Submitted to the Annals of Applied Probability

INVARIANT TIMES*

By Stéphane Crépey and Shiqi Song

Université d'Évry Val d'Essonne
Laboratoire de Mathématiques et Modélisation d'Évry
91037 Évry Cedex, France

Keywords: Random time, Progressive enlargement of filtration, Pseudo-stopping times.

Motivated by counterparty and credit risk applications, we define an invariant time as a stopping time such that local martingales with respect to a smaller filtration and a possibly changed probability measure, once stopped right before that time, are local martingales with respect to the original model filtration and probability measure. The possibility to change the measure provides an additional degree of freedom with respect to other classes of random times such as Cox times or pseudo-stopping times that are commonly used to model default times. We provide an Azéma supermartingale characterization of invariant times and we characterize the positivity of the stochastic exponential involved in a tentative measure change. We study the avoidance properties of invariant times and their connections with pseudo-stopping times.

1. Introduction. We define an invariant time θ as a stopping time in a bigger filtration $\mathbb{G}$ such that local martingales with respect to a smaller filtration $\mathbb{F}$ and a possibly changed probability measure, once stopped at $(\theta-)$ ("right before θ"), stay local martingales with respect to the original (bigger) model filtration and probability measure. As shown in Crépey and Song (2014a,2014b), invariant times offer a great flexibility and tractability for modeling default times in counterparty and credit risk applications. In this paper we provide an Azéma supermartingale characterization of invariant times. The core of the idea is to exploit the formal similarity between, on the one hand, the Girsanov martingale decomposition formula in the context of measure change and, on the other hand, the Jeulin-Yor martingale decomposition formula in the context of progressive enlargement of filtration, for devising a measure change that compensates in some sense a progressive enlargement of filtration. But this is only possible under suitable integrability conditions on the density process of a tentative measure change, which is exactly our main characterization result (see Theorem 3.1 and Corollary 4.1). Note that this Azéma supermartingale characterization of invariant times is not only of theoretical interest, it is used in Crépey and Song (2014a, 2014b) for studying the well posedness and numerical solution of counterparty risk related backward stochastic differential equations.

The above definition is evocative of the concept of pseudo-stopping times of Nikeghbali and Yor (2005), except that invariant times have an additional degree of freedom provided by the possibility of changing the measure in the martingale invariance condition (and also for the fact that we stop local martingales at $(\theta-)$ in this definition, instead of θ in

*This research benefited from the support of the "Chair Markets in Transition" under the aegis of Louis Bachelier laboratory, a joint initiative of École polytechnique, Université d'Évry Val d'Essonne and Fédération Bancaire Française. The authors are grateful to Monique Jeanblanc for her comments on a preliminary version of the manuscript.

MSC 2010 subject classifications: Primary 60G07; secondary 60G44.

the case of pseudo-stopping times; of course, this makes no difference in the case of times that have the avoidance property). Due to this possibility to change the measure, invariant times offer more flexibility than Cox times or pseudo-stopping times that are commonly used to model default times (see e.g. Bielecki and Rutkowski (2001) or Crépey, Bielecki, and Brigo (2014, Sect. 13.7)). Another conceptual difference is that pseudo-stopping times were devised in a spirit of progressive enlargement of filtration (studying in an enlarged filtration objects defined in a reference filtration $\mathbb{F}$), whereas we view invariant times in a spirit of reduction of filtration, i.e., given a stopping time θ relative to a full model filtration $\mathbb{G}$, separating the information that comes from θ from a reference information (cf. the problem of filtration shrinkage of Föllmer and Protter (2011) regarding the stability of local martingales by projection onto a smaller filtration). The idea of coupling a measure change with an enlargement (or reduction) of filtration can already be found in the literature about initial enlargement of filtration in relation with random times satisfying the density hypothesis of Jacod (1987)—times also used in the context of progressive enlargement of filtration as so called initial times in Jeanblanc and Le Cam (2009) and El Karoui, Jeanblanc, and Jiao (2010). But the measure change makes a density (or initial) time independent from the reference filtration, whereas in the case of invariant times it only produces the above mentioned martingale invariance property. From this point of view, invariant times seem to be less constrained than initial times.

The paper is organized as follows. In Sect. 2, we revisit the Barlow-Jeulin-Yor theory of progressive enlargement of filtration under a condition (B) relative to a subfiltration $\mathbb{F}$ of $\mathbb{G}$. In Sect. 3 we study a stronger condition (A) also involving a changed probability measure $\mathbb{P}$ and we characterize the Radon-Nikodym density $\frac{d\mathbb{P}}{d\mathbb{Q}}$ in terms of the Azéma supermartingale of θ. In Sect. 4 invariant times are formally introduced and studied based on the condition (A). We compare invariant times with pseudo-stopping times (Sect. 4.1) and we characterize the positivity of the Doléans-Dade exponential involved in a tentative measure change density $\frac{d\mathbb{P}}{d\mathbb{Q}}$ (Sect. 4.2). Appendix A provides an alternative proof of Theorem 3.1. An index of symbols is provided after the bibliography.

1.1. *Standing Assumptions and Notation.* The real line, half-line and the nonnegative integers are respectively denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$; $\frac{0}{0}=0$; $\mathcal{B}(\mathbb{R}^k)$ is the Borel σ algebra on $\mathbb{R}^k$ ($k \in \mathbb{N}$); $\boldsymbol{\lambda}$ is the Lebesgue measure on $\mathbb{R}_+$. We work on a space Ω equipped with a σ-field $\mathcal{A}$, a probability measure $\mathbb{Q}$ on $\mathcal{A}$ and a filtration $\mathbb{G}=(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ of sub-σ-fields of $\mathcal{A}$, satisfying the usual conditions. We use the terminology of the general theory of processes and of filtrations as given in the books by Dellacherie and Meyer (1975) and He, Wang, and Yan (1992). Footnotes are used for referring to comparatively standard results. We denote by $\mathcal{P}(\mathbb{F})$ and $\mathcal{O}(\mathbb{F})$ the predictable and optional σ-fields with respect to a filtration $\mathbb{F}$. The continuous and discontinuous components of a local martingale are denoted by $\cdot^c$ and $\cdot^d$. For any semimartingale Y and predictable, Y integrable process L, the stochastic integral process of L with respect to Y is denoted by $\int_0^{\cdot} L_t dY_t = \int_{(0,\cdot]} L_t dY_t = L\centerdot Y$, with the usual precedence convention $KL\centerdot Y = (KL)\centerdot Y$ if K is another predictable process such that KL is Y integrable. The stochastic exponential of a semimartingale X is denoted by $\mathcal{E}(X)$. For any càdlàg process Y, for any random time τ (nonnegative random variable), $\Delta_\tau Y$ represents the jump of Y at τ. As Dellacherie and Meyer (1975) or He et al. (1992), we use the convention that $Y_{0-}=Y_0$ (hence $\Delta_0 Y=0$) and we write Y^τ and $Y^{\tau-}$ for the process Y stopped at τ and at $\tau-$ ("right before τ"), i.e., respectively,

$$Y^\tau = Y\mathbb{1}_{[0,\tau)} + Y_\tau \mathbb{1}_{[\tau,+\infty)},\ \ Y^{\tau-} = Y\mathbb{1}_{[0,\tau)} + Y_{\tau-}\mathbb{1}_{[\tau,+\infty)}. \tag{1.1}$$

In particular, if τ' is another random time, one can check from the definition that

$$(Y^{\tau-})^{\tau'-} = Y^{\tau\wedge\tau'-}. \tag{1.2}$$

We also work with semimartingales on a predictable set of interval type $\mathcal{I}$ as of He et al. (1992, Sect. VIII.3) and, occasionally, with stochastic integrals on $\mathcal{I}$, where $Z = L \centerdot Y$ on $\mathcal{I}$, for semimartingales Y and Z on $\mathcal{I}$, means that

$$Z^{\tau_n} = L \centerdot (Y^{\tau_n}) \tag{1.3}$$

holds for at least one, or equivalently any, nondecreasing sequence of stopping times such that $\cup[0,\tau_n] = \mathcal{I}$ (the existence of at least one such sequence is ensured by He et al. (1992, Theorem 8.18 3))). We call compensator of a stopping time τ the compensator of $\mathbb{1}_{[\tau,\infty)}$. For $A \in \mathcal{G}_\tau$, we denote $\tau_A = \mathbb{1}_A\tau + \mathbb{1}_{A^c}\infty$, a $\mathbb{G}$ stopping time[1]. Unless otherwise stated, a function (or process) is real-valued; order relationships between random variables (respectively processes) are meant almost surely (respectively in the indistinguishable sense); a time interval is random (in particular, the graph of a random time τ is simply written $[\tau]$). We don't explicitly mention the domain of definition of a function when it is implied by the measurability, e.g. we write "a $\mathcal{B}(\mathbb{R}^k)$ measurable function h (or $h(x)$)" rather than "a $\mathcal{B}(\mathbb{R}^k)$ measurable function h defined on $\mathbb{R}$". For a function $h(\omega, x)$ defined on a product space $\Omega \times E$, we usually write $h(x)$ without ω (or h_t in the case of a stochastic process).

2. Condition (B).

Throughout the paper θ denotes a $\mathbb{G}$ stopping time with $\mathsf{J} = \mathbb{1}_{[0,\theta)}$, hence $\mathsf{J}_- = \mathbb{1}_{0<\theta}\mathbb{1}_{[0,\theta]}$. Let $\mathbb{F}$ be a subfiltration of $\mathbb{G}$ satisfying the usual conditions. We consider the following:

Condition (B). For any $\mathbb{G}$ predictable process L, there exists an $\mathbb{F}$ predictable process K, called the $\mathbb{F}$ predictable reduction[2] of L, such that $\mathbb{1}_{(0,\theta]}K = \mathbb{1}_{(0,\theta]}L$.

Note that the equality $\mathbb{1}_{(0,\theta]}K = \mathbb{1}_{(0,\theta]}L$ is in the sense of indistinguishability. But, as $\mathbb{F}$ satisfies the usual condition, we can find a version of K such that the equality holds everywhere[3]. The condition (B) is a relaxation of the classical progressive enlargement of filtration setup, where the bigger filtration $\mathbb{G}$ is simply the smaller reference filtration $\mathbb{F}$ progressively enlarged by θ, which implies (B). As an immediate consequence of this condition[4],

$$\{0 < \theta < \infty\} \cap \mathcal{G}_{\theta-} = \{0 < \theta < \infty\} \cap \mathcal{F}_{\theta-}.$$

But we can say more. The result that follows establishes the connection between the condition (B) and a classical condition in the theory of enlargement of filtrations, stated in terms of the auxiliary right-continuous[5] filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\in\mathbb{R}_+}$, where

$$\overline{\mathcal{F}}_t = \{B \in \mathcal{A} : \exists A \in \mathcal{F}_t, A \cap \{t < \theta\} = B \cap \{t < \theta\}\} \tag{2.1}$$

(see Dellacherie, Maisonneuve, and Meyer (1992)).

Lemma 2.1 *The subfiltration $\mathbb{F}$ satisfies the condition (B) if and only if $\mathbb{G}$ is a subfiltration of $\overline{\mathbb{F}}$.*

[1] Cf. Theorem 3.9 in He et al. (1992).
[2] Also known as pre-default process in the credit risk literature such as Bielecki and Rutkowski (2001).
[3] Cf. He et al. (1992, Theorem 4.26)).
[4] Cf. He et al. (1992, Corollary 3.23 2)).
[5] And complete under our assumption that $\mathbb{F}$ satisfies the usual conditions.

Proof. Suppose the condition (B). For any $t \in \mathbb{R}_+$, for any $B \in \mathcal{G}_t$, $\mathbb{1}_B \mathbb{1}_{(t,\infty)}$ is a $\mathbb{G}$ predictable process, with $\mathbb{F}$ predictable reduction K such that $\mathbb{1}_{(0,\theta]} \mathbb{1}_B \mathbb{1}_{(t,\infty)} = \mathbb{1}_{(0,\theta]} K \mathbb{1}_{(t,\infty)}$. Then $\mathbb{1}_B \mathbb{1}_{\{t<s\leq\theta\}} = K_s \mathbb{1}_{\{t<s\leq\theta\}}$, hence $\liminf_{s\downarrow t} \mathbb{1}_B \mathbb{1}_{\{t<s\leq\theta\}} = \liminf_{s\downarrow t} K_s \mathbb{1}_{\{t<s\leq\theta\}}$. But $\liminf_{s\downarrow t} \mathbb{1}_B \mathbb{1}_{\{t<s\leq\theta\}} = \mathbb{1}_B \mathbb{1}_{\{t<\theta\}}$ and $\liminf_{s\downarrow t} K_s \mathbb{1}_{\{t<s\leq\theta\}} = (\liminf_{s\downarrow t} K_s) \mathbb{1}_{\{t<\theta\}}$, which proves $B \in \overline{\mathcal{F}}_t$. Conversely (cf. Lemma 1 in Jeulin and Yor (1978)), suppose that $\mathbb{G}$ is a subfiltration of $\overline{\mathbb{F}}$. For any $t > 0$, for any $B \in \mathcal{G}_t$, let $A \in \mathcal{F}_t$ satisfy $B \cap \{t < \theta\} = A \cap \{t < \theta\}$, so that

$$\mathbb{1}_{(0,\theta]} \mathbb{1}_B \mathbb{1}_{(t,\infty)} = \mathbb{1}_{(0,\theta]} \mathbb{1}_A \mathbb{1}_{(t,\infty)}.$$

Note that $\mathbb{1}_A \mathbb{1}_{(t,\infty)}$ is an $\mathbb{F}$ predictable process. For any $B \in \mathcal{G}_0$, $\mathbb{1}_{(0,\theta]} \mathbb{1}_B \mathbb{1}_{\{0\}} = 0$, again an $\mathbb{F}$ predictable process. Since the processes $\mathbb{1}_B \mathbb{1}_{(t,\infty)}$, for $t > 0$ and $B \in \mathcal{G}_t$, and $\mathbb{1}_B \mathbb{1}_{\{0\}}$, for $B \in \mathcal{G}_0$, generate the $\mathbb{G}$ predictable σ-algebra[6], this proves the condition (B). ∎

The proofs of the progressive of enlargement results in Jeulin and Yor (1978) or Chapitre XX in Dellacherie, Maisonneuve, and Meyer (1992) only require that $\mathbb{G}$ be a subfiltration of $\overline{\mathbb{F}}$. Hence, in view of Lemma 2.1, all their results and proofs apply under the condition (B), which we postulate henceforth. Let ${}^o\cdot$ and ${}^p\cdot$ denote the $\mathbb{F}$ optional and predictable projections. In particular, $\mathsf{S} = {}^o\mathsf{J}$ represents the $\mathbb{F}$ Azéma supermartingale of θ, with canonical Doob-Meyer decomposition $\mathsf{S} = \mathsf{Q} - \mathsf{D}$, where Q denotes the martingale part (with $\mathsf{Q}_0 = \mathsf{S}_0$) and D represents the $\mathbb{F}$ drift of S (also the $\mathbb{F}$ dual predictable projection of $\mathbb{1}_{\{0<\theta\}} \mathbb{1}_{[\theta,\infty)}$). We recall the classical identities

$$({\rm 2.2}) \qquad {}^p(\mathsf{J}_-) = \mathsf{S}_- \text{ on } (0,\infty)$$

(see Jeulin (1980, page 63)) and (by predictable projection of the Doob-Meyer decomposition of S)

$$({\rm 2.3}) \qquad {}^p\mathsf{S} = \mathsf{Q}_- - \mathsf{D} = \mathsf{S}_- - \Delta\mathsf{D} = \mathsf{S} - \Delta\mathsf{Q}.$$

In particular, if θ has an intensity, then $\Delta\mathsf{D} = 0$ and ${}^p\mathsf{S} = \mathsf{S}_-$. Let

$$({\rm 2.4}) \qquad \varsigma = \inf\{s > 0; \mathsf{S}_s = 0\} = \inf\{s > 0; \mathsf{S}_{s-} = 0\}$$

(since S is a nonnegative supermartingale[7]). Beyond ς, one has $\mathsf{S} = \mathsf{S}_- = {}^p\mathsf{S} = 0$ (see Dellacherie and Meyer (1975, Chap. 6 no 17) regarding S and S_- and see Jacod (1979, Corollary 6.28) for the case of ${}^p\mathsf{S}$). Moreover, one has

$$({\rm 2.5}) \qquad \{\mathsf{S} > 0\} \subseteq \{{}^p\mathsf{S} > 0\} \subseteq \{\mathsf{S}_- > 0\}$$

and

$$({\rm 2.6}) \qquad \mathsf{S}_{\theta-} > 0 \text{ on } \{0 < \theta\}$$

(cf. Yor (1978, page 63)). Let

$$({\rm 2.7}) \qquad \varsigma_n = \inf\left\{s > 0; \mathsf{S}_s \leq \frac{1}{n}\right\} \ (n > 0),$$

so that (using the definitions)

$$({\rm 2.8}) \qquad \varsigma = \sup_n \varsigma_n, \ \ \{\mathsf{S}_- > 0\} \cup [0] = \cup_n [0, \varsigma_n].$$

[6]Cf. Theorem 3.21 in He et al. (1992).
[7]Cf. n°17 Chapitre VI in Dellacherie and Meyer (1975).

In particular,

$$\text{(2.9)}\qquad \begin{array}{l}\text{on } \{\mathsf{S}_0>0\}\text{ , we have } 0\in\{\mathsf{S}_->0\}\text{ , hence } \cup_n[0,\varsigma_n]=\{\mathsf{S}_->0\},\\ \text{on } \{\mathsf{S}_0=0,\}\text{ we have } \{\mathsf{S}_->0\}=\emptyset \text{ and } \cup_n[0,\varsigma_n]=[0].\end{array}$$

The next lemma assembles the main results that we need under the condition (B). In particular, the Jeulin formula (2.10) yields the $(\mathbb{G},\mathbb{Q})$ martingale part of an $(\mathbb{F},\mathbb{Q})$ martingale Q, in a classical spirit of enlargement of filtration (studying in an enlarged filtration objects defined in a reference filtration). For the reduction of filtration studied in this paper, the part 4) of the lemma addresses the inverse problem of knowing when an $\mathbb{F}$ semimartingale K is such that $K^{\theta-}$ is a $\mathbb{G}$ local martingale. On other multiplicative decompositions of the Azéma supermartingale S such as the one of part 5), see Nikeghbali and Yor (2006), Kardaras (2014) or Penner and Reveillac (2014).

Lemma 2.2 *Under the condition (B):*

1) *For any $\mathbb{G}$ stopping time τ, there exists an $\mathbb{F}$ stopping time σ, which we call the $\mathbb{F}$ reduction of τ, such that $\{\tau<\theta\}=\{\sigma<\theta\}\subseteq\{\tau=\sigma\}$.*
2) *Let $(E,\mathcal{E})$ be a measurable space. Any $\mathcal{P}(\mathbb{G})\otimes\mathcal{E}$ (respectively $\mathcal{O}(\mathbb{G})\otimes\mathcal{E}$) measurable function $g_t(\omega,x)$ admits a $\mathcal{P}(\mathbb{F})\otimes\mathcal{E}$ (respectively $\mathcal{O}(\mathbb{F})\otimes\mathcal{E}$) reduction, i.e. a $\mathcal{P}(\mathbb{F})\otimes\mathcal{E}$ (respectively $\mathcal{O}(\mathbb{F})\otimes\mathcal{E}$) measurable function $f_t(\omega,x)$ such that $\mathbb{1}_{(0,\theta]}f=\mathbb{1}_{(0,\theta]}g$ (respectively $\mathbb{1}_{[0,\theta)}f=\mathbb{1}_{[0,\theta)}g$, i.e. $\mathsf{J}f=\mathsf{J}g$) everywhere.*
3) *For any $\mathbb{F}$ martingale Q with integrable $\sqrt{[Q,Q]_\infty}$, the process*

$$\text{(2.10)}\qquad Q^{\theta-}-\frac{\mathsf{J}_-}{\mathsf{S}_-}\centerdot\langle Q,\mathsf{Q}\rangle$$

is a $\mathbb{G}$ uniformly integrable martingale, where $\langle Q,\mathsf{Q}\rangle$ is computed with respect to $(\mathbb{F},\mathbb{Q})$.
4) *Let M be a $\mathbb{G}$ local martingale on $\mathbb{R}_+$ with $\Delta_\theta M=0$. For any $\mathbb{F}$ optional reduction K of M, K is an $\mathbb{F}$ semimartingale on $\{\mathsf{S}_->0\}$, $\mathbb{1}_{\{\mathsf{S}_->0\}}K_-$ is an $\mathbb{F}$ predictable reduction of M_- and $\mathsf{S}_-\centerdot K+[\mathsf{S},K]$ is an $\mathbb{F}$ local martingale on $\{\mathsf{S}_->0\}$. Conversely, for any $\mathbb{F}$ semimartingale K on $\{\mathsf{S}_->0\}$ such that $\mathsf{S}_-\centerdot K+[\mathsf{S},K]$ is an $\mathbb{F}$ local martingale on $\{\mathsf{S}_->0\}$, $K^{\theta-}$ is a $\mathbb{G}$ local martingale on $\mathbb{R}_+$.*
5) *The Azéma supermartingale S admits the multiplicative decomposition*

$$\mathsf{S}=\mathsf{S}_0\mathcal{E}(-\frac{1}{\mathsf{S}_-}\centerdot\mathsf{D})\mathcal{E}(\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})\ \textit{on}\ \{{}^p\mathsf{S}>0\},$$

where $\mathcal{E}(\cdot)$ stands for the stochastic exponential of a semimartingale.

Proof. 1) is proven in Chapitre XX, n°75 a), in Dellacherie, Maisonneuve, and Meyer (1992); 2) is a consequence of Chapitre XX, n°75 d) of the same reference in the case of processes, the parameterized extension being deduced by monotone class theorem; 3) is proven in Chapitre XX, n°77 b), in Dellacherie et al. (1992); 4) is proven in Song (2014, Lemmas 6.5 and 6.8); 5) is Song (2014, Lemma 3.5) (in particular, $\frac{1}{\mathsf{S}_-}\centerdot\mathsf{D}$ is well defined on $\{\mathsf{S}_->0\}$, by (2.9), and, by Jacod (1979, Corollary 6.28), $\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q}$ is well defined on $\{{}^p\mathsf{S}>0\}$). ∎

In view of Lemma 2.2 3) and 4), the transform $\cdot^{\theta-}$ is more natural than $\cdot^{\theta}$ in the context of reduction of filtration, i.e. when the problem is, given a stopping time θ relative to a full model filtration $\mathbb{G}$, to separate the information that comes from θ from a reference information $\mathbb{F}$ in order to simplify the computations. By contrast, $\cdot^{\theta}$ is more commonly used in the progressive enlargement of filtration literature, i.e. for constructing a stopping time in a larger filtration $\mathbb{G}$ given a reference filtration $\mathbb{F}$. Note that Lemma 2.2 3) and 4) are progressive enlargement analogs of formally similar results regarding the transformation of martingales through measure change, the Azéma supermartingale S playing the role of the measure change density. Specifically, the Jeulin formula (2.10) is a counterpart to the Girsanov formula and Lemma 2.2 4) is a counterpart to He et al. (1992, Theorem 12.18 4)). This analogy can be pushed further by effectively representing S as a subdensity (see Yoeurp (1985) and Song (1987,2013)).

Lemma 2.3 *Two $\mathbb{F}$ predictable (resp. optional) processes K and K' undistinguishable until θ (resp. before θ) are undistinguishable on $\{\mathsf{S}_- > 0\}$ (resp. $\{\mathsf{S} > 0\}$).*

Proof. Otherwise, the optional section theorem would imply the existence of an $\mathbb{F}$ stopping time σ such that $\mathbb{E}[\mathbb{1}_{K_\sigma \neq K'_\sigma}\mathsf{S}_{\sigma-}] \neq 0$ (resp. $\mathbb{E}[\mathbb{1}_{K_\sigma \neq K'_\sigma}\mathsf{S}_{\sigma}] \neq 0$), in contradiction with

$$\mathbb{E}\left[\mathbb{1}_{K_\sigma \neq K'_\sigma}\mathsf{S}_{\sigma-}\right] = \mathbb{E}\left[\mathbb{1}_{K \neq K'}\mathsf{S}_{-} \centerdot \mathbb{1}_{[\sigma,+\infty)}\right] = \mathbb{E}\left[\mathbb{1}_{K \neq K'}\mathsf{J}_{-} \centerdot \mathbb{1}_{[\sigma,+\infty)}\right] = \mathbb{E}\left[\mathbb{1}_{K_\sigma \neq K'_\sigma}\mathsf{J}_{\sigma-}\right] = 0$$

(resp.

$$\mathbb{E}\left[\mathbb{1}_{K_\sigma \neq K'_\sigma}\mathsf{S}_{\sigma}\right] = \mathbb{E}\left[\mathbb{1}_{K \neq K'}\mathsf{S} \centerdot \mathbb{1}_{[\sigma,+\infty)}\right] = \mathbb{E}\left[\mathbb{1}_{K \neq K'}\mathsf{J} \centerdot \mathbb{1}_{[\sigma,+\infty)}\right] = \mathbb{E}\left[\mathbb{1}_{K_\sigma \neq K'_\sigma}\mathsf{J}_{\sigma}\right] = 0, \ )$$

where Theorems 5.4 and 5.16 1) (resp. 2)) in He et al. (1992) and the formula (2.2) (resp. the definition $\mathsf{S} = {}^{o}\mathsf{J}$) were used in the next-to-last equality.∎

3. Condition (A).

Let

$$\eta = \inf\{s > 0; {}^{p}\mathsf{S}_s = 0,\, \mathsf{S}_{s-} > 0\} = \inf\{s > 0; \mathsf{S}_{s-} = \Delta_s\mathsf{D} > 0\}, \tag{3.1}$$

by (2.3).

Lemma 3.1 *We have*

$$\eta = \inf\{s \in \{\mathsf{S}_- > 0\}; \mathcal{E}(-\frac{1}{\mathsf{S}_-} \centerdot \mathsf{D})_s = 0\}. \tag{3.2}$$

Proof. The stochastic exponential $\mathcal{E}(-\frac{1}{\mathsf{S}_-} \centerdot \mathsf{D})$ cancels at $t \in \{\mathsf{S}_- > 0\}$ if and only if

$$\Delta_t\Big(-\frac{1}{\mathsf{S}_-} \centerdot \mathsf{D}\Big) = \frac{-1}{\mathsf{S}_{t-}}\Delta_t\mathsf{D} = -1, \quad \text{i.e. } \mathsf{S}_{t-} = \Delta_t\mathsf{D}.$$

Hence,

$$\begin{aligned}\inf\{s \in \{\mathsf{S}_- > 0\}; \mathcal{E}(-\frac{1}{\mathsf{S}_-} \centerdot \mathsf{D})_s = 0\} &= \inf\{s \in \{\mathsf{S}_- > 0\}; \mathsf{S}_{s-} = \Delta_s\mathsf{D}\} \\ &= \inf\{s \in \{\mathsf{S}_- > 0\}; \mathsf{S}_{s-} = \Delta_s\mathsf{D} > 0\} = \inf\{s > 0; \mathsf{S}_{s-} = \Delta_s\mathsf{D} > 0\} = \eta. \ ∎\end{aligned}$$

Using the results of Jacod (1979, Chapter 6), we check that

$$\{\mathsf{S}_- > 0\} \setminus \{{}^{p}\mathsf{S} > 0\} = [\eta] \tag{3.3}$$

and

$$\eta = \varsigma \text{ on } \{\eta < \infty\} \text{ , hence } \eta \geq \varsigma. \tag{3.4}$$

In addition, the interval $\{{}^p\mathsf{S} > 0\}$ is predictable and there exists a nondecreasing sequence $(\zeta_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$ stopping times such that (cf. the formula (6.24) in Jacod (1979))

$$\{{}^p\mathsf{S} > 0\} \cup [0] = \cup_n [0, \zeta_n] \text{ and } {}^p\mathsf{S} \text{ is bounded away from 0 on } [0, \zeta_n] \text{ for every } n. \tag{3.5}$$

In particular (note $\mathsf{S}_0 = {}^p\mathsf{S}_0$),

$$\begin{array}{l} \text{on } \{\mathsf{S}_0 > 0\} \text{ , we have } 0 \in \{{}^p\mathsf{S} > 0\}, \text{ hence } \cup_n [0, \zeta_n] = \{{}^p\mathsf{S} > 0\}, \\ \text{on } \{\mathsf{S}_0 = 0\} \text{ (hence } \{{}^p\mathsf{S} > 0\} = \emptyset), \text{ it holds that } \cup_n [0, \zeta_n] = [0]. \end{array} \tag{3.6}$$

Letters of the "m" family are used to denote $\mathbb{G}$ local martingales, which are all defined in reference to the original probability measure $\mathbb{Q}$. Regarding $\mathbb{F}$, we will also deal with another probability measure $\mathbb{P}$, so that letters of the "q" (including Q that was introduced above as the $(\mathbb{F},\mathbb{Q})$ martingale part of S) and "p" family are used to denote $(\mathbb{F},\mathbb{Q})$ and $(\mathbb{F},\mathbb{P})$ local martingales, respectively. Given a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$, we denote by q the $(\mathbb{F},\mathbb{Q})$ martingale of the density functions $\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_{t\wedge T}}, t \in \mathbb{R}_+$. We also introduce $p = \frac{1}{q}$ and the stochastic logarithms p and q such that

$$p = p_0\mathcal{E}(\mathsf{p}),\ q = q_0\mathcal{E}(\mathsf{q}),\ \ \mathsf{p}_0 = \mathsf{q}_0 = 0, \tag{3.7}$$

so that p and p (resp. q and q) are $(\mathbb{F},\mathbb{P})$ (resp. $(\mathbb{F},\mathbb{Q})$) local martingales on $[0,T]$. Recall that all our stochastic integrals start from 0 at time 0.

Lemma 3.2 *For any probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$, we have*

$$\mathbb{1}_{\{{}^p\mathsf{S}>0\}} \centerdot (\frac{q}{q_0}) = \mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}} \centerdot \mathsf{q}) - 1 \text{ on } [0,T]. \tag{3.8}$$

In addition, we have

$$q = q_0\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q}) \text{ on } \{{}^p\mathsf{S} > 0\} \cap [0,T] \tag{3.9}$$

if and only if

$$\mathbb{1}_{\{{}^p\mathsf{S}>0\}} \centerdot \mathsf{q} = \mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q} \text{ on } [0,T] \tag{3.10}$$

if and only if

$$\mathbb{1}_{\{{}^p\mathsf{S}>0\}} \centerdot (\frac{q}{q_0}) = \mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}} \centerdot \mathsf{q}) - 1 = \mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q}) - 1 \text{ on } [0,T] \tag{3.11}$$

if and only if

$${}^p\mathsf{S} \centerdot \mathsf{q} = \mathsf{Q} - \mathsf{Q}_0 \text{ on } [0,T]. \tag{3.12}$$

***Proof*.** The $\mathbb{F}$ predictable and bounded process $\mathbb{1}_{\{^p\mathsf{S}>0\}}$ integrates any $(\mathbb{F},\mathbb{Q})$ local martingale on $[0,T]$. Therefore, by the dominated convergence theorem for stochastic integrals[8], the identity

$$\mathbb{1}_{[0,\zeta_n]} \centerdot \frac{q}{q_0} = \frac{q^{\zeta_n}}{q_0} - 1 = \mathcal{E}(\mathsf{q}^{\zeta_n}) - 1 = \mathcal{E}(\mathbb{1}_{[0,\zeta_n]} \centerdot \mathsf{q}^{\zeta}) - 1 \ (n \in \mathbb{N})$$

passes to the limit, as $n \to \infty$, into (3.8). Next we prove that (3.9), i.e.

$$\mathsf{q} = \frac{1}{^p\mathsf{S}} \centerdot \mathsf{Q} \text{ on } \{^p\mathsf{S} > 0\} \cap [0,T], \tag{3.13}$$

is not only obviously implied (cf. (1.3)) by (3.10), but is actually equivalent to it. By definition, (3.13) means that (cf. (3.6))

$$\mathbb{1}_{[0,\zeta_n \wedge T]} \centerdot \mathsf{q} = \mathbb{1}_{[0,\zeta_n \wedge T]} \frac{1}{^p\mathsf{S}} \centerdot \mathsf{Q} \ (n \in \mathbb{N}). \tag{3.14}$$

For $t \in [0,T]$, we have by monotone convergence in $[0,+\infty]$

$$\sqrt{\int_0^t \mathbb{1}_{\{^p\mathsf{S}_s>0\}} \frac{1}{^p\mathsf{S}_s^2} d[\mathsf{Q},\mathsf{Q}]_s} = \lim_{n\to\infty} \sqrt{\int_0^{t\wedge\zeta_n} \frac{1}{^p\mathsf{S}_s^2} d[\mathsf{Q},\mathsf{Q}]_s} \le \sqrt{[\mathsf{q},\mathsf{q}]_t},$$

by (3.14). Since the process $\sqrt{[\mathsf{q},\mathsf{q}]}$ is integrable in $\mathbb{R}_+$, so is in turn $\sqrt{\mathbb{1}_{\{^p\mathsf{S}>0\}} \frac{1}{^p\mathsf{S}^2} \centerdot [\mathsf{Q},\mathsf{Q}]}$, i.e. $\mathbb{1}_{\{^p\mathsf{S}>0\}} \frac{1}{^p\mathsf{S}}$ is Q integrable in $\mathbb{F}$ on $[0,T]$ (see He et al. (1992, Theorem 9.2)), which implies (3.10), obviously equivalent to the right-hand side identity in (3.11), where the left-hand side identity is simply (3.8). Last, (3.10) is equivalent to

$$\mathbb{1}_{\{^p\mathsf{S}>0\}} {^p\mathsf{S}} \centerdot \mathsf{q} = \mathbb{1}_{\{^p\mathsf{S}>0\}} \centerdot \mathsf{Q} \text{ on } [0,T]. \tag{3.15}$$

Moreover,

$$\mathbb{1}_{\{^p\mathsf{S}=0\}} {^p\mathsf{S}} \centerdot \mathsf{q} = 0 = \mathbb{1}_{\{^p\mathsf{S}=0\}} \centerdot \mathsf{Q} \text{ on } [0,T], \tag{3.16}$$

because Q is constant on $\{^p\mathsf{S} = 0\} \subseteq \{\mathsf{S} = 0\}$ (cf. (2.5)). Hence, (3.15) is equivalent to (3.12). ∎

Definition 3.1 Given a positive constant T, we say that $(\mathbb{F},\mathbb{P})$ is a reduced stochastic basis of $(\mathbb{G},\mathbb{Q})$ if $\mathbb{F}$ is a subfiltration of $\mathbb{G}$ satisfying the usual conditions and the condition (B) and if $\mathbb{P}$ is a probability measure equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$.

We introduce the following:

Condition (A). $(\mathbb{F},\mathbb{P})$ is a reduced stochastic basis of $(\mathbb{G},\mathbb{Q})$ such that for any $(\mathbb{F},\mathbb{P})$ local martingale P, $P^{\theta-}$ is a $(\mathbb{G},\mathbb{Q})$ local martingale on $[0,T]$.

If θ is $\mathbb{G}$ predictable, the reduced stochastic basis $(\mathbb{F},\mathbb{P}) = (\mathbb{G},\mathbb{Q})$ obviously satisfies the condition (A). But we are mostly interested in the case where θ has a nontrivial totally inaccessible component. The condition (A) is nonstandard in the enlargement of filtration

[8]Cf. He et al. (1992, Theorem 9.30).

literature. This condition raises questions such as the materiality of stopping at $(\theta-)$ rather than at θ in its definition (in other words, can the condition (A) be satisfied in cases where $(\mathbb{F},\mathbb{P})$ martingales really jump at θ). Another natural question is the connection between the condition (A) and the notion of pseudo-stopping time in Nikeghbali and Yor (2005). Our next result provides a characterization of the condition (A) in terms of the Azéma supermartingale S of θ. Here we provide a concise proof using auxiliary results of Song (2014) and Jacod (1979, Chapter 6). A longer but more elementary proof is given in Appendix A.

Theorem 3.1 *1) A reduced stochastic basis* $(\mathbb{F},\mathbb{P})$ *of* $(\mathbb{G},\mathbb{Q})$ *satisfies the condition (A) if and only if* (3.9) *(i.e.* (3.10)*,* (3.11) *or* (3.12)*) holds.*

2) Given a subfiltration $\mathbb{F}$ *of* $\mathbb{G}$ *satisfying the usual conditions and the condition (B), there exists a probability measure* $\mathbb{P}$ *equivalent to* $\mathbb{Q}$ *on* $\mathcal{F}_T$ *such that the reduced stochastic basis* $(\mathbb{F},\mathbb{P})$ *of* $(\mathbb{G},\mathbb{Q})$ *satisfies the condition (A) if and only if*

(3.17)
$$\begin{cases} \mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}} \text{ is } \mathsf{Q} \text{ integrable on } [0,T] \text{ with respect to } (\mathbb{F},\mathbb{Q}) \text{ and} \\ \mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q}) \text{ is a positive true martingale on } [0,T] \text{ with respect to } (\mathbb{F},\mathbb{Q}). \end{cases}$$

In this case, a probability measure $\mathbb{P}$ *such that* $(\mathbb{F},\mathbb{P})$ *satifies the condition (A) is given by the* $\mathbb{Q}$ *density* $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})_T$ *on* $\mathcal{F}_T$*.*

Proof. 1) The condition (A) says that, for any $(\mathbb{F},\mathbb{P})$ local martingale P, $(P^{\theta-})^T=(P^T)^{\theta-}$ is a $(\mathbb{G},\mathbb{Q})$ local martingale. By Lemma 2.2 4), this property holds if and only if

$$\mathsf{S}_-\centerdot P^T+[\mathsf{S},P^T] \text{ is an } (\mathbb{F},\mathbb{Q}) \text{ local martingale on } \{\mathsf{S}_->0\}.$$

By the integration by parts formula, this is equivalent to

$$P^T\mathsf{S}+P^T_-\centerdot\mathsf{D} \text{ is an } (\mathbb{F},\mathbb{Q}) \text{ local martingale on } \{\mathsf{S}_->0\}. \tag{3.18}$$

Note that P is an $(\mathbb{F},\mathbb{P})$ local martingale on $[0,T]$ if and only if there exists an $(\mathbb{F},\mathbb{Q})$ local martingale Q such that $P=Qp$ on $[0,T]$. Hence, (3.18) is equivalent to $Q^T(p^T\mathsf{S}+p^T_-\centerdot\mathsf{D})=(Qp)^T\mathsf{S}+(Qp)^T_-\centerdot\mathsf{D}+(p^T_-\centerdot\mathsf{D})\centerdot Q^T$ (by integration by parts) being an $(\mathbb{F},\mathbb{Q})$ local martingale on $\{\mathsf{S}_->0\}$, for any $(\mathbb{F},\mathbb{Q})$ local martingale Q. As a consequence, the condition (A) is equivalent to $p\mathsf{S}+p_-\centerdot\mathsf{D}$ being an $(\mathbb{F},\mathbb{Q})$ local martingale on $\{\mathsf{S}_->0\}\cap[0,T]$ orthogonal to all the others, i.e.

$$p\mathsf{S}+p_-\centerdot\mathsf{D} \text{ constant on } \{\mathsf{S}_->0\}\cap[0,T]. \tag{3.19}$$

Noting that $p\mathsf{S}+p_-\centerdot\mathsf{D}=p\mathsf{S}+(p\mathsf{S})_-\frac{1}{\mathsf{S}_-}\centerdot\mathsf{D}$, (3.19) is equivalent to

$$p\mathsf{S}=p_0\mathsf{S}_0\mathcal{E}(-\frac{1}{\mathsf{S}_-}\centerdot\mathsf{D}) \text{ on } \{\mathsf{S}_->0\}\cap[0,T]. \tag{3.20}$$

Recall (3.1) through (3.4). If η is finite, then $\mathsf{S}_\eta=0$, by (3.4), whereas (3.2) yields $\mathcal{E}(-\frac{1}{\mathsf{S}_-}\centerdot\mathsf{D})_\eta=0$, so that one has the trivial equality $(p\mathsf{S})_\eta=p_0\mathsf{S}_0\mathcal{E}(-\frac{1}{\mathsf{S}_-}\centerdot\mathsf{D})_\eta=0$, independent of the condition (A). Hence, the identity (3.20) is equivalent to the identity on the "smaller" set (cf. (3.3))

$$p\mathsf{S}=p_0\mathsf{S}_0\mathcal{E}(-\tfrac{1}{\mathsf{S}_-}\centerdot\mathsf{D}) \text{ on } \{{}^p\mathsf{S}>0\}\cap[0,T], \tag{3.21}$$

i.e. in view of Lemma 2.2 5)

$$p\mathsf{S}_0\mathcal{E}(-\frac{1}{\mathsf{S}_-}\centerdot\mathsf{D})\mathcal{E}(\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q}) = p_0\mathsf{S}_0\mathcal{E}(-\frac{1}{\mathsf{S}_-}\centerdot\mathsf{D}) \text{ on } \{{}^p\mathsf{S}>0\}\cap[0,T],$$

i.e. (3.9) since η, the first zero of $\mathcal{E}(-\frac{1}{\mathsf{S}_-}\centerdot\mathsf{D})$ on $\{\mathsf{S}_->0\}$, does not belong to $\{{}^p\mathsf{S}>0\}$ (if $\eta<\infty$; cf. (3.2) and (3.3)).

2) We use $\mathcal{E}^\star$ as shorthand notation for $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})$. Assuming the existence of a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that $(\mathbb{F},\mathbb{P})$ satisfies (A), then (3.9) holds (by part 1)), meaning by definition that, for each $n\in\mathbb{N}$ (cf. (1.3) and (3.6)),

$$\mathbb{1}_{[0,\zeta_n]}\centerdot\frac{q}{q_0} = \frac{q^{\zeta_n}}{q_0}-1 = (\mathcal{E}^\star)^{\zeta_n}-1 = \mathbb{1}_{[0,\zeta_n]}\centerdot\mathcal{E}^\star.$$

By the dominated convergence theorem for stochastic integrals, for each $s\ge 0$:

$$\lim_n \mathbb{1}_{[0,\zeta_n]}\centerdot q_s = \mathbb{1}_{\{{}^p\mathsf{S}>0\}}\centerdot q_s.$$

Hence $\mathbb{1}_{[0,\zeta_n]}\centerdot\mathcal{E}^\star_s$ converges, so that $\mathbb{1}_{\{{}^p\mathsf{S}>0\}}$ is $\mathcal{E}^\star$ integrable on $\mathbb{R}_+$ and

$$\lim_n \mathbb{1}_{[0,\zeta_n]}\centerdot\mathcal{E}^\star_s = \mathbb{1}_{\{{}^p\mathsf{S}>0\}}\centerdot\mathcal{E}^\star_s = \mathcal{E}^\star_s-1. \tag{3.22}$$

Moreover, since q is a true martingale, these convergences hold not only almost surely, but also in L^1. Therefore, by taking $\mathcal{F}_t$ conditional expectations for $t\le s$ in (3.22), we obtain:

$$\begin{aligned}&\mathbb{E}[q_0+q_0\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\centerdot\mathcal{E}^\star_s|\mathcal{F}_t] = \lim\nolimits_n \mathbb{E}[q_0+q_0\mathbb{1}_{[0,\zeta_n]}\centerdot\mathcal{E}^\star_s|\mathcal{F}_t] = q_0\lim\nolimits_n\mathbb{E}[\mathcal{E}^\star_{\zeta_n\wedge s}|\mathcal{F}_t]\\ &= q_0\lim\nolimits_n\mathcal{E}^\star_{\zeta_n\wedge t} = q_0(1+\lim\nolimits_n\mathbb{1}_{[0,\zeta_n]}\centerdot\mathcal{E}^\star_t) = q_0\mathcal{E}^\star_t,\end{aligned}$$

where the true martingality of $q_0(\mathcal{E}^\star)^{\zeta_n}=q^{\zeta_n}$ was used for passing to the second line. Hence, $\mathcal{E}^\star$ is an $(\mathbb{F},\mathbb{Q})$ true martingale. In addition, (3.11) yields

$$\mathcal{E}^\star = 1+\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\centerdot(\frac{q}{q_0}) = \mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\centerdot\mathsf{q})>0 \text{ on } [0,T],$$

where the inequality holds because, for any $t\ge 0$, $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\centerdot\mathsf{q})_t=0$ if and only if $\mathbb{1}_{\{{}^p\mathsf{S}_t>0\}}\Delta_t\mathsf{q}=-1$, i.e. $\Delta_t\mathsf{q}=-1$, in contradiction with $q_t=\mathcal{E}(\mathsf{q})_t>0$. In conclusion, (3.17) holds. Conversely, assuming (3.17), defining a probability measure $\mathbb{P}$ by the $\mathbb{Q}$ density $\mathcal{E}^\star_T$ on $\mathcal{F}_T$, then $(\mathbb{F},\mathbb{P})$ satifies the condition (A), by part 1). ∎

Note that supposing (3.17), unless $\{{}^p\mathsf{S}>0\}$ is a positive process, there is no uniqueness of $\mathbb{P}$ such that $(\mathbb{F},\mathbb{P})$ satisfies the condition (A). In fact, for any $(\mathbb{F},\mathbb{Q})$ local martingale Q such that $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}=0\}}\centerdot Q)$ is a positive $(\mathbb{F},\mathbb{Q})$ true martingale on $[0,T]$,

$$\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})_T\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}=0\}}\centerdot Q)_T$$

yields the density of another probability measure $\mathbb{P}$ such that $(\mathbb{F},\mathbb{P})$ satisfies the condition (A).

Corollary 3.1 *Let there be given a subfiltration $\mathbb{F}$ of $\mathbb{G}$ satisfying the usual conditions and the condition (B).*

1) *If θ has an intensity and $\mathbb{Q}(\theta \le T) > 0$, then $(\mathbb{F} = \mathbb{G}, \mathbb{P})$ doesn't satisfy the condition (A), whatever the probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$.*

2) *$(\mathbb{F}, \mathbb{P} = \mathbb{Q})$ satisfies the condition (A) for all $T > 0$ if and only if $\mathsf{Q} = \mathsf{S}_0$.*

Proof. 1) In the case where $\mathbb{F} = \mathbb{G}$ and θ has an intensity, we have:

$$\mathsf{S} = \mathsf{J},\ \ \mathsf{D} = \mathsf{D}^\theta \text{ continuous },\ \ {}^p\mathsf{S} = \mathsf{J}_-,\ \ \mathsf{Q} = \mathsf{J} + \mathsf{D}.$$

Hence

$$\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q})_t = \mathcal{E}(\mathsf{Q})_t = e^{\mathsf{Q}_t - \mathsf{Q}_0}\prod_{s\le t}(1+\Delta_s\mathsf{Q})e^{-\Delta_s\mathsf{Q}} = e^{\mathsf{J}_t+\mathsf{D}_t-1}\mathsf{J}_t e^{(1-\mathsf{J}_t)} = e^{D_t}\mathsf{J}_t,$$

which vanishes at θ on $\{\theta \le T\}$. Therefore, in view of Theorem 3.1 2), the condition (A) cannot hold unless $\mathbb{Q}(\theta \le T) = 0$.
2) In the case where $\mathbb{P} = \mathbb{Q}$, we have $q = 1$. Hence, in view of Theorem 3.1 1), the condition (A) holds for all $T > 0$ if and only if Q is constant on $\{{}^p\mathsf{S} > 0\}$. But, as a general fact, if Q is constant on $\{{}^p\mathsf{S} > 0\}$, then ${}^p\mathsf{S} = \mathsf{S}$ and Q is constant. In fact, if Q is constant on $\{{}^p\mathsf{S} > 0\}$, then ${}^p\mathsf{S} = \mathsf{S}$ holds on $\{{}^p\mathsf{S} > 0\}$, therefore $\{{}^p\mathsf{S} > 0\} \cap \{\mathsf{S} > 0\} = \{{}^p\mathsf{S} > 0\}$, i.e. $\{{}^p\mathsf{S} > 0\} \subseteq \{\mathsf{S} > 0\}$, whereas the converse inclusion holds in general (cf. (2.5)). Hence, $\{{}^p\mathsf{S} > 0\} = \{\mathsf{S} > 0\}$ and ${}^p\mathsf{S} = \mathsf{S}$. Therefore, $\mathsf{Q} = \mathsf{Q}_0 = \mathsf{S}_0$ on $\{\mathsf{S} > 0\}$. In addition, Q (like D) is constant on $\{\mathsf{S} = 0\}$. In view of the continuity of Q that follows from (2.3) in case ${}^p\mathsf{S} = \mathsf{S}$, we conclude that $\mathsf{Q} = \mathsf{Q}_0 = \mathsf{S}_0$ everywhere. ∎

Corollary 3.2 *Assume the condition (A).*

1) *A process P is an $(\mathbb{F}, \mathbb{P})$ local martingale on $\{{}^p\mathsf{S} > 0\} \cap [0, T]$ if and only if ${}^p\mathsf{S}\centerdot P + [\mathsf{Q}, P]$ is an $(\mathbb{F}, \mathbb{Q})$ local martingale on $\{{}^p\mathsf{S} > 0\} \cap [0, T]$,*

2) *In the special case where θ has an intensity, this condition reduces to $\mathsf{S}_-\centerdot P + [\mathsf{S}, P]$ being an $(\mathbb{F}, \mathbb{Q})$ local martingale on $\{\mathsf{S}_- > 0\} \cap [0, T]$. In addition, we have $\varsigma_n < \varsigma$ $(n \in \mathbb{N})$ and*

$$\{\mathsf{S} > 0\} = \{{}^p\mathsf{S} > 0\} = \{\mathsf{S}_- > 0\} = [0, \varsigma). \tag{3.23}$$

Proof. 1) On $\{{}^p\mathsf{S} > 0\} \cap [0, T]$,

$$\begin{aligned} qP &= P_-\centerdot q + q_-\centerdot P + [q, P] \\ &= P_-\centerdot q + q_-\centerdot P + q_-\tfrac{1}{{}^p\mathsf{S}}\centerdot[\mathsf{Q}, P], \end{aligned}$$

where the second equality comes from Theorem 3.1 1) and (3.12). Hence, qP is an $(\mathbb{F}, \mathbb{Q})$ local martingale on $\{{}^p\mathsf{S} > 0\} \cap [0, T]$, i.e. P is an $(\mathbb{F}, \mathbb{P})$ local martingale on $\{{}^p\mathsf{S} > 0\} \cap [0, T]$, if and only if $q_-\centerdot P + q_-\frac{1}{{}^p\mathsf{S}}\centerdot[\mathsf{Q}, P]$ is an $(\mathbb{F}, \mathbb{Q})$ local martingale on $\{{}^p\mathsf{S} > 0\} \cap [0, T]$. Recalling the sequence ζ_n introduced in (3.5) and introducing a nondecreasing sequence of $\mathbb{F}$ stopping times σ_n tending to infinity and reducing q_- and its inverse to bounded processes on $[0, T]$, this means that $q_-\centerdot P^{\zeta_n\wedge\sigma_n} + q_-\frac{1}{{}^p\mathsf{S}}\centerdot[\mathsf{Q}, P]^{\zeta_n\wedge\sigma_n}$ is an $(\mathbb{F}, \mathbb{Q})$ local martingale on $[0, T]$ for every n, i.e. (recalling from (3.5) that ${}^p\mathsf{S}$ is bounded away from 0 on $[0, \zeta_n]$) ${}^p\mathsf{S}\centerdot P^{\zeta_n\wedge\sigma_n} + [\mathsf{Q}, P]^{\zeta_n\wedge\sigma_n}$ is an $(\mathbb{F}, \mathbb{Q})$ local martingale on $[0, T]$ for every n, i.e. ${}^p\mathsf{S}\centerdot P + [\mathsf{Q}, P]$ is an $(\mathbb{F}, \mathbb{Q})$ local martingale on $\{{}^p\mathsf{S} > 0\} \cap [0, T]$.

2) In the special case where θ has an intensity, then D is continuous, hence $[\cdot,\mathsf{D}]=0$ and (2.3) implies ${}^p\mathsf{S}=\mathsf{S}_-$. so that the equivalence of part 1) is rewritten as in part 2). Let $P=\mathbb{1}_{0<\varsigma_n=\varsigma<C}\mathbb{1}_{[\varsigma,\infty)}$ (C constant positive). We compute

$$\mathsf{S}_-\centerdot P+[\mathsf{S},P]=\mathsf{S}_{\varsigma-}\mathbb{1}_{0<\varsigma_n=\varsigma<C}\mathbb{1}_{[\varsigma,\infty)}+\Delta_\varsigma\mathsf{S}\mathbb{1}_{0<\varsigma_n=\varsigma<C}\mathbb{1}_{[\varsigma,\infty)}=0$$

(for $\mathsf{S}_\varsigma=0$). In particular, by application of the just proven equivalence, P is an $(\mathbb{F},\mathbb{P})$ local, hence true (as bounded), martingale. Therefore, noting $P_0=0$ and $P_C=\mathbb{1}_{0<\varsigma_n=\varsigma<C}\mathbb{1}_{C\geq\varsigma}=\mathbb{1}_{0<\varsigma_n=\varsigma<C}$,

$$0=\mathbb{E}^{\mathbb{P}}[P_C]=\mathbb{P}[0<\varsigma_n=\varsigma<C],$$

for every positive constant C. Hence $\mathbb{P}[0<\varsigma_n=\varsigma]=0$, thus $\varsigma_n<\varsigma$ a.s., under $\mathbb{P}$ as under $\mathbb{Q}$. Hence, in view of (2.9), on $\{\mathsf{S}_0>0\}$, we have

$$\{\mathsf{S}_->0\}=\cup_n[0,\varsigma_n]\subseteq[0,\varsigma)\subseteq\{\mathsf{S}_->0\}$$

(where the last inclusion holds by (2.4)), whereas on $\{\mathsf{S}_0=0\}$, we have

$$\{\mathsf{S}_->0\}=\emptyset=[0,\varsigma).$$

In both cases, $\{\mathsf{S}_->0\}=[0,\varsigma)$ and (3.23) follows in view of (2.5). ∎

4. Invariant Times.

The condition (A) is stated relative to a fixed reduced stochastic basis $(\mathbb{F},\mathbb{P})$ of $(\mathbb{G},\mathbb{Q})$. In applications (see e.g. Crépey and Song (2014b)), the choice of a reduced stochastic basis $(\mathbb{F},\mathbb{P})$ is a degree of freedom of the modeler. Thus, we are interested in the stopping times θ such that the condition (A) holds for at least one reduced stochastic basis $(\mathbb{G},\mathbb{Q})$. This motivates the following:

Definition 4.1 A $\mathbb{G}$ stopping time θ is called invariant if there exists a reduced stochastic basis $(\mathbb{F},\mathbb{P})$ of $(\mathbb{G},\mathbb{Q})$ satisfying the condition (A).

Beyond the obvious reference to the martingale invariance property defined by the condition (A), this terminology also fits the invariance by reduction of filtration of certain backward stochastic differential equations with random terminal time given as an invariant time that appear in the study of counterparty risk in finance (see Crépey and Song (2014a,2014b)). We can restate Theorem 3.1 in the form of the following characterization of invariant times.

Corollary 4.1 *A $\mathbb{G}$ stopping time θ is invariant if and only if there exists a filtration $\mathbb{F}$ satisfying the usual conditions, the inclusions $\mathbb{F}\subseteq\mathbb{G}\subseteq\overline{\mathbb{F}}$ and the integrability and positivity conditions* (3.17). *In this case, a probability measure $\mathbb{P}$ on $(\Omega,\mathcal{F}_T)$ such that $(\mathbb{F},\mathbb{P})$ satisfies the condition (A) is defined by the $\mathbb{Q}$ density $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})_T$.*

This section is devoted to the study of invariant times. The proof of Theorem 3.1 (or Corollary 4.1) given in Sect. 3 uses auxiliary results of Song (2014) and Jacod (1979, Chapter 6). A self-contained proof is given in Appendix A. The core of the argument is to combine the Jeulin formula (2.10) with the Girsanov formula (A.1), so that the change of measure "compensates" the enlargement of filtration (the interaction between them is directly visible in the formula (A.9)).

4.1. *Avoidance Property and Pseudo-Stopping Times.*

According to Nikeghbali and Yor (2005), a random time θ is called an $(\mathbb{F},\mathbb{Q})$ pseudo-stopping time if and only if θ is finite and, for any bounded $\mathbb{F}$ martingale X,

$$\mathbb{E}[X_\theta]=\mathbb{E}[X_0].$$

This section studies some connections between the notions of invariant time and pseudo-stopping time. Let A denote the $\mathbb{F}$ dual optional projection of $\mathbb{1}_{[\theta,\infty)}$ and

$$N = (1 - \mathsf{S}_0) + \mathsf{A} + \mathsf{S}. \tag{4.1}$$

Pseudo-stopping times admit several equivalent characterizations, e.g. $(\mathbb{F},\mathbb{Q})$ local martingales stopped at θ are $(\mathbb{G},\mathbb{Q})$ local martingales, or $\mathsf{A}_\infty = 1$, or $N = 1$ (see Nikeghbali and Yor (2005)). Consistent with the obvious fact that a pseudo-stopping time θ that avoids $\mathbb{F}$ stopping times (hence $\mathsf{A} = \mathsf{D}$ continuous as well known) satisfies (P2) below, we have:

Theorem 4.1 *Let there be given a subfiltration $\mathbb{F}$ of $\mathbb{G}$ satisfying the usual conditions and the condition (B). Suppose $0 < \theta < \infty$. We consider the two following properties:*
(P1) *θ is an $\mathbb{F}$ pseudo-stopping;*
(P2) *$(\mathbb{F}, \mathbb{P} = \mathbb{Q})$ satifies the condition (A) for any positive constant T.*

1) One property being satisfied, the other also holds if and only if $\mathsf{A} = \mathsf{D}$.

2) Both properties hold simultaneously if and only if

$$\mathsf{Q} = \mathsf{S}_0 \text{ and } \mathsf{A} = \mathsf{D}. \tag{4.2}$$

Proof. 2) follows immediately from 1) that we show. (P1) is equivalent to $N = 1$ and, in view of Corollary 3.1, (P2) is equivalent to $\mathsf{Q} = \mathsf{S}_0$. Assuming (P1), (4.1) yields

$$1 = (1 - \mathsf{S}_0) + \mathsf{A} + \mathsf{S} \text{ , hence } (\mathsf{Q} - \mathsf{S}_0) + (\mathsf{A} - \mathsf{D}) = 0,$$

so that (P2) is equivalent to $\mathsf{A} = \mathsf{D}$. Conversely, assuming (P2), we have

$$N = (1 - \mathsf{S}_0) + \mathsf{A} + \mathsf{S} = 1 + \mathsf{A} - \mathsf{D},$$

so that (P1) is equivalent to $\mathsf{A} = \mathsf{D}$. ∎

Theorem 4.1 2) shows that, letting aside times θ with the avoidance property, the times θ for which $(\mathbb{F}, \mathbb{P} = \mathbb{Q})$ satisfies the condition (A) are in a sense "orthogonal" to pseudo-stopping times. But avoidance typically holds in applications (see Crépey and Song (2014b)) and a pseudo-stopping time θ with the avoidance property is such that $(\mathbb{F}, \mathbb{P} = \mathbb{Q})$ satisfies the condition (A). But, with respect to $(\mathbb{F}, \mathbb{Q})$ pseudo-stopping times that are defined with respect to the fixed probability measure $\mathbb{Q}$, the additional flexibility of invariant times lies in the possibility to use a changed measure $\mathbb{P}$ on top of a reduced filtration $\mathbb{F}$. For instance, a pseudo-stopping time without the avoidance property can still be invariant in view of a probability measure $\mathbb{P} \neq \mathbb{Q}$ in the condition (A). This is illustrated in the following examples, which also address the first issue that was raised after the condition (A) was introduced, i.e. the materiality or not of stopping at $(\theta-)$ rather than θ in its definition. In fact, we construct examples of invariant times θ (also a pseudo-stopping time in the first case but not in the second one) that do intersect $\mathbb{F}$ stopping times in a reduced basis $(\mathbb{F}, \mathbb{P})$ satisfying (A). Hence, the "$-$"is material in $(\theta-)$.

Example 4.1 Fix a filtration $\mathbb{F}$ satisfying the usual conditions. For $i = 1, 2$, let $\sigma_i > 0$ be a finite $\mathbb{F}$ stopping time with bounded compensator $\mathbf{v}_i$. Assuming $\sigma_2 > T$, define $\theta = \mathbb{1}_A \sigma_1 + \mathbb{1}_{A^c} \sigma_2$, for some random event A independent of $\mathcal{F}_\infty$ such that $\alpha = \mathbb{Q}(A) \in (0, 1)$.

Let $\mathbb{G}$ be the progressive enlargement of $\mathbb{F}$ with θ. Hence, θ intersects the $\mathbb{F}$ stopping times σ_i. By independence of A, on $[0,T]$,

$$\begin{aligned}
&\mathsf{S} = \mathbb{1}_{[0,\sigma_1]}\alpha + \mathbb{1}_{[0,\sigma_2]}(1-\alpha),\\
&\mathsf{D} = \alpha\mathbf{v}_1 + (1-\alpha)\mathbf{v}_2,\\
&\mathsf{Q} = \alpha(\mathbb{1}_{[0,\sigma_1)} + \mathbf{v}_1) + (1-\alpha)(\mathbb{1}_{[0,\sigma_2)} + \mathbf{v}_2).
\end{aligned}$$

Hence, since $\sigma_2 \geq T$, we have $\mathsf{S} \geq 1-\alpha$, ${}^p\mathsf{S} \geq 1-\alpha$ on $[0,T]$ and, by Theorem I.8 in Lepingle and Mémin (1978), $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q})$ is an $(\mathbb{F},\mathbb{Q})$ true martingale on $[0,T]$. It is also positive because $\frac{\Delta\mathsf{Q}}{{}^p\mathsf{S}} = \frac{\mathsf{S}}{{}^p\mathsf{S}} - 1 > -1$ on $[0,T]$. Hence the conditions of Corollary 4.1 are fulfilled and θ is an invariant time. In addition, for every bounded $\mathbb{F}$ optional process K, by independence of A,

$$\mathbb{E}[K_\theta] = \mathbb{E}[\mathbb{1}_A K_{\sigma_1} + \mathbb{1}_{A^c} K_{\sigma_2}] = \mathbb{E}[\alpha K_{\sigma_1} + (1-\alpha)K_{\sigma_2}],$$

hence

$$\mathsf{A} = (\mathbb{1}_{[\theta,\infty)})^o = \alpha\mathbb{1}_{[\sigma_1,\infty)} + (1-\alpha)\mathbb{1}_{[\sigma_2,\infty)}.$$

As the σ_i are finite, $\mathsf{A}_\infty = 1$ and, by application of Theorem 1 (3) in Nikeghbali and Yor (2005), θ is a pseudo-stopping time.

Example 4.2 To obtain an invariant time θ intersecting $\mathbb{F}$ stopping times without being a pseudo-stopping time, we set

$$\theta = \mathbb{1}_{A_1}\sigma_1 + \mathbb{1}_{A_2}\sigma_2 + \mathbb{1}_{A_3}\sigma_3,$$

for a non pseudo-stopping time σ_3 and a partition $A_i, i = 1,2,3$, independent of $\mathcal{F}_\infty$ and of σ_3. With $\alpha_i = \mathbb{Q}(A_i) > 0$, we have

$$\mathsf{A} = (\mathbb{1}_{[\theta,\infty)})^o = \alpha_1\mathbb{1}_{[\sigma_1,\infty)} + \alpha_2\mathbb{1}_{[\sigma_2,\infty)} + \alpha_3(\mathbb{1}_{[\sigma_3,\infty)})^o,$$

where $(\mathbb{1}_{[\sigma_3,\infty)})^o_\infty \neq 1$, hence $\mathsf{A}_\infty \neq 1$, with positive $\mathbb{Q}$ probability. Thus, by the converse part in Theorem 1 (3) in Nikeghbali and Yor (2005), θ is not a pseudo-stopping time. But the Azéma supermartingale of θ is given by

$$\mathsf{S} = \mathbb{1}_{[0,\sigma_1]}\alpha_1 + \mathbb{1}_{[0,\sigma_2]}\alpha_2 + {}^o(\mathbb{1}_{[0,\sigma_3)})\alpha_3 \geq \alpha_2 \text{ on } [0,T].$$

The reasoning in the example 4.1 remains valid to prove that θ is an invariant time.

4.2. *Positivity of the Doléans-Dade exponential.* In Theorem 3.1 2) and Corollary 4.1 the strongest requirements are the true martingality and positivity conditions on $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q})$ on the interval $[0,T]$. This section studies the role of the positivity condition in these results, i.e. the role of the equivalence between $\mathbb{P}$ and $\mathbb{Q}$ on $\mathcal{F}_T$ in the condition (A).

Example 4.3 We take for $\mathbb{G}$ the augmentation of the natural filtration of a Poisson process stopped at its first time of jump θ relative to some probability measure $\mathbb{Q}$.

For $\mathbb{F} = \mathbb{G}$ (so that the condition (B) holds trivially), Corollary 3.1 1) shows that the condition (A) doesn't hold, whatever the probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$. In the present Poisson case, this can also be recovered directly from the definition of the condition (A). In fact, for any probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$, any $(\mathbb{G},\mathbb{P})$ local martingale P can be represented as a $(\mathbb{G},\mathbb{Q})$ local martingale minus some continuous bracket deterministic until θ. Thus, $P^{\theta-} = P^\theta$ is a nonconstant finite variation continuous process, hence not a $(\mathbb{G},\mathbb{P})$ local martingale. Therefore, $(\mathbb{F}=\mathbb{G},\mathbb{P})$ does not satisfy the condition (A), whatever the probability measure $\mathbb{P}$.

For $\mathbb{F}$ trivial, any $\mathbb{G}$ predictable process coincides with a Borel function before θ, hence the condition (B) is satisfied. The constants are the only $(\mathbb{F}=\{\emptyset,\Omega\},\mathbb{Q})$ local martingales, so that the condition (A) is satisfied by $(\mathbb{F},\mathbb{P})=(\{\emptyset,\Omega\},\mathbb{Q})$ (hence θ is an invariant time). Consistent with this conclusion in regard of Theorem 3.1 1), S is deterministic (equal to the survival function of θ), Q is constant and $q=1$, hence (3.9) is satisfied.

In the sequel, we show that the positivity of $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})$ (assuming that $\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q}$ is Q integrable on $[0,T]$ with respect to $(\mathbb{F},\mathbb{Q})$) reduces to the predictability of the stopping time $\varsigma_{\{\varsigma\leq T\}}$ (in $\mathbb{F}$), where ς is the first zero of S in (2.4).

Lemma 4.1 *Let σ be an $\mathbb{F}$ predictable stopping time. Then ${}^p\mathsf{S}_\sigma=0$ if and only if $\sigma\geq\varsigma$.*

Proof. By definition of the predictable projection and nonnegativity of S,

$$^p\mathsf{S}_\sigma=0\Leftrightarrow\mathbb{E}[\mathsf{S}_\sigma|\mathcal{F}_{\sigma-}]=0\Leftrightarrow\mathsf{S}_\sigma=0\Leftrightarrow\sigma\geq\varsigma\ \text{ on the set }\{\sigma<\infty\}.\ \blacksquare$$

Lemma 4.2 *$\varsigma_{\{{}^p\mathsf{S}_\varsigma=0\}}$ is a predictable stopping time.*

Proof. Since ${}^p\mathsf{S}=\mathsf{S}_--\Delta\mathsf{D}$, therefore $\mathsf{S}_{\varsigma-}=\Delta_\varsigma\mathsf{D}>0$ on $\{\varsigma<\infty,{}^p\mathsf{S}_\varsigma=0,\Delta_\varsigma\mathsf{D}>0\}$. The set

$$\Theta=\{{}^p\mathsf{S}=0,\Delta\mathsf{D}>0\}$$

is thin and predictable. Let $\Theta_\omega=\{s>0;(s,\omega)\in\Theta\}$, so that $\{\Theta_\cdot\neq\emptyset\}=\pi(\Theta)$, the projection of Θ onto Ω. If $\mathbb{Q}[\pi(\Theta)]>0$, then, by predictable section theorem, there exists a sequence of predictable stopping times σ_n such that $[\sigma_n]\subseteq\Theta$ and $\lim_{n\to\infty}\mathbb{Q}[\sigma_n<\infty]=\mathbb{Q}[\pi(\Theta)]>0$. Since $\Delta\mathsf{D}=0$ on (ς,∞), $\sigma_n(\omega)<\infty$ implies $\sigma_n(\omega)\leq\varsigma(\omega)$. But, by Lemma 4.1, $\sigma_n(\omega)\geq\varsigma(\omega)$. We conclude that $\sigma_n=\varsigma$ on $\{\sigma_n<\infty\}$, i.e. $[\sigma_n]\subseteq[\varsigma]$. Set $\sigma_\infty=\inf_n\sigma_n$. Since $\sigma_n(\omega)$ can only take two values (∞ and $\varsigma(\omega)$), σ_∞ is a stationary infimum of predictable stopping times, hence a predictable stopping time[9]. Clearly, $\Delta_{\sigma_\infty}\mathsf{D}>0$ on $\{\sigma_\infty<\infty\}$, $[\sigma_\infty]\subseteq[\varsigma]$, $\{\sigma_\infty<\infty\}\subseteq\pi(\Theta)$ and $\mathbb{Q}[\sigma_\infty<\infty]=\mathbb{Q}[\pi(\Theta)]$. Therefore,

$$\begin{aligned}\varsigma<\infty,{}^p\mathsf{S}_\varsigma=0,\Delta_\varsigma\mathsf{D}>0\ &\Rightarrow\ \varsigma<\infty,\Theta_\cdot\neq\emptyset,{}^p\mathsf{S}_\varsigma=0,\mathsf{S}_{\varsigma-}>0\\ &\Rightarrow\ \varsigma<\infty,\sigma_\infty<\infty,{}^p\mathsf{S}_\varsigma=0,\mathsf{S}_{\varsigma-}>0\\ &\Rightarrow\ \varsigma=(\sigma_\infty)_{\{\varsigma<\infty,{}^p\mathsf{S}_\varsigma=0,\mathsf{S}_{\varsigma-}>0\}}<\infty\end{aligned}$$

and

$$\begin{aligned}\varsigma<\infty,{}^p\mathsf{S}_\varsigma=0,\Delta_\varsigma\mathsf{D}=0\ &\Rightarrow\ \varsigma<\infty,{}^p\mathsf{S}_\varsigma=0,\mathsf{S}_{\varsigma-}=0\\ &\Rightarrow\ \varsigma=\varsigma_{\{\varsigma<\infty,{}^p\mathsf{S}_\varsigma=0,\mathsf{S}_{\varsigma-}=0\}}<\infty.\end{aligned}$$

Therefore,

$$\varsigma_{\{\varsigma<\infty,{}^p\mathsf{S}_\varsigma=0\}}=(\sigma_\infty)_{\{\varsigma<\infty,{}^p\mathsf{S}_\varsigma=0,\mathsf{S}_{\varsigma-}>0\}}\wedge\varsigma_{\{\varsigma<\infty,{}^p\mathsf{S}_\varsigma=0,\mathsf{S}_{\varsigma-}=0\}}.$$

As $\varsigma\leq\sigma_\infty$, therefore $\{\varsigma<\infty,{}^p\mathsf{S}_\varsigma=0,\mathsf{S}_{\varsigma-}>0\}\in\mathcal{F}_{\sigma_\infty-}$, hence $(\sigma_\infty)_{\{\varsigma<\infty,{}^p\mathsf{S}_\varsigma=0,\mathsf{S}_{\varsigma-}>0\}}$ is predictable, like σ_∞, by Theorem 3.29 in He et al. (1992). The stopping time $\varsigma_{\{\varsigma<\infty,{}^p\mathsf{S}_\varsigma=0,\mathsf{S}_{\varsigma-}=0\}}$ is predictable by the proof of Theorem 9.41 in He et al. (1992). Hence, $\varsigma_{\{\varsigma<\infty,{}^p\mathsf{S}_\varsigma=0\}}$ is predictable as the minimum of two predictable stopping times. $\blacksquare$

The next result characterizes the positivity of the Doléans-Dade exponential $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})$ (whenever well defined). Note that it is consistent with the findings of the example 4.3, where, in the first considered case $\mathbb{F}=\mathbb{G}$, $\varsigma=\theta$ is the first time of jump of a Poisson process (hence, $\varsigma_{\{\varsigma\leq T\}}$ is not predictable) and $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})$ vanishes at θ on $\{\theta\leq T\}$.

[9] Cf. He et al. (1992, Theorem 3.29).

Theorem 4.2 *Assuming that $\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}$ is Q integrable on $[0,T]$ with respect to $(\mathbb{F},\mathbb{Q})$, we have: $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})>0$ on $[0,T] \Longleftrightarrow {}^p\mathsf{S}_\varsigma=0$ on $\{\varsigma\le T\} \Longleftrightarrow \varsigma_{\{\varsigma\le T\}}$ is a predictable stopping time.*

Proof. $\mathcal{E}(\mathbb{1}_{\{{}^p\mathsf{S}>0\}}\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q})$ is positive on $[0,T]$ if and only if

$$\mathbb{1}_{\{{}^p\mathsf{S}_t>0\}}\Delta_t\Big(\frac{1}{{}^p\mathsf{S}}\centerdot\mathsf{Q}\Big)=\mathbb{1}_{\{{}^p\mathsf{S}_t>0\}}\frac{1}{{}^p\mathsf{S}_t}\Delta_t\mathsf{Q}>-1,\ \ t\in[0,T]. \tag{4.3}$$

Since $\mathsf{S}-{}^p\mathsf{S}=\Delta\mathsf{Q}$ (cf. (2.3)), for $t\in[0,\varsigma)$,

$$\frac{1}{{}^p\mathsf{S}_t}\Delta_t\mathsf{Q}=\frac{\mathsf{S}_t-{}^p\mathsf{S}_t}{{}^p\mathsf{S}_t}=\frac{\mathsf{S}_t}{{}^p\mathsf{S}_t}-1>-1.$$

If ς is finite with ${}^p\mathsf{S}_\varsigma>0$, then the same computation at ς yields

$$\frac{1}{{}^p\mathsf{S}_\varsigma}\Delta_\varsigma\mathsf{Q}=-1.$$

If ς is finite with ${}^p\mathsf{S}_\varsigma=0$, then $\Delta_\varsigma\mathsf{Q}=\mathsf{S}_\varsigma-{}^p\mathsf{S}_\varsigma=0$. For $t>\varsigma$, $\Delta_t\mathsf{Q}=0$. Putting together these observations, the condition (4.3) is equivalent to

$${}^p\mathsf{S}_\varsigma=0 \text{ on } \{\varsigma\le T\}. \tag{4.4}$$

If so, then $\{\varsigma\le T\}=\{\varsigma_{\{{}^p\mathsf{S}_\varsigma=0\}}\le T\}$ is $\mathcal{F}_{(\varsigma_{\{{}^p\mathsf{S}_\varsigma=0\}})-}$ measurable. Therefore, by Lemma 4.2 and He et al. (1992, Theorem 9.41 7)), $\varsigma_{\{\varsigma\le T\}}=(\varsigma_{\{{}^p\mathsf{S}_\varsigma=0\}})_{\{\varsigma\le T\}}$ is predictable. Conversely, if $\varsigma_{\{\varsigma\le T\}}$ is predictable, as it is $\ge\varsigma$, the condition (4.4) holds by Lemma 4.1. ∎

5. Conclusion.

Given a subfiltration $\mathbb{F}$ of $\mathbb{G}$ satisfying the condition (B) with respect to a $\mathbb{G}$ stopping time θ, the results of this paper reduce the question of the existence of a probability measure $\mathbb{P}$ such that $(\mathbb{F},\mathbb{P})$ satisfies the condition (A), i.e. the question of the invariance of the time θ, to suitable integrability conditions (essentially; see Theorem 3.1 and Corollary 4.1). The results of this paper are used in Crépey and Song (2014a, 2014b) to study the well posedness and numerical solution of a backward stochastic differential equation that appears in the modeling of counterparty risk in finance. From a broader mathematical finance perspective, this paper is a contribution to the question of reduction of filtration, or separability of a full model filtration $\mathbb{G}$ in which a stopping time θ is given, i.e., given a stopping time θ relative to a full model filtration $\mathbb{G}$, when and how one can separate the information that comes from θ from a reference information (filtration) in order to simplify the computations (the inverse problem of progressive enlargement of filtration). Our results say that this is possible whenever θ is an invariant time, switching if need be from the original probability measure $\mathbb{Q}$ to a changed probability measure $\mathbb{P}$ such that the change of measure compensates in some sense the change of filtration. The additional degree of freedom provided by the possibility of changing the measure makes invariant times more flexible than Cox times or pseudo-stopping times that are commonly used to model default times. To complete this study it would be interesting to compare invariant times with other classical classes of random times, such as honest times or initial times satisfying Jacod's density hypothesis.

APPENDIX A: AN ALTERNATIVE PROOF OF THEOREM 3.1

In this section we provide a longer but relatively elementary proof of Theorem 3.1, independent of the results of Song (2014). One specific interest of this proof is to make directly visible the compensation, which is the core idea of that paper (see Sect. 1), between the Girsanov measure change formula and the Jeulin-Yor progressive enlargement of filtration formula. In fact, the interaction between the two formulas is directly visible in (A.9).

The computations of this section rely a lot on the properties of projections, e.g. ${}^pM = M_-$ in the case of a martingale M. Superscripts ^{o} and ^{p} are used for the $(\mathbb{F},\mathbb{Q})$ projections; $\mathbb{E}^{\mathbb{Q}}$ represents the $\mathbb{Q}$ expectation, whereas the $\mathbb{P}$ expectation is denoted by $\mathbb{E}^{\mathbb{P}}$. All our predictable brackets are computed with respect to $\mathbb{F}$. The $(\mathbb{F},\mathbb{Q})$ and $(\mathbb{F},\mathbb{P})$ predictable brackets are respectively denoted by $\langle\cdot,\cdot\rangle^{\mathbb{Q}}$ (as in (2.10)) and $\langle\cdot,\cdot\rangle^{\mathbb{P}}$. By definition, if one argument is continuous (resp. discontinuous), then brackets reduce to the brackets of the continuous (resp. discontinuous) parts. Predictable brackets coincide with the square bracket and are continuous if one argument is continuous; in this case $\langle\cdot,\cdot\rangle^{\mathbb{Q}} = \langle\cdot,\cdot\rangle^{\mathbb{P}}$ and we simply denote $\langle\cdot,\cdot\rangle$. Finite variation predictable processes don't contribute to predictable brackets against local martingales (this can be seen as consequence of Yoeurp's formula). Given a probability measure $\mathbb{P}$ equivalent to the probability measure $\mathbb{Q}$ on $\mathcal{F}_T$, we use the notation $p, \mathsf{p}, q, \mathsf{q}$ introduced in (3.7). In particular, for any bounded $(\mathbb{F},\mathbb{P})$ martingale P null at the origin,

$$\widetilde{P} = P - q_- \centerdot \langle p, P\rangle^{\mathbb{P}} = P - \langle \mathsf{p}, P\rangle^{\mathbb{P}} \tag{A.1}$$

is an $(\mathbb{F},\mathbb{Q})$ local martingale on $[0,T]$, by the Girsanov theorem. The next lemma expresses the denseness of the class of such Girsanov transform martingales among all the $(\mathbb{F},\mathbb{Q})$ martingales.

Lemma A.1 *Let Q be an $(\mathbb{F},\mathbb{Q})$ uniformly integrable martingale null at the origin such that $\widetilde{P}Q$ is an $(\mathbb{F},\mathbb{Q})$ local martingale, for any bounded $(\mathbb{F},\mathbb{P})$ martingale P null at the origin. Then $Q = 0$ on $[0,T]$.*

***Proof*.** For any $\mathbb{F}$ stopping time $\sigma \le T$ reducing the concerned processes to integrability,

$$\begin{aligned} 0 &= \mathbb{E}^{\mathbb{Q}}[\widetilde{P}_\sigma Q_\sigma] = \mathbb{E}^{\mathbb{P}}[\widetilde{P}_\sigma Q_\sigma p_\sigma] = \mathbb{E}^{\mathbb{P}}[(P - q_- \centerdot \langle p, P\rangle^{\mathbb{P}})_\sigma Q_\sigma p_\sigma] \\ &= \mathbb{E}^{\mathbb{P}}[[Qp, P]_\sigma - Q_- \centerdot \langle p, P\rangle^{\mathbb{P}}_\sigma] = \mathbb{E}^{\mathbb{P}}[[Qp - Q_- \centerdot p, P]_\sigma], \end{aligned} \tag{A.2}$$

where the $(\mathbb{F},\mathbb{P})$ martingale properties of P^σ and $(Qp)^\sigma$ were used to pass to the second line (in combination with the predictable projection formula, yielding $\mathbb{E}^{\mathbb{P}}[q_- \centerdot \langle p, P\rangle^{\mathbb{P}}_\sigma Q_\sigma p_\sigma] = \mathbb{E}^{\mathbb{P}}[Q_- \centerdot \langle p, P\rangle^{\mathbb{P}}_\sigma]$). By optional section theorem, (A.2) shows that $[Qp - Q_- \centerdot p, P] = 0$ on $[0,T]$, for any bounded $(\mathbb{F},\mathbb{P})$ martingale P null at the origin. Hence

$$0 = Qp - Q_- \centerdot p = Qp - (Qp)_- \centerdot (q_- \centerdot p) \text{ on } [0,T].$$

By uniqueness of the solution to an exponential stochastic differential equation, we conclude that $Q = 0$ on $[0,T]$. ∎

Lemma A.2 *For any bounded $(\mathbb{F},\mathbb{P})$ martingale P null at the origin, $\widetilde{P^c}$ and $\widetilde{P^d}$ are a continuous $(\mathbb{F},\mathbb{Q})$ martingale and a purely discontinuous $(\mathbb{F},\mathbb{Q})$ martingale on $[0,T]$, respectively. Hence, $\widetilde{P^c} = (\widetilde{P})^c$ and $\widetilde{P^d} = (\widetilde{P})^d$.*

***Proof*.** We prove the first assertion regarding $\widetilde{P^d}$ (the first assertion regarding $\widetilde{P^c}$ is obvious and the last statement is then clear). Let Q be any continuous $(\mathbb{F},\mathbb{Q})$ martingale null at the origin, hence an $(\mathbb{F},\mathbb{P})$ continuous semimartingale on $[0,T]$. For any $\mathbb{F}$ stopping time $\sigma \leq T$ reducing the concerned processes to integrability, as in the proof of Lemma A.1, we can write:

$$\mathbb{E}^{\mathbb{Q}}[\widetilde{P^d_\sigma} Q_\sigma] = \mathbb{E}^{\mathbb{P}}[[Qp - Q_- \centerdot p, P^d]_\sigma],$$

where, by the integration by parts formula on $[0,T]$,

$$[Qp - Q_- \centerdot p, P^d] = [p_- \centerdot Q + [Q,p], P^d] = 0,$$

because $p_- \centerdot Q + [Q,p]$ is continuous. ∎

Note that, for any bounded $\mathbb{F}$ predictable process K and $\mathbb{F}$ predictable stopping time $\sigma \leq T$,

$$\mathbb{E}^{\mathbb{Q}}[K_\sigma q_\sigma \Delta_\sigma p] = \mathbb{E}^{\mathbb{P}}[K_\sigma \Delta_\sigma p] = 0,$$

so that, by Theorem 7.42 in He et al. (1992), there exists an $(\mathbb{F},\mathbb{Q})$ purely discontinuous local martingale $\mathbf{q}$ on $[0,T]$ such that $\Delta_s \mathbf{q} = q_s \Delta_s p$.

Lemma A.3 *We have*

$$\mathsf{q}^c = -\widetilde{\mathsf{p}^c},\ \ \mathsf{q}^d = -\mathbf{q}. \tag{A.3}$$

For any bounded $(\mathbb{F},\mathbb{P})$ *martingale* P *null at the origin,*

$$\langle p, P\rangle^{\mathbb{P}} = \langle p^c, P^c\rangle + p_- \centerdot \langle \mathbf{q}, P^d\rangle^{\mathbb{Q}}. \tag{A.4}$$

Proof. We prove (A.4) first. For any bounded $\mathbb{F}$ predictable process K, for any $\mathbb{F}$ stopping time $\sigma \leq T$ reducing the concerned processes to integrability,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[K \centerdot \langle p, P\rangle^{\mathbb{P}}_\sigma] &= \mathbb{E}^{\mathbb{P}}[K \centerdot [p,P]_\sigma] = \mathbb{E}^{\mathbb{Q}}[K \centerdot [p,P]_\sigma\, q_\sigma] \\
&= \mathbb{E}^{\mathbb{Q}}[Kq \centerdot [p,P]_\sigma] = \mathbb{E}^{\mathbb{Q}}[K \centerdot (q \centerdot [p,P])^p_\sigma] = \mathbb{E}^{\mathbb{P}}[K \centerdot (q \centerdot [p,P])^p_\sigma\, p_\sigma] \\
&= \mathbb{E}^{\mathbb{P}}[Kp_- \centerdot (q \centerdot [p,P])^p_\sigma] = \mathbb{E}^{\mathbb{P}}\Big[Kp_- \centerdot \big(q_- \centerdot \langle p^c, P^c\rangle_\sigma + (\sum_{0<s\leq \cdot} q_s \Delta_s p \Delta_s P)^p_\sigma\big)\Big],
\end{aligned}$$

where the optional (resp. predictable) projection formula and the smoothing property of projections were used to pass to the second line (resp. third) line. By predictable section theorem, this implies

$$\langle p, P\rangle^{\mathbb{P}} = \langle p^c, P^c\rangle + p_- \centerdot \big(\sum_{0<s\leq \cdot} q_s \Delta_s p \Delta_s P\big)^p,$$

which yields (A.4).

By Lemma 3.4 in Karatzas and Kardaras (2007), we have the following relation between q and p:

$$\mathsf{q} = -\mathsf{p} + \langle \mathsf{p}^c, \mathsf{p}^c\rangle + \sum_{s\leq \cdot} \frac{(\Delta_s \mathsf{p})^2}{1+\Delta_s \mathsf{p}}. \tag{A.5}$$

Moreover, on the time interval $[0,T]$,

$$\Delta_t \mathbf{q} = \frac{\Delta_t p}{p_t} = \frac{p_{t-}\Delta_t \mathsf{p}}{p_{t-} + \Delta_t p} = \frac{\Delta_t \mathsf{p}}{1+\Delta_t \mathsf{p}}\ \text{, hence } [\mathbf{q}, \mathsf{p}^d] = \sum_{s\leq \cdot} \frac{(\Delta_s \mathsf{p})^2}{1+\Delta_s \mathsf{p}}. \tag{A.6}$$

Using (A.5) and (A.6),

$$\mathsf{q} \quad = \quad -\mathsf{p} + \langle \mathsf{p}^c, \mathsf{p}^c \rangle + [\mathbf{q}, \mathsf{p}^d] = -\mathsf{p}^c + \langle \mathsf{p}^c, \mathsf{p}^c \rangle - \mathsf{p}^d + [\mathbf{q}, \mathsf{p}^d] = -\widetilde{\mathsf{p}^c} - \mathsf{p}^d + [\mathbf{q}, \mathsf{p}^d],$$

by the Girsanov formula (A.1). As a consequence, $\mathsf{q}^c = -\widetilde{\mathsf{p}^c}$. In addition, by (A.5),

$$\Delta_t \mathsf{q}^d = \Delta_t \mathsf{q} = -\Delta_t \mathsf{p} + \frac{(\Delta_t \mathsf{p})^2}{1 + \Delta_t \mathsf{p}} = -\frac{\Delta_t \mathsf{p}}{1 + \Delta_t \mathsf{p}} = -\Delta_t \mathbf{q},$$

by (A.6). Since q^d and $(-\mathbf{q})$ are both $(\mathbb{F}, \mathbb{Q})$ purely discontinuous local martingales, they coincide by Corollary 7.23 in He et al. (1992). This proves (A.3). ∎

Lemma A.4 *For any bounded* $(\mathbb{F}, \mathbb{P})$ *martingale* P *null at the origin, the* $(\mathbb{G}, \mathbb{Q})$ *compensator of the process* $\mathsf{J} \centerdot \langle \mathsf{p}, P \rangle^{\mathbb{P}}$ *is given by*

$$\mathsf{J}_- \centerdot \langle \mathsf{p}^c, P^c \rangle + \mathsf{J}_- \frac{{}^p\mathsf{S}}{\mathsf{S}_-} \centerdot \langle \mathbf{q}, P^d \rangle^{\mathbb{Q}}. \tag{A.7}$$

The process $P^{\theta-}$ *is a* $(\mathbb{G}, \mathbb{Q})$ *local martingale if and only if*

$$\mathsf{S}_- \centerdot \langle \mathsf{p}^c, P^c \rangle + {}^p\mathsf{S} \centerdot \langle \mathbf{q}, P^d \rangle^{\mathbb{Q}} + \langle \mathsf{Q}, P \rangle^{\mathbb{Q}} = 0 \tag{A.8}$$

on $[0, T]$.

***Proof*.** First we compute the $(\mathbb{G}, \mathbb{Q})$ compensator of $\mathsf{J} \centerdot \langle \mathbf{q}, P^d \rangle^{\mathbb{Q}}$ on $[0, T]$. For any bounded $\mathbb{G}$ predictable process L null outside of $[0, T]$, with $\mathbb{F}$ predictable reduction denoted by K, and for any $\mathbb{F}$ stopping time σ,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[L \mathsf{J} \centerdot \langle \mathbf{q}, P^d \rangle^{\mathbb{Q}}_\sigma] &= \mathbb{E}^{\mathbb{Q}}[K \mathsf{J} \centerdot \langle \mathbf{q}, P^d \rangle^{\mathbb{Q}}_\sigma] \\ &= \mathbb{E}^{\mathbb{Q}}[K \mathsf{S} \centerdot \langle \mathbf{q}, P^d \rangle^{\mathbb{Q}}_\sigma] \\ &= \mathbb{E}^{\mathbb{Q}}[K {}^p\mathsf{S} \centerdot \langle \mathbf{q}, P^d \rangle^{\mathbb{Q}}_\sigma] \\ &= \mathbb{E}^{\mathbb{Q}}[K \mathsf{J}_- \tfrac{{}^p\mathsf{S}}{\mathsf{S}_-} \centerdot \langle \mathbf{q}, P^d \rangle^{\mathbb{Q}}_\sigma] = \mathbb{E}^{\mathbb{Q}}[L \mathsf{J}_- \tfrac{{}^p\mathsf{S}}{\mathsf{S}_-} \centerdot \langle \mathbf{q}, P^d \rangle^{\mathbb{Q}}_\sigma], \end{aligned}$$

where the optional (resp. predictable) projection formula was used to pass to the second line (resp. third and fourth lines, in combination with (2.2) in the second case). By definition, this shows that the $(\mathbb{G}, \mathbb{Q})$ compensator of $\mathsf{J} \centerdot \langle \mathbf{q}, P^d \rangle^{\mathbb{Q}}$ is given by

$$\mathsf{J}_- \frac{{}^p\mathsf{S}}{\mathsf{S}_-} \centerdot \langle \mathbf{q}, P^d \rangle^{\mathbb{Q}}$$

on $[0, T]$, from which the formula (A.7) follows through (A.4). Combining the Jeulin formula (2.10) with the Girsanov formula (A.1), we obtain that

$$P^{\theta-} - \mathsf{J} \centerdot \langle \mathsf{p}, P \rangle^{\mathbb{P}} - \mathsf{J}_- \tfrac{1}{\mathsf{S}_-} \centerdot \langle \mathsf{Q}, P \rangle^{\mathbb{Q}} \tag{A.9}$$

is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$. If $P^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$, so is in turn the $\mathbb{G}$ optional process with finite variation

$$\mathsf{J} \centerdot \langle \mathsf{p}, P \rangle^{\mathbb{P}} + \mathsf{J}_- \frac{1}{\mathsf{S}_-} \centerdot \langle \mathsf{Q}, P \rangle^{\mathbb{Q}}, \tag{A.10}$$

the $(\mathbb{G},\mathbb{Q})$ compensator of which must therefore vanish, i.e., given the formula (A.7) for the $(\mathbb{G},\mathbb{Q})$ compensator of $\mathsf{J}\centerdot\langle \mathsf{p}, P\rangle^{\mathbb{P}}$:

$$\mathsf{J}_{-}\centerdot\langle \mathsf{p}^c, P^c\rangle + \mathsf{J}_{-}\frac{{}^p\mathsf{S}}{\mathsf{S}_{-}}\centerdot\langle \mathbf{q}, P^d\rangle^{\mathbb{Q}} + \mathsf{J}_{-}\frac{1}{\mathsf{S}_{-}}\centerdot\langle \mathsf{Q}, P\rangle^{\mathbb{Q}} = 0$$

on $[0,T]$. By $(\mathbb{F},\mathbb{Q})$ predictable projection using (2.2)[10], we obtain (A.8). ∎

We are now in a position to prove Theorem 3.1 1) (which implies Theorem 3.1 2) in the way already established in Sect. 3). In view of the second assertion in Lemma A.4, the condition (A) holds if and only if any bounded $(\mathbb{F},\mathbb{P})$ martingale P null at the origin satisfies (A.8) on $[0,T]$, or, equivalent to (A.8),

$$\begin{aligned} 0 &= \mathsf{S}_{-}\centerdot\langle \widetilde{\mathsf{p}^c}, \widetilde{P^c}\rangle + {}^p\mathsf{S}\centerdot\langle \mathbf{q}, \widetilde{P^d}\rangle^{\mathbb{Q}} + \langle \mathsf{Q}^c, (\widetilde{P})^c\rangle + \langle \mathsf{Q}^d, (\widetilde{P})^d\rangle^{\mathbb{Q}} \\ &= \mathsf{S}_{-}\centerdot\langle \widetilde{\mathsf{p}^c}, \widetilde{P^c}\rangle + {}^p\mathsf{S}\centerdot\langle \mathbf{q}, \widetilde{P^d}\rangle^{\mathbb{Q}} + \langle \mathsf{Q}^c, \widetilde{P^c}\rangle + \langle \mathsf{Q}^d, \widetilde{P^d}\rangle^{\mathbb{Q}}, \end{aligned} \tag{A.11}$$

by Lemma A.2. For any bounded $(\mathbb{F},\mathbb{P})$ local martingale P null at the origin, the formula (A.11) applied with P^c instead of P is rewritten as

$$0 = \mathsf{S}_{-}\centerdot\langle \widetilde{\mathsf{p}^c}, \widetilde{P^c}\rangle + \langle \mathsf{Q}^c, \widetilde{P^c}\rangle = \langle \mathsf{S}_{-}\centerdot\widetilde{\mathsf{p}^c} + \mathsf{Q}^c, \widetilde{P^c}\rangle = \langle \mathsf{S}_{-}\centerdot\widetilde{\mathsf{p}^c} + \mathsf{Q}^c, (\widetilde{P})^c\rangle = \langle \mathsf{S}_{-}\centerdot\widetilde{\mathsf{p}^c} + \mathsf{Q}^c, \widetilde{P}\rangle^{\mathbb{Q}},$$

by Lemma A.2, i.e. the $(\mathbb{F},\mathbb{Q})$ local martingales $\mathsf{S}_{-}\centerdot\widetilde{\mathsf{p}^c} + \mathsf{Q}^c$ and $\widetilde{P}$ are $\mathbb{Q}$ orthogonal on $[0,T]$. In view of Lemma A.1, this holding for any bounded $(\mathbb{F},\mathbb{P})$ local martingale P null at the origin is equivalent to

$$\mathsf{S}_{-}\centerdot\widetilde{\mathsf{p}^c} + \mathsf{Q}^c - \mathsf{Q}^c_0 = 0 \text{ on } [0,T]. \tag{A.12}$$

Likewise, the formula (A.11) applied with P^d instead of P is rewritten as

$$0 = {}^p\mathsf{S}\centerdot\langle \mathbf{q}, \widetilde{P^d}\rangle^{\mathbb{Q}} + \langle \mathsf{Q}^d, \widetilde{P^d}\rangle^{\mathbb{Q}} = \langle {}^p\mathsf{S}\centerdot\mathbf{q} + \mathsf{Q}^d, \widetilde{P^d}\rangle^{\mathbb{Q}} = \langle {}^p\mathsf{S}\centerdot\mathbf{q} + \mathsf{Q}^d, (\widetilde{P})^d\rangle^{\mathbb{Q}} = \langle {}^p\mathsf{S}\centerdot\mathbf{q} + \mathsf{Q}^d, \widetilde{P}\rangle^{\mathbb{Q}},$$

by Lemma A.2, i.e. the $(\mathbb{F},\mathbb{Q})$ local martingales ${}^p\mathsf{S}\centerdot\mathbf{q} + \mathsf{Q}^d$ and $\widetilde{P}$ are $\mathbb{Q}$ orthogonal on $[0,T]$. By Lemma A.1 again, this holding for any bounded $(\mathbb{F},\mathbb{P})$ local martingale P null at the origin is equivalent to

$${}^p\mathsf{S}\centerdot\mathbf{q} + \mathsf{Q}^d - \mathsf{Q}^d_0 = 0 \text{ on } [0,T]. \tag{A.13}$$

Finally, in view of the identities (A.3) in Lemma A.3, (A.12) and (A.13) can be rewritten as

$$\mathsf{S}_{-}\centerdot\mathsf{q}^c = \mathsf{Q}^c - \mathsf{Q}^c_0 \text{ and } {}^p\mathsf{S}\centerdot\mathsf{q}^d = \mathsf{Q}^d - \mathsf{Q}^d_0 \text{ on } [0,T],$$

which is (3.12). ∎

[10]The formula (2.2) is an identity on $(0,+\infty)$ (only), but our stochastic integrals in this paper, starting from 0 at time 0, only use values of their integrands at positive times.

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INDEX OF SYMBOLS

S. Crépey and S. Song
Université d'Évry Val d'Essonne
Laboratoire de Mathématiques et Modélisation d'Évry
91037 Évry Cedex, France
E-mail: stephane.crepey@univ-evry.fr; shiqi.song@univ-evry.fr