UP AND DOWN CREDIT RISK

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1 Introduction

Presently, most if not all credit portfolio derivatives have cash flows that are determined solely by the evolution of the cumulative loss process generated by the underlying portfolio. Thus, as of today, credit portfolio derivatives can be considered as derivatives of the cumulative loss process L. The consequence of this is that, as of today, most of the models of portfolio credit risk, and related derivatives, focus on eventual modeling of the dynamics of the process L, or, directly on modeling of the dynamics of the related conditional probabilities, such as

$$\text{Prob}(L \text{ takes some values at future time(s) given present information}).$$

In this paper we shall study various methodologies that have been developed for this purpose (see for instance, among so many others, the references in the bibliography; specific comments will be given in the course of the paper). In addition, we shall discuss the issue of *hedging* of loss process derivatives, and we shall argue that loss process may not provide a sufficient basis for this, in the sense described later in the paper. In fact, we shall engage in some in depth study of the role of information with regard to valuation and hedging of derivatives written on the loss process.

1.1 Outline of the Paper

In Section 2 we provide an overview of the main modeling approaches that have been developed so far for handling portfolio credit derivatives. In Section 3 we provide some mathematical insights to the fact that information (namely, the choice of a relevant model filtration) is the major issue in this regard. In Section 4 we illustrate this on simple mathematical examples. In Section 5 we illustrate further by means of numerical simulations (semi-static hedging experiments) that the loss process L is not a sufficient statistics for the purpose of valuation and hedging of portfolio credit risk, as soon as L is not a Markov process. Conclusions are drawn in Section 6.

1.2 Standing Notation

Considering a pool (portfolio) of n credit names, we denote by τ_i the default time corresponding to the i^{th} name, by $H^i_t = \mathbb{1}_{\tau_i \le t}$ and $J^i_t = 1 - H^i_t = \mathbb{1}_{t<\tau_i}$ the related default and non-default indicator processes (raw processes in the sense of random functions, without reference to any filtration yet), and by R_i a related (possibly random) recovery at default. We define the *cumulative default process* N and the *cumulative loss process* L by $N_t = \sum_{i=1}^n H^i_t$ and $L_t = \sum_{i=1}^n (1 - R_i)H^i_t$, respectively. Note that except in Section 5 we shall assume that $R_i = 0$ for each i, so that $L_t = \sum_{i=1}^n H^i_t$ (the cumulative loss process L reduces to the cumulative default process N).

From now on, t will denote the present time, and $T > t$ will denote some future time. Suppose that ξ^t represents a cumulative ex-dividend cash flow on the time interval $(t, T]$,[1] which will be derived from the evolution of the loss process L, and representing a specific credit portfolio derivative claim. We may have at least two tasks at hand:

- to compute the time-t price of the claim, given the information that we may have available and we are willing to use at time t;
- to hedge the claim at time t. By this, we mean computing hedging sensitivities of the claim with respect to hedging instruments that are available and that we may want to use.

For simplicity we shall assume that we use spot martingale measure, say $\mathbb{P}$, for pricing, and that the interest rate is zero. Thus, denoting by $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ a filtration that represents flow of

[1] So $\xi^t = \xi$ if there are no dividends but only a terminal payment; note however that most credit products are swapped and involve therefore coupon streams.

information we use for pricing, and by $\mathbb{E}$ expectation relative to $\mathbb{P}$ (ξ^t being assumed $\mathcal{F}_T$-measurable and $\mathbb{P}$-integrable), the pricing task amounts to computation of the conditional expectation $\mathbb{E}(\xi^t|\mathcal{F}_t)$. Finally, if X is a given process, we denote by $\mathbb{F}^X$ its natural filtration satisfying usual conditions (perhaps after completion and augmentation).

We denote $\mathbb{N}_k = \{0, \cdots, k\}$, for any positive integer k.

2 Top and Bottom Approaches: an Overview

Various approaches to valuation of derivatives written on credit portfolios differ between themselves depending on what is the content of filtration $\mathbb{F}$. Thus, loosely speaking, these approaches differ between themselves depending on what they take (presume) to be sufficient information so to price (and consequently to hedge) credit portfolio derivatives.

2.1 Top and Top-Down Approaches

The approach, that we dub the *pure top* approach takes as $\mathbb{F}$ the filtration generated by the loss process alone. Thus, in the pure top approach we have that $\mathbb{F} = \mathbb{F}^L$. Examples are Laurent, Cousin and Fermanian [40], Cont and Minca [15], most of Herbertsson [35] or van der Voort [48].

The approach that we dub the *top* approach takes as $\mathbb{F}$ the filtration generated by the loss process and by some additional relevant (preferably low dimensional) *auxiliary factor process*, say Y. Thus, in this case, $\mathbb{F} = \mathbb{F}^L \vee \mathbb{F}^Y$. Examples are Bennani [3], Schönbucher [46], Sidenius, Piterbarg and Andersen [47], Arnsdorf and Halperin [2], Lopatin and Misirpashaev [43] or Ehlers and Schönbucher [22].

It appears that works devoted to pure–top/top approaches focus on valuation issues alone, and that they fail to address the key issue of hedging, in particular the issue of hedging of credit portfolio derivatives by vanilla individual contracts (such as default swaps).

To address this issue, the so-called *top-down* approach starts from *top*, that is, it starts with modeling of evolution of the portfolio loss process subject to information structure $\mathbb{F}$. Then, it attempts to 'decompose' the dynamics of the portfolio loss process *down* on the individual constituent names of the portfolio, so to deduce the dynamics of processes H^i. This is done by a method of random thinning formalized in Giesecke and Goldberg [29]. Further illustration is given in Ding, Giesecke and Tomecek [20] and Errais, Giesecke and Goldberg [24]. This approach is also advocated by Halperin in [32].

2.2 Bottom-Up Approaches

The approach that we dub the *pure bottom-up* approach takes as $\mathbb{F}$ the filtration generated by the state of the pool process $H = (H^1, \dots, H^n)$, i.e., $\mathbb{F} = \mathbb{F}^H$ (see, for instance, Herbertsson [34]).

The approach that we dub the *bottom-up* approach takes as $\mathbb{F}$ the filtration generated by process H and by an *auxiliary factor process* Z. Thus, in this case, $\mathbb{F} = \mathbb{F}^H \vee \mathbb{F}^Z$. Examples are Bielecki, Crépey, Jeanblanc and Rutkowski [4], Bielecki, Vidozzi and Vidozzi [10], Frey and Backhaus [25, 26], Duffie and Garleanu [21] or Gaspar and Schmidt [28].

Remark 2.1 A bottom-up model may be such that

$$\mathbb{F}^H \subseteq \mathbb{F}^Z. \tag{1}$$

For example, take $n = 2$, and take three positive random variables: $\sigma_j, j = 1, 2, 3$. Next, define $Z^j_t = \mathbb{1}_{\sigma_j \le t}$, $j = 1, 2, 3$. Finally, let $\tau_1 = \sigma_1 \wedge \sigma_3$ and $\tau_2 = \sigma_2 \wedge \sigma_3$. We can interpret σ_1 and

σ_2 as 'idiosyncratic default times,' and we can interpret σ_3 as a 'systemic default time.' Letting $Z = (Z^j)_{1 \le j \le 3}$, we see that (1) holds.

2.2.1 Interacting Particles Approaches

As an aside to bottom-up approaches, let us mention the interacting particles approaches (see Liggett [41] and [42] for a general reference, and Giesecke and Weber [30] or Frey and Backhaus [27] for applications to portfolio credit derivatives). Experience seems to show however that interacting particle models are not appropriate for risk management of portfolio credit derivatives. We can see two reasons for this:
• Firstly, interacting particle models ultimately rely on *homogeneity* assumptions which are obviously not satisfied in the case of credit portfolios, in general. Attempts to turn round this shortcoming by considering sub-group of homogeneous obligors face the difficulty that there is no way to determine such groups in a manner which would be consistent across time; for instance, economic sectors do not define groups of obligors which would be homogeneous in terms of credit risk, whereas homogeneous groups which would be defined by tranching the range of CDS spreads would vary over time (note however that a homogeneous groups set-up will be fruitfully used for numerical illustration purposes in Section 5);
• Secondly, the kind of contagion typically embedded in interacting particle systems (*nearest neighbor interaction* as of Liggett [41]) is not appropriate, neither quantitatively (not enough contagion and frailty) nor qualitatively, for portfolio credit derivatives management.
It is possible that interacting particle approaches might be of interest for *large portfolio credit value at risk assessment* (rather than credit derivatives management), however (see Dai Pra, Runggaldier, Sartori and Tolotti [17]).

Remark 2.2 On a different note, interacting particles approaches also lead to generic importance sampling techniques that can fruitfully be applied to simulation in the context of dynamic Markovian models of portfolio credit risk (see Crépey and Carmona [16]).

2.3 Discussion

To discuss the previous approaches a prerequisite is to provide analysis criteria as for what a good credit basket model (or credit portfolio model) should be:
• Firstly, a good model should of course contain *the right inputs*, namely the inputs with respect to which the trader wishes to compute sensitivities or *Greeks* (typically sensitivities with respect to index and/or CDS spreads in the case of CDO tranches, etc.);
• Secondly, a good model should be *calibrable* to the market consistently over time, since consistency or *robustness* of calibrated parameters over time effectively means that a model produces the right Greeks (this can be considered as a heuristic principle largely valid in practice: in any class of models achieving consistent calibration to the market, one gets essentially the same Greeks);
• Thirdly, pricing and calibration (the latter is of course the most demanding) should be doable *in real time*.

Now, from the pricing perspective, the pure top approach is undoubtedly the best suited for fast calibration and fast valuation, as it only refers to a single driver – the loss process itself. However, it probably produces incorrect pricing results, as it is rather unlikely that financial market evaluates derivatives of the loss process based only on the history of evolution of the loss process alone. Note in particular that loss process is not a traded instrument. Thus, it seems to be necessary to work with a larger amount of information than the one carried by filtration $\mathbb{F}^L$ alone. This is quite likely the reason why several versions of the top approach have been developed. Enlarging filtration from $\mathbb{F}^L$ to $\mathbb{F}^L \vee \mathbb{F}^Y$ may lead to increased computational complexity, but at the same time it is rather sure to increase accuracy in calculation of important quantities, such as CDO tranche spreads and/or CDO prices.

From the hedging perspective both the pure top approach and the top approach appear to be far inadequate. Since the loss process is not a traded security, a user of the top approaches is forced to hedge one derivative of the loss process, say ξ, with another loss process derivative, say χ, which is available for (liquid) trading. This may not be such a good idea since, for one, it is only possible to compute sensitivities of ξ with respect to χ indirectly, via sensitivities of ξ and χ with respect to L, so that hedging may not be quite precise. Moreover, this kind of hedging may be quite expensive (e.g., hedging a CDO tranche using iTraxx).

Remark 2.3 It is possible however that effective pricing and hedging of derivatives written on derivatives of loss process, such as CDO options, can be achieved using the top approaches (this is actually the most common market practice in this regard) . Yet this statement should be considered with caution and this issue should be thoroughly investigated.

Operating on the top level definitely prohibits computing sensitivities of a loss process derivative with respect to constituents of the portfolio of credits generating the loss process in question. So, for example, when operating just on top level one can't compute sensitivities of CDO tranche prices with respect to prices of the CDS contracts underlying the portfolio. This is of course the problem that led to the idea of the *top-down approach*, that is the idea of *thinning*. However one of the purpose of this paper (see Section 3.6 in particular) is to show that the top-down approach is quite misguided. In fact the opinion developed in this paper is that only the *bottom-up approaches* allow adequate hedging of portfolio credit derivatives with respect to the constituents of the portfolio.

3 Thinning

Note that processes H^i are sub-martingales with respect to any filtration for which they are adapted, as non-decreasing processes, and therefore they can be compensated with respect to any filtration for which they are adapted *Thinning* refers to the recovery of individual compensators, starting from the loss compensator as input data. Since the compensator is an information- (filtration-) dependent quantity, thinning of course depends on the filtration under consideration.

3.1 Set-up

We fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions. On this space we consider an $\mathbb{F}$-adapted point process $\widetilde{L}$ (a non-decreasing process that takes values in $\mathbb{N}$ and whose jumps are of size one, see, e.g., Brémaud [13] or Last and Brandt [39]), and we denote by $\widetilde{\Lambda}$ the $\mathbb{F}$-*compensator* (See section A.2 in the Appendix) of $\widetilde{L}$. We impose mild technical conditions on $\widetilde{\Lambda}$ to the effect that process $\widetilde{L}$ is cadlag, so that in particular, it only admits a finite number of jumps in any bounded time interval and does not admit jumps of the second kind (since it has left limits and is right continuous). In particular, we assume that $\widetilde{\Lambda}$ continuous.

In credit portfolio applications we only deal with a finite number, say n of credit names. Because of this, we are only interested in studying process $\widetilde{L}$ through its n-th jump. Towards this end, let us denote by ϑ_i, $i = 1, 2, \ldots, n, \ldots,$ the consecutive jump times of process $\widetilde{L}$, and let us denote by L process $\widetilde{L}$ stopped at ϑ_n, that is

$$L = \widetilde{L}_{\cdot \wedge \vartheta_n} = \sum_{i=1}^{n} \mathbb{1}_{\vartheta_i \leq t} \ .$$

The $\mathbb{F}$-compensator of process L, say Λ, is then given as

$$\Lambda = \widetilde{\Lambda}_{\cdot \wedge \vartheta_n},$$

and the process

$$M := L - \Lambda \tag{2}$$

is a uniformly integrable $\mathbb{F}$-martingale (see, e.g., Theorem 11 page 112 of Protter [45]).

3.1.1 Unordered versus Ordered Default Times

Next, we denote by $\tau_i,\ i = 1, 2, \ldots, n,$ an arbitrary collection of (mutually avoiding) random times on $(\Omega, \mathcal{F}, \mathbb{P})$, and by $\tau_{(i)},\ i = 1, 2, \ldots, n,$ we denote the corresponding ordered sequence, that is $\tau_{(1)} < \tau_{(2)} < \cdots < \tau_{(n)}$. We denote as usual $H^i_t = \mathbb{1}_{\tau_i \le t}$. Accordingly, we set $H^{(i)}_t = \mathbb{1}_{\tau_{(i)} \le t}$. So, obviously, $\sum_{i=1}^n H^i = \sum_{i=1}^n H^{(i)}$. Note that, since $\widetilde{L}$ (hence L) is assumed to be $\mathbb{F}$-adapted in this section, thus the $\tau_{(i)}$'s are $\mathbb{F}$-stopping times.

In what follows process L models the loss process. In this regard observe that at this point we do not define process L as

$$L_t = \sum_{i=1}^n H^i_t. \tag{3}$$

In the top approaches such representation is secondary, and not always needed. Note however that the question of existence of representation of the form (3), for an $\mathbb{F}$-adapted point process L stopped at level n (supplemented by suitable Markov property requirements), is by no means obvious.

The following Remark is, of course, elementary,

Remark 3.1 We have

$$L = \sum_{i=1}^n H^i \tag{4}$$

if and only if

$$\vartheta_i = \tau_{(i)},\ i = 1, 2, \ldots, n. \tag{5}$$

It is clear that for a given process L there may be multiple families of random times $\tau_i,\ i = 1, 2, \ldots, n,$ for which equation (4) is satisfied. For example, in the case where $n = 2$ and ϑ_1 and ϑ_2 are *constants* with $\vartheta_1 < \vartheta_2$, the particular choice

$$\tau_1 = \vartheta_1 \mathbb{1}_{\omega \in \Omega_1} + \vartheta_2 \mathbb{1}_{\omega \in \Omega_2}\ ,\ \tau_2 = \vartheta_2 \mathbb{1}_{\omega \in \Omega_1} + \vartheta_1 \mathbb{1}_{\omega \in \Omega_2},$$

where $\{\Omega_1, \Omega_2\}$ is any measurable partition of Ω, gives a family of times $\tau_i, i = 1, 2$, such that

$$\tau_{(1)} = \tau_1 \wedge \tau_2 = \vartheta_1,\ \tau_{(2)} = \tau_1 \vee \tau_2 = \vartheta_2.$$

It is important to note that any of the random times $\tau_i,\ i = 1, 2, \ldots, n,$ may or may not be an $\mathbb{F}$-stopping time. For example, if $\mathbb{F}$ coincides with the filtration generated by process L, then, unless all times $\tau_i,\ i = 1, 2, \ldots, n,$ are ordered, at least one of them is not an $\mathbb{F}$-stopping time. Of course, all of the times $\tau_{(i)},\ i = 1, 2, \ldots, n,$ are $\mathbb{F}$-stopping times in this case. As a matter of fact, since knowledge of the loss process L is equivalent to knowledge of its jump times, it follows from Remark 3.1 that the knowledge of loss process is equivalent to the knowledge of the ordered sequence $\tau_{(i)},\ i = 1, \ldots, n$, as long as (5) holds for this sequence.

From now on we shall only consider sequences $\tau_i,\ i = 1, 2, \ldots, n,$ for which (4) holds.

3.2 Motivation

A preliminary question regarding thinning is why would one wish to know the individual compensators. The answer depends on one's objectives.

3.2.1 Pricing

Suppose that all one wants to do is to compute the expectation $\mathbb{E}(\xi|\mathcal{F}_t)$ for $0 \leq t < T$, where the integrable random variable $\xi = \pi(L_T)$ represents the stylized payoff of a portfolio loss derivative. In general, this is not an easy task. Sometimes, exact formulas may be available for $\mathbb{E}(\xi|\mathcal{F}_t)$. But in general, computation of such expectations will need to be done by simulation. Since the value of ξ does not depend on identities of defaulting names, computing of the expectation $\mathbb{E}(\xi|\mathcal{F}_t)$ by simulation will only require simulation of the process L, which is the same as simulation of the sequence $\tau_{(i)}$, $i = 1, 2, \ldots, n$. If one additionally makes Markovian assumptions, or conditionally Markovian assumptions (assuming further factors Y), about process L with respect to the filtration $\mathbb{F}$, then, in principle, the expectation $\mathbb{E}(\xi|\mathcal{F}_t)$ can be computed (at least numerically). The point is that for computation of $\mathbb{E}(\xi|\mathcal{F}_t)$, one does not really need to know the individual compensators. So, with regard to the problem of pricing of derivatives of the loss process, a top model may be fairly adequate. In particular, the filtration $\mathbb{F}$ may not necessarily contain the pool filtration $\mathbb{H}$. Also, the representation $L = \sum_{i=1}^{n} H^i$ need not be considered at all in this context.

3.2.2 Hedging

But there is a fundamental reason why one may need to know the individual $\mathbb{F}$-compensators Λ^i's of the τ^i's (assuming here τ^i are $\mathbb{F}$ stopping times). Computing the price $\mathbb{E}(\xi|\mathcal{F}_t)$ is just one task of interest, which of course is important in the context of valuation of derivatives written on credit portfolio. Yet the key task is *hedging*. From the mathematical point of view hedging relies on the derivation of a *martingale representation* of $\mathbb{E}(\xi|\mathcal{F}_t)$, which is useful in the context of computing sensitivities of the price of ξ with respect to changes in prices of (liquid) instruments, such as CDS contracts, corresponding to the credit names composing the credit pool underlying the loss process L. Sensitivities computed in this way account for both spread risk and jump-to-default risk.

Typically, one will seek a martingale representation in the form

$$\mathbb{E}\left(\xi|\mathcal{F}_t\right) = \mathbb{E}\xi + \sum_{i=1}^{n} \int_0^t \zeta_s^i dM_s^i + \sum_{j=1}^{m} \int_0^t \eta_s^j dN_s^j, \tag{6}$$

where the M^i's are some fundamental martingales associated with the non-decreasing processes H^i's, and the N^j's are some fundamental martingales associated with all relevant auxiliary factors included in the model. The coefficients ζ^i's and η^j's can, in principle, be computed given a particular model specification; now, for this, one will typically need to know the compensators Λ^i's (see Section 4 for illustrative examples).

3.2.3 Multi-Name versus Single-Name Credit

We now assume that, for some i, $\mathbb{F}$ can be decomposed as $\mathbb{F}^i \vee \mathbb{H}^i$, where $\mathbb{H}^i = (\mathcal{H}_t^i)_{t\geq 0}$ is the filtration generated by $H_t^i = \mathbb{1}_{\tau_i \leq t}$ (in particular, τ_i is an $\mathbb{H}^i$, hence an $\mathbb{F}$-stopping time) and $\mathbb{F}^i$ is a reference filtration. Let further ξ stand for a $\mathcal{F}_T^i$ measurable, integrable random variable. Thinking of *single-name* credit (see, e.g., Bielecki et al. [6]), one might think, under this assumption, that for the computation of quantities like $\mathbb{E}(J_T^i \xi|\mathcal{F}_t)$ (recall $J^i = 1 - H^i$) for $t < T$, the knowledge of the compensator Λ^i of τ_i with respect to $\mathbb{F}$ may be quite useful.[2] To see why, let us first denote $G_t^i = \mathbb{P}(\tau_i > t|\mathcal{F}_t^i)$. Assuming (w.l.o.g.) that G^i is strictly positive, we define the corresponding hazard process

$$\Gamma_t^i = -\ln G_t^i \ . \tag{7}$$

The importance of the hazard process comes, among other reasons, from the fact that using this process we can provide the following convenient representation:

$$\mathbb{E}(J_T^i \xi|\mathcal{F}_t) = J_t^i \mathbb{E}(e^{\Gamma_t^i - \Gamma_T^i} \xi|\mathcal{F}_t^i). \tag{8}$$

[2] By the compensator of a random time τ we mean the compensator of the one point process $\mathbb{1}_{\tau \leq t}$.

Now, under some additional assumptions (see discussion below), the $\mathbb{F}$-compensator of τ_i coincides with the $\mathbb{F}$-hazard process of τ_i, that is $\Gamma^i = \Lambda^i$. This would be one of the reasons why sometimes one may want to compute processes Λ^i.

Sufficient conditions ensuring $\Gamma^i = \Lambda^i$ are that Γ^i is a continuous and non-decreasing process, where these requirements are typically met by postulating that τ^i is an $\mathbb{F}^i$-pseudo-stopping time which avoids $\mathbb{F}^i$-stopping times (conversely, the continuity of Γ^i implies that τ^i avoids $\mathbb{F}^i$-stopping times; see Coculescu and Nikeghbali [14]).

Recall that the $\mathbb{F}^i$-random time τ^i being an $\mathbb{F}^i$-pseudo-stopping means that $\mathbb{F}^i$-martingales stopped at τ^i are $\mathbb{F}$-martingales (see Nikeghbali and Yor [44]). This is of course satisfied when $\mathbb{F}^i$-martingales are $\mathbb{F}$-martingales, namely, when *immersion* (also referred to as *the* $(\mathcal{H})$ *Hypothesis*) holds between $\mathbb{F}^i$ and $\mathbb{F}$. Now, in the case of multi-name credit risk, the typical situation is that $\mathbb{H}^j \subset \mathbb{F}^i$ for $j \neq i$. In this case immersion typically does *not* hold between $\mathbb{F}^i$ and $\mathbb{F}$ (unless we are in degenerate situations like the τ_i's being either ordered or independent, cf. Ehlers and Schönbucher [22]; see Proposition 4.2 and the comments following it for a concrete example). Moreover, τ_i is typically not an $\mathbb{F}^i$-pseudo-stopping time either. So the identity $\Gamma^i = \Lambda^i$ may not hold and identity (8) may not be exploited, fault of knowing Γ^i.

3.3 Ordered Thinning of Λ

Let $\Lambda^{(i)}$ denote the $\mathbb{F}$-compensator of $\tau_{(i)}$ (recall that the $\tau_{(i)}$ are $\mathbb{F}$-stopping times).

Proposition 3.1 *Assuming* Λ *continuous,*[3] *we have, for* $t \geq 0$,

$$\Lambda^{(i)}_t = \Lambda_{t\wedge\tau_{(i)}} - \Lambda_{t\wedge\tau_{(i-1)}}. \tag{9}$$

So in particular $\Lambda^{(i)} = 0$ *on the set* $t \leq \tau_{(i-1)}$.

Proof. Note first that

$$L_{t\wedge\tau_{(i)}} - \Lambda_{t\wedge\tau_{(i)}} \tag{10}$$

is an $\mathbb{F}$-martingale for every $i = 1, 2, \ldots, n,$ as it is equal to the $\mathbb{F}$-martingale M (cf. (2)) stopped at the $\mathbb{F}$-stopping time $\tau_{(i)}$. Taking the difference between expression in (10) for i and $i-1$ yields that $H^{(i)}_t - \widehat{\Lambda}^{(i)}_t$, with $\widehat{\Lambda}^{(i)}_t$ defined as the RHS of (9), is an $\mathbb{F}$-martingale (stopped at $\tau_{(i)}$). Hence (9) follows, due to uniqueness of compensators. □

Formula (9) represents the 'ordered thinning' of Λ. Note that Remark 3.1 and (for Λ continuous) formula (9) are true regardless of whether τ_i's are $\mathbb{F}$-stopping times or not. This reflects the simple truth that modeling the loss process L is the same as modeling the ordered sequence $\tau_{(i)}$, $i = 1, \ldots, n,$ no matter what is the informational context of the model otherwise.

3.4 $\mathbb{F}$-Thinning of Λ

In this section and in the next one we only consider the case that each random time τ_i is an $\mathbb{F}$-stopping time, and we are interested in (unordered) thinning, that is computing the compensators of τ_i relative to various sub-filtrations of $\mathbb{F}$, which respect to which τ_i is a stopping time, starting from the process Λ.

Suppose that τ_i, $i = 1, 2, \ldots, n,$ are $\mathbb{F}$-stopping times and let $\mathbb{H}^i$ denote the filtration generated by H^i. Assume further, for every $i = 1, \cdots, n$, a decomposition $\mathbb{F} = \mathbb{H}^i \vee \mathbb{F}^i$, where $\mathbb{F}^i$ is a strict

[3] This is equivalent to the $\tau_{(i)}$'s being *totally inaccessible* $\mathbb{F}$-stopping times; see, for instance, Dellacherie and Meyer [19].

sub-filtration of $\mathbb{F}$. Now, let us denote by Λ^i the $\mathbb{F}$-compensator of τ_i. We of course have that

$$\Lambda = \sum_{i=1}^{n} \Lambda^i. \tag{11}$$

Moreover, the following is true.

Proposition 3.2 (Giesecke and Goldberg [29]) *An $\mathbb{F}$-predictable, non-negative and non-decreasing process Λ is the compensator of L if and only if there exist $\mathbb{F}$-predictable non-negative processes Z^i, $i = 1, 2, \ldots, n$, such that $Z^1 + Z^2 + \cdots + Z^n = 1$ and*

$$\Lambda^i = \int_0^{\cdot} Z_t^i d\Lambda_t,\ i = 1, 2, \ldots, n. \tag{12}$$

Proof (Sketched). Use (11) and set $Z^i = \frac{d\Lambda^i}{d\Lambda}$ for the "only if" part; $\mathbb{F}$-predictability of Z^i is apparent in the Airault–Föllmer representation of Z^i, see Theorem 4.7 in Airault and Föllmer [1] or Giesecke and Goldberg [29]. The "if" part is obvious. □

In the special case where random times τ_i, $i = 1, 2, \ldots, n$, constitute an ordered sequence, then the ordered thinnning formula (9) yields that $Z_t^i = \mathbb{1}_{\tau_{i-1} < t \le \tau_i}$. In general, it is possible to interpret Z^i as the *probability that i is the next name to default, conditional on the imminence of a default* (this is also apparent in the Airault–Föllmer representation of Z^i).

Proposition 3.2 tells us that, if one starts building a model from top, that is, if one starts building the model by first modeling the $\mathbb{F}$-compensator Λ of the loss process L, then the only way to go down relative to the information carried by $\mathbb{F}$, i.e., to obtain $\mathbb{F}$-compensators Λ^i, is to do thinning in the sense of equation (12). We shall refer to this as to $\mathbb{F}$-*thinning* of Λ.

Remark 3.2 : When Top-Down becomes to Bottom-Up? Given that all τ_i's are $\mathbb{F}$-stopping times, this thinning is of course equivalent to building the model from the bottom up. That is, modeling processes Λ, Z^i, $i = 1, 2, \ldots, n$, is equivalent to modeling processes Λ^i, $i = 1, 2, \ldots, n$.

3.5 $\mathbb{F}^i$-Thinning of Λ

Now, suppose that $\mathbb{F}^i$ is some sub-filtration of $\mathbb{F}$ and that τ_i is also an $\mathbb{F}^i$-stopping time. We want to compute the $\mathbb{F}^i$-compensator Γ^i of τ_i, starting with Λ.

The first step is to do the $\mathbb{F}$-thinning of Λ, that is, to obtain the $\mathbb{F}$-compensator Λ^i of τ_i (cf. Section 3.4, formula (12)). The second step is to obtain the $\mathbb{F}^i$-compensator Γ^i of τ_i from Λ^i. Towards this end we denote by ${}^{o_i}\Lambda^i$ the optional projection of Λ^i on the sub-filtration $\mathbb{F}^i$ (see Section A.1). Since Λ^i is non-decreasing, ${}^{o_i}\Lambda^i$ is a $\mathbb{F}^i$-submartingale, thus it admits a Doob-Meyer decomposition. Denoting[4] by $\left({}^{o_i}\Lambda^i\right)^{p_i}$ the $\mathbb{F}^i$-compensator (in the sense of the predictable non-decreasing component) of the $\mathbb{F}^i$-submartingale ${}^{o_i}\Lambda^i$, we thus have the following result, by application of Proposition A.1 in the Appendix.

Proposition 3.3 *Γ^i and Λ^i denoting the $\mathbb{F}^i$- and the $\mathbb{F}$- compensators of τ_i, one has,*

$$\Gamma^i = \left({}^{o_i}\Lambda^i\right)^{p_i} . \tag{13}$$

Moreover, in case Γ^i and Λ^i are time-differentiable with related $\mathbb{F}^i$- and $\mathbb{F}$- intensity processes γ^i and λ^i, then γ^i is the $\mathbb{F}^i$-optional projection of λ^i (see Section A.1), so

$$\gamma^i = {}^{o_i}\lambda^i . \tag{14}$$

[4] with a slight abuse of notation, see Section A.2.

3.5.1 Calibration Issues

The above relations are important regarding the issue of *calibration* of a model to marginal data, one of the key issues in financial modeling.

For example, one may want to calibrate the credit portfolio model to spreads on individual CDS contracts. If the spread on the i^{th} CDS contract is computed using conditioning with respect to $\mathbb{F}^i$, then the $\mathbb{F}^i$-compensator Γ^i of τ_i will typically be used in calibration, solving (14) in λ^i with γ^i observed on the market. We refer the reader to the comments following Proposition 4.2 for an illustration in a pure bottom-up situation where the individual CDS spreads reflect only information relevant to the given obligor, so in this case $\mathbb{F}^i = \mathbb{H}^i$.

3.6 The case when τ_i's are not $\mathbb{F}$-stopping times

Finally, let us consider the case that at least one τ_i is not an $\mathbb{F}$-stopping time.

The point we want to make here is that if the model is built from top, and if the filtration $\mathbb{F}$ does not provide information about τ_i, then no credit derivative, say ξ, built off the loss process L, in the sense that $\xi \in \mathcal{F}^L_T$, can be hedged only by instruments which derive their value solely from τ_i, such as CDS contracts on τ_i (assuming deterministic recovery).

Giesecke and Goldberg [29] introduce a notion of (we call it *top-down*) intensity of τ_i (even though τ_i is not $\mathbb{F}$-stopping time), defined as the time-derivative, assumed to exist, of the compensator (in the sense of the $\mathbb{F}$-predictable non-increasing component) of the optional projection ${}^{o_i}H^i$ of H^i on $\mathbb{F}$. However, except in the case where τ_i is an $\mathbb{F}$-stopping time and top-down boils down to bottom-up (see Remark 3.2), this notion of intensity cannot be identified with the intensity of name i as extracted from the related marginal market data (market CDS curve on name i). Indeed the market intensity of name i obviously corresponds to an intensity in a filtration adapted to τ_i. So the top-down intensity of τ_i is not represented in the market, and therefore there is no way one may hope to calibrate such a top-down intensity. The most striking illustration of this corresponds to the simple fact that outside the special bottom-up case of τ_i being an $\mathbb{F}$-stopping time, *a top-down intensity typically fails to vanish after* τ_i.

4 Explicit Examples

There are two major (and rather natural) messages in the previous section:
• The concept of thinning of the compensator of the loss process so to obtain compensators of the individual default times makes sense only if there is enough information to do so. However, the 'enough information' requirement renders the thinning really irrelevant;
• Insufficient information about the pool of credit names does not allow for hedging with respect to individual names.

It is also crucial to emphasize the following observations:
— Representation (6) is key to computing hedging ratios for $\mathbb{E}(\phi(L_T)|\mathcal{F}_t)$ with respect to instruments derived from the sub-pools of the pool of given n credits (in particular, with respect to individual instruments, such as CDS contracts);
— Such a representation can't be obtained if the model is not a bottom-up type model, since in this case the fundamental martingales M^i are no more available, so they cannot be used in (6).

We shall illustrate these points in simple set-ups of the pure bottom-up, bottom-up, pure top and top models, generally involving only two random times τ_1 and τ_2, for simplicity of presentation. The extension of the results to n random times is straightforward.
In particular, we are going to provide various ways of shedding a dynamic perspective on two random times τ_1 and τ_2, introducing in each case related $\mathbb{F}$-adapted point processes assumed to admit ($\mathbb{F}$-predictable, without loss of generality [13, 45]) $\mathbb{F}$-*intensity processes*. The dynamic perspective is

important if one is interested in hedging credit portfolio derivatives (e.g. CDOs), as well as if one is interested in pricing and hedging derivatives on credit portfolio derivatives (e.g. an option on a CDO tranche).

In every set-up (pure bottom-up, bottom-up, pure top and top) there will be basically two practical ways of 'dynamizing' τ_1 and τ_2: the *Markovian* approach, mainly, but also, under certain circumstances, a *distributional* approach (also exploited for various purposes in Jiao [38], El Karoui et al. [23] or Jeanblanc and Le Cam [37]).
The Markovian approach relies on the possibility to perform suitable Markovian change of measure, starting from models in which all the ingredients (default related point processes, auxiliary factor process if any) are independent. The model primitives are the *generator* of the related Markovian factor process, or equivalently, in pure (top or bottom) approaches, the $\mathbb{F}$-*intensities* of the point processes at hand. In the distributional approach the model primitives are related marginal and/or joint distributions (in pure top or bottom approaches) or conditional distributions (when there are auxiliary factor processes involved) functions, and relies on suitable regularity assumptions on these distributions. Incidentally, the connection between the two approaches is a delicate issue only partially addressed in this paper (we can only conclude from our examples that the two approaches are non-inclusive).

4.1 Notation

In order to discuss hedging, we introduce (Borel-measurable and bounded, say) *loss payoff functionals* π, ϕ and ψ. Here the idea is to hedge the claim with payoff $\pi(L_T)$ at T by the one with payoff $\phi(L_T)$ at T, and, possibly also, the one with payoff $\psi(L_T)$ at T. We denote the pricing process $\Pi_t = \mathbb{E}(\pi(L_T)|\mathcal{F}_t)$, and we introduce likewise Φ and Ψ. The *tracking error* $e = e^\zeta$ of the (self-financing) dynamic hedging strategy $\zeta = (\zeta^1, \zeta^2)$ based on Φ and Ψ (and the riskless, constant asset) satisfies, for $t \in [0,T]$

$$de_t = d\Pi_t - \zeta_t d \begin{pmatrix} \Phi_t \\ \Psi_t \end{pmatrix} \tag{15}$$

(and $e_0 = 0$). In particular, restricting oneself to single-instrument hedges, one can min-variance hedge the π-claim by the ϕ-claim and the riskless asset (so $\zeta^2 = 0$, here) by using the strategy ζ^1 in Φ defined by, for $t \in [0,T]$ (cf. [7] page 85):

$$\zeta_t^1 = \frac{d\langle \Pi, \Phi \rangle_t}{d\langle \Phi \rangle_t} \; . \tag{16}$$

Of course, the analysis of the tracking error will depend, in particular, on the information (filtration $\mathbb{F}$) which is used.

4.2 Bottom-Up Approaches

In the bottom-up approaches, τ_1 and τ_2 are $\mathbb{F}$-stopping times, and H_1 and H_2 are therefore $\mathbb{F}$-adapted processes. We denote by M^1 and M^2 the $\mathbb{F}$ – compensated martingales of H^1 and H^2, so

$$M^1 = H^1 - \int_0^{\cdot} J_t^1 \widetilde{\lambda}_t^1 dt \; , \; M^2 = H^2 - \int_0^{\cdot} J_t^2 \widetilde{\lambda}_t^2 dt \tag{17}$$

where $\widetilde{\lambda}^1$ and $\widetilde{\lambda}^2$ are the *pre-default* $\mathbb{F}$-*intensities* of τ^1 and τ^2.

We denote by $\imath = (i_1, i_2)$ a generic pair in $\{0,1\}^2$. Moreover for every $\imath = (i_1, i_2) \in \{0,1\}^2$ we denote $\jmath = (j_1, j_2) = (1 - i_1, 1 - i_2)$.

4.2.1 Pure Bottom-Up Approaches

Here we assume that available information is carried by the filtration $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$. So in this case $\mathbb{F} = \mathbb{F}^H = \mathbb{H}$.

Markovian Approach To cast the model in a Markovian framework, in the sense that the pair $H = (H^1, H^2)$ is an $\mathbb{F}$ − Markov process, one starts with the generator matrix

$$\mathcal{A}_t = \begin{pmatrix} -(\lambda^1_{0,0}(t) + \lambda^2_{0,0}(t)) & \lambda^1_{0,0}(t) & \lambda^2_{0,0}(t) & 0 \\ 0 & -\lambda^2_{1,0}(t) & 0 & \lambda^2_{1,0}(t) \\ 0 & 0 & -\lambda^1_{0,1}(t) & \lambda^1_{0,1}(t) \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{18}$$

In $\lambda^l_\imath(t)$ the superscript l refers to 'which obligors defaults' and the subscript $\imath = (i_1, i_2)$ to 'from which (bivariate) state'. The $\mathbb{F}$-*intensity functions* $\lambda^1_\imath(t)$ and $\lambda^2_\imath(t)$, sometimes also denoted $\lambda^1(t,\imath)$ and $\lambda^2(t,\imath)$, are of the form, with $(j_1, j_2) = (1 - i_1, 1 - i_2)$:

$$\lambda^1_\imath(t) = j_1 \widetilde{\lambda}^1_{i_2}(t) \, , \; \lambda^2_\imath(t) = j_2 \widetilde{\lambda}^2_{i_1}(t) \tag{19}$$

for (non-negative) *pre-default intensity functions* $\widetilde{\lambda}^1_i(t)$ and $\widetilde{\lambda}^2_i(t)$, or $\widetilde{\lambda}^1(t,i)$ and $\widetilde{\lambda}^2(t,i)$, such that (cf. (17))

$$\widetilde{\lambda}^1_t = \widetilde{\lambda}^1(t, H^2_t) \, , \; \widetilde{\lambda}^2_t = \widetilde{\lambda}^2(t, H^1_t) \, .$$

Note that, given that there are no common jumps between processes H^i, so individual pre-default intensity functions $\widetilde{\lambda}^l$'s are in one-to-one correspondence with the generator $\mathcal{A}$. They can thus be considered as the *model primitives*.

In this paragraph we shall consider hedging of claims that are of the form $\bar{\pi}(H_T)$ by means of trading of the claims of the form $\bar{\psi}(H_T)$ and $\bar{\phi}(H_T)$. Taking $\bar{\pi}(\iota) = \pi(i_1 + i_2)$, and likewise for $\bar{\psi}$ and $\bar{\phi}$, we can specialize these hedging results to claim depending solely on the loss process.

Since H is here a Markov process, we have that

$$\Pi_t = u(t, H_t) \, , \tag{20}$$

where $u(t,\imath)$ (or $u_\imath(t)$), for $t \in [0,T]$ and $\imath \in \{0,1\}^2$ is the *pricing function* (system of time-functionals $u_\imath$). Using the Itô formula in conjunction with the martingale property of Π, the pricing function can then be characterized as the solution to the following *pricing equation* (system of ODEs):

$$(\partial_t + \mathcal{A}_t) u = 0 \text{ on } [0,T) \, , \; u_\imath(T) = \bar{\pi}(\iota) \, . \tag{21}$$

Moreover we also get the following martingale representation, for $t \in [0,T]$:

$$\Pi_t = \mathbb{E}(\bar{\pi}(H_T)) + \int_0^t \delta^1 u(s, H_{s-}) \, dM^1_s + \int_0^t \delta^2 u(s, H_{s-}) \, dM^2_s \, , \tag{22}$$

where the delta functions $\delta^1 u$ and $\delta^2 u$ are defined by

$$\delta^1 u_\imath(t) = u_{1,i_2}(t) - u_{0,i_2}(t) \, , \; \delta^2 u_\imath(t) = u_{i_1,1}(t) - u_{i_1,0}(t) \, .$$

or in a short-hand notation immediately extendable to the case of n obligors, for every $l = 1$ or 2,

$$\delta^l u_\imath(t) = u_{\bar{\imath}^l}(t) - u_{\underline{\imath}^l}(t)$$

where $\underline{\imath}^l$ and $\bar{\imath}^l$ denote the vector $\imath$ with the l^{th} component replaced by 0 and 1, respectively.

Introducing likewise the pricing functions v and w of the ϕ and ψ-claims, and plugging all this in (15)–(16), the following hedging results follow.

Proposition 4.1 **(i)** *One can replicate $\bar{\pi}(H_T)$ at T by using the strategy $\zeta = (\zeta^1, \zeta^2)$ based on $\bar{\phi}$ and $\bar{\psi}$ (and the riskless, constant asset) defined by, for $t \in [0, T]$ (under the related matrix-invertibility assumption):*

$$\zeta_t = (\delta^1 u, \delta^2 u) \begin{pmatrix} \delta^1 v & \delta^2 v \\ \delta^1 w & \delta^2 w \end{pmatrix}^{-1} (t, H_{t-}) .$$

(ii) *Alternatively, it is possible to min-variance hedge the $\bar{\pi}$-claim by the $\bar{\phi}$-claim and the riskless asset using the strategy ζ such that $\zeta^2 = 0$ and, for $t \in [0, T]$:*

$$\zeta_t^1 = \frac{\lambda^1 (\delta^1 u)(\delta^1 v) + \lambda^2 (\delta^2 u)(\delta^2 v)}{\lambda^1 (\delta^1 v)^2 + \lambda^2 (\delta^2 v)^2} (t, H_{t-}) . \tag{23}$$

Distributional Approach Let G^i (for $i = 1, 2$) and G denote the marginal and joint *survival functions* of τ_1, τ_2, so for every $u, v \in \mathbb{R}_+$,

$$G^i(u) = \mathbb{P}(\tau_i > u) \ , \ G(u, v) = \mathbb{P}(\tau_1 > u, \tau_2 > v).$$

We assume the G^i's of class $\mathcal{C}^1$ and G of class $\mathcal{C}^2$. In particular there is therefore no common jump in the distributional model, consistently with our standing assumptions in this paper.

Let here and henceforth $\partial_i^j f$ (or simply $\partial_i f$ in case $j = 1$) denote the partial derivative of order j of a function f with respect to its i^{th} argument.

Remark 4.1 Since G is continuous, there exists a unique *(survival) copula function* $C(\cdot, \cdot)$ such that

$$G(u, v) = C(G^1(u), G^2(v)) .$$

So in particular, since G, G^1 and G^2 are differentiable:

$$\partial_1 G(u, v) = \partial_1 G^1(u) \partial_1 C(G^1(u), G^2(v))$$

and likewise for $\partial_2 G$.

The following proposition can be established by using standard conditioning techniques. Note that in the present approach, the *model primitive* is G, which determines the joint distribution of τ_1 and τ_2 under $\mathbb{P}$. The $\mathbb{H}$-intensities of τ_1 and τ_2 are then deduced as follows. More generally, the following proposition establishes the relation between the $\mathbb{H}^1$- and the $\mathbb{H}$- intensities of τ^1 (of course symmetric results obey for τ^2). Recall $J = 1 - H$.

Proposition 4.2 **(i)** *Let*

$$\widetilde{\gamma}_t^1 = -\frac{\partial_1 G^1(t)}{G^1(t)} .$$

Then the process $N_t^1 = H_t^1 - \Gamma_t^1$, with $\Gamma^1 = \int_0^{\cdot} J_t^1 \widetilde{\gamma}_t^1 \, dt$, is an $\mathbb{H}^1$-martingale.
(ii) *Let*

$$\widetilde{\lambda}_t^1 = -J_t^2 \frac{\partial_1 G(t, t)}{G(t, t)} - H_t^2 \frac{\partial_1 \partial_2 G(t, \tau_2)}{\partial_2 G(t, \tau_2)} = -\frac{\partial_1 \partial_2^{H_t^2} G(t, t \wedge \tau_2)}{\partial_2^{H_t^2} G(t, t \wedge \tau_2)} . \tag{24}$$

Then the process

$$M_t^1 = H_t^1 - \Lambda_t^1, \tag{25}$$

with $\Lambda^1 = \int_0^{\cdot} J_t^1 \widetilde{\lambda}_t^1 \, dt$, is an $\mathbb{H}$-martingale.

The (predictable versions of the) $\mathbb{H}^1$- and $\mathbb{H}$- intensities of τ_1 are thus given as, respectively,

$$\gamma_t^1 = J_{t-}^1 \widetilde{\gamma}_{t-}^1 \ , \ \lambda_t^1 = J_{t-}^1 \widetilde{\lambda}_{t-}^1 . \tag{26}$$

For comparison with the Markovian case discussed in the previous paragraph, observe that the process $H = (H^1, H^2)$ is not a Markov process here, unless $\partial_1\partial_2 G(t, \tau_2)$ does not depend on τ_2 in (24).

Proposition 4.2 shows explicitly how the pre-default intensity of τ_1 depends on the underlying filtration. In particular, since γ^1 and λ^1 obviously differ in (26), thus the $\mathbb{H}$-martingale N^1 of Proposition 4.2(i) is therefore *not* an $\mathbb{H}$-martingale, and we recover the fact that immersion (of $\mathbb{H}^1$ into $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$, here) typically does not hold in multi-name credit (cf. the discussion at the end of section 3.2.3).

Proposition 4.2 is also interesting with respect to the calibration issue risen in Section 3.5.1. In the notation introduced therein we should have by application of Proposition A.1 that

$$\left({}^{o_1}\Lambda^1\right)^{p_1} = \Gamma^1 \ , \ {}^{o_1}\lambda^1 = \gamma^1 \ . \tag{27}$$

This can indeed be verified directly using the forms of λ^1 and γ^1 derived in Proposition 4.2 and the definitions of the optional and dual predictable projections (see Appendix A and Proposition 3.3).

Remark 4.2 In a practical situation (cf. section 3.5.1), relation (27) would be used in the reverse-engineering fashion. By this we mean that (27) would provide a calibration constraint for the model, so that $\left({}^{o_1}\Lambda^1\right)^{p_1}$ computed from the model, meets Γ^1, which is extracted from the CDS market on name one.

In this paragraph we shall consider hedging of claims that are of the form $\widehat{\pi}(\tau_1, \tau_2)$ by means of trading of the claims of the form $\widehat{\phi}(\tau_1, \tau_2)$ and $\widehat{\psi}(\tau_1, \tau_2)$. Taking

$$\widehat{\pi}(\tau_1, \tau_2) = \pi(\mathbb{1}_{\tau_1 \le T} + \mathbb{1}_{\tau_2 \le T}) = \pi(H_T^1 + H_T^2) = \pi(L_T) \ , \tag{28}$$

and likewise for $\widehat{\psi}$ and $\widehat{\phi}$, we can specialize these hedging results to claim depending solely on the loss process.

Let us introduce the following notation:

$$\Pi_{1,1}(s,t) = \widehat{\pi}(s,t) \ , \ \Pi_{1,0}(s,t) = \frac{\int_t^\infty \widehat{\pi}(s,v)\partial_2 G(s,v)dv}{\int_t^\infty \partial_2 G(s,v)dv}$$

$$\Pi_{0,1}(t,s) = \frac{\int_t^\infty \widehat{\pi}(u,s)\partial_1 G(u,s)du}{\int_t^\infty \partial_1 G(u,s)du} \ , \ \Pi_{0,0}(t,t) = \frac{\int_t^\infty \int_t^\infty \widehat{\pi}(u,v)\partial_1\partial_2 G(u,v)dudv}{G(t,t)} \ ,$$

so in short-hand notation, for every $\theta = (t_1, t_2) \in \mathbb{R}_+^2$ and $\imath = (i_1, i_2) \in \{0,1\}^2$, with $\jmath = (j_1, j_2) = (1 - i_1, 1 - i_2)$:

$$\Pi_\imath(\theta) \ = \ \frac{\left(\int_{t_1}^\infty\right)^{j_1} \left(\int_{t_2}^\infty\right)^{j_2} \ \widehat{\pi}\, \partial_1^{j_1}\partial_2^{j_2} G\ (u_1^{j_1} t_1^{i_1}, u_2^{j_2} t_2^{i_2})\ (du_1)^{j_1} (du_2)^{j_2}}{\left(\int_{t_1}^\infty\right)^{j_1} \left(\int_{t_2}^\infty\right)^{j_2} \ \partial_1^{j_1}\partial_2^{j_2} G\ (u_1^{j_1} t_1^{i_1}, u_2^{j_2} t_2^{i_2})\ (du_1)^{j_1} (du_2)^{j_2}} \tag{29}$$

Lemma 4.3 *The following decomposition holds true, for every $t \ge 0$:*

$$\begin{aligned} \Pi_t &= \Pi_{1,1}(\tau_1, \tau_2) H_t^1 H_t^2 + \Pi_{1,0}(\tau_1, t) H_t^1 J_t^2 \\ &\quad + \Pi_{0,1}(t, \tau_2) J_t^1 H_t^2 + \Pi_{0,0}(t,t) J_t^1 J_t^2 \\ &= \Pi_{H_t}(t \wedge \tau_1, t \wedge \tau_2). \end{aligned}$$

We are now ready to derive the following martingale representation.

Proposition 4.4 *We have that*

$$\Pi_t = \mathbb{E}(\widehat{\pi}(\tau_1, \tau_2)) + \int_0^t \delta^1\Pi_s \, dM_s^1 + \int_0^t \delta^2\Pi_s \, dM_s^2 \, , \tag{30}$$

where

$$\delta^1\Pi_t = \Pi_{1,H_t^2}(t, t\wedge\tau_2) - \Pi_{0,H_t^2}(t, t\wedge\tau_2)$$
$$\delta^2\Pi_t = \Pi_{H_t^1,1}(t\wedge\tau_1, t) - \Pi_{H_t^1,0}(t\wedge\tau_1, t)$$

or in short-hand notation, for every $l = 1, 2$:

$$\delta^l\Pi_t = \Pi_{\overline{H}_t^l}(\theta_t^l) - \Pi_{\underline{H}_t^l}(\theta_t^l)$$

where $\underline{H}^l$ *and* $\overline{H}^l$ *denote the vector* H *with the* l^{th} *component replaced by 0 and 1, respectively, and where* θ_t^l *denotes the vector with entries* $t\wedge\tau_k$ *for* $k \neq l$ *and* t *for* $k = l$.

Remark 4.3 A direct proof of this result may be derived at not too much harm by using Proposition 4.2 in combination with Lemma 4.3. Note that the existence of a martingale representation of the general form (39) for Π is well known by standard results (see, e.g., Brémaud [13] or Last and Brandt [39]). Under conditions on G stated above, the expression for the coefficient $\delta^1\Pi$ (and likewise for $\delta^2\Pi$) is natural in view of Lemma 4.3, noting that:
• in case $\tau_1 < \tau_2$, the process Π has a jump at time τ_1 equal to $\Pi_{1,0}(\tau_1, t) - \Pi_{0,0}(t, t)$ at time $t = \tau_1$;
• in case $\tau_1 > \tau_2$, the process Π has a jump at time τ_1 equal to $\Pi_{1,1}(\tau_1, \tau_2) - \Pi_{0,1}(t, \tau_2)$ at time $t = \tau_1$.

Using similar decompositions for processes Φ and Ψ, the following analog to Proposition 4.1 may then be formulated,

Proposition 4.5 **(i)** *One can replicate* $\widehat{\pi}(\tau_1, \tau_2)$ *at* T *by using the hedging strategy* ζ *in the* $\widehat{\phi}$*- and* $\widehat{\psi}$*- claims (and the riskless asset) defined by, for* $t \in [0, T]$ *(under the related matrix-invertibility assumption):*

$$\zeta_t = (\delta^1\Pi_t, \delta^2\Pi_t) \begin{pmatrix} \delta^1\Phi_t & \delta^2\Phi_t \\ \delta^1\Psi_t & \delta^2\Psi_t \end{pmatrix}^{-1} .$$

(ii) *Alternatively, it is possible to min-variance hedge the* $\widehat{\pi}$*-claim by the* $\widehat{\phi}$*-claim and the riskless asset using the strategy* ζ *such that* $\zeta^2 = 0$ *and, for* $t \in [0, T]$*:*

$$\zeta_t^1 = \frac{\lambda_t^1(\delta^1\Pi_t)(\delta^1\Phi_t) + \lambda_t^2(\delta^2\Pi_t)(\delta^2\Phi_t)}{\lambda_t^1(\delta^1\Phi_t)^2 + \lambda_t^2(\delta^2\Phi_t)^2} \, . \tag{31}$$

It is worth stressing that the explicit formulas of this paragraph are derived in a dynamic *non-Markovian* model of credit risk.

4.2.2 Adding a Reference Filtration

Let us now assume, more generally, that $\mathbb{F} = \mathbb{F}^H \vee \mathbb{F}^Z$, where Z is a suitable factor process (to be specified later). We call the filtration $\mathbb{F}^Z$ the *reference filtration.*

Markovian Set-up To cast the model in a Markovian framework, in the sense that the pair (H, Z) is an $\mathbb{F}$ − Markov process, one starts with a generator, say

$$\mathcal{A} = \Lambda + \Gamma,$$

where Λ corresponds to H and where Γ corresponds to Z. Since we assume that there are no common jumps between processes H^i, so individual pre-default intensities are in one-to-one correspondence with Λ (cf. (18)–(19)). To determine Λ it thus suffices to specify pre-default individual intensities, say

$$\widetilde{\lambda}^1_t = \widetilde{\lambda}^1(t, H^2_t, Z_t)\ ,\ \widetilde{\lambda}^2_t = \widetilde{\lambda}^2(t, H^1_t, Z_t)\ , \tag{32}$$

so for $l = 1$ or 2:

$$\lambda^l_t = J^l_t \widetilde{\lambda}^l_t =: \lambda^l(t, H_t, Z_t)\ . \tag{33}$$

The construction of a Markovian model (H, Z) of stopping times τ_1 and τ_2 with $\mathbb{F}$-intensity processes λ^1_t and λ^2_t satisfying (33) can for example be realized by Markovian change of probability measure, starting from a model with independent default times and factor process (see [4]).

Setting $\mathcal{Z}_t = (t, H_t, Z_t)$, one may then define the pricing function $u = u_\imath(t, z)$ for the claim $\pi(L_T)$ (or, more generally, $\bar{\pi}(\mathcal{Z}_T)$), and, respectively, pricing functions v, w for the claims $\phi(L_T)$ and $\psi(L_T)$ (or, more generally, $\bar{\phi}(\mathcal{Z}_T)$ and $\bar{\psi}(\mathcal{Z}_T)$), characterized as the solutions to the related pricing equations with generator $\mathcal{A}$. The delta functions $\delta^1 u$ and $\delta^2 u$ are defined as in Section 4.2.1, except for the fact that they involve an additional argument z.

Moreover we have the following hedging results (we refer to [5], Section 3.3, and [36], page 109, for mathematical details behind these results),

Proposition 4.6 **(i)** *Assume Z satifies the following d-dimensional SDE*

$$dZ_t = b(\mathcal{Z}_t)dt + \sigma(\mathcal{Z}_t)dB_t\ ,$$

for suitable coefficients b and σ and a d-dimensional standard $\mathbb{F}$ – Brownian motion B. Then, denoting by ∂ the row-gradient of a function with respect to the argument z, we have:

$$\begin{aligned} de_t =& \Big(\delta^1 u - \zeta_t \begin{pmatrix} \delta^1 v \\ \delta^1 w \end{pmatrix}\Big)(\mathcal{Z}_{t-})dM^1_t + \Big(\delta^2 u - \zeta_t \begin{pmatrix} \delta^2 v \\ \delta^2 w \end{pmatrix}\Big)(\mathcal{Z}_{t-})dM^2_t \\ &+ \Bigg(\Big(\partial u - \zeta_t \begin{pmatrix} \partial v \\ \partial w \end{pmatrix}\Big)\sigma\Bigg)(\mathcal{Z}_t)dB_t. \end{aligned}$$

In particular one can min-variance hedge the $\bar{\pi}$-claim by the $\bar{\phi}$-claim and the riskless asset using the strategy ζ such that $\zeta^2 = 0$ and, for $t \in [0, T]$:

$$\zeta^1_t = \frac{\lambda^1(\delta^1 u)(\delta^1 v) + \lambda^2(\delta^2 u)(\delta^2 v) + (\partial u)\sigma\sigma^\mathsf{T}(\partial v)}{\lambda^1(\delta^1 v)^2 + \lambda^2(\delta^2 v)^2 + (\partial v)\sigma\sigma^\mathsf{T}(\partial v)}(\mathcal{Z}_{t-}). \tag{34}$$

(ii) *Assume Z is given as a pure jump process with finite state space E of cardinality d, jump times disjoint from τ_1 and τ_2, jump intensity vector-process $\Gamma(\mathcal{Z}_{t-}, z)_{z\in E}$ and compensated jump vector-martingale $(M_t(z),\ z \in E)$. Then, denoting by $\Delta u_\imath(t, z)$ the d-dimensional row-vector $(u_\imath(t, z') - u_\imath(t, z))_{z'\in E}$ and likewise for Δv and Δw, we have:*

$$\begin{aligned} de_t =& \Big(\delta^1 u - \zeta_t \begin{pmatrix} \delta^1 v \\ \delta^1 w \end{pmatrix}\Big)(\mathcal{Z}_{t-})dM^1_t + \Big(\delta^2 u - \zeta_t \begin{pmatrix} \delta^2 v \\ \delta^2 w \end{pmatrix}\Big)(\mathcal{Z}_{t-})dM^2_t \\ &+ \Big(\Delta u - \zeta_t \begin{pmatrix} \Delta v \\ \Delta w \end{pmatrix}\Big)(\mathcal{Z}_{t-})dM_t \end{aligned}$$

In particular one can min-variance hedge the $\bar{\pi}$-claim by the $\bar{\phi}$-claim and the riskless asset using the strategy ζ such that $\zeta^2 = 0$ and, for $t \in [0, T]$:

$$\zeta^1_t = \frac{\lambda^1(\delta^1 u)(\delta^1 v)(\mathcal{Z}_{t-}) + \lambda^2(\delta^2 u)(\delta^2 v)(\mathcal{Z}_{t-}) + \sum_{z\in E}\Gamma(\mathcal{Z}_{t-}, z)\Delta u(\mathcal{Z}_{t-}, z)\Delta v(\mathcal{Z}_{t-}, z)}{\lambda^1(\delta^1 v)^2(\mathcal{Z}_{t-}) + \lambda^2(\delta^2 v)^2(\mathcal{Z}_{t-}) + \sum_{z\in E}\Gamma(\mathcal{Z}_{t-}, z)\Delta v(\mathcal{Z}_{t-}, z)\Delta v(\mathcal{Z}_{t-}, z)}\ . \tag{35}$$

Remark 4.4 Of course the auxiliary factor process Z introduces potentially several additional sources of randomness that need to be hedged. This can be dealt with by taking as a hedging instrument (on top of the bank account) a possibly multidimensional claim ϕ. The results of Proposition 4.6 can then be easily extended to the case of multidimensional claim ϕ by formulating appropriate systems of linear equations.

Distributional Approach Let G^i_t (for $i = 1, 2$) and G_t denote the marginal and joint *conditional survival survival functions* of τ_1 and τ_2, so for every $t, u, v \geq 0$,

$$G^i_t(u) = \mathbb{P}(\tau_i > u \,|\, \mathcal{F}^Z_t)\ ,\ G_t(u, v) = \mathbb{P}(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}^Z_t)\ .$$

In particular G^i_0 and G_0 reduces to the (unconditional) marginal and joint survival function G, for $\mathcal{F}^Z_0$ trivial. Assuming the G^i_t's of class $\mathcal{C}^1$ and G_t of class $\mathcal{C}^2$ with respect to u and v, we may then easily derive formal extensions of the initial times approach of the pure bottom case of Section 4.2.1. Then, we have (see, for instance, [23]),

Proposition 4.7 *The pre-default $\mathbb{H}^1 \vee \mathbb{F}^Z$- and $\mathbb{H} \vee \mathbb{F}^Z$- intensities of τ^1 are given by, respectively:*

$$\widetilde{\gamma}^1_t = -\frac{\partial_1 G^1_t(t)}{G^1_t(t)}\ ,\ \widetilde{\lambda}^1_t = -\frac{\partial_1 \partial_2^{H^2_t} G_t(t, t \wedge \tau_2)}{\partial_2^{H^2_t} G_t(t, t \wedge \tau_2)}\ .$$

However it seems difficult to derive explicit and constructive martingale representations in this set-up (so hedging cannot be implemented either), unless we are in the case of an auxiliary factor process Z given as a pure jump process with finite state space. In this case it is possible to derive an elementary martingale representation and a suitable analog to Proposition 4.6(ii), valid for payoffs $\widehat{\pi}, \widehat{\phi}, \widehat{\psi}\,(\tau_1, \tau_2)$. We leave this to the reader.

4.3 Top Approaches

We now work with a top filtration $\mathbb{F}$. In this context it does not make the notation heavier to consider directly n stopping times, which we do. Since we work with a top filtration $\mathbb{F}$, the τ_i's are not $\mathbb{F}$-stopping times, as opposed to the *ordered* default times $\tau_{(i)}$'s. The loss process is therefore an $\mathbb{F}$-adapted, non-decreasing process. We denote by M its $\mathbb{F}-$ compensated martingale, so

$$M = L - \int_0^{\cdot} \lambda_t dt$$

where λ is the (predictable version of the) $\mathbb{F}$-intensity, assumed to exist, of L.

4.3.1 Pure Top Approaches

Here $\mathbb{F} = \mathbb{F}^L$.

Markovian Set-up In the Markovian case L is a pure birth process, or *local intensity process* (cf. Laurent, Cousin and Fermanian [40] or Cont and Minca [15]), with $\mathbb{F}$-intensity $\lambda_t = \lambda(t, L_t)$, for a suitable $\mathbb{F}$-*intensity function* $\lambda(t, i)$ (vanishing for $i \geq n$, consistently with the fact that the loss process L is stopped at level n).

In this set-up, we obtain that, for $t \in [0, T]$

$$\Pi_t = \mathbb{E}(\pi(L_T) \,|\, \mathcal{F}_t) = u(t, L_t)\ , \tag{36}$$

where $u(t, i)$ or $u_i(t)$ for $(t, i) \in [0, T] \times \mathbb{N}_n$ (with $\mathbb{N}_n = \{0, \cdots, n\}$), is the pricing function (system of time-functionals u_i), solution to the related system of backward Kolmogorov differential equations. Moreover we have the following martingale representation, for $t \in [0, T]$:

$$\Pi_t = \mathbb{E}(\pi(L_T)) + \int_0^t \delta u(s, L_{s-})\, dM_s \tag{37}$$

where the delta function δu is defined by, for $t \in [0, T]$ and $i \in \mathbb{N}_{n-1}$:

$$\delta u_i(t) = u_{i+1}(t) - u_i(t)\ . \tag{38}$$

It is rather clear that in this present case it is enough to use just one claim, say ϕ (and the riskless account) so to replicate claim π. So, using the analogous martingale representation for the ϕ claim, the following result follows (in view of (15)),

Proposition 4.8 *One can replicate $\pi(L_T)$ at T by using the strategy ζ based on the ϕ-claim (and on the riskless asset) defined by, for $t \in [0,T]$ (assuming $\delta v \neq 0$):*

$$\zeta_t = \frac{\delta u}{\delta v}(t, L_{t-}) \ .$$

Remark 4.5 Let us emphasize here that the (theoretical) perfect replication can be achieved as the present model in the local intensity model. We shall see later (cf. Section 5.2.3) that in more realistic set-ups one may not be able to hedge a claim written on L_T with another claim written on L_T.

Distributional Approach We denote $G^i_t(u) = \mathbb{P}(L_u = i \,|\, \mathcal{F}_t)$, for $i \in \mathbb{N}_n$.

As an $\mathbb{F}^L$-martingale, the process $G^i_t(u)$ admits a representation of the form

$$G^i_t(u) = G^i_0(u) + \int_0^t \delta G^i_s(u)\, dM_s$$

for some integrand $\delta G^i_s(u)$. We are now ready to write the representation for Π in terms of δG.

Proposition 4.9 *We have, for $t \in [0,T]$:*

$$\Pi_t = \sum_{0 \leq i \leq n} \pi(i) G^i_t(T) = \mathbb{E}(\pi(L_T)) + \int_0^t \delta\Pi_s\, dM_s \ , \tag{39}$$

where we set

$$\delta\Pi_t = \textstyle\sum_{0 \leq i \leq n} \pi(i) \delta G^i_t(T) \ .$$

Using the analogous representation regarding the ϕ claim, and plugging all these expressions in (15)), one gets the following,

Proposition 4.10 *One can replicate π at T by using the strategy ζ based on the ϕ-claim (and on the riskless asset) defined by, for $t \in [0,T]$ (assuming $\delta\Phi_t \neq 0$):*

$$\zeta_t = \frac{\delta\Pi_t}{\delta\Phi_t} \ .$$

To illustrate the feasibility (and the limits) of the approach of this paragraph we need to provide specific examples in which $G^i_t(u)$ and $\delta G^i_t(u)$ are computable. For this it is enough that the joint cumulative distribution of the consecutive default times ϑ_l's of L be computable and continuous. Indeed we have, since $\mathbb{F} = \mathbb{F}^L$:

$$G^i_t(u) = \mathbb{P}(L_u = i \,|\, \mathcal{F}_t) = \mathbb{P}(L_u = i \,|\, \mathcal{F}_t \vee \sigma(L_t)) \ , \tag{40}$$

which on the random time interval $t \in [\vartheta_l, \vartheta_{l+1})$ (or, equivalently, on the event $\{L_t = l\}$), is easily seen to coincide with $\mathbb{P}(\vartheta_i \leq u < \vartheta_{i+1} \,|\, \vartheta_1, \cdots, \vartheta_l \,;_, \vartheta_l \leq t < \vartheta_{l+1})$. Now, by the Bayes rule, the latter quantity is determined by the joint law of the ϑ_l's. Moreover, if the joint cumulative distribution of the ϑ_l's is continuous, then $G^i_t(u)$ is thus continuous on every interval $[\vartheta_l, \vartheta_{l+1})$, and the only possible jumps of $G^i_t(u)$ occur at the ϑ_l's, where they are given by

$$G^i_{\vartheta_l}(u) - G^i_{\vartheta_l -}(u) = \mathbb{P}(\vartheta_i \leq u < \vartheta_{i+1} \,|\, \vartheta_1, \cdots, \vartheta_l) - \mathbb{P}(\vartheta_i \leq u < \vartheta_{i+1} \,|\, \vartheta_1, \cdots, \vartheta_{l-1}) \ , \tag{41}$$

which can also be evaluated by the Bayes rule given the joint cumulative distribution of the ϑ_l's. So $\delta G_t^i(u)$ is computable too.

As an explicit example, let us specifically consider two ordered random times ϑ_1 and ϑ_2 defined by, given IID unit exponential random variables $\mathcal{E}_1$ and $\mathcal{E}_2$:

$$\vartheta_1 = \inf\{t > 0 \,|\, \in \vartheta_0^t \mu_1(s)ds > \mathcal{E}_1\} \ , \ \vartheta_2 = \inf\{t > \vartheta_1 \,|\, \in \vartheta_{\vartheta_1}^t \mu_2(s, \vartheta_1)ds > \mathcal{E}_2\}$$

with $\mu_1(t) = 1$, $\mu_2(t, \vartheta_1) = t\vartheta_1$. So

$$\vartheta_1 = \mathcal{E}_1 \ , \ \vartheta_2 = \vartheta_1 + \frac{\mathcal{E}_2}{\vartheta_1} \ .$$

One can show that the related loss process L is a non-Markovian *Hawkes process* (see Errais, Giesecke and Goldberg [24], Hawkes [33]). Moreover, we have in this case

$$\mathbb{P}(\vartheta_1 > u_1, \, \vartheta_2 > u_2) = \mathbb{P}(u_1 < \mathcal{E}_1, \, u_2 < \mathcal{E}_1 + \frac{\mathcal{E}_2}{\mathcal{E}_1}) \ ,$$

which is explicitly given by $\in \vartheta_{x>u_1} \in \vartheta_{y>x(u_2-x)^+} e^{-(x+y)} dxdy$.

4.3.2 Adding a Reference Filtration

We now assume $\mathbb{F} = \mathbb{F}^L \vee \mathbb{F}^Y$, for a suitable factor process Y.

Markovian Set-up We suppose that the pair (L, Y) is an $\mathbb{F}$ – Markov process with generator $\mathcal{A}$, assuming more specifically that the $\mathbb{F}$-intensity of L satisfies

$$\lambda_t = \lambda(t, L_t, Y_t) \tag{42}$$

for a given *intensity function* $\lambda_i(t, y)$ (vanishing for $i \geq n$). The construction of such a model (L, Y) can be realized by a Markovian change of probability measure, starting from an auxiliary model with independent loss and factor processes.

Setting $\mathcal{Y}_t = (t, L_t, Y_t)$, one may then define the pricing function $u, v, w = u, v, w_i(t, z)$ for the $\pi, \phi, \psi(L_T)$-claims (or more general $\breve{\pi}, \breve{\phi}, \breve{\psi}(\mathcal{Y}_T)$-claims), characterized as the solutions to the related pricing equations with generator $\mathcal{A}$.

The loss delta function δu is defined by (cf. (38))

$$\delta u_i(t, y) = u_{i+1}(t, y) - u_i(t, y) \ .$$

Moreover, we have the following hedging result, which is an analog to Proposition 4.6 (and the analog of Remark 4.4 also holds),

Proposition 4.11 **(i)** *Assume Y satisfies the following d-dimensional SDE*

$$dY_t = b(\mathcal{Y}_t)dt + \sigma(\mathcal{Y}_t)dB_t \ ,$$

for suitable coefficients b and σ and a d-dimensional standard $\mathbb{F}$ – Brownian motion B. Then, denoting by ∂ the row-gradient of a function with respect to the argument y, we have:[5]

$$de_t = \Big(\delta u - \zeta_t \delta v\Big)(\mathcal{Y}_{t-})dM_t + \Big(\partial u \sigma - \zeta_t \partial v \sigma\Big)(\mathcal{Y}_t)dB_t$$

[5] with $\delta u(\mathcal{Y}_{t-}) = u_{L_{t-}+1}(t, Y_t) - u_{L_{t-}}(t, Y_t)$, and likewise for δv.

In particular one can min-variance hedge the $\breve{\pi}$-claim by the $\breve{\phi}$-claim and the riskless asset using the strategy ζ such that $\zeta^2 = 0$ and, for $t \in [0,T]$:

$$\zeta^1_t = \frac{\lambda(\delta u)(\delta v) + (\partial u)\sigma\sigma^{\mathsf{T}}(\partial v)}{\lambda(\delta v)^2 + (\partial v)\sigma\sigma^{\mathsf{T}}(\partial v)}(\mathcal{Y}_{t-}). \tag{43}$$

(ii) *Assume Y given as a pure jump process with finite state space E of cardinality d, jump times disjoint from L, jump intensity vector-process $\Gamma(\mathcal{Y}_{t-}, y)_{y\in E}$ and compensated jump vector-martingale $(N_t(y),\ y \in E)$. Then, denoting by $\Delta u_i(t,y)$ the d-dimensional row-vector $(u_i(t,y') - u_i(t,y))_{y'\in E}$ and likewise for Δv, we have:*

$$de_t = \Big(\delta u - \zeta_t \delta v\Big)(\mathcal{Y}_{t-})dM_t + \Big(\Delta u - \zeta_t \Delta v\Big)(\mathcal{Y}_{t-})dN_t$$

In particular one can min-variance hedge the $\breve{\pi}$-claim by the $\breve{\phi}$-claim and the riskless asset using the strategy ζ such that $\zeta^2 = 0$ and, for $t \in [0,T]$:

$$\zeta^1_t = \frac{\lambda(\delta u)(\delta v)(\mathcal{Y}_{t-}) + \sum_{y\in E}\Gamma(\mathcal{Y}_{t-}, y)\Delta u(\mathcal{Y}_{t-}, y)\Delta v(\mathcal{Y}_{t-}, y)}{\lambda(\delta v)^2(\mathcal{Y}_{t-}) + \sum_{y\in E}\Gamma(\mathcal{Y}_{t-}, y)(\Delta v(\mathcal{Y}_{t-}, y))^2}\ . \tag{44}$$

Distributional Approach As in the bottom-up initial times approach of Section 4.2.2, there is little hope to obtain a constructive martingale representation (with computable integrands) in this set-up, unless maybe we consider a factor process Y taking a finite number of values. The detail is left to the reader.

5 Numerical Examples

For credit derivatives with stylized payoff given as $\xi = \pi(L_T)$ at maturity time T, it is tempting to adopt a Black–Scholes like approach, modeling L as a Markov point process and performing factor hedging of one derivative by another as in Proposition 4.8, balancing the related sensitivities computed by the Itô-Markov formula. However, since the loss process L is far from being Markovian in the market (unless maybe additional factors are considered to form a Markovian vector state-process), it is quite likely that L a not a sufficient statistics for the purpose of valuation and hedging of portfolio credit risk. In other words, ignoring the potentially non-Markovian dynamics of L for pricing and/or hedging may cause huge model risk, even though the payoffs of the products at hand are given as functions of L_T.

In this section we want to illustrate this point further by means of numerical hedging simulations. We use a homogeneous and constant recovery $R = 40\%$ (rather than 0 above). We thus need to distinguish the cumulative default process $N = \sum_{i=1}^n H^i_t$ and the cumulative loss process $L_t = (1-R)N_t$.

We shall consider the benchmark problem of pricing and hedging a stylized loss derivative. Specifically, for simplicity, we only consider protection legs of *of equity tranches* , resp. *super-senior tranches* (i.e. detachment of 100%) with stylized payoffs

$$\pi(N_T) = \frac{(1-R)N_T}{n} \wedge k = \frac{L_T}{n} \wedge k\ ,\ \text{resp.}\ (\frac{L_T}{n} - k)^+$$

at a maturity time T. The 'strike' (detachment, resp. attachment point) k belongs to $[0,1]$. In this formalism the (stylized) *credit index* corresponds to the equity tranche with $k = 100\%$ (or senior tranche with $k = 0$). With a slight abuse of terminology, we shall refer to our stylized loss derivatives as to *tranches* and *index*, respectively.

We shall now consider the problem of hedging the tranches with the index, using a simplified bottom-up market model of credit risk.

5.1 Homogeneous Groups Model

For ease of implementation we consider a Markov chain model of credit risk as of Frey and Backhaus [26] (see also Bielecki et al. [4]). Namely, the n names of a pool are grouped in d classes of $\nu-1=\frac{n}{d}$ homogeneous obligors (assuming $\frac{n}{d}$ integer), and the cumulative default processes N^l, $l=1,\cdots,d$, of the different groups (so $N=\sum N^l$) are jointly modeled as a d-variate Markov point process $\mathcal{N}$, with $\mathbb{F}^{\mathcal{N}}$-intensity of N^l given as

$$\lambda_t^l=(\nu-1-N_t^l)\widetilde{\lambda}^l(t,\mathcal{N}_t)\ , \tag{45}$$

for some *pre-default individual intensity functions* $\widetilde{\lambda}^l=\widetilde{\lambda}^l(t,\imath)$, where $\imath=(i_1,\cdots,i_d)\in\mathbb{N}_{\nu-1}^d$ (recall $\mathbb{N}_{\nu-1}=\{0,\cdots,\nu-1\}$). The related generator (spatial generator at time t) may then be written in the form of a ν^d-dimensional (sparse) matrix Λ_t.

Remark 5.1 We are not claiming here that this kind of models should necessarily be used for dealing with credit derivatives (cf. the reservation made in Section 2.2.1). In particular note that *simultaneous defaults* are implicitly excluded from our modeling, which may not be such an innocuous restriction in practice (on this issue, see, for instance, Bielecki et al. [10] or Brigo et al. [9], or the comments at the beginning of section 6.1 of Schönbucher [46]).

For $d=1$, we recover the well-known *local intensity model* already considered in the first paragraph of Section 4.3.1 (pure top approach with N modeled as a Markov birth point process stopped at level n; see, for instance, Laurent, Cousin and Fermanian [40], Cont and Minca [15] or van der Voort [48]. At the other extreme, for $d=n$, we are in effect modeling the vector of the default indicator processes of the pool names. As d varies between 1 and n, we thus get a variety of models of credit risk, ranging from pure top models for $d=1$ to pure bottom-up models for $d=n$.

Remark 5.2 Observe that in the *homogeneous case* where $\widetilde{\lambda}^l(t,\imath)=\widehat{\lambda}(t,\sum_j i_j)$ for some function $\widehat{\lambda}=\widehat{\lambda}(t,i)$ (independent of l), the model (whatever the nominal value of d / structure of the matrix generator used for encoding the model) effectively reduces to a local intensity model (with $d=1$ and pre-default individual intensity $\widehat{\lambda}(t,i)$ therein).
Further specifying the model to $\widehat{\lambda}$ independent of i corresponds to the situation of homogeneous and independent obligors.
In general, introducing parsimonious parameterizations of the intensities allows one to account for inhomogeneity between groups and/or defaults contagion. It is also possible to extend this set-up to more general credit migrations models, or to generic bottom-up models of credit migrations influenced by macro-economic factors (see Bielecki et al. [4, 10] or Frey and Backhaus [27]).

5.1.1 Pricing

Since $\mathcal{N}$ is a Markov process and N is a function of $\mathcal{N}$, the related tranche price process writes, for $t\in[0,T]$ (assuming $\pi(N_T)$ integrable):

$$\Pi_t=\mathbb{E}(\pi(N_T)|\mathcal{F}_t^{\mathcal{N}})=u(t,\mathcal{N}_t)\ , \tag{46}$$

where $u(t,\imath)$ or $u_\imath(t)$ for $t\in[0,T]$ and $\imath\in\mathbb{N}_{\nu-1}^d$, is the *pricing function* (system of time-functions $u_\imath$). Using the Itô formula in conjunction with the martingale property of Π, the pricing function can then be characterized as the solution to the following *pricing equation* (system of ODEs):

$$(\partial_t+\Lambda_t)u=0 \text{ on } [0,T) \tag{47}$$

with terminal condition $u_\imath(T)=\pi(\imath)$, for $\imath\in\mathbb{N}_{\nu-1}^d$. In particular, in the case of a time-homogeneous generator Λ (independent of t), we have the following semi-closed formula, for $t\in[0,T]$:

$$u(t)=\exp[(T-t)\Lambda]\pi\ . \tag{48}$$

Pricing in this model can be achieved by various means, like numerical resolution of the ODE system (47), numerical matrix exponentiation based on (48) (in the time-homogeneous case) or Monte Carlo simulation. However solution of (47) or computation of (48) by deterministic numerical schemes is typically precluded by the curse of dimensionality for d greater than a few units (depending on ν). So for high d simulation is the only way to go, which can be done quite efficiently by suitable importance sampling techniques (see Crépey and Carmona [16]).

The distribution of the vector of time-t losses (for each group), that is, $q_\imath(t) = \mathbb{P}(\mathcal{N}_t = \imath)$ for $t \in [0,T]$ and $\imath \in \mathbb{N}^d_{\nu-1}$, and portfolio loss distribution, $p = p_i(t) = \mathbb{P}(N_t = i)$ for $t \in [0,T]$ and $i = 0, \cdots, n$, can be computed likewise by numerical solution of the associated forward Kolmogorov equations (for more detail, see, e.g., Crépey and Carmona [16]).

5.1.2 Hedging

In general, in the Markovian model described above, it is possible to replicate dynamically in continuous time any payoff provided d non-redundant hedging instruments are available (see Frey and Backhaus [25] or Bielecki, Vidozzi and Vidozzi [10]; see also Laurent, Cousin and Fermanian [40] for results in the special case where $d = 1$). From the mathematical side this corresponds to the fact that the model is of (Davis-Varaiya) *multiplicity* d [18], in general. So, in general, it is not possible to replicate a payoff, such as tranche, *by the index alone* in this model, unless the model dimension d is equal to one (or reducible to one, cf. Remark 5.2). Now our point is that this potential lack of replicability is not purely speculative, but can be very significant in practice.

Since delta-hedging in continuous time is expensive in terms of transaction costs, and because main changes occur at default times in this model (in fact, default times are the only events in this model, if not for time flow and the induced time-decay effects), we shall focus on *semi-static hedging* in what follows, only updating at default times the composition of the hedging portfolio. More specifically, denoting by $\tau_{(1)}$ the first default time of a reference obligor, we shall examine the result at $\tau_{(1)}$ of a static hedging strategy on the random time interval $[0, \tau_{(1)}]$.
Let Π and Θ denote the tranche and index model price processes, respectively. Using a constant hedge ratio δ_0 over the time interval $[0, \tau_{(1)}]$, the *tracking error* or *profit-and-loss* of the delta-hedged tranche at $\tau_{(1)}$ writes:

$$e_{\tau_{(1)}} = (\Pi_{\tau_{(1)}} - \Pi_0) - \delta_0(\Theta_{\tau_{(1)}} - \Theta_0) \; . \tag{49}$$

The question we want to consider is whether it is possible to make this quantity 'small' (in terms, say, of risk-neutral variance) by a suitable choice of δ_0. It is expected that this should depend on the model dimension d and on the characteristics of the products at hand (value of the strike k in particular). However, it is intuitively clear that for too large values of $\tau_{(1)}$ time-decay effects matter and the hedge should be rebalanced at some intermediate points of the time interval $[0, \tau_{(1)}]$ (even though no default occurred yet). To keep it as simple as possible we shall merely apply a cutoff and restrict our attention to the random set $\{\omega \,:\, \tau_{(1)}(\omega) < T_1\}$ for some fixed $T_1 \in [0,T]$.

5.2 Numerical Results

We work with the above model for $d = 2$ and $\nu = 5$ (we thus consider a stylized credit portfolio of $n = 8$ obligors), and $\widetilde{\lambda}^l$ given by, for $l = 1, 2$ (cf. (45)):

$$\widetilde{\lambda}^l(t, \imath) = l\left(\frac{1 + i_l}{n}\right) \; . \tag{50}$$

So the pre-default individual intensities of the obligors of group 1 and 2 are given as $\frac{1+i_1}{n} = \frac{1+i_1}{8}$ and $\frac{1+i_2}{4}$, where i_1 and i_2 represent the number of currently defaulted obligors in groups 1 and 2, respectively.

Recall that the computation time for exact pricing (using matrix exponentiation based on (48)) in such model grows as ν^d, which motivated the previous modest choices for d and ν. So in this case the model is two-dimensional and the model generator is a matrix of dimension $\nu^d = 25$.

We set the maturity T equal to 5 years and the cutoff T_1 equal to 1 year. We thus make a focus on the random set of trajectories for which $\tau_{(1)} < 1$, meaning that a default occurred during the first year of hedging.

5.2.1 Model Simulation

In this toy model the simulation takes the following very simple form (see also [25] or [4] for more details in more general set-ups):
Compute Π_0 (for the tranche) and Θ_0 (for the index) by numerical matrix exponentiation based on (48), and then for every $j = 1, \cdots, m$:

- draw a pair $(\widetilde{t}_1^j, \widehat{t}_1^j)$ of independent exponential random variables with parameter $4\times(\frac{1}{8}, \frac{1}{4}) = (\frac{1}{2}, 1)$;
- set $\tau_{(1)}^j = \min(\widetilde{t}_1^j, \widehat{t}_1^j)$ and $\mathcal{N}_{\tau_{(1)}} = (1,0)$ or $(0,1)$ depending on whether $\tau_{(1)}^j = \widetilde{t}_1^j$ or $\widehat{t}_1^j$;
- compute $\Pi_{\tau_{(1)}^j}$ (for the tranche) and $\Theta_{\tau_{(1)}^j}$ (for the index) by (48).

Doing[6] this for $m = 10^4$, we got 9994 draws with $\tau_{(1)} < 5$, among which 7701 ones with $\tau_{(1)} < 1$, subdividing themselves into 2628 defaults in the first group of obligors and 5073 defaults in the second one.

5.2.2 Pricing

We consider two $T = 5y$-tranches in the above model: an 'equity tranche' with $k = 20\%$, corresponding to a payoff $\frac{(1-R)N_T}{n} \wedge k = (\frac{60N_T}{8} \wedge 20)\%$ (of a unit nominal amount), and a 'senior tranche' defined simply as the complement of the equity tranche to the index, thus with payoff $(\frac{(1-R)N_T}{n} - k)^+ = (\frac{60N_T}{8} - 20)^+\%$.

We also computed the portfolio loss distribution at maturity by numerical matrix exponentiation corresponding to explicit solution of the associated forward Kolmogorov equations (see, e.g., [16]).

Note that there is virtually no error involved in the previous computations, in the sense that our simulation is exact (without simulation bias), and the prices and loss probabilities were computed by matrix exponentiation.

The left pane of Figure 1 represents the histogram of the loss distribution at the time horizon T; we indicate by a vertical line the loss level x beyond which the equity tranche is exhausted, and the senior tranche starts being hit (so $\frac{(1-R)x}{n} = k$, e.g. $x = 2.66$).
The right pane of Figure 1 displays the equity (labeled by $+$), senior ($\times$) and index ($\circ$) tranche prices at $\tau_{(1)}$ versus $\tau_{(1)}$, for all the points in the simulated data with $\tau_{(1)} < 5$ (9994 points). Blue and red points correspond to defaults in the first and in the second group of obligors, respectively. We also represented in black the points $(0, \Pi_0)$ (for the tranches) and $(0, \Theta_0)$ (for the index).

Note that in the case of the senior tranche and of the index, there is a clear difference between prices at $\tau_{(1)}$ depending on whether $\tau_{(1)}$ corresponds to a default in the first or in the second group of obligors, whereas in the case of the equity tranche there seems to be little difference in this regard. In view of the portfolio loss distribution in the left pane, this can be explained by the fact that in the case of the equity tranche, the probability conditional on $\tau_{(1)}$ that the tranche will be exhausted before maturity is essentially one unless $\tau_{(1)}$ is close to T. Therefore the equity tranche price at $\tau_{(1)}$ is essentially equal to $k = 20\%$ for $\tau_{(1)}$ not to close to T. Moreover for $\tau_{(1)}$ close to T the intrinsic value of the tranche at $\tau_{(1)}$ constitutes the major part of the equity tranche price at $\tau_{(1)}$ (since the tranche has low time-value close to maturity). In conclusion the state of $\mathcal{N}$ at $\tau_{(1)}$ has a low impact on $\Pi_{\tau_{(1)}}$, whatever the value of $\tau_{(1)}$ may be (far or close to T).

[6] All the numerical conmputations were made using the free statistical software R (see www.r-project.org).

On the other hand, in the case of the senior tranche or in case of the index, the state of $\mathcal{N}$ at $\tau_{(1)}$ has a high impact on the corresponding price, unless $\tau_{(1)}$ is close to T (in which case intrinsic value effects are dominant). This explains the 'two-track' pictures seen on the right pane of Figure 1 (whereas the two-tracks are essentially superimposed in the case of the equity tranche).

Looking at these results in terms of price changes $\Pi_0 - \Pi_{\tau_{(1)}}$ of a tranche versus the corresponding index price changes $\Theta_0 - \Theta_{\tau_{(1)}}$, we obtain the graphs of Figure 2 for the equity tranche and 3 for the senior tranche. We consider all points with $\tau_{(1)} < T$ on the left panes and focus on the points with $\tau_{(1)} < T_1$ on the right ones. We use the same blue/red color code as above, and we further highlight in green on the left panes the points with $\tau_{(1)} < 1$, which are focused upon on the right panes.
Figure 2 gives a further graphical illustration of the low level of correlation between price changes of the equity tranche and of the index. Indeed the cloud of points on the right pane is obviously "far from a straight line", due to the partitioning of points between blue points / defaults in group one on one segment versus red points / defaults in group two on a different segment.
On the opposite (Figure 3), price changes of the senior tranche and of the index evidently exhibit a high degree of correlation, since in this case the blue and the red segments are essentially on a common line.

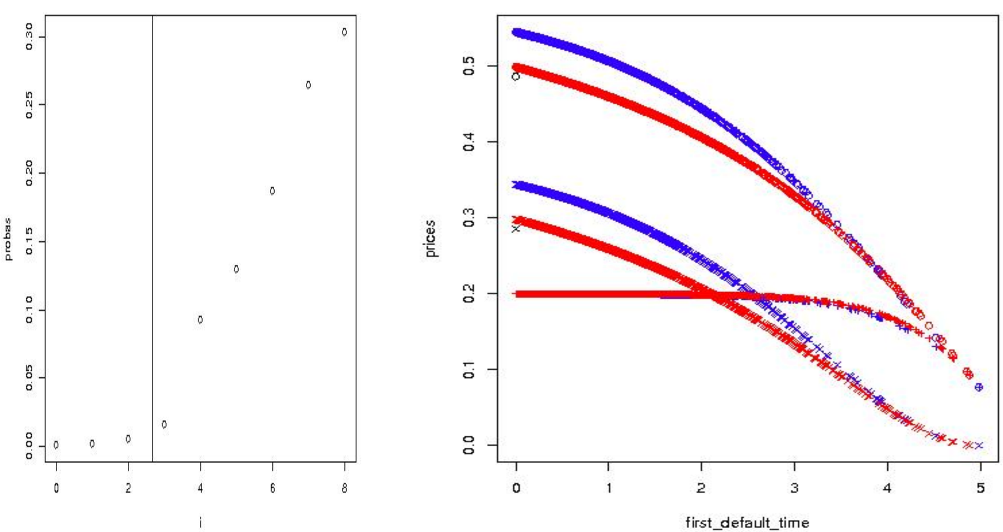

Figure 1: *(Left) Portfolio loss distribution at maturity $T = 5yr$; (Right) Tranche Prices at $\tau_{(1)}$ for $\tau_{(1)} < T = 5$ (equity tranche (+), senior tranche (×) and index (○)).*

5.2.3 Hedging

We then computed the (empirical, risk-neutral) variance of the profit-and-loss $e_{\tau_{(1)}}$ in (49) (restricting attention to the subset $\tau_{(1)} < T_1 = 1$), using for δ_0 the empirical regression delta of the tranche with respect to the index at time 0, so

$$\delta_0 = \frac{\widehat{\mathbb{C}\mathrm{ov}}(\Pi_{\tau_{(1)}} - \Pi_0, \Theta_{\tau_{(1)}} - \Theta_0)}{\widehat{\mathbb{V}\mathrm{ar}}(\Theta_{\tau_{(1)}} - \Theta_0)} .$$

Moreover, we also did these computations restricting further attention to the subsets of $\tau_{(1)} < 1$ corresponding to defaults in the first and in the second group of obligors (blue and red points on the figures), respectively. The latter results are to be understood as giving proxies of the situation that would prevail in a one-dimensional complete model of credit risk (local intensity model for N).

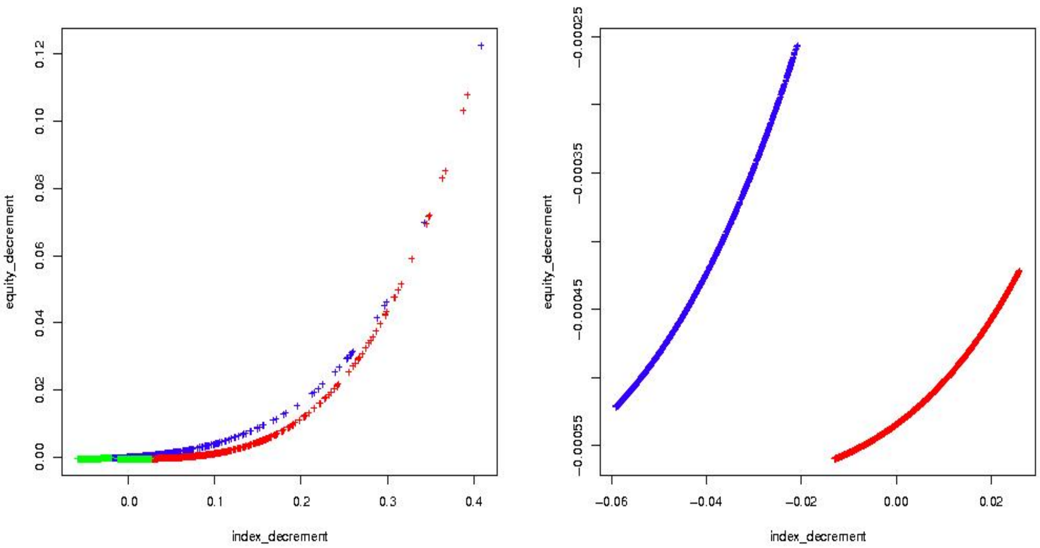

Figure 2: *Equity vs Index Price Changes between 0 and* $\tau_{(1)}$ ($\tau_{(1)} < T = 5$, *last pane; zoom on* $\tau_{(1)} < T_1 = 1$, *right pane).*

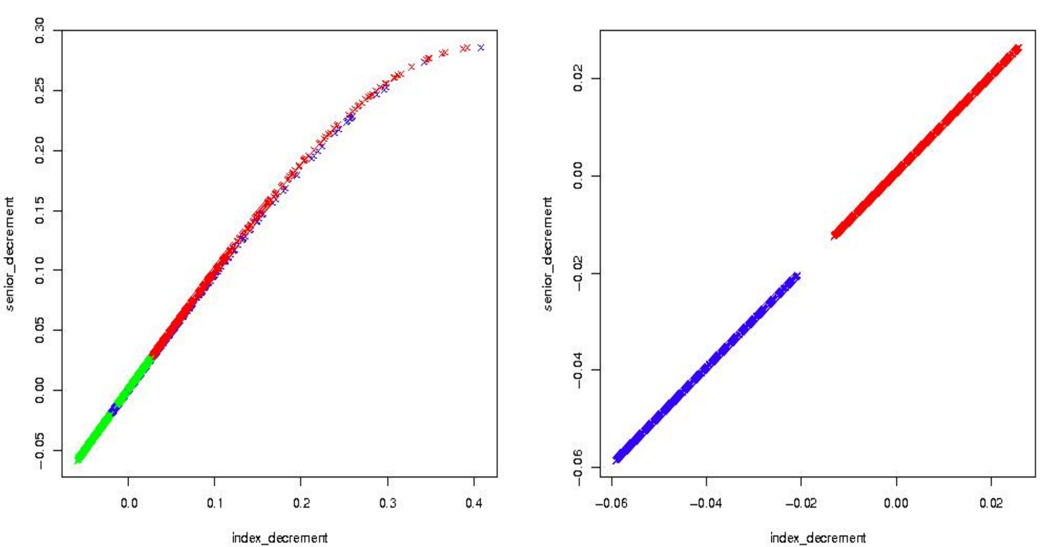

Figure 3: *Senior vs Index Price Changes between 0 and* $\tau_{(1)}$ ($\tau_{(1)} < T = 5$, *last pane; zoom on* $\tau_{(1)} < T_1 = 1$, *right pane).*

The results are displayed in Table 1. In this table we denoted by $\Sigma_0 = \frac{10^4}{kT}\Pi_0$ or $\frac{10^4}{(1-R-k)T}\Pi_0$ (for the equity or senior tranche) or $\Sigma_0 = \frac{10^4}{(1-R)T}\Theta_0$ (for the index) stylized 'bp spreads' corresponding to the time zero prices Π_0 and Θ_0 of the equity or senior tranche and of the index. Note that by construction the hedging variance reduction factor $\frac{\widehat{\mathbb{V}\mathrm{ar}}(\Pi_{\tau_{(1)}}-\Pi_0)}{\widehat{\mathbb{V}\mathrm{ar}}(e_{\tau_{(1)}})}$ (displayed in the last column of Table 1) is equal to $\frac{1}{1-\rho^2}$, where ρ (displayed in column four) is the empirical correlation of the tranche price increments $\Pi_{\tau_{(1)}} - \Pi_0$ versus the index price increments $\Theta_{\tau_{(1)}} - \Theta_0$ (so ρ^2 is the coefficient of determination $R2$ of the regression).

	Π_0 or Θ_0	Σ_0 or S_0	δ_0	ρ	$\widehat{\mathrm{dev}}(\Pi_{\tau_{(1)}} - \Pi_0)$	$\frac{\widehat{\mathbb{V}\mathrm{ar}}(\Pi_{\tau_{(1)}}-\Pi_0)}{\widehat{\mathbb{V}\mathrm{ar}}(e_{\tau_{(1)}})}$
Eq	0.1994141	1994.141	-0.0005463	-0.2282511	0.0002934755	**1.054962**
Eq1	–	–	0.006361217	0.9896867	0.0003284477	48.73252
Eq2	–	–	0.003125651	0.9806078	0.0001670460	26.03603
Sen	0.2862444	1431.222	1.000546	0.9999973	0.08546279	**184335.6**
Sen1	–	–	0.9936388	0.9999973	0.03537296	1164639
Sen2	–	–	0.9968743	0.9999998	0.03639573	2546628
Ind	0.4856585	1618.861	–	–	–	–

Table 1: *Hedging Tranches by the Index in the Semi-Homogeneous Model (*dev=*standard deviation).*

Recall that the senior tranche's behaviour is very close to that of the index itself (see Section 5.2.2). Accordingly, we find that hedging the senior tranche with the index works extremely well (huge variance reduction factor in bold blue in the last column). This case thus seems to be supportive of the claim according to which one could use the index for hedging a loss derivative, even in a non Markovian model of portfolio loss process L.

But in the case of the equity tranche we get the exact opposite message: the index is essentially useless for hedging the equity tranche (variance reduction factor equal to 1.05 in bold red in the table).
Moreover, the equity tranche variance reduction factors conditional on defaults in the first and in the second group of obligors (in purple in the table) amount to 48.73 and 26.03, respectively. This supports the interpretation that the impossibility of hedging the equity tranche with the index really comes from the fact that the full model dynamics is not represented in the loss process.

We conclude that in general, unless specific values of the model parameters and tranche characteristics are considered, hedging tranches with the index may not be possible in a non Markovian model of portfolio loss process L.

Note that the equity and the senior tranche sum-up to the index, by construction. Therefore a perfect replication of the equity tranche (say) is provided by a long position in the index and a short position in the senior tranche. As a reality-check of this statement, we performed a bilinear regression of the equity price increments versus the index and the senior tranche price increments, in order to minimize over $(\delta_0^{ind}, \delta_0^{sen})$ the (risk-neutral) variance of

$$\widetilde{e}_{\tau_{(1)}} = (\Pi^{eq}_{\tau_{(1)}} - \Pi^{eq}_0) - \delta^{ind}_0(\Theta_{\tau_{(1)}} - \Theta_0) - \delta^{sen}_0(\Pi^{sen}_{\tau_{(1)}} - \Pi^{sen}_0) \ . \tag{51}$$

The results are displayed in Table 2. We recover the perfect two-instruments replication strategy mentioned above, whereas a single-instrument hedge using only the index was essentially useless in this case (cf. red entry in Table 1).

Observe that these results were derived in a very simple two-dimensional model of portfolio credit risk (for computational cost issues). And we got the message that even in a market that would be given by this simple (but not Markovian in L) model, a pure top approach can be doomed to failure

δ_0^{ind}	δ_0^{sen}	$\widehat{\mathbf{dev}}(\widetilde{e}_{\tau_{(1)}})$	$\frac{\widehat{\mathbb{V}\mathrm{ar}}(\Pi_{\tau_{(1)}}-\Pi_0)}{\widehat{\mathbb{V}\mathrm{ar}}(e_{\tau_{(1)}})}$
1	-1	1.968e-17	8.842381e+24

Table 2: *Replicating the equity tranche by the index and the senior tranche.*

(since there may be virtually no correlation between the tranche and the index price increments). We let the reader imagine what the situation can be in a real market of credit derivatives.

5.2.4 Fully Homogeneous Case

Admittedly, the previous example is an extreme case, since the equity tranche is almost bound to be exhausted at T given the value of the strike k and the structure of the portfolio loss distribution at the horizon T (cf. the left pane of Figure 1).

For confirmation of the previous analysis and interpretation of the results, we thus redid the computations using the same values as before for all the model, products and simulation parameters, except for the fact that the following pre-default individual intensities were used, for $l = 1, 2$:

$$\widetilde{\lambda}^l(\imath) = \frac{1}{n} + \frac{\sum_{1\leq\ell\leq d} i_\ell}{nd} =: \widehat{\lambda}(\sum_{1\leq\ell\leq d} i_\ell) \; . \qquad (52)$$

We are thus in the case of homogeneous obligors mentioned in Remark 5.2, reducible to a local intensity model (with $d = 1$ and pre-default individual intensity $\widehat{\lambda}(i)$ therein). So in this case we expect that hedging tranches by the index should work, including in the case of the equity tranche. This is exactly what happens numerically, as it is evident from the following figures and tables (which are the analogs of those in previous sections, using the same notation everywhere).

Out of our new 10^4 draws using the intensities given in (52), we got 9922 draws with $\tau_{(1)} < 5$, among which 6267 ones with $\tau_{(1)} < 1$, subdividing themselves into 3186 defaults in the first group of obligors and 3081 defaults in the second one.

Note that all red and blue curves are superimposed, which is consistent with the fact that the identity of a defaulted name has no bearing in this case, given the present specification of the identities.

Looking at Table 3, we find as before that hedging the senior tranche with the index works extremely well (huge variance reduction factor in bold blue in the last column). But as opposed to the previous parameterization, hedging the equity tranche with the index also works very well (variance reduction factor of 84.40411 in bold purple in the last column). Moreover the equity unconditional variance reduction factor and variance reduction factor conditional on defaults in the first and in the second group of obligors are now roughly the same.

These results confirm our previous analysis that the impossibility of hedging the equity tranche by the index in the previous case was due to the non-Markovianity of the loss process L.

6 Conclusions

Even for basket credit derivatives which can be considered as derivatives on the (non-traded) loss process L in the sense that their payoff processes are given as functions of L, this loss process L is not a sufficient statistic for pricing and hedging them. This effectively means that their prices depend on factors others than L, like the identity (and not only the number) of the defaulted names, the ratings (or implied ratings, and not only the identities) of survivors, etc. This makes of course perfect sense since it is rather clear that the default of a major name in the index does not bear the

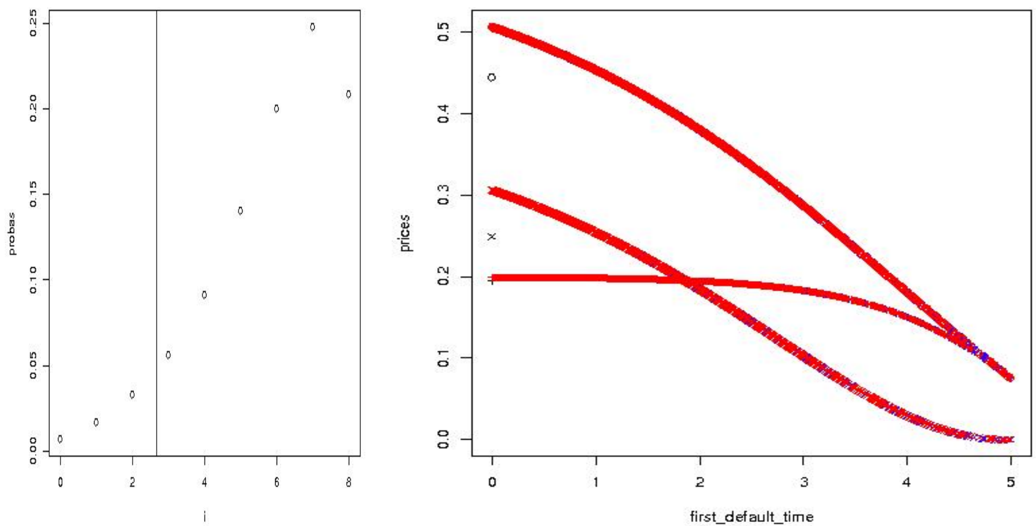

Figure 4: *(Left) Portfolio loss distribution at maturity $T = 5y$ (Right) Tranche Prices at $\tau_{(1)}$ (for $\tau_{(1)} < T$).*

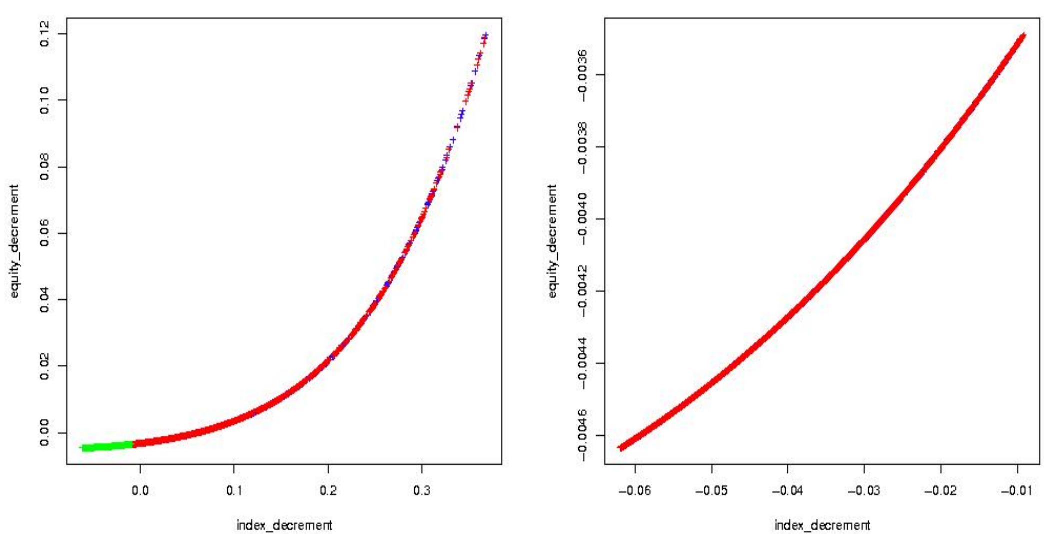

Figure 5: *Equity vs Index Price Decrements between 0 and $\tau_{(1)}$ ($\tau_{(1)} < T$, last pane; zoom on $\tau_{(1)} < 1$, right pane).*

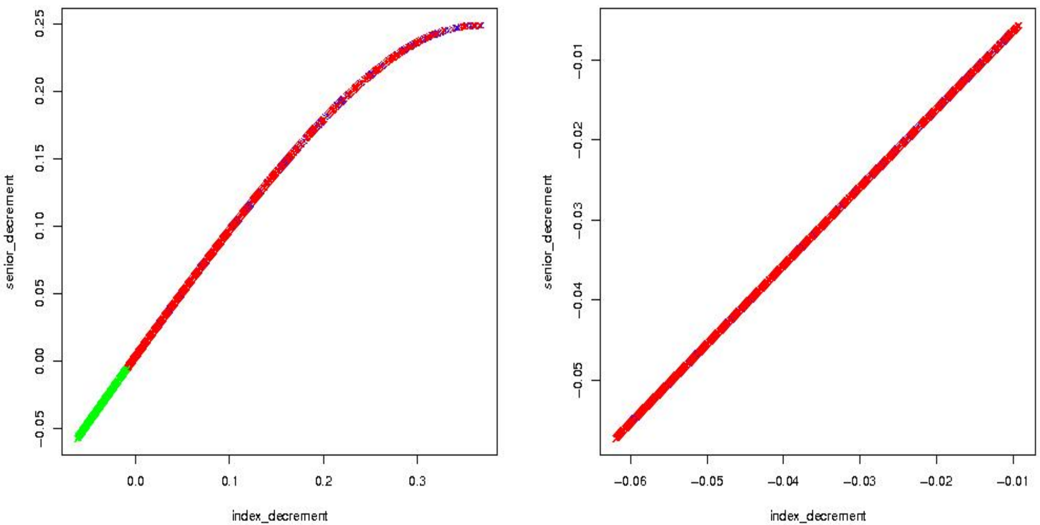

Figure 6: *Senior vs Index Price Decrements between 0 and* $\tau_{(1)}$ *(*$\tau_{(1)} < T$*, last pane; zoom on* $\tau_{(1)} < 1$*, right pane).*

	Π_0 or Θ_0	Σ_0 or S_0	δ_0	ρ	$\widehat{\text{dev}}(\Pi_{\tau_{(1)}} - \Pi_0)$	$\frac{\widehat{\mathbb{V}\text{ar}}(\Pi_{\tau_{(1)}} - \Pi_0)}{\widehat{\mathbb{V}\text{ar}}(e_{\tau_{(1)}})}$
Eq	0.1948869	1948.869	0.02067877	0.9940585	0.001549888	**84.40411**
Eq1	–	–	0.02071242	0.9940464	0.001563447	84.23423
Eq2	–	–	0.02064252	0.9940747	0.00153585	84.63472
Sen	0.2488361	1244.180	0.9793212	0.9999973	0.05714564	**187064.2**
Sen1	–	–	0.9792876	0.9999973	0.05754927	186064.3
Sen2	–	–	0.9793575	0.9999973	0.05673051	188254.7
Ind	0.443723	1479.076	–	–	–	–

Table 3: *Hedging Tranches by the Index in the Fully-Homogeneous Model (*dev*=standard deviation).*

same informational content as that of an arbitrary firm, and, moreover, pricing is done by agents with regard to the quality of the remaining names in the portfolio rather than with regard to the defaulted names.
As a consequence, the use of *pure top and top approaches* should be considered with caution.

As for *top-down approaches* possibly used for hedging of basket credit derivatives by single-name derivatives, we saw in Section 3.6 that they eventually boil down to the bottom-up approach, since except for the case of a full 'down' filtration (which effectively corresponds to ending 'bottom-up'), there is no way to establish connection between a top-down model and real-life single-name default markets. Recall for instance that the 'down' intensity of a name in the sense of a general top-down approach typically fails to vanish after that name's default.

Our conclusion is that only the *bottom-up approaches* allow adequate risk management of portfolio credit derivatives. To further support of the bottom-up approaches, note that it is quite likely that sooner or later the financial markets will introduce credit portfolio derivatives that will derive their cash flows from the evolution of the credit ratings in the underlying portfolio of credits. Then, of course, methodologies capable of only modeling the dynamics of the loss process L and/or the dynamics of the related conditional probabilities, will not be adequate. It is also clear that such methodologies are inadequate for pricing and hedging of credit portfolio derivatives whose cash-flows depend not only on the evolution of the cumulative loss process, but also on the identity of the defaulting names in the portfolio.

6.1 Bottom-Up Approaches, Curse of dimensionality and Markovian copulae

At this point one may raise the issue of the so called *curse of dimensionality* that is commonly associated with the bottom-up approaches: for example, if considered as a Markov chain, the process H lives in a 'n-dimensional' state space of the size of 2^n. However, recent developments in the bottom up modeling enable one to efficiently cope with this curse of dimensionality. It is thus possible to specify high-dimensional bottom-up dynamic Markovian models of portfolio credit risk *with automatically calibrated model marginals* (to the individual CDS curves, say), see Bielecki, Vidozzi and Vidozzi [10].
Much like in the standard static copula framework, this effectively reduces the main computational cost issue, that relative to model calibration, to calibration of the few *dependence parameters* in the model at hand. This calibration can thus be achieved in a very reasonable time, including by pure simulation procedures if need be (without using any closed pricing formulae, if there aren't any in the model under consideration). Appropriate reduction variance methods may help in this regard (see Crépey and Carmona [16]) .

A A glimpse of General Theory

In this Appendix we recall well known definitions and results from the theory of processes that we use repeatedly in this paper (see, e.g., Dellacherie and Meyer [19]).

A.1 Optional Projections

Let X be an integrable process (not necessarily adapted). Then there exists a unique adapted process $({}^pX_t)_{t\geq 0}$, called the *optional projection* of X, such that, for any stopping time τ,

$$\mathbb{E}\left({}^pX_\tau \,|\, \mathcal{F}_\tau\right) = \mathbb{E}\left(X_\tau \,|\, \mathcal{F}_\tau\right) \ .$$

A.2 Dual Predictable Projections and Compensators

Let A be an integrable non-decreasing process (not necessarily adapted). Then there exists a unique predictable non-decreasing process $(A_t^p)_{t\geq 0}$, called the *dual predictable projection* of A, such that, for any positive predictable process H:

$$\mathbb{E}\left(\int_0^\infty H_s dA_s\right) = \mathbb{E}\left(\int_0^\infty H_s dA_s^p\right) .$$

In case A is adapted, hence a sub-martingale, and admitting as such a unique Doob-Meyer decomposition

$$A_t = M_t + \widetilde{A}_t$$

where M is a local martingale and the *compensator* $\widetilde{A}$ is a predictable finite variation process, then $A^p = \widetilde{A}$.

Moreover (see Dellacherie and Meyer [19] or Last and Brandt [39], Brémaud [13]), these definitions and results admit straightforward extensions to *integer-valued random measures* (rather than processes) A, viewing such a measure A as a family, parameterized by α, of increasing processes $A_t(\omega, \alpha)$, counting the jumps with mark α in the mark space $\mathcal{A}$ of an underlying marked point process.

A.3 A General Result

Let τ denote an $\widehat{\mathbb{F}}$-stopping time where $\widehat{\mathbb{F}} \subseteq \mathbb{F}$. Let Λ and $\widehat{\Lambda}$ denote the $\mathbb{F}$-compensator and the $\widehat{\mathbb{F}}$-compensator of τ, respectively. We denote by ${}^o\Lambda$ the optional projection of Λ on the sub-filtration $\widehat{\mathbb{F}}$ (see Section A.1). Since Λ is non-decreasing, ${}^o\Lambda$ is an $\widehat{\mathbb{F}}$-submartingale, thus it admits a Doob-Meyer decomposition. With a slight abuse of notation (see Section A.2), we denote by $({}^o\Lambda)^p$ the $\widehat{\mathbb{F}}$-compensator (in the sense of the predictable non-decreasing component) of the $\widehat{\mathbb{F}}$-submartingale ${}^o\Lambda$.

Proposition A.1 *$\widehat{\Lambda}$ and Λ denoting the $\widehat{\mathbb{F}}$- and the $\mathbb{F}$- compensators of τ, one has,*

$$\widehat{\Lambda} = ({}^o\Lambda)^p \ . \tag{53}$$

Moreover, in case $\widehat{\Lambda}$ and Λ are time-differentiable with related $\widehat{\mathbb{F}}$- and $\mathbb{F}$- intensity processes $\widehat{\lambda}$ and λ, then $\widehat{\lambda}$ is the $\widehat{\mathbb{F}}$-optional projection of λ (see Section A.1), so

$$\widehat{\lambda} = {}^o\lambda \ . \tag{54}$$

Proof. $\widehat{\Lambda} := ({}^o\Lambda)^p$ is an $\widehat{\mathbb{F}}$-predictable non-decreasing process. Moreover the tower property of iterated conditional expectations yields, for $0 \leq t \leq T$,

$$\begin{aligned}\mathbb{E}(\int_t^T dH_u - d\widehat{\Lambda}_u|\widehat{\mathcal{F}}_t) &= \mathbb{E}(\int_t^T dH_u - d({}^o\Lambda)_u|\widehat{\mathcal{F}}_t) \\ &= \mathbb{E}(\int_t^T dH_u - d\Lambda_u|\widehat{\mathcal{F}}_t) = \mathbb{E}\Big(\mathbb{E}(\int_t^T dH_u - d\Lambda_u|\mathcal{F}_t)|\widehat{\mathcal{F}}_t\Big) = 0 \,,\end{aligned}$$

since $H - \Lambda$ is an $\mathbb{F}$-martingale. This proves (13). Now, in case Λ and $\widehat{\Lambda}$ are time-differentiable with related intensity processes λ and $\widehat{\lambda}$, (13) means that

$$\int_0^t \widehat{\lambda}_s ds - \mathbb{E}(\int_0^t \lambda_s ds|\widehat{\mathcal{F}}_t) \tag{55}$$

is an $\widehat{\mathbb{F}}$-martingale. Moreover it is immediate to check, using the tower property of iterated conditional expectations, that

$$\mathbb{E}(\int_0^t \lambda_s ds|\widehat{\mathcal{F}}_t) - \int_0^t \mathbb{E}(\lambda_s|\widehat{\mathcal{F}}_s)ds \tag{56}$$

is an $\widehat{\mathbb{F}}$-martingale as well. By addition between (55) and (56),

$$\int_0^t \widehat{\lambda}_s ds - \int_0^t \mathbb{E}(\lambda_s|\widehat{\mathcal{F}}_s)ds$$

is in turn an $\widehat{\mathbb{F}}$-martingale. Since it is also a predictable (as continuous) finite variation process, it is thus in fact identically equal to 0, so for $t \geq 0$,

$$\widehat{\lambda}_t = \mathbb{E}(\lambda_t|\widehat{\mathcal{F}}_t) ,$$

and (54) follows. □

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