BSDEs of Counterparty Risk and Invariant Times

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Abstract

We study a BSDE with random terminal time that appears in the modeling of counterparty risk in finance. We proceed by reduction of the original BSDE into a simpler BSDE posed with respect to a smaller filtration and a changed probability measure. This is done under a relaxation of the classical immersion hypothesis, stated in terms of the changed probability measure, of which we characterize the Radon-Nikodym derivative. Our study reveals the importance of a new class of so-called invariant times that we characterize in terms of their Azéma supermartingale.

Keywords: Random time, Enlargement of filtration, BSDE, Counterparty risk.

Mathematics Subject Classification: 60G07,60G44,60H10,91G40

1 Introduction

We study a BSDE with random terminal time $\vartheta = T \wedge \theta$, where T is a positive constant and the stopping time θ has an intensity. As developed in the companion papers by Crépey and Song (2014a, 2014b) (following up on Crépey (2012)), this BSDE is key to the modeling of counterparty risk in finance (see Brigo, Morini, and Pallavicini (2013) for general counterparty risk references and Crépey, Bielecki, and Brigo (2014) for a more mathematical perspective). But this BSDE comes in a rather unusual form. Our approach in this work is to reduce the problem to simpler BSDEs relative to a smaller filtration and a possibly changed probability measure. Moreover, we want to achieve this under minimal assumptions on θ, so that the model stays as flexible and can fit as many counterparty risk scenarios and features as possible (notably bilateral counterparty risk and the related nonlinear funding issue, wrong-way risk and gap risk). This leads us to introduce and study a new class of so-called invariant times that appear in the follow-up papers Crépey and Song (2014a, 2014b) as the appropriate notion to model an arbitrary nature and level of dependence between the

reference filtration and the default time θ (as opposed to the weak or indirect dependence that would be implied by a basic immersion setup).

The paper is organized as follows. Sect. 2 deals with the compensator of ϑ and measurability issues in relation with a single step martingale that corresponds to the compensation of a jump process associated with the terminal condition of our BSDE. In Sect. 3, the original BSDE is rewritten in terms of an auxiliary BSDE with solution continuous at ϑ. The two BSDEs are posed with respect to a common stochastic basis $(\Omega, \mathbb{G}, \mathbb{Q})$. After revisiting the Barlow-Jeulin-Yor theory under a condition (B) relative to a subfiltration $\mathbb{F}$ of $\mathbb{G}$, Sect. 4 reduces the auxiliary $(\mathbb{G}, \mathbb{Q})$ BSDE with random terminal time ϑ to an $(\mathbb{F}, \mathbb{Q})$ BSDE with a null terminal condition at the fixed time T. Next, an even simpler $(\mathbb{F}, \mathbb{P})$ BSDE is obtained under an additional condition (A) on a changed probability measure $\mathbb{P}$. In Sect. 5 invariant times are introduced and studied based on a characterization of the condition (A) in terms of the Radon-Nikodym density $\frac{d\mathbb{P}}{d\mathbb{Q}}$ and of the Azéma supermartingale S of θ.

1.1   Standing Assumptions and Notation

We work on a space Ω equipped with a σ-field $\mathcal{B}$, with a probability measure $\mathbb{Q}$ on $\mathcal{B}$ and with a filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$ of sub-σ-fields of $\mathcal{B}$, satisfying the usual conditions. We use the terminology of the general theory of processes and of filtrations as given in the books by Dellacherie and Meyer (1975) and He, Wang, and Yan (1992). In particular, we use the notions of predictable and optional processes, and of predictable and optional projections and dual projections (or compensator, in case of the dual predictable projection of a finite variation adapted process). Sometimes we say projection of a stopping time τ for projection of its indicator process $\mathbb{1}_{[\tau,\infty)}$. As also implicit in all these notions of projections, we work with the notion of generalized conditional expectation that is applicable to any locally integrable random variable (see Sect. I.4 in He, Wang, and Yan (1992)). We denote by $\mathcal{P}(\mathbb{F})$ and $\mathcal{O}(\mathbb{F})$ the predictable and optional σ-fields with respect to a filtration $\mathbb{F}$. All semimartingales are taken in a càdlàg version. For any semimartingale Y and predictable Y-integrable process L, the stochastic integral process of L with respect to Y is denoted by $L \boldsymbol{.} Y$ (with the usual precedence convention $KL \boldsymbol{.} Y = (KL) \boldsymbol{.} Y$, if K is another predictable Y-integrable process). For any càdlàg process Y, for any random time (nonnegative random variable) τ, we write $\Delta_\tau Y$ for the jump of Y at τ (with the convention $\Delta_0 Y = Y_0$). We use the Dellacherie and Meyer (1975) notation $Y^{\tau-}$ to represent the process Y stopped "right before τ" or "at $\tau-$", i.e.

$$Y^{\tau-} = Y \mathbb{1}_{[0,\tau)} + Y_{\tau-} \mathbb{1}_{[\tau,+\infty)}; \tag{1.1}$$

$\boldsymbol{\delta}_\tau(dt)$ denotes the Dirac measure at (a possibly random) time τ; a Lebesgue measure is denoted by $\boldsymbol{\lambda}$; the real line and half-line and the nonnegative integers are respectively denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$. Unless otherwise stated, a function (or process) is real-valued; order relationships between random variables (respectively processes) are meant almost surely (respectively in the indistinguishable sense). We don't explicitly mention the domain of definition of a variable x in $h(x)$ when it is implied by the measurability, writing e.g. "a $\mathcal{B}(\mathbb{R})$ measurable function h (or $h(x)$)" rather than "a $\mathcal{B}(\mathbb{R})$ measurable function h defined on $\mathbb{R}$". As is usual, for a function $h(\omega, x)$ defined on a product space $\Omega \times E$, we often write $h(x)$ without ω.

2 Preparatory Results

Throughout the paper ϑ denotes a $\mathbb{G}$ stopping time with indicator process $H = \mathbb{1}_{[\vartheta, +\infty)}$ and $J = 1 - H$, so that $Y^{\vartheta-} = YJ + Y_{\vartheta-}H$ (cf. (1.1)).

2.1 Stopping Time and Compensator

For $A \in \mathcal{G}_\vartheta$, we denote by ϑ_A the following random time (see Definition 3.8 in He, Wang, and Yan (1992)):

$$\vartheta_A(\omega) = \begin{cases} \vartheta(\omega), & \text{if } \omega \in A, \\ \\ \infty, & \text{if } \omega \notin A. \end{cases}$$

Note that ϑ_A is again a $\mathbb{G}$ stopping time. Therefore, so is also ϑ_{A^c}. According to Theorem 4.20 in He, Wang, and Yan (1992), there exists $A \in \mathcal{G}_{\vartheta-}$ such that ϑ_A is accessible and ϑ_{A^c} is totally inaccessible. Writing $\vartheta^a = \vartheta_A, \vartheta^i = \vartheta_{A^c}$, we have, for any bounded $\mathbb{G}$ predictable process h,

$$\mathbb{E}[h_\vartheta \mathbb{1}_{\{\vartheta < \infty\}}] = \mathbb{E}[h_{\vartheta^a \wedge \vartheta^i} \mathbb{1}_{\{\vartheta^a \wedge \vartheta^i < \infty\}}] = \mathbb{E}[h_{\vartheta^a} \mathbb{1}_{\{\vartheta^a < \infty\}}] + \mathbb{E}[h_{\vartheta^i} \mathbb{1}_{\{\vartheta^i < \infty\}}], \tag{2.1}$$

i.e. in terms of the compensators $\mathbf{v}$ of ϑ, $\mathbf{v}^a$ of ϑ_A and $\mathbf{v}^i$ of ϑ_{A^c},

$$\mathbb{E}[\int_{[0,\infty)} h_s \, d\mathbf{v}_s] = \mathbb{E}[\int_{[0,\infty)} h_s \, d\mathbf{v}_s^a] + \mathbb{E}[\int_{[0,\infty)} h_s \, d\mathbf{v}_s^i] = \mathbb{E}[\int_{[0,\infty)} h_s \, d(\mathbf{v}^a + \mathbf{v}^i)_s],$$

so that $\mathbf{v} = \mathbf{v}^a + \mathbf{v}^i$. More precisely, we have the following result regarding the structure of the compensator $\mathbf{v}$ of ϑ.

Lemma 2.1 *The processes $\mathbf{v}^i$ and $\mathbf{v}^a$ are the continuous component and the pure jump component of $\mathbf{v}$.*

Proof. In view of Corollary 5.28 in He, Wang, and Yan (1992), $\mathbf{v}^i$ is continuous. As for $\mathbf{v}^a$, by definition of an accessible time, there exists an at most countable family of predictable stopping times $(\tau_i)_{i \in I}$ such that $[\![\vartheta_A]\!] \subset \cup_{i \in I} [\![\tau_i]\!]$ and $[\![\tau_i]\!] \cap [\![\tau_j]\!] = \emptyset$ for $i \neq j$. Consequently,

$$\mathbb{E}[h_{\vartheta_A} \mathbb{1}_{\{\vartheta_A < \infty\}}] = \sum_{i \in I} \mathbb{E}[h_{\tau_i} \mathbb{1}_{\{\vartheta_A = \tau_i\}} \mathbb{1}_{\{\tau_i < \infty\}}] = \sum_{i \in I} \mathbb{E}[h_{\tau_i} \Delta_{\tau_i} \mathbf{v} \mathbb{1}_{\{\tau_i < \infty\}}], \tag{2.2}$$

where we make use of

$$\mathbb{Q}[\{\vartheta_A = \tau_i\} | \mathcal{G}_{\tau_i-}] = \mathbb{Q}[\{\vartheta = \tau_i\} | \mathcal{G}_{\tau_i-}] = \Delta_{\tau_i} \mathbf{v},$$

which holds by Theorem 5.27 in He, Wang, and Yan (1992). By Theorem 3.33 in He, Wang, and Yan (1992), the process $\sum_{i \in I} \Delta_{\tau_i} \mathbf{v} \mathbb{1}_{[\tau_i, \infty)}$ is predictable. Moreover, being nondecreasing (recall $\mathbf{v}$ itself is nondecreasing as the compensator of the nondecreasing process H), it is integrable since

$$\sum_{s \le t} \Delta_s \mathbf{v} \le \int_0^t |d\mathbf{v}| = \mathbf{v},$$

where $\mathbf{v}$ itself is integrable as the compensator of the bounded, hence integrable process H. Now, the identity (2.2), for a predictable integrable process $\sum_{i \in I} \Delta_{\tau_i} \mathbf{v} \mathbb{1}_{[\tau_i, \infty)}$, proves that this process is equal to $\mathbf{v}^a$. $\qquad \square$

2.2 Parameterized Conditioning

To solve our BSDEs we need to compute conditional expectations of the form $\mathbb{E}[h(\xi)|\mathcal{G}_{\vartheta-}]$ for a nonnegative measurable function $h(\omega, x)$ and a $\mathcal{G}_{\vartheta-}$ measurable random variable ξ depending on a real x. The intuition suggests that, under the conditioning, the random variable ξ can be treated as a constant, so that the computation can be performed in two steps: first for a constant x instead of ξ, then by substituting ξ for x, i.e.

$$\text{``}\mathbb{E}[h(\xi)|\mathcal{G}_{\vartheta-}] = \mathbb{E}[h(x)|\mathcal{G}_{\vartheta-}]_{x=\xi}\text{''}.$$

However, this is not well defined because the conditional expectation $\mathbb{E}[h(x)|\mathcal{G}_{\vartheta-}]$ is an equivalence class depending on the real x. A "bad" choice of the class (one for each x) in the first step may result in a nonmeasurable expression in the second step.

Example 2.1 Let $\mathcal{B}$ denote the Borel σ-field over $[0,1]$, considered as a sub-σ-field of the Borel σ-field over $\Omega = [0,1]^2$ equipped with the Lebesgue measure ("sub-σ-field" through the inverse of the first coordinate projection p). Let $h(\omega, x)$ be a nonnegative Borel function on $\Omega \times \mathbb{R}$. By Fubini's theorem, there exists a Borel function h' on $\Omega \times \mathbb{R}_+$ such that, for any x, $\omega \to h'(\omega, x)$ is a version of $\mathbb{E}[h(x)|\mathcal{B}](\omega)$ ("expectation with respect to v for u frozen in $\omega = (u,v) \in \Omega = [0,1]^2$"). Let

$$h''(\omega, x) = h'(\omega, x) + \mathbb{1}_V(x)\mathbb{1}_{\{x\}}(p(\omega)),$$

where V is the Vitali set in $[0,1]$ (assuming the axiom of choice). Since x is fixed here, $h'(\cdot, x)$ and $h''(\cdot, x)$ are almost surely equal, i.e. $h''(\cdot, x)$ is a version of $\mathbb{E}[h(x)|\mathcal{B}]$ (for fixed x). However, since the Vitali set is not Lebesgue measurable, the function

$$h''(\omega, p(\omega)) = h'(\omega, \omega) + \mathbb{1}_V(p(\omega))$$

is not Borel.

Several appoaches exist to deal with this version choice problem. By the same monotone class argument as in Stricker and Yor (1978), there exists a $\mathcal{G}_{\vartheta-} \otimes \mathcal{B}(\mathbb{R})$ measurable function $\widehat{h}(\omega, x)$ such that
$$\mathbb{E}[h(\xi)|\mathcal{G}_{\vartheta-}] = \widehat{h}(\xi), \ x \in \mathbb{R}.$$

But the function $\widehat{h}$ obtained in this way only works for a fixed random variable ξ (i.e. the function $\widehat{h}$ depends on ξ), whereas we need it below in an equation for an unknown random variable ξ (i.e. we need a common function $\widehat{h}$ that works for all ξ). This motivates the following development.

Definition 2.1 For any measurable space $(E, \mathcal{E})$, for any nonnegative $\mathcal{B} \otimes \mathcal{E}$ measurable function $h(\omega, x)$, we say that $\widehat{h}_t(\omega, x)$ exists and that a nonnegative $\mathcal{P}(\mathbb{G}) \otimes \mathcal{E}$ measurable function $g_t(\omega, x)$ is a version of $\widehat{h}_t(\omega, x)$ ($\widehat{h} = g$ in a shorthand notation) if, for any E valued $\mathcal{G}_{\vartheta-}$ measurable random variable ξ,

$$\mathbb{E}[h(\xi)|\mathcal{G}_{\vartheta-}]\mathbb{1}_{\{\vartheta<\infty\}} = g_\vartheta(\xi)\mathbb{1}_{\{\vartheta<\infty\}}.$$

For a general (not necessarily nonnegative) $\mathcal{B} \otimes \mathcal{E}$ measurable function h, we say that $\widehat{h}$ exists and equals $\widehat{h_1} - \widehat{h_2}$ if there exist two nonnegative functions h_1 and h_2 such that $h = h_1 - h_2$, $\widehat{h_1}$ and $\widehat{h_2}$ exist and $\widehat{h_1} - \widehat{h_2}$ is well defined.

The following result establishes the existence of $\widehat{h}$ by monotone class arguments. Under suitable topological conditions, $\widehat{h}$ can be computed using the notion of regular conditional distribution with respect to $\mathcal{G}_{\vartheta-}$ (see the paragraphs 88 to 90, Sect. II.6 in Rogers and Williams (2000)).

Lemma 2.2 *For any measurable space $(E, \mathcal{E})$, for any nonnegative $\mathcal{B} \otimes \mathcal{E}$ measurable function $h(\omega, x)$, $\widehat{h}$ exists.*

Proof. Let $\mathfrak{C}$ denote the class of all bounded $\mathcal{B} \otimes \mathcal{E}$ measurable function h for which $\widehat{h}$ exists. Obviously, the class $\mathfrak{C}$ contains the constants and is stable by multiplication by constants. Given two nonnegative functions $h, h' \in \mathfrak{C}$, we can check directly from the definition that $\widehat{h} + \widehat{h'}$ is a version of $\widehat{h + h'}$. Given two real valued bounded functions $h, h' \in \mathfrak{C}$, let h_1 and h_2 (respectively h'_1 and h'_2) be two nonnegative functions such that $h = h_1 - h_2$ and $\widehat{h_1}$ and $\widehat{h_2}$ exist (respectively same properties with $'$). Hence $h_1 + h'_1 \geq 0, h_2 + h'_2 \geq 0$, $\widehat{h_1 + h'_1}$ and $\widehat{h_2 + h'_2}$ exist, and

$$h + h' = h_1 + h'_1 - (h_2 + h'_2).$$

This shows that $h + h' \in \mathfrak{C}$. The class $\mathfrak{C}$ is a linear space of real functions.

Let $(h_n)_{n \in \mathbb{N}}$ be a nondecreasing uniformly bounded sequence of nonnegative functions in $\mathfrak{C}$. For any $n \in \mathbb{N}$, for any E valued $\mathcal{G}_{\vartheta-}$ measurable random variable ξ,

$$\mathbb{E}[h_n(\xi)|\mathcal{G}_{\vartheta-}]\mathbb{1}_{\{\vartheta<\infty\}} = (\widehat{h_n})_\vartheta(\xi)\mathbb{1}_{\{\vartheta<\infty\}}.$$

By the monotone convergence theorem,

$$\mathbb{E}[\sup_{n \in \mathbb{N}} h_n(\xi)|\mathcal{G}_{\vartheta-}]\mathbb{1}_{\{\vartheta<\infty\}} = \sup_{n \in \mathbb{N}} \mathbb{E}[h_n(\xi)|\mathcal{G}_{\vartheta-}]\mathbb{1}_{\{\vartheta<\infty\}} = (\sup_{n \in \mathbb{N}} \widehat{h_n})_\vartheta(\xi)\mathbb{1}_{\{\vartheta<\infty\}}.$$

This formula shows that $\sup_{n \in \mathbb{N}} h_n$ is an element in $\mathfrak{C}$ and

$$\widehat{\sup_{n \in \mathbb{N}} h_n} = \sup_{n \in \mathbb{N}} \widehat{h_n}.$$

Finally, let's consider $A \in \mathcal{B}, B \in \mathcal{E}$. The random variable $\mathbb{Q}[A|\mathcal{G}_{\vartheta-}]$ is $\mathcal{G}_{\vartheta-}$ measurable. There exists a $\mathbb{G}$ predictable process L such that (cf. Corollary 3.22 in He, Wang, and Yan (1992))

$$L_\vartheta = \mathbb{Q}[A|\mathcal{G}_{\vartheta-}]\mathbb{1}_{\{\vartheta<\infty\}}.$$

We check directly that $L_t(\omega)\mathbb{1}_B(x)$ is a version of $\widehat{\mathbb{1}_A\mathbb{1}_B}$, which shows that $\mathbb{1}_A(\omega)\mathbb{1}_B(x)$ is an element in $\mathfrak{C}$. We can now apply the monotone class theorem (cf. Theorem 1.4 in He, Wang, and Yan (1992)) to say that $\mathfrak{C}$ contains all bounded $\mathcal{B} \otimes \mathcal{E}$ measurable functions. Last, by taking suprema over sequences, the result is extended to general (non necessarily bounded) nonnegative $\mathcal{B} \otimes \mathcal{E}$ measurable functions. $\square$

In this paper the function $\widehat{h}$ is mainly used to compute compensation martingales through the following lemma (classical after the above clarification regarding the definition of $\widehat{h}$).

Lemma 2.3 *Let $\mathbf{v}_t, t \in \mathbb{R}_+$, be the compensator of the nondecreasing process H. Given a $\mathcal{B} \otimes \mathcal{B}(\mathbb{R}^d)$ measurable function $h(\omega, x)$, we set $\widehat{h} = \widehat{h^+} - \widehat{h^-}$. Let Y be a $\mathbb{G}$ adapted*

càdlàg process in $\mathbb{R}$. *If* $h \geq 0$ *or* $|h|(Y_{\vartheta-})\mathbb{1}_{\{\vartheta<\infty\}}$ *is integrable, then, for every* $\mathbb{G}$ *predictable bounded process* L,

$$\mathbb{E}[L_\vartheta h(Y_{\vartheta-})\mathbb{1}_{\{\vartheta<\infty\}}] = \mathbb{E}[\int_{[0,\infty)} L_v \widehat{h}_v(Y_{v-})d\mathbf{v}_v].$$

If $|h|(Y_{\vartheta-})\mathbb{1}_{\{\vartheta<\infty\}}$ *is integrable, the process*

$$h(Y_{\vartheta-})H - \mathbb{1}_{[0,\vartheta]}\widehat{h}.(Y_-) \bullet \mathbf{v}$$

is a $\mathbb{G}$ *uniformly integrable martingale.* $\qquad\qquad\square$

The following result gives an explicit formula for $\widehat{h}$ in the case, used in the application paper Crépey and Song (2014b), of a so-called marked stopping time. Namely, we consider a stopping time $\vartheta(\omega)$ taking, for every ω, the value of one of the $\vartheta^e(\omega)$, where for every e in an at most countable space of marks E endowed with some σ-algebra $\mathcal{E}$, ϑ^e represents a totally inaccessible stopping time with intensity γ_t^e, such that $\mathbb{Q}[\vartheta^e \neq \vartheta^{e'}] = 1$ for $e' \neq e$.

Lemma 2.4 *Let* $\mathfrak{r}_t(\omega, e, x)$ *be a nonnegative* $\mathcal{P}(\mathbb{G}) \otimes \mathcal{E} \otimes \mathcal{B}(\mathbb{R})$ *measurable function. Writing* $\zeta(\omega) = \sum_{e\in E} \mathbb{1}_{\{\vartheta^e(\omega)=\vartheta(\omega)\}}e$, *which is* $\mathcal{G}_\vartheta$ *measurable, let*

$$h(\omega, x) = \mathfrak{r}_{\vartheta(\omega)}(\omega, \zeta(\omega), x). \tag{2.3}$$

A version of $\widehat{h}_t(\omega, x)$ *is given by* $\sum_{e\in E} q_t^e \mathfrak{r}_t(e, x)$, *where, for every* $e \in E$, q^e *is a* $[0,1]$ *valued* $\mathbb{G}$ *predictable process (which exists) such that* $q_\vartheta^e = \mathbb{Q}[\{\vartheta^e = \vartheta\}|\mathcal{G}_{\vartheta-}]$ *on* $\{\vartheta < \infty\}$.

Proof. On $\{\vartheta < \infty\}$, we have

$$\mathbb{E}[h(x)|\mathcal{G}_{\vartheta-}] = \mathbb{E}[\mathfrak{r}_\vartheta(\zeta, x)|\mathcal{G}_{\vartheta-}] = \sum_{e\in E} \mathbb{Q}[\{\vartheta^e = \vartheta\}|\mathcal{G}_{\vartheta-}]\mathfrak{r}_\vartheta(e, x) = \sum_{e\in E} q_\vartheta^e \mathfrak{r}_\vartheta(e, x).$$

$\qquad\qquad\square$

The following result shows that the shape (2.3) that is postulated for the function $h(\omega, x)$ in the example 2.4 is, in fact, general under the structural condition (2.4) below on $\mathcal{G}_\vartheta$ and $\mathcal{G}_{\vartheta-}$.

Lemma 2.5 *Let* $(E, \mathcal{E})$ *be a measurable space. Suppose that there exists a random variable* ζ *taking at most countably many values in* E *such that*

$$\mathcal{G}_\vartheta = \mathcal{G}_{\vartheta-} \vee \sigma(\zeta). \tag{2.4}$$

Then, for any nonnegative $\mathcal{G}_\vartheta \otimes \mathcal{B}(\mathbb{R})$ *measurable function* $h(\omega, x)$, *there exists a nonnegative* $\mathcal{P}(\mathbb{G}) \otimes \mathcal{E} \otimes \mathcal{B}(\mathbb{R})$ *measurable function* $\mathfrak{r}_t(\omega, e, x)$ *such that*

$$h(\omega, x) = \mathfrak{r}_{\vartheta(\omega)}(\omega, \zeta(\omega), x).$$

A version of $\widehat{h}_t(\omega, x)$ *is given by* $\sum_{e\in \mathtt{E}} q_t^e(\omega)\mathfrak{r}_t(\omega, e, x)$, *where* $\mathtt{E} = \zeta(\Omega)$ *and, for every* $e \in \mathtt{E}$, q^e *is a* $[0,1]$ *valued* $\mathbb{G}$ *predictable process such that* $q_\vartheta^e = \mathbb{Q}[\{\zeta = e\}|\mathcal{G}_{\vartheta-}]$ *on* $\{\vartheta < \infty\}$.

Proof. The existence of $\mathfrak{x}_t(\omega, e, x)$ is a consequence of the monotone class theorem. The second part of the result follows by an application of Lemma 2.4. $\qquad\square$

We now give a concrete specification for (2.4) in case where $\mathbb{G}$ is the progressive enlargement of a reference filtration $\mathbb{F}$ by n random times $\vartheta_1, \ldots, \vartheta_n$, assuming the optional splitting formula of Song (2012) holds (as is the case under the density hypothesis or under the immersion hypothesis), i.e. for any $\mathbb{G}$ optional process Y, there exist $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}[0, \infty]^n$-measurable functions $Y^{(0)}, Y^{(1)}, \ldots, Y^{(n)}$ such that

$$Y = \sum_{i=0}^{n} Y^{(i)}(\vartheta_1 \dagger \vartheta_{(i)}, \ldots, \vartheta_n \dagger \vartheta_{(i)}) \mathbb{1}_{[\vartheta_{(i)}, \vartheta_{(i+1)})}, \tag{2.5}$$

where the $\vartheta_{(i)}$ are a nondecreasing reordering of the ϑ_i with $\vartheta_{(0)} = 0$ and $\vartheta_{(n+1)} = \infty$.

Lemma 2.6 *Suppose that the ϑ_i don't intercept each other. Let $\vartheta = \vartheta_1 \wedge \vartheta_2$. Then*

$$\mathcal{G}_\vartheta = \mathcal{G}_{\vartheta-} \vee \sigma(\{\vartheta = \vartheta_1\}, \{\vartheta = \vartheta_2\}).$$

Proof. *For $a, b \in [0, \infty]$, let $a \dagger b$ denote a is $a \le b$ and ∞ if $a > b$. Let $\mathtt{I} = \{1, 2, 3, \ldots, n\}$. By the optional splitting formula (2.5),*

$$
\begin{aligned}
Y_\vartheta \mathbb{1}_{\{\vartheta = \vartheta_{(i)}\}} &= Y_\vartheta^{(i)}(\vartheta_1 \dagger \vartheta, \ldots, \vartheta_n \dagger \vartheta) \mathbb{1}_{\{\vartheta = \vartheta_{(i)}\}} \\
&= \sum_{I \subseteq \mathtt{I}: \#I = i-1} Y_\vartheta^{(i)}(\vartheta_1 \dagger \vartheta, \vartheta_2 \dagger \vartheta, \ldots, \vartheta_n \dagger \vartheta) \mathbb{1}_{\{\forall j \in I, \vartheta_j < \vartheta\}} \mathbb{1}_{\{\forall j \in \mathtt{I} \setminus I, \vartheta \le \vartheta_j\}} \\
&= \sum_{k=1}^{2} \sum_{I \subseteq \mathtt{I}: \#I = i-1} Y_\vartheta^{(i,I,k)}(\vartheta, \vartheta_j, j \in I) \mathbb{1}_{\{\forall j \in I, \vartheta_j < \vartheta\}} \mathbb{1}_{\{\forall j \in \mathtt{I} \setminus I, \vartheta \le \vartheta_j\}} \mathbb{1}_{\{\vartheta = \vartheta_k\}},
\end{aligned}
$$

where $Y_t^{(i,I,k)}(\omega, x, y_j, j \in I)$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}[0, \infty] \otimes \mathcal{B}[0, \infty]^{i-1}$ measurable and such that

$$Y_\vartheta^{(i,I,k)}(\vartheta, \vartheta_j, j \in I) \mathbb{1}_{\{\forall j \in I, \vartheta_j < \vartheta\}} \mathbb{1}_{\{\forall j \in \mathtt{I} \setminus I, \vartheta \le \vartheta_j\}}$$

is $\mathcal{G}_{\vartheta-}$ measurable. $\qquad\square$

3 Full BSDEs

In addition to our standing $\mathbb{G}$ stopping time ϑ, let $g_t(\omega, x)$ be a $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$ measurable function and $G(\omega, x)$ be a $\mathcal{G}_\vartheta \otimes \mathcal{B}(\mathbb{R})$ measurable function. We consider the backward martingale problem consisting of the following integrability, equation and terminal conditions:

$$
\begin{cases}
\int_0^t |g_s(X_{s-})| ds < \infty \quad \text{for } t \in [0, \vartheta], \\[2mm]
X_t^\vartheta + \int_0^{t \wedge \vartheta} g_s(X_{s-}) ds \quad \text{defines a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale,} \\[2mm]
X_\vartheta = G(X_{\vartheta-}).
\end{cases}
\tag{3.1}
$$

The motivation for such problems is the study of backward stochastic differential applications (BSDEs) that arise in counterparty risk applications (see Sect. 1). Therefore we refer to such problems as BSDEs henceforth. Of course, (3.1) is a rather unusual BSDE, due in particular to the generality of the terminal condition $G(X_{\vartheta-})$, which depends on the solution and with, for fixed x, $G(\omega, x)$ only $\mathcal{G}_\vartheta$ measurable, as opposed to $\mathcal{G}_{\vartheta-}$ measurable in standard credit risk problems (where, in addition, the terminal condition would

not depend on x). But this generality is needed to fit all the specifications of counterparty risk problems. Our goal is to reduce the "full" BSDE (3.1) to simpler BSDEs relative to a reduced filtration $\mathbb{F}$, with constant terminal time T and exogenous terminal condition (a given random variable at time T). Moreover, we want to achieve this under minimal assumptions on ϑ, so that the model stays as flexible and can fit as many situations as possible.

Remark 3.1 At the abstract level of this paper, with unspecified structure of the $(\mathbb{G}, \mathbb{Q})$ local martingales, considering a more general (or more standard from a BSDE point of view) coefficient "$g_t(x, w)$", where w would correspond to some integrand in a martingale representation of the martingale part of X, would make no difference (except for the fact that we would need to define a solution as a pair process (X, W) in a suitable space).

Remark 3.2 BSDEs with random terminal time (given as first exit time of a domain by a diffusion) were first introduced in Darling and Pardoux (1997), in order to give a BSDE formulation to semilinear elliptic PDE. This and the present work are completely unrelated.

The main findings of this section can be summarized in the form of the following result, which is an immediate consequence of lemmas 3.2 and 3.3.

Theorem 3.1 *If the BSDE (3.1) has a solution X, then the process $Z = XJ$ is a solution to the following BSDE:*

$$\begin{cases} \int_0^t |g_s(Z_{s-})|ds + \int_0^t |\widehat{G}_s(Z_{s-})|d\mathbf{v}_s < \infty \quad \text{for } t \in [0, \vartheta], \\\\ Z_t^{\vartheta-} + \int_0^{t \wedge \vartheta} g_s(Z_{s-})ds + \int_0^{t \wedge \vartheta} (\widehat{G}_s(Z_{s-}) - Z_{s-})d\mathbf{v}_s \quad \text{defines a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale,} \end{cases}$$

or equivalently when the accessible component ϑ^a of ϑ is predictable:

$$\begin{cases} \int_0^t |g_s(Z_{s-})|ds + \int_0^t |\widehat{G}_s(Z_{s-})|d\mathbf{v}_s < \infty \quad \text{for } t \in [0, \vartheta], \\\\ Z_t^{\vartheta-} + \int_0^{t \wedge \vartheta} g_s(Z_{s-})ds + \int_0^{t \wedge \vartheta} (\widehat{G}_s(Z_{s-}) - Z_{s-})d\mathbf{v}_s^i \quad \text{defines a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale,} \\\\ \mathbb{1}_{\{\vartheta^a < \infty\}}(\widehat{G}_{\vartheta^a}(Z_{\vartheta^a-}) - Z_{\vartheta^a-}) = 0. \end{cases}$$

Conversely, if Z is a solution to the above BSDE such that the process $G(Z_{\vartheta-})H$ has locally integrable total variation, then the process

$$X = ZJ + G(Z_{\vartheta-})H$$

is a solution to the BSDE (3.1). □

3.1 BSDE Transformations

Assuming (3.1), we denote by

$$R_t = X_t + \int_0^{t \wedge \vartheta} g_s(X_{s-})ds, t \in \mathbb{R}_+ \tag{3.2}$$

the $(\mathbb{G}, \mathbb{Q})$ local martingale component of X, so that

$$\Delta_\vartheta R = \Delta_\vartheta X = G(X_{\vartheta-}) - X_{\vartheta-}. \tag{3.3}$$

Lemma 3.1 *Assuming (3.1), the processes $\Delta_\vartheta RH$ and $G(X_{\vartheta-})H$ have locally integrable total variation, the dual predictable projection of $\Delta_\vartheta RH$ is given by $(\widehat{G}.(X_-) - X_-) \bullet \mathbf{v}$ and we have*

$$\int_0^t |\widehat{G}_s(X_{s-})| d\mathbf{v}_s < \infty \quad \text{for } t \in [0, \vartheta]. \tag{3.4}$$

In particular, we can decompose R as the sum of the two following local martingales:

$$R^\bullet = \Delta_\vartheta RH - (\widehat{G}.(X_-) - X_-) \bullet \mathbf{v}, \quad R^\circ = R - R^\bullet.$$

Proof. Let (τ_n) denote an increasing sequence of stopping times such that each R^{τ_n} is a uniformly integrable martingale and each R^{τ_n-} is bounded. We have

$$\begin{aligned}
(\Delta_\vartheta RH)^{\tau_n} &= \mathbb{1}_{\vartheta \leq \tau_n} \Delta_\vartheta RH \\
&= \mathbb{1}_{\vartheta \leq \tau_n} \Delta_\vartheta (R^{\tau_n}) H \\
&= \mathbb{1}_{\vartheta \leq \tau_n} (R_{\vartheta_n} - R_{\vartheta_n-}) H,
\end{aligned}$$

which is integrable by construction of (τ_n). Therefore, the process $\Delta_\vartheta RH$ has locally integrable total variation and admits as such a dual predictable projection. For any $\mathbb{G}$ stopping time τ such that

$$(\Delta_\vartheta RH)^\tau = (G(X_{\vartheta-}) - X_{\vartheta-}) \mathbb{1}_{\{\vartheta \leq \tau\}} H$$

has integrable total variation, Lemma 2.3 implies that

$$(G(X_{\vartheta-}) - X_{\vartheta-}) \mathbb{1}_{\{\vartheta \leq \tau\}} H - (\widehat{G}.(X_-) - X_-) \mathbb{1}_{[0,\tau)} \bullet \mathbf{v}$$

is a $\mathbb{G}$ uniformly integrable martingale. This shows that $(\widehat{G}.(X_-) - X_-) \bullet \mathbf{v}$ is the dual predictable projection of $\Delta_\vartheta RH$. As a consequence, for any $\mathbb{G}$ predictable bounded process L,

$$\mathbb{E}[L_\vartheta \Delta_\vartheta R \mathbb{1}_{\{\vartheta < \infty\}}] = \mathbb{E}\Big[\int_{[0,\infty)} L_s (\widehat{G}_s(X_{s-}) - X_{s-}) d\mathbf{v}_s\Big].$$

Since $\int_0^{\cdot} |X_{s-}| d\mathbf{v}_s$ is finite (by left-continuity of X_-), (3.4) follows by taking $L = \text{sgn}(\widehat{G}.(X_-) - X_-)$. $\square$

We consider the following BSDE:

$$\begin{cases}
\int_0^t |g_s(Y_{s-})| ds + \int_0^t |\widehat{G}_s(Y_{s-})| d\mathbf{v}_s < \infty \quad \text{for } t \in [0, \vartheta], \\[2mm]
Y_t^\vartheta + \int_0^{t \wedge \vartheta} g_s(Y_{s-}) ds + \int_0^{t \wedge \vartheta} \widehat{G}_s(Y_{s-}) d\mathbf{v}_s \quad \text{defines a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale.} \\[2mm]
Y_\vartheta = 0.
\end{cases} \tag{3.5}$$

Lemma 3.2 *If X is a solution to the BSDE (3.1), then $Y = XJ$ is a solution to the BSDE (3.5). Conversely, if Y is a solution to the BSDE (3.5) such that the process $G(Y_{\vartheta-})H$ has locally integrable total variation, then the process*

$$X = YJ + G(Y_{\vartheta-})H.$$

is a solution to the BSDE (3.1).

Proof. Assuming (3.1), the Itô formula yields (recall $X^{\vartheta-} = XJ + X_{\vartheta-}H$)

$$
\begin{aligned}
dX_t^{\vartheta-} &= d(XJ)_t + d(X_{\vartheta-}H)_t \\
&= J_{t-}dX_t - X_t\boldsymbol{\delta}_\vartheta(dt) + X_{\vartheta-}\boldsymbol{\delta}_\vartheta(dt) \\
&= J_{t-}dR_t - J_{t-}g_t(X_{t-})dt - X_\vartheta\boldsymbol{\delta}_\vartheta(dt) + X_{\vartheta-}\boldsymbol{\delta}_\vartheta(dt) \\
&= J_{t-}dR_t^\circ + J_{t-}dR_t^\bullet - J_{t-}g_t(X_{t-})dt - \Delta_\vartheta X\boldsymbol{\delta}_\vartheta(dt) \\
&= J_{t-}dR_t^\circ - J_{t-}(\widehat{G}_t(X_{t-}) - X_{t-})d\mathbf{v}_t - J_{t-}g_t(X_{t-})dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
dY_t &= d(XJ)_t \\
&= J_{t-}dR_t^\circ - d(X_{\vartheta-}H)_t - J_{t-}(\widehat{G}_t(X_{t-}) - X_{t-})d\mathbf{v}_t - J_{t-}g_t(X_{t-})dt \\
&= J_{t-}dR_t^\circ - (d(X_{\vartheta-}H)_t - J_{t-}X_{t-}d\mathbf{v}_t) - J_{t-}\widehat{G}_t(X_{t-})d\mathbf{v}_t - J_{t-}g_t(X_{t-})dt \\
&= J_{t-}dR_t^\circ - J_{t-}X_{t-}d(H - \mathbf{v})_t - J_{t-}\widehat{G}_t(X_{t-})d\mathbf{v}_t - J_{t-}g_t(X_{t-})dt.
\end{aligned}
$$

The process $(H - \mathbf{v})$ being a local martingale, this computation and (3.4) show that the process $Y = XJ$ solves (3.5).

Conversely, given Y solving (3.5) such that $G(Y_{\vartheta-})H$ has locally integrable total variation, let

$$X = YJ + G(Y_{\vartheta-})H.$$

Hence, by Lemma 2.3,

$$\mathsf{R} = G(Y_{\vartheta-})H - J_-\widehat{G}.(Y_-)\boldsymbol{\cdot}\mathbf{v} \tag{3.6}$$

is a $(\mathbb{G}, \mathbb{Q})$ local martingale. Set

$$R' = Y^\vartheta + J_-g.(X_-)\boldsymbol{\cdot}\boldsymbol{\lambda} + J_-\widehat{G}.(Y_-)\boldsymbol{\cdot}\mathbf{v}, \tag{3.7}$$

another $(\mathbb{G}, \mathbb{Q})$ local martingale. The Itô formula gives

$$
\begin{aligned}
dX_t & \\
&= d(YJ)_t + d(G(Y_{\vartheta-})H)_t \\
&= J_{t-}dY_t - Y_t\boldsymbol{\delta}_\vartheta(dt) + d\mathsf{R}_t + J_{t-}\widehat{G}_t(Y_{t-})d\mathbf{v}_t \\
&= J_{t-}dR_t' - J_{t-}g_t(Y_{t-})dt - J_{t-}\widehat{G}_t(Y_{t-})d\mathbf{v}_t - Y_\vartheta\boldsymbol{\delta}_\vartheta(dt) \\
&\qquad + d\mathsf{R}_t + J_{t-}\widehat{G}_t(Y_{t-})d\mathbf{v}_t \\
&= J_{t-}dR_t' - J_{t-}g_t(Y_{t-})dt + d\mathsf{R}_t,
\end{aligned}
$$

thus

$$dX_t + J_{t-}g_t(X_{t-})dt = J_{t-}dR_t' + d\mathsf{R}_t, \tag{3.8}$$

which shows that X is a solution to the BSDE (3.1). $\qquad\square$

For the application of the previous lemma in the reduced form models of Sect. 4.1, we need another form of the BSDE (3.5).

Lemma 3.3 *Given a $(\mathbb{G}, \mathbb{Q})$ semimartingale Z, the process $Y = ZJ$ solves the BSDE (3.5) if and only if*

$$
\begin{cases}
\int_0^t |g_s(Z_{s-})|ds + \int_0^t |\widehat{G}_s(Z_{s-})|d\mathbf{v}_s < \infty & \text{for } t \in [0, \vartheta], \\[2mm]
Z_t^{\vartheta-} + \int_0^{t\wedge\vartheta} g_s(Z_{s-})ds + \int_0^{t\wedge\vartheta} \widehat{G}_s(Z_{s-})d\mathbf{v}_s - \int_0^{t\wedge\vartheta} Z_{s-}d\mathbf{v}_s & \text{defines a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale.}
\end{cases}
\tag{3.9}
$$

When the accessible component ϑ^a of ϑ is predictable, the BSDE (3.9) for Z becomes

$$
\begin{cases}
\int_0^t |g_s(Z_{s-})|ds + \int_0^t |\widehat{G}_s(Z_{s-})|d\mathbf{v}_s < \infty \quad \text{for } t \in [0, \vartheta], \\[2mm]
Z_t^{\vartheta-} + \int_0^{t\wedge\vartheta} g_s(Z_{s-})ds + \int_0^{t\wedge\vartheta}(\widehat{G}_s(Z_{s-}) - Z_{s-})d\mathbf{v}_s^i \quad \text{defines a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale,} \\[2mm]
\mathbb{1}_{\{\vartheta^a<\infty\}}(\widehat{G}_{\vartheta^a}(Z_{\vartheta^a-}) - Z_{\vartheta^a-}) = 0.
\end{cases}
$$
(3.10)

Remark 3.3 The BSDE (3.9) has a solution if and only if the following BSDE has a solution (with $\bar{Z} = Z^{\vartheta-}$ as a particular solution below if Z solves (3.9))

$$
\begin{cases}
\int_0^t |g_s(\bar{Z}_{s-})|ds + \int_0^t |\widehat{G}_s(\bar{Z}_{s-})|d\mathbf{v}_s < \infty \quad \text{for } t \in [0, \vartheta], \\[2mm]
\bar{Z}_t + \int_0^{t\wedge\vartheta} g_s(\bar{Z}_{s-})ds + \int_0^{t\wedge\vartheta} \widehat{G}_s(\bar{Z}_{s-})d\mathbf{v}_s - \int_0^{t\wedge\vartheta} \bar{Z}_{s-}d\mathbf{v}_s \quad \text{defines a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale,} \\[2mm]
\Delta_\vartheta \bar{Z} = 0.
\end{cases}
$$
(3.11)

In particular, the integrability conditions (on $[0, \vartheta]$) for Z to solve (3.9) and for $\bar{Z} = Z^{\vartheta-}$ to solve (3.11) are clearly equivalent.

Proof. Let R_t'' denote the semimartingale

$$
R_t'' = Z_t^{\vartheta-} + \int_0^{t\wedge\vartheta} g_s(Z_{s-})ds + \int_0^{t\wedge\vartheta} \widehat{G}_s(Z_{s-})d\mathbf{v}_s - \int_0^{t\wedge\vartheta} Z_{s-}d\mathbf{v}_s.
$$
(3.12)

The Itô formula applied to $Y = ZJ$ yields

$$
\begin{aligned}
dY_t &= d(ZJ)_t = d(Z^{\vartheta-}J)_t \\
&= J_{t-}dZ_t^{\vartheta-} - Z_t^{\vartheta-}dH_t \\
&= J_{t-}dR_t'' - J_{t-}g_t(Z_{t-})dt - J_{t-}\widehat{G}_t(Z_{t-})d\mathbf{v}_t + J_{t-}Z_{t-}d\mathbf{v}_t - Z_{\vartheta-}dH_t \\
&= J_{t-}dR_t'' - J_{t-}g_t(Z_{t-})dt - J_{t-}\widehat{G}_t(Z_{t-})d\mathbf{v}_t - J_{t-}Z_{t-}d(H - \mathbf{v})_t,
\end{aligned}
$$

which proves the first part of the lemma. If ϑ^a is predictable, then $\mathbf{v}^a = \mathbb{1}_{[\vartheta^a,\infty)}$. Hence,

$$
R_t'' = Z_t^{\vartheta-} + \int_0^{t\wedge\vartheta} g_s(Z_{s-})ds + \int_0^{t\wedge\vartheta}(\widehat{G}_s(Z_{s-}) - Z_{s-})d\mathbf{v}_s^i + (\widehat{G}_{\vartheta^a}(Z_{\vartheta^a-}) - Z_{\vartheta^a-})\mathbb{1}_{[\vartheta^a,\infty)}.
$$

As a consequence,

$$
\Delta_{\vartheta^a} R'' \mathbb{1}_{\{\vartheta^a<\infty\}} = \mathbb{1}_{\{\vartheta^a<\infty\}}(\widehat{G}_{\vartheta^a}(Z_{\vartheta^a-}) - Z_{\vartheta^a-}).
$$

Therefore (3.10) obviously implies (3.9). Conversely, assuming (3.9), so that R_t'' is a local martingale, taking the (generalized) conditional expectation given $\mathcal{G}_{\vartheta^a-}$ both sides of the above identity, we obtain since ϑ^a is predictable:

$$
\mathbb{1}_{\{\vartheta^a<\infty\}}(\widehat{G}_{\vartheta^a}(Z_{\vartheta^a-}) - Z_{\vartheta^a-}) = 0,
$$

hence (3.10) follows. □

Remark 3.4 By combining (3.7) and (3.12), if Z solves (3.9), so that $Y = ZJ$ solves (3.5) and $X = YJ + G(Y_{\vartheta-})H$ solves (3.1) (assuming $G(Y_{\vartheta-})H$ has locally integrable total variation), then

$$R'_t = R''_t - Z_{\vartheta-}H_t + \int_0^{t\wedge\vartheta} Z_{s-}d\mathbf{v}_s, \ \ t \geq 0. \tag{3.13}$$

Therefore, in view of (3.2), (3.8) and (3.6), the local martingale component R of X satisfies

$$
\begin{aligned}
dR_t &= J_{t-}dR'_t + d\mathsf{R}_t \\
&= J_{t-}dR''_t - J_{t-}Z_{t-}(dH - d\mathbf{v}_t) + G(Z_{\vartheta-})dH_t - J_{t-}\widehat{G}_t(Z_t)d\mathbf{v}_t,
\end{aligned}
$$

which, in case ϑ^a is predictable, reduces, in view of the terminal condition in (3.10), to:

$$dR_t = J_{t-}\left(dR''_t - Z_{t-}(dH - d\mathbf{v}^i_t) + G(Z_{\vartheta-})dH_t - \widehat{G}_t(Z_t)d\mathbf{v}^i_t\right). \tag{3.14}$$

4 Reduction of Filtration

In Sect. 4.1 the auxiliary BSDE (3.10) (in the case of Sect. 4.1 where ϑ^a is predictable) will be transformed into a reduced BSDE relative to a smaller filtration $\mathbb{F} \subseteq \mathbb{G}$, under the following assumption on $\mathbb{F}$.

Condition (B). For any $\mathbb{G}$ predictable process L, there exists an $\mathbb{F}$ predictable process K, which we call the $\mathbb{F}$ predictable representative of L, such that K coincides with L until ϑ, i.e. $J_-K = J_-L$.

Note that J_- could equivalently be replaced by J in this condition, since the predictable σ-field is generated by left-continuous processes. Also, if L is bounded, K may and will be chosen bounded. A similar statement applies and will not be repeated regarding the various notions of $\mathbb{F}$ representative introduced later.

The following is an immediate consequence of the condition (B):

$$\{\vartheta < \infty\} \cap \mathcal{G}_{\vartheta-} = \{\vartheta < \infty\} \cap \mathcal{F}_{\vartheta-}.$$

But we can say more. The next result establishes the connection between the condition (B) and a classical condition in the theory of enlargement of filtrations, stated in terms of the auxiliary filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\in\mathbb{R}_+}$ (Jeulin (1980) writes $\overline{\mathbb{G}}$, whereas Dellacherie, Maisonneuve, and Meyer (1992) use $\mathfrak{F}^L$), where

$$\overline{\mathcal{F}}_t = \{B_{(t)} \in \mathcal{B} : \exists A_{(t)} \in \mathcal{F}_t, A_{(t)} \cap \{t < \vartheta\} = B_{(t)} \cap \{t < \vartheta\}\}. \tag{4.1}$$

Lemma 4.1 *The filtration $\mathbb{F}$ satisfies the condition (B) if and only if $\mathbb{G}$ is a subfiltration of $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\in\mathbb{R}_+}$.*

Proof. As seen in n°75 Chapitre XX in Dellacherie, Maisonneuve, and Meyer (1992), $\overline{\mathbb{F}}$ is a filtration satisfying the usual conditions. Suppose the condition (B). For any $t \in \mathbb{R}_+$, for any $B_{(t)} \in \mathcal{G}_t$, $\mathbb{1}_{B_{(t)}}\mathbb{1}_{(t,\infty)}$ is a $\mathbb{G}$ predictable process, with $\mathbb{F}$ predictable representative L such that $J_-\mathbb{1}_{B_{(t)}}\mathbb{1}_{(t,\infty)} = J_-L\mathbb{1}_{(t,\infty)}$. In particular, $\mathbb{1}_{B_{(t)}}\mathbb{1}_{\{t<s\leq\vartheta\}} = L_s\mathbb{1}_{\{t<s\leq\vartheta\}}$, or

$\liminf_{s\downarrow t}\mathbb{1}_{B_{(t)}}\mathbb{1}_{\{t<s\leq\vartheta\}} = \liminf_{s\downarrow t} L_s\mathbb{1}_{\{t<s\leq\vartheta\}}$. But $\liminf_{s\downarrow t}\mathbb{1}_{B_{(t)}}\mathbb{1}_{\{t<s\leq\vartheta\}} = \mathbb{1}_{B_{(t)}}\mathbb{1}_{\{t<\vartheta\}}$ while $\liminf_{s\downarrow t} L_t\mathbb{1}_{\{t<s\leq\vartheta\}} = (\liminf_{s\downarrow t} L_s)\mathbb{1}_{\{t<\vartheta\}}$, which proves $B_{(t)}\in\overline{\mathcal{F}}_{(t)}$.

Conversely (note that this converse is already known as Lemma 1 in Jeulin and Yor (1978)), suppose that $\mathbb{G}$ is a subfiltration of $\overline{\mathbb{F}}$. For any $t\in\mathbb{R}_+$, for any $B_{(t)}\in\mathcal{G}_{(t)}$, let $A_{(t)}\in\mathcal{F}_{(t)}$ satisfy $B_{(t)}\cap\{t<\vartheta\} = A_{(t)}\cap\{t<\vartheta\}$, so that

$$J_-\mathbb{1}_{A_{(t)}}\mathbb{1}_{(t,\infty)} = J_-\mathbb{1}_{B_{(t)}}\mathbb{1}_{(t,\infty)}.$$

Note that $\mathbb{1}_{A_{(t)}}\mathbb{1}_{(t,\infty)}$ is an $\mathbb{F}$ predictable process. Similarly, for $B_{(0)}\in\mathcal{G}_{(0)}$, there exists $A_{(0)}\in\mathcal{F}_{(0)}$ such that $B_{(0)}\cap\{0<\vartheta\} = A_{(0)}\cap\{0<\vartheta\}$ and therefore $J_-\mathbb{1}_{B_{(0)}}\mathbb{1}_{\{0\}} = J_-\mathbb{1}_{A_{(0)}}\mathbb{1}_{\{0\}}$. Again we note that $\mathbb{1}_{A_{(0)}}\mathbb{1}_{\{0\}}$ is an $\mathbb{F}$ predictable process. Since the processes $\mathbb{1}_{B_{(t)}}\mathbb{1}_{(t,\infty)}$ and $\mathbb{1}_{B_{(0)}}\mathbb{1}_{\{0\}}$ generate the $\mathbb{G}$ predictable σ-algebra (cf. Theorem 3.21 in He, Wang, and Yan (1992)), this proves the condition (B). □

The condition (B) is henceforth postulated. Let ^{o} denote the $\mathbb{F}$ optional projection. In particular, let $\mathsf{S} = {}^oJ$ denote the Azéma supermartingale of ϑ with respect to $\mathbb{F}$, with $\mathbb{F}$ canonical nondecreasing predictable component A. The part 2 in the lemma below is known as the "key lemma" (see Bielecki, Jeanblanc, and Rutkowski (2009)). The part 3 is stated as Remarque (4,5)3) in Jeulin (1980). The semimartingale decomposition formula in the part 5, in which the jump of Q at ϑ is removed by stopping Q at $\vartheta-$, is a celebrated result, sometimes called the Jeulin-Yor theorem, stated as no 77 Remarques b) in Dellacherie, Maisonneuve, and Meyer (1992).

Lemma 4.2   *1. For any $\mathbb{G}$ stopping time τ, there exists an $\mathbb{F}$ stopping time σ, which we call the $\mathbb{F}$ representative of τ, such that $\{\tau<\vartheta\} = \{\sigma<\vartheta\}\subseteq\{\tau=\sigma\}$.*

2. For any positive t and nonnegative $\mathcal{G}_\infty$ measurable random variable χ,

$$J_t\mathbb{E}[\chi|\mathcal{G}_t] = J_t\frac{1}{\mathsf{S}_t}\mathbb{E}[J_t\chi|\mathcal{F}_t].$$

3. The $\mathbb{G}$ compensator of H is given by $J_-\frac{1}{\mathsf{S}_-}\centerdot\mathsf{A}$. The $\mathbb{F}$ dual predictable projection of H is given by A.

4. For any $\mathbb{G}$ optional process Y, there exists an $\mathbb{F}$ optional process X, which we call the $\mathbb{F}$ optional representative of Y, such that X coincides with Y before ϑ, i.e. $JX = JY$.

5. For any bounded $\mathbb{F}$ martingale Q, the process

$$Q^{\vartheta-} - J_-\frac{1}{\mathsf{S}_-}\centerdot\langle Q,\mathsf{S}\rangle \qquad (4.2)$$

is a $\mathbb{G}$ uniformly integrable martingale. In particular, any $\mathbb{F}$ semimartingale stopped at ϑ is $\mathbb{G}$ semimartingale.

Proof. We notice that, even if the filtration $\overline{\mathbb{F}}$ was introduced in Jeulin and Yor (1978) or Chapitre XX in Dellacherie, Maisonneuve, and Meyer (1992) for a classical progressive enlargement setting, their study on the filtration $\overline{\mathbb{F}}$ depends only on the relation $\mathbb{G}\subseteq\overline{\mathbb{F}}$. Hence, in view of Lemma 4.1, we can repeat the arguments in Jeulin and Yor (1978) or Chapitre XX in Dellacherie, Maisonneuve, and Meyer (1992) to get all the results. However, regarding the part 4, they state it without demonstration. Since this result is key in the

sequel, we give a proof as follows. Let χ be a bounded $\mathcal{G}_\infty$ measurable random variable. Let Y be any (càdlàg) $\mathbb{G}$ martingale with terminal variable χ. By the key lemma, for every $t \in \mathbb{R}_+$, we can write the almost sure identity

$$J_t Y_t = J_t \mathbb{E}[\chi | \mathcal{G}_t] = J_t \frac{1}{\mathsf{S}_t} \mathbb{Q}[J_t \chi | \mathcal{F}_t] = J_t X_t,$$

where

$$X_t = \mathbb{1}_{\{\mathsf{s}_t > 0\}} \frac{{}^{o}(J\chi)_t}{\mathsf{S}_t}.$$

By Lemma 3.1 in Jeanblanc and Song (2010) and Lemma 3.1 in Jeanblanc and Song (2012), the process $J\mathbb{1}_{\{\mathsf{s}>0\}}\frac{1}{\mathsf{s}}$ is indistinguishably a càdlàg process. So is also the process ${}^{o}(J\chi)$, by Problem 4.13 in He, Wang, and Yan (1992). Therefore, $JY = JX$ holds in the indistinguishable sense. This shows that the $\mathbb{G}$ martingale Y has an $\mathbb{F}$ optional representative, which is also true for any deterministic process (obviously). Now, let $\mathfrak{C}$ denote the family of all the bounded $\mathcal{B}[0, \infty) \otimes \mathcal{G}_\infty$ measurable functions Y such that there exists a version of ${}^{\bar{o}}Y$ admitting an $\mathbb{F}$ optional representative, where ${}^{\bar{o}}$ denotes the $\mathbb{G}$ optional projection. We can verify that $\mathfrak{C}$ is a functional monotone class in the sense of Theorem 1.4 in He, Wang, and Yan (1992). It results from above that $\mathfrak{C}$ contains all the random variables $\mathbb{1}_{(a,b]}\chi$, where $a, b \in \mathbb{R}_+$ and χ is a bounded $\mathcal{G}_\infty$ measurable random variable. Therefore, by the monotone class theorem, $\mathfrak{C}$ contains all the bounded $\mathcal{B}[0, \infty) \otimes \mathcal{G}_\infty$-measurable random variables, so that every bounded $\mathbb{G}$ optional process Y admits a (bounded) $\mathbb{F}$ optional representative. By taking suprema over sequences and then differences between positive and negative parts, the result is extended to nonnegative and in turn general $\mathbb{G}$ optional processes. $\qquad\square$

Lemma 4.3 *Any $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$ (respectively $\mathcal{O}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$) measurable function $h_t(\omega, x)$ admits a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ (respectively $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$) representative, i.e. a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ (respectively $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ measurable) function $f_t(\omega, x)$ such that $\mathbb{1}_{[0, \vartheta(\omega)]} f_t(\omega, x) = \mathbb{1}_{[0, \vartheta(\omega)]} h_t(\omega, x)$ everywhere.*

Proof. Since $\mathbb{F}$ satisfies the usual conditions, any evanescent measurable process is $\mathbb{F}$ predictable (Theorem 4.26 in He, Wang, and Yan (1992)). Therefore, all indistinguishable identities between processes can be established everywhere for suitably modified versions. In particular, in the condition (B), one can achieve $J_- K = J_- L$ everywhere (not only outside an evanescent set) by taking another version of K. Likewise, in Lemma 4.2.4, one can achieve $JX = JY$ everywhere (not only outside an evanescent set) by taking another version of X. Therefore, the statement in the lemma holds for any function h given as the product of a $\mathbb{G}$ predictable (respectively optional) process by a Borel function. The proof is completed by a standard monotone class argument. $\qquad\square$

The Jeulin-Yor theorem stated as Lemma 4.2.5 above establishes the relation between $\mathbb{F}$ martingales and $\mathbb{G}$ semimartingales. The next result addresses the partial "inverse problem" of knowing when an $\mathbb{F}$ semimartingale P is such that $= P^{\vartheta-}$ is a $\mathbb{G}$ local martingale. Note that p below yields an $\mathbb{F}$ predictable process by n°90 Chapitre IV in Dellacherie and Meyer (1975), whereas the left-limit process of P does not need to exist on the right of ϑ, for lack of regularity of P that is only the $\mathbb{F}$ optional representative of R.

Lemma 4.4 *Let R be a $\mathbb{G}$ local martingale without jump at ϑ, i.e. $\Delta_\vartheta R = 0$, with $\mathbb{F}$ optional representative of R denoted by P. For $t > 0$, we write $p_t = \liminf_{s \uparrow\uparrow t} P_s$. There exists an increasing sequence $(\sigma_n)_{n \in \mathbb{N}}$ of $\mathbb{F}$ stopping times such that $\mathsf{S}_{\sigma_\infty} = 0$ and $({}^{o}R)^{\sigma_n}, (P\mathsf{S} + p.\mathtt{A})^{\sigma_n}.$*

are $\mathbb{F}$ martingales, for every n. Conversely, if P is an $\mathbb{F}$ semimartingale such that $\mathtt{S}_- \centerdot P + [P,\mathtt{S}]$ is an $\mathbb{F}$ local martingale, then the process $R = P^{\vartheta-}$ is a $\mathbb{G}$ local martingale.

Proof.

$\Rightarrow$ Let $(\tau_n)_{n\geq 0}$ be a sequence of bounded $\mathbb{G}$ stopping times reducing R to a martingale in $\mathtt{H}^1$ (the space of local martingales with $\frac{1}{2}$-times integrable bracket) and R_- to a bounded process, with $\mathbb{F}$ representatives of the τ_n denoted by σ_n, which can be assumed to form an nondecreasing sequence with a limit σ_∞. Omitting the subscript n, we have

$$^o(R^{\vartheta\wedge\tau}) = {}^o(R^{\vartheta\wedge\sigma}) = {}^o((R^\sigma)^\vartheta) = {}^o(R^\sigma J + R_\vartheta^\sigma H) = {}^o(R^\sigma J + R_{\vartheta-}^\sigma H)$$

because $\Delta_\vartheta R = 0$. For $t > 0$, we write $\rho_t = \liminf_{s\uparrow\uparrow t} P_s^\sigma$, and $\mathtt{R} = \rho_\vartheta H - \rho J_- \frac{1}{\mathtt{S}_-} \centerdot \mathtt{A}$, a $\mathbb{G}$ martingale. We compute

$$
\begin{aligned}
^o(R^{\vartheta\wedge\tau}) &= {}^o(P^\sigma J + \rho_\vartheta H) \\
&= P^\sigma \mathtt{S} + {}^o(\mathtt{R}) + ({}^o(\rho J_- \tfrac{1}{\mathtt{S}_-} \centerdot \mathtt{A}) - \rho \centerdot \mathtt{A}) + \rho \centerdot \mathtt{A}, \\
&= P^\sigma \mathtt{S}^\sigma + {}^o(\mathtt{R}) + ({}^o(\rho J_- \tfrac{1}{\mathtt{S}_-} \centerdot \mathtt{A}) - \rho \centerdot \mathtt{A}) + \mathbb{1}_{(0,\sigma]} \rho \centerdot \mathtt{A} \\
&\quad + P^\sigma(\mathtt{S} - \mathtt{S}^\sigma) + \mathbb{1}_{(\sigma,\infty)} \rho \centerdot \mathtt{A} \\
&= {}^o(\mathtt{R}^\sigma) + ({}^o(\rho J_- \tfrac{1}{\mathtt{S}_-} \centerdot \mathtt{A}) - \rho \centerdot \mathtt{A}) + (P\mathtt{S} + p \centerdot \mathtt{A})^\sigma \\
&\quad + P_\sigma(\mathtt{S} - \mathtt{S}^\sigma + \mathtt{A} - \mathtt{A}^\sigma),
\end{aligned}
$$

where the last equality holds because $\rho = p$ on $(0,\sigma]$ and $\rho = P_\sigma$ on (σ,∞). Since the processes $^o(R^{\vartheta\wedge\tau})$, $^o\mathtt{R}$, $({}^o(\rho J_- \frac{1}{\mathtt{S}_-} \centerdot \mathtt{A}) - \rho \centerdot \mathtt{A})$, and $P_\sigma(\mathtt{S} - \mathtt{S}^\sigma + \mathtt{A} - \mathtt{A}^\sigma)$ are $\mathbb{F}$ martingales, so is the process $(P\mathtt{S} + p \centerdot \mathtt{A})^\sigma$. Note that

$$\mathbb{E}[\mathtt{S}_{\sigma_n}] = \mathbb{Q}[\sigma_n < \vartheta] = \mathbb{Q}[\tau_n < \vartheta] \to 0,$$

which, since $\mathtt{S}$ is a nonnegative $\mathbb{F}$ supermartingale, implies $\mathtt{S}_{\sigma_\infty} = 0$. This completes the proof of the first assertion of the lemma.

$\Leftarrow$ Let R be defined as in the second assertion of the lemma. For any $\mathbb{F}$ stopping time σ and $t \in \mathbb{R}_+$, we have $R_t^\sigma = (P^\sigma)_t^{\vartheta-}$. Hence, we may and do restrict attention to the case where $\mathtt{S}_- \centerdot P + [P,\mathtt{S}]$ is an $\mathbb{F}$ martingale in $\mathtt{H}^1$ and P_- is a bounded process. Using both statements in Lemma 4.2.3,

$$\mathbb{E}[P_{\vartheta-}H_\sigma - P_- \centerdot \mathtt{A}_\sigma] = \mathbb{E}[P_{\vartheta-}H_\sigma - P_- J_- \frac{1}{\mathtt{S}_-} \centerdot \mathtt{A}_\sigma + P_- J_- \frac{1}{\mathtt{S}_-} \centerdot \mathtt{A}_\sigma - P_- \centerdot \mathtt{A}_\sigma] = 0.$$

As a consequence, for a bounded $\mathbb{G}$ stopping time τ with $\mathbb{F}$ reduction σ,

$$
\begin{aligned}
\mathbb{E}[R_\tau] &= \mathbb{E}[P_\tau J_\tau + P_{\vartheta-}H_\tau] \\
&= \mathbb{E}[P_\sigma J_\sigma + P_{\vartheta-}H_\sigma] \\
&= \mathbb{E}[(P\mathtt{S} + P_- \centerdot \mathtt{A})_\sigma] \\
&= \mathbb{E}[(\mathtt{S}_- \centerdot P + [P,\mathtt{S}])_\sigma] \\
&= \mathbb{E}[(\mathtt{S}_- \centerdot P + [P,\mathtt{S}])_0] = \mathbb{E}[P_0 \mathtt{S}_0],
\end{aligned}
$$

a constant, so that by Theorem 4.40 in He, Wang, and Yan (1992), R is a $\mathbb{G}$ uniformly integrable martingale. $\square$

Remark 4.1 In view of this result and of the Jeulin-Yor Theorem stated as Lemma 4.2.5 above (see the comments preceding the lemma), the transform $\cdot^{\vartheta-}$ seems more natural than $\cdot^\vartheta$ (though the latter is more commonly used in the enlargement of filtration literature).

This explains the relevance of Lemma 3.3 for our BSDE problem. In particular, working with $\cdot^{\vartheta-}$ allows us to consistently work with the dual predictable projection $\mathtt{A}$ of H, which is more practical than the dual optional projection also used in Jeulin (1980) (and not only also, but even exclusively in the pseudo-stopping times theory of Nikeghbali and Yor (2005)).

4.1 Reduced BSDEs

We are now in the position to derive reduced forms of the BSDE (3.1) relative to the filtration $\mathbb{F}$. Let θ be a $\mathbb{G}$ stopping time with $\mathbb{F}$ dual predictable projection of the form $\alpha \centerdot \boldsymbol{\lambda}$, for some $\mathbb{F}$ predictable process α, which implies that θ is a $\mathbb{G}$ totally inaccessible stopping time with $\mathbb{G}$ compensator $\mathbb{1}_{(0,\theta]}\gamma \centerdot \boldsymbol{\lambda}$, where $\gamma = \mathbb{1}_{\mathtt{S}_->0}\frac{\alpha}{\mathtt{S}_-}$. We assume the condition (B) for θ, so that it also holds for $\vartheta = \theta \wedge T \leq \theta$, where $T > 0$ is any fixed real number. In the notation already used in (2.1), $\vartheta^i = \theta_{\{\theta<T\}}, \vartheta^a = T_{\{T\leq\theta\}}$ (which is predictable) and

$$\mathbf{v}^i = \mathbb{1}_{(0,\theta]}\mathbb{1}_{(0,T)}\gamma \centerdot \boldsymbol{\lambda}, \ \mathbf{v}^a = \mathbb{1}_{\{T\leq\theta\}}\mathbb{1}_{[T,\infty)}.$$

In the sequel we suppose a terminal function G of the form $\mathbb{1}_{\{\theta<T\}}F(x)$ in (3.1), so that $\widehat{G}_t(x) = \mathbb{1}_{t<T}\widehat{F}_t(x)$. Since $\vartheta^a = T_{\{T\leq\theta\}}$, this yields the terminal condition

$$\mathbb{1}_{\{T\leq\theta\}}Z_{T-} = 0 \tag{4.3}$$

in (3.10), which is equivalent to (3.1) (as ϑ^a is predictable). Now, if the BSDE (3.10) has a solution Z, then

$$R_t'' = Z_t^{\theta\wedge T-} + \int_0^{t\wedge\theta\wedge T} g_s(Z_{s-})ds + \int_0^{t\wedge\theta\wedge T} (\widehat{G}_s(Z_{s-}) - Z_{s-})d\mathbf{v}_s^i$$

is a $(\mathbb{G}, \mathbb{Q})$ local martingale. Let f and $\overline{F}$ be $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ representatives of g and $\widehat{G}$. Denoting by U an $\mathbb{F}$ optional representative of Z, for $t \in \mathbb{R}_+$,

$$R_t''\mathbb{1}_{\{t<\theta\wedge T\}} = \left(U_t + \int_0^t f_s(U_{s-})ds + \int_0^t (\overline{F}_s(U_{s-}) - U_{s-})\mathbb{1}_{[0,T]}\gamma_s ds\right)\mathbb{1}_{\{t<\theta\wedge T\}} .$$

In view of Lemma 4.4, this suggests to consider the following $\mathbb{F}$ BSDE for an $\mathbb{F}$ semimartingale U:

$$\begin{cases} \int_0^{t\wedge T} |f_s(U_{s-})| + |\overline{F}_s(U_{s-})\gamma_s|ds < \infty, \ t \in \mathbb{R}_+, \\[2mm] \mathtt{S}_- \centerdot P + [P, \mathtt{S}] \text{ is an } \mathbb{F} \text{ local martingale}, \\[2mm] U_{T-} = 0, \end{cases} \tag{4.4}$$

where, in the second line,

$$P_t = U_t^T + \int_0^{t\wedge T} \left(f_s(U_{s-}) + (\overline{F}_s(U_{s-}) - U_{s-})\gamma_s\right) ds. \tag{4.5}$$

Theorem 4.1 *If an $\mathbb{F}$ semimartingale U solves the $\mathbb{F}$ BSDE (4.4), then the process $Z = U^{\theta\wedge T-}$ is a solution, stopped at $\theta \wedge T = \vartheta$, to the BSDE (3.10). As a consequence, if the process $\mathbb{1}_{[\theta\wedge T,\infty)}\mathbb{1}_{\{\theta<T\}}F(U_{\theta\wedge T-})$ is locally integrable, the process*

$$X = \mathbb{1}_{[0,\theta\wedge T)}U + \mathbb{1}_{[\theta\wedge T,\infty)}\mathbb{1}_{\{\theta<T\}}F(U_{\theta\wedge T-})$$

solves the BSDE (3.1) with the data $\vartheta = \theta \wedge T, g = f$ and $G = \mathbb{1}_{\theta<T}F$.

Proof. Clearly, the process Z satisfies the integrability condition in (3.10). According to Lemma 4.4, $P^{\theta \wedge T-}$ is a $\mathbb{G}$ local martingale, such that (with $H = \mathbb{1}_{[\vartheta,\infty)} = \mathbb{1}_{[\theta \wedge T,\infty)}$)

$$
\begin{aligned}
P_t^{\theta \wedge T-} &= P_t^{\theta \wedge T} - (\Delta_{\theta \wedge T} P) H_t \\
&= \left(U_t^{\theta \wedge T} + \int_0^{t \wedge \theta \wedge T} \left(f_s(U_{s-}) + (\overline{F}_s(U_{s-}) - U_{s-}) \gamma_s \right) ds \right) - (\Delta_{\theta \wedge T} U) H_t \\
&= U_t^{\theta \wedge T-} + \int_0^{t \wedge \theta \wedge T} \left(g_s(U_{s-}) + (\widehat{G}_s(U_{s-}) - U_{s-}) \gamma_s \right) ds \\
&= Z_t^{\theta \wedge T-} + \int_0^{t \wedge \theta \wedge T} \left(g_s(Z_{s-}) + (\widehat{G}_s(Z_{s-}) - Z_{s-}) \gamma_s \right) ds,
\end{aligned}
$$

so that the martingale condition in (3.10) is satisfied. Finally,

$$
\mathbb{1}_{\{T \le \theta\}} Z_{T-} = \mathbb{1}_{\{T \le \theta\}} U_{T-} = 0,
$$

which is the terminal condition (4.3) in (3.10). □

In the $\mathbb{F}$ BSDE (4.4), the martingale condition is quite involved. Next, we study the reduction of the BSDE (3.1) under the following additional assumption (on top of the condition (B)), given a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$.

Condition (A). For any $(\mathbb{F}, \mathbb{P})$ local martingale P, $P^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$-local martingale on $[0, \theta \wedge T]$ (or $[0, T]$).

The condition (A) is nonstandard in the enlargement of filtration literature. Its statement immediately raises questions such as the materiality of stopping at $\theta-$ rather than at θ in this definition (in other words, can the condition (A) be satisfied in cases where $(\mathbb{F}, \mathbb{P})$ martingales really jump at θ) and its connection with the classical notions of pseudo-stopping times (see Nikeghbali and Yor (2005)), or initial times satisfying Jacod's condition (see e.g. Jeanblanc and Le Cam (2009)). To address these issues, we need a characterization of this condition in terms of the Azéma supermartingale of θ. This will be the topic of Sect. 5. For now, we consider the following $(\mathbb{F}, \mathbb{P})$ BSDE for an $(\mathbb{F}, \mathbb{P})$ semimartingale V :

$$
\begin{cases}
\int_0^t |f_s(V_{s-})| ds + \int_0^t |\overline{F}_s(V_{s-})| \gamma_s ds < \infty \quad \text{for } t \in [0, \theta \wedge T], \\[2mm]
V_t^T + \int_0^{t \wedge T} \left(f_s(V_{s-}) + (\overline{F}_s(V_{s-}) - V_{s-}) \gamma_s \right) ds \text{ is an } (\mathbb{F}, \mathbb{P}) \text{ local martingale,} \qquad (4.6) \\[2mm]
V_T = 0.
\end{cases}
$$

Theorem 4.2 *If V solves the $(\mathbb{F}, \mathbb{P})$ BSDE (4.6), then $Z = V^{\theta \wedge T-}$ is a solution, stopped at $\theta \wedge T$, to the BSDE (3.10). As a consequence, if the process $\mathbb{1}_{[\theta \wedge T,\infty)} \mathbb{1}_{\{\theta < T\}} F(V_{\theta \wedge T-})$ is locally integrable, then the process*

$$
X = V \mathbb{1}_{[0, \theta \wedge T)} + \mathbb{1}_{\{\theta < T\}} \mathbb{1}_{[\theta \wedge T,\infty)} F(V_{\theta \wedge T-})
$$

solves the BSDE (3.1) with the data $\vartheta = \theta \wedge T$, $g = f$ and $G = \mathbb{1}_{\theta < T} F$.

Proof. By the martingale condition in (4.6),

$$
P = V_t^T + \int_0^{t \wedge T} \left(f_s(V_{s-}) + (\overline{F}_s(V_{s-}) - V_{s-}) \gamma_s \right) ds
$$

is an $(\mathbb{F}, \mathbb{P})$ local martingale. Hence, the condition (A) implies that $P^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, \theta \wedge T]$. Note that

$$\Delta_T P = \Delta_T V = -V_{T-}.$$

As T is predictable and P is an $(\mathbb{F}, \mathbb{P})$ local martingale, taking conditional expectation with respect to $\mathcal{F}_{T-}$ yields $V_{T-} = 0$ and in turn $\Delta_T P = 0$, so that

$$P^{\theta-} = P^{\theta \wedge T-} = Z^{\theta \wedge T-} + \mathbb{1}_{(0, \theta \wedge T]}(g.(Z_-) + (\widehat{G}.(Z_-) - Z_-)\gamma \boldsymbol{.} \boldsymbol{\lambda}.$$

Therefore, the martingale condition is satisfied in the BSDE (3.10), where the integrability condition is clearly satisfied, while the terminal condition, which reduces to (4.3), holds because $\mathbb{1}_{\{T \leq \theta\}} V_{T-} = 0$. $\square$

Corollary 4.1 *Under the conditions of Theorem 4.2,*

$$dR_t = J_{t-} \left(dP_t^{\theta \wedge T-} + (\mathbb{1}_{\{\theta < T\}} F(Z_{\vartheta-}) - Z_{\vartheta-})dH_t - \mathbb{1}_{\{t < T\}}(\widehat{F}_t(Z_t) - Z_t)\gamma_t dt \right). \quad (4.7)$$

Proof. In the present setup, (3.14) implies

$$dR_t = J_{t-} \left(dR_t'' - Z_{t-}(dH_t - \gamma_t dt) + \mathbb{1}_{\{\theta < T\}} F(Z_{\theta-})dH_t - \mathbb{1}_{\{t < T\}}\widehat{F}_t(Z_t)\gamma_t dt \right),$$

which is (4.7) since $R'' = P^{\theta \wedge T-}$. $\square$

Remark 4.2 Formally, the BSDE (4.6) is the BSDE (3.10) with ϑ there replaced by T (i.e. θ replaced by $+\infty$ in $\vartheta = \theta \wedge T$). From the financial point of view, Theorem 4.2 can be interpreted as an invariance principle, stating a consistency relation between a non-arbitrable pre-default pricing model $(\mathbb{F}, \mathbb{P})$ and a non-arbitrable "full" pricing model $(\mathbb{G}, \mathbb{Q})$.

5 Invariant Times

The condition (A) is nonstandard in the enlargement of filtration literature. The aim of this section is to introduce invariant times, based on a characterization of this condition in terms of the $(\mathbb{F}, \mathbb{Q})$ Azéma supermartingale S of θ in $\vartheta = \theta \wedge T$ (so that $\mathsf{S} = \mathbb{1}_{[0,T)}S$). As before, superscripts ^{p} and ^{o} (respectively $^{\bar{p}}$ and $^{\bar{o}}$) are used for the $(\mathbb{F}, \mathbb{Q})$ (respectively $(\mathbb{G}, \mathbb{Q})$) projections; $\mathbb{E}$ represents the $\mathbb{Q}$ expectation, whereas the $\mathbb{P}$ expectation will be denoted by $\widetilde{\mathbb{E}}$; the $(\mathbb{F}, \mathbb{Q})$ predictable bracket is denoted by $\langle \cdot, \cdot \rangle$ (cf. (4.2)), whereas the $(\mathbb{F}, \mathbb{P}))$ predictable bracket will be denoted by $\langle \cdot, \cdot \rangle$ (in fact, it is enough to know that all the predictable brackets in this paper are relative to $(\mathbb{F}, \mathbb{P})$ when both arguments are $(\mathbb{F}, \mathbb{P})$ local martingales and relative to $(\mathbb{F}, \mathbb{Q})$ otherwise; of course, when both arguments are continuous, the two brackets coincide). The continuous and discontinuous components of a local martingale (with respect to a given stochastic basis) are denoted by ^{c} and ^{d}. Let $q = \frac{1}{p}$ denote the $(\mathbb{F}, \mathbb{Q})$ martingale of the density functions $\frac{d\mathbb{P}}{d\mathbb{Q}}_{|\mathcal{F}_{t \wedge T}}, t \in \mathbb{R}_+$. Let P be a bounded $(\mathbb{F}, \mathbb{P})$ martingale null at the origin. By the Girsanov theorem,

$$\widetilde{P} = P - q_- \boldsymbol{.} \langle p, P \rangle \qquad (5.1)$$

is an $(\mathbb{F}, \mathbb{Q})$ local martingale on $[0, T]$.

Lemma 5.1 *Let Q be an $(\mathbb{F}, \mathbb{Q})$ uniformly integrable martingale null at the origin such that $\widetilde{P}Q$ is an $(\mathbb{F}, \mathbb{Q})$ local martingale, for any bounded $(\mathbb{F}, \mathbb{P})$ martingale P null at the origin. Then $Q = 0$ on $[0, T]$.*

Proof. For any $\mathbb{F}$ stopping time $\sigma \leq T$ reducing the concerned processes into integrable processes,

$$
\begin{aligned}
0 = \mathbb{E}[\widetilde{P}_\sigma Q_\sigma] &= \widetilde{\mathbb{E}}[\widetilde{P}_\sigma Q_\sigma p_\sigma] = \widetilde{\mathbb{E}}[(P - q_- \cdot \langle p, P \rangle)_\sigma Q_\sigma p_\sigma] \\
&= \widetilde{\mathbb{E}}[\langle Qp, P \rangle_\sigma - Q_- \cdot \langle p, P \rangle_\sigma] = \widetilde{\mathbb{E}}[[Qp - Q_- \cdot p, P]_\sigma],
\end{aligned}
$$

where the $(\mathbb{F}, \mathbb{P})$ local martingale property of Qp was used to pass to the second line. This computation shows that

$$
Qp - Q_- \cdot p = Qp - (Qp)_- q_- \cdot p = 0, \quad \text{on } [0, T].
$$

By uniqueness of the solution to the exponential stochastic differential equation, we conclude that $Q = 0$ on $[0, T]$. □

Lemma 5.2 *For any bounded $(\mathbb{F}, \mathbb{P})$ martingale P null at the origin, $\widetilde{P^c}$ and $\widetilde{P^d}$ are a continuous $(\mathbb{F}, \mathbb{Q})$ martingale and a purely discontinuous $(\mathbb{F}, \mathbb{Q})$ martingale on $[0, T]$, respectively.*

Proof. We prove the assertion about $\widetilde{P^d}$ (the one about $\widetilde{P^c}$ is clear). Let Q be any continuous $(\mathbb{F}, \mathbb{Q})$ martingale null at the origin, hence an $(\mathbb{F}, \mathbb{P})$ continuous semimartingale on $[0, T]$. For any $\mathbb{F}$ stopping time $\sigma \leq T$ reducing the concerned processes into integrable processes, as in the proof of Lemma 5.1, we can write

$$
\mathbb{E}[\widetilde{P^d_\sigma} Q_\sigma] = \widetilde{\mathbb{E}}[[Qp - Q_- \cdot p, P^d]_\sigma],
$$

where, by the integration by parts formula on $[0, T]$,

$$
[Qp - Q_- \cdot p, P^d] = [p_- \cdot Q + [Q, p], P^d] = 0,
$$

because $p_- \cdot Q + [Q, p]$ is continuous. □

Let Q denote the martingale part of the $(\mathbb{F}, \mathbb{Q})$ Azéma supermartingale S of θ (with $Q_0 = 0$). Note that, for any $\mathbb{F}$ predictable stopping time $\sigma \leq T$,

$$
\mathbb{E}[K_\sigma q_\sigma \Delta_\sigma p] = \widetilde{\mathbb{E}}[K_\sigma \Delta_\sigma p] = 0,
$$

so that by Theorem 7.42 in He, Wang, and Yan (1992), there exists an $(\mathbb{F}, \mathbb{Q})$ purely discontinuous local martingale $\mathbf{q}$ on $[0, T]$ such that $\Delta_s \mathbf{q} = q_s \Delta_s p$.

Lemma 5.3 *For any bounded $(\mathbb{F}, \mathbb{P})$ martingale P null at the origin, we have*

$$
\langle p, P \rangle = \langle p^c, P^c \rangle + p_- \cdot \langle \mathbf{q}, P^d \rangle \tag{5.2}
$$

and the $(\mathbb{G}, \mathbb{Q})$ dual predictable projection of the process $\mathbb{1}_{[0,\theta)} q_- \cdot \langle p, P \rangle$ is given by

$$
\left(\mathbb{1}_{[0,\theta)} q_- \cdot \langle p, P \rangle \right)^{\bar{p}} = \mathbb{1}_{[0,\theta]} q_- \cdot \langle p^c, P^c \rangle + \mathbb{1}_{[0,\theta]} \frac{{}^p S}{S_-} \cdot \langle \mathbf{q}, P^d \rangle. \tag{5.3}
$$

The process $P^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale if and only if

$$
S_- q_- \cdot \langle p^c, P^c \rangle + {}^p S \cdot \langle \mathbf{q}, P^d \rangle + \langle Q, P \rangle = 0 \tag{5.4}
$$

on $[0, T]$.

Proof. For any bounded $\mathbb{F}$ predictable process K, for any $\mathbb{F}$ stopping time $\sigma \le T$ reducing the concerned processes into integrable processes,

$$
\begin{aligned}
\widetilde{\mathbb{E}}[K \cdot \langle p, P \rangle_\sigma] &= \widetilde{\mathbb{E}}[K \cdot [p, P]_\sigma] \\
&= \mathbb{E}[Kq \cdot [p, P]_\sigma] \\
&= \mathbb{E}[K \cdot (q \cdot [p, P])^p_\sigma] \\
&= \widetilde{\mathbb{E}}[Kp_- \cdot (q \cdot [p, P])^p_\sigma] \\
&= \widetilde{\mathbb{E}}[Kp_- \cdot \left(q_- \cdot \langle p^c, P^c \rangle_\sigma + (\sum_{0 < s \le \cdot} q_s \Delta_s p \Delta_s P)^p_\sigma \right)].
\end{aligned}
$$

This computation implies

$$
\langle p, P \rangle = \langle p^c, P^c \rangle + p_- \cdot (\sum_{0 < s \le \cdot} q_s \Delta_s p \Delta_s P)^p,
$$

which yields (5.2). Now we compute the $(\mathbb{G}, \mathbb{Q})$ dual predictable projection of $\mathbb{1}_{[0,\theta)} \cdot \langle \mathbf{q}, P^d \rangle$ on $[0, T]$. For any bounded $\mathbb{G}$ predictable process L null outside of $[0, T]$, with $\mathbb{F}$ predictable representative (until θ) denoted by K, and for any $\mathbb{G}$ stopping time τ:

$$
\begin{aligned}
\mathbb{E}[L\mathbb{1}_{[0,\theta)} \cdot \langle \mathbf{q}, P^d \rangle_\tau] &= \mathbb{E}[K\mathbb{1}_{[0,\theta)} \cdot \langle \mathbf{q}, P^d \rangle_\tau] \\
&= \mathbb{E}[KS \cdot \langle \mathbf{q}, P^d \rangle_\tau] \\
&= \mathbb{E}[K^pS \cdot \langle \mathbf{q}, P^d \rangle_\tau] \\
&= \mathbb{E}[K\mathbb{1}_{[0,\theta]}\frac{^pS}{S_-} \cdot \langle \mathbf{q}, P^d \rangle_\tau] \\
&= \mathbb{E}[L\mathbb{1}_{[0,\theta]}\frac{^pS}{S_-} \cdot \langle \mathbf{q}, P^d \rangle_\tau].
\end{aligned}
$$

This shows that the $(\mathbb{G}, \mathbb{Q})$ dual predictable projection of $\mathbb{1}_{[0,\theta)} \cdot \langle \mathbf{q}, P^d \rangle$ on $[0, T]$ is given by

$$
\mathbb{1}_{[0,\theta]}\frac{^pS}{S_-} \cdot \langle \mathbf{q}, P^d \rangle
$$

on the time interval $[0, T]$, hence (5.3) follows from (5.2). Combining Lemma 4.2.5 with the Girsanov formula (5.1), we obtain that

$$
P^{\theta-} - \mathbb{1}_{[0,\theta)}q_- \cdot \langle p, P \rangle - \mathbb{1}_{[0,\theta]}\frac{1}{S_-} \cdot \langle Q, P \rangle
$$

is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$. If $P^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$, so is in turn the $\mathbb{G}$ optional process with finite variation

$$
\mathbb{1}_{[0,\theta)}q_- \cdot \langle p, P \rangle + \mathbb{1}_{[0,\theta]}\frac{1}{S_-} \cdot \langle Q, P \rangle, \tag{5.5}
$$

the $(\mathbb{G}, \mathbb{Q})$ dual predictable projection of which must therefore vanish, i.e. by (5.3)

$$
\mathbb{1}_{[0,\theta]}q_- \cdot \langle p^c, P^c \rangle + \mathbb{1}_{[0,\theta]}\frac{^pS}{S_-} \cdot \langle \mathbf{q}, P^d \rangle + \mathbb{1}_{[0,\theta]}\frac{1}{S_-} \cdot \langle Q, P \rangle = 0
$$

on $[0, T]$ under both probability measures $\mathbb{P}$ and $\mathbb{Q}$. Taking the $(\mathbb{F}, \mathbb{Q})$ dual predictable projection of the above equation, we obtain the equivalent equation (5.4) (under the condition (B)). $\qquad \square$

Let us write the processes p and $q = 1/p$ on $[0, T]$ in stochastic exponential form

$$
p = \mathcal{E}(\mathfrak{p}), \quad q = \mathcal{E}(\mathfrak{q}), \tag{5.6}
$$

where $\mathfrak{p}$ is an $(\mathbb{F}, \mathbb{P})$ local martingale on $[0, T]$ and $\mathfrak{q}$ is an $(\mathbb{F}, \mathbb{Q})$ local martingale on $[0, T]$. By Lemma 3.4 in Karatzas and Kardaras (2007), we have the following relation between $\mathfrak{q}$ and $\mathfrak{p}$:

$$\mathfrak{q} = -\mathfrak{p} + \langle \mathfrak{p}^c, \mathfrak{p}^c \rangle + \sum_{s \leq \cdot} \frac{(\Delta_s \mathfrak{p})^2}{1 + \Delta_s \mathfrak{p}}. \tag{5.7}$$

We are now in the position to derive the following characterization of the condition (A).

Lemma 5.4 *The condition (A) holds if and only if on $[0, T]$*

$$S_- \centerdot \mathfrak{q}^c = Q^c, \ {}^p S \centerdot \mathfrak{q}^d = Q^d. \tag{5.8}$$

Proof. In view of the second assertion in Lemma 5.3, the condition (A) holds if and only if any bounded $(\mathbb{F}, \mathbb{P})$ martingale P null at the origin satisfies (5.4), or, equivalently in view of Lemma 5.2,

$$S_- q_- \centerdot \langle \widetilde{p^c}, \widetilde{P^c} \rangle + {}^p S \centerdot \langle \mathbf{q}, \widetilde{P^d} \rangle + \langle Q^c, \widetilde{P^c} \rangle + \langle Q^d, \widetilde{P^d} \rangle = 0. \tag{5.9}$$

For any bounded $(\mathbb{F}, \mathbb{P})$ local martingale P null at the origin, the formula (5.9) applied with P^c instead of P yields

$$0 = S_- q_- \centerdot \langle \widetilde{p^c}, \widetilde{P^c} \rangle + \langle Q^c, \widetilde{P^c} \rangle = \langle S_- q_- \centerdot \widetilde{p^c} + Q^c, \widetilde{P^c} \rangle = \langle S_- q_- \centerdot \widetilde{p^c} + Q^c, \widetilde{P} \rangle$$

on $[0, T]$ under the probability measure $\mathbb{P}$ and also $\mathbb{Q}$. So, under the probability measure $\mathbb{Q}$, the $(\mathbb{F}, \mathbb{Q})$ local martingales $S_- q_- \centerdot \widetilde{p^c} + Q^c$ is orthogonal to $\widetilde{P}$ on $[0, T]$. Since this holds for any P bounded $(\mathbb{F}, \mathbb{P})$ local martingale P null at the origin, an application of Lemma 5.1 yields that

$$S_- q_- \centerdot \widetilde{p^c} + Q^c = 0. \tag{5.10}$$

Likewise, the formula (5.9) applied with P^d instead of P yields

$$0 = {}^p S \centerdot \langle \mathbf{q}, \widetilde{P^d} \rangle + \langle Q^d, \widetilde{P^d} \rangle = \langle {}^p S \centerdot \mathbf{q} + Q^d, \widetilde{P^d} \rangle = \langle {}^p S \centerdot \mathbf{q} + Q^d, \widetilde{P} \rangle$$

on $[0, T]$ under the probability measure $\mathbb{P}$ and $\mathbb{Q}$. So, under the probability measure $\mathbb{Q}$, the $(\mathbb{F}, \mathbb{Q})$ local martingales ${}^p S \centerdot \mathbf{q} + Q^d$ is orthogonal to $\widetilde{P}$ on $[0, T]$, for any P bounded $(\mathbb{F}, \mathbb{P})$ local martingale P null at the origin, which implies as above that

$$^p S \centerdot \mathbf{q} + Q^d = 0. \tag{5.11}$$

We note that p is a positive $(\mathbb{F}, \mathbb{Q})$ semimartingale and q is a positive $(\mathbb{F}, \mathbb{Q})$ martingale on $[0, T]$. In terms of the process $\mathfrak{p}$, we can write, on the time interval $[0, T]$,

$$S_- q_- \centerdot \widetilde{p^c} = S_- q_- p_- \centerdot \widetilde{\mathfrak{p}^c} = S_- \centerdot \widetilde{\mathfrak{p}^c},$$

$$\Delta_t \mathbf{q} = \frac{\Delta_t p}{p_t} = \frac{p_{t-} \Delta_t \mathfrak{p}}{p_{t-} + \Delta_t p} = \frac{\Delta_t \mathfrak{p}}{1 + \Delta_t \mathfrak{p}}, \tag{5.12}$$

and the Girsanov formula (5.1) appears as

$$\widetilde{P} = P - q_- \centerdot \langle p, P \rangle = P - \langle \mathfrak{p}, P \rangle,$$

where, using (5.2) in the second line,

$$
\begin{aligned}
\langle \mathfrak{p}, P \rangle &= q_- \cdot \langle p, P \rangle \\
&= q_- \cdot \langle p^c, P^c \rangle + \langle \mathbf{q}, \widetilde{P^d} \rangle = \langle \mathfrak{p}^c, P^c \rangle + \langle \mathbf{q}, \widetilde{P^d} \rangle.
\end{aligned}
$$

Now, by (5.7),

$$
\begin{aligned}
\mathfrak{q} &= -\mathfrak{p}^c + \langle \mathfrak{p}^c, \mathfrak{p}^c \rangle - \mathfrak{p}^d + [\mathbf{q}, \mathfrak{p}^d] \\
&= -\widetilde{\mathfrak{p}^c} - \widetilde{\mathfrak{p}^d} - \langle \mathbf{q}, \widetilde{\mathfrak{p}^d} \rangle + [\mathbf{q}, \mathfrak{p}^d].
\end{aligned}
$$

As a consequence, $\mathfrak{q}^c = -\widetilde{\mathfrak{p}^c}$, whereas, by (5.7),

$$
\Delta_t \mathfrak{q}^d = \Delta_t \mathfrak{q} = -\Delta_t \mathfrak{p} + \frac{(\Delta_t \mathfrak{p})^2}{1 + \Delta_t \mathfrak{p}} = -\frac{\Delta_t \mathfrak{p}}{1 + \Delta_t \mathfrak{p}} = -\Delta_t \mathbf{q},
$$

by (5.12). Since $\mathfrak{q}^d$ and $\mathbf{q}$ are both $(\mathbb{F}, \mathbb{Q})$ purely discontinuous local martingales, according to Corollary 7.23 in He, Wang, and Yan (1992),

$$
\mathfrak{q}^d = -\mathbf{q}.
$$

Hence, $\mathfrak{q}^c = -\widetilde{\mathfrak{p}^c}$ and $\mathfrak{q}^d = -\mathbf{q}$, so that (5.10) and (5.11) can be rewritten as (5.8). $\qquad \square$

The condition (A) is stated relative to a fixed reduced stochastic basis $(\mathbb{F}, \mathbb{P})$. But, in applications, such as the one of Theorem 4.2 further developed in Crépey and Song (2014a) and Crépey and Song (2014b), the choice of a reduced stochastic basis $(\mathbb{F}, \mathbb{P})$ is a degree of freedom of the modeler. Thus, our next aim is to characterize stopping times θ such that the conditions (A) and (B) (recall (B) was postulated everywhere before in this section) hold for at least one reduced basis $(\mathbb{F}, \mathbb{P})$. First, echoing the remark 4.2, we state the following:

Definition 5.1 A $\mathbb{G}$ stopping time θ is said to be invariant if there exists a filtration $\mathbb{F}$ satisfying the usual conditions, along with a probability measure $\mathbb{P}$ equivalent to the probability measure $\mathbb{Q}$ on $\mathcal{F}_T$, such that the conditions (A) and (B) are satisfied.

Comparing invariant times with other classical notions of random times, such as pseudo-stopping times (see Nikeghbali and Yor (2005)) or initial times (see Jeanblanc and Le Cam (2009) or El Karoui, Jeanblanc, and Jiao (2010)), is not easy in general. Pseudo-stopping times are defined with respect to a fixed probability measure $\mathbb{Q}$ and admit several equivalent characterizations, e.g. $(\mathbb{F}, \mathbb{Q})$ local martingales stopped at θ are $(\mathbb{G}, \mathbb{Q})$ local martingale, or $\mathsf{H}_\infty^o \equiv 1$, where $\mathsf{H} = \mathbb{1}_{[\theta, \infty)}$. It's only in the case where $\mathbb{P} = \mathbb{Q}$ that a few things can be said regarding the corresponding condition (A) (also assuming (B) as before) versus pseudo-stopping times. So, for θ avoiding $\mathbb{F}$-stopping times, the two notions are equivalent (and obviously hold under immersion from $\mathbb{F}$ into $\mathbb{G}$). In the case where $\mathbb{P} = \mathbb{Q}$, we have $\mathfrak{q} \equiv 0$, so that, by (5.8), the condition (A) is equivalent to $Q \equiv 0$, which means that S is an $\mathbb{F}$ nonincreasing predictable process, i.e. S has no $\mathbb{F}$ local martingale component. Hence, still in the case where $\mathbb{P} = \mathbb{Q}$, we can also say that when S is nonincreasing, the predictability of S is equivalent to the condition (A), whereas it is the continuity of S that implies (only) that θ is a pseudo-stopping time (see Nikeghbali and Yor (2005)). But all these remarks are of limited interest since the flexibility of invariant times lies precisely in the possibility to use a changed measure $\mathbb{P}$ on top of a reduced filtration $\mathbb{F}$ (see Crépey and Song (2014b) for concrete illustrations). The connection with initial times seems even less clear. However, based on Lemma 5.4, we can state the following theoretical characterization of invariance.

Theorem 5.1 *A $\mathbb{G}$ stopping time θ is invariant if and only if there exists a filtration $\mathbb{F}$ satisfying the usual conditions such that $\mathbb{F} \subseteq \mathbb{G} \subseteq \overline{\mathbb{F}}$, $\frac{1}{S_-}$ $(\mathbb{F}, \mathbb{Q})$ integrates Q^c, $\frac{1}{{}^pS}$ $(\mathbb{F}, \mathbb{Q})$ integrates Q^d and $q = \mathcal{E}(\frac{1}{S_-} \cdot Q^c)\mathcal{E}(\frac{1}{{}^pS} \cdot Q^d)$ is a positive $(\mathbb{F}, \mathbb{Q})$ true martingale on $[0, T]$. In this case, a probability measure $\mathbb{P}$ such that $(\mathbb{F}, \mathbb{P})$ fulfills all the required conditions is uniquely characterized on $\mathcal{F}_T$ by the $\mathbb{Q}$ density q given as above.*

Proof. In view of Lemma 4.1 and Definition 5.1, one is reduced to show that, given a filtration $\overline{\mathbb{F}}$ satisfying the usual conditions and $\mathbb{F} \subseteq \mathbb{G} \subseteq \overline{\mathbb{F}}$ (i.e. the condition (B)), there exists a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ and satisfying the condition (A) if and only if $\frac{1}{S_-}$ $(\mathbb{F}, \mathbb{Q})$ integrates Q^c, $\frac{1}{{}^pS}$ $(\mathbb{F}, \mathbb{Q})$ integrates Q^d and $q = \mathcal{E}(\frac{1}{S_-} \cdot Q^c)\mathcal{E}(\frac{1}{{}^pS} \cdot Q^d)$ is a positive $(\mathbb{F}, \mathbb{Q})$ true martingale on $[0, T]$ (a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ granting the condition (A) being then characterized on $\mathcal{F}_T$ by the $\mathbb{Q}$ density q in view of Lemma 5.4). Assuming some $\mathbb{P} \sim \mathbb{Q}$ on $\mathcal{F}_T$ satisfies (A), by the necessity in Lemma 5.4, $\mathfrak{q}$, the stochastic logarithm of the $\mathbb{Q}$ density q of $\mathbb{P}$ has to solve the equation (5.8). By Theorem 9.3 in He, Wang, and Yan (1992), this implies the stated integrabilities, hence the stochastic exponential formula for q follows from (5.8). Therefore, this formula must yield an $(\mathbb{F}, \mathbb{Q})$ true martingale on $[0, T]$, positive for the sake of the equivalence of $\mathbb{P}$ and $\mathbb{Q}$ on $\mathcal{F}_T$. Conversely, if $\mathfrak{q}$ is given as the solution to (5.8) under the stated integrabilities, then, by the sufficiency in Lemma 5.4, the condition (A) holds for the equivalent probability measure $\mathbb{P}$ on $\mathcal{F}_T$ defined by the $\mathbb{Q}$ density $q = \mathcal{E}(\mathfrak{q})$ (under the postulated positivity and $(\mathbb{F}, \mathbb{Q})$ true martingality of $q = \mathcal{E}(\mathfrak{q})$ on $[0, T]$). $\qquad\square$

What follows illustrates the role of the positivity condition, studied theoretically in Sect. 5.1, on the Doleans-Dade exponential in Theorem 5.1, i.e. the role of the equivalence between $\mathbb{P}$ and $\mathbb{Q}$ on $\mathcal{F}_T$.

Example 5.1 Take for $\mathbb{F} = \mathbb{G}$ (so that the condition (B) holds trivially) the augmentation of the natural filtration of a Poisson process stopped at its first time of jump θ, with survival process $\mathsf{J} = \mathbb{1}_{[0,\theta)}$, relative to some probability measure $\mathbb{Q}$. For any $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{G}_T$, any $(\mathbb{G}, \mathbb{P})$ local martingale P can be represented as a $(\mathbb{G}, \mathbb{Q})$ local martingale minus some predictable bracket deterministic until θ, so that $P^{\theta-}$ is a nonconstant finite variation process, predictable as a càdlàg predictable process stopped before a totally inaccessible stopping time (Theorem 3.33 in He, Wang, and Yan (1992)), hence not a $(\mathbb{G}, \mathbb{Q})$ local martingale. Therefore, $(\mathbb{G}, \mathbb{P})$ does not satisfy the condition (A). Consistent with this conclusion in regard of Lemma 5.4, note (this is general) ${}^pS = \mathsf{J}_- + \Delta A = \mathsf{J}_-$ (since $A_t = t \wedge \theta$ here). Also, in this case, $S = \mathsf{J} = \mathsf{J} + A - A$, so that $Q = Q^d = \mathsf{J} + A$, which is stopped at θ, whereas ${}^pS = \mathsf{J}_- = 1$ on $[0, \theta]$. Consequently,

$$\mathcal{E}(\frac{1}{{}^pS} \cdot Q^d)_t = \mathcal{E}(Q)_t = e^{Q_t - Q_0} \prod_{s \leq t}(1 + \Delta_s Q)e^{-\Delta_s Q} = e^{\mathsf{J}_t + t \wedge \theta - 1}\mathsf{J}_t e^{(1 - \mathsf{J}_t)} = e^{t \wedge \theta}\mathsf{J}_t,$$

a $(\mathbb{G}, \mathbb{Q})$ true martingale but vanishing on $[\theta \wedge T, T]$.

Now, for $\mathbb{F}$ trivial and $\mathbb{P} = \mathbb{Q}$ in this example, any $\mathbb{G}$ predictable process coincides with a Borel function before θ, hence the condition (B) is satisfied. The constants are the only $(\mathbb{F}, \mathbb{Q})$ local martingales, so that the condition (A) holds. In particular, θ is an invariant time. Consistent with these conclusions in regard of Lemma 5.4 and Theorem 5.1, S is deterministic (equal to the survival function of θ), Q is constant, $\mathfrak{q} \equiv 0$ and $q \equiv 1$.

Theorem 5.1 can help us to answer the questions we were asking after the statement of the condition (A), i.e. what is the materiality of stopping at $\theta-$ rather than θ in this

condition, and to understand a little bit more of the above-mentioned connections between invariance, the condition (A) and pseudo-stopping times. The following stopping time θ is an invariant time that intersects $\mathbb{F}$ stopping times in a reduced basis $(\mathbb{F}, \mathbb{P})$ satisfying the condition (A). This invariant time θ is a also a pseudo-stopping time, a feature that is removed by considering the modified time $\widetilde{\theta}$.

Example 5.2 For $i = 1, 2$, let $\mu_i = \nu_i \wedge C_i$, where C_i is a positive constant and ν_i is a totally inaccessible $\mathbb{F}$ stopping time with bounded compensator. Assuming $\mu_2 > T$, define $\theta = \mathbb{1}_A \mu_1 + \mathbb{1}_{A^c} \mu_2$, which intersects the $\mathbb{F}$ stopping times μ_i, for some $A \in \mathcal{G}_\infty$ independent from $\mathcal{F}_\infty$ such that $\alpha = \mathbb{Q}(A) \in (0, 1)$. On $[0, T]$,

$$S_t = \mathbb{1}_{t < \mu_1} \alpha + \mathbb{1}_{t < \mu_2}(1 - \alpha) \geq (1 - \alpha), \quad S_- \geq 1 - \alpha.$$

Moreover, we have $Q = Q^d$ and, by Theorem I.8 in Lepingle and Mémin (1978), $\mathcal{E}(\frac{1}{pS}\bullet Q^d)$ is a positive $\mathbb{Q}$ true martingale that defines the $\mathbb{Q}$ density of a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$. By construction, the reduced basis $(\mathbb{F}, \mathbb{P})$ satisfies the condition (A), which shows that θ is an invariant time. Note that, writing $\mathsf{H} = \mathbb{1}_{[\theta,\infty)}$,

$$\mathsf{H}^o = \mathbb{1}_{[\mu_1,\infty)} \alpha + \mathbb{1}_{[\mu_2,\infty)}(1 - \alpha), \quad \mathsf{H}^o_\infty \equiv 1,$$

so that, by application of Theorem 1 (3) in Nikeghbali and Yor (2005), θ is also a pseudo-stopping time.

Now, to obtain an invariant time τ intersecting $\mathbb{F}$ stopping times without being a pseudo-stopping time, set
$$\tau = \mathbb{1}_{A_1} \mu_1 + \mathbb{1}_{A_2} \mu_2 + \mathbb{1}_{A_3} \eta,$$

for a non pseudo-stopping time η and a partition $A_i, i = 1, 2, 3$, independent from $\mathcal{F}_\infty$ and η. Writing $\mathsf{K} = \mathbb{1}_{[\tau,\infty)}$, $\mathsf{L} = \mathbb{1}_{[\eta,\infty)}$, $\alpha_i = \mathbb{Q}(A_i)$, we have

$$\mathsf{K}^o = \mathbb{1}_{[\mu_1,\infty)} \alpha_1 + \mathbb{1}_{[\mu_2,\infty)} \alpha_2 + \mathsf{L}^o \alpha_3, \quad \mathsf{L}^o_\infty \neq \alpha_1 + \alpha_2 + \alpha_3 = 1 \text{ with positive } \mathbb{Q} \text{ probability},$$

so that, by the converse part in the above-quoted theorem, τ is not a pseudo-stopping time. But the Azéma supermartingale of τ is given by

$$\widetilde{S}_t = \mathbb{1}_{t < \mu_1} \alpha_1 + \mathbb{1}_{t < \mu_2} \alpha_2 + Z_t \alpha_3 \geq \alpha_2,$$

where Z_t is the Azéma supermartingale of η. Hence, the other computations above do not change, which shows that τ is an invariant time.

5.1 Positivity of the Dolean-Dade exponential

In Theorem 5.1 the strongest requirements are the martingality and positivity conditions on the Dolean-Dade exponential $\mathcal{E}(\frac{1}{S_-}\bullet Q^c)\mathcal{E}(\frac{1}{pS}\bullet Q^d)$. We assume the existence of the stochastic integrals $\frac{1}{S_-} \bullet Q^c$ and $\frac{1}{pS} \bullet Q^d$. In this concluding section we show that the positivity of $\mathcal{E}(\frac{1}{pS} \bullet Q^d)$ (the one of $\mathcal{E}(\frac{1}{S_-} \bullet Q^c)$ is obvious) reduces to the predictability of the stopping time $\varsigma_{\{\varsigma \leq T\}}$, where $\varsigma = \inf\{s > 0; S_s = 0\}$.

Lemma 5.5 Let σ be a predictable stopping time. Then $^pS_\sigma = 0$ if and only if $\sigma \geq \varsigma$.

Proof. $^pS_\sigma = 0 \Leftrightarrow {}^pS_\sigma = \mathbb{E}[S_\sigma | \mathcal{F}_{\sigma-}] = 0 \Leftrightarrow S_\sigma = 0 \Leftrightarrow \sigma \geq \varsigma.$ $\qquad\square$

Lemma 5.6 $\varsigma_{\{\varsigma<\infty,{}^pS_\varsigma=0\}}$ *is a predictable stopping time.*

Proof. Let A be the drift (nondecreasing predictable component) in the canonical decomposition of S. Since ${}^pS = S_- - \Delta A$, therefore $S_{\varsigma-} = \Delta_\varsigma A > 0$ on $\{\varsigma < \infty, {}^pS_\varsigma = 0, \Delta_\varsigma A > 0\}$. The set

$$\mathfrak{A} = \{{}^pS = 0, \Delta A > 0\}$$

is thin and predictable. If $\mathbb{Q}[\mathfrak{A} \neq \emptyset] > 0$, by the section theorem 4.8 in He, Wang, and Yan (1992), there exists a sequence of predictable stopping times σ_n such that $[\![\sigma_n]\!] \subseteq \mathfrak{A}$ and $\lim_{n\to\infty} \mathbb{Q}[\sigma_n < \infty] = \mathbb{Q}[\pi(\mathfrak{A})] > 0$, where $\pi(\mathfrak{A}) = \{\mathfrak{A} \neq \emptyset\}$ is the projection of $\mathfrak{A}$ onto Ω. Note that $\Delta A \equiv 0$ on (ς, ∞), thus $\sigma_n(\omega) < \infty$ implies $\sigma_n(\omega) \leq \varsigma(\omega)$. But by Lemma 5.5, $\sigma_n(\omega) \geq \varsigma(\omega)$. We conclude that $\sigma_n = \varsigma$ on $\{\sigma_n < \infty\}$, i.e. $[\![\sigma_n]\!] \subseteq [\![\varsigma]\!]$. Set $\sigma_\infty = \inf_n \sigma_n$. Since $\sigma_n(\omega)$ can only take two values (∞ and $\varsigma(\omega)$), σ_∞ is a stationary infimum of predictable stopping times, hence a predictable stopping time by Theorem 3.29 in He, Wang, and Yan (1992). Clearly, $\Delta_{\sigma_\infty} A > 0$ on $\{\sigma_\infty < \infty\}$, $[\![\sigma_\infty]\!] \subset [\![\varsigma]\!]$, $\{\sigma_\infty < \infty\} \subset \{\mathfrak{A} \neq \emptyset\}$ and $\mathbb{Q}[\sigma_\infty < \infty] = \mathbb{Q}[\mathfrak{A} \neq \emptyset]$. As $\varsigma \leq \sigma_\infty$, $(\sigma_\infty)_{\{\varsigma<\infty,{}^pS_\varsigma=0,S_{\varsigma-}>0\}}$ is predictable by Theorem 3.29 in He, Wang, and Yan (1992). Therefore,

$$\varsigma < \infty, {}^pS_\varsigma = 0, \Delta_\varsigma A > 0 \quad \Rightarrow \quad \varsigma < \infty, \mathfrak{A} \neq \emptyset, {}^pS_\varsigma = 0, S_{\varsigma-} > 0$$
$$\Rightarrow \quad \varsigma < \infty, \sigma_\infty < \infty, {}^pS_\varsigma = 0, S_{\varsigma-} > 0$$
$$\Rightarrow \quad \varsigma = (\sigma_\infty)_{\{\varsigma<\infty,{}^pS_\varsigma=0,S_{\varsigma-}>0\}} < \infty$$

and

$$\varsigma < \infty, {}^pS_\varsigma = 0, \Delta_\varsigma A = 0 \quad \Rightarrow \quad \varsigma < \infty, {}^pS_\varsigma = 0, S_{\varsigma-} = 0$$
$$\Rightarrow \quad \varsigma = \varsigma_{\{\varsigma<\infty,{}^pS_\varsigma=0,S_{\varsigma-}=0\}} < \infty.$$

As a consequence,

$$\varsigma_{\{\varsigma<\infty,{}^pS_\varsigma=0\}} = (\sigma_\infty)_{\{\varsigma<\infty,{}^pS_\varsigma=0,S_{\varsigma-}>0\}} \wedge \varsigma_{\{\varsigma<\infty,{}^pS_\varsigma=0,S_{\varsigma-}=0\}}.$$

The stopping time $(\sigma_\infty)_{\{\varsigma<\infty,{}^pS_\varsigma=0,S_{\varsigma-}>0\}}$ is predictable because σ_∞ is predictable and $\{\varsigma < \infty, {}^pS_\varsigma = 0, S_{\varsigma-} > 0\} \in \mathcal{F}_{\sigma_\infty-}$. The stopping time $\varsigma_{\{\varsigma<\infty,{}^pS_\varsigma=0,S_{\varsigma-}=0\}}$ is predictable by the proof of Theorem 9.41 in He, Wang, and Yan (1992) (to clarify the connection with their setup, note that $\varsigma = \inf\{s > 0; S_s = 0\} = \inf\{s > 0; S_s = S_{s-}\}$, because S is a nonnegative supermartingale). Hence, $\varsigma_{\{\varsigma<\infty,{}^pS_\varsigma=0\}}$ is predictable as the minimum of two predictable stopping times. □

Last, we can state the following characterization (consistent with the findings of example 5.1) of the positivity of the Dolean-Dade exponential in Theorem 5.1.

Theorem 5.2 $\mathcal{E}(\frac{1}{{}^pS} \boldsymbol{\cdot} Q^d) > 0$ *on* $[0,T] \iff {}^pS_\varsigma = 0$ *on* $\{\varsigma \leq T\} \iff \varsigma_{\{\varsigma\leq T\}}$ *is a predictable stopping time.*

Proof. We know that $\mathcal{E}(\frac{1}{{}^pS} \boldsymbol{\cdot} Q^d) > 0$ on $[0,T]$ if and only if

$$\frac{1}{{}^pS_t} \Delta_t Q > -1, \quad t \in [0,T] \tag{5.13}$$

(with the convention $\frac{0}{0} = 0$). We know that $S - {}^pS = \Delta Q$ (see page 79 in Jeulin and Yor (1978)). Hence, for $t \in [0, \varsigma)$,

$$\frac{1}{{}^pS_t} \Delta_t Q = \frac{S_t - {}^pS_t}{{}^pS_t} = \frac{S_t}{{}^pS_t} - 1 > -1.$$

At ς, whenever it is finite and that $^pS_\varsigma > 0$, the above computation again applies, which implies

$$\frac{1}{^pS_\varsigma}\Delta_\varsigma Q = -1.$$

If $^pS_\varsigma = 0$, $\Delta_\varsigma Q = S_\varsigma - {}^pS_\varsigma = 0$. For $t > \varsigma$, $\Delta_t Q = 0$. Putting together these three observations yields that the condition (5.13) is equivalent to

$$^pS_\varsigma = 0 \text{ on } \{\varsigma \leq T\}. \tag{5.14}$$

By Lemma 5.6, the condition (5.14) implies that $\varsigma_{\{\varsigma \leq T\}}$ is predictable. Conversely, if $\varsigma_{\{\varsigma \leq T\}}$ is predictable, as it is greater than ς, the condition (5.14) holds by Lemma 5.5. □

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