Doubly Reflected BSDEs with Call Protection and their Approximation

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1 Introduction

In this work and in the follow-up paper [16], we consider the issue of numerical solution of a doubly reflected backward stochastic differential equation, with an upper barrier which is only active on random time intervals (doubly reflected BSDE with an intermittent upper barrier, or RIBSDE for short henceforth, where the 'I' in RIBSDE stands for 'intermittent').

From the mathematical point of view, such RIBSDEs and, in the Markovian case, the related variational inequality (VI for short henceforth) approach, were first introduced in Crépey [18]. From the point of view of financial interpretation, RIBSDEs arise as pricing equations of game options (like convertible bonds) with call protection, in which the call times of the option's issuer are subject to constraints preventing the issuer from calling the bond on certain random time intervals. Moreover, in the standing example of convertible bonds, this protection is typically monitored at discrete times in a possibly very path-dependent way. Calls may thus be allowed or not at a given time depending on the past values of the underlying stock S, which leads, after extension of the state space to markovianize the problem, to highly-dimensional pricing problems. Deterministic pricing schemes are then ruled out by the curse of dimensionality, and simulation methods appear to be the only viable alternative.

The ***purpose of this paper*** is to propose a practical and mathematically justified approach to the problem of solving numerically by simulation the RIBSDEs that arise as pricing equations of game options with call protection. The ***main result*** is Theorem 3.3, which establishes convergences rates for a discrete time approximation scheme by simulation to an RIBSDE. The practical value of this scheme is assessed in the follow-up paper [16].

1.1 Standing Notation

Let us be given a continuous time stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where in the financial interpretation $\mathbb{P}$ denotes a risk-neutral pricing measure. We assume that the filtration $\mathbb{F}$ satisfies the usual completeness and right-continuity conditions, and that all semimartingales are càdlàg. Also, since our practical concern consists in pricing a contingent claim with maturity T, we set $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F}_0$ trivial and $\mathcal{F}_T = \mathcal{F}$. Moreover, we declare that a *process* on $[0,T]$ (resp. *a random variable*) has to be $\mathbb{F}$-adapted (resp. $\mathcal{F}$*-measurable*), by definition. By default in the sequel, all (in)equalities between random variables or processes are to be understood $d\mathbb{P}$ – almost surely or $d\mathbb{P} \otimes dt$ – almost everywhere, respectively.

We shall denote:
- ${}^c\widetilde{\Omega}$, the complement of an event $\widetilde{\Omega} \subseteq \Omega$,
- $\mathbb{N}_n = \{0, 1, \ldots, n\}$, for every non-negative integer n,
- R^q and $R^{1\otimes q}$, the set of q-dimensional vectors and row-vectors with real components,
- $|\cdot|_p$ for $p \in [1, +\infty)$, or simply $\|\cdot\|$ in case $p = 2$ or $|\cdot|$ in case $p = 1$, the p-norm of an element of R^q or $R^{1\otimes q}$,
- $^\mathsf{T}$, the transposition operator.

2 Results in the Continuous-Time Setting

2.1 Diffusion Set-Up with Marker Process

Given a q-dimensional Brownian motion W, let X be the solution on $[0,T]$ of the following SDE:

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dW_s\ , \tag{1}$$

where $X_0 \in \mathbb{R}^q$ and the coefficients $b : [0,T] \times \mathbb{R}^q \to \mathbb{R}^q$ and $\sigma : [0,T] \times \mathbb{R}^q \to \mathbb{R}^{q\otimes q}$ are such that

(Hx) b, σ are Λ-Lipschitz continuous in x, uniformly in t, for some positive constant Λ, and $b(t, 0)$ and $\sigma(t, 0)$ are bounded by Λ over $[0,T]$.

Let the *time-state space* $\mathcal{E} = [0,T] \times \mathbb{R}^q \times \mathcal{K}$ for some finite set $\mathcal{K}$. Given a function u of three arguments t, x, k where the third argument k takes its values in a discrete set, so that k can be thought of as referring to the index of a vector or system of functions of time t and the spatial variable x, we shall denote either $u(t, x, k)$, or $u^k(t, x)$, depending on what is more convenient in the context at hand. Moreover we denote by ∇u, ∂u and $\mathcal{H}u$ the *gradient*, the *row-gradient* and the *Hessian* of u with respect to x. We also let $\partial_t u = \frac{\partial u}{\partial t}$.

Let us further be given a set $\mathfrak{T} = \{T_0, T_1 \ldots, T_N\}$ of fixed times with $0 = T_0 < T_1 < \cdots < T_{N-1} < T_N = T$. On the state-space $\mathcal{E}$, we then consider the *factor process* $\mathcal{X} = (X, H)$, where X is defined by (1), and where the $\mathcal{K}$-valued pure jump *marker process* H is supposed to be constant except for deterministic jumps at the (strictly) positive T_Is, from H_{T_I-} to

$$H_{T_I} = \kappa_I(X_{T_I}, H_{T_I-})\ , \tag{2}$$

for *jump functions* $\kappa_I : \mathbb{R}^q \times \mathcal{K} \to \mathcal{K}$, starting from an initial condition

$$H_0 = k \in \mathcal{K} \tag{3}$$

(note that H does not jump at time $T_0 = 0$).

Remark 2.1 In the financial interpretation (see section 2.5), the function u typically represents a *pricing function*, and $\mathfrak{T}$, a set of *call protection monitoring times*. The marker process H is used for keeping track of the path-dependence of the call protection clauses, in view of 'markovianizing' the model.

We suppose that the jump function κ_I is given as

$$\kappa_I^k(x) = \kappa_I^{k,-}(x)\mathbf{1}_{\{\mathrm{d}(x)<0\}} + \kappa_I^{k,+}(x)\mathbf{1}_{\{\mathrm{d}(x)\geq 0\}} ,$$

with $\kappa_I^{\pm} : \mathbb{R}^q \times \mathcal{K} \to \mathcal{K}$, and where d is the algebraic distance function to a closed domain $\mathcal{O} = \{x \in \mathbb{R}^q \mid \mathrm{d}(x) \leq 0\}$ of $\mathbb{R}^q$. Note that the function $\kappa(T_I, \cdot, k)$ is continuous at every $x \notin \partial\mathcal{O}$. One shall work under the following regularity assumption on $\mathcal{O}$.

(Ho) The distance function d is of class C_b^3.

Let us finally be given a non-decreasing sequence of stopping times $\vartheta = (\vartheta_l)_{0\leq l\leq N+1}$ defined by $\vartheta_0 = 0$ and, for every $l \geq 0$:

$$\vartheta_{2l+1} = \inf\{t > \vartheta_{2l}\,;\, H_t \notin K\} \wedge T\,,\ \vartheta_{2l+2} = \inf\{t > \vartheta_{2l+1}\,;\, H_t \in K\} \wedge T\,, \tag{4}$$

relatively to a given subset K of $\mathcal{K}$. Observe that the ϑ_ls, to be interpreted as *times of switching of call protection* in the financial interpretation, reduce to $\mathfrak{T}$-valued stopping times, and that $\vartheta_{N+1} = T$.

2.2 Markovian RIBSDE

We denote by (P) the class of functions u on $\mathbb{R}^q$, $[0,T] \times \mathbb{R}^q$ or $\mathcal{E}$ such that u is Borel-measurable, with polynomial growth in its spatial argument $x \in \mathbb{R}^q$. Let us further be given real-valued and continuous *cost functions* $g(x)$, $\ell(t,x)$, $h(t,x)$ and $f(t,x,y,z)$ in (P), with $y \in \mathbb{R}$ and $z \in \mathbb{R}^{1\otimes q}$ in f, such that:
• the *running payoff function* $f(t,x,y,z)$ is Lipschitz in (y,z);
• the *payoff function at maturity* $g(x)$ and the *put and call payoff functions* $\ell(t,x)$ and $h(t,x)$ satisfy $\ell \leq h$, $\ell(T,\cdot) \leq g \leq h(T,\cdot)$.
In the sequel, we shall sometimes use the following assumptions

(Hℓ) $\ell(t,x) = \lambda(t,x) \vee c$, for a constant $c \in \mathbb{R} \cup \{-\infty\}$ and a function λ of class $\mathcal{C}^{1,2}$ on $[0,T] \times \mathbb{R}^q$ such that

$$\lambda,\, \mathcal{G}\lambda,\, \partial\lambda\sigma\, \in (P)\,, \tag{5}$$

(Hh) $h(t,x)$ is jointly Lipschitz in (t,x).

The Markovian *RIBSDE* with data

$$f(t,X_t,y,z)\,,\ \xi = g(X_T)\,,\ \ell(t,X_t)\,,\ h(t,X_t)\,,\ \vartheta\,, \tag{6}$$

denoted in the sequel by $(\mathcal{E})$, is a doubly reflected BSDE (see, e.g., [20, 18]) with lower and upper barriers respectively given by, for $t \in [0,T]$,

$$L_t = \ell(t,X_t)\,,\ U_t = \sum_{l=0}^{[N/2]} \mathbf{1}_{[\vartheta_{2l},\vartheta_{2l+1})}\infty + \sum_{l=1}^{[N+1/2]} \mathbf{1}_{[\vartheta_{2l-1},\vartheta_{2l})}h(t,X_t)\,. \tag{7}$$

With respect to standard, 'continuously reflected' doubly reflected BSDEs, the peculiarity of RIBSDEs is thus that the 'nominal' upper obstacle $h(t,X_t)$ is only active on the 'odd' random time intervals $[\vartheta_{2l-1},\vartheta_{2l})$, $l>0$.

Let us introduce the following Banach (or Hilbert, in case of $\mathcal{L}^2$ or $\mathcal{H}^2_q$) spaces of random variables or processes, where p denotes here and henceforth a real number in $[1,\infty)$:
• $\mathcal{L}^p$, the space of real valued random variables ξ such that

$$\|\xi\|_{\mathcal{L}^p}=\Big(\mathbb{E}[|\xi|^p]\Big)^{\frac{1}{p}}<+\infty\,;$$

• $\mathcal{S}^p_q$, for any real $p\geq 2$ (or $\mathcal{S}^p$, in case $q=1$), the space of $\mathbb{R}^q$-valued càdlàg processes Y such that

$$\|Y\|_{\mathcal{S}^p_d}:=\Big(\mathbb{E}\Big[\sup_{t\in[0,T]}|Y_t|^p\Big]\Big)^{\frac{1}{p}}<+\infty\,;$$

• $\mathcal{H}^p_q$ (or $\mathcal{H}^p$, in case $d=1$), the space of $\mathbb{R}^{1\otimes q}$-valued predictable processes Z such that

$$\|Z\|_{\mathcal{H}^p_q}=\Big(\mathbb{E}\Big[\Big(\int_0^T\|Z_t\|^2\,dt\Big)^{\frac{p}{2}}\Big]\Big)^{\frac{1}{p}}<+\infty\,;$$

• $\mathcal{A}^2$, the space of finite variation processes A with (non-decreasing) Jordan components $A^\pm\in\mathcal{S}^2$ null at time 0.

Under (Hx), one thus has $\|X\|_{\mathcal{S}^2}\leq C_\Lambda$, where from now on C_Λ is a generic constant whose value may change from line to line but which depends only on Λ, T, X_0 and q (in case this constant depends on some extra parameter, say ρ, we shall write C^ρ_Λ). $p\geq 1$, $\varepsilon>0$ we shall write C^p_Λ or C^ε_Λ if it).

Definition 2.2 An $(\Omega,\mathbb{F},\mathbb{P})$-*solution* $\mathcal{Y}$ to $(\mathcal{E})$ is a triple $\mathcal{Y}=(Y,Z,A)$, such that:

(i) $Y\in\mathcal{S}^2, Z\in\mathcal{H}^2_q, A\in\mathcal{A}^2, A^+$ is continuous,
and $\{(\omega,t)\,;\,\Delta Y_t\neq 0\}=\{(\omega,t)\,;\,\Delta A^-_t\neq 0\}\subseteq\bigcup_{l=0}^{[N/2]}[\![\vartheta_{2l}]\!]$,
(ii) $Y_t=\xi+\int_t^T f(s,X_s,Y_s,Z_s)ds+A_T-A_t-\int_t^T Z_s dW_s\ ,\ t\in[0,T]$,
(iii) $L_t\leq Y_t$ on $[0,T]\,,\ Y_t\leq U_t$ on $[0,T]$
and $\int_0^T(Y_t-L_t)dA^+_t=\int_0^T(U_{t-}-Y_{t-})dA^-_t=0$,

where L and U are defined by (7), and with the convention that $0\times\pm\infty=0$ in (iii).

Note that this definition admits an obvious extension to a random terminal time θ, instead of constant $\theta=T$ above. This extension will be used in the next results, in the special case of simply reflected and (continuously) doubly reflected BSDEs.

Also note that $(\mathcal{E})$ is implicitly parameterized by the initial condition $(t=0,x,k)$ of $\mathcal{X}$. In the sequel we let the superscript ^{t} (whenever necessary) stand in reference to an initial condition (t,x,k) of $\mathcal{X}$ (with in particular $t\in[0,T]$, rather than $t=0$ implicitly above).[1] By application of the results of [18], one thus has,

[1] However it is convenient to extend all our processes to $[0,T]$ 'in a natural way' so that they live in spaces of functions defined over $[0,T]$, which do not change with t, see Crépey [18].

Proposition 2.1 (Crépey [18]) *We assume (Hℓ).*
(i) *The following iterative construction is well-defined, for I decreasing from N to 0: $\mathcal{Y}^{I,t} = (Y^{I,t}, Z^{I,t}, A^{I,t})$ is the unique solution, with $A^{I,t}$ continuous, to the reflected BSDE with random terminal time ϑ^t_{I+1} (for I even) or the doubly reflected BSDE with random terminal time ϑ^t_{I+1} (for I odd) on $[t, \vartheta^t_{I+1}]$ with data*

$$\left\{ \begin{array}{ll} f(s, \mathcal{X}^t_s, y, z)\,,\ Y^{I+1,t}_{\vartheta^t_{I+1}}\,,\ \ell(s, \mathcal{X}^t_s)\,,\ \vartheta^t_{I+1} & (I \text{ even}) \\ f(s, \mathcal{X}^t_s, y, z)\,,\ \min(Y^{I+1,t}_{\vartheta^t_{I+1}}, h(\vartheta^t_{I+1}, \mathcal{X}^t_{\vartheta^t_{I+1}}))\,,\ \ell(s, \mathcal{X}^t_{st})\,,\ h(s, \mathcal{X}^t_s)\,,\ \vartheta^t_{I+1} & (I \text{ odd}) \end{array} \right. \tag{8}$$

where, in case $I = N$, $Y^{I+1,t}_{\vartheta^t_{I+1}}$ is to be understood as $g(\mathcal{X}^t_T)$.

(ii) *Let us define $\mathcal{Y}^t = (Y^t, Z^t, A^t)$ on $[t, T]$ by, for every $I = 0, \ldots, N$:*
- *$(Y^t, Z^t) = (Y^{I,t}, Z^{I,t})$ on $[\vartheta^t_I, \vartheta^t_{I+1})$, and also at $\vartheta^t_{I+1} = T$ in case $I = N$,*
- *$dA^t = dA^{I,t}$ on $(\vartheta^t_I, \vartheta^t_{I+1})$,*

$$\Delta K^t_{\vartheta^t_I} = Y^{I,t}_{\vartheta^t_I} - \min(Y^{I,t}_{\vartheta^t_I}, U^t_{\vartheta^t_I}) = \Delta Y^t_{\vartheta^t_I} (= 0 \text{ for } I \text{ odd})$$

and $\Delta K^t_T = \Delta Y^t_T = 0$. So in particular

$$Y^t_t = \left\{ \begin{array}{ll} Y^{0,t}_t, & k \in K \\ Y^{1,t}_t, & k \notin K\ , \end{array} \right. \tag{9}$$

where k is the index which is implicit in the condition initial (t, x, k) of $\mathcal{X}$ referred to by the superscript ^{t}.
Then $\mathcal{Y}^t = (Y^t, Z^t, A^t)$ is the unique solution to the RIBSDE $(\mathcal{E}^t)$. Moreover, $A^{t,+}$ is a continuous process, and

$$\{(\omega, s)\,;\ \Delta A^{t,+}_s \neq 0\} \subseteq \bigcup_{\{I \text{ even}\}} [\![\vartheta^t_I]\!]\,,\ \Delta Y^t = \Delta K^{t,-} \text{ on } \bigcup_{\{I \text{ even}\}} [\![\vartheta^t_I]\!]\,.$$

Note in particular that existence and uniqueness of solutions ***with a continuous reflecting process component*** (process $A^{I,t}$ above) to the auxiliary reflected BSDEs and doubly reflected BSDEs with random terminal time, that appear in point (i) above, is granted by the results of [19, 18].

One will need further stability results on $\mathcal{Y}^t$, or, more precisely, on the $\mathcal{Y}^{I,t}$s. Toward this end a suitable stability assumption on ϑ^t is needed. Our next result is thus a càdlàg property of ϑ, viewed as a random function of the initial condition (t, x, k) of $\mathcal{X}$.

Proposition 2.2 *At every (t, x, k) in $\mathcal{E}$, ϑ is, almost surely:*
(i) *continuous at (t, x, k) if $t \notin \mathfrak{T}$, and right-continuous at (t, x, k) if $t \in \mathfrak{T}$,*
(ii) *left-limited at (t, x, k) if $t = T_I \in \mathfrak{T}$ and $x \notin \partial\mathcal{O}$.*

By the above statement, we mean that:
- $\vartheta^{t_n} \to \vartheta^t$ if $(t_n, x_n, k) \to (t, x, k)$ with $t \notin \mathfrak{T}$, or, for $t = T_I \in \mathfrak{T}$, if $\mathcal{E}_{I+1} \ni (t_n, x_n, k) \to (T_I, x, k)$;
- if $\mathcal{E}^*_I \ni (t_n, x_n, k) \to (t = T_I, x \notin \partial\mathcal{O}, k)$, then ϑ^{t_n} converges to some non-decreasing sequence $\widetilde{\vartheta}^t = (\widetilde{\vartheta}^t_I)_{0 \le l \le N+1}$ of $[0, T]$-valued stopping times, with in particular $\widetilde{\vartheta}^t_{N+1} = T$.

Definition 2.3 One denotes by $\widetilde{\mathcal{Y}}^t = (\widetilde{\mathcal{Y}}^{I,t})_{0\le I\le N}$, with $\widetilde{\mathcal{Y}}^{I,t} = (\widetilde{Y}^{I,t}, \widetilde{Z}^{I,t}, \widetilde{A}^{I,t})$ and $\widetilde{A}^{I,t}$ continuous for every $I = 0, \ldots, N$, the sequence of solutions of the BSDEs with random terminal times which is obtained by substituting $\widetilde{\vartheta}^t$ to ϑ^t in the construction of $\mathcal{Y}^t$ in Proposition 2.1(i).

Observe that since the ϑ_Is are in fact $\mathfrak{T}$-valued stopping times:
• The continuity assumption effectively means that $\vartheta_I^{t_n} = \vartheta_I^t$ for n large enough, almost surely, for every $I = 1, \ldots, N+1$ and $\mathcal{E} \ni (t_n, x_n, k) \to (t, x, k) \in \mathcal{E}$ with $t \notin \mathfrak{T}$;
• The right-continuity, resp. left-limit assumption, effectively means that for n large enough $\vartheta_I^{t_n} = \vartheta_I^t$, resp. $\widetilde{\vartheta}_I^t$, almost surely, for every $I = 1, \ldots, N+1$ and $\mathcal{E}_{I+1} \ni$, resp. $\mathcal{E}_I^* \ni$ $(t_n, x_n, k) \to (T_I, x, k) \in \mathcal{E}$.

Proposition 2.3 (Crépey [18]) *We assume (Hℓ) and (Hh). Let $\mathcal{Y}^t = (\mathcal{Y}^{I,t})_{0\le I\le N}$ and $\widetilde{\mathcal{Y}}^t = (\widetilde{\mathcal{Y}}^{I,t})_{0\le I\le N}$ be defined as in Proposition 2.1(i) and Definition 2.3, respectively. Then, for every $I = N, \ldots, 0$:*
(i) *One has the following estimate on $\mathcal{Y}^{I,t}$,*

$$\|Y^{I,t}\|^2_{\mathcal{S}^2} + \|Z^{I,t}\|^2_{\mathcal{H}^2_q} + \|A^{I,t}\|^2_{\mathcal{S}^2} \le C(1 + |x|^{2q}) \ . \tag{10}$$

Moreover, an analogous bound estimate is satisfied by $\widetilde{\mathcal{Y}}^{I,t}$;
(ii) *^{t_n} referring to a perturbed initial condition (t_n, x_n, k) of $\mathcal{X}$, then:*
• *in case $t \notin \mathfrak{T}$, $\mathcal{Y}^{I,t_n}$ $\mathcal{S}^2 \times \mathcal{H}^2_q \times \mathcal{S}^2$ – converges to $\mathcal{Y}^{I,t}$ as $\mathcal{E} \ni (t_n, x_n, k) \to (t, x, k)$;*
• *in case $t = T_{I'} \in \mathfrak{T}$:*
— $\mathcal{Y}^{I,t_n}$ $\mathcal{S}^2 \times \mathcal{H}^2_q \times \mathcal{S}^2$ – converges to $\mathcal{Y}^{I,t}$ as $\mathcal{E}_{J+1} \ni (t_n, x_n, k) \to (t, x, k)$;
— if κ_J is continuous at (x, k) for some $J \in \mathbb{N}_N$, then $\mathcal{Y}^{I,t_n}$ $\mathcal{S}^2 \times \mathcal{H}^2_q \times \mathcal{S}^2$ – converges to $\widetilde{\mathcal{Y}}^{I,t}$ as $\mathcal{E}_J^* \ni (t_n, x_n, k) \to (t, x, k)$.

2.3 Analytic Approach

The main contribution of this article consists in a simulation scheme, shown to be convergent in theory and efficient in practice, for solving the RIBSDE ($\mathcal{E}$). However, for the sake of the numerical validation of the results of the simulation scheme, it will be useful to compare these results with those of an alternative, deterministic numerical scheme. A deterministic scheme for the Markovian RIBSDE ($\mathcal{E}$) is based on the analytic characterization of ($\mathcal{E}$), or, more precisely, of a related *value function* u, in terms of an associated system of VIs.

We denote by $\mathcal{G}$ the generator of X, so, with $a(t,x) = \sigma(t,x)\sigma(t,x)^\mathsf{T}$,

$$\mathcal{G}u(t,x) = \partial_t u(t,x) + \partial u(t,x) b(t,x) + \frac{1}{2}\mathrm{Tr}[a(t,x)\mathcal{H}u(t,x)] \ , \tag{11}$$

where ∂u and $\mathcal{H}u$ denote the *row-gradient* and the *Hessian* of a function $u = u(t,x)$ with respect to x. We also introduce, for $I = 1, \ldots, N$,

$$\mathcal{E}_I = \mathcal{E} \cap \{T_{I-1} \le t \le T_I\} \times \mathbb{R}^q \times \mathcal{K} \ , \ \mathcal{E}_I^* = \mathcal{E} \cap \{T_{I-1} \le t < T_I\} \times \mathbb{R}^q \times \mathcal{K} \ . \tag{12}$$

Note that the sets $\mathcal{E}_I^*$s and $\{T\} \times \mathbb{R}^q \times \mathcal{K}$ partition $\mathcal{E}$.

We denote for short $\kappa(T_I, \cdot, k) = \kappa_I^k$. In view of introducing the value function u in Proposition 2.4, it is convenient to state the following definition.

Definition 2.4 **(i)** A *Cauchy cascade* g, ν on $\mathcal{E}$ is pair made of a terminal condition g of class (P) at T, along with a sequence $\nu = (u_I)_{1 \le l \le N}$ of functions u_Is of class (P) on the $\mathcal{E}_I$s, satisfying the following jump condition, at every point of continuity of κ_I^k in x:

$$u_I^k(T_I, x) = \begin{cases} \min(u_{I+1}(T_I, x, \kappa_I^k(x)), h(x)) & \text{if } k \notin K \text{ and } \kappa_I^k(x) \in K, \\ u_{I+1}(T_I, x, \kappa_I^k(x)) & \text{else }, \end{cases} \tag{13}$$

where, in case $I = N$, u_{I+1} is to be understood as g.
A *continuous Cauchy cascade* is a Cauchy cascade with continuous ingredients g at T and u_Is on the $\mathcal{E}_I$s, except maybe for discontinuities of the u_Is at the points (T_I, x, k) of discontinuity of κ_I^k in x;
(ii) The function defined by a Cauchy cascade is the function on $\mathcal{E}$ given as the concatenation on the $\mathcal{E}_I^*$s of the u_Is, and by the terminal condition g at T.

Remark 2.5 Recall that κ_I^k is continuous outside $\partial\mathcal{O}$. Yet, at a discontinuity point x of κ_I^k, $u_I^k(t_n, x_n)$ may fail to converge to $u_I^k(T_I, x)$ as $\mathcal{E}_I \ni (t_n, x_n, k) \to (T_I, x, k)$.

One then has,

Proposition 2.4 *Assuming (Hℓ) and (Hh), the state-process Y of $\mathcal{Y}$ satisfies, $\mathbb{P}$-a.s.,*

$$Y_t = u(t, \mathcal{X}_t)\,,\ t \in [0, T]\,, \tag{14}$$

for a deterministic pricing function u, *defined by a continuous Cauchy cascade* $g, \nu = (u_I)_{1 \le I \le N}$ *on* $\mathcal{E}$.

The next step consists in deriving an analytic characterization of the value function u, or, more precisely, of the cascade $\nu = (u_I)_{1 \le I \le N}$, in terms of solutions to a related analytic problem.

Note that except in the (too specific, thinking for instance of the situation of Example 2.6 below) case of κ not depending on x, κ presents discontinuities in x, and the function u_Is typically present discontinuities at the points (T_I, x, k) of discontinuity of the κ_I^ks (cf. Remark 2.5).
It would be possible however, though we shall not develop this further in this article, to characterize ν in terms of a suitable notion of *discontinuous viscosity solution* [17, 18] to the following *Cauchy cascade* of VIs:

For I decreasing from N to 1,
- At $t = T_I$, for every $k \in \mathcal{K}$ and $x \in \mathbb{R}^q$,

$$u_I^k(T_I, x) = \begin{cases} \min(u_{I+1}(T_I, x, \kappa_I^k(x)), h(x)), & k \notin K \text{ and } \kappa_I^k(x) \in K \\ u_{I+1}(T_I, x, \kappa_I^k(x)), & \text{else }\ , \end{cases} \tag{15}$$

with u_{I+1} in the sense of g in case $I = N$,
- On the time interval $[T_{I-1}, T_I)$, for every $k \in \mathcal{K}$,

$$\begin{cases} \min\Big(-\mathcal{G}u_I^k - f^{u_I^k}, u_I^k - \ell \Big) = 0\,,\ k \in K \\ \max\Big(\min\Big(-\mathcal{G}u_I^k - f^{u_I^k}, u_I^k - \ell \Big), u_I^k - h \Big) = 0\,,\ k \notin K \end{cases} \tag{16}$$

where $\mathcal{G}$ is given by (11) and where we set, for any function $\phi = \phi(t, x)$,

$$f^\phi = f^\phi(t, x) = f(t, x, \phi(t, x))\ . \tag{17}$$

It would also be possible to state related convergence results for standard deterministic (like finite differences) schemes to the viscosity solution of the Cauchy cascade (15)–(16). Note however that (15)–(16) involves Card($\mathcal{K}$) equations in the u^ks. From a deterministic computational point of view the Cauchy cascade (15)–(16) can thus be considered as a $q+d$ – dimensional pricing problem, with $d = \log(\mathrm{Card}(\mathcal{K}))$. For 'very large' sets $\mathcal{K}$, like for instance in the case of Example 2.6(ii), the use of deterministic schemes is thus precluded by the curse of dimensionality, and simulation schemes are the only viable alternative.

2.4 Case of an Affine Coefficient

We consider in this section the special case of an affine coefficient

$$f = f(t,x,y) = c(t,x) - \mu(t,x)y + \eta(t,x)z^{\mathsf{T}} \;, \tag{18}$$

for continuous bounded real-valued and $\mathbb{R}^{1\otimes q}$-valued functions $\mu(t,x)$ and $\eta(t,x)$. In this case, it is straightforward to verify the following classic

Lemma 2.5 *$\mathcal{Y} = (Y, Z, A)$ denoting a solution to ($\mathcal{E}$), the triple*

$$\Big(\beta Y, \beta(Z + Y\eta), \int_0^{\cdot} \beta_t dA_t\Big)$$

solves the RIBSDE with data (cf. (6))

$$\beta_t c(t,X_t)\;,\; \beta_T g(X_T)\;,\; \beta_t \ell(t,X_t)\;,\; \beta_t h(t,X_t)\;,\; \vartheta\;, \tag{19}$$

where the adjoint process β *is the solution of the following linear (forward) SDE:*

$$d\beta_t = \beta_t\Big(\eta(t,X_t)dW_t - \mu(t,X_t)dt\Big)\;,\; t\in[0,T] \tag{20}$$

with initial condition $\beta_0 = 1$. In particular, $\beta > 0$ on $[0,T]$.

2.4.1 Verification Principle

Using Lemma 2.5, the following *verification principle* can be established in a standard way (see, e.g., [5, 18]). This result establishes the connection between a solution $\mathcal{Y} = (Y, Z, A)$ of the RIBSDE ($\mathcal{E}$) with an affine coefficient f as of (18), and a related *Dynkin Game*, or *optimal game problem* (see [22]).

Let $\mathcal{T}_t$ and $\mathcal{T}_t^{\vartheta}$ denote the sets of the $[t,T]$-valued and of the $\cup_{l>0}[\vartheta_{2l-1}\vee t, \vartheta_{2l}\vee t)\cup\{T\}$-valued stopping times, respectively. Let $\zeta = \tau\wedge\theta$, for any τ, $\theta\in\mathcal{T}_t$.

Proposition 2.6 *Let $\mathcal{Y} = (Y, Z, A)$ denote a solution to ($\mathcal{E}$).*
(i) *Y is the conditional value process of the Dynkin game with cost criterion $\mathbb{E}_t\big(\pi^t(\tau,\theta)\big)$ on $\mathcal{T}_t\times\mathcal{T}_t^{\vartheta}$, where $\pi^t(\tau,\theta)$ is the $\mathcal{F}_\zeta$-measurable random variable defined by*

$$\beta_t\pi^t(\tau,\theta) = \int_t^{\zeta}\beta_s c(s,X_s)ds + \beta_\zeta\big(\mathbb{1}_{\{\zeta=\tau<T\}}L_\tau + \mathbb{1}_{\{\zeta=\theta<\tau\}}U_\theta + \mathbb{1}_{\{\zeta=T\}}\xi\big),$$

with $\zeta = \tau \wedge \theta$. One thus has $\mathbb{P}$ – almost surely, for every $t \in [0,T]$,

$$\mathrm{esssup}_{\tau\in\mathcal{T}_t}\mathrm{essinf}_{\theta\in\mathcal{T}_t^\vartheta}\mathbb{E}_t\pi^t(\tau,\theta) = Y_t = \mathrm{essinf}_{\theta\in\mathcal{T}_t^\vartheta}\mathrm{esssup}_{\tau\in\mathcal{T}_t}\mathbb{E}_t\pi^t(\tau,\theta)\ . \tag{21}$$

More precisely, for any $t \in [0,T]$ and for any $\varepsilon > 0$, the pair of stopping times $(\tau^\varepsilon, \theta^\varepsilon) \in \mathcal{T}_t \times \mathcal{T}_t^\vartheta$ given by

$$\begin{aligned} \tau^\varepsilon &= \inf\Big\{\, u \in [t,T]\,;\, Y_u \le \ell(u, X_u) + \varepsilon \Big\} \wedge T \\ \theta^\varepsilon &= \inf\Big\{\, u \in \cup_{l\ge 0}[\vartheta_{2l+1} \vee t, \vartheta_{2l+2} \vee t)\,;\, Y_u \ge U_u - \varepsilon \Big\} \wedge T, \end{aligned} \tag{22}$$

is an ε – saddle-point for this Dynkin game at time t, in the sense that one has, for any $(\tau,\theta) \in \mathcal{T}_t \times \mathcal{T}_t^\vartheta$,

$$\mathbb{E}_t\big(\pi^t(\tau,\theta^\varepsilon)\big) - \varepsilon \le Y_t \le \mathbb{E}_t\big(\pi^t(\tau^\varepsilon,\theta)\big) + \varepsilon. \tag{23}$$

(ii) *If the component A of $\mathcal{Y}$ is continuous, then the pair of stopping times $(\tau^*, \theta^*) \in \mathcal{T}_t \times \mathcal{T}_t^\vartheta$ obtained by setting $\varepsilon = 0$ in (22), is a saddle-point of the game. One thus has in this case, for any $(\tau,\theta) \in \mathcal{T}_t \times \mathcal{T}_t^\vartheta$,*

$$\mathbb{E}_t\big(\pi^t(\tau,\theta^*)\big) \le Y_t \le \mathbb{E}_t\big(\pi^t(\tau^*,\theta)\big).$$

2.5 Connection with Finance

In the case of risk-neutral pricing problems in finance, the driver coefficient function f is typically given as

$$f = f(t,x,y) = c(t,x) - \mu(t,x)y\ , \tag{24}$$

for *dividend and interest-rate* related functions c and μ. Note that f in (24) is affine in y and does not depend on z. We are thus in the sub-case of section 2.4 corresponding to $\eta = 0$, and therefore, in view of (20),

$$\beta = \exp\big(-\int_0^\cdot \mu(t, X_t)dt\big)\ .$$

Modeling the pricing problem under the historical probability, as opposed to the risk-neutral probability by default in this article, would lead to a '*z-dependent*' driver coefficient function f. Moreover we tacitly assume in this paper a perfect, frictionless financial market. Accounting for market imperfections would lead to a *nonlinear* coefficient f (see, e.g., El Karoui et al. [24]).
Also note that in a context of *vulnerable claims* (*defaultable game options* [4]), it is enough, to account for counterparty risk, to work with suitably *credit-spread adjusted interest-rates* μ and *recovery-adjusted dividend-yields* c in (24), and to amend accordingly the dynamics of the factor process X (see, e.g., [18]).

Moreover, in the financial interpretation:
• $g(\mathcal{X}_T^t)$ corresponds to a *terminal payoff* that is paid by the issuer to the holder at time T if the contract was not exercised before T;
• $\ell(\mathcal{X}_t^t)$, resp. $h(\mathcal{X}_t^t)$, corresponds to a *lower*, resp. *upper payoff* that is paid by the issuer to the holder of the claim in the event of early termination of the contract at the initiative

of the holder, resp. issuer;

• The sequence of stopping time ϑ is interpreted as a sequence of *times of switching of a call protection*. More precisely, the issuer of the claim is allowed to call it back (enforcing early exercise) on the 'odd' (random) time intervals $[\vartheta_{2l-1}, \vartheta_{2l})$. At other times call is not possible.

The contingent claims under consideration are thus general *game contingent claims* [30], covering convertible bonds, American options (and also European options) as special cases.

Now, in view of a rather standard verification principle and of the arbitrage theory for game options (see, e.g., [18]), $\Pi = Y$ is an arbitrage price process for the game option, the arbitrage price relative to the pricing measure $\mathbb{P}$, which can be thought of as 'the pricing measure chosen by the market'.
Given a suitable set of hedging instruments, Π is also a *bilateral super-hedging price* (see, e.g., [18, 6]), in the sense that there exists a self-financing super-hedging strategy for the issuer of the claim starting from any issuer initial wealth greater then Π and a self-financing super-hedging strategy for the holder of the claim starting from any holder initial wealth greater than $-\Pi$. Finally Π is also the infimum of the initial wealths of all the issuer's self-financing super-hedging strategies.

2.5.1 Model Specifications

A rather typical specification of the terminal cost functions is given by, for constants $\bar{P} \leq \bar{N} \leq \bar{C}$,

$$\ell(t,x) = \bar{P} \vee S \,,\ h(t,x) = \bar{C} \vee S \,,\ g(x) = \bar{N} \vee S \,, \tag{25}$$

where $S = x_1$ denotes the first component of x. Note that this specification satisfies assumptions (Hℓ)-(Hh), as well as all the standing assumptions of this paper. In particular, one then has (cf. (1), (11)),

$$\lambda(t,x) = x_1 = S \,,\ \mathcal{G}\lambda = b_1 \,,\ \partial\lambda\sigma = \sigma_1 \,,$$

so that condition (5) in (Hh) reduces to b_1, $\sigma_1 \in (P)$, which holds by the Lipschitz property of b and σ.

As for ϑ, one may consider the following specifications, which are commonly found in the case of convertible bonds on an underlying stock S.

Example 2.6 Given a constant *trigger level* $\bar{S}$ and a constant $l \leq N$:
(i) $\mathcal{K} = \mathbb{N}_l$, $K = \mathbb{N}_{l-1}$ and κ defined by

$$\kappa_I^k(x) = \begin{cases} (k+1) \wedge l, & S \geq \bar{S} \\ 0, & S < \bar{S} \end{cases}$$

(independently of I). With the initial condition $H_0 = 0$, H_t then represents the number of consecutive monitoring dates T_Is with $S_{T_I} \geq \bar{S}$ from time t backwards, capped at l. Call is possible whenever $H_t \geq l$, which means that S has been $\geq \bar{S}$ at the last l monitoring times; Otherwise call protection is in force;
(ii) $\mathcal{K} = \{0,1\}^d$ for some given integer $d \in \{l, \ldots, N\}$, $K = \{k \in \mathcal{K}\,;\, |k| < l\}$ with $|k| = \sum_{1 \leq p \leq d} k_p$, and κ defined by

$$\kappa_I^k(x) = (\mathbf{1}_{S \geq \bar{S}}, k_1, \ldots, k_{d-1}) \,.$$

With the initial condition $H_0 = \mathbf{0}_d$, H_t then represents the vector of the indicator functions of the events $S_{T_I} \geq \bar{S}$ at the last d monitoring dates preceding time t. Call is possible whenever $|H_t| \geq l$, which means that S has been $\geq \bar{S}$ on at least l of the last d monitoring times; Otherwise call protection is in force.

3 Discrete-Time Approximation Results

In this section, we propose an approximation scheme for a solution $\mathcal{Y} = (Y, Z, A)$, ***assumed to exist***, to $(\mathcal{E})$ (for instance because assumption (Hℓ) holds, see Proposition 2.1), and we provide an upper bound for the convergence rate of this scheme. The proofs are deferred to section 4.

3.1 Approximation of the Forward Process

When the diffusion X in (1) cannot be perfectly simulated, we use the Euler scheme approximation $\widehat{X}$ defined for a grid $\mathfrak{t} = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ of $[0, T]$, $n \geq 1$, by $\widehat{X}_0 = X_0$, and for $i \leq n - 1$,

$$\widehat{X}_{t_i} + b(t_i, \widehat{X}_{t_i})(t_{i+1} - t_i) + \sigma(t_i, \widehat{X}_{t_i})(W_{t_{i+1}} - W_{t_i}) \ .$$

In the sequel, we shall denote by $|\mathfrak{t}| = \max_{i \leq n-1}(t_{i+1} - t_i)$ the modulus of $\mathfrak{t}t$ and assume that

$$n|\mathfrak{t}| \leq \Lambda \ .$$

As usual, we define a continuous-time version of $\widehat{X}$ by setting, for every $i \leq n - 1$ and $t \in [t_i, t_{i+1})$,

$$\widehat{X}_t = \widehat{X}_{t_i} + b(t_i, \widehat{X}_{t_i})(t - t_i) + \sigma(t_i, \widehat{X}_{t_i})(W_t - W_{t_i}) \ , \tag{26}$$

or in an equivalent differential notation, for $t \in [0, T]$,

$$d\widehat{X}_t = b(\bar{t}, \widehat{X}_{\bar{t}})dt + \sigma(\bar{t}, \widehat{X}_{\bar{t}})dW_t \ , \tag{27}$$

where we set $\bar{t} = \sup\{s \in \mathfrak{t} | s \leq t\}$.

Under the Lipschitz continuity assumption (Hx), one has, for every $p \geq 1$ (see, for instance, Kloeden and Platen [31]),

$$\| \sup_{t \leq T} \|X_t - \widehat{X}_t\| \, \|_{\mathcal{L}^p} + \max_{i<n} \| \sup_{t \in [t_i, t_{i+1}]} \|X_t - \widehat{X}_{t_i}\| \, \|_{\mathcal{L}^p} \leq C^p_\Lambda \, |\mathfrak{t}|^{\frac{1}{2}} \ . \tag{28}$$

3.2 Approximation of the Upper Barrier

The lower barrier is naturally approximated by $\ell(t, \widehat{X}_t)$.

We now present the approximation of the upper barrier, which involves the approximation of the activation/deactivation times. We first define the approximation of the marker process H, denoted by $\widehat{H}$, which is naturally given by

$$\widehat{H}_0 = H_0 \text{ and } \widehat{H}_{T_I} = \kappa(T_I, \widehat{X}_{T_I}, \widehat{H}_{T_I-}) \ , \text{ for } 1 \le I \le N.$$

We then define the approximation $\widehat{\vartheta}$ of ϑ as the sequence of $\mathfrak{T}$-valued stopping times obtained by using $\widehat{\mathcal{X}} = (\widehat{X}, \widehat{H})$ instead of $\mathcal{X}$ in (4).

In order to control the error between the call protection times and their approximation, we make the following assumption on the coefficients of X,

(Hxo) σ and b are bounded and of class $\mathcal{C}^{2,b}$ on the following set (union of cylinders),

$$\mathcal{Q} = \{(t,x) \in \cup_{1 \le I \le N-1} [T_I^\Lambda, T_I] \times \mathbb{R}^q \, ; \, |\mathrm{d}(x)| \le \Lambda^{-1}\},$$

where we set, for $1 \le I \le N-1$,

$$T_I^\Lambda = T_I - \Lambda^{-1} > T_{I-1} \ .$$

Moreover, for every $(t,x) \in \mathcal{Q}$,

$$a(t,x) \ge \Lambda^{-1} I_q \ . \tag{29}$$

The proof of the following Proposition is postoned to the Section 4.1.

Proposition 3.1 *Under (Hxo), for every $\varepsilon > 0$, there exists a constant C_Λ^ε such that for every $l \le N+1$,*

$$\mathbb{E}\Big[|\vartheta_l - \widehat{\vartheta}_l|\Big] \le C_\Lambda^\varepsilon |\mathfrak{t}|^{\frac{1}{2}-\varepsilon} \ .$$

The upper barrier is then approximated by the processes $\widetilde{U}$ and $\widehat{U}$ defined by, for $t \in [0,T]$,

$$\widetilde{U}_t = \sum_{l=0}^{[N/2]} \mathbf{1}_{[\widehat{\vartheta}_{2l}, \widehat{\vartheta}_{2l+1})} \infty + \sum_{l=1}^{[N+1/2]} \mathbf{1}_{[\widehat{\vartheta}_{2l-1}, \widehat{\vartheta}_{2l})} h(t, \widehat{X}_t)$$

$$\widehat{U}_t = \left(\mathbf{1}_{\{0\}} + \sum_{l=0}^{[N/2]} \mathbf{1}_{(\widehat{\vartheta}_{2l}, \widehat{\vartheta}_{2l+1})} \right) \infty + \sum_{l=1}^{[N+1/2]} \mathbf{1}_{[\widehat{\vartheta}_{2l-1}, \widehat{\vartheta}_{2l}]} h(t, \widehat{X}_t) \ .$$

3.3 Approximation of the RIBSDE

In the sequel, we shall use one of the following regularity assumptions:

(Hb) h and ℓ are Λ-Lipschitz continuous with respect to (t,x).

(Hb)' h and ℓ verify for a constant Λ and some $\Lambda_1, \Lambda_2 : \mathbb{R}^q \to \mathbb{R}^{1 \otimes q}$ and $\Lambda_3 : \mathbb{R}^q \to \mathbb{R}^+$,

$$|\Lambda_1(x)| + |\Lambda_2(x)| + |\Lambda_3(x)| \le \Lambda(1 + |x|^\Lambda)$$
$$\ell(t,x) - \ell(t,y) \le \Lambda_1(x)(y-x) + \Lambda_3(x)|x-y|^2 \ \ , \ \forall \, x,y \in \mathbb{R}^q \ .$$
$$h(t,y) - h(t,x) \le \Lambda_2(x)(y-x) + \Lambda_3(x)|x-y|^2 \ \ , \ \forall \, x,y \in \mathbb{R}^q \ .$$

Remark 3.1 (i) Assumption (Hb)' is slightly weaker than the semi-convexity assumption of Definition 1 in [1].

(ii) Observe that (Hb)' implies (Hb).

Given $\varrho = \vartheta$ or $\widehat{\vartheta}$, let the *projection operator* $\mathcal{P}_\varrho$ be defined by

$$\mathcal{P}_\varrho(t,x,y) = y + [\ell(t,x) - y]^+ - [y - h(t,x)]^+ \sum_{l=1}^{[N+1/2]} \mathbf{1}_{\{\varrho_{2l-1} \le t \le \varrho_{2l}\}} \ . \tag{30}$$

To tackle the reflection issue, we introduce a discrete set of reflection times defined by

$$\mathfrak{r} = \{0 = r_0 < r_1 < \cdots < r_\nu = T\} \text{ with } \mathfrak{T} \subseteq \mathfrak{r} \subseteq \mathfrak{t} \tag{31}$$

$$\text{and } |\mathfrak{r}| := \max_{\imath \le \nu - 1}(r_{\imath+1} - r_\imath) \le C_\Lambda \min_{\imath \le \nu-1}(r_{\imath+1} - r_\imath) \ , \tag{32}$$

for some $\nu \ge 1$.

The idea is that in the approximation scheme the reflection will operate only on $\mathfrak{r}$. The components Y and Z of a solution $\mathcal{Y} = (Y, Z, A)$ to the RIBSDE $(\mathcal{E})$ are then approximated by a triplet of processes $(\widehat{Y}, \widetilde{Y}, \bar{Z})$ defined on $\mathfrak{t}$ by the terminal condition

$$\widehat{Y}_T = \widetilde{Y}_T = g(\widehat{X}_T) \ ,$$

and then for i decreasing from $n-1$ to 0,

$$\left\{ \begin{array}{rcl} \bar{Z}_{t_i} & = & (t_{i+1} - t_i)^{-1} \, \mathbb{E}\left[\widehat{Y}_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\ \widetilde{Y}_{t_i} & = & \mathbb{E}\left[\widehat{Y}_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + (t_{i+1} - t_i) f(t_i, \widehat{X}_{t_i}, \widetilde{Y}_{t_i}, \bar{Z}_{t_i}) \\ \widehat{Y}_{t_i} & = & \widetilde{Y}_{t_i} \mathbf{1}_{\{t_i \notin \mathfrak{r}\}} + \mathcal{P}_{\widehat{\vartheta}}(t_i, \widehat{X}_{t_i}, \widetilde{Y}_{t_i}) \mathbf{1}_{\{t_i \in \mathfrak{r}\}} \ , \ i \le n-1 \ . \end{array} \right. \tag{33}$$

We also let $\bar{Z}_T = 0$.

Using an induction argument and the Lipschitz-continuity assumption on f, g, l, h, one easily checks that the above processes are square integrable. It follows that the conditional expectations are well defined at each step of the algorithm.

We also consider a piecewise continuous version of the scheme. Using the martingale representation theorem, we define $\widehat{Z}$ on $[t_i, t_{i+1})$ by

$$\widehat{Y}_{t_{i+1}} = \mathbb{E}_{t_i}\Big[\widehat{Y}_{t_{i+1}}\Big] + \int_{t_i}^{t_{i+1}} \widehat{Z}_s dW_s \ .$$

We then define $\widetilde{Y}$ on $[t_i, t_{i+1})$ by

$$\widetilde{Y}_t = \widehat{Y}_{t_{i+1}} + (t_{i+1} - t) f(t_i, \widehat{X}_{t_i}, \widetilde{Y}_{t_i}, \bar{Z}_{t_i}) - \int_t^{t_{i+1}} \widehat{Z}_s dW_s$$

and we let finally, for $t \in [0, T]$,

$$\widehat{Y}_t = \widetilde{Y}_t \mathbf{1}_{\{t \notin \mathfrak{r}\}} + \mathcal{P}_{\widehat{\vartheta}}(t, \widehat{X}_t, \widetilde{Y}_t) \mathbf{1}_{\{t \in \mathfrak{r}\}} \ . \tag{34}$$

Observe that one has, for $i \le n-1$,

$$\bar{Z}_{t_i} = \frac{1}{t_{i+1}-t_i}\mathbb{E}_{t_i}\Big[\int_{t_i}^{t_{i+1}} \widehat{Z}_s ds\Big] \ .$$

For later use, we also define $\bar{Z}_t = \bar{Z}_{\bar{t}}$, for $t \in [0,T]$.

When there is no call or no call protection, the convergence of the scheme is given by Theorem 6.2 in [14] in the general setting where f depends on z.

Theorem 3.2 *Under (Hb), the following holds*

$$\max_{i\le n-1}\sup_{t\in[t_i,t_{i+1})}\mathbb{E}\Big[|Y_t-\widetilde{Y}_{t_i}|^2\Big] + \max_{i\le n-1}\sup_{t\in[t_i,t_{i+1})}\mathbb{E}\Big[|Y_t-\widehat{Y}_{t_i}|^2\Big] \le C_\Lambda|\mathfrak{t}|^{\frac{1}{2}} \ .$$

Under stronger assumption on the boundaries, it is possible to obtain a better control of the convergence rate of the approximation see Theorem 6.2 in [14].

Regarding call protection, our main result is the following

Theorem 3.3 *Under (Hb) and if f does not depend on z, the following holds*

$$\max_{i\le n-1}\sup_{t\in[t_i,t_{i+1})}\mathbb{E}\Big[|Y_t-\widetilde{Y}_{t_i}|^2\Big] + \max_{i\le n-1}\sup_{t\in[t_i,t_{i+1})}\mathbb{E}\Big[|Y_{t-}-\widehat{Y}_{t_i}|^2\Big] \le C^\varepsilon_\Lambda|\mathfrak{t}|^{\frac{1}{2}-\varepsilon} \ .$$

Under (Hb)' and if f does not depend on z, the following holds

$$\max_{i\le n-1}\sup_{t\in[t_i,t_{i+1})}\mathbb{E}\Big[|Y_t-\widetilde{Y}_{t_i}|^2\Big] + \max_{i\le n-1}\sup_{t\in[t_i,t_{i+1})}\mathbb{E}\Big[|Y_{t-}-\widehat{Y}_{t_i}|^2\Big] \le C_\Lambda|\mathfrak{t}| \ .$$

4 Proofs

We denote by χ a positive random variable which may change from line to line but satisfies $\mathbb{E}[\chi^p] \le C^p_\Lambda$, for $p \ge 1$.

4.1 Stability of Call Protection Monitoring Times

In the following, we consider two diffusions. The first one X starts at (t,x) $(t<T)$ and is the solution of the following SDE:

$$X_s = x + \int_t^s b(s,X_s)ds + \int_t^s \sigma(s,X_s)dW_s \ , \quad \text{for } s\in[t,T] \ .$$

The second one ξ starts at (t',x') $(t'<T)$ and can be written:

$$\xi_s = x' + \int_{t'}^s b_\xi(s)ds + \int_{t'}^s \sigma_\xi(s)dW_s \ , \quad \text{for } s\in[t',T] \ .$$

In practice, ξ will be the solution of an SDE with coefficient (b,σ) or the Euler scheme associated to this SDE.

We consider the following 'monitoring grid' for X (resp. ξ) $\mathfrak{T}^t := \{s \in \mathfrak{T} \,|\, s > t\}$, $\mathcal{T}^t = \inf \mathfrak{T}^t$ (resp. $\mathfrak{T}^{t'} = \{s \in \mathfrak{T} \,|\, s > t'\}$, $\mathcal{T}^{t'} = \inf \mathfrak{T}^{t'}$).

As in Section 2.1, we are given a finite set $\mathcal{K}$, a subset K of $\mathcal{K}$ and an initial condition $k \in \mathcal{K}$. We introduce for ξ the marker process $\mathcal{X}'$, recalling Section 2.1. Namely, $\mathcal{X}' = (\xi, H')$, with H' defined by

$$H'_{t'} = k \in \mathcal{K} \text{ and } H'_{\mathcal{T}} = \kappa(\mathcal{T}, \xi_{\mathcal{T}}, H'_{\mathcal{T}-}),\ \forall \mathcal{T} \in \mathfrak{T}^{t'}$$

and H' is constant between two dates of $\{t'\} \cup \mathfrak{T}^{t'}$. Observe that H' does not jump at t'.

We also consider a non-decreasing sequence of stopping times $\vartheta' = (\vartheta'_l)_{0 \le l \le N+1}$, representing the call protection monitoring times, defined by $\vartheta'_0 = t'$ and, for every $l \ge 0$:

$$\vartheta'_{2l+1} = \inf\{t > \vartheta'_{2l}\,;\, H'_t \notin K\} \wedge T\,,\ \vartheta_{2l+2} = \inf\{t > \vartheta'_{2l+1}\,;\, H'_t \in K\} \wedge T\,. \tag{35}$$

The ϑ'_ls effectively reduce to $\{t'\} \cup \mathfrak{T}^{t'}$-valued stopping times, and that $\vartheta_{N+1} = T$.

To the process X, we associate two different marker processes $\mathcal{X}$ ans $\widetilde{\mathcal{X}}$. The first one $\mathcal{X} = (X, H)$ is defined as above, replacing ξ by X. Observe that H does not jump at t and that $H_t = H'_{t'} = k$. We also consider the sequence of call protection monitoring times ϑ, defined as in (35) with t and H instead of t' and H'.

The second marker process $\widetilde{\mathcal{X}} = (X, \widetilde{H})$ is given by

$$\widetilde{H}_t = \kappa(t, x, k) \text{ and } \widetilde{H}_{\mathcal{T}} = \kappa(\mathcal{T}, X_{\mathcal{T}}, \widetilde{H}_{\mathcal{T}-}),\ \forall \mathcal{T} \in \mathfrak{T}^t \setminus \{t\}$$

and $\widetilde{H}$ is constant between two dates of $\{t\} \cup \mathfrak{T}^t$. Observe that, contrary to H, $\widetilde{H}$ may not jump at t.

We also consider the corresponding call protection monitoring times sequence $\widetilde{\vartheta}$ defined as in (35) with t and $\widetilde{H}$ instead of t' and H'.

We are then interested in two different cases regarding the initial set of data (t, x) and (t', x').

Case 1: $\mathcal{T}^t = \mathcal{T}^{t'}$.

Case 2: $\mathcal{T}^{t'} = t$ and $\mathrm{d}(x') \neq 0$.

Let us finally introduce, for $0 < h < \min_{r \neq s \in \mathfrak{T}^t} |r - s|$, and $\mathcal{T} \in \mathfrak{T}^t$, the sets

$$\Omega^{\mathrm{h}}_{\mathcal{T}} = \{ \sup_{\mathcal{T} - \mathrm{h} \le u \le \mathcal{T}} |X_u - X_T| \le \frac{1}{3\Lambda} \}\,,\ \Omega^{\mathrm{h}} = \bigcap_{\mathcal{T} \in \mathfrak{T}^t} \Omega^{\mathrm{h}}_{\mathcal{T}}$$

$$\widehat{\Omega}^{\delta} = \{ \sup_{u \in [\mathcal{T}^t, T]} |\xi_u - X_u| < \delta \}\,.$$

The proof of the following Lemma is deferred to Appendix A.

Lemma 4.1 *Assume (Hxo).*
(i) *One has, for $\mathcal{T} \in \mathfrak{T}^t$,*

$$\mathbb{P}(\Omega^{\mathrm{h}}_{\mathcal{T}} \cap \{|\mathrm{d}(X_{\mathcal{T}})| \le \delta\}) \le C_{\Lambda} \frac{\delta}{\mathrm{h}}\,. \tag{36}$$

(ii) *For $p, \bar{p} > 0$, for $l \geq 0$, one has in Case 1,*

$$\mathbb{E}\big[|\vartheta_l - \vartheta'_l|\big] \leq |t - t'| + C_\Lambda \frac{\delta}{\mathrm{h}} + C_\Lambda^{\bar{p}} \mathrm{h}^{\bar{p}} + C_\Lambda^p \frac{\mathbb{E}\Big[\sup_{u \in [\mathcal{T}^t, T]} |\xi_u - X_u|^p\Big]}{\delta^p}$$

and in Case 2,

$$\mathbb{E}\Big[|\widetilde{\vartheta}_l - \vartheta'_l|\Big] \leq |t - t'| + C_\Lambda \frac{\delta}{\mathrm{h}} + C_\Lambda^{\bar{p}} \mathrm{h}^{\bar{p}} + C_\Lambda^p \frac{\mathbb{E}\Big[\sup_{u \in [\mathcal{T}^t, T]} |\xi_u - X_u|^p\Big]}{\delta^p} .$$

4.1.1 Proof of Proposition 3.1

We set $\xi = \widehat{X}$, in the framework introduced at the beginning of this section. We have here that $t = t' = 0$, so we shall use the results of Case 1 above. Observing that

$$\mathbb{E}\left[\sup_{u \in [0,T]} |\widehat{X}_u - X_u|^p\right] \leq C_\Lambda^p |\mathfrak{t}|^{\frac{p}{2}},$$

and applying the results of Case 1 in Lemma 4.1(ii), we thus get

$$\mathbb{E}\Big[\vartheta_l - \widehat{\vartheta}_l\Big] \leq C_\Lambda \frac{\delta}{\mathrm{h}} + C_\Lambda^{\bar{p}} \mathrm{h}^{\bar{p}} + C_\Lambda^p \frac{|\mathfrak{t}|^{\frac{p}{2}}}{\delta^p}$$

The proof is then concluded setting $\delta = |\mathfrak{t}|^{\frac{1}{2} - \frac{\varepsilon}{2}}$, $\bar{p} = p = \frac{1}{\varepsilon} - 2$, for ε and $|\mathfrak{t}|$ small enough.

4.1.2 Proof of Proposition 2.2

For $(t_n, x_n) \in [0, T[\times \mathbb{R}^q$, we set $\xi = X^{t_n, x_n}$.

(i) When $t_n \downarrow t$, we want to control the difference between ϑ^t and ϑ^{t_n} to prove the càd property. We shall use here the result of Case 1.

(ii) When $t_n \uparrow t$, we want to control the difference between $\widetilde{\vartheta}^t$ and ϑ^{t_n} to prove the làg property, under the assumption $\mathrm{d}(x) \neq 0$. Since $x_n \to x$, we have for some $n \geq 0$, $\mathrm{d}(x_n) \neq 0$. We shall use here the result of Case 2.

To prove (i), we observe that

$$\mathbb{E}\left[\sup_{u \in [0,T]} |X_u^{t_n, x_n} - X_u^{t,x}|^p\right] \leq C_\Lambda^p (|x - x_n|^p + |t - t_n|^{\frac{p}{2}}).$$

We then obtain, applying Lemma 4.1(ii), that

$$\mathbb{E}[\vartheta_l^t - \vartheta_l^{t_n}] \leq |t - t_n| C_\Lambda \frac{\delta_n}{\mathrm{h}_n} + C_\Lambda^{\bar{p}} \mathrm{h}_n^{\bar{p}} + C_\Lambda^p \frac{|x - x_n|^p + |t - t_n|^{\frac{p}{2}}}{\delta_n^p}.$$

The proof is then concluded taking $\delta_n^2 = |x - x_n| \vee |t - t_n|^{\frac{1}{2}}$, $h_n^2 = \delta_n$, $\bar{p} = p = 2$ and letting n go to ∞.

Arguing exactly as above, we obtain the proof of (ii).

4.2 Proof of Proposition 2.4

By standard semi-group properties of $\mathcal{X}$ and $\mathcal{Y}$ immediately resulting from the uniqueness of solutions to the related SDEs, one gets, for every $I = 1, \ldots, N$ and $T_{I-1} \le t \le r < T_I$ (see Crépey [18] for the detail),

$$Y_r^t = u_I(r, \mathcal{X}_r^t)\,,\ \mathbb{P}{-}a.s. \tag{37}$$

for a deterministic function u_I on $\mathcal{E}_l^*$. In particular,

$$Y_t^t = u^k(t,x)\ \text{, for any } (t,x,k) \in \mathcal{E}\ , \tag{38}$$

where u is the function defined on $\mathcal{E}$ by the concatenation of the u_Is and the terminal condition g at T. In view of (9), the fact that u is of class $\mathcal{P}$ then directly follows from the bound estimates (10) on $\mathcal{Y}^{0,t}$ and $\mathcal{Y}^{1,t}$.

Let us show that the u_Is are continuous over the $\mathcal{E}_l^*$s. Given $\mathcal{E} \ni (t_n, x_n, k) \to (t,x,k)$ with $t \notin \mathfrak{T}$ or $t_n \ge T_I = t$, one decomposes by (9):

$$|u^k(t,x) - u^k(t_n,x_n)| = |Y_t^t - Y_{t_n}^{t_n}| \le$$
$$\begin{cases} |\mathbb{E}(Y_t^{0,t} - Y_{t_n}^{0,t})| + \mathbb{E}|Y_{t_n}^{0,t} - Y_{t_n}^{0,t_n}|, & k \in K \\ |\mathbb{E}(Y_t^{1,t} - Y_{t_n}^{1,t})| + \mathbb{E}|Y_{t_n}^{1,t} - Y_{t_n}^{1,t_n}|, & i \notin K \end{cases}$$

In either case we conclude classically by using Proposition 2.3 as a main tool, as for instance in the proof of Theorem 9.3(i) of Crépey [18, Part II], that $|u^k(t,x) - u^k(t_n,x_n)|$ goes to zero as $n \to \infty$.

It remains to show that the u_Is can be extended by continuity over the $\mathcal{E}_I$s, except maybe at the boundary points (T_I, x, k) such that κ_I^k is discontinuous at x. Given $\mathcal{E}_I^* \ni (t_n, x_n, k) \to (T_I, x, k)$ with κ_I continuous at (x,k), one needs to show that $u_I^k(t_n,x_n) = u^k(t_n,x_n) \to u_l^k(T_I,x)$, where $u_l^k(T_I,x)$ is given by (13)). We distinguish four cases.

- In case $k \notin K$ and $\kappa_I^k(x) \in K$, one has, denoting $\widetilde{u}^j(s,y) = \min(u(s,y,\kappa_I^j(y)), h(y))$,

$$|\widetilde{u}^k(T_I,x) - u^k(t_n,x_n)|^2 = |\widetilde{u}^k(T_I,x) - Y_{t_n}^{1,t_n}|^2 \le$$
$$2\mathbb{E}|\widetilde{u}^k(T_I,x) - \widetilde{u}(T_I,\mathcal{X}_{T_I}^{t_n})|^2 + 2|\mathbb{E}\big(\widetilde{u}(T_I,\mathcal{X}_{T_I}^{t_n}) - Y_{t_n}^{1,t_n}\big)|^2\ . \tag{39}$$

By continuity of κ_I at (x,k), one has $\kappa_I(\mathcal{X}_{T_I}^{t_n}) = \kappa_I^k(x) \in K$ for $\mathcal{X}_{T_I}^{t_n}$ close enough to x, say $||\mathcal{X}_{T_I}^{t_n} - x|| \le c$. In this case $T_I = \tau_2^{t_n}$, therefore (cf. (8)) $Y_{T_I}^{1,t_n} = \widetilde{u}(T_I, \mathcal{X}_{T_I}^{t_n})$. So

$$\mathbb{E}|\mathbf{1}_{||\mathcal{X}_{T_I}^{t_n} - x|| \le c}\big(\widetilde{u}(T_I,\mathcal{X}_{T_I}^{t_n}) - Y_{t_n}^{1,t_n}\big)|^2 \le \mathbb{E}|Y_{T_I}^{1,t_n} - Y_{t_n}^{1,t_n}|^2\ ,$$

which can be shown to converge to zero as $n \to \infty$ by using the R2BSDE satisfied by Y^{1,t_n} and the convergence of $\mathcal{Y}^{1,t_n}$ to $\widetilde{\mathcal{Y}}^{1,t}$. Moreover $\mathbb{E}|\mathbf{1}_{||\mathcal{X}_{T_I}^{t_n} - x|| > c}\big(\widetilde{u}(T_I,\mathcal{X}_{T_I}^{t_n}) - Y_{t_n}^{1,t_n}\big)|^2$ goes to zero as $n \to \infty$ by the a priori estimates on X and Y^{1,t_n} and the continuity of $\widetilde{u}$ already established over $\mathcal{E}_{l+1}^*$. Moreover by this continuity and the a priori estimates on X the first term in (39) also goes to zero as $n \to \infty$. So, as $n \to \infty$,

$$u^k(t_n,x_n) \to \widetilde{u}^k(T_I,x) = \min(u(T_I,x,\kappa_I^k(x)), h(x)) = u_l^k(T_I,x)\ .$$

• In case $k \in K$ and $\kappa_I^k(x) \notin K$, one can show likewise, using $\widehat{u}^j(s,y) := u(s,y,\kappa_I^j(y))$ instead of $\widetilde{u}^j(s,y)$ and Y^0 instead of Y^1 above, that

$$u^k(t_n,x_n) \to u(T_I,x,\kappa_I^k(x)) = u_l^k(T_I,x) \tag{40}$$

as $n \to \infty$.
• If $k, \kappa_I^k(x) \notin K$, it comes,

$$\begin{aligned}|\widehat{u}^k(T_I,x) - u^k(t_n,x_n)|^2 &= |\widehat{u}^k(T_I,x) - Y_{t_n}^{1,t_n}|^2\\ &\le 2\mathbb{E}|\widehat{u}^k(T_I,x) - \widehat{u}(T_I,\mathcal{X}_{T_I}^{t_n})|^2 + 2|\mathbb{E}\big(\widehat{u}(T_I,\mathcal{X}_{T_I}^{t_n}) - Y_{t_n}^{1,t_n}\big)|^2\\ &\le 2\mathbb{E}|\widehat{u}^k(T_I,x) - \widehat{u}(T_I,\mathcal{X}_{T_I}^{t_n})|^2 + 2|\mathbb{E}\big(Y_{T_I}^{1,t_n} - Y_{t_n}^{1,t_n}\big)|^2\,.\end{aligned}$$

which goes to zero as $\to \infty$ by an analysis similar to (but simpler than) that of the first bullet point. Hence (40) follows.
• If $k, \kappa_I^k(x) \in K$, (40) can be shown as in the above bullet point.

4.3 Discretely reflected BSDEs

As in [14, 7], the study of the convergence of the scheme will be done in several steps, using a suitable concept of *discretely reflected BSDEs* that we introduce now. Given $\varrho = \vartheta$ or $\widehat{\vartheta}$, the solution of the *discretely reflected BSDE* is a triplet $(\mathfrak{Y}^\varrho, \widetilde{\mathfrak{Y}}^\varrho, \mathfrak{Z}^\varrho)$ defined by the terminal condition

$$\mathfrak{Y}_T^\varrho = \widetilde{\mathfrak{Y}}_T^\varrho = g(X_T)\,,$$

and then for $\imath$ decreasing from $\nu - 1$ to 0 and $t \in [r_\imath, r_{\imath+1})$,

$$\begin{cases}\widetilde{\mathfrak{Y}}_t^\varrho &= \mathfrak{Y}_{r_{\imath+1}}^\varrho + \int_t^{r_{\imath+1}} f(X_u, \widetilde{\mathfrak{Y}}_u^\varrho)du - \int_t^{r_{\imath+1}} \mathfrak{Z}_u^\varrho dW_u \;,\\ \mathfrak{Y}_t^\varrho &= \widetilde{\mathfrak{Y}}_t^\varrho \mathbf{1}_{\{t\notin\mathfrak{r}\}} + \mathcal{P}_\varrho(t, X_t, \widetilde{\mathfrak{Y}}_t^\varrho)\mathbf{1}_{\{t\in\mathfrak{r}\}}\,.\end{cases} \tag{41}$$

Under (Hx)-(Hb), such a solution can be defined by backward induction. At each step, existence and uniqueness of a solution in $\mathcal{S}^2 \times \mathcal{H}_q^2$ follow from [24].

Remark 4.1 **(i)** $\widetilde{\mathfrak{Y}}^\varrho$ is a càdlàg process whereas $\mathfrak{Y}^\varrho$ is a *càglàd* process. By convention, we set $Y_{0-} = Y_0$.
(ii) One has, for $r \in \mathfrak{r}$,

$$Y_{r-} = \mathcal{P}_\vartheta(r, X_r, Y_r)\,,\; \mathfrak{Y}_r^\vartheta = \mathcal{P}_\vartheta(r, X_r, \widetilde{\mathfrak{Y}}_r^\vartheta)\,,\; \mathfrak{Y}_r^{\widehat{\vartheta}} = \mathcal{P}_{\widehat{\vartheta}}(r, \widehat{X}_r, \widetilde{\mathfrak{Y}}_r^{\widehat{\vartheta}})\,. \tag{42}$$

We now present two properties of discretely reflected BSDEs which will be useful to prove the bound of convergence rate of the approximation scheme. We first show that under suitable conditions the discretely reflected BSDE with $\varrho = \vartheta$ is a 'good' approximation of the RIBSDE ($\mathcal{E}$). In view of Definition 2.2(i), the component Y of $\mathcal{Y}$ may be discontinuous at ϑ_{2l}. This discontinuity is problematic and the fact that $\mathfrak{T} \subseteq \mathfrak{r}$ is essential to obtain the following result.

Proposition 4.2 *Let $\alpha = \frac{1}{2}$ or $\alpha = 1$ under (Hb) or (Hb)', respectively. If f does not depend on z, then*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|Y_t - \widetilde{\mathfrak{I}}^\vartheta_t|^2\Big] + \sup_{t\in[0,T]} \mathbb{E}\Big[|Y_{t-} - \mathfrak{I}^\vartheta_t|^2\Big] + \mathbb{E}\Big[\int_0^T |Z_s - \mathfrak{Z}^\vartheta_s|^2 ds\Big] \leq |\mathfrak{r}|^\alpha .$$

Proof. Let for $t \leq T$,

$$\delta\widetilde{Y}_t = Y_t - \widetilde{\mathfrak{I}}^\vartheta_t \,,\; \delta Y_t = Y_{t-} - \mathfrak{I}^\vartheta_t \,,\; \delta Z_t = Z_{t-} - \mathfrak{Z}^\vartheta_t \,,\; \delta f_t = f(t, X_t, Y_t) - f(t, X_t, \widetilde{\mathfrak{I}}^\vartheta_t) \,.$$

Observe that is continuous outside $\mathfrak{r}$ and that $\delta\widetilde{Y}_{t-} = \delta Y_t$ for $t \in (0, T]$. Applying Itô's formula to the càdlàg process $|\delta\widetilde{Y}|^2$ and observing that the local martingale term is in fact a martingale, we compute,

$$\mathbb{E}_{r_\imath}\Big[|\delta\widetilde{Y}_t|^2 + \int_t^{r_{\imath+1}} |\delta Z_u|^2 du\Big] = \mathbb{E}_{r_\imath}\Big[|\delta\widetilde{Y}_{r_{\imath+1}-}|^2 + 2\int_t^{r_{\imath+1}} \delta\widetilde{Y}_s \delta f_s ds + 2\int_{(t,r_{\imath+1})} \delta\widetilde{Y}_s dA_s\Big] \,,$$

for $t \in [r_\imath, r_{\imath+1})$. Moreover, one has by (42), for $r \in \mathfrak{r}$,

$$|\delta Y_r| = |Y_{r-} - \mathfrak{I}^\vartheta_r| \leq |\delta\widetilde{Y}_r| \,.$$

Using then usual arguments, we obtain, for $t \in [r_\imath, r_{\imath+1})$,

$$\mathbb{E}_{r_\imath}\Big[|\delta\widetilde{Y}_t|^2 + \int_t^{r_{\imath+1}} |\delta Z_s|^2 ds\Big] \leq (1 + C_\Lambda|\mathfrak{r}|)\mathbb{E}_{r_\imath}\Big[|\delta\widetilde{Y}_{r_{\imath+1}}|^2 + 2\int_{(t,r_{\imath+1})} \delta\widetilde{Y}_s dA^+_s - 2\int_{(t,r_{\imath+1})} \delta\widetilde{Y}_s dA^-_s\Big] \,.$$

We first study the term related to the upper barrier. One has,

$$\begin{aligned} -\mathbb{E}_{r_\imath}\Big[\int_{(t,r_{\imath+1})} \delta\widetilde{Y}_s dA^-_s\Big] &= \mathbb{E}_{r_\imath}\Big[\int_{(t,r_{\imath+1})} (\widetilde{\mathfrak{I}}^\vartheta_s - h(s, X_s)) dA^-_s\Big] \\ &= \mathbb{E}_{r_\imath}\Big[\int_{(t,r_{\imath+1})} (\mathfrak{I}^\vartheta_{r_{\imath+1}} - h(s, X_s)) dA^-_s + \int_{(t,r_{\imath+1})} \int_s^{r_{\imath+1}} f(u, X_u, \widetilde{\mathfrak{I}}^\vartheta_u) du dA^-_s\Big] \end{aligned}$$

where in particular the upper barrier minimality condition in $(\mathcal{E})$ was used in the first identity. The second term is bounded by

$$\mathbb{E}_{r_\imath}\Big[\chi|\mathfrak{r}|(A^-_{r_{\imath+1}-} - A^-_{r_\imath})\Big] \leq \mathbb{E}_{r_\imath}\Big[\chi|\mathfrak{r}|(A^-_{r_{\imath+1}} - A^-_{r_\imath})\Big] \,,$$

since f does not depend on z and A^- is increasing. For the first term, we use the fact $dA^- \mathbf{1}_{]\!]\vartheta_{2l}, \vartheta_{2l+1}[\![} = 0$, $0 \leq l \leq [N + 1/2]$, to obtain that

$$\begin{aligned} \mathbb{E}_{r_\imath}\Big[\int_{(t,r_{\imath+1})} (\mathfrak{I}^\vartheta_{r_{\imath+1}} - h(s, X_s)) dA^-_s\Big] &= \mathbb{E}_{r_\imath}\Big[\sum_{l=1}^{[N+1/2]} \int_{(t,r_{\imath+1})} (\mathfrak{I}^\vartheta_{r_{\imath+1}} - h(s, X_s)) \mathbf{1}_{\{\vartheta_{2l-1} \leq s \leq \vartheta_{2l}\}} dA^-_s\Big] \\ &\leq \mathbb{E}_{r_\imath}\Big[\sum_{l=1}^{[N+1/2]} \int_{(t,r_{\imath+1})} (h(r_{\imath+1}, X_{r_{\imath+1}}) - h(s, X_s)) \mathbf{1}_{\{\vartheta_{2l-1} \leq s \leq \vartheta_{2l}\}} dA^-_s\Big] \\ &\leq \mathbb{E}_{r_\imath}\Big[\int_{(t,r_{\imath+1})} (h(r_{\imath+1}, X_{r_{\imath+1}}) - h(s, X_s)) dA^-_s\Big] \,. \end{aligned}$$

The proof is then concluded using the same argument as in the proof of Propositions 2.6.1 and 1.4.1 in [13]. $\square$

We now give a control of the difference between the two discretely reflected BSDEs $(\mathfrak{I}^{\vartheta},\widetilde{\mathfrak{I}}^{\vartheta},\mathfrak{Z}^{\vartheta})$ and $(\mathfrak{I}^{\widehat{\vartheta}},\widetilde{\mathfrak{I}}^{\widehat{\vartheta}},\mathfrak{Z}^{\widehat{\vartheta}})$.

Proposition 4.3 *Set $\alpha = 0$ or 1 under (Hb) or (Hb)', respectively. If f does not depend on z, then for every $\varepsilon > 0$, there exists $C^{\varepsilon}_{\Lambda}$ such that*

$$\sup_{t\in[0,T]}\mathbb{E}\Big[|\mathfrak{I}^{\vartheta}_t-\mathfrak{I}^{\widehat{\vartheta}}_t|^2\Big]+\sup_{t\in[0,T]}\mathbb{E}\Big[|\widetilde{\mathfrak{I}}^{\vartheta}_t-\widetilde{\mathfrak{I}}^{\widehat{\vartheta}}_t|^2\Big]+\|\mathfrak{Z}^{\vartheta}-\mathfrak{Z}^{\widehat{\vartheta}}\|^2_{\mathcal{H}^2}\le C^{\varepsilon}_{\Lambda}|\mathfrak{r}|^{\alpha}\sum_{l=1}^{N}\Big(\mathbb{E}\Big[|\vartheta_l-\widehat{\vartheta}_l|\Big]\Big)^{1-\varepsilon}\ .$$

Proof. Let, for $t\le T$,

$$\begin{aligned}&\delta\widetilde{\mathfrak{I}}_t=\widetilde{\mathfrak{I}}^{\vartheta}_t-\widetilde{\mathfrak{I}}^{\widehat{\vartheta}}_t\ ,\ \delta\mathfrak{I}_t=\mathfrak{I}^{\vartheta}_t-\mathfrak{I}^{\widehat{\vartheta}}_t\ ,\ \delta\mathfrak{Z}_t=\mathfrak{Z}^{\vartheta}_t-\mathfrak{Z}^{\widehat{\vartheta}}_t\\&\eta_t=|\delta\mathfrak{I}_t|^2-|\delta\widetilde{\mathfrak{I}}_t|^2\ ,\ \delta f_t=f(t,X_t,\widetilde{\mathfrak{I}}^{\vartheta}_t)-f(t,X_t,\widetilde{\mathfrak{I}}^{\widehat{\vartheta}}_t)\ .\end{aligned}$$

Step 1 Applying Itô's formula to the càdlàg process $|\delta\widetilde{\mathfrak{I}}|^2$, we compute for $t\in[r_\imath,r_{\imath+1})$

$$\mathbb{E}_{r_\imath}\Big[|\delta\widetilde{\mathfrak{I}}_t|^2+\int_t^{r_{\imath+1}}|\delta\mathfrak{Z}_u|^2du\Big]=\mathbb{E}_{r_\imath}\Big[|\delta\widetilde{\mathfrak{I}}_{r_{\imath+1}}|^2+\eta_{r_{\imath+1}}+2\int_t^{r_{\imath+1}}\delta\widetilde{\mathfrak{I}}_s\delta f_s ds\Big]\ .$$

Usual arguments then yield that

$$\sup_{s\in[t,T]}\mathbb{E}\Big[|\delta\widetilde{\mathfrak{I}}_s|^2+\int_s^T|\delta\mathfrak{Z}_s|^2ds\Big]\le C_\Lambda\mathbb{E}\Big[\sum_{r\in\mathfrak{r}}\eta_r\Big]\ . \tag{43}$$

Step 2a In order to study the right-hand side term of (43), we introduce the processes defined by, for $r\in[0,T]$,

$$\mathrm{I}_r=\sum_{l=1}^{[N+1/2]}\mathbf{1}_{\{\vartheta_{2l-1}\le r\le\vartheta_{2l}\}}\ ,\ \widehat{\mathrm{I}}_r=\sum_{l=1}^{[N+1/2]}\mathbf{1}_{\{\widehat{\vartheta}_{2l-1}\le r\le\widehat{\vartheta}_{2l}\}}\ ,\ {}^c\mathrm{I}_r=1-\mathrm{I}_r\ ,\ {}^c\widehat{\mathrm{I}}_r=1-\widehat{\mathrm{I}}_r\ . \tag{44}$$

Observe that $\mathrm{I}=1$ (or $\widehat{\mathrm{I}}=1$) means that the upper barrier is activated for reflection. Also notice that one has, for $r\in\mathfrak{r}$,

$$\eta_r\le([\widetilde{\mathfrak{I}}^{\vartheta}_r-h(r,X_r)]^+)^2\mathrm{I}_r{}^c\widehat{\mathrm{I}}_r+([\widetilde{\mathfrak{I}}^{\widehat{\vartheta}}_r-h(r,X_r)]^+)^2\widehat{\mathrm{I}}_r{}^c\mathrm{I}_r\ . \tag{45}$$

The two terms at the right-hand side of (45) are treated similarly, we thus concentrate on the first one. We have here to take into account the fact that a reflection date may be a deactivation date for the upper boundary, i.e., for $r\in\mathfrak{r}$,

$$([\widetilde{\mathfrak{I}}^{\vartheta}_r-h(r,X_r)]^+)^2\mathrm{I}_r{}^c\widehat{\mathrm{I}}_r=([\widetilde{\mathfrak{I}}^{\vartheta}_r-h(r,X_r)]^+)^2\,{}^c\widehat{\mathrm{I}}_r(\sum_{l=1}^{[N+1/2]}\mathbf{1}_{\{r=\vartheta_{2l}\}}+\sum_{l=1}^{[N+1/2]}\mathbf{1}_{\{\vartheta_{2l-1}\le r<\vartheta_{2l}\}}) \tag{46}$$

Step 2b We first study the first term in the right hand side of (46). We obviously have that $([\widetilde{\mathfrak{I}}^\vartheta_r - h(r,X_r)]^+)^2 \le \chi$, thus, since the ϑ_ls are $\mathfrak{T}$-valued stopping-times,

$$\sum_{r\in\mathfrak{r}}([\widetilde{\mathfrak{I}}^\vartheta_r - h(r,X_r)]^+)^2\, {}^c\widehat{\mathrm{I}}_r \sum_{l=1}^{[N+1/2]} \mathbf{1}_{\{r=\vartheta_{2l}\}} \le \sum_{r\in\mathfrak{T}} \chi\, {}^c\widehat{\mathrm{I}}_r \sum_{l=1}^{[N+1/2]} \mathbf{1}_{\{r=\vartheta_{2l}\}}.$$

Moreover, by definition of I and $\widehat{\mathrm{I}}$,

$$\begin{aligned}\sum_{r\in\mathfrak{T}} \chi\, {}^c\widehat{\mathrm{I}}_r \sum_{l=1}^{[N+1/2]} \mathbf{1}_{\{r=\vartheta_{2l}\}} &= \sum_{r\in\mathfrak{T}} \chi\, {}^c\widehat{\mathrm{I}}_r \sum_{l=1}^{[N+1/2]} \mathbf{1}_{\{r=\vartheta_{2l}, r\neq\widehat{\vartheta}_{2l}\}}\\ &\le \sum_{r\in\mathfrak{T}} \sum_{l=1}^{[N+1/2]} \chi \mathbf{1}_{\{|\vartheta_{2l}-\widehat{\vartheta}_{2l}|\ge \min_{i\le n-1}|T_{i+1}-T_i|\}}\,.\end{aligned}$$

Using the Cauchy-Schwartz inequality with $\frac{1}{p} = 1-\varepsilon$ and the Markov inequality, we obtain

$$\mathbb{E}\left[\sum_{r\in\mathfrak{T}} \sum_{l=1}^{[N+1/2]} \chi \mathbf{1}_{\{|\vartheta_{2l}-\widehat{\vartheta}_{2l}|\ge \inf_{i\neq j}|T_i-T_j|\}}\right] \le C^\varepsilon_\Lambda \sum_{l=1}^{[N+1/2]} \mathbb{E}\Big[|\vartheta_{2l}-\widehat{\vartheta}_{2l}|\Big]^{1-\varepsilon}\,. \tag{47}$$

Step 2c We now study the last term in the right hand side of (46). On the event $\{\vartheta_{2l-1} \le r < \vartheta_{2l}\}$, which is $\mathcal{F}_r$-measurable, the upper barrier is active on $[\vartheta_{2l-1}, \vartheta_{2l}]$, thus

$$\widetilde{\mathfrak{I}}^\vartheta_r - h(r,X_r) \le \mathbb{E}_r\left[h(r^+, X_{r^+}) - h(r,X_r) + \int_r^{r^+} |f(s,X_s,\mathfrak{I}^\vartheta_s)|ds\right]$$

where we set $r^+ = \inf\{s\in\mathfrak{r} | s>r\}\wedge T$. One thus gets,

$$([\widetilde{\mathfrak{I}}^\vartheta_r - h(r,X_r)]^+)^2 \mathbf{1}_{\{\vartheta_{2l-1}\le r<\vartheta_{2l}\}} \le \chi|\mathfrak{r}|^{\alpha+1}\,. \tag{48}$$

This leads to

$$\begin{aligned}&\sum_{r\in\mathfrak{r}}\Big(([\widetilde{\mathfrak{I}}^\vartheta_r - h(r,X_r)]^+)^2\, {}^c\widehat{\mathrm{I}}_r \sum_{l=1}^{[N+1/2]} \mathbf{1}_{\{\vartheta_{2l-1}\le r<\vartheta_{2l}\}}\Big)\\ &\quad\le |\mathfrak{r}|^{\alpha+1}\chi \sum_{r\in\mathfrak{r}}\Big({}^c\widehat{\mathrm{I}}_r \sum_{l=1}^{[N+1/2]} \mathbf{1}_{\{\vartheta_{2l-1}\le r<\vartheta_{2l}\}}\Big)\,.\end{aligned} \tag{49}$$

Moreover,

$$\begin{aligned}\sum_{r\in\mathfrak{r}}\sum_{l=1}^{[N+1/2]} {}^c\widehat{\mathrm{I}}_r \mathbf{1}_{\{\vartheta_{2l-1}\le r<\vartheta_{2l}\}} &\le \sum_{r\in\mathfrak{r}}\sum_{l=1}^{[N+1/2]} \mathbf{1}_{\{\vartheta_{2l-1}\le r<\vartheta_{2l}\}}(\mathbf{1}_{\{\widehat{\vartheta}_{2l-1}>r\}} + \mathbf{1}_{\{r>\widehat{\vartheta}_{2l}\}})\\ &\le \sum_{r\in\mathfrak{r}}\sum_{l=1}^{[N+1/2]} (\mathbf{1}_{\{\vartheta_{2l-1}\le r<\widehat{\vartheta}_{2l-1}\}} + \mathbf{1}_{\{\widehat{\vartheta}_{2l}<r<\vartheta_{2l}\}})\end{aligned}$$

Since the quantity

$$\sum_{r\in\mathfrak{r}}(\mathbf{1}_{\{\vartheta_{2l-1}\le r<\widehat{\vartheta}_{2l-1}\}} + \mathbf{1}_{\{\widehat{\vartheta}_{2l}<r<\vartheta_{2l}\}})$$

counts the number of reflection date r between ϑ_{2l-1} and $\widehat{\vartheta}_{2l-1}$ and between ϑ_{2l} and $\widehat{\vartheta}_{2l}$, we have the following control,

$$\sum_{r\in\mathfrak{r}}(\mathbf{1}_{\{\vartheta_{2l-1}\le r<\widehat{\vartheta}_{2l-1}\}}+\mathbf{1}_{\{\widehat{\vartheta}_{2l}<r<\vartheta_{2l}\}})\le C_\Lambda(1+\frac{|\vartheta_{2l-1}-\widehat{\vartheta}_{2l-1}|+|\vartheta_{2l}-\widehat{\vartheta}_{2l}|}{|\mathfrak{r}|})\ .$$

Using (49), this leads to

$$\begin{aligned}
&\mathbb{E}\Bigg[\sum_{r\in\mathfrak{r}}\Big(([\widetilde{\mathfrak{I}}^{\vartheta}_r-h(r,X_r)]^+)^2\,{}^c\widehat{\mathrm{I}}_r\sum_{l=1}^{[N+1/2]}\mathbf{1}_{\{\vartheta_{2l-1}\le r<\vartheta_{2l}\}}\Big)\Bigg]\\
&\le C_\Lambda\Big(|\mathfrak{r}|^{\alpha+1}+|\mathfrak{r}|^{\alpha}\sum_{l=0}^{N}\mathbb{E}\Big[\chi|\vartheta_l-\widehat{\vartheta}_l|\Big]\Big)\\
&\le C_\Lambda\Big(|\mathfrak{r}|^{\alpha+1}+|\mathfrak{r}|^{\alpha}\sum_{l=0}^{N}\mathbb{E}\Big[|\vartheta_l-\widehat{\vartheta}_l|\Big]^{1-\varepsilon}\Big)\ ,
\end{aligned}$$

where for the last inequality we used the Cauchy-Schwartz inequality with $\frac{1}{p}=1-\varepsilon$ and the fact that $|\vartheta_l-\widehat{\vartheta}_l|^p\le C^{\varepsilon}_{\Lambda}|\vartheta_l-\widehat{\vartheta}_l|$. Combining this last inequality with (47), it comes that

$$\mathbb{E}\Bigg[\sum_{r\in\mathfrak{r}}\eta_r\Bigg]\le C_\Lambda\Big(|\mathfrak{r}|^{\alpha+1}+|\mathfrak{r}|^{\alpha}\sum_{l=0}^{N}\mathbb{E}\Big[|\vartheta_l-\widehat{\vartheta}_l|\Big]^{1-\varepsilon}\Big)\ .$$

Step 3 Since $|\delta\mathfrak{I}_s|^2=\eta_r+|\delta\widetilde{\mathfrak{I}}_s|^2$, the proof is concluded by combining the last inequality with (43). □

4.4 Proof of Theorem 3.3

Step 1 We first study the error between $(\mathfrak{I}^{\widehat{\vartheta}},\widetilde{\mathfrak{I}}^{\widehat{\vartheta}},\mathfrak{Z}^{\widehat{\vartheta}})$ and the continuous-time Euler scheme for $\mathcal{Y}$ defined in section 3.3. Observe that these have the same activation/desactivation time sequence $\widehat{\vartheta}$. We are thus going to show that

$$\sup_{t\in[0,T]}\mathbb{E}\Big[|\widetilde{\mathfrak{I}}^{\widehat{\vartheta}}_t-\widetilde{Y}_t|^2\Big]+\sup_{t\in[0,T]}\mathbb{E}\Big[|\mathfrak{I}^{\widehat{\vartheta}}_t-\widehat{Y}_t|^2\Big]\le C_\Lambda|\mathfrak{t}|\text{ and }\|\mathfrak{Z}^{\widehat{\vartheta}}-\bar{Z}\|^2_{\mathcal{H}^2}\le C_\Lambda\nu|\mathfrak{t}|\ . \tag{50}$$

Toward this end, arguing as in the proof of Lemma 2.1 in [14] (See also Remark 5.2 in [14]), one could show that under (Hb), for $t\in\mathfrak{t}$, there exists S_t, Q_t in $\mathcal{F}_t$ such that $S_t\cap Q_t=\emptyset$ and

$$|\mathfrak{I}^{\widehat{\vartheta}}_t-\widehat{Y}_t|^2\le|\widetilde{\mathfrak{I}}^{\widehat{\vartheta}}_t-\widetilde{Y}_t|^2\mathbf{1}_{S_t}+C_\Lambda|X_t-\widehat{X}_t|^2\mathbf{1}_{Q_t}\ . \tag{51}$$

Observe in particular that for $t\in\mathfrak{t}\setminus\mathfrak{r}$, one can take $S_t=\Omega$ and $Q_t=\emptyset$ in (51) since, for $t\in[0,T]\setminus\mathfrak{r}$, we have $|\mathfrak{I}^{\widehat{\vartheta}}_t-\widehat{Y}_t|=|\widetilde{\mathfrak{I}}^{\widehat{\vartheta}}_t-\widetilde{Y}_t|$.

The proof of (50) is then similar to the proof of Proposition 5.1 (steps ia and ii) in [14]. Note that since f does not depend on z, the expression of B_i in equation (5.5) of [14] reduces in the present case to

$$B_i=\int_{t_{i-1}}^{t_i}(|X_u-\widehat{X}_{t_{i-1}}|^2+|\widetilde{\mathfrak{I}}^{\widehat{\vartheta}}_u-\widetilde{\mathfrak{I}}^{\widehat{\vartheta}}_{t_{i-1}}|^2)du\ .$$

Observing that, for $u \in [t_{i-1}, t_i)$,

$$\mathbb{E}\Big[|\widetilde{\mathfrak{I}}^{\widehat{\vartheta}}_u - \widetilde{\mathfrak{I}}^{\widehat{\vartheta}}_{t_{i-1}}|^2\Big] \le C\mathbb{E}\left[\int_{t_{i-1}}^{t_i} |f(s, X_s, \widetilde{\mathfrak{I}}^{\widehat{\vartheta}}_s)|^2 ds + \int_{t_{i-1}}^{t_i} |\mathfrak{Z}^{\widehat{\vartheta}}_s|^2 du\right] ,$$

we obtain $\mathbb{E}[\sum_i B_i] \le C|\mathfrak{t}|$. Inequalities (50) then follow from exactly the same arguments as in the proof of Proposition 5.1.

Step 2 Since

$$|Y_{t-} - \widehat{Y}_t|^2 \le C_\Lambda(|Y_{t-} - \mathfrak{I}^{\vartheta}_t|^2 + |\mathfrak{I}^{\vartheta}_t - \mathfrak{I}^{\widehat{\vartheta}}_t|^2 + |\mathfrak{I}^{\widehat{\vartheta}}_t - \widehat{Y}_t|^2) ,$$

we obtain using (50), Proposition 4.2 and Proposition 4.3 that

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|Y_{t-} - \widehat{Y}_t|^2\Big] \le C_\Lambda \left(|\mathfrak{t}| + |\mathfrak{r}|^\alpha + C^\varepsilon_\Lambda |\mathfrak{r}|^{\alpha-1} \sum_{l=1}^N \mathbb{E}\Big[|\widehat{\vartheta}_l - \vartheta_l|\Big]^{1-\varepsilon} \right) .$$

Similarly, one has that

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|Y_t - \widetilde{Y}_t|^2\Big] \le C_\Lambda \left(|\mathfrak{t}| + |\mathfrak{r}|^\alpha + C^\varepsilon_\Lambda |\mathfrak{r}|^{\alpha-1} \sum_{l=1}^N \mathbb{E}\Big[|\widehat{\vartheta}_l - \vartheta_l|\Big]^{1-\varepsilon} \right)$$

$$\|\mathfrak{Z} - \bar{Z}\|^2_{\mathcal{H}^2} \le C_\Lambda \left(\nu|\mathfrak{t}| + |\mathfrak{r}|^\alpha + C^\varepsilon_\Lambda |\mathfrak{r}|^{\alpha-1} \sum_{l=1}^N \mathbb{E}\Big[|\widehat{\vartheta}_l - \vartheta_l|\Big]^{1-\varepsilon} \right) .$$

Under (Hb)', the proof is concluded by using the last inequality together with Proposition 3.1 and letting $|\mathfrak{r}| \sim |\mathfrak{t}|^{\frac{1}{2}}$.

Under (Hb), one chooses $|\mathfrak{r}| \sim |\mathfrak{t}|^{\frac{1}{3}}$.

A Proof of Lemma 4.1

A.1 Proof of Part (i)

Step 1a In order to prove the result, we define C^2_b extensions $\widetilde{\sigma}$ and $\widetilde{b}$ of σ and b from $\mathcal{Q}$ to $[0,T] \times \mathbb{R}^q$, such that $\widetilde{\sigma}$ satisfies (29) on $[0,T] \times \mathbb{R}^q$. We then introduce, for every $t \le s \le T$,

$$\widetilde{X}^t_s = X_t + \int_t^s \widetilde{b}(u, \widetilde{X}^t_u) du + \int_t^s \widetilde{\sigma}(u, \widetilde{X}^t_u) dW_u , \ \xi^t_s = \mathrm{d}(\widetilde{X}^t_s) .$$

Since d, $\widetilde{b}$ and $\widetilde{\sigma}$ are smooth enough, we may introduce the *gradient process* of $\widetilde{X}$ with respect to the initial condition x of X, which is defined by, for $t \le s \le T$,

$$\nabla \widetilde{X}^t_s = I_q + \int_t^s \partial\widetilde{b}(u, \widetilde{X}^t_u) \nabla \widetilde{X}^t_u du + \int_t^s \partial\widetilde{\sigma}(u, \widetilde{X}^t_u) \nabla \widetilde{X}^t_u dW_u . \quad (52)$$

Note the following standard estimates,

$$\|\nabla \widetilde{X}^t\|_{\mathcal{S}^p} + \|(\nabla \widetilde{X}^t)^{-1}\|_{\mathcal{S}^p} \le C_\Lambda . \quad (53)$$

Observe moreover that one has, for $t \leq r \leq s \leq T$,

$$D_r \widetilde{X}_s^t = \widetilde{\sigma}(t, \widetilde{X}_s^t) + \int_t^s \partial \widetilde{b}(u, \widetilde{X}_u^t) D_r \widetilde{X}_u^t du + \int_t^s \partial \widetilde{\sigma}(u, \widetilde{X}_u^t) D_r \widetilde{X}_u^t dW_u$$

and therefore

$$D_r \widetilde{X}_s^t = \nabla \widetilde{X}_s^t (\nabla \widetilde{X}_r^t)^{-1} \widetilde{\sigma}(r, \widetilde{X}_r^t) \mathbf{1}_{t \leq r \leq s} \,. \tag{54}$$

Step 1b We now prove that (36) is satisfied by $\widetilde{X}^{T_I - \mathrm{h}}$, namely, for every $1 \leq I \leq N$,

$$\mathbb{P}\{a \leq \mathrm{d}(\widetilde{X}_{T_I}^{T_I - \mathrm{h}}) \leq b\} \leq C_\Lambda \frac{b-a}{\mathrm{h}} \,. \tag{55}$$

For this step, integration with respect to dW has to be understood in the Skorohod sense. Without loss of generality we fix $I \in \{1, \ldots, N\}$. For $x \in \mathbb{R}$, let

$$x \mapsto \phi(x) = \int_a^b \mathbf{1}_{\{x > z\}} dz = \int_{-\infty}^x \mathbf{1}_{[a,b]} dz$$

and let $\widetilde{\phi}$ stand for a regularization of ϕ. For every $T_I - \mathrm{h} \leq r \leq T_I$, we compute

$$\begin{aligned} D_r \widetilde{\phi}(\mathrm{d}(\widetilde{X}_{T_I}^{T_I - \mathrm{h}})) &= \nabla \widetilde{\phi}(\mathrm{d}(\widetilde{X}_{T_I}^{T_I - \mathrm{h}})) D_r \mathrm{d}(\widetilde{X}_{T_I}^{T_I - \mathrm{h}}) \\ &= \nabla \widetilde{\phi}(\mathrm{d}(\widetilde{X}_{T_I}^{T_I - \mathrm{h}})) \nabla \mathrm{d}(\widetilde{X}_{T_I}^{T_I - \mathrm{h}}) D_r \widetilde{X}_{T_I}^{T_I - \mathrm{h}} \\ &= \nabla \widetilde{\phi}(\mathrm{d}(\widetilde{X}_{T_I}^{T_I - \mathrm{h}})) \nabla \mathrm{d}(\widetilde{X}_{T_I}^{T_I - \mathrm{h}}) \nabla \widetilde{X}_{T_I}^{T_I - \mathrm{h}} (\nabla \widetilde{X}_r^{T_I - \mathrm{h}})^{-1} \widetilde{\sigma}(r, \widetilde{X}_r^{T_I - \mathrm{h}}) \mathbf{1}_{T_I - \mathrm{h} \leq r \leq T_I} \,. \end{aligned}$$

Multiplying by $\psi_r^I = \widetilde{\sigma}^\mathsf{T}(r, \widetilde{X}_r^{T_I - \mathrm{h}}) \nabla \widetilde{X}_r^{T_I - \mathrm{h}} (\nabla \widetilde{X}_{T_I}^{T_I - \mathrm{h}})^{-1} a(r, \widetilde{X}_r^{T_I - \mathrm{h}})^{-1}$, it comes,

$$\begin{aligned} \mathbb{E}\Big[\nabla \widetilde{\phi}(\mathrm{d}(\widetilde{X}_{T_I}^{T_I - \mathrm{h}}))\Big] &= \mathrm{h}^{-1} \mathbb{E}\Big[\int_{T_I - \mathrm{h}}^{T_I} D_r \widetilde{\phi}(\mathrm{d}(\widetilde{X}_{T_I}^{T_I - \mathrm{h}})) \psi_r^I dr\Big] \\ &= \mathrm{h}^{-1} \mathbb{E}\Big[\widetilde{\phi}(\mathrm{d}(\widetilde{X}_{T_I}^{T_I - \mathrm{h}})) \int_{T_I - \mathrm{h}}^{T_I} \psi_r^I dW_r\Big] \,. \end{aligned}$$

Recalling that $\widetilde{\phi}$ is the regularization of ϕ, the last equality leads, together with Fubini theorem and (53), to,

$$\mathbb{E}\Big[\nabla_x \phi(\widetilde{X}_{T_I}^{T_I - \mathrm{h}})\Big] = \mathrm{h}^{-1} \sum_{k=1}^q \int_a^b \mathbb{E}\Big[\mathbf{1}_{\{\widetilde{X}_{T_I}^{T_I - \mathrm{h}} > x\}} \int_{T_I - \mathrm{h}}^{T_I} \psi_r^I dW_r\Big] .$$

which concludes the proof of (55).

Step 2 We now prove that we can replace $\widetilde{X}_{T_I}^{T_I - \mathrm{h}}$ by X_{T_I} on Ω_I^{h}.

Let $B_I^{\mathrm{h}} = \{|\mathrm{d}(X_{T_I - \mathrm{h}})| \leq \frac{1}{3\Lambda}\}$, and

$$\tau_I^{\mathrm{h}} = \inf\{s \geq T_I - \mathrm{h} \; ; \; |\mathrm{d}(X_s)| \mathbf{1}_{B_I^{\mathrm{h}}} \geq \Lambda^{-1}\} \wedge T \,.$$

For every $u \in [T_I - \mathrm{h}, T]$, one has,

$$\widetilde{X}_{u \wedge \tau_I^{\mathrm{h}}}^{T_I - \mathrm{h}} \mathbf{1}_{B_I^{\mathrm{h}}} = X_{u \wedge \tau_I^{\mathrm{h}}} \mathbf{1}_{B_I^{\mathrm{h}}}. \tag{56}$$

Observe that $\tau_I^{\mathrm{h}} > T_I - \mathrm{h}$ and that for $T_I - \mathrm{h} \le u \le T$, one has,

$$(b,\sigma)(X_{u\wedge\tau_I^{\mathrm{h}}})\mathbf{1}_{B_I^{\mathrm{h}}} = (\widetilde{b},\widetilde{\sigma})(X_{u\wedge\tau_I^{\mathrm{h}}})\mathbf{1}_{B_I^{\mathrm{h}}} \ .$$

Let, for $u \ge t$, $\phi(u) = \mathbb{E}\Big[|\widetilde{X}^t_{u\wedge\tau_I^{\mathrm{h}}} - X_{u\wedge\tau_I^{\mathrm{h}}}|^2\mathbf{1}_{B_I^{\mathrm{h}}}\Big]$. Straightforward computations based on the Itô formula yield,

$$\phi(u) \le C_\Lambda \int_t^u \phi(s)ds \ ,$$

Identity (56) then follows by an application of Gronwall's Lemma.

The proof of the Lemma is then concluded by combining (56) and (55).

A.2 Proof of Part (ii)

We consider the two different cases.

Case 1: (i) By definition of ϑ, ϑ', we have that $\mathbb{E}[|\vartheta_0 - \vartheta'_0|] = |t - t'|$, and obviously, for $l \ge 1$,

$$\mathbb{E}\big[|\vartheta_l - \vartheta'_l|\big] = \mathbb{E}\big[|\vartheta_l - \vartheta'_l|\mathbf{1}_{\,^c\Omega^{\mathrm{h}}\cup\,^c\widehat{\Omega}^\delta}\big] + \mathbb{E}\Big[|\vartheta_l - \vartheta'_l|\mathbf{1}_{\Omega^{\mathrm{h}}\cap\widehat{\Omega}^\delta\cap\{\vartheta_l\neq\vartheta'_l\}}\Big] \,. \tag{57}$$

It follows then from Tchebytchev's inequality applied on ${}^c\Omega^{\mathrm{h}}_{\mathcal{T}}$, for $\mathcal{T} \in \mathfrak{T}^t$, on ${}^c\widehat{\Omega}^\delta$ and the fact that $|\vartheta_l - \vartheta'_l| \le T$

$$\mathbb{E}\big[|\vartheta_l - \vartheta'_l|\mathbf{1}_{\,^c\Omega^{\mathrm{h}}\cup\,^c\widehat{\Omega}^\delta}\big] \le C_\Lambda^{\bar{p}}\mathrm{h}^{\bar{p}} + C_\Lambda^p \frac{\mathbb{E}\Big[\sup_{u\in[\mathcal{T}^t,T]}|\xi_u - X_u|^p\Big]}{\delta^p} \ , \tag{58}$$

for p, $\bar{p} > 0$.

(ii) We now work on the second term of the right-hand side of (57).

By definition of ϑ, ϑ', if $k \notin K$, we have $\mathbb{E}\big[|\vartheta_1 - \vartheta'_1|\mathbf{1}_{\{k\notin K\}}\big] = |t-t'|$. We are going to prove a control between ϑ and ϑ', for $l \ge 2$, and for $l = 1$, $k \in K$.

To this end, we observe that

$$\mathbf{1}_{\{\mathrm{d}(X_{\mathcal{T}})\ge 0\}} = \mathbf{1}_{\{\mathrm{d}(\xi_{\mathcal{T}})\ge 0\}}, \forall\, \mathcal{T} \in \mathfrak{T}^t \implies H = H', \tag{59}$$

thus for $l \ge 2$, $\vartheta_l = \vartheta'_l$ and if $k \in K$, $\vartheta_1 = \vartheta'_1$.

We then introduce the set

$$\Omega_1 = \bigcup_{\mathcal{T}\in\mathfrak{T}^t} \left(\{\mathrm{d}(X_{\mathcal{T}}) \ge 0\} \cap \{\mathrm{d}(\xi_{\mathcal{T}}) < 0\}\right) \cup \left(\{\mathrm{d}(X_{\mathcal{T}}) < 0\} \cap \{\mathrm{d}(\xi_{\mathcal{T}}) \ge 0\}\right)$$

Since d is 1-Lipschitz continuous, by definition of $\widehat{\Omega}^\delta$, we have

$$\Omega_1 \subset \bigcup_{\mathcal{T}\in\mathfrak{T}^t} \{|\mathrm{d}(X_{\mathfrak{T}})| \le \delta\}$$

This leads, by definition of Ω^{h}, to

$$\Omega^{\mathrm{h}} \cap \widehat{\Omega}^{\delta} \cap \Omega_1 \subset \bar{\Omega}\ ,\ \text{with } \bar{\Omega} := \bigcup_{\mathcal{T} \in \mathfrak{T}^{T*}} \Omega^{\mathrm{h}}_{\mathcal{T}} \cap \{|\mathrm{d}(X_{\mathcal{T}})| < \delta\}$$

Using (59), we have that, for $l \geq 2$, $\{\vartheta_l \neq \vartheta'_l\} \subset \Omega_1$ and if $k \in K$, $\{\vartheta_1 \neq \vartheta'_1\} \subset \Omega_1$. Thus, for $l \geq 2$, $\Omega^{\mathrm{h}} \cap \widehat{\Omega}^{\delta} \cap \{\vartheta_l \neq \vartheta'_l\} \subset \bar{\Omega}$ and if $k \in K$, $\Omega^{\mathrm{h}} \cap \widehat{\Omega}^{\delta} \cap \{\vartheta_1 \neq \vartheta'_1\} \subset \bar{\Omega}$.

Using the result of part (i), one then gets,

$$\mathbb{E}\Big[|\vartheta_l - \vartheta'_l| \mathbf{1}_{\Omega^{\mathrm{h}}_{\mathcal{T}} \cap \widehat{\Omega}^{\delta} \cap \{\vartheta_l \neq \vartheta'_l\}}\Big] \leq C_\Lambda \frac{\delta}{\mathrm{h}}\ ,$$

for $l \geq 2$ and $l = 1$, if $k \in K$. In this case, the proof is concluded combining the last inequality with (58) and (57).

Case 2: In this case, $\mathfrak{T}^{t'} = \mathfrak{T}^t \cup \{t\}$.

As in Case 1 (i) above, we compute

$$\mathbb{E}\Big[|\widetilde{\vartheta}_l - \vartheta'_l|\Big] \leq C^{\bar{p}}_\Lambda \mathrm{h}^{\bar{p}} + C^p_\Lambda \frac{\mathbb{E}\Big[\sup_{u \in [\mathcal{T}^t, T]} |\xi_u - X_u|^p\Big]}{\delta^p} + \mathbb{E}\Big[|\widetilde{\vartheta}_l - \vartheta'_l| \mathbf{1}_{\Omega^{\mathrm{h}} \cap \widehat{\Omega}^{\delta} \cap \{\widetilde{\vartheta}_l \neq \vartheta'_l\}}\Big]\ , \qquad (60)$$

for $l \geq 0$ and p, $\bar{p} > 0$.

Recall that by definition of $\widetilde{\vartheta}$, ϑ', $\mathbb{E}\Big[|\widetilde{\vartheta_0 - \vartheta'_0}|\Big] = |t - t'|$ and if $k \notin K$, $\mathbb{E}\Big[|\widetilde{\vartheta_1 - \vartheta'_1}|\Big] = |t - t'|$.

Regarding the last term of (60), we observe here that

$$\mathbf{1}_{\{\mathrm{d}(X_{\mathcal{T}}) \geq 0\}} = \mathbf{1}_{\{\mathrm{d}(\xi_{\mathcal{T}}) \geq 0\}}, \forall \mathcal{T} \in \mathfrak{T}^t \cup \{t\} \implies \widetilde{H} = H'.$$

The set Ω_1 is now replaced by

$$\Omega_2 = \bigcup_{\mathcal{T} \in \mathfrak{T}^t \cup \{t\}} (\{\mathrm{d}(X_{\mathcal{T}}) \geq 0\} \cap \{\mathrm{d}(\xi_{\mathcal{T}}) < 0\}) \cup (\{\mathrm{d}(X_{\mathcal{T}}) < 0\} \cap \{\mathrm{d}(\xi_{\mathcal{T}}) \geq 0\})$$

The difference with the last step is that the reunion is on $\mathfrak{T}^t \cup \{t\}$. But, since for δ small enough $\{|\mathrm{d}(X_t)| < \delta\} = \emptyset$, we have

$$\Omega^{\mathrm{h}} \cap \widehat{\Omega}^{\delta} \cap \Omega_2 \subset \bar{\Omega}\ .$$

The proof is then concluded arguing as in the last step.

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