \delta, K, N) = N\delta(L_T(T, T + \delta) - K),$$

where $L_T(T, T + \delta)$ is the T -spot LIBOR. Thus, the value of the FRA at time t is calculated as the conditional expectation with respect to the forward measure $\mathbb{P}^{T+\delta}$ associated to the

date $T + \delta$ and is given by

$$P^{FRA}(t; T, T + \delta, K, N) = N\delta B_t(T + \delta)\mathbb{E}^{\mathbf{P}^{T+\delta}}[L_T(T, T + \delta) - K|\mathcal{E}_t].$$

We emphasize again that the forward rate implied by this FRA, that is the rate K_0 such that $P^{FRA}(t; T, T + \delta, K_0, N) = 0$, is different in the multiple-curve setup from the forward LIBOR.

Let us derive the value of the FRA and calculate the forward rate K_0 in our setup. Using definition (4) of the LIBOR $L_T(T, T + \delta)$ we have

$$P^{FRA}(t; T, T + \delta, K, N) = NB_t(T + \delta)\mathbb{E}^{\mathbf{P}^{T+\delta}}\left[\frac{1}{\bar{B}_T^*(T + \delta)} - \bar{K}|\mathcal{E}_t\right], \quad (47)$$

where $\bar{K} = 1 + \delta K$. The key issue is thus to compute the conditional expectation

$$v_t^{T, T+\delta} := \mathbb{E}^{\mathbf{P}^{T+\delta}}\left[\frac{1}{\bar{B}_T^*(T + \delta)} \mid \mathcal{E}_t\right]. \quad (48)$$

Inserting (23) into (48) we obtain

$$\begin{aligned} v_t^{T, T+\delta} &= \frac{\bar{B}_0^*(T)}{\bar{B}_0^*(T + \delta)} \exp\left(\int_0^T (\bar{A}_s^*(T + \delta) - \bar{A}_s^*(T))ds\right) \\ &\quad \times \mathbb{E}^{\mathbf{P}^{T+\delta}}\left[\exp\left(\int_0^T (\bar{\Sigma}_s^*(T + \delta) - \bar{\Sigma}_s^*(T))dY_s\right) \mid \mathcal{E}_t\right] \\ &= c^{T, T+\delta} \exp\left(\int_0^t (\bar{\Sigma}_s^*(T + \delta) - \bar{\Sigma}_s^*(T))dY_s\right) \\ &\quad \times \mathbb{E}^{\mathbf{P}^{T+\delta}}\left[\exp\left(\int_t^T (\bar{\Sigma}_s^*(T + \delta) - \bar{\Sigma}_s^*(T))dY_s\right)\right], \end{aligned} \quad (49)$$

with

$$\begin{aligned} c^{T, T+\delta} &= \frac{\bar{B}_0^*(T)}{\bar{B}_0^*(T + \delta)} \exp\left(\int_0^T (\bar{A}_s^*(T + \delta) - \bar{A}_s^*(T))ds\right) \\ &= \frac{\bar{B}_0^*(T)}{\bar{B}_0^*(T + \delta)} \exp\left(\int_0^T (\kappa(-\bar{\Sigma}_s^*(T + \delta)) - \kappa(-\bar{\Sigma}_s^*(T))) ds\right), \end{aligned}$$

where we used the drift condition (27). For the second equality in (49) we use the fact that $\int_0^t (\bar{\Sigma}_s^*(T + \delta) - \bar{\Sigma}_s^*(T))dY_s$ is \mathcal{E}_t -measurable. Moreover, since Y is a time-inhomogeneous Lévy process under the measure $\mathbf{P}^{T+\delta}$, its increments are independent (cf. Proposition 2.3 and Lemma 2.5 in Kluge (2005)). This combined with the deterministic volatility structure which is integrated with respect to Y yields the equality.

The remaining expectation can be calculated making use of Proposition 3.1 in Eberlein and Kluge (2006b), which yields

$$\begin{aligned} &\mathbb{E}^{\mathbf{P}^{T+\delta}}\left[\exp\left(\int_t^T (\bar{\Sigma}_s^*(T + \delta) - \bar{\Sigma}_s^*(T))dY_s\right)\right] \\ &= \exp\left(\int_t^T \kappa_s^{\mathbf{P}^{T+\delta}}(\bar{\Sigma}_s^*(T + \delta) - \bar{\Sigma}_s^*(T))ds\right), \end{aligned} \quad (50)$$

where $\kappa_s^{\mathbf{P}^{T+\delta}}$ denotes the cumulant function of Y under the measure $\mathbf{P}^{T+\delta}$. However, to obtain the expression for this expectation using directly the cumulant function κ of Y under the measure \mathbf{P} , we have the following sequence of equalities

$$\begin{aligned}
& \mathbb{E}^{\mathbf{P}^{T+\delta}} \left[\exp \left(\int_t^T (\bar{\Sigma}_s^*(T+\delta) - \bar{\Sigma}_s^*(T)) dY_s \right) \right] \\
&= \exp \left(- \int_0^T A_s(T+\delta) ds \right) \\
&\quad \times \mathbb{E} \left[\exp \left(\int_t^T (\bar{\Sigma}_s^*(T+\delta) - \bar{\Sigma}_s^*(T)) dY_s - \int_0^T \Sigma_s(T+\delta) dY_s \right) \right] \\
&= \exp \left(- \int_0^T \kappa(-\Sigma_s(T+\delta)) ds \right) \mathbb{E} \left[\exp \left(\int_0^t (-\Sigma_s(T+\delta)) dY_s \right) \right] \\
&\quad \times \mathbb{E} \left[\exp \left(\int_t^T (\bar{\Sigma}_s^*(T+\delta) - \bar{\Sigma}_s^*(T) - \Sigma_s(T+\delta)) dY_s \right) \right] \\
&= \exp \left(\int_t^T (\kappa(\bar{\Sigma}_s^*(T+\delta) - \bar{\Sigma}_s^*(T) - \Sigma_s(T+\delta)) - \kappa(-\Sigma_s(T+\delta))) ds \right),
\end{aligned} \tag{51}$$

where we have used equation (46) for the first equality, and the drift condition (18) plus the independence of the increments of Y for the second one. The third equality follows by Eberlein and Kluge (2006b, Proposition 3.1). Finally, we obtain

$$\begin{aligned}
v_t^{T,T+\delta} &= c^{T,T+\delta} \exp \left(\int_0^t (\bar{\Sigma}_s^*(T+\delta) - \bar{\Sigma}_s^*(T)) dY_s \right) \\
&\quad \times \exp \left(\int_t^T (\kappa(\bar{\Sigma}_s^*(T+\delta) - \bar{\Sigma}_s^*(T) - \Sigma_s(T+\delta)) - \kappa(-\Sigma_s(T+\delta))) ds \right).
\end{aligned} \tag{52}$$

Note that as a by-product we obtain a formula for the cumulant function $\kappa_s^{\mathbf{P}^{T+\delta}}$

$$\kappa_s^{\mathbf{P}^{T+\delta}}(z) = \kappa(z - \Sigma_s(T+\delta)) - \kappa(-\Sigma_s(T+\delta)), \tag{53}$$

for $z \in \mathbb{R}^n$ such that $\kappa(z - \Sigma_s(T+\delta))$ is well-defined. This follows by combining (50) and (51), for every $t \in [0, T]$ and for $\bar{\Sigma}_s^*(T+\delta) - \bar{\Sigma}_s^*(T)$ replaced with z .

Let us sum-up our findings in the form of the following

Proposition 4.1 *The value of the FRA at time $t = 0$ is given by*

$$P^{FRA}(0; T, T+\delta, K, N) = NB_0(T+\delta) \left[v_0^{T,T+\delta} - \bar{K} \right],$$

with

$$\begin{aligned}
v_0^{T,T+\delta} &= \frac{\bar{B}_0^*(T)}{\bar{B}_0^*(T+\delta)} \exp \left(\int_0^T (\kappa(-\bar{\Sigma}_s^*(T+\delta)) - \kappa(-\bar{\Sigma}_s^*(T)) - \kappa(-\Sigma_s(T+\delta))) ds \right) \\
&\quad \times \exp \left(\int_0^T \kappa(\bar{\Sigma}_s^*(T+\delta) - \bar{\Sigma}_s^*(T) - \Sigma_s(T+\delta)) ds \right).
\end{aligned}$$

The forward rate K_0 implied by this FRA is given by

$$K_0 = \frac{1}{\delta} \left[v_0^{T,T+\delta} - 1 \right]. \tag{54}$$

The spread with respect to the one-curve forward rate given by $\frac{1}{\delta} \left(\frac{B_0(T)}{B_0(T+\delta)} - 1 \right)$, is equal to

$$\text{Spread}_0^{FRA} = \frac{1}{\delta} \left[v_0^{T, T+\delta} - \frac{B_0(T)}{B_0(T+\delta)} \right]. \quad (55)$$

As soon as the driving process Y and the parameters of the model are specified, all these values can be easily computed. We provide an example in Section 5.

4.2 Interest rate swaps

An interest rate swap is a financial contract between two parties to exchange one stream of future interest payments for another, based on a specified notional amount N . Here we consider a fixed-for-floating swap, where a fixed payment is exchanged for a floating payment linked to the LIBOR. We assume that, as typical, the LIBOR is set in advance and the payments are done in arrears. The swap is initiated at time $T_0 \geq 0$. Denote by $T_1 < \dots < T_n$, where $T_1 > T_0$, a collection of the payment dates and by S the fixed rate. Then the time- t value of the swap for the receiver of the floating rate is given by

$$\begin{aligned} P^{Sw}(t; T_1, T_n) &= N \sum_{k=1}^n \delta_{k-1} B_t(T_k) \mathbf{E}^{\mathbf{P}^{T_k}} [L_{T_{k-1}}(T_{k-1}, T_k) - S | \mathcal{E}_t] \\ &= N \sum_{k=1}^n P^{FRA}(t; T_{k-1}, T_k, S, 1) \\ &= N \sum_{k=1}^n B_t(T_k) \left(v_t^{T_{k-1}, T_k} - \bar{S}_{k-1} \right), \end{aligned}$$

where $\delta_{k-1} = T_k - T_{k-1}$, $\bar{S}_{k-1} = 1 + \delta_{k-1} S$, and $v_t^{T_{k-1}, T_k}$ is given by (52), for every $k = 1, \dots, n$. This formula follows directly from (47) and (48).

The swap rate $S(t; T_1, T_n)$ is the rate that makes the time- t value $P^{Sw}(t; T_1, T_n)$ of the swap equal to zero. Therefore,

Proposition 4.2 *The swap rate $S(t; T_1, T_n)$ is given by*

$$S(t; T_1, T_n) = \frac{\sum_{k=1}^n B_t(T_k) (v_t^{T_{k-1}, T_k} - 1)}{\sum_{k=1}^n \delta_{k-1} B_t(T_k)}. \quad (56)$$

4.3 Basis swaps

A basis swap is an interest rate swap, where two floating payments linked to the LIBORs of different tenors are exchanged. For example, a buyer of such a swap receives semiannually a 6m-LIBOR and pays quarterly a 3m-LIBOR, both set in advance and paid in arrears. Note that there exist also other conventions regarding the payments on the two legs of a basis swap. A more detailed account on basis swaps can be found in Mercurio (2010, Section 5.2) and Filipović and Trolle (2011, Section 2.4 and Appendix F). Let us consider a basis swap with the two tenor structures denoted by $\mathcal{T}^1 = \{T_0^1 < \dots < T_{n_1}^1\}$ and $\mathcal{T}^2 = \{T_0^2 < \dots < T_{n_2}^2\}$, where $T_0^1 = T_0^2 \geq 0$, $T_{n_1}^1 = T_{n_2}^2 = \hat{T}$, and $\mathcal{T}^1 \subset \mathcal{T}^2$. The notional amount is denoted by N and the swap is initiated at time T_0^1 , where the first payments are due at T_1^1 and T_1^2 . The

time- t value of such an agreement is given by

$$P^{BSw}(t; \widehat{T}, N) = N \left(\sum_{i=1}^{n_1} \delta_{i-1}^1 B_t(T_i^1) \mathbb{E}^{\mathbf{P}^{T_i^1}} [L_{T_{i-1}^1}(T_{i-1}^1, T_i^1) | \mathcal{E}_t] - \sum_{j=1}^{n_2} \delta_{j-1}^2 B_t(T_j^2) \mathbb{E}^{\mathbf{P}^{T_j^2}} [L_{T_{j-1}^2}(T_{j-1}^2, T_j^2) | \mathcal{E}_t] \right).$$

Making use of (47) and (48) we obtain

Proposition 4.3 *The value of the basis swap at time t is given by*

$$P^{BSw}(t; \widehat{T}, N) = N \left(\sum_{i=1}^{n_1} B_t(T_i^1) (v_t^{T_{i-1}^1, T_i^1} - 1) - \sum_{j=1}^{n_2} B_t(T_j^2) (v_t^{T_{j-1}^2, T_j^2} - 1) \right), \quad (57)$$

where $v_t^{T_{k-1}^i, T_k^i}$ is given by (52), for each tenor structure \mathcal{T}^i , $i = 1, 2$.

Note that before the 2007-09 credit crisis the value of such a swap was zero at any time t . Since the crisis, markets quote positive basis swap spreads that have to be added to the smaller tenor leg, which is consistently accounted for in our setup; see Section 5 for a numerical example.

Let us check that the value of the basis swap in the one-curve setup is indeed zero. We recall that in this setup the forward LIBORs, which were defined using the default-free zero coupon bonds as $(L_t(T, T + \delta) = \frac{1}{\delta} \left(\frac{B_t(T)}{B_t(T+\delta)} - 1 \right))_{0 \leq t \leq T}$, are martingales under the corresponding forward measures. We thus have

$$\begin{aligned} P^{BSw}(t; \widehat{T}, N) &= N \left(\sum_{i=1}^{n_1} \delta_{i-1}^1 B_t(T_i^1) \mathbb{E}^{\mathbf{P}^{T_i^1}} [L_{T_{i-1}^1}(T_{i-1}^1, T_i^1) | \mathcal{E}_t] - \sum_{j=1}^{n_2} \delta_{j-1}^2 B_t(T_j^2) \mathbb{E}^{\mathbf{P}^{T_j^2}} [L_{T_{j-1}^2}(T_{j-1}^2, T_j^2) | \mathcal{E}_t] \right) \\ &= N \left(\sum_{i=1}^{n_1} \delta_{i-1}^1 B_t(T_i^1) L_t(T_{i-1}^1, T_i^1) - \sum_{j=1}^{n_2} \delta_{j-1}^2 B_t(T_j^2) L_t(T_{j-1}^2, T_j^2) \right) \\ &= N \left((B_t(T_0^1) - B_t(T_{n_1}^1)) - (B_t(T_0^2) - B_t(T_{n_2}^1)) \right) = 0, \end{aligned}$$

by initial assumptions on the tenor structures.

In the multiple-curve setup we cannot use the same calculation, since now the LIBORs are not martingales under the classical forward measures. Hence, one ends up with formula (57), which in general yields a non-zero value of the basis swap and this value is exactly the basis swap spread (cf. Tables 4–6 in Section 5).

4.4 Caps and floors

Recall that an interest rate cap (respectively floor) is a financial contract in which the buyer receives payments at the end of each period in which the interest rate exceeds (respectively falls below) a mutually agreed strike level. The payment that the seller has to make covers

exactly the difference between the strike K and the interest rate at the end of each period. Every cap (respectively floor) is a series of caplets (respectively floorlets). The time- t price of a caplet with strike K and maturity T , which is settled in arrears, is given by

$$\begin{aligned} P^{Cpl}(t; T, K) &= \delta B_t(T + \delta) \mathbb{E}^{\mathbf{P}^{T+\delta}} \left[(L_T(T, T + \delta) - K)^+ \mid \mathcal{E}_t \right] \\ &= B_t(T + \delta) \mathbb{E}^{\mathbf{P}^{T+\delta}} \left[\left(\frac{1}{\bar{B}_T^*(T + \delta)} - \bar{K} \right)^+ \mid \mathcal{E}_t \right] \end{aligned}$$

where $\bar{K} = 1 + \delta K$.

It is worthwhile mentioning that the classical transformation of a caplet into a put option on a bond does not work in the multiple-curve setup. More precisely, the (still valid) fact that the payoff $((1 + \delta L_T(T, T + \delta)) - \bar{K})^+$ settled at time $T + \delta$ is equivalent to the payoff $B_T(T + \delta) ((1 + \delta L_T(T, T + \delta)) - \bar{K})^+$ settled at time T will not yield the desired cancelation of discount factors. Since the LIBOR depends on the $\bar{B}^*(T)$ bonds and the default-free $B(T)$ bonds are used for discounting, we have

$$B_T(T + \delta) ((1 + \delta L_T(T, T + \delta)) - \bar{K})^+ = B_T(T + \delta) \left(\frac{1}{\bar{B}_T^*(T + \delta)} - \bar{K} \right)^+,$$

which cannot be simplified further as in the one-curve case.

Let us now calculate the value of the caplet at time $t = 0$ using the Fourier transform method. We have

$$\begin{aligned} P^{Cpl}(0; T, K) &= B_0(T + \delta) \mathbb{E}^{\mathbf{P}^{T+\delta}} \left[\left(\frac{1}{\bar{B}_T^*(T + \delta)} - \bar{K} \right)^+ \right] \\ &= B_0(T + \delta) \mathbb{E}^{\mathbf{P}^{T+\delta}} \left[(e^X - \bar{K})^+ \right], \end{aligned}$$

where X is a random variable given by (see (23))

$$X := \log \frac{\bar{B}_0^*(T)}{\bar{B}_0^*(T + \delta)} + \int_0^T (\bar{A}_s^*(T + \delta) - \bar{A}_s^*(T)) ds + \int_0^T (\bar{\Sigma}_s^*(T + \delta) - \bar{\Sigma}_s^*(T)) dY_s.$$

Let us denote by $M_X^{T+\delta}$ the moment generating function of X under the measure $\mathbf{P}^{T+\delta}$, i.e.

$$M_X^{T+\delta}(z) = \mathbb{E}^{\mathbf{P}^{T+\delta}} [e^{zX}],$$

for $z \in \mathbb{R}$ such that the above expectation is finite. We have

$$\begin{aligned} M_X^{T+\delta}(z) &= \exp \left(- \int_0^T \kappa(-\Sigma_s(T + \delta)) ds \right) \\ &\quad \times \exp \left(z \left(\log \frac{\bar{B}_0^*(T)}{\bar{B}_0^*(T + \delta)} + \int_0^T (\kappa(-\bar{\Sigma}_s^*(T + \delta)) - \kappa(-\bar{\Sigma}_s^*(T))) ds \right) \right) \\ &\quad \times \exp \left(\int_0^T \kappa(z (\bar{\Sigma}_s^*(T + \delta) - \bar{\Sigma}_s^*(T)) - \Sigma_s(T + \delta)) ds \right), \end{aligned} \tag{58}$$

where κ is the cumulant function of Y under the measure \mathbb{P} . The derivation of this formula follows along similar lines as the computations in Section 4.1. In particular, we have used equations (46), (18), (27), and Proposition 3.1 in Eberlein and Kluge (2006b).

Let us impose some conditions on the boundedness of the volatility structures Σ and Σ^* for the sake of the next result. We assume that there exists a positive constant $\tilde{\mathcal{K}} < \frac{\mathcal{K}}{3}$ such that $\Sigma_s(T) \leq \tilde{\mathcal{K}}$ and $\Sigma_s^*(T) \leq \tilde{\mathcal{K}}$ componentwise and for all $s, T \in [0, \bar{T}]$ (note that this is a slightly stronger boundedness condition than the one in Assumption 3.1).

Now, applying Theorem 2.2 and Example 5.1 in Eberlein, Glau, and Papapantoleon (2010) we obtain

Proposition 4.4 *The time-0 price of a caplet with strike K and maturity T is given by*

$$P^{Cpl}(0; T, K) = \frac{B_0(T + \delta)}{2\pi} \int_{\mathbb{R}} \frac{\bar{K}^{1+iv-R} M_X^{T+\delta}(R - iv)}{(iv - R)(1 + iv - R)} dv, \quad (59)$$

for any $R \in (1, \frac{\mathcal{K} - \tilde{\mathcal{K}}}{2\tilde{\mathcal{K}}})$.

Proof. One has to apply Theorem 2.2 in Eberlein, Glau, and Papapantoleon (2010) with the Fourier transform of the caplet payoff function derived in Example 5.1 of the same paper, where other prerequisites for Theorem 2.2 related to the payoff function are also checked. Note that the Fourier transform of the caplet payoff function is well-defined for any $R \in (1, +\infty)$. To ensure that $M_X^{T+\delta}(R - iv)$ is finite, it suffices to take any $R \in (1, \frac{\mathcal{K} - \tilde{\mathcal{K}}}{2\tilde{\mathcal{K}}})$. More precisely, for every $i = 1, \dots, n$,

$$\begin{aligned} |R(\bar{\Sigma}_s^{i,*}(T + \delta) - \bar{\Sigma}_s^{i,*}(T)) - \Sigma_s^i(T + \delta)| &\leq R|\bar{\Sigma}_s^{i,*}(T + \delta) - \bar{\Sigma}_s^{i,*}(T)| + |\Sigma_s^i(T + \delta)| \\ &\leq R2\tilde{\mathcal{K}} + \tilde{\mathcal{K}} \\ &\leq \frac{\mathcal{K} - \tilde{\mathcal{K}}}{2\tilde{\mathcal{K}}} 2\tilde{\mathcal{K}} + \tilde{\mathcal{K}} < \mathcal{K}, \end{aligned}$$

and thus $M_X^{T+\delta}(R) < \infty$ (compare (58) and recall that κ is well-defined for all $z \in [-(1 + \varepsilon)\mathcal{K}, (1 + \varepsilon)\mathcal{K}]^n$). \square

4.5 Swaptions

A swaption is an option to enter an interest rate swap with swap rate S and maturity T_n at a pre-specified date T . Let us consider the swap from Section 4.2. Recall that a swaption can be seen as a sequence of fixed payments $\delta_{j-1}(S(T; T_1, T_n) - S)^+$, $j = 1, \dots, n$, that are received at payment dates T_1, \dots, T_n , where $S(T; T_1, T_n)$ is the swap rate of the underlying swap at time $T \leq T_1$. Hence, the value at time t of the swaption is given by

$$P^{Swn}(t; T, T_n, S) = B_t(T) \sum_{j=1}^n \delta_{j-1} \mathbb{E}^{\mathbf{P}^T} [B_T(T_j) (S(T; T_1, T_n) - S)^+ | \mathcal{E}_t];$$

see Musiela and Rutkowski (2005, Section 13.1.2, p.482). At time $t = 0$ we have

$$\begin{aligned} P^{Swn}(0; T, T_n, S) &= B_0(T) \mathbb{E}^{\mathbf{P}^T} \left[\sum_{j=1}^n \delta_{j-1} B_T(T_j) (S(T; T_1, T_n) - S)^+ \right] \\ &= B_0(T) \mathbb{E}^{\mathbf{P}^T} \left[\left(\sum_{j=1}^n B_T(T_j) v_T^{T_j-1, T_j} - \sum_{j=1}^n B_T(T_j) \bar{S}_{j-1} \right)^+ \right], \end{aligned}$$

which follows by inserting (56). Recall that $\bar{S}_{j-1} = 1 + \delta_{j-1}S$ and $v_T^{T_{j-1}, T_j}$ is given by (49).

To proceed we assume in addition that the conditions of Example 3.5 are satisfied, i.e. the driving process Y is two-dimensional and we assume the Vasicek volatility structures (42). Note that since a swaption is defined on one fixed tenor structure, a two-dimensional process is sufficient as a driving process. Recall that for each j , $B_T(T_j)$ is given by (equation (17))

$$B_T(T_j) = \frac{B_0(T_j)}{B_0(T)} \exp \left(\int_0^T (A_s(T) - A_s(T_j)) ds + \int_0^T (\Sigma_s(T) - \Sigma_s(T_j)) dY_s \right)$$

and $v_T^{T_{j-1}, T_j}$ is given by (equation (49) and (50))

$$\begin{aligned} v_T^{T_{j-1}, T_j} &= c^{T_{j-1}, T_j} \exp \left(\int_0^T (\bar{\Sigma}_s^*(T_j) - \bar{\Sigma}_s^*(T_{j-1})) dY_s \right) \\ &\quad \times \exp \left(\int_T^{T_{j-1}} \kappa_s^{\mathbf{P}^{T_j}} (\bar{\Sigma}_s^*(T_j) - \bar{\Sigma}_s^*(T_{j-1})) ds \right), \end{aligned}$$

where $\kappa_s^{\mathbf{P}^{T_j}}$ is given by (53). Since the volatilities are Vasicek, we have the following separation of variables:

$$\begin{aligned} \Sigma_s(T) - \Sigma_s(T_j) &= \left(\frac{\sigma}{a} e^{as} (e^{-aT_j} - e^{-aT}), 0 \right) \\ \bar{\Sigma}_s^*(T_j) - \bar{\Sigma}_s^*(T_{j-1}) &= \left(\frac{\sigma}{a} e^{as} (e^{-aT_{j-1}} - e^{-aT_j}), \frac{\sigma^*}{a^*} e^{a^*s} (e^{-a^*T_{j-1}} - e^{-a^*T_j}) \right), \end{aligned}$$

which motivates us to introduce the following \mathcal{E}_T -measurable random vector

$$X_T = \left(\int_0^T e^{as} dY_s^1, \int_0^T e^{a^*s} dY_s^2 \right).$$

Consequently, for each j we can rewrite $B_T(T_j)$ and $v_T^{T_{j-1}, T_j}$ as follows

$$B_T(T_j) = c^{j,0} e^{c^{j,1} X_T^1} \quad \text{and} \quad v_T^{T_{j-1}, T_j} = \bar{c}^{j,0} e^{\bar{c}^j X_T},$$

where

$$\begin{aligned} c^{j,0} &= \frac{B_0(T_j)}{B_0(T)} \exp \left(\int_0^T (A_s(T) - A_s(T_j)) ds \right), \\ c^{j,1} &= \frac{\sigma}{a} (e^{-aT_j} - e^{-aT}), \\ \bar{c}^{j,0} &= c^{T_{j-1}, T_j} \exp \left(\int_T^{T_{j-1}} \kappa_s^{\mathbf{P}^{T_j}} (\bar{\Sigma}_s^*(T_j) - \bar{\Sigma}_s^*(T_{j-1})) ds \right) \\ \bar{c}^j &= \left(\frac{\sigma}{a} (e^{-aT_{j-1}} - e^{-aT_j}), \frac{\sigma^*}{a^*} (e^{-a^*T_{j-1}} - e^{-a^*T_j}) \right) \end{aligned}$$

are deterministic constants. Hence, the value at time $t = 0$ of the swaption depends only on the distribution of the random vector X_T under the measure \mathbb{P}^T :

$$\begin{aligned} P^{Sw}n(0; T, T_n, S) &= B_0(T) \mathbb{E}^{\mathbf{P}^T} \left[\left(\sum_{j=1}^n c^{j,0} e^{c^{j,1} X_T^1} \bar{c}^{j,0} e^{\bar{c}^j X_T} - \sum_{j=1}^n \bar{S}_{j-1} c^{j,0} e^{c^{j,1} X_T^1} \right)^+ \right] \\ &= B_0(T) \mathbb{E}^{\mathbf{P}^T} \left[\left(\sum_{j=1}^n a^{j,0} e^{a^{j,1} X_T^1 + a^{j,2} X_T^2} - \sum_{j=1}^n b^{j,0} e^{b^{j,1} X_T^1} \right)^+ \right], \quad (60) \end{aligned}$$

where $a^{j,0} = c^{j,0}\bar{c}^{j,0}$, $a^{j,1} = c^{j,1} + \bar{c}^{j,1}$, $a^{j,2} = \bar{c}^{j,2}$, $b^{j,0} = \bar{S}_{j-1}c^{j,0}$ and $b^{j,1} = c^{j,1}$. To calculate this expectation we shall use the moment generating function $M_{X_T}^T$ of X_T under the measure \mathbb{P}^T , which is given explicitly in terms of the characteristics of Y by

$$\begin{aligned} M_{X_T}^T(z) &= \mathbb{E}^{\mathbb{P}^T} \left[e^{z_1 X_T^1 + z_2 X_T^2} \right] \\ &= \mathbb{E}^{\mathbb{P}^T} \left[e^{\int_0^T z_1 e^{as} dY_s^1 + \int_0^T z_2 e^{a^*s} dY_s^2} \right] \\ &= \exp \left(- \int_0^T \kappa \left(\left(\frac{\sigma}{a} (1 - e^{-a(T-s)}), 0 \right) \right) ds \right) \\ &\quad \times \exp \left(\int_0^T \kappa \left(z_1 e^{as} - \frac{\sigma}{a} (1 - e^{-a(T-s)}), z_2 e^{a^*s} \right) ds \right), \end{aligned} \quad (61)$$

for any $z \in \mathbb{R}^2$ such that the expectation above is finite. This follows along the same lines as in (51), for a deterministic function $U(s) = (z_1 e^{as}, z_2 e^{a^*s})$, the forward measure \mathbb{P}_T and inserting the Vasicek volatility specifications.

Next, to compute the expectation in (60), one has to use a two-dimensional version of the Jamshidian trick and apply the Fourier transform method, similarly to Section 4.4. More precisely, let us introduce deterministic functions $\tilde{f}, f : \mathbb{R}_+^2 \rightarrow \mathbb{R}$

$$\begin{aligned} \tilde{f}(x_1, x_2) &= \sum_{j=1}^n a^{j,0} e^{a^{j,1}x_1 + a^{j,2}x_2} - \sum_{j=1}^n b^{j,0} e^{b^{j,1}x_1} \\ f(x_1, x_2) &= \tilde{f}(x_1, x_2)^+. \end{aligned}$$

Then

$$P^{Sw_n}(0; T, T_n, S) = B_0(T) \mathbb{E}^{\mathbb{P}^T} [f(X_T^1, X_T^2)],$$

and making use of Theorem 3.2 in Eberlein, Glau, and Papapantoleon (2010), one obtains

Proposition 4.5 *The time-0 price of a swaption with swap rate S and maturity T_n is given by the following semi-closed formula:*

$$P^{Sw_n}(0; T, T_n, S) = \frac{B_0(T)}{(2\pi)^2} \int_{\mathbb{R}^2} M_{X_T}^T(R + iu) \hat{f}(iR - u) du, \quad (62)$$

where $R \in \mathbb{R}^2$ is such that $M_{X_T}^T(R)$ given in (61) exists and the function $g(x) := e^{-Rx} f(x)$ satisfies prerequisites of Theorem 3.2 in Eberlein, Glau, and Papapantoleon (2010) (R is the so-called dampening coefficient).

A closed analytic expression for the Fourier transform \hat{f} is not available in this case. However, it can be computed numerically in a quite efficient way. Observe thus that for each fixed $x_1 \in \mathbb{R}_+$, the function $x_2 \mapsto \tilde{f}(x_1, x_2)$ is continuous, strictly increasing in x_2 and $\lim_{x_2 \rightarrow \infty} \tilde{f}(x_1, x_2) = +\infty$, since $a^{j,0}, a^{j,2} > 0$. Hence, let us define

$$q(x_1) = \inf\{x_2 \in \mathbb{R}_+ : \tilde{f}(x_1, x_2) \geq 0\}.$$

Note that if $\tilde{f}(x_1, \cdot)$ has a zero, this zero is unique since the function is strictly increasing (for every fixed x_1). Consequently,

$$f(x_1, x_2) = \tilde{f}(x_1, x_2)^+ = \tilde{f}(x_1, x_2) \mathbf{1}_{\{x_2 \geq q(x_1)\}}.$$

The Fourier transform of f is therefore given, for $z \in \mathbb{C}$, by

$$\begin{aligned}\widehat{f}(z) &= \int_{\mathbb{R}^2} e^{izx} f(x) dx \\ &= \int_{\mathbb{R}^2} e^{izx} \widetilde{f}(x_1, x_2) \mathbf{1}_{\{x_2 \geq q(x_1)\}} dx \\ &= \int_0^\infty \int_{q(x_1)}^\infty e^{izx} \left(\sum_{j=1}^n a^{j,0} e^{a^{j,1}x_1 + a^{j,2}x_2} - \sum_{j=1}^n b^{j,0} e^{b^{j,1}x_1} \right) dx_2 dx_1.\end{aligned}\quad (63)$$

Since $q(x_1)$ is obtained by numerically solving $\widetilde{f}(x_1, x_2) = 0$, we shall not obtain a closed formula for \widehat{f} . However, based on (63), \widehat{f} can be efficiently valued numerically. We refer the reader to a follow-up numerical paper for every detail and numerical illustration, as well as for discussion of conditions of the prerequisites of Theorem 3.2 in Eberlein, Glau, and Papapantoleon (2010) regarding function g in Proposition 4.5.

5 Numerical example

Implementation and numerical issues will be dealt with in detail in a follow-up paper. However, to give a flavor of the practical behavior of the model, we consider in this section a toy example, which illustrates the ability of the model to produce a wide range of FRA and basis swap spreads. We work with a one-dimensional driving process Y , which is an IG process with parameter b (see Example 3.6). Let us consider a one-dimensional Vasicek volatility structure given by

$$\Sigma_t(T) = \frac{\sigma}{a} \left(1 - e^{-a(T-t)} \right) \quad \text{and} \quad \Sigma_t^*(T) = \frac{\sigma^*}{a^*} \left(1 - e^{-a^*(T-t)} \right).$$

The initial bond term structure is assumed to be given by

$$B_0(T) = e^{-\bar{r}T} \quad \text{and} \quad B_0^*(T) = e^{-(\bar{r} + \bar{\lambda})T},$$

where $\bar{r} > 0$ and $\bar{\lambda} > 0$ are some given constants.

5.1 FRAs

First let us consider an FRA and calculate the spread (55) (“FRA spread” henceforth) between the forward rate in our model and a classical, one-curve forward rate. Figure 1 in Morini (2009) shows 2007-09 market data for FRA spreads surging at more than 170bps at the peak of the 2007-09 crisis. As for spot rates at that time, Filipović and Trolle (2011) report that the spread between the 3m-LIBOR and the 3m-OIS rate attained “366 basis points on Oct 10, 2008” (see also their Figure 1).

Tables 1 to 3 display the FRA spread in our model with $\widehat{T} = 2$, $\delta = 0.5$, $N = 1$, for three parameter-sets for $(\bar{r}, \bar{\lambda})$ and for b going over a range of values from 7 to 1000 in each case. The risk-free (one-curve classical setup) FRA rates for $\bar{r} = 2\%$ (Tables 1 and 2) and $\bar{r} = 4\%$ (Table 3) respectively amount to 2.01% and 4.04%. The results in Table 1 finely cover the ranges of FRA rates observed in the crisis. Tables 2 and 3 show that even wider ranges of values are obtainable by playing with the model parameters, illustrating further the ability of the model in fitting various market regimes.

$b/\bar{\lambda}$	10	50	300
7	126.84	167.54	423.74
8	89.60	130.22	385.96
10	51.63	92.17	347.43
12	34.40	74.91	329.95
15	22.66	63.15	318.04
20	15.44	55.91	310.71
30	11.69	52.16	306.91
50	10.45	50.91	305.65
1000	10.10	50.57	305.30

Table 1: bp-FRA spreads for $\bar{r} = 2\%$, $a = 0.025$, $a^* = 0.02$, $\sigma = \sigma^* = 0.5$ ($\bar{\lambda}$ in bps).

$b/\bar{\lambda}$	10	50	300
7	427.41	468.71	728.70
8	297.70	338.74	597.09
10	162.64	203.41	460.07
12	100.18	140.82	396.69
15	57.02	97.58	352.90
20	30.17	70.67	325.66
30	16.11	56.58	311.39
50	11.41	51.87	306.62
1000	10.10	50.57	305.30

Table 2: bp-FRA spreads for $\bar{r} = 2\%$, $a = 0.05$, $a^* = 0.04$, $\sigma = \sigma^* = 1$ ($\bar{\lambda}$ in bps).

$b/\bar{\lambda}$	10	50	300
7	431.70	473.42	736.02
8	300.69	342.14	603.10
10	164.28	205.45	464.69
12	101.19	142.24	400.68
15	57.59	98.56	356.45
20	30.47	71.38	328.93
30	16.27	57.15	314.52
50	11.52	52.39	309.70
1000	10.20	51.07	308.37

Table 3: bp-FRA spreads for $\bar{r} = 4\%$, $a = 0.05$, $a^* = 0.04$, $\sigma = \sigma^* = 1$ ($\bar{\lambda}$ in bps).

5.2 Basis swap spreads

Now let us consider a 6m-LIBOR versus 12m-LIBOR basis swap with maturity $\hat{T} = 10$ years. One can read on page 8 of Morini (2009) (see also Figure 3 therein): “From August 2008 to April 2009, the basis swap spread to exchange 6m-LIBOR with 12m-LIBOR over 1 year was strongly positive and averaged 40bps.” We calculate the model spread $P^{BSw}(0; 10, 1)$ (“basis swap spread” henceforth) using formula (57), for the same sets of model parameters as in the previous subsection. The results are displayed in Tables 4 to 6. The results in Table 4 largely cover the ranges of spreads observed in the 2007-09 crisis. Again one can see in Tables 5 and 6 that even wider ranges of values can be obtained by playing further with model parameters.

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$b/\bar{\lambda}$	10	50	300
7	67.44	69.58	84.85
8	44.00	45.54	56.96
10	21.91	22.83	30.32
12	12.53	13.17	18.88
15	6.38	6.83	11.31
20	2.71	3.03	6.77
30	0.83	1.10	4.44
50	0.22	0.46	3.67
1000	0.05	0.28	3.46

Table 4: bp-6m/12m-basis swap spreads for $\bar{r} = 2\%$, $a = 0.025$, $a^* = 0.02$, $\sigma = \sigma^* = 0.5$. ($\bar{\lambda}$ in bps).

$b/\bar{\lambda}$	10	50	300
7	289.19	296.12	341.90
8	179.83	184.64	216.96
10	84.13	86.78	105.34
12	46.45	48.11	60.35
15	22.98	23.96	31.86
20	9.50	10.06	15.26
30	2.82	3.15	6.93
50	0.64	0.90	4.21
1000	0.05	0.29	3.46

Table 5: bp-6m/12m-basis swap spreads $\bar{r} = 2\%$, $a = 0.05$, $a^* = 0.04$, $\sigma = \sigma^* = 1$ ($\bar{\lambda}$ in bps).

$b/\bar{\lambda}$	10	50	300
7	285.99	292.68	336.81
8	179.02	183.72	215.15
10	84.54	87.20	105.65
12	46.95	48.67	61.17
15	23.35	24.43	32.84
20	9.71	10.39	16.25
30	2.91	3.38	7.90
50	0.69	1.10	5.16
1000	0.08	0.47	4.41

Table 6: bp-6m/12m-basis swap spreads $\bar{r} = 2\%$, $a = 0.05$, $a^* = 0.04$, $\sigma = \sigma^* = 1$ ($\bar{\lambda}$ in bps).

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