Bilateral Counterparty Risk under Funding Constraints – Part II: CVA.

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Abstract

The correction in value of an OTC derivative contract due to counterparty risk under funding constraints, is represented as the value of a dividend-paying option on the value of the contract clean of counterparty risk and excess funding costs. This representation allows one to analyze the structure of this correction, the so-called Credit Valuation Adjustment (CVA for short), in terms of replacement cost/benefits, credit cost/benefits and funding cost/benefits. We develop a reduced-form backward stochastic differential equations (BSDE) approach to the problem of pricing and hedging the CVA. In the Markov setup, explicit CVA pricing and hedging schemes are formulated in terms of semilinear PDEs.

Keywords: Counterparty Risk, Funding Costs, Credit Valuation Adjustment (CVA), Backward Stochastic Differential Equation (BSDE).

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1 Introduction

We pursue the study of valuation and hedging of bilateral counterparty risk on OTC derivatives under funding constraints initiated in the companion paper Crépey (2012), to which we refer the reader for all the background. We consider a netted portfolio of OTC derivatives between two defaultable counterparties, generically referred to in our papers as the "contract between the bank and the investor". Within the bank, every particular business trading desk has only a precise view on its own activity, lacking the global view, and specifically the aggregated data, needed to properly value the CSA cash-flows related to the defaultability of either party (margin calls and CSA close-out cash-flow on the netted portfolio), as well as to fairly assess the related funding costs. Therefore in major investment banks today the trend is to have a central CVA desk in charge of collecting the global information and of valuing and hedging counterparty risk, also accounting for any excess funding costs involved. Here CVA stands for Credit Value Adjustment. The value-and-hedge of a contract is then obtained as the difference between the "clean" value-and-hedge provided by the trading desk (clean of counterparty risk and excess funding costs), and a value-and-hedge adjustment computed by the CVA desk.

This allocation of tasks between the various business trading desks of an investment bank, and a central CVA desk, motivates the present mathematical CVA approach to the problem of valuing and hedging counterparty risk. Moreover this is done in a multiple-curve setup accounting for the various funding costs involved, allowing one to investigate the question of interaction between (bilateral in particular) counterparty risk and funding.

In the previous paper we identified a non standard backward stochastic differential equation (BSDE) which was key in the pricing of a counterparty risky contract (or portfolio of contracts) under funding constraints. Interestingly enough, the notion of CVA, which emerged for practical reasons in banks, will also be useful mathematically. In a sense this paper tells the story of the reduction of a non standard price BSDE, to an ultimately quite classical pre-default CVA BSDE.

1.1 Outline of the Paper

Since the pioneering works of Brigo and Pallavicini (2008) for unilateral counterparty risk and Brigo and Capponi (2010) in a context of bilateral counterparty risk, it is well understood that the CVA can be viewed as an option, the so-called Contingent Credit Default Swap (CCDS), on the clean value of the contract. Section 2 extends to a nonlinear multiple-curve setup the representation of the CVA as the price of a CCDS. Our CVA accounts not only for counterparty risk, but also for funding costs. The CCDS is then a dividend-paying option, where the dividends correspond to these costs. We then develop in Section 3 a practical reduced-form CVA BSDE approach, to the problem of pricing and hedging bilateral counterparty risk under funding constraints. Counterparty risk and funding corrections to the clean price-and-hedge of the contract are represented as the solution of a pre-default CVA BSDE stated with respect to a reference filtration, in which defaultability of the two parties only shows up through their default intensities. In the Markovian setup of Section 4, explicit CVA pricing and hedging schemes are formulated in terms of semilinear pre-default CVA PDEs.

The paper sheds some light onto the structure of the CVA (see Examples 2.1 and 3.1) and on the debate about unilateral versus bilateral counterparty risk (see Subsection 4.4). Our main results are Proposition 4.2 and Corollary 4.2, which yield concrete recipes for risk-managing the contract as a whole or its CVA component, according to the following objective of the bank: minimizing the (risk-neutral) variance of the cost process (which is essentially the hedging error) of the contract or of its CVA component, whilst achieving a perfect hedge of the jump-to-default exposure.

A take-away message is that the counterparty risk two stages valuation and hedging methodology (counterparty risky price obtained as clean price minus CVA) which is currently emerging for practical reasons in banks, is also useful in the mathematical analysis of the problem. This makes the CVA not only a very important and legitimate financial object, but also a valuable mathematical tool.

1.2 Setup

This Subsection is a brief recap of the companion paper Crépey (2012). We refer the reader to Sections 2 and 3 of this previous paper[1] for the detailed notation, standing assumptions and all the financial interpretation. All cash-flows that appear in the paper are assumed to be integrable under the prevailing pricing measure. "Martingale" should be understood everywhere as local martingale, a genuine martingale being called "true martingale".

We consider a generic contract of time horizon T with promised dividends dD_t from the bank to the investor. The two parties are defaultable, with respective default times denoted by θ and $\overline{\theta}$. This results in an effective dividend stream $dC_t = J_t dD_t$, where $J_t = \mathbb{1}_{t<\tau}$ with $\tau = \theta \wedge \overline{\theta}$. Moreover if $\tau < T$ there is a CSA close-out cash-flow R^i at time τ from the bank to the investor. One denotes by $\bar{\tau} = \tau \wedge T$ the effective time horizon of our problem (there are no cash-flows after $\bar{\tau}$). The case of unilateral counterparty risk (from the perspective of the bank, as everything in the paper) can be recovered by letting $\theta = \infty$.

After having sold the contract to the investor at time 0, the bank sets-up a collateralization, funding and hedging portfolio ("hedging portfolio" for short). Let $\mathcal{M}$ denote the $\mathbb{R}^d$-valued gain process of a buy-and-hold position into the hedging assets traded in swapped form (that is, when every hedging instrument is either a genuine swap, or exchanged on a repo market). A standing stochastic basis $(\Omega, \mathcal{G}_T, \mathcal{G}, \mathbb{P})$, where $\mathcal{G} = (\mathcal{G}_t)_{t\in[0,T]}$, is interpreted as a risk-neutral pricing model on the primary market, in the sense that

Assumption 1.1 The primary risky gain process $\mathcal{M}$ is a $(\mathcal{G}, \mathbb{P})$-martingale.

Accounting for funding costs, the self-financing condition imposes the following dynamics for the wealth of the hedging portfolio, for $t \in [0, \bar{\tau}]$:

$$d\mathcal{W}_t = -dC_t + (r_t + g_t(\mathcal{W}_t, \zeta_t)dt + \zeta_t d\mathcal{M}_t.$$

Here r_t is the risk-free interest rate (in the abstract economic sense of time-value of money, without necessarily a related funding asset); a "hedge" ζ denotes a left-continuous locally bounded $\mathbb{R}^d$-valued row-vector process representing the number of units of the hedging assets which are held in the bank's hedging portfolio; and a random function $g_t(\pi, \varsigma)$ represents a funding benefit coefficient of the bank in excess over the risk-free rate. The funding of the bank's position is ensured by an external risk-free funder. The (algebraic) debt of the bank to this funder at time t is given as $\mathfrak{X}^+_{t-}(\mathcal{W}_{t-}, \zeta_{t-})$, for an external debt random function $\mathfrak{X}^+_t(\pi, \varsigma)$. In case the bank defaults at time $\tau = \theta < T$, this results in an additional, external

[1] Subsections 2.2 and 3.1 therein actually refer to a particular example of funding specification which can be ignored to begin with.

funding close-out cash-flow, from the funder to the bank (so this is a funding benefit of the bank at her own default), worth

$$(1-\mathfrak{r})\mathfrak{X}^+_{\tau-}(\mathcal{W}_{\tau-},\zeta_{\tau-})$$

in which a $\mathcal{G}_\theta$-measurable random variable $\mathfrak{r}$ represents the recovery rate of the bank towards its external funder. In case $\mathfrak{r}<1$ the bank defaults at time θ not only on its commitments in the contract with regard to the investor, but also on its related funding debt. The case $\mathfrak{r}=1$ corresponds to a partial default in which at time θ the bank only defaults on its contractual commitments with regard to the investor, but not on its funding debt with respect to its funder. It can be used for rendering the situation of a bank in a global net lender position, so that it actually does not need any external lender. In case cash is needed for funding its position, the bank simply uses its own cash. By convention we also let $\mathfrak{r}=1$ in case of unilateral counterparty risk of the bank where $\theta=\infty$.

Given a $\mathcal{G}$-semimartingale Π and a hedge ζ, we denote $R=R^i-\mathbb{1}_{\tau=\theta}R^f$, in which

$$R^f:=(1-\mathfrak{r})\mathfrak{X}^+_{\tau-}(\Pi_{\tau-},\zeta_{\tau-}).$$

Note R implicitly depends on $(\Pi_{\tau-},\zeta_{\tau-})$ in this notation.

Definition 1.1 Let a pair (Π,ζ) made of a $\mathcal{G}$-semimartingale Π and a hedge ζ, satisfy the following BSDE on $[0,\bar{\tau}]$:

$$\begin{aligned}&\Pi_{\bar{\tau}}=\mathbb{1}_{\tau<T}R \text{ and for } t\in[0,\bar{\tau}]:\\ &d\Pi_t+dC_t-\big(r_t\Pi_t+g_t(\Pi_t,\zeta_t)\big)dt=d\nu_t\end{aligned} \tag{1.1}$$

for some $\mathcal{G}$-martingale ν null at time 0. Process (Π,ζ) is then said to be a price-and-hedge of the contract. The related cost process is the $\mathcal{G}$-martingale ε defined by $\varepsilon_0=0$ and for $t\in[0,\bar{\tau}]$

$$d\varepsilon_t=d\nu_t-\zeta_t d\mathcal{M}_t. \tag{1.2}$$

As shown in the previous paper, the cost process ε essentially corresponds to the hedging error of a price-and-hedge (Π,ζ). Equivalently to the BSDE (1.1) in differential form, one can write in integral form, letting also $\beta_t=e^{-\int_0^t r_s ds}$ and denoting $\mathbb{E}_t=\mathbb{E}\big(\cdot\,\big|\,\mathcal{G}_t\big)$

$$\beta_t\Pi_t=\mathbb{E}_t\Big(\int_t^{\bar{\tau}}\beta_s dC_s-\int_t^{\bar{\tau}}\beta_s g_s(\Pi_s,\zeta_s)ds+\beta_{\bar{\tau}}\mathbb{1}_{\tau<T}R\Big). \tag{1.3}$$

The BSDE (1.1) is made non-standard by the random terminal time $\bar{\tau}$, the dependence of the terminal condition R in $(\Pi_{\tau-},\zeta_{\tau-})$, the dividend term dC_t, and the fact that it is not driven by an explicit set of fundamental martingales like Brownian motions and/or compensated Poisson measures.

From the BSDE point of view, a particularly simple situation will be the one where

$$\mathfrak{X}^+_t(\pi,\varsigma)=\mathfrak{X}^+_t(\pi)\,,\;\; g_t(\pi,\varsigma)=g_t(\pi). \tag{1.4}$$

We call it the fully swapped (as opposed to externally funded) hedge case in reference to its financial interpretation seen in the previous paper. This is the most common case in practice, see however (Burgard and Kjaer 2011a; Burgard and Kjaer 2011b) and Section 5 of the previous paper for a case of an externally funded hedge.

2 CVA

Since the pioneering works of Damiano Brigo and his coauthors, it is well understood that the CVA can be viewed as an option, the so-called Contingent Credit Default Swap (CCDS), on the clean value of the contract. This Section extends to a nonlinear multiple-curve setup the notion of CVA and its representation as the price of a CCDS. In our setup the CVA actually accounts not only for counterparty risk, but also for excess funding costs. The CCDS is then a dividend-paying option, where the dividends correspond to these costs.

2.1 Bilateral Reduced Form Setup

We assume henceforth that the model filtration $\mathcal{G}$ can be decomposed into $\mathcal{G} = \mathcal{F} \vee \mathcal{H}^{\theta} \vee \mathcal{H}^{\overline{\theta}}$, where $\mathcal{F}$ is some reference filtration and $\mathcal{H}^{\theta}$ and $\mathcal{H}^{\overline{\theta}}$ stand for the natural filtrations of θ and $\overline{\theta}$. Let also $\bar{\mathcal{G}} = \mathcal{F} \vee \mathcal{H}$, where $\mathcal{H}$ is the natural filtration of $\bar{\tau}$ (or equivalently, of τ). We refer the reader to Bielecki and Rutkowski (2002) for the standard material regarding the reduced-form approach in credit risk modeling. The Azéma supermartingale associated with τ is the process G defined by, for $t \in [0, T]$,

$$G_t = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t). \tag{2.1}$$

We assume that G is a positive, continuous and non-increasing process. This is a classical, slight relaxation of the so-called immersion or ($\mathcal{H}$)-hypothesis of $\mathcal{F}$ into $\bar{\mathcal{G}}$. In particular,

Lemma 2.1 **(i)** *An $\mathcal{F}$-martingale stopped at τ is a $\bar{\mathcal{G}}$-martingale, and a $\bar{\mathcal{G}}$-martingale stopped at τ is a $\mathcal{G}$-martingale.*
(ii) *An $\mathcal{F}$-adapted càdlàg process cannot jump at τ. One thus has that $\Delta X_\tau = 0$ almost surely, for every $\mathcal{F}$-adapted càdlàg process X.*

Proof. **(i)** Since τ has a positive, continuous and non-increasing Azéma supermartingale, it is known from Elliot et al. (2000) that an $\mathcal{F}$-martingale stopped at τ, is a $\bar{\mathcal{G}}$-martingale. Besides, two successive applications of the Dellacherie-Meyer Key Lemma (see for instance Bielecki and Rutkowski (2002)) yield that for every $\bar{\mathcal{G}}$-adapted integrable process M, one has for every $0 \le s \le t \le T$

$$\begin{aligned}\mathbb{E}\left(M_{t\wedge\tau} \,|\, \mathcal{G}_s\right) &= \mathbb{1}_{s\ge\tau} M_\tau \,+\, \mathbb{1}_{s<\tau} \frac{\mathbb{E}\left(M_{t\wedge\tau}\mathbb{1}_{s<\tau} \,|\, \mathcal{F}_s\right)}{\mathbb{P}\left(s<\tau \,|\, \mathcal{F}_s\right)} \\ &= \mathbb{1}_{s\ge\tau} M_{s\wedge\tau} \,+\, \mathbb{1}_{s<\tau}\mathbb{E}\left(M_{t\wedge\tau} \,|\, \bar{\mathcal{G}}_s\right),\end{aligned}$$

which, in case M is a $\bar{\mathcal{G}}$-true martingale, boils down to $M_{s\wedge\tau}$. A $\bar{\mathcal{G}}$-true martingale stopped at τ is thus a $\mathcal{G}$-true martingale. A standard localization argument then yields that a $\bar{\mathcal{G}}$-(local) martingale stopped at τ is a $\mathcal{G}$-(local) martingale.
(ii) As G is continuous, τ avoids $\mathcal{F}$-stopping times in the sense that $\mathbb{P}(\tau = \sigma) = 0$ for any $\mathcal{F}$-stopping time σ (see for instance Coculescu and Nikeghbali (2012)). The results then follows from the fact that by Theorem 4.1 page 120 in He et al. (1992), there exists a sequence of $\mathcal{F}$-stopping times exhausting the jump times of an $\mathcal{F}$-adapted càdlàg process. □

2.2 Clean Price

In the sequel, the risk-free short rate process r, or equivalently the risk-free discount factor process $\beta = e^{-\int_0^\cdot r_t dt}$, and the clean dividend process D, are assumed to be $\mathcal{F}$-adapted. In order to define the CVA process Θ, one first needs to introduce the clean price process P of the contract. This is a fictitious, instrumental value process, which would correspond to the price of the contract without counterparty risk nor excess funding costs. In the present reduced-form setup, the clean price process P of the contract is naturally defined as, for $t \in [0, T]$,

$$\beta_t P_t = \mathbb{E}\left(\int_t^T \beta_s dD_s \,\Big|\, \mathcal{F}_t \right). \tag{2.2}$$

The discounted cumulative clean price,

$$\beta P + \int_{[0,\cdot]} \beta_t dD_t, \tag{2.3}$$

is thus an $\mathcal{F}$-martingale. The corresponding clean $\mathcal{F}$-martingale M on $[0, T]$, to be compared with the $\mathcal{G}$-martingale component ν of Π in the price BSDE (1.1), is defined by, for $t \in [0, T]$,

$$dM_t = dP_t + dD_t - r_t P_t dt, \tag{2.4}$$

along with the terminal condition $P_T = 0$.

Lemma 2.2 **(i)** *The clean price process P satisfies for $t \in [0, \bar{\tau}]$*

$$\beta_t P_t = \mathbb{E}_t\big[\int_t^{\bar{\tau}} \beta_s dD_s + \beta_{\bar{\tau}} P_{\bar{\tau}} \big]. \tag{2.5}$$

(ii) *There can be no promised dividend of the contract nor jump of the clean price process at the default time τ, so $\Delta D_\tau = \Delta P_\tau = 0$ almost surely.*

Proof. **(i)** Since the discounted cumulative clean price (2.3) is an $\mathcal{F}$-martingale, by Lemma 2.1(i), this process stopped at τ is a $\mathcal{G}$-martingale, integrable by standing assumption in all the paper, thus (2.5) follows.
(ii) Since all our semimartingales are taken in a càdlàg version, then by Lemma 2.1(ii) the $\mathcal{F}$-semimartingales D and P cannot jump at τ. □

Remark 2.1 (Immersion) A reduced-form approach draws its computational power from, essentially, an immersion hypothesis between the reference filtration "ignoring" the default times of the two parties, and the filtration progressively enlarged by the latter. This immersion hypothesis implies a kind of weak or indirect dependence between the reference contract and the default times of the two parties (see Jeanblanc and Le Cam (2008), Morini and Brigo (2011) or Jamshidian (2002)). In other words, in the language of counterparty risk, the immersion hypothesis (at least in the basic form of this paper, see Remark 3.5) precludes major right/wrong-way-risk effects such as the ones that are observed for instance with counterparty risk on credit derivatives. A contrario, in the case without strong dependence between the contract and the default of the parties, this "advantage" should be "pushed" in the model, and this is precisely the object of a reduced-form approach (see also Remark 2.4 for results which hold true in a general case without immersion).

Moreover, with credit derivatives, a reduced-form approach to counterparty risk, besides losing in relevance from the point of view of financial modeling, also loses from its computational appeal. With credit derivatives the discontinuous and high-dimensional nature of the problem is such that the gain in tractability resulting from the above reduction of filtration, is not so tangible. As a consequence, a reduced-form approach, at least in the basic form of this paper, is inappropriate to deal with counterparty risk on credit derivatives. We refer the reader to Brigo and Chourdakis (2008), Brigo and Capponi (2010), Lipton and Sepp (2009), Bielecki and Crépey (2011), or Blanchet-Scalliet and Patras (2008) regarding possible approaches to appropriately deal with CVA on credit derivatives (or strong wrong-way risk more generally). Immersion may also be a concern in some cases with FX derivatives since it has been shown empirically that there can be some rather strong dependence between the default risk of an obligor and an exchange rate (see Remark 3.1 about related aspects regarding the collateral).

Of course ideally counterparty risk should not be considered at the level of a specific class of assets, but at the level of all the contracts between two counterparties under a given CSA. The construction of a global model and methodology for valuing and hedging a CSA hybrid book of derivatives, including credit derivatives, will be dealt with in future research.

On a related line of thought note that continuous collateralization is a very efficient way of mitigating counterparty risk in a reduced-form model (see Example 3.1), but it may not be enough in a model[2] without immersion (see Bielecki and Crépey (2011)).

2.3 CSA Close-Out Cash-Flow

Before moving to CVA we now need to specify R^i in the CSA close-out cash-flow $\mathbb{1}_{\tau<T}R^i$. Toward this end we define a $\mathcal{G}_\tau$-measurable random variable χ as

$$\chi = Q_\tau - \Gamma_\tau \tag{2.6}$$

where Q denotes the so-called CSA close-out valuation process of the contract, expectation of future cash-flows or so, in a sense defined by the CSA (see Example 3.1 for possible specifications). From the point of view of financial interpretation, χ represents the (algebraic) debt of the bank to the investor at time τ, given as the CSA close-out price Q_τ less the margin amount Γ_τ (since the latter is 'instantaneously transferred' to the investor at time τ). We then set

$$R^i = \Gamma_\tau + \mathbb{1}_{\tau=\theta}\big(\rho\chi^+ - \chi^-\big) - \mathbb{1}_{\tau=\overline{\theta}}\big(\overline{\rho}\chi^- - \chi^+\big) - \mathbb{1}_{\theta=\overline{\theta}}\chi \tag{2.7}$$

in which the $[0,1]$-valued $\mathcal{G}_\theta$- and $\mathcal{G}_{\overline{\theta}}$-measurable random variables ρ and $\overline{\rho}$ denote the recovery rates of the bank and the investor to each other. So:

- If the investor defaults at time $\overline{\theta} < \theta \wedge T$, then $R^i = \Gamma_\tau - \big(\overline{\rho}\chi^- - \chi^+\big)$,
- If the bank defaults at time $\theta < \overline{\theta} \wedge T$, then $R^i = \Gamma_\tau + \rho\chi^+ - \chi^-$,
- If the bank and the investor default simultaneously at time $\theta = \overline{\theta} < T$, then $R^i = \Gamma_\tau + \rho\chi^+ - \overline{\rho}\chi^-$.

Note that the margin amount Γ typically depends on Q, often in a rather path dependent way. We refer the reader to Brigo et al. (2011) or Bielecki and Crépey (2011) regarding this and other, theoretically minor, yet practically important issues, like haircut, re-hypothecation risk and segregation, or the cure period. All these can also be accommodated in our setup.

[2]Or a market, as recurrently observed with credit derivatives.

Remark 2.2 The practical behavior of the counterparty risk is of course quite sensitive to the choice of a precise close-out formulation. We refer the reader to Example 3.1 for various concrete specifications of Q and Γ. See also Brigo and Morini (2010) for an analysis of the consequences of different possible choices of Q. Our abstract formulation (2.7) for the CSA close-out cash-flow R^i offers a good paradigm for the theoretical study of this paper, but it is not the only possible one, is still a stylized payoff, and does not cover the totality of cases encountered in practice. Capponi (2011) thus considers a more general payoff with different level of recoveries applying to different components of the exposure χ, in order to account for the fact that the collateral which is lent can in certain cases be recovered at a better rate than the rest of the exposure (for instance in the case of a collateral segregated by a third-party, on this see also Bielecki and Crépey (2011)).

2.4 CVA Representation

With R^i thus specified in $R = R^i - \mathbb{1}_{\tau=\theta}R^f$, we are now ready to introduce the CVA process Θ of the bank. Recall from Definition 1.1 that unless $\mathfrak{r} = 1$, the terminal condition $R = R^i - \mathbb{1}_{\tau=\theta}R^f$ in a solution (Π, ζ) to the price BSDE (1.1), implicitly depends on $(\Pi_{\tau-}, \zeta_{\tau-})$, via $R^f = (1-\mathfrak{r})\widehat{\mathfrak{X}}^+_{\theta-}$, where $\widehat{\mathfrak{X}}_t$ is used as a shorthand for $\mathfrak{X}_t(\Pi_t, \zeta_t)$. Also note that

$$\begin{aligned} &P_\tau - R \\ &\quad = P_\tau - Q_\tau + \chi - \mathbb{1}_{\tau=\theta}\big(\rho\chi^+ - \chi^-\big) + \mathbb{1}_{\tau=\bar{\theta}}\big(\bar{\rho}\chi^- - \chi^+\big) + \mathbb{1}_{\theta=\bar{\theta}}\chi + \mathbb{1}_{\tau=\theta}(1-\mathfrak{r})\widehat{\mathfrak{X}}^+_{\theta-} \\ &\quad = P_\tau - Q_\tau + \mathbb{1}_{\tau=\theta}\Big((1-\rho)\chi^+ + (1-\mathfrak{r})\widehat{\mathfrak{X}}^+_{\tau-}\Big) - \mathbb{1}_{\tau=\bar{\theta}}(1-\bar{\rho})\chi^-. \end{aligned} \tag{2.8}$$

One can then state the following

Definition 2.1 *Given a solution* (Π, ζ) *to the price BSDE* (1.1)*, the corresponding CVA process* Θ *is defined by* $\Theta = P - \Pi$ *on* $[0, \bar{\tau}]$. *In particular,* $\Theta_{\bar{\tau}} = \mathbb{1}_{\tau<T}\xi$*, where*

$$\begin{aligned} \xi &:= P_\tau - R \\ &= P_\tau - Q_\tau + \mathbb{1}_{\tau=\theta}\Big((1-\rho)\chi^+ + (1-\mathfrak{r})\widehat{\mathfrak{X}}^+_{\tau-}\Big) - \mathbb{1}_{\tau=\bar{\theta}}(1-\bar{\rho})\chi^-. \end{aligned} \tag{2.9}$$

Remark 2.3 The clean contract is assumed to be funded at the risk-free rate r_t. The clean price P is thus not only clean of counterparty risk, but also of excess funding costs. Our Credit Valuation Adjustment (CVA) should thus rather be called Credit and Funding Value Adjustment. We stick to the name Credit Valuation Adjustment (CVA) for simplicity.

The following result extends to the multiple-curve setup, the one-curve bilateral CVA representation result of Brigo and Capponi (2010). Note that in a multiple-curve setup this representation, in the form of Equation (2.10) below, is implicit. Namely, the right-hand side of (2.10) involves Θ and ζ, via R in ξ and via g in the integral term. This is at least the case unless $\mathfrak{r} = 1$ and a funding coefficient $g(\pi, \varsigma) = g(\pi)$ is linear in π, so that one can get rid of these dependencies by a suitable adjustment of the discount factor (see Remark 3.2).

Proposition 2.1 *Let be given a hedge* ζ *and* $\mathcal{G}$*-semimartingales* Π *and* Θ *such that* $\Theta = P - \Pi$ *on* $[0, \bar{\tau}]$. *The pair-process* (Π, ζ) *is a solution to the price BSDE* (1.1) *if and only if* Θ *satisfies for* $t \in [0, \bar{\tau}]$

$$\beta_t\Theta_t = \mathbb{E}_t\Big[\beta_{\bar{\tau}}\mathbb{1}_{\tau<T}\xi + \int_t^{\bar{\tau}} \beta_s g_s(P_s - \Theta_s, \zeta_s)ds\Big]. \tag{2.10}$$

Proof. Recall $P_T = 0$ and $\Delta D_\tau = 0$, so $P_{\bar\tau} = \mathbb{1}_{\tau<T} P_\tau$ and $\mathbb{1}_{\tau<T}\Delta D_{\bar\tau} = 0$. Taking the difference between (2.5) and (1.3), one gets for $t \in [0, \bar\tau]$

$$\beta_t \left(P_t - \Pi_t\right) = \mathbb{E}_t\Big[\beta_{\bar\tau}\mathbb{1}_{t<\bar\tau}\big(\Delta D_{\bar\tau} - \mathbb{1}_{\bar\tau<\tau}\Delta D_{\bar\tau}\big) + \int_t^{\bar\tau} \beta_s g_s(\Pi_s, \zeta_s)ds + \beta_{\bar\tau}\mathbb{1}_{\tau<T}\left(P_\tau - R\right)\Big]$$
$$= \mathbb{E}_t\Big[\int_t^{\bar\tau} \beta_s g_s(\Pi_s, \zeta_s)ds + \beta_{\bar\tau}\mathbb{1}_{\tau<T}\xi\Big]$$

which is Equation (2.10) in Θ. □

One thus recovers in the multiple-curve setup, the general interpretation of the CVA as the price of the so-called contingent credit default swap (CCDS, see Brigo and Pallavicini (2008), Brigo and Capponi (2010)), which is an option on the debt χ (sitting via R in ξ) of the bank to the investor at time τ. However, in a multiple-curve setup, this is a dividend-paying option, paying not only the amount ξ at time $\tau < T$, but also dividends at rate $g_t(P_t - \Theta_t, \zeta_t) - r_t\Theta_t$ between times 0 and $\bar\tau$.

Example 2.1 In the fully swapped hedge case (1.4) and under the funding specifications of Subsection 2.3 in the previous paper, we saw there that

$$g_t(\pi, \varsigma) = g_t(\pi) = b_t\Gamma_t^+ - \bar b_t\Gamma_t^- + \lambda_t\big(\pi - \Gamma_t\big)^+ - \bar\lambda_t\big(\pi - \Gamma_t\big)^-, \tag{2.11}$$

where b and $\bar b$ stand for bases over the risk-free rate related to the remuneration of the collateral Γ, and λ and $\bar\lambda$ for the bases related to the remuneration of the external funding debt of the bank. The CVA representation of Equation (2.10) then reads as follows:

$$\begin{aligned}\beta_t\Theta_t &= \mathbb{E}_t\big[\mathbb{1}_{\tau<T}\beta_{\bar\tau}\left(P_\tau - Q_\tau\right)\big]\\ &+ \mathbb{E}_t\Big[\mathbb{1}_{\tau=\theta<T}\beta_{\bar\tau}(1-\rho)(Q_\tau - \Gamma_\tau)^+\Big]\\ &- \mathbb{E}_t\Big[\mathbb{1}_{\tau=\bar\theta<T}\beta_{\bar\tau}(1-\bar\rho)(Q_\tau - \Gamma_\tau)^-\Big]\\ &+ \mathbb{E}_t\Big[\mathbb{1}_{\tau=\theta<T}\beta_{\bar\tau}(1-\mathfrak{r})(P_{\tau-} - \Theta_{\tau-} - \Gamma_{\tau-})^- + \int_t^{\bar\tau}\beta_s\Big(b_s\Gamma_s^+ + \lambda_s\left(P_s - \Theta_s - \Gamma_s\right)^+\Big)ds\\ &\quad - \int_t^{\bar\tau}\beta_s\Big(\bar b_s\Gamma_s^- + \bar\lambda_s\left(P_s - \Theta_s - \Gamma_s\right)^-\Big)ds\Big].\end{aligned} \tag{2.12}$$

From the perspective of the bank, the four terms in this decomposition of the (net) CVA Θ, can respectively be interpreted as a replacement benefit/cost (depending on the sign of $P_\tau - Q_\tau$), a beneficial debt value adjustment, a costly (non-algebraic, strict) credit value adjustment, and an excess funding benefit/cost. We shall dwell more about such decompositions in Example 3.1. See also Section 5 of the previous paper for a decomposition that arises in the context of an externally funded hedge (as opposed to the most common case of a fully swapped hedge with $g(\pi, \varsigma) = g(\pi)$ in (2.11)).

Remark 2.4 In a model without immersion (see Remark 2.1), defining the clean price P by

$$\beta_t P_t = \mathbb{E}\left(\int_t^T \beta_s dD_s \,\Big|\, \mathcal{G}_t\right) \tag{2.13}$$

(instead of expectation given $\mathcal{F}_t$ in (2.2)), then (2.5) is satisfied by definition, and one can check that all the results of this Subsection still hold true provided one adds a further term ΔD_τ (which can be non-zero without immersion) to the exposure χ in (2.6), so for an exposure redefined as

$$\chi = Q_\tau + \Delta D_\tau - \Gamma_\tau \tag{2.14}$$

(see Bielecki and Crépey (2011)).

2.4.1 CCDS Static Hedging Interpretation

Let us temporarily assume just for the sake of the argument that the clean contract with price process P and the CCDS were traded assets. It is interesting to note that in that case, Definition 1.1 of a price-and-hedge (Π, ζ) would make perfect sense provided static hedging is used, and this even in case $\zeta = 0$. Indeed, given a price process Π solving the price BSDE (1.1) for $\zeta = 0$, a static replication scheme of the bank shortening the contract to the investor and funding it by its external funder would consist in:

- At time 0, using the proceeds Π_0 from the shortening of the contract and $\Theta_0 = P_0 - \Pi_0$ from the shortening of a CCDS to buy the clean contract at price P_0,
- On the time interval $(0, \bar\tau)$, holding P and $(-\Theta)$, transferring to the investor all the dividends dD_t which are perceived by the bank through its owning of P, and incurring dt-costs at rate $r_t P_t + g(\Pi_t, 0) - r_t\Theta_t = g_t(\Pi_t, 0) + r_t\Pi_t$. These costs exactly match the dt-funding benefits from the short naked (non dynamically hedged) position in the contract.

Thus, at time $\bar\tau$:

- If $\bar\tau = \tau < T$, the bank is left with an amount $P_\tau - \Theta_\tau = P_\tau - \xi = R$, which is exactly the close-out cash-flow it must deliver to the investor and to its funder,
- If $\bar\tau = T$, there are no cash-flows at $\bar\tau$.

In both cases the bank is left break-even at $\bar\tau$.

But of course this static buy-and-hold replication strategy is not practical, since neither the clean contract nor the CCDS are traded assets. One is thus led to active management of the cost and error of the trading strategy through dynamic hedging. Here a question arises whether one should try to hedge the contract globally, or (if any freedom in this is left by the internal organization of the bank) to hedge the clean contract P separately from the CVA component Θ of Π. In order to address these issues one needs to dig further into the analysis of the cost process $d\varepsilon = d\nu - \zeta d\mathcal{M}$ of a price-and-hedge (Π, ζ).

3 Pre-Default BSDE Modeling

We develop in this Section a reduced-form CVA BSDE approach to the problem of pricing and hedging counterparty risk under funding constraints. Counterparty risk and funding corrections to the clean price-and-hedge of the portfolio are obtained as the solution of a pre-default BSDE stated with respect to the reference filtration, in which defaultability of the two parties only shows up through their default intensities.

3.1 Reduction of Filtration

Let us call the CVA BSDE of the bank, the $\mathcal{G}$-BSDE on the random time interval $[0, \bar{\tau}]$, with terminal condition $\mathbb{1}_{\tau<T}\xi$ at $\bar{\tau}$, and driver coefficient $g_t(P_t - \vartheta, \varsigma) - r_t\vartheta$, $\vartheta \in \mathbb{R}, \varsigma \in \mathbb{R}^d$. The following Lemma rephrases Proposition 2.1 in BSDE terms.

Lemma 3.1 *Given a hedge ζ and $\mathcal{G}$-semimartingales Π and Θ summing-up to P, (Π, ζ) solving the price BSDE is equivalent to (Θ, ζ) solving the corresponding CVA BSDE.*

Passing from the price BSDE in (Π, ζ) to the CVA BSDE in (Θ, ζ), allows one to get rid of the dC_t-term (contract's dividend) in (1.1). This makes the CVA BSDE more tractable than the price BSDE (1.1). We assume in the sequel that:

- The $\mathcal{G}$-semimartingale (collateral) Γ_t is $\mathcal{F}$-adapted. By Lemma 2.1(ii), one then almost surely has that $\Delta\Gamma_\tau = 0$;
- The CSA close-out price process Q is left-continuous, and thus $\mathcal{G}$-predictable. This makes financial sense since what Q_τ is really meant to be is a notion of fair value of the contract right before the default time τ of either party;
- The recovery rates ρ, $\overline{\rho}$ and $\mathfrak{r}$ can be represented as ρ_θ, $\overline{\rho}_{\overline{\theta}}$ and $\mathfrak{r}_\theta$, for some $\mathcal{G}$-predictable processes ρ_t, $\overline{\rho}_t$ and $\mathfrak{r}_t$.

Remark 3.1 Assuming $\mathcal{F}$-adaptedness of Γ_t also makes financial sense since securities eligible as collateral are only cash or very basic securities which should not be affected by the default of either party. There is one reservation however. It is often so that collateral can be posted in different currencies. The choice of the collateral currency is actually a debated problem in the industry, see Fujii et al. (2010). Now, as already mentioned in Remark 2.1, it has been shown empirically that there can be some situations of strong dependence between the default risk of an obligor and an exchange rate, yielding to models accounting for the possibility of default of a large firm feeding an instantaneous jump in the FX rate of the related economy that is driven by the default time (see Ehlers and Schönbucher (2006)). In case of such strong dependence, and even for collateral posted as cash (but in this other currency), Γ_t would typically jump upon default of this obligor, and it would therefore not be $\mathcal{F}$-adapted.

By Theorem 67.b in Dellacherie and Meyer (1975), the $\mathcal{G}_{\tau-}$-measurable random variables $\mathbb{P}(\tau = \theta \,|\, \mathcal{G}_{\tau-})$ and $\mathbb{P}(\tau = \overline{\theta} \,|\, \mathcal{G}_{\tau-})$ can be represented as p_τ and $\overline{p}_\tau$, for some $\mathcal{G}$-predictable process p and $\overline{p}$. Since $\Delta\Gamma_\tau = 0$, there exists likewise an $\mathcal{F}$-predictable process with the same value as Γ at τ, so that one can assume that the collateral process Γ is in fact $\mathcal{F}$-predictable. The debt χ of the bank to the investor, and, given a price-and-hedge (Π, ζ), the terminal payoff ξ of a CCDS, are then the values at time τ of the $\mathcal{G}$-predictable process χ_t and of the $\mathcal{G}$-progressively measurable process ξ_t defined as, for $t \in [0, T]$

$$\begin{aligned} \chi_t &= Q_t - \Gamma_t \\ \xi_t &= (P_t - Q_t) + \mathbb{1}_{t\geq\theta}\Big((1-\rho_t)\chi_t^+ + (1-\mathfrak{r}_t)\widehat{\mathfrak{X}}_{t-}^+\Big) - \mathbb{1}_{t\geq\overline{\theta}}(1-\bar{\rho}_t)\chi_t^- \end{aligned} \tag{3.1}$$

where $\widehat{\mathfrak{X}}_t$ is a shorthand for $\mathfrak{X}_t(\Pi_t, \zeta_t)$. Let for $t \in [0, T]$, $\pi \in \mathbb{R}$ and $\varsigma \in \mathbb{R}^d$

$$\bar{\xi}_t(\pi, \varsigma) = (P_t - Q_t) + p_t\Big((1-\rho_t)\chi_t^+ + (1-\mathfrak{r}_t)\mathfrak{X}_{t-}^+(\pi, \varsigma)\Big) - \overline{p}_t(1-\overline{\rho}_t)\chi_t^- . \tag{3.2}$$

Let also J denote the non-default indicator process such that $J_t = \mathbb{1}_{t<\tau}$ and $J_{t-} = \mathbb{1}_{t\leq\tau}$ for $t \in [0, \bar{\tau}]$. Observe that given a hedge ζ, a process Θ solving Equation (2.10) over $[0, \bar{\tau}]$ is equivalent to $\Theta = J\bar{\Theta} + (1-J)\mathbb{1}_{\tau<T}\xi$, for a process $\bar{\Theta}$ such that for $t \in [0, \bar{\tau}]$

$$\beta_t\bar{\Theta}_t = \mathbb{E}_t\Big[\beta_{\bar{\tau}}\mathbb{1}_{\tau<T}\bar{\xi}_\tau(P_{\tau-} - \bar{\Theta}_{\tau-}, \zeta_{\tau-}) + \int_t^{\bar{\tau}} \beta_s g_s(P_s - \bar{\Theta}_s, \zeta_s)ds\Big]. \tag{3.3}$$

To simplify the problem further, we now introduce an equivalent pre-default CVA BSDE over $[0, T]$, relative to the pre-default filtration $\mathcal{F}$. The following result is classical, see for instance Bielecki et al. (2009) for precise references.

Lemma 3.2 *For any $\mathcal{G}$-adapted, respectively $\mathcal{G}$-predictable process X over $[0, T]$, there exists a unique $\mathcal{F}$-adapted, respectively $\mathcal{F}$-predictable, process $\widetilde{X}$ over $[0, T]$, called the pre-default value process of X, such that $JX = J\widetilde{X}$, respectively $J_-X = J_-\widetilde{X}$ over $[0, T]$.*

Given the structure of the data, we may therefore assume without loss of generality that process $g_t(P_t - \vartheta, \varsigma)$ is $\mathcal{F}$-progressively measurable for every $\vartheta \in \mathbb{R}$, $\varsigma \in \mathbb{R}^d$, and that all the processes (including for instance p and $\overline{p}$) which appear as building blocks in $\bar{\xi}$, are $\mathcal{F}$-predictable. We assume further that the Azéma supermartingale G of τ is time-differentiable. This allows one to define the hazard intensity $\gamma_t = -\frac{d\ln G_t}{dt}$ of τ, so $G_t = e^{-\int_0^t \gamma_s ds}$. We then define the credit-risk-adjusted-interest-rate $\widetilde{r}$ and the credit-risk-adjusted-discount-factor $\widetilde{\beta}$ as, for $t \in [0, T]$,

$$\widetilde{r}_t = r_t + \gamma_t\,,\ \widetilde{\beta}_t = \beta_t G_t = \beta_t \exp(-\int_0^t \gamma_s ds) = \exp(-\int_0^t \widetilde{r}_s ds).$$

One can then state the following

Definition 3.1 The pre-default CVA BSDE of the bank is the $\mathcal{F}$-BSDE in $(\widetilde{\Theta}, \zeta)$ on $[0, T]$ with a null terminal condition at T, and with driver coefficient

$$\widetilde{g}_t(P_t - \vartheta, \varsigma) = g_t(P_t - \vartheta, \varsigma) + \gamma_t\widetilde{\xi}_t(P_t - \vartheta, \varsigma) - \widetilde{r}_t\vartheta \tag{3.4}$$

where $\widetilde{\xi}_t(\pi, \varsigma)$ denotes for every $\pi \in \mathbb{R}$ and $\varsigma \in \mathbb{R}^d$ the $\mathcal{F}$-progressively measurable process defined by, for $t \in [0, T]$

$$\widetilde{\xi}_t(\pi, \varsigma) = (P_t - Q_t) + p_t\Big((1-\rho_t)\chi_t^+ + (1-\mathfrak{r}_t)\mathfrak{X}_t^+(\pi, \varsigma)\Big) - \overline{p}_t(1-\overline{\rho}_t)\chi_t^-. \tag{3.5}$$

An $\mathcal{F}$-special semimartingale $\widetilde{\Theta}$ and a hedge ζ to the contract, thus solve the pre-default CVA BSDE if and only if

$$\begin{cases} \widetilde{\Theta}_T = 0, \text{ and for } t \in [0, T]: \\ -d\widetilde{\Theta}_t = \widetilde{g}_t(P_t - \widetilde{\Theta}_t, \zeta_t)dt - d\widetilde{\mu}_t \end{cases} \tag{3.6}$$

where $\widetilde{\mu}$ is the $\mathcal{F}$-martingale component of $\widetilde{\Theta}$. Or equivalently to the second line in (3.6): For $t \in [0, T]$

$$-d(\widetilde{\beta}_t\widetilde{\Theta}_t) = \widetilde{\beta}_t\Big(g_t(P_t - \widetilde{\Theta}_t, \zeta_t) + \gamma_t\widetilde{\xi}_t(P_t - \widetilde{\Theta}_t, \zeta_t)\Big)dt - \widetilde{\beta}_t d\widetilde{\mu}_t. \tag{3.7}$$

Or equivalently to (3.6), in integral form: For $t \in [0, T]$

$$\widetilde{\beta}_t\widetilde{\Theta}_t = \mathbb{E}\Big[\int_t^T \widetilde{\beta}_s\big(g_s(P_s - \widetilde{\Theta}_s) + \gamma_s\widetilde{\xi}_s(P_s - \widetilde{\Theta}_s)\big)ds \,\Big|\, \mathcal{F}_t\Big]. \tag{3.8}$$

Remark 3.2 (Linear Case) In the linear case with $\mathfrak{r}=1$ and $g_t(P-\vartheta,\varsigma)=g_t^*(P)-\lambda_t^*\vartheta$, the CVA equations (2.10) and (3.8) respectively boil down to the explicit representations

$$\beta_t^*\Theta_t=\mathbb{E}\Big[\beta_{\bar\tau}^*\mathbb{1}_{\tau<T}\xi+\int_t^{\bar\tau}\beta_s^*g_s^*(P_s)ds\,\Big|\,\mathcal{G}_t\Big] \tag{3.9}$$

$$\widetilde\beta_t^*\widetilde\Theta_t=\mathbb{E}\Big[\int_t^T\widetilde\beta_s^*\Big(g_s^*(P_s^*)+\gamma_s\widetilde\xi_s^*\Big)ds\,\Big|\,\mathcal{F}_t\Big] \tag{3.10}$$

for the funding-adjusted discount factors

$$\beta_t^*=\exp(-\int_0^t(r_s+\lambda_s^*)ds)\,,\;\;\widetilde\beta_t^*=\exp(-\int_0^t(\widetilde r_s+\lambda_s^*)ds)$$

and with in (3.10)

$$\widetilde\xi_t^*=(P_t-Q_t)+p_t(1-\rho_t)\chi_t^+-\bar p_t(1-\bar\rho_t)\chi_t^-\,.$$

On the numerical side such explicit representations allow one to estimate these "linear CVAs" by standard Monte Carlo loops (provided P and Q can be computed explicitly), as opposed to time-discretisation BSDE techniques that must be used in general (and as soon as $\mathfrak{r}<1$).

Remark 3.3 (CSA Close-Out Pricing and Collateralization Schemes) It is implicitly understood above that the CSA close-out price process Q is an exogenous process, as in the standard clean CSA close-out pricing scheme $Q=P_-$. An a priori unusual situation from this point of view, yet one which is sometimes considered in the counterparty risk literature, is the so-called pre-default CSA close-out pricing scheme $Q=\Pi_-$ (see Crépey et al. (2010); see also Example 3.1 below and Brigo and Morini (2010) on the impact of alternative CSA close-out pricing schemes). It's interesting to note that from a mathematical point of view such an "implicit" scheme $Q=\Pi_-$ can be accounted for at no harm, simply by letting $Q=P_--\widetilde\Theta_-$ everywhere in the coefficient $\widetilde g_t$ of the pre-default CVA BSDE (3.6).

Note however that in order to meet ISDA requirements, a real-life collateralization scheme Γ is typically path dependent in Q (see Section 3.2 of Bielecki and Crépey (2011)). Under the pre-default CSA close-out pricing scheme, and in case of a path dependent collateralization, one ends-up with a time-delayed BSDE with a coefficient depending on the past of $\widetilde\Theta$. This raises a mathematical difficulty of the pre-default CSA close-out pricing scheme since even for a Lipschitz coeffficient, a time-delayed BSDE may only have a solution for T small enough, depending on the Lipschitz constant of the coefficient (see Delong and Imkeller (2010)).

Example 3.1 Under the fully swapped hedge funding specification of Example 2.1 with $g(\pi,\varsigma)=g(\pi)$ given by (2.11), one obtains by plugging (2.11) into (3.4) and reordering terms that

$$\begin{aligned}\widetilde g_t(P_t-\vartheta)+r_t\vartheta=\gamma_t\,&(P_t-\vartheta-Q_t)\\&+\gamma_tp_t\big((1-\rho)(Q_t-\Gamma_t)^+\\&-\gamma_t\bar p_t(1-\bar\rho)(Q_t-\Gamma_t)^-\\&+\big(b_t\Gamma_t^+ +\lambda_t\,(P_t-\vartheta-\Gamma_t)^+\big)-\big(\bar b_t\Gamma_t^- +\widetilde\lambda_t\,(P_t-\vartheta-\Gamma_t)^-\big)\end{aligned} \tag{3.11}$$

where the coefficient $\widetilde\lambda_t:=\bar\lambda_t-\gamma_tp_t(1-\mathfrak{r})$ of $(P_t-\vartheta-\Gamma_t)^-$ can be interpreted as an external borrowing rate adjusted for credit risk, or liquidity (as opposed to credit) external borrowing funding basis (see Remark 4.5). The four terms (lines) in this decomposition can be interpreted as those of (2.12).

In case of a clean CSA recovery scheme $Q = P_-$, (3.11) rewrites as follows

$$\begin{aligned}\widetilde{g}_t(P_t - \vartheta) + \widetilde{r}_t\vartheta = \gamma_t p_t\big((1-\rho)(P_t - \Gamma_t)^+ \\ - \gamma_t\overline{p}_t(1-\overline{\rho})(P_t - \Gamma_t)^- \\ + \big(b_t\Gamma_t^+ + \lambda_t\,(P_t - \vartheta - \Gamma_t)^+\big) - \big(\bar{b}_t\Gamma_t^- + \widetilde{\lambda}_t\,(P_t - \vartheta - \Gamma_t)^-\big)\end{aligned} \tag{3.12}$$

which in case of no collateralization ($\Gamma = 0$), respectively continuous collateralization with $\Gamma = Q = P_-$, reduces to

$$\gamma_t p_t(1-\rho)P_t^+ - \gamma_t\overline{p}_t(1-\overline{\rho})P_t^- + \lambda_t\,(P_t - \vartheta)^+ - \widetilde{\lambda}_t\,(P_t - \vartheta)^- \tag{3.13}$$

respectively

$$b_t P_t^+ - \bar{b}_t P_t^- + \lambda_t\vartheta^- - \widetilde{\lambda}_t\vartheta^+. \tag{3.14}$$

In case of a pre-default CSA recovery scheme $Q = \Pi_- = P - \Theta_-$, (3.11) rewrites as follows (with r_t in the left-hand side as opposed to $\widetilde{r}_t$ above)

$$\begin{aligned}\widetilde{g}_t(P_t - \vartheta) + r_t\vartheta = \gamma_t p_t\big((1-\rho)(P_t - \vartheta - \Gamma_t)^+ \\ - \gamma_t\overline{p}_t(1-\overline{\rho})(P_t - \vartheta - \Gamma_t)^- \\ + \big(b_t\Gamma_t^+ + \lambda_t\,(P_t - \vartheta - \Gamma_t)^+\big) - \big(\bar{b}_t\Gamma_t^- + \widetilde{\lambda}_t\,(P_t - \vartheta - \Gamma_t)^-\big)\end{aligned} \tag{3.15}$$

which in case of no collateralization ($\Gamma = 0$), respectively continuous collateralization with $\Gamma = Q = P - \Theta_-$, reduces to

$$\big(\gamma_t p_t(1-\rho) + \lambda_t\big)(P_t - \vartheta)^+ - \big(\gamma_t\overline{p}_t(1-\overline{\rho}) + \widetilde{\lambda}_t\big)\,(P_t - \vartheta)^- \tag{3.16}$$

respectively

$$b_t(P_t - \vartheta)^+ - \bar{b}_t(P_t - \vartheta)^-. \tag{3.17}$$

In view of (3.14) and (3.17), it is under the pre-default CSA recovery scheme $Q = \Pi_-$ that continuous collateralization is the most efficient. In this case continuous collateralization works almost perfectly (in the present reduced-form setup, see end of Remark 2.1), the corresponding CVA vanishing up to a term related to the excess-remuneration of the collateral. In the line of the discussion in Brigo and Morini (2010), this would plead in favor of the scheme $Q = \Pi_-$ as the less "intrusive" CSA recovery convention. But on the other hand this scheme induces more asymmetry than the clean scheme $Q = P_-$ between the (even bilateral) CVAs computed from the perspective of the two parties (see Remarks 2.3 and 4.9 in the previous paper).

3.2 Modeling Assumption

From now on, our approach to deal with the price BSDE (1.1) will consist in modeling the counterparty risky price process Π via the corresponding pre-default CVA process $\widetilde{\Theta}$. In this Section we work under the following

Assumption 3.1 The pre-default CVA BSDE (3.6) admits a solution $(\widetilde{\Theta}, \zeta)$.

Remark 3.4 At this stage this may seem quite an assumption. However this is only a temporary one, which is made for examining its consequences in terms of existence of a solution (Π,ζ) to the price BSDE (1.1) in this Subsection, and of analysis of the cost process ε of (Π,ζ) in Subsection 3.3. We refer the reader to Section 4 for more about the issue of existence and uniqueness of a solution to (3.6) (or an equivalent BSDE (4.1)), which will ultimately hold under mild regularity and square-integrability conditions on the data (after specification of a jump-diffusion setup endowed with a martingale representation property).

Lemma 3.3 *Under Assumption 3.1:*
(i) *The pair* (Θ,ζ) *with* Θ *defined over* $[0,\bar\tau]$ *as*

$$\Theta := J\widetilde{\Theta} + (1-J)\mathbb{1}_{\tau<T}\xi \tag{3.18}$$

solves the CVA BSDE (2.10) *over* $[0,\bar\tau]$. *Therefore, the pair* (Π,ζ) *with*

$$\Pi := P - \Theta = J(P-\widetilde{\Theta}) + (1-J)\mathbb{1}_{\tau<T}R \tag{3.19}$$

solves the price BSDE (1.1) *over* $[0,\bar\tau]$*;*
(ii) *The* $\mathcal{G}$*-martingale component* ν *of the counterparty risky price* $\Pi = P-\Theta$ *and the* $\mathcal{G}$*-martingale component* $\mu = M-\nu$ *of* Θ,[3] *satisfy for* $t\in[0,\bar\tau]$*:*

$$\begin{aligned} d\mu_t &= d\widetilde{\mu}_t - \Big((\xi_t-\widetilde{\Theta}_t)dJ_t + \gamma_t(\widehat{\xi}_t - \widetilde{\Theta}_t)dt\Big) \\ d\nu_t &= d\widetilde{\nu}_t - \Big((R_t-\widetilde{\Pi}_t)dJ_t + \gamma_t(\widetilde{R}_t - \widetilde{\Pi}_t)dt\Big)\,. \end{aligned} \tag{3.20}$$

Here $\widehat{\xi}_t$ *is a shorthand for* $\widetilde{\xi}_t(P_t-\widetilde{\Theta}_t,\zeta_t)$; $\widetilde{\Pi} := P-\widetilde{\Theta}$ *is the pre-default value process of* Π; $\widetilde{\nu} := M-\widetilde{\mu}$ *is an* $\mathcal{F}$*-martingale component of* $\widetilde{\Pi}$; *the* $\mathcal{G}$*-progressively measurable process* R_t *and the* $\mathcal{F}$*-progressively measurable process* $\widetilde{R}_t$ *are defined by, for* $t\in[0,T]$,

$$\begin{aligned} R_t &= \Gamma_t + \mathbb{1}_{t\ge\theta}\Big((\rho_t\chi_t^+ - \chi_t^-) - (1-\mathfrak{r}_t)\widehat{\mathfrak{X}}_{t-}^+\Big) - \mathbb{1}_{t\ge\bar\theta}(\bar\rho_t\chi_t^- - \chi_t^+) - \mathbb{1}_{t\ge\theta=\bar\theta}\chi_t \\ \widetilde{R}_t &= \Gamma_t + p_t\Big((\rho_t\chi_t^+ - \chi_t^-) - (1-\mathfrak{r}_t)\widehat{\mathfrak{X}}_t^+\Big) - \bar p_t(\bar\rho_t\chi_t^- - \chi_t^+) - q_t\chi_t \end{aligned} \tag{3.21}$$

in which $\widehat{\mathfrak{X}}_t$ *stands as a shorthand for* $\mathfrak{X}_t(\widetilde{\Pi}_t,\zeta_t)$, *and* q *in* $\widetilde{R}$ *is an* $\mathcal{F}$*-predictable process such that* $q_\tau = \mathbb{P}(\theta=\bar\theta\,|\,\mathcal{G}_{\tau-})$.

Proof. **(i)** Using the pre-default CVA BSDE (3.6) which is solved by $(\widetilde{\Theta},\zeta)$ over $[0,T]$, reduction-of-filtration computations similar to those of Bielecki et al. (2009) show via (3.3) that (Θ,ζ) solves the CVA BSDE (2.10) over $[0,\bar\tau]$. By Lemma 3.1, the pair (Π,ζ), where $\Pi := P-\Theta$, thus solves the price BSDE (1.1). Also recall $P_T=0$, which justifies the right-hand side identity in (3.19).
(ii) The proof is similar to the one of Lemma 4.1 in Bielecki et al. (2009), and is thus omitted. □

Remark 3.5 The jump-to-default exposure corresponding to the dJ-term in either line of (3.20) can be seen as a marked process, where the mark corresponds to the default being a default of the investor alone, of the bank alone, or a joint default. Consistently with this interpretation, the compensator of either dJ-term in (3.20) corresponds to the "average jump size" given by the dt-term in the same line, where the average is taken with respect to the probabilities of the marks, conditionally on the fact that a jump occurs at time τ.

Enriching further the mark space of τ would be a way of going beyond the basic immersion setup of this paper, as we shall illustrate in further work.

[3] Recall (1.1) and (2.4) for the definition of ν and M.

3.3 Cost Processes Analysis

Let us now postulate for the $\mathcal{G}$-martingale component $\mathcal{M}$ of the primary risky assets price process denoted by $\mathcal{P}$, with pre-default value process $\widetilde{\mathcal{P}}$, a structure analogous to the one which is apparent in the second line of (3.20) for the $\mathcal{G}$-martingale component ν of Π. One thus assumes that on $[0, \bar{\tau}]$

$$d\mathcal{M}_t = d\widetilde{\mathcal{M}}_t - \Big((\mathcal{R}_t - \widetilde{\mathcal{P}}_t) dJ_t + \gamma_t (\widetilde{\mathcal{R}}_t - \widetilde{\mathcal{P}}_t) dt \Big) \tag{3.22}$$

for an $\mathcal{F}$-martingale $\widetilde{\mathcal{M}}$, a $\mathcal{G}$-progressively measurable primary recovery process $\mathcal{R}_t$, and an $\mathcal{F}$-progressively measurable process $\widetilde{\mathcal{R}}_t$ such that $\gamma_t(\widetilde{\mathcal{R}}_t - \widetilde{\mathcal{P}}_t)dt$ compensates $\big(\mathcal{R}_t - \widetilde{\mathcal{P}}_t\big)dJ_t$ over $[0, \bar{\tau}]$.

For every hedges ϕ and ζ, to be understood as hedges of the contract clean price P and price Π, let $\eta = \phi - \zeta$ denote the corresponding hedge of the CVA component Θ of Π. Let then the cost processes $\varepsilon^{P,\phi}$, $\varepsilon^{\Theta,\eta}$ and $\varepsilon^{\Pi,\zeta}$ be defined by $\varepsilon_0^{P,\phi} = \varepsilon_0^{\Theta,\eta} = \varepsilon_0^{\Pi,\zeta} = 0$, and for $t \in [0, \bar{\tau}]$

$$d\varepsilon_t^{P,\phi} = dM_t - \phi_t d\mathcal{M}_t\,, \;\; d\varepsilon_t^{\Theta,\eta} = d\mu_t - \eta_t d\mathcal{M}_t\,, \;\; d\varepsilon_t^{\Pi,\zeta} = d\varepsilon_t^{P,\phi} - d\varepsilon_t^{\Theta,\eta} = d\nu_t - \zeta_t d\mathcal{M}_t. \tag{3.23}$$

One retrieves in particular $\varepsilon^{\Pi,\zeta} = \varepsilon$, the cost process of a price-and-hedge (Π, ζ) in (1.2). An immediate application of (3.20) and (3.22) yields,

Proposition 3.1 *For $t \in [0, \bar{\tau}]$,*

$$d\varepsilon_t^{P,\phi} = \Big(dM_t - \phi_t d\widetilde{\mathcal{M}}_t\Big) + \phi_t\big(\mathcal{R}_t - \widetilde{\mathcal{P}}_t\big)dJ_t + \gamma_t\phi_t\big(\widetilde{\mathcal{R}}_t - \widetilde{\mathcal{P}}_t\big)dt \tag{3.24}$$

$$d\varepsilon_t^{\Theta,\eta} = \Big(d\widetilde{\mu}_t - \eta_t d\widetilde{\mathcal{M}}_t\Big) - \Big(\big(\xi_t - \widetilde{\Theta}_t\big) - \eta_t\big(\mathcal{R}_t - \widetilde{\mathcal{P}}_t\big)\Big)dJ_t - \gamma_t\Big(\big(\widehat{\xi}_t - \widetilde{\Theta}_t\big) - \eta_t\big(\widetilde{\mathcal{R}}_t - \widetilde{\mathcal{P}}_t\big)\Big)dt \tag{3.25}$$

$$d\varepsilon_t^{\Pi,\zeta} = \Big(d\widetilde{\nu}_t - \zeta_t d\widetilde{\mathcal{M}}_t\Big) - \Big(\big(R_t - \widetilde{\Pi}_t\big) - \zeta_t\big(\mathcal{R}_t - \widetilde{\mathcal{P}}_t\big)\Big)dJ_t - \gamma_t\Big(\big(\widetilde{R}_t - \widetilde{\Pi}_t\big) - \zeta_t\big(\widetilde{\mathcal{R}}_t - \widetilde{\mathcal{P}}_t\big)\Big)dt. \tag{3.26}$$

We thus get decompositions of the related cost processes as $\mathcal{F}$-martingales stopped at τ, hence $\mathcal{G}$-martingales, plus $\mathcal{G}$-compensated jump-to-default exposures. These decompositions can then be used for devising specific pricing and hedging schemes, such as pricing at the cost of hedging by replication (if possible), or of hedging only pre-default risk, or of hedging only the jump-to-default risk (dJ-terms), or of min-variance hedging, etc. This will now be made practical in a Markovian setup.

4 Markovian Case

In a Markovian setup, explicit CVA pricing and hedging schemes can be formulated in terms of semilinear pre-default CVA PDEs. More precisely, we shall relate suitable notions of orthogonal solutions to the pre-default CVA BSDE to:

- From a financial point of view, corresponding min-variance hedging strategies of the bank, based on the cost processes analysis of Subsection 3.3;

- From a mathematical point of view, classical Markovian BSDEs driven by an explicit set of fundamental martingales given in the form of a multi-variate Brownian motion and a compensated jump measure.

These Markovian BSDEs will be well posed under mild conditions, yielding related orthogonal solutions to the pre-default CVA BSDE, and providing in turn the corresponding min-variance hedges to the bank. This approach will be developed for three different min-variance hedging objectives, respectively considered in Subsections 4.2, 4.3 and 4.4. In the end the preferred criterion (we mainly see the analysis of Subsection 4.2 as preparatory to those of Subsections 4.3 and 4.4) can be optimized by solving (numerically if need be) the related Markovian BSDE, or (if more efficient) by solving an equivalent semilinear parabolic PDE. Also we shall see that this methodology can be applied to either the risk-management of the overall contract, or of its CVA component in isolation. But in all cases the pre-default CVA BSDE will be key in the mathematical analysis of the problem. Our main results are Proposition 4.2 and Corollary 4.2, which yield concrete recipes for risk-managing the contract as a whole or its CVA component, according to the following objective of the bank: minimizing the variance of the cost process of the contract or of its CVA component, whilst achieving a perfect hedge of the jump-to-default exposure.

As explained in the introduction, a clean price-and-hedge (P,ϕ) is typically determined by the business trading desks of the bank. The central CVA desk is then left with the task of devising a CVA price-and-hedge (Θ,η). Consistently with this logic, given a clean price-and-hedge (P,ϕ), a solution $(\widetilde{\Theta},\zeta)$ to the pre-default CVA BSDE will be sought henceforth in the form $(\widetilde{\Theta},\phi-\eta)$, where an $\mathcal{F}$-adapted triplet $(\widetilde{\Theta},\eta,\epsilon)$ solves

$$\begin{cases} \widetilde{\Theta}_T = 0, \text{ and for } t \in [0,T]: \\ -d\widetilde{\Theta}_t = \widetilde{g}_t(P_t - \widetilde{\Theta}_t, \phi_t - \eta_t)dt - \big(\eta_t d\widetilde{\mathcal{M}}_t + d\epsilon_t\big) \end{cases} \tag{4.1}$$

for an $\mathcal{F}$-predictable[4] integrand η and an $(\mathcal{F},\mathbb{P})$-martingale ϵ. The pre-default CVA BSDE in form (4.1) is indeed equivalent to the original pre-default CVA BSDE (3.6), letting $\eta=\phi-\zeta$, and ϵ be defined through the second line of (4.1) (and $\epsilon_0=0$). Henceforth, accordingly,

Definition 4.1 We call CVA price-and hedge, any pair-process (Θ,η) such that $(\widetilde{\Theta},\eta,\epsilon)$, with $\widetilde{\Theta}=J\Theta$ and ϵ defined through $(\widetilde{\Theta},\eta)$ by the second line of (4.1) (and $\epsilon_0=0$), solves (4.1), meaning that ϵ thus defined is an $(\mathcal{F},\mathbb{P})$-martingale.

4.1 Factor Process

We assume further that the pre-default CVA BSDE thus redefined as (4.1) is Markovian, in the sense that any of its input data of the form $\mathcal{D}_t$ is given as a deterministic function $\mathcal{D}(t,X_t)$ of an $\mathcal{F}$-Markov factor process X. So in particular $(P_t,\phi_t)=(P(t,X_t),\phi(t,X_t))$. Consequently, one has for an obviously defined deterministic function $\widetilde{g}(t,x,\pi,\varsigma)$:

$$\widetilde{g}_t(P_t-\widetilde{\Theta}_t,\phi_t-\eta_t)dt = \widetilde{g}\big(t,X_t,P(t,X_t)-\widetilde{\Theta}_t,\phi(t,X_t)-\eta_t\big)dt. \tag{4.2}$$

We shall use as drivers of the pre-default factor process X an $\mathbb{R}^q$-valued $\mathcal{F}$-Brownian motion W and an $\mathcal{F}$-compensated jump measure N on $[0,T]\times\mathbb{R}^q$ (see for instance Jacod

[4] Typically left-continuous in a Markov setup.

and Shiryaev (2003)), for some integer q. Given coefficients $b(t,x)$, $\sigma(t,x)$, $\delta(t,x,y)$ and $F(t,x,dy)$ to be specified depending on the application at hand, we assume that the pre-default factor process X satisfies the following Markovian (forward) SDE in $\mathbb{R}^q$: $X_0 = x$ given as an observable or calibratable constant, and for $t \in [0,T]$

$$dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t + \delta(t,X_{t-})\cdot dN_t \tag{4.3}$$

with a (random) jump intensity measure given by $F(t,X_t,dx)$. We denote for every matrix-valued function $f = f(t,x,y)$ on $[0,T]\times\mathbb{R}^q\times\mathbb{R}^q$ (like $f=\delta$ in (4.3))

$$f(t,X_{t-})\cdot dN_t = \int_{\mathbb{R}^q} f(t,X_{t-},x)N(dt,dx)\,,\ \ \big(f\cdot F\big)(t,x) \ \ = \int_{\mathbb{R}^q} f(t,x,y)F(t,x,dy).$$

The matrix-integrals are performed entry by entry of f, so that one ends up with matrices of the same dimensions as f.

Example 4.1 (Systemic counterparty risk) In the aftermath of the 2007–09 financial crisis, a variety of spreads have developed between rates that had been essentially the same until then, notably LIBOR-OIS spreads. By the end of 2011, with the sovereign credit crisis, these spreads were again significant (close to 100 basis points). Interestingly enough, this can also be interpreted as a manifestation of a form of counterparty risk, we call it systemic counterparty risk, in reference to a default and/or liquidity risk of the banking sector as a whole (see Filipović and Trolle (2011)).

The resulting discrepancy between risk-free rates which are used for discounting and LIBORs which are used as underlyings of interest rate derivatives, must be reflected in a clean valuation model P of interest rate derivatives. We refer the reader to Crépey et al. (2012) for a defaultable HJM clean valuation methodology which is developed in this regard, ending up with simple Markovian short-term specifications X of the form (4.3) such that

$$P_t = P(t,X_t) \tag{4.4}$$

(a prerequisite to (4.2)) for most vanilla interest rate derivatives (including IR swaps, basis swaps, cap/floors and swaptions). The vector factor process X consists of the risk-free short rate process r, a "systemic" short credit spread process λ of the LIBOR banks, and auxiliary processes which may be needed for the sake of (4.4). The most tractable specification found in Crépey et al. (2012) consists of the following Lévy Hull-White model for a two-dimensional $X_t = (r_t,\ell_t)$

$$\begin{aligned} dr_t &= a(h(t)-r_t)dt + \sigma dL^r_t \\ d\ell_t &= a^*(h^*(t)-\ell_t)dt + \sigma^* dL^\ell_t \end{aligned} \tag{4.5}$$

in which L^r and L^ℓ denote Lévy subordinators (non-decreasing Lévy processes starting at 0) and where the coefficients are defined in connection with a defaultable HJM setup.

Remark 4.1 Since the jumps of L^r and L^ℓ only affect r and ℓ linearly (via σ and σ^*) in this model, and since one only deals with finite variation processes here, therefore in this example an explicit representation (4.5) of X in terms of the driving Lévy noises is available (explicit representation as opposed to a representation in terms of the related compensated jump measure N in a generic jump-diffusion setup (4.3); see also Example 4.2).

Remark 4.2 In case one of the two parties in an OTC interest-rate derivative contract is a LIBOR bank (or peer to a LIBOR bank), her default risk and the above-mentioned systemic counterparty risk have some dependence, which should be reflected in a a realistic model of counterparty risk.

Further analysis of the cost processes (3.24)-(3.26) depends on a hedging criterion of the bank. In the following Subsections we shall propose three tractable approaches, all of them involving, to some extent, min-variance hedging. In case of a complete primary market, min-variance hedging of course reduces to hedging by replication. Moreover we shall consider the two issues of hedging the contract globally, or to only hedge its CVA. In all cases the mathematical analysis will ultimately rely on the pre-default CVA BSDE (4.1).

Remark 4.3 We refer the reader to Schweizer (2001) for a survey about various quadratic hedging approaches which can be used in an incomplete market. For the sake of tractability we only consider in this work minimization under the martingale pricing measure $\mathbb{P}$, whereas the main difficulty with quadratic hedging usually comes from the fact that one aims at minimizing the hedging error under the historical probability measure. To emphasize this difference we write in our case min-variance hedging instead of mean-variance hedging. This min-variance hedging will be performed with respect to the reference filtration $\mathcal{F}$, on the top of a given choice of a hedging strategy regarding the jump-to-default exposure of the bank: no hedge in Subsection 4.2, perfect hedge in Subsection 4.3 and hedge of an isolated default of the investor in Subsection 4.4.

Given a vector-valued function $u = u(t,x)$ on $[0,T] \times \mathbb{R}^q$, let $\nabla u(t,x)$ denote the Jacobian matrix of u with respect to x at (t,x), and let δu be the function on $[0,T] \times \mathbb{R}^q \times \mathbb{R}^q$ such that for every $(t,x,y) \in [0,T] \times \mathbb{R}^q \times \mathbb{R}^q$

$$\delta u(t,x,y) = u(t,x+\delta(t,x,y)) - u(t,x).$$

Given another vector-valued function $v = v(t,x)$ on $[0,T] \times \mathbb{R}^q$, we denote likewise

$$(u,v)(t,x) = (\nabla u \sigma)(\nabla v \sigma)^{\mathsf{T}}(t,x) \; + \; \big((\delta u \delta v^{\mathsf{T}}) \cdot F\big)(t,x), \tag{4.6}$$

in which $^{\mathsf{T}}$ stands for “transposed”. So, if u and v are n- and m-dimensional vector-functions of (t,x), one ends-up with an $\mathbb{R}^{n \times m}$-valued matrix-function (u,v) of (t,x).

We assume further that $\widetilde{\mathcal{P}}_t = \widetilde{\mathcal{P}}(t,X_t)$ for some pre-default primary risky assets pricing function $\widetilde{\mathcal{P}}$, so that the dynamics of the $\mathcal{F}$-martingale component $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ in (3.22) write as follows

$$d\widetilde{\mathcal{M}}_t = (\nabla \widetilde{\mathcal{P}} \sigma)(t,X_t)dW_t + \delta \widetilde{\mathcal{P}}(t,X_{t-}) \cdot dN_t.$$

4.2 Min-Variance Hedging of Market Risk

Our first objective will be to min-variance hedge the market risk corresponding to the term $d\widetilde{\mu}_t - \eta_t d\widetilde{\mathcal{M}}_t$ in the CVA cost process $\varepsilon^{\Theta,\eta}$ in (3.25), or $d\widetilde{\nu}_t - \zeta_t d\widetilde{\mathcal{M}}_t$ in the overall contract cost process $\varepsilon^{\Pi,\zeta} = \varepsilon$ in (3.26).

Regarding (3.25), this is tantamount to seeking for a solution $(\widetilde{\Theta}, \eta, \epsilon)$ to the pre-default CVA BSDE (4.1) in which ϵ is $\mathcal{F}$-orthogonal to $\widetilde{\mathcal{M}}$ (cf. Proposition 5.2 in El Karoui et al. (1997)). Given such an orthogonal solution $(\widetilde{\Theta}, \eta, \epsilon)$ to (4.1), and if moreover $\widetilde{\Theta}_t = \widetilde{\Theta}(t,X_t)$, one then has by a standard min-variance oblique bracket formula,[5] in the $(\cdot,\cdot)$ notation of (4.6):

$$\eta_t = \frac{d\langle\widetilde{\mu},\widetilde{\mathcal{M}}\rangle_t}{dt}\left(\frac{d\langle\widetilde{\mathcal{M}}\rangle_t}{dt}\right)^{-1} = \left(\left(\widetilde{\Theta},\widetilde{\mathcal{P}}\right)\Lambda\right)(t,X_{t-}) =: \eta(t,X_{t-}) \tag{4.7}$$

where we let $\Lambda = \left(\widetilde{\mathcal{P}},\widetilde{\mathcal{P}}\right)^{-1}$. Here invertibility of the $\mathcal{F}_t$-conditional covariance matrix $\frac{d\langle\widetilde{\mathcal{M}}\rangle_t}{dt}$ is assumed.

This leads to the following Markovian BSDE in $(\widetilde{\Theta}(t,X_t),(\nabla\widetilde{\Theta}\sigma)(t,X_t),\delta\widetilde{\Theta}(t,X_{t-},\cdot))$ over $[0,T]$:

$$\begin{cases}\widetilde{\Theta}(T,X_T)=0, \text{ and for } t\in[0,T]:\\ -d\widetilde{\Theta}(t,X_t)=\widehat{g}\left(t,X_t,\widetilde{\Theta}(t,X_t),(\nabla\widetilde{\Theta}\sigma)(t,X_t),\left((\delta\widetilde{\Theta}\delta\widetilde{\mathcal{P}}^{\mathsf{T}})\cdot F\right)(t,X_t)\right)dt\\ \qquad -(\nabla\widetilde{\Theta}\sigma)(t,X_t)dW_t-\delta\widetilde{\Theta}(t,X_{t-})\cdot dN_t,\end{cases} \tag{4.8}$$

with for every $(t,x,\vartheta,z,w)\in[0,T]\times\mathbb{R}^q\times\mathbb{R}\times\mathbb{R}^q\times\mathbb{R}^d$ (for row-vectors z,w)

$$\widehat{g}\,(t,x,\vartheta,z,w)=\widetilde{g}\,(t,x,P(t,x)-\vartheta,\phi(t,x)-\widehat{\eta}(t,x,\vartheta,z,w))$$

where we let

$$\widehat{\eta}(t,x,\vartheta,z,w)=\left(z(\nabla\widetilde{\mathcal{P}}\sigma)^{\mathsf{T}}(t,x)+w\right)\Lambda(t,x).$$

Indeed one then has in view of (4.6) and (4.7):

$$\begin{aligned}&\widehat{\eta}(t,X_t,\widetilde{\Theta}(t,X_t),(\nabla\widetilde{\Theta}\sigma)(t,X_t),\left((\delta\widetilde{\Theta}\delta\widetilde{\mathcal{P}}^{\mathsf{T}})\cdot F\right)(t,X_t))=\eta(t,X_t)\\ &\widehat{g}\left(t,X_t,\widetilde{\Theta}(t,X_t),(\nabla\widetilde{\Theta}\sigma)(t,X_t),\left((\delta\widetilde{\Theta}\delta\widetilde{\mathcal{P}}^{\mathsf{T}})\cdot F\right)(t,X_t)\right)dt=\\ &\quad\widetilde{g}\left(t,X_t,P(t,x)-\widetilde{\Theta}(t,X_t),\phi(t,X_t)-\eta(t,X_t)\right)dt.\end{aligned}$$

We refer the reader to the literature (see for instance Parts II and III of Crépey (2011)) regarding the fact that under mild regularity and square-integrability conditions on the coefficient $\widehat{g}$, the Markovian BSDE (4.8) has a unique square-integrable solution; moreover, the pre-default CVA function $\widetilde{\Theta}=\widetilde{\Theta}(t,x)$ in this solution can be characterized as the unique solution in suitable spaces to the following semilinear partial integro-differential equation (PDE for short):

$$\begin{cases}\widetilde{\Theta}(T,x)=0\,,\ x\in\mathbb{R}^q\\ (\partial_t+\mathcal{X})\,\widetilde{\Theta}(t,x)+\widehat{g}(t,x,\widetilde{\Theta}(t,x),(\nabla\widetilde{\Theta}\sigma)(t,x),\big((\delta\widetilde{\Theta}\delta\widetilde{\mathcal{P}}^{\mathsf{T}})\cdot F\big)(t,x))=0 \text{ on } [0,T)\times\mathbb{R}^q,\end{cases} \tag{4.9}$$

where $\mathcal{X}$ stands for the infinitesimal generator of X.

Remark 4.4 In Crépey and Matoussi (2008), it is postulated that the driver coefficient, $\widehat{g}$ in the case of the Markovian BSDE (4.8), only depends on $\delta\widetilde{\Theta}(t,X_{t-},\cdot)$ through one average of $\delta\widetilde{\Theta}(t,X_{t-},\cdot)$ against some jump measure, rather than through d such averages in (4.8) (the last argument w of $\widehat{g}$ is a row-vector in $\mathbb{R}^d$). By inspection of the proof in Crépey and Matoussi (2008), the comparison principle which is established there, and which is key in the connection between a BSDE and a PDE approach to a semilinear parabolic equation

[5] See for instance Part I of Crépey (2011).

(see for instance Part III of Crépey (2011)), can be elevated from the scalar case to the case of any finite number of averages. However this comparison is, as already in the scalar case, subject to a monotonicity condition of $\widehat{g}$ with respect to w, so

$$\widehat{g}\left(t,x,\vartheta,z,w\right) \le \widehat{g}\left(t,x,\vartheta,z,w'\right) \text{ if } w_i \le w'_i,\ i=1,\dots,d.$$

See also Royer (2006) for such technicalities. Of course these vanish in the (most common) case of a fully swapped hedge satisfying (1.4) so that $\widetilde{g}\big(t,x,\pi,\varsigma\big) = \widetilde{g}\big(t,x,\pi\big)$ (see also Remark 4.6 for the corresponding Markovian BSDEs and semilinear PDEs).

Proposition 4.1 *Assuming invertibility of the primary-risky-assets-covariance-matrix* $\left(\widetilde{\mathcal{P}},\widetilde{\mathcal{P}}\right)$, *the solution* $\widetilde{\Theta} = \widetilde{\Theta}(t,x)$ *to* (4.9) *yields, via* (4.7) *for* η *and in turn* (4.1) *for* ϵ, *an orthogonal solution* $(\widetilde{\Theta},\eta,\epsilon)$ *to the pre-default CVA BSDE* (4.1).

The CVA-market-risk-min-variance hedge is thus given by Formula (4.7) *as*

$$\eta_t = \eta(t,X_{t-}) = \left(\left(\widetilde{\Theta},\widetilde{\mathcal{P}}\right)\left(\widetilde{\mathcal{P}},\widetilde{\mathcal{P}}\right)^{-1}\right)(t,X_{t-}).$$

Process ϵ *in the solution to* (4.1) *is the residual CVA market risk under the CVA hedge* η.

Hedging of the Contract as a Whole We now consider hedging of the market risk $d\widetilde{\nu}_t - \zeta_t d\widetilde{\mathcal{M}}_t$ of the overall contract cost process $\varepsilon^{\Pi,\zeta} = \varepsilon$ in (3.26). Let the clean hedge ϕ be specifically given here as the coefficient of regression in an $\mathcal{F}$-orthogonal decomposition $dM = \phi d\widetilde{\mathcal{M}} + de$. By the min-variance oblique bracket formula, one thus has for $t \in [0,\bar{\tau}]$

$$\phi_t = \frac{d\langle M,\widetilde{\mathcal{M}}\rangle_t}{dt}\left(\frac{d\langle\widetilde{\mathcal{M}}\rangle_t}{dt}\right)^{-1} = \left(\left(P,\widetilde{\mathcal{P}}\right)\Lambda\right)(t,X_{t-}) =: \phi(t,X_{t-}). \tag{4.10}$$

Besides, in view of (3.24)-(3.26), it holds that

$$d\widetilde{\nu}_t - \zeta_t d\widetilde{\mathcal{M}}_t = \big(dM_t - \phi_t d\widetilde{\mathcal{M}}_t\big) - \big(d\widetilde{\mu}_t - \eta_t d\widetilde{\mathcal{M}}_t\big).$$

Therefore, $dM - \phi d\widetilde{\mathcal{M}}$ and $d\widetilde{\mu} - \eta d\widetilde{\mathcal{M}}$ being $\mathcal{F}$-orthogonal to $d\widetilde{\mathcal{M}}$, implies the same property for $d\widetilde{\nu} - \zeta d\widetilde{\mathcal{M}}$. In other words, Proposition 4.1 admits the following

Corollary 4.1 *For* ϕ *given as the regression coefficient of* M *against* $\widetilde{\mathcal{M}}$, *the strategy* $\zeta_t := (\phi-\eta)(t,X_{t-})$ *is a min-variance hedge of the market risk component* $d\widetilde{\nu} - \zeta d\widetilde{\mathcal{M}}$ *of the contract cost process* $\varepsilon^{\Pi,\zeta} = \varepsilon$. *The residual market risk of the contract hedged in this way is given by* $e - \epsilon$.

4.3 Min-Variance Hedging Constrained to Perfect Hedging of Jump-to-Default Risk

The previous approach disregards the jump-to-default risk corresponding to the dJ-terms in (3.25) or (3.26). We now wish to min-variance hedge the market risk corresponding to the term $d\widetilde{\mu}_t - \eta_t d\widetilde{\mathcal{M}}_t$ in the CVA cost process $\varepsilon^{\Theta,\eta}$ in (3.25) (respectively $d\widetilde{\nu}_t - \zeta_t d\widetilde{\mathcal{M}}_t$ in the overall contract cost process $\varepsilon^{\Pi,\zeta} = \varepsilon$ in (3.26)), under the constraint that one perfectly hedges the jump-to-default risk corresponding to the dJ-term in (3.25) (respectively (3.26)). Note that in view of the marked point process interpretation provided in Remark 3.5, cancelation of the dJ-term in any of Equation (3.24) to (3.26), implies cancelation of the

dt-driven process which compensates it in the same equation. We are thus equivalently minimizing the variance of the cost processes $\varepsilon^{\Theta,\eta}$ or $\varepsilon^{\Pi,\zeta} = \varepsilon$ under the constraint of perfectly hedging the jump-to-default exposure.

Let us re-order if need be the primary risky assets so that the first ones (if any) cannot jump at time τ, and the last ones (if any) can jump at time τ. We then let a superscript ^{0} refer to the subset of the hedging instruments with price processes which cannot jump at time τ, so $\mathcal{R}^0 = \widetilde{\mathcal{R}}^0 = \widetilde{\mathcal{P}}^0$, and we let ^{1} refer to the subset, complement of ^{0}, of the hedging instruments which can jump at time τ.[6] The CVA cost equation (3.25) can thus be rewritten as, for $t \in [0,\bar{\tau}]$:

$$\begin{aligned} d\varepsilon_t^{\Theta,\eta} =& \Big(d\widetilde{\mu}_t - \eta_t^0 d\widetilde{\mathcal{M}}_t^0 - \eta_t^1 d\widetilde{\mathcal{M}}_t^1\Big) - \Big(\big(\xi_t - \widetilde{\Theta}_t\big) - \eta_t^1\big(\mathcal{R}_t^1 - \widetilde{\mathcal{P}}_t^1\big)\Big) dJ_t \\ & - \gamma_t\Big(\big(\widehat{\xi}_t - \widetilde{\Theta}_t\big) - \eta_t^1\big(\widetilde{\mathcal{R}}_t^1 - \widetilde{\mathcal{P}}_t^1\big)\Big) dt. \end{aligned} \tag{4.11}$$

The condition that a CVA price-and-hedge (Θ,η) perfectly hedges the dJ-term in (4.11) reads as follows:

$$\xi_t - \widetilde{\Theta}_{t-} = \eta_t^1\big(\mathcal{R}_t^1 - \widetilde{\mathcal{P}}_t^1\big),\ t \in [0,\bar{\tau}] \tag{4.12}$$

where it should be noted in view of (3.1) that ξ_t is, via $(1-\mathfrak{r}_t)\widehat{\mathfrak{X}}_{t-}^+$, a random function of $\widetilde{\Theta}_{t-}$ and $\zeta_{t-} = \phi_{t-} - \eta_{t-}$. Condition (4.12) is thus implicitly a nonlinear equation in η_t^1, unless one is in the special case where (in the present Markov setup)

$$(1-\mathfrak{r}(t,X_t))\mathfrak{X}^+(t,X_t,\pi,\varsigma) = (1-\mathfrak{r}(t,X_t))\mathfrak{X}^+(t,X_t,\pi) \tag{4.13}$$

does not depend on ς, so that ξ_t does not depend on η_{t-}. In this case, in view of the expression of ξ_t in (3.1), depending on whether one considers a model of unilateral counterparty risk ($\theta = \infty$), of bilateral counterparty risk without joint default of the bank and of the investor ($\theta, \overline{\theta} < \infty$ with $\theta \neq \overline{\theta}$ almost surely), or of bilateral counterparty risk with a possible joint default of the bank and of the investor, then Equation (4.12) respectively boils down to a system of one, two or three linear equations in η_t^1.

Remark 4.5 (Discussion of Condition (4.13)**)** Condition (4.13) holds in the (quite typical) case of a fully swapped hedge, as well as in the partial default case (covering the case of unilateral counterparty risk) where $\mathfrak{r} = 1$. Besides in specific cases a solution to Equation (4.12) may be found without condition (4.13), see Section 5 of the previous paper for an example based on (Burgard and Kjaer 2011a; Burgard and Kjaer 2011b).

If condition (4.13) does not hold, a possible idea to recover it if need be however, could be to forget about the close-out funding cash-flow $R^f = (1-\mathfrak{r}_\theta)\widehat{\mathfrak{X}}_{\theta-}^+$ in R, thus working everywhere as if $\mathfrak{r}$ was equal to one, whilst using a dt-funding-coefficient $g_t(\pi,\varsigma)$ adjusted to

$$g_t^\sharp(\pi,\varsigma) = g_t(\pi,\varsigma) - \gamma_t p_t (1-\mathfrak{r}_t)\mathfrak{X}_t^+(\pi,\varsigma). \tag{4.14}$$

The problem thus modified then satisfies (4.13). The adjusted funding benefit coefficient $g_t^\sharp(\pi,\varsigma)$ represents a pure liquidity (as opposed to credit risk) funding benefit coefficient. Using this approach also allows one to decouple the credit risk ingredients in the model,

[6] This is to an harmless abuse of notation that Y^0 and Y^1 do not represent anymore the "coordinates ^{0} and ^{1}" of an $\mathbb{R}^d$-valued vector Y, or these are now "group-coordinates".

represented by θ and $\overline{\theta}$, from the liquidity funding ingredients, represented by the adjusted funding coefficient $g^\sharp$. Note that simply ignoring the close-out funding cash-flow R^f without adjusting g as in (4.14), would induce a valuation and hedging bias. In contrast, accordingly adjusting g as in (4.14) makes it at least correct from the valuation point of view, for every fixed ζ. But this correctness in value is only for a given hedge process ζ. Since a central point in all this (particularly without (4.13)) is precisely on how to choose ζ, we believe this adjustment approach is in the end fallacious.

We work henceforth in this Subsection under the assumption that Equation (4.12) has a solution of the form

$$\eta^1_t = \eta^1_t(\widetilde{\Theta}_{t-}) = \eta^1(t, X_{t-}, \widetilde{\Theta}_{t-}). \tag{4.15}$$

Again, under condition (4.13), this is satisfied under a mild non-redundancy condition on the hedging instruments in group ^{1}, with η^1 typically univariate in case $\theta = \infty$, bivariate in case $\theta, \overline{\theta} < \infty$ with $\theta \neq \overline{\theta}$, and trivariate otherwise; and we also refer the reader to Section 5 in the previous paper for a case where this is satisfied without condition (4.13).

For any CVA hedge η with components η^1 of η in group ^{1} given as $\eta^1_t(\widetilde{\Theta}_{t-})$ in (4.15), the CVA cost process (4.11) reduces to

$$d\varepsilon^{\Theta,\eta}_t = d\widetilde{\mu}_t - \eta^0_t d\widetilde{\mathcal{M}}^0_t - \eta^1_t d\widetilde{\mathcal{M}}^1_t. \tag{4.16}$$

This leads us to seek for a solution (Θ, η) to the problem of min-variance hedging of the CVA constrained to perfect hedging of CVA jump-to-default risk, with η_t of the form

$$\eta_t = \left(\eta^0_t, \eta^1_t(\widetilde{\Theta}_{t-})\right), \tag{4.17}$$

and with $(\widetilde{\Theta}, \eta, \epsilon)$ solving the pre-default CVA BSDE (4.1), where ϵ is defined through $\widetilde{\Theta}$ and η by the second line of (4.1). Note in view of the pre-default CVA BSDE (4.1) that $d\epsilon_t$ then boils down to $d\varepsilon^{\Theta,\eta}_t$ in (4.16), the variance of which one would like to minimize. Now, in order to minimize the variance of $d\varepsilon^{\Theta,\eta}_t = d\epsilon_t$ among all solutions $(\widetilde{\Theta}, \eta, \epsilon)$ of (4.1) such that $\eta^1_t = \eta^1_t(\widetilde{\Theta}_{t-})$, one must choose η^0 as the coefficient of regression of $d\bar{\mu}_t := d\widetilde{\mu}_t - \eta^1_t(\widetilde{\Theta}_{t-})d\widetilde{\mathcal{M}}^1_t$ against $d\widetilde{\mathcal{M}}^0_t$. In other words we are now looking for a solution $(\widetilde{\Theta}, \eta, \epsilon)$ to the pre-default CVA BSDE (4.1), with $\eta^1_t = \eta^1_t(\widetilde{\Theta}_{t-})$ and with $d\widetilde{\mu}_t - \eta^1_t(\widetilde{\Theta}_{t-})d\widetilde{\mathcal{M}}^1_t - \eta^0_t d\widetilde{\mathcal{M}}^0_t$ orthogonal to $d\widetilde{\mathcal{M}}^0_t$. In such a solution, assuming further a deterministic $\widetilde{\Theta}_t = \widetilde{\Theta}(t, X_t)$, it comes by the min-variance oblique bracket formula:

$$\begin{aligned}
\eta^0_t &= \frac{d < \bar{\mu}, \widetilde{\mathcal{M}}^0 >_t}{dt} \left(\frac{d < \widetilde{\mathcal{M}}^0 >_t}{dt} \right)^{-1} \\
&= \Big(\Big(\widetilde{\Theta}, \widetilde{\mathcal{P}}^0 \Big) \Lambda^0 \Big)(t, X_{t-}) - \eta^1(t, X_{t-}, \widetilde{\Theta}(t, X_{t-})) \Big(\Big(\widetilde{\mathcal{P}}^1, \widetilde{\mathcal{P}}^0 \Big) \Lambda^0 \Big)(t, X_{t-}) \\
&=: \eta^0(t, X_{t-})
\end{aligned} \tag{4.18}$$

where we let $\Lambda^0 = \left(\widetilde{\mathcal{P}}^0, \widetilde{\mathcal{P}}^0 \right)^{-1}$, assumed to exist. This leads us to the following Markovian BSDE in $(\widetilde{\Theta}(t, X_t), (\nabla\widetilde{\Theta}\sigma)(t, X_t), \delta\widetilde{\Theta}(t, X_{t-}, \cdot))$ over $[0, T]$:

$$\begin{cases}
\widetilde{\Theta}(T, X_T) = 0, \text{ and for } t \in [0, T] : \\
- d\widetilde{\Theta}(t, X_t) = \bar{g}\left(t, X_t, \widetilde{\Theta}(t, X_t), (\nabla\widetilde{\Theta}\sigma)(t, X_t), \big((\delta\widetilde{\Theta}\delta(\widetilde{\mathcal{P}}^0)^\mathsf{T}) \cdot F\big)(t, X_t)\right) dt \\
\qquad - (\nabla\widetilde{\Theta}\sigma)(t, X_t) dW_t - \delta\widetilde{\Theta}(t, X_{t-}) \cdot dN_t,
\end{cases} \tag{4.19}$$

with for every $(t,x,\vartheta,z,w)\in[0,T]\times\mathbb{R}^q\times\mathbb{R}\times\mathbb{R}^q\times\mathbb{R}^{d_0}$, in which d_0 is the number of assets in group ^{0}:

$$\bar{g}\left(t,x,\vartheta,z,w\right)=\widetilde{g}\left(t,x,P(t,x)-\vartheta,\phi(t,x)-\left(\bar{\eta}^0(t,x,\vartheta,z,w),\eta^1(t,x,\vartheta)\right)\right)$$

where we let

$$\bar{\eta}^0(t,x,\vartheta,z,w)=\Big(z(\nabla\widetilde{\mathcal{P}}^0\sigma)^{\mathsf{T}}(t,x)+w\Big)\,\Lambda^0(t,x)-\eta^1(t,x,\vartheta)\left(\left(\widetilde{\mathcal{P}}^1,\widetilde{\mathcal{P}}^0\right)\Lambda^0\right)(t,x).$$

Indeed one then has in view of (4.6) and (4.18)

$$\begin{aligned}&\bar{\eta}^0\left(t,X_t,\widetilde{\Theta}(t,X_t),(\nabla\widetilde{\Theta}\sigma)(t,X_t),\big((\delta\widetilde{\Theta}\delta(\widetilde{\mathcal{P}}^0)^{\mathsf{T}})\cdot F\big)(t,X_t)\right)=\eta^0(t,X_t)\\ &\bar{g}\left(t,X_t,\widetilde{\Theta}(t,X_t),(\nabla\widetilde{\Theta}\sigma)(t,X_t),\big((\delta\widetilde{\Theta}\delta(\widetilde{\mathcal{P}}^0)^{\mathsf{T}})\cdot F\big)(t,X_t)\right)dt\\ &\qquad=\widetilde{g}\left(t,X_t,P(t,X_t)\widetilde{\Theta}(t,X_t),\phi(t,X_t)-\left(\eta^0(t,X_t),\eta^1(t,X_t,\widetilde{\Theta}(t,X_t))\right)\right)dt.\end{aligned}$$

Now, under mild technical conditions, the Markovian BSDE (4.19) has a unique solution, and[7] the pre-default CVA function $\widetilde{\Theta}=\widetilde{\Theta}(t,x)$ in this solution can be characterized as the unique solution to the following semilinear PDE:

$$\begin{cases}\widetilde{\Theta}(T,x)=0\,,\ x\in\mathbb{R}^q\\ (\partial_t+\mathcal{X})\,\widetilde{\Theta}(t,x)+\bar{g}(t,x,\widetilde{\Theta}(t,x),(\nabla\widetilde{\Theta}\sigma)(t,x),\big((\delta\widetilde{\Theta}\delta(\widetilde{\mathcal{P}}^0)^{\mathsf{T}})\cdot F\big)(t,x))=0\text{ on }[0,T)\times\mathbb{R}^q.\end{cases}\tag{4.20}$$

One then has by virtue of the above analysis,

Proposition 4.2 *Assume existence of a solution $\eta^1_t=\eta^1(\widetilde{\Theta}_{t-})$ to Equation* (4.12) *and invertibility of the group ^{0}-primary-risky-assets-covariance-matrix $\left(\widetilde{\mathcal{P}}^0,\widetilde{\mathcal{P}}^0\right)$. Then the solution $\widetilde{\Theta}=\widetilde{\Theta}(t,x)$ to* (4.20) *yields, via* (4.17)-(4.18) *for η and* (4.1) *for ϵ, a solution $(\widetilde{\Theta},\eta,\epsilon)$ to the pre-default CVA BSDE* (4.1)*, such that $\eta^1_t=\eta^1_t(\widetilde{\Theta}_{t-})$ and $d\widetilde{\mu}-\eta^1_t(\widetilde{\Theta}_{t-})d\widetilde{\mathcal{M}}^1_t-\eta^0_t d\widetilde{\mathcal{M}}^0_t$ is orthogonal to $d\widetilde{\mathcal{M}}^0_t$.*

The min-variance hedge of the CVA (market risk) constrained to perfect hedge of the CVA jump-to-default risk, is thus given as $\big(\eta^0_t,\eta^1_t(\widetilde{\Theta}_{t-})\big)$, where $\eta^1_t(\widetilde{\Theta}_{t-})$ is the assumed solution to (4.12)*, and where $\eta^0_t=\eta^0(t,X_{t-})$ is in turn given by Formula* (4.18)*:*

$$\eta^0_t=\left(\left(\widetilde{\Theta},\widetilde{\mathcal{P}}^0\right)\left(\widetilde{\mathcal{P}}^0,\widetilde{\mathcal{P}}^0\right)^{-1}\right)(t,X_{t-})-\eta^1_t(\widetilde{\Theta}_{t-})\left(\left(\widetilde{\mathcal{P}}^1,\widetilde{\mathcal{P}}^0\right)\left(\widetilde{\mathcal{P}}^0,\widetilde{\mathcal{P}}^0\right)^{-1}\right)(t,X_{t-}).$$

Process $\epsilon=\varepsilon^{\Theta,\eta}$ represents the residual CVA (market) risk under this CVA hedge η.

Hedging of the Contract as a Whole We now consider the constrained min-variance hedging problem of the contract as a whole, rather than simply of its CVA component. We assume further that the hedge ϕ of the contract clean price P, only involves the primary assets in group ^{0}, and that ϕ^0 is given as the coefficient of regression in an $\mathcal{F}$-orthogonal decomposition $dM=\phi^0 d\widetilde{\mathcal{M}}^0+d\bar{e}$, so

$$\phi^0_t=\frac{d{<}M,\widetilde{\mathcal{M}}^0{>}_t}{dt}\left(\frac{d{<}\widetilde{\mathcal{M}}^0{>}_t}{dt}\right)^{-1}=\left(\left(P,\widetilde{\mathcal{P}}^0\right)\Lambda^0\right)(t,X_{t-})=:\phi^0(t,X_{t-}).$$

[7]Up to the monotonicity condition of Remark 4.4, applying here to $\bar{g}$.

For $(\widetilde{\Theta}, \eta, \epsilon)$ as in Proposition 4.2 and for $\zeta := \phi - \eta$, the cost equations (3.24)-(3.26) boil down to

$$\begin{aligned} d\varepsilon_t^{P,\phi} =& dM_t - \phi_t^0 d\widetilde{\mathcal{M}}_t^0 = d\bar{e}_t \\ d\varepsilon_t^{\Theta,\eta} =& d\widetilde{\mu}_t - \eta_t^0 d\widetilde{\mathcal{M}}_t^0 - \eta_t^1(\widetilde{\Theta}_{t-}) d\widetilde{\mathcal{M}}_t^1 = d\epsilon_t \\ d\varepsilon_t^{\Pi,\zeta} =& d\varepsilon_t = d\varepsilon_t^{P,\phi} - d\varepsilon_t^{\Theta,\eta} \\ =& d\widetilde{\nu}_t - \zeta_t^0 d\widetilde{\mathcal{M}}_t^0 + \eta_t^1(\widetilde{\Theta}_{t-}) d\widetilde{\mathcal{M}}_t^1. \end{aligned}$$

Therefore $dM_t - \phi_t^0 d\widetilde{\mathcal{M}}_t^0$ and $d\widetilde{\mu}_t - \eta_t^0 d\widetilde{\mathcal{M}}_t^0 - \eta_t^1(\widetilde{\Theta}_{t-}) d\widetilde{\mathcal{M}}_t^1$ being $\mathcal{F}$-orthogonal to $d\widetilde{\mathcal{M}}_t^0$, implies the same property for $d\widetilde{\nu}_t - \zeta_t^0 d\widetilde{\mathcal{M}}_t^0 + \eta_t^1(\widetilde{\Theta}_{t-}) d\widetilde{\mathcal{M}}_t^1$. Proposition 4.2 thus admits the following

Corollary 4.2 *For ϕ^0 given as the regression coefficient of M against $\widetilde{\mathcal{M}}^0$, the strategy $\zeta_t = \big(\phi^0(t, X_{t-}) - \eta^0(t, X_{t-}), -\eta_t^1(\widetilde{\Theta}_{t-})\big)$, is a min-variance hedge of the contract (market risk), under the contract jump-to-default perfect hedge constraint that $\zeta_t^1 = -\eta_t^1(\widetilde{\Theta}_{t-})$. The residual (market) risk of the contract hedged in this way is given by $\varepsilon^{\Pi,\zeta} = \varepsilon = \bar{e} - \epsilon$.*

Remark 4.6 Under the fully swapped hedge condition (1.4), which in the current Markov setup implies (4.13) through a more specific $\widetilde{g}\big(t, x, \pi, \varsigma\big) = \widetilde{g}\big(t, x, \pi\big)$, the Markovian BSDEs (4.8) and (4.19) both boil down to:

$$\begin{cases} \widetilde{\Theta}(T, X_T) = 0, \text{ and for } t \in [0, T] : \\ -d\widetilde{\Theta}(t, X_t) = \widetilde{g}\left(t, X_t, P(t, X_t) - \widetilde{\Theta}(t, X_t)\right) dt - (\nabla\widetilde{\Theta}\sigma)(t, X_t) dW_t - \delta\widetilde{\Theta}(t, X_{t-}) \cdot dN_t, \end{cases} \tag{4.21}$$

with a related semilinear PDE given as

$$\begin{cases} \widetilde{\Theta}(T, x) = 0\,, \ x \in \mathbb{R}^q \\ (\partial_t + \mathcal{X})\,\widetilde{\Theta}(t, x) + \widetilde{g}(t, x, P(t, x) - \widetilde{\Theta}(t, x)) = 0 \text{ on } [0, T) \times \mathbb{R}^q. \end{cases} \tag{4.22}$$

Note that even though the value $\widetilde{\Theta}$ of the CVA is uniquely defined through (4.21)-(4.22), the hedges η related to it via Propositions 4.1 or 4.2 (resp. ζ via Corollaries 4.1 or 4.2) differ.

Example 4.2 Assuming (4.2) for X given as the pair (r, ℓ) in (4.5), and in the fully swapped hedge case $\widetilde{g}\big(t, x, \pi, \varsigma\big) = \widetilde{g}\big(t, x, \pi\big)$, the generator $\mathcal{X}$ of X in (4.22) writes as follows

$$\begin{aligned} \mathcal{X}\widetilde{\Theta}\,(t, x) =& (a(h(t) - r))\partial_r\widetilde{\Theta} + (a^*(h^*(t) - \ell))\partial_\ell\widetilde{\Theta} + \\ & \int_{\delta,\epsilon>0} \left(\widetilde{\Theta}(t, r + \sigma\delta, \ell + \sigma^*\epsilon) - \widetilde{\Theta}(t, r, \ell)\right) F(d(\delta, \epsilon)) \end{aligned} \tag{4.23}$$

where F stands for the Lévy measure of (L^r, L^ℓ).[8] The CVA Markovian BSDE (4.21) writes: $\widetilde{\Theta}(T, X_T) = 0$, and for every $t \in [0, T]$:

$$\begin{aligned} -d\widetilde{\Theta}(t, X_t) =& \widetilde{g}(t, X_t, \widetilde{\Theta}(t, X_t))dt \\ & - \int_{\delta,\epsilon>0} \left(\widetilde{\Theta}(t, r_{t-} + \sigma\delta, \ell_{t-} + \sigma^*\epsilon) - \widetilde{\Theta}(t, X_{t-})\right) N(dt, d(\delta, \epsilon)) \end{aligned} \tag{4.24}$$

[8] The integral in (4.23) converges under technical conditions stated in Crépey et al. (2012).

where N stands for the compensated jump measure of (L^r, L^ℓ). In case of path dependence, for instance via the collateral Γ (see Subsection 2.3), further state variables can be added to X for the sake of markovianity. From the point of view of numerical solution, deterministic PDE schemes can be used provided the dimension of X is less than 3 or 4; otherwise simulation BSDE schemes are the only viable alternative.

4.4 Unilateral or Bilateral in the End?

The importance of hedging counterparty risk in terms not only of market risk, but also of jump-to-default exposure, was revealed in the aftermath of the 2007–09 credit crisis. But, since selling one's own CDS is illegal, whether it is practically possible to hedge one's own jump-to-default exposure is rather dubious, due to lacking of suitable hedging instruments (apart from the possibility considered in Burgard and Kjaer or Section 5 of the previous paper to repurchase its own bond; and as opposed to hedging its own credit spread, which is possible by factor hedging through peers).

If not possible and/or wished (or in case of unilateral counterparty risk), the bank can resort to a variant of the approach of Subsection 4.3 consisting in min-variance hedging of market risk constrained to perfect hedging of the investor's jump-to-default risk, whilst not hedging its own default. Only hedging the investor's jump-to-default risk means hedging the dJ-term in (4.11) on the random set $\{\bar{\theta} < \theta \wedge T\}$. In view of the CVA cost equation (4.11) and given the specification (3.1) of ξ_t, this boils down to the following univariate explicit linear equation to be satisfied by a scalar process η^1_t, for $t \in [0, T]$:

$$P_t - Q_t - (1 - \bar{\rho}_t)\chi^-_t - \widetilde{\Theta}_{t-} = \eta^1_t\big(\mathcal{R}^1_t - \widetilde{\mathcal{P}}^1_t\big). \tag{4.25}$$

Min-variance hedging the CVA market risk of the contract (or of the contract as a whole) subject to perfect hedge of the investor's isolated jump-to-default, thus boils down to min-variance hedging the CVA market risk of the contract (or of the contract as a whole) subject to $\eta^1_t = \eta^1_t(\widetilde{\Theta}_{t-})$ solving (4.25). This min-variance hedging can be implemented along similar lines as in Subsection 4.3, yielding easily derived analogs of Proposition 4.2 and Corollary 4.2. Note this involves no technical condition like (4.13).

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