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Counterparty Risk and Funding: Immersion and Beyond

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Abstract In Crépey (2015, Part II), a basic reduced-form counterparty risk modeling approach was introduced, under a rather standard immersion hypothesis between a reference filtration and the filtration progressively enlarged by the default times of the two parties, also involving the continuity of some of the data at default time. This basic approach is too restrictive for application to credit derivatives, which are characterized by strong wrong-way risk, i.e. adverse dependence between the exposure and the credit riskiness of the counterparties, and gap risk, i.e. slippage between the portfolio and its collateral during the so called cure period that separates default from liquidation. This paper shows how a suitable extension of the basic approach can be devised so that it can be applied in dynamic copula models of counterparty risk on credit derivatives. More generally, this method is applicable in any marked default times intensity setup satisfying a suitable integrability condition. The integrability condition expresses that no mass is lost in a related measure change. The changed probability measure is not needed algorithmically. All one needs in practice is an explicit expression for the intensities of the marked default times.

Keywords: Counterparty risk, funding, BSDE, reduced-form credit modeling, immersion, wrong-way risk, gap risk, collateral, credit derivatives, marked default times, Gaussian copula, Marshall-Olkin copula, dynamic copula.

Mathematics Subject Classification: 91G40, 60H10, 60G07.

1 Introduction

Counterparty risk is the risk of default of a party in an OTC derivative transaction, a topical issue since the global financial crisis. As a significant part of the market is moving to central counterparties (also called clearinghouses), nowadays the problem must be analysed in two different setups, bilateral versus centrally cleared. In this paper, we focus on the bilateral case. See Brigo, Morini, and Pallavicini (2013) and Crépey, Bielecki, and Brigo (2014), respectively in a more financial and mathematical perspective, for recent bilateral counterparty risk references in book form, and see Armenti and Crépey (2014) for the case of centrally cleared trading. The case of centrally cleared trading is also considered in Brigo and Pallavicini (2014), but from the point of view of a client, as opposed to a member, of a clearinghouse. From the point of view of a client of the clearinghouse, the analysis of the present paper applies as well, as explained in the remark 4.2. With respect to Brigo and Pallavicini (2014), the present paper is more mathematical and yields a special focus on credit derivatives.

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To mitigate counterparty risk, a margining procedure is set up according to what is called in the context of bilateral trading a master agreement between the two parties, also referred to as CSA, for credit support annex. However, accounting for various frictions and delays, notably the so called cure period that separates default and liquidation, there is gap risk, i.e. risk of slippage between the portfolio and its collateral. This is why another layer of collateralization, called initial margins as opposed to the variation margin that only accounts for market risk, is now maintained in both centrally cleared transactions and bilateral transactions under a sCSA (standard CSA). Gap risk is magnified in the presence of wrong-way risk, i.e. adverse dependence between the underlying exposure and the credit risk of the counterparties. This is a special case of concern regarding counterparty risk on credit derivatives, given the default contagion and frailty effects between the two parties and the underlying credit names. In fact, to properly deal with counterparty risk embedded in credit derivatives, one needs a credit portfolio model with the following features. First, the model should be calibratable to relevant data sets: CDS data if the targeted application consists of counterparty risk computations on CDS contracts and, additionally, tranches data for computations involving CDO contracts. In particular, one needs a bottom-up model of portfolio credit risk, with efficient pricing schemes for vanillas (CDS contracts and/or CDO tranches) as well as a copula separation property between the individual and the dependence model parameters. Second, as counterparty risk and related funding valuation adjustments price options on future values of the underlyings, it should be a dynamic model. One possibility is to use dynamic copula models resulting from the introduction of a suitable filtration on top of a static copula model for the default times of the two parties and of underlying credit names. The simplest example is the dynamic Gaussian copula (DGC) model of Crépey, Jeanblanc, and Wu (2013), which can be sufficient to deal with counterparty risk on CDS contracts. If there are also CDO tranches in the portfolio, then a Gaussian copula dependence structure is not rich enough for calibration purposes. Instead, one can use the dynamic Marshall-Olkin (DMO) common-shock model of Bielecki, Cousin, Crépey, and Herbertsson (2014b).

However, this dynamic copula methodology, reviewed in Crépey et al. (2014, Part IV), does not immediately extend to the case of bilateral counterparty risk combined with the related funding issue. As is well known since the seminal papers by Korn (1995), Cvitanic and Karatzas (1993) or El Karoui, Peng, and Quenez (1997), in presence of different borrowing and lending rates, pricing rules become nonlinear. This question has known a revival of interest in recent years in connection with the post-crisis multi-curve issue (see Piterbarg (2010), Mercurio (2014) and Bielecki and Rutkowski (2014)). Accordingly, in the case of bilateral counterparty risk combined with the related funding issue, the pricing equations for the corresponding valuation adjustment, dubbed TVA for total valuation adjustment, are implicit and nonlinear (see Crépey (2015, Part I), Brigo and Pallavicini (2014) or M. Bichuch and Sturm (2015)). Moreover, they are posed over random time intervals and may involve nonstandard, implicit terminal conditions at the first default time of a party. To deal with such equations, a first reduced-form counterparty risk modeling approach was introduced in Crépey (2015, Part II), in a rather basic immersion setup between the reference filtration of the underlying market exposure and the full model filtration progressively enlarged by the default times of the two parties. But this basic immersion setup, with a related continuity assumption on the data at default time, is too restrictive for wrong-way and gap risk applications such as counterparty risk on credit derivatives, in which case one also faces specific dependence and dimensionality challenges.

1.1 Contributions and Outline

To tackle this issue, this paper shows how an extended reduced-form approach can be applied, beyond the first approach of Crépey (2015, Part II) dubbed "basic approach" henceforth, in the above-mentioned dynamic copula models. In the first part of the paper (Sect. 2 through 5), we generalize, resorting to the notion of invariant times in Crépey and Song (2014c), the basic reduced-form approach of Crépey (2015, Part II), in view of a proper wrong-way and gap risk modeling in applications. With respect to our previous works, we also introduce a positive cure period. The TVA is modeled in terms of solutions to backward stochastic differential equations (BSDEs): the exact TVA BSDE (2.7), the full BSDE (2.8) and the reduced BSDE (3.1). The exact TVA BSDE (2.7) is derived from hedging principles. It is then approximated by the full BSDE (2.8), which can also be viewed as the exact BSDE for slightly simplified data. The reduced TVA is an auxiliary, simpler equation that, if solved, yields a solution to the full BSDE. In this sense, solving the reduced BSDE is sufficient in practice. This is

Theorem 3.1, which can also be digged out from the results of Crépey and Song (2014a), but in a more abstract setup there, mainly motivated by the converse to this theorem, i.e. any solution to the full BSDE is based on a solution to the reduced BSDE. Beyond self-containedness, the main reason why we include a direct proof of Theorem 3.1 in this paper is to show how easily it flows once the right framework has been set up, namely the condition (C) in this paper. In the easiness of the result lies its power here. But, when one thinks of it, Theorem 3.1 leaves several questions open, namely:

1) how strong is the condition (C)?

1)' Theorem 3.1 says that a solution to the reduced BSDE always yields a solution to the full BSDE, but what is then the gap between the full BSDE and the reduced BSDE?

2) how can one compute the data (coefficient $\widetilde{f}$) of the reduced BSDE, in particular the term $cdva = \gamma\widehat{\xi}$ (cf. (4.6))?

2)' can one formally establish the well posedness of the reduced BSDE, in which exact meaning?

2)" once the reduced BSDE is well posed and its data can be computed, how can one solve it numerically? is the changed measure $\mathbb{P}$ needed algorithmically?

3) what is the usefulness of all this, namely can one find concrete models of counterparty risk where the two degrees of freedom left by the condition (C) can be exploited to compute numbers more relevant than without these?

3)' better than without these?

Crépey and Song (2014c) and Crépey and Song (2014a) address the questions 1) and 1)' at a theoretical level by respectively reducing the condition (C) to a suitable integrability condition and by showing that, under an additional condition (B), the full and reduced BSDEs are essentially equivalent. Hence, solving the reduced BSDE is not an arbitrary way of approaching the full BSDE and well-posedness of the reduced BSDE, if it can be proved, implies well-posedness of the full BSDE. The present paper addresses these and the other questions in the list by checking that all the required conditions hold and why and how in concrete models of counterparty risk on credit derivatives. As a prerequisite, a marked default times methodology is developed as a standard machinery to produce concrete models where all conditions are satisfied. The general reduced-form approach of Theorem 3.1 is then implemented through marked default times in dynamic extensions of two well known copula models: the Gaussian copula and the exponential (or Marshall Olkin) copula. In each case, the dynamic copula model full filtration $\mathbb{G}$ is decomposed as $\mathbb{G} = \mathbb{F} \vee \mathbb{K}$, where $\mathbb{F}$ is a reference filtration and $\mathbb{K}$ is a filtration carrying information about the default times of the two counterparties. However, this is done for a default filtration $\mathbb{K}$ possibly larger than the filtration $\mathbb{H}$, generated by the default indicator processes of the counterparties, used in the basic approach. In particular, $\mathbb{K}$ can encode a mark conveying some additional information about the defaults, in order to account for various possible wrong-way and gap risk scenarios and features. Then, a reduced-form approach is developed in $\mathbb{F}$, but with respect to a possibly changed probability measure $\mathbb{P}$, under an integrability condition expressing that no mass is lost in the tentative measure change. The marks methodology of this paper allows addressing the dependence modeling and the high-dimensional nonlinear computational challenges posed by bilateral counterparty risk and funding costs on credit derivatives. We emphasize that the changed probability measure $\mathbb{P}$ is not needed algorithmically. All one needs in practice is an explicit expression for the intensities of the marked stopping times. Back to the above list, specific answers to the respective questions are provided by:

1) Theorems 6.2, 6.3 and 7.2,

2) Lemmas 5.3 and 7.1,

2)' Theorem 4.1 and Corollaries 6.1, 7.1 and 7.2,

2)" Lemmas 8.1 and 8.2,

3) Lemma 5.2, Theorems 6.2 and 7.2, Figures 3 and 5 compared with Figure 7, Table 5,

3)' Figures 2, 4, 5, 6 and Tables 2 through 4.

Another contribution is the introduction and study of the cure period (even if a positive cure period is also considered in Brigo and Pallavicini (2014)). Theorems 6.1 and 7.1 formally establish the main properties of the dynamic Gaussian copula and Marshall-Olkin copula models, which is not done anywhere else. In particular, Sect. B.1 provides a number of explicit formulas in the dynamic Gaussian copula model. Even if we mainly derive these formulas for the proof of Theorem 6.3, they are of independent interest.

The detailed outline of the paper is as follows. Sect. 2 fixes the setup and derives the exact and full TVA BSDEs with respect to the full model filtration $\mathbb{G}$. Sect. 3 develops an extended reduced-form approach for the full BSDE, applicable whenever the first default time of the two parties satisfies the condition (C) in this paper (i.e., essentially, is an invariant time satisfying the condition (A) in Crépey and Song (2014c), where the condition (A) means (C) but also (B) in this paper). In Sect. 4 we establish the well-posedness of the reduced BSDE under a standard CSA specification of the data. In the marked default times framework of Sect. 5, we derive a CVA/DVA (credit/debit valuation adjustment) and LVA (funding liquidity valuation adjustment) decomposition of an all-inclusive TVA. Sect. 6 and Sect. 7 apply the proposed approach in two dynamic copula models of counterparty risk on credit derivatives. In Sect. 8 numerical results are presented. An index of symbols is provided after the bibliography.

1.2 Standing Notation

We use the notation $\cdot$ for stochastic integration and $\boldsymbol{\lambda}$ for the Lebesgue measure on $\mathbb{R}_+$. We write $\int_a^b = \int_{(a,b]}$ with, in particular, $\int_a^b = 0$ whenever $a \geq b$; $x^+ = \max(x, 0)$, $x^- = \max(-x, 0) = (-x)^+$. The Borel σ field on a topological space $\mathcal{R}$ is denoted by $\mathcal{B}(\mathcal{R})$. Any "deterministic function" of real (resp. discrete) arguments is measurable with respect to the corresponding Borel (resp. powerset) σ field. Any function involving discrete arguments is continuous with respect to these, in reference to the discrete topology. We denote by $\mathcal{P}(\mathbb{F})$, $\mathcal{O}(\mathbb{F})$ and $\mathcal{R}(\mathbb{F})$ the predictable, optional and progressive σ fields with respect to a filtration $\mathbb{F}$. When a process f_t can be represented in terms of a function of some factor process X, we typically write $f(t, X_t)$, i.e. the function is denoted by the same letter as the related process. Order relationships between random variables (resp. processes) are meant almost surely (resp. in the indistinguishable sense).

Part I

Wrong Way and Gap Risks Modeling: A Marked Default Times Perspective

2 Counterparty Risk Setup

In this section we offer a concise introduction to bilateral counterparty risk under funding constraints, following the replication approach developed originally in Crépey (2015) and recently revised in book form, using the same unified notation and same sign convention as the present paper, in Crépey et al. (2014). For the reader convenience we systematically refer to both book and papers[1] in the sequel. With respect to these references, we add consideration of a positive cure period. Given the current regulatory trend of extensive collateralization, this is important in practice since, once a position is fully collateralized in terms of variation margins, the gap risk related to the cure period becomes the first order residual risk.

We consider a netted portfolio of OTC derivatives traded between two defaultable counterparties, generically referred to as the "contract between the bank and its counterparty". We adopt the perspective of the bank. In particular, a cash-flow of ± 1 means ± 1 to the bank. After having "bought" the contract from its counterparty at time 0 (see the beginning of Sect. A), the bank sets-up a collateralization, hedging and funding portfolio. Collateral consists of cash or various possible eligible securities posted through margin calls as default guarantee by the two parties. The margin requirements are specified by the CSA. Regarding hedging, for simplicity, we restrict ourselves to the situation of a securely funded hedge, entirely implemented by means of swaps, short sales or repurchase agreements,

[1] Preprint versions that can be downloaded from the webpage http://screpey.free.fr.

at no upfront payment. As explained in Crépey et al. (2014, Section 4.2.1 page 87)[2], this assumption encompasses the vast majority of hedges that are used in practice. In particular, it includes (counterparty-risk-free) CDS contracts that can be used for hedging the counterparty jump-to-default exposure. We call "funder" of the bank a third party insuring funding of the bank's strategy. In practice, the funder can be composed of several entities or devices. Assumed default-free for simplicity, it plays the role of lender/borrower of last resort after exhaustion of the internal sources of funding provided to the bank through its hedge and its collateral.

Let $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, with $\mathbb{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$ satisfying the usual conditions and $\mathbb{Q}$ expectation denoted by $\mathbb{E}$, represent a prevailing pricing stochastic basis, such that all our processes are $\mathbb{G}$ adapted and all the random times of interest are $\mathbb{G}$ stopping times. The meaning of a risk-neutral pricing measure in our setup, with different funding rates in particular, is specified by a martingale condition introduced in the form of the pricing BSDE (A.2) for the contract. But, in the first place, a pricing measure in our sense must be such that the gain processes related to the trading of the hedging assets, processes denoted in vector form by $\mathcal{M}$ in Sect. A, are local martingales. As explained in the comments following Assumption 4.4.1 in Crépey et al. (2014, page 96)[3], this rules out arbitrage opportunities in the market of hedging instruments (provided one restricts attention to hedging strategies resulting in a wealth process bounded from below; see Bielecki and Rutkowski (2014, Corollary 3.1) for a formal statement).

For reasons pertaining to division and specialisation of tasks in banks, the price of the contract is computed as the difference (2.5) between a reference price P and a counterparty risk and funding adjustment Θ. The reference or "clean" price P is the mark-to-market ignoring counterparty risk and assuming a risk-free funding rate. Specifically, we denote by r_t a progressively measurable OIS rate process and by $\beta_t = e^{-\int_0^t r_s ds}$ the corresponding discount factor. This OIS rate, where OIS stands for overnight indexed swap rate, stands together as the best market proxy of the risk-free rate and as the typical reference rate for the remuneration of the collateral. Let a finite variation process D represent the cumulative promised dividend process of the contract, where by promised we mean contractual cash-flows ignoring counterparty risk. Hence,

$$\beta_t P_t = \mathbb{E}\left(\int_t^T \beta_s dD_s \,\middle|\, \mathcal{G}_t \right), \ \ t \in [0, T]. \tag{2.1}$$

Here T is a relevant time horizon, typically the one of the CSA. If there is some residual value in the contract at that time, it is treated as a dividend $(D_T - D_{T-})$ at time T.

But the two parties are defaultable. Let τ_b and τ_c stand for the default times of the bank and of the counterparty, modeled as $\mathbb{G}$ stopping times with $(\mathbb{G}, \mathbb{Q})$ intensities γ^b and γ^c. As a consequence, the first counterparty default time $\tau = \tau_b \wedge \tau_c$ is a stopping time with intensity γ such that $\max(\gamma^b, \gamma^c) \le \gamma \le \gamma^b + \gamma^c$, with indistinguishable equality in the right hand side if $\tau_b \neq \tau_c$ a.s.. Note that in such an intensity setup, any event $\{\tau_b = t\}$ or $\{\tau_c = t\}$, for any fixed time t, has zero probability and can be ignored in the analysis. An additional feature is a time lag $\delta \ge 0$, called the cure period, typically taken as ten (resp. five) days in the case of bilateral (resp. centrally cleared) transactions, between the first to default time τ of a party and the liquidation of the contract. We assume that a risk-free liquidator takes over the hedge of a defaulted party during the cure period. For every time t, we write

$$\bar{t} = t \wedge T, \ \ t^\delta = t + \delta, \ \ \bar{t}^\delta = \mathbb{1}_{t<T} t^\delta + \mathbb{1}_{\{t \ge T\}} T.$$

The cure period results in an effective time horizon $\bar{\tau}^\delta$ for the TVA problem. If $\tau < T$, the contractual dividends dD_t cease to be paid from time τ onwards and a terminal cash-flow $\mathfrak{R}$ paid to the bank at time τ^δ closes out its position.

Until $\bar{\tau}$, the bank needs to fund its position, i.e. the contract and its collateral (the cost of funding the hedge is already accounted for in the primary gain martingale $\mathcal{M}$). We denote by $g = g_t(\pi)$ an $\mathcal{R}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$ measurable funding coefficient such that

$$(-r_t \mathcal{W}_t + g_t(-\mathcal{W}_t))dt \tag{2.2}$$

represents the bank's funding cost over $(t, t + dt)$, where $\mathcal{W}$ is the value process of the hedging, collateralization and funding portfolio of the bank. We refer the reader to Sect. 4 for concrete specifications

[2] Or Crépey (2015, Part I, Section 2.1) in the journal version.

[3] Or Crépey (2015, Part I, Assumption 4.1) in the journal version.

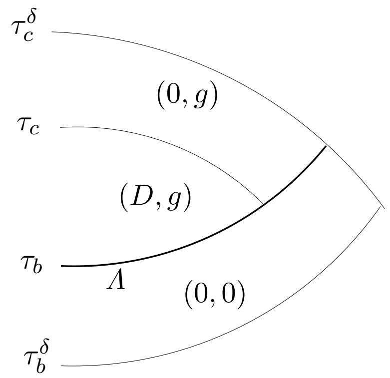

Fig. 1 Representation of the data of the TVA pricing problem in a (t,ω) state space representation, focusing on the default and liquidation times and ignoring T to alleviate the picture, i.e. "for $T=\infty$". The data in parentheses refer to the effective dividends and funding costs of the bank depending on the time t and scenario ω.

of the data, including $\mathfrak{R}$ and g, and to Lemma A.1 for the resulting SDE satisfied by $\mathcal{W}$. If a DVA is acknowledged, one may want to include likewise a funding windfall benefit of the bank at its default, in the form of an additional cashflow to the bank by the amount

$$(-\mathcal{W}_{\tau_b-}-\mathcal{C}_{\tau_b-})^+\Lambda \text{ at } \tau_b \text{ if } \tau_b<\bar{\tau}^\delta. \tag{2.3}$$

Here the process $(-\mathcal{C})$ represents the amount of collateral funded by the bank (see Sect. 4), so that $(-\mathcal{W}_{\tau_b-}-\mathcal{C}_{\tau_b-})^+$ represents the debt of the bank to its funder right before τ_b. The constant Λ corresponds to the fractional loss of the external funder (one minus the recovery rate of the bank to its funder).

2.1 TVA BSDEs

Figure 1 provides a stylized (t,ω) state space representation of the data, focusing on the default and liquidation times and ignoring T to alleviate the picture, i.e. "for $T=\infty$". Let

$$Q_t=P_t+\Delta_t, \text{ where } \beta_t\Delta_t=\int_{[\tau,t]}\beta_s dD_s \tag{2.4}$$

(in particular, $\Delta_t=0$ and $Q_t=P_t$ for $t<\tau$), and let

$$f_t(\vartheta)=g_t(Q_t-\vartheta)-r_t\vartheta \quad (\vartheta\in\mathbb{R})$$
$$\tau'=\tau_b\wedge\tau_c^\delta,\ \ \bar{\tau}'=\mathbb{1}_{\tau<T}\tau'+\mathbb{1}_{\{\tau\geq T\}}T.$$

In Sect. A, we start from a TVA primarily defined over $[0,\bar{\tau}^\delta]$ as

$$\Theta_t=Q_t-\Pi_t, \tag{2.5}$$

where Π represents the overall cost of hedging the contract (inclusive of counterparty risk and funding costs, as opposed to the clean price P). Let

$$\xi=Q_{\tau^\delta}-\mathfrak{R},\ \ \bar{\xi}_t=\mathbb{E}(\beta_t^{-1}\beta_{\tau^\delta}\xi\,|\,\mathcal{G}_t)\ (t\leq\bar{\tau}^\delta), \tag{2.6}$$

assuming integrability of the so called counterparty risk exposure ξ. Under a replication assumption, Lemma A.2 derives the following exact TVA BSDE for the $(\mathbb{G},\mathbb{Q})$ semimartingale Θ on $[0,\bar{\tau}']$:

$$\begin{aligned}&\Theta_{\bar{\tau}'}=\mathbb{1}_{\{\tau<T\}}\left(\bar{\xi}_{\tau'}-(Q_{\tau_b-}-\mathcal{C}_{\tau_b-}-\Theta_{\tau_b-})^+\mathbb{1}_{\{\tau'=\tau_b\}}\Lambda\right) \text{ and}\\&d\mu_t:=d\Theta_t+f_t(\Theta_t)dt \text{ is a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } [0,\bar{\tau}'].\end{aligned} \tag{2.7}$$

This equation implicitly includes the time integrability of $f_\cdot(\Theta_\cdot)$ on $[0,\bar{\tau}']$. Similar convention is in force and will not be repeated regarding the various BSDEs introduced below. Note that the TVA problem is originally posed over the domain $[0,\bar{\tau}^{\delta}]$, which corresponds to the union of the three subdomains in Figure 1, with effective dividend and funding data respectively abbreviated as (D,g), $(0,g)$ and $(0,0)$. But since data $(0,0)$ simply means, in terms of pricing, taking conditional expectation of the terminal condition discounted at the risk-free rate, the computations of Sect. A show that the TVA BSDE can be formulated as (2.7) on the smaller domain $[0,\bar{\tau}'] \subseteq [0,\bar{\tau}^{\delta}]$, modulo the modified terminal condition $\bar{\xi}$ at $\bar{\tau}'$ instead of ξ at $\bar{\tau}^{\delta}$.

The replication assumption that underlies the TVA BSDE (2.7) here, or (2.8) for slightly modified data below, implies that the bank actively risk manages its position. Otherwise, the risk-neutral equations of this paper are not enough conservative. In view of the incompleteness of the TVA market, our approach is only a first step. But any incomplete market approach, adding in some way or another one layer of optimization to the analysis (e.g. utility maximisation), would most likely be hardly feasible in practice, especially on real-life portfolios with tens to hundreds of thousands of contracts. The TVA problem entails some nonlinearities but banks can't really deal with nonlinear pricing rules, because these are too complicated to manage at the portfolio level. Hence, one should aim in the end for "the best linear TVA approximation".

Toward this aim, it is useful to derive an equivalent but simpler reduced BSDE, which will be the topic of Sect. 3. But the reduction of the exact TVA BSDE (2.7) would lead to nested BSDEs, where the coefficient of a reduced BSDE on $[0,T]$ is defined in terms of solutions to auxiliary reduced BSDEs "conditional on $\mathcal{G}_{\bar{\tau}}$" (see Crépey and Song (2014b, Section 4) for such a treatment in a preprint version of the present paper). The nested feature arises for, in the scenarios where $\tau_c \le \tau_b \wedge T$, the nonlinear funding coefficient g of the bank is still in force, hence the funding data of the problem remain nonlinear, on the time interval $[\bar{\tau},\bar{\tau}'] = [\tau_c,\bar{\tau}']$ (upper right subdomain in Figure 1). However, because the cure period δ is only a few days, the quantitative impact on Θ of the coefficient g on the time interval $[\bar{\tau},\bar{\tau}']$ can only be very limited. Hence, in order to avoid the numerical burden of nested BSDEs, we work henceforth with the following full TVA BSDE:

$$\begin{aligned}
&\Theta_{\bar{\tau}} = \mathbb{1}_{\{\tau<T\}}\left(\bar{\xi}_{\tau} - (P_{\tau_b-} - \mathcal{C}_{\tau_b-} - \Theta_{\tau_b-})^{+}\mathbb{1}_{\{\tau=\tau_b\}}\Lambda\right),\\
&d\mu_t = d\Theta_t + f_t(\Theta_t)dt \text{ is a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } [0,\bar{\tau}].
\end{aligned} \tag{2.8}$$

This equation is obtained if one replaces g and Λ by 0 on the time interval $[\bar{\tau},\bar{\tau}']$ in the exact TVA BSDE (2.7) or everywhere in Sect. A, which means replacing $(0,g)$ by $(0,0)$ and Λ by 0 in the upper right subdomain in Figure 1. Proceeding in this way, the computations of Sect. A show that the domain of computation of the TVA can be reduced further, from $[0,\bar{\tau}^{\delta}]$ initially to $[0,\bar{\tau}']$ as explained above regarding the exact TVA BSDE, to even $[0,\bar{\tau}]$ now for the appropriately modified terminal condition regarding the full TVA BSDE (2.8), with all data set to zero outside $[0,\bar{\tau}]$. Reformulating the problem on the smaller domain $[0,\bar{\tau}]$ makes it simpler in view of the reduced-form analysis of Sect. 3, for τ, as opposed to τ', has an intensity. Note that (2.8) is also, essentially, the so called master equation in Brigo and Pallavicini (2014).

Of course, a TVA BSDE modeling approach based on (2.8) supposes the existence (at the very least) of a solution to (2.8), which is still a quite nonstandard BSDE. Even if a bit simpler than (2.7), (2.8) does not fall in the classical scope of BSDEs with random but explicit terminal conditions (see e.g. Kruse and Popier (2014, Sect. 5)). But the results of the next section allow reducing the nonstandard $(\mathbb{G},\mathbb{Q})$ BSDE (2.8) to the simpler BSDE (3.5) with a null terminal condition at the fixed time horizon T, stated with respect to a smaller filtration $\mathbb{F}$ and a possibly changed probability measure $\mathbb{P}$. Hence, well-posedness for the full BSDE (2.8) can be established by showing well-posedness for the reduced BSDE (3.5), which is done in Theorem 4.1. Even in the case where $\Lambda = 0$ and the terminal condition of (2.8) is explicit (does not depend on Θ_{τ_b-}), having a null terminal condition as in (3.5) is key in the numerical scheme used in Sect. 8.

3 Reduced Form Approach

In this section we develop a reduced-form approach for the full BSDE (2.8), beyond the basic immersion setup of Crépey (2015, Part II), in view of application to credit derivatives in the second part of this paper. By Corollary 3.23 2) in He, Wang, and Yan (1992), for any $\mathcal{G}_{\tau}$ measurable random variable ζ, there exists a $\mathbb{G}$ predictable process $\widehat{\zeta}$ such that

$$\mathbb{1}_{\{\tau<\infty\}}\mathbb{E}[\zeta|\mathcal{G}_{\tau-}] = \mathbb{1}_{\{\tau<\infty\}}\widehat{\zeta}_{\tau}. \tag{3.1}$$

If ζ is integrable, then the Lebesgue integral $\gamma_t \widehat{\zeta}_t dt$ exists and is independent of the choice of a version of $\widehat{\zeta}$, and

$$\zeta_t dJ_t + \gamma_t \widehat{\zeta}_t dt \tag{3.2}$$

is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $\mathbb{R}_+$(cf. Corollary 5.31 1) in He et al. (1992)). In particular, with the abuse of notation that $\widehat{\xi}$ below refers to $\bar{\xi}$ instead of ξ, let $\widehat{\xi}$ be a $\mathbb{G}$ predictable process such that, on $\{\tau < \infty\}$,

$$\widehat{\xi}_\tau = \mathbb{E}(\bar{\xi}_\tau \,|\, \mathcal{G}_{\tau-}) = \mathbb{E}(\beta_\tau^{-1} \beta_{\tau+\delta} \xi \,|\, \mathcal{G}_{\tau-}). \tag{3.3}$$

For $t \in [0, \bar{\tau}]$ and $\vartheta \in \mathbb{R}$, we write

$$\begin{aligned} \widehat{f}_t(\vartheta) &= f_t(\vartheta) + \gamma_t \widehat{\xi}_t - (P_t - \mathcal{C}_t - \vartheta)^+ \gamma_t^b \Lambda - \gamma_t \vartheta \\ &= \gamma_t \widehat{\xi}_t + g_t(P_t - \vartheta) - (P_t - \mathcal{C}_t - \vartheta)^+ \gamma_t^b \Lambda - (r_t + \gamma_t)\vartheta. \end{aligned} \tag{3.4}$$

Let $J = \mathbb{1}_{[0,\tau)}$ and $Y^{\tau-} = JY + (1-J)Y_-$, for any left-limited process Y. Conditions of the following kind are studied at the theoretical level in Crépey and Song (2014a,2014c).

Condition (C). There exist:

(C.1) a subfiltration $\mathbb{F}$ of $\mathbb{G}$ satisfying the usual conditions such that $\mathbb{F}$ semimartingales stopped at τ are $\mathbb{G}$ semimartingales,

(C.2) a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that any $(\mathbb{F}, \mathbb{P})$ local martingale stopped at $(\tau-)$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$,

(C.3) an $\mathbb{F}$ progressive reduction $\widetilde{f}_t(\vartheta)$ of $\widehat{f}_t(\vartheta)$, i.e. an $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ function $\widetilde{f}_t(\vartheta)$ such that $\int_0^\cdot \widehat{f}_t(\vartheta)dt = \int_0^\cdot \widetilde{f}_t(\vartheta)dt$ on $[0, \bar{\tau}]$.

Let S denote the Azéma supermartingale of τ, i.e. the process such that $S_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t)$, $t \geq 0$.

Condition (B). Any $\mathbb{G}$ predictable process Y admits an $\mathbb{F}$ predictable process $\widetilde{Y}$ that coincides with Y on $(0, \tau]$.

The process $\widetilde{Y}$, uniquely determined until the first zero of S, is called the $\mathbb{F}$ predictable reduction of Y.

The condition (C.1) relates to the $(\mathcal{H}')$ hypothesis between $\mathbb{F}$ and $\mathbb{G}$, i.e. $\mathbb{F}$ semimartingales are $\mathbb{G}$ semimartingales (see Bielecki, Jeanblanc, and Rutkowski (2009)). The condition (C.3) is a mild technical condition, which holds in particular under the condition (B). See Crépey and Song (2014c), where it is shown that the condition (B) also implies the second part in (C.1) (which is also encompassed by (C.2)). If $(\mathbb{F}, \mathbb{P})$ local martingales don't jump at τ, then the condition (C.2) says that $(\mathbb{F}, \mathbb{P})$ local martingales stopped at τ are $(\mathbb{G}, \mathbb{Q})$ local martingales. The conditions (C.1-2-3) with (C.1) and (C.3) reinforced as (B) above yield the condition (A) in Crépey and Song (2014c). In the case where $\mathbb{P} = \mathbb{Q}$, these properties are related to the notions of immersion of $\mathbb{F}$ into $\mathbb{G}$, i.e. $(\mathbb{F}, \mathbb{Q})$ local martingales are $(\mathbb{G}, \mathbb{Q})$ local martingales (see Bielecki et al. (2009)), and of an $\mathbb{F}$ pseudo-stopping time τ, i.e. $(\mathbb{F}, \mathbb{Q})$ local martingales stopped at τ are $(\mathbb{G}, \mathbb{Q})$ local martingales (see Nikeghbali and Yor (2005)). However, even in this "immersion" case where $\mathbb{P} = \mathbb{Q}$, the condition (C) offers a richer framework than a standard reduced-form intensity model of credit risk, where the full model filtration $\mathbb{G}$ is given as the reference filtration $\mathbb{F}$ progressively enlarged by τ, i.e. "$\mathbb{G} = \mathbb{F} \vee \mathbb{H}$" in the standard notation of Bielecki et al. (2009). In fact, under (C.1-2-3), the full filtration $\mathbb{G}$ can be bigger than $\mathbb{F} \vee \mathbb{H}$. In particular, even for $\mathbb{P} = \mathbb{Q}$, the conditions (C.1-2-3) do not exclude a jump of an $\mathbb{F}$ adapted càdlàg process at time τ, which happens for instance with a nonvanishing random variable $\Delta_\tau = D_\tau - D_{\tau-}$ in the DMO model of Sect. 7. By contrast, a jump of an $\mathbb{F}$ adapted càdlàg process at time τ cannot happen in a basic reduced-form credit risk setup (see Crépey et al. (2014, Lemma 13.7.3(ii) page 331)[4]). Above all, the flexibility of the condition (C) comes from the possibility to choose $(\mathbb{F}, \mathbb{P})$ ensuring (C.1-2-3). See Sect. 6 and Sect. 7 for concrete examples, with $\mathbb{P} \neq \mathbb{Q}$ and $\Delta_\tau = 0$ in the first case and $\mathbb{P} = \mathbb{Q}$ but

[4] Or Lemma 2.1(ii) in the journal version Crépey (2015, Part II).

$\Delta_\tau \neq 0$ in the second case, which we view as respective wrong-way risk and gap risk stylized examples (as our concluding figure 7 will illustrate).

The result that follows can also be retrieved from Crépey and Song (2014a), but in a much more abstract setup there, mainly motivated by the converse to this result. Hence, we provide a self-contained proof. Here this result is derived under the "minimal condition" (C), instead of (A) in Crépey and Song (2014a), where (A) in Crépey and Song (2014a) means (B) and (C) in this paper. But, as explained above, the condition (B) is latent in (C.3) and (C.1), hence the random time τ here is essentially an invariant time in the sense of the condition (A) in Crépey and Song (2014c).

Theorem 3.1 *Under the condition (C), assume that an* $(\mathbb{F},\mathbb{P})$ *semimartingale* $\widetilde{\Theta}$ *satisfies the following reduced TVA BSDE on* $[0,T]$*:*

$$\widetilde{\Theta}_T = 0 \text{ and } d\widetilde{\mu}_t := d\widetilde{\Theta}_t + \widetilde{f}_t(\widetilde{\Theta}_t)dt \text{ is an } (\mathbb{F},\mathbb{P}) \text{ local martingale on } [0,T]. \tag{3.5}$$

Let $\Theta = \widetilde{\Theta}$ *on* $[0,\bar{\tau})$ *and* $\Theta_{\bar{\tau}} = \mathbb{1}_{\{\tau<T\}}\left(\bar{\xi}_\tau - (P_{\tau-} - \mathcal{C}_{\tau-} - \widetilde{\Theta}_{\tau-})^+\mathbb{1}_{\{\tau=\tau_b\}}\Lambda\right)$*. Then* Θ *satisfies the full TVA equation (2.8) on* $[0,\bar{\tau}]$ *and we have, for* $t\in[0,\bar{\tau}]$*,*

$$\begin{aligned} d\mu_t = d\widetilde{\mu}_t^{\tau-} &- \big(\bar{\xi}_\tau - (P_{\tau-} - \mathcal{C}_{\tau-} - \widetilde{\Theta}_{\tau-})^+\mathbb{1}_{\{\tau=\tau_b\}}\Lambda - \widetilde{\Theta}_{\tau-}\big)dJ_t \\ &- \big((\widehat{\xi}_t - \widetilde{\Theta}_t)\gamma_t - (P_t - \mathcal{C}_t - \widetilde{\Theta}_t)^+\gamma_t^b\Lambda\big)dt. \end{aligned} \tag{3.6}$$

Proof. By definition of Θ here, a $(\mathbb{G},\mathbb{Q})$ semimartingale by (C.1), we have, for $t\in[0,\bar{\tau}]$:

$$\begin{aligned} d\Theta_t &= d(J_t\widetilde{\Theta}_t) - \left(\bar{\xi}_\tau - (P_{\tau-} - \mathcal{C}_{\tau-} - \widetilde{\Theta}_{\tau-})^+\mathbb{1}_{\{\tau=\tau_b\}}\Lambda\right)dJ_t \\ &= d\widetilde{\Theta}_t^{\tau-} + \widetilde{\Theta}_{\tau-}\,dJ_t - \left(\bar{\xi}_\tau - (P_{\tau-} - \mathcal{C}_{\tau-} - \widetilde{\Theta}_{\tau-})^+\mathbb{1}_{\{\tau=\tau_b\}}\Lambda\right)dJ_t. \end{aligned} \tag{3.7}$$

Then by (3.5), for $t\in[0,\bar{\tau}]$:

$$\begin{aligned} -d\Theta_t &= \widetilde{f}_t(\widetilde{\Theta}_t)dt - d\widetilde{\mu}_t^{\tau-} + \left(\bar{\xi}_\tau - (P_{\tau-} - \mathcal{C}_{\tau-} - \widetilde{\Theta}_{\tau-})^+\mathbb{1}_{\{\tau=\tau_b\}}\Lambda - \widetilde{\Theta}_{\tau-}\right)dJ_t \\ &= f_t(\Theta_t)dt - d\widetilde{\mu}_t^{\tau-} + \left(\bar{\xi}_\tau - (P_{\tau-} - \mathcal{C}_{\tau-} - \widetilde{\Theta}_{\tau-})^+\mathbb{1}_{\{\tau=\tau_b\}}\Lambda - \widetilde{\Theta}_{\tau-}\right)dJ_t \\ &\quad + \left((\widehat{\xi}_t - \widetilde{\Theta}_t)\gamma_t - (P_t - \mathcal{C}_t - \Theta_t)^+\gamma_t^b\Lambda\right)dt, \end{aligned}$$

by (C.3). By (C.2), $\widetilde{\mu}_t^{\tau-}$ is a $(\mathbb{G},\mathbb{Q})$ local martingale, as is also on $[0,\bar{\tau}]$

$$\left(\bar{\xi}_\tau - (P_{\tau-} - \mathcal{C}_{\tau-} - \widetilde{\Theta}_{\tau-})^+\mathbb{1}_{\{\tau=\tau_b\}}\Lambda - \widetilde{\Theta}_{\tau-}\right)dJ_t + \left((\widehat{\xi}_t - \widetilde{\Theta}_t)\gamma_t - (P_t - \mathcal{C}_t - \Theta_t)^+\gamma_t^b\Lambda\right)dt$$

(cf. (3.2)). This yields the decomposition (3.6) of $d\mu_t = d\Theta_t + f_t(\Theta_t)dt$, which implies the martingale condition in (2.8), where the terminal condition holds by definition of $\Theta_{\bar{\tau}}$ in the theorem. ∎

Summarizing so far, assuming the condition (C) on an invariant default time τ, based on Theorem 3.1, we can design a TVA process in terms of a solution to the reduced BSDE (3.5). Moreover, this approach is not arbitrary. In fact, under the additional condition (B) that is latent in (C.3) and (C.1), the results of Crépey and Song (2014a) show that the full and reduced BSDEs are equivalent (assuming a positive Azéma supermartingale S of τ, as typical in applications). In particular, as will be seen in the final statements of Theorems 6.2 and 7.2, the condition (B) holds and S is positive in the concrete models of Sect. 6 and 7, so that this equivalence applies to each of the reduced TVA BSDEs considered in Corollaries 6.1, 7.1 and 7.2. Moreover, in Theorem 4.1, the reduced BSDE (3.5) will be seen to well-posed under mild technical conditions on the data. As a consequence of the equivalence between (2.8) and (3.5), the full BSDE (2.8) is well-posed too.

4 Specification of the Data

For applications, we need to specify the close-out cash flow $\mathfrak{R}$, hence the counterparty risk exposure $\xi = Q_{\tau^\delta} - \mathfrak{R}$, and the funding coefficient $g_t(\pi)$, conformly with usual CSA specifications. Let V denote the variation margin, where $V \geq 0$ (resp. ≤ 0) means collateral posted by the counterparty and received by the bank (resp. posted by the bank and received by the counterparty). Let $I \geq 0$ (resp. $\mathfrak{I} \leq 0$) represent the initial margin posted by the counterparty (resp. the negative of the initial margin posted by the bank). Let

$$C = V + I \quad \text{and} \quad \mathfrak{C} = V + \mathfrak{I}, \tag{4.1}$$

respectively the total collateral guarantee for the bank and the negative of the total collateral guarantee for the counterparty.

Remark 4.1 In practice, the variation and initial margin calls are executed according to a discrete schedule (t_l). Specifically, the variation margin V tracks the mark-to-market P at grid times $t_l < \tau$. The initial margins I and $\mathfrak{I}$ are also updated at some of the times $t_l < \tau$, based on risk measures of the profit-and-loss of the variation-margined position at the horizon δ of the cure period (see Brigo and Pallavicini (2014) and Armenti and Crépey (2014) for detailed specifications). Hence, the margin processes are stopped at the greatest $t_l < \tau$ (or $= \tau$, but the corresponding event has zero probability in a default intensity setup).

In particular, we assume all the margin processes are stopped at τ. The bank's close-out cash-flow $\mathfrak{R}$ is derived from the debts of the counterparty to the bank and vice versa, respectively modeled at time τ^δ as

$$\chi = (Q_{\tau^\delta} - C_\tau)^+, \;\; \mathfrak{X} = (Q_{\tau^\delta} - \mathfrak{C}_\tau)^-. \tag{4.2}$$

Note that $\chi \times \mathfrak{X} \equiv 0$, by nonnegativity of I and $(-\mathfrak{I})$. The close-out cash-flow is modeled as

$$\mathfrak{R} = \begin{cases} C_\tau + R_c \chi \text{ if } \chi > 0 \text{ and } \tau_c \leq \tau_b^\delta, \\ \mathfrak{C}_\tau - R_b \mathfrak{X} \text{ if } \mathfrak{X} > 0 \text{ and } \tau_b \leq \tau_c^\delta, \\ \quad Q_{\tau^\delta} \text{ otherwise.} \end{cases} \tag{4.3}$$

Here R_c and R_b stand for constant recovery rates of the counterparty to the bank and vice versa. The ensuing counterparty risk exposure of the bank results from the left hand side in (2.6) as

$$\xi = Q_{\tau^\delta} - \mathfrak{R} = \mathbb{1}_{\{\tau_c \leq \tau_b^\delta\}}(1 - R_c)\chi - \mathbb{1}_{\{\tau_b \leq \tau_c^\delta\}}(1 - R_b)\mathfrak{X}. \tag{4.4}$$

Remark 4.2 The expression (4.4) for the counterparty risk exposure ξ is consistent with the one in Brigo and Pallavicini (2014), where it is also explained that, if the bank is a client of a counterparty acting as member of a clearinghouse, the above formalism can still be used just by setting $R_c = 1$ and $I = 0$. In fact, as members of a clearinghouse are backed-up by other members if they default, their actual recovery rate is immaterial and everything happens for their clients as if R_c would be one. In addition, as clearing members don't post initial margins to their clients but to the clearinghouse, the initial margin posted by the counterparty has no impact on the funding costs of the bank in this case, so that everything happens for the bank as if I would be zero. The situation where the bank itself is a member (as opposed to a client) of a clearinghouse is dealt with in Armenti and Crépey (2014).

The collateral funded by the bank is $(-\mathcal{C})$, where $\mathcal{C} = \mathfrak{C} + I = V + I + \mathfrak{I}$ (assuming all the margins re-hypothecable). We assume that posted collateral is remunerated at a rate $(r_t + c_t)$ and that the bank can invest (resp. get some unsecured funding) at a rate $(r_t + \lambda_t)$ (resp. $(r_t + \bar{\lambda}_t)$). This corresponds to the following dt-funding cost of the bank, for $t < \bar{\tau}'$:

$$\underbrace{J_t(r_t + c_t)\mathcal{C}_t}_{\text{remuneration of the collateral}} \qquad + \underbrace{(r_t + \bar{\lambda}_t)\,(-\mathcal{W}_t - \mathcal{C}_t)^+ - (r_t + \lambda_t)\,(-\mathcal{W}_t - \mathcal{C}_t)^-}_{\text{funding costs / benefits}},$$

which is of the form (2.2) with in particular, for $t < \tau$,

$$g_t(\pi) = c_t \mathcal{C}_t + \bar{\lambda}_t\,(\pi - \mathcal{C}_t)^+ - \lambda_t\,(\pi - \mathcal{C}_t)^-\,. \tag{4.5}$$

Remark 4.3 This expression of g corresponds to the case of re-hypothecable margins, as is typical for the variation margin, whereas the initial margins are sometimes segregated. In the case of segregated initial margins, the expression for g changes slightly, but the overall structure (2.2) of the funding costs is preserved.

Assuming (4.5), $\widehat{f}_t(\vartheta)$ in (3.4) is such that, on $[0, \bar{\tau}]$,

$$\begin{aligned}\widehat{f}_t(\vartheta) + (r_t + \gamma_t)\vartheta &= \gamma_t\widehat{\xi}_t + c_t\mathcal{C}_t + \bar{\lambda}_t\,(P_t - \mathcal{C}_t - \vartheta)^+ - \lambda_t\,(P_t - \mathcal{C}_t - \vartheta)^- - (P_t - \mathcal{C}_t - \vartheta)^+\gamma_t^b\Lambda \\ &= \underbrace{\gamma_t\widehat{\xi}_t}_{cdva_t} + \underbrace{c_t\mathcal{C}_t + \widetilde{\lambda}_t\,(P_t - \mathcal{C}_t - \vartheta)^+ - \lambda_t\,(P_t - \mathcal{C}_t - \vartheta)^-}_{lva_t(\vartheta)},\end{aligned} \tag{4.6}$$

where $\widetilde{\lambda}_t = \bar{\lambda}_t - \gamma_t^b\Lambda$ can be interpreted as a liquidity borrowing spread of the bank, net of its credit spread to its external funder. From the perspective of the bank, the term $\gamma_t\widehat{\xi}_t$ represents the counterparty risk component of $\widehat{f}_t(\vartheta)$, whereas the remaining terms can be interpreted as a funding liquidity component. The positive (resp. negative) components of $\widehat{f}_t(\vartheta)$ can be considered as deal adverse (resp. deal friendly) as they increase (resp. decrease) the TVA Θ of the bank. Depending on the sign of $\Pi = Q - \Theta$, a "less positive" Π is interpreted as a lower buyer price by the bank and a "more negative" Π as a higher seller price by the bank.

Remark 4.4 The materiality of a debit benefit at own default represented by a DVA proportional to $(1 - R_b)$ and of a funding benefit at own default proportional to Λ are clearly subject to caution, unless corresponding hedges allow the bank to monetize these benefits before it defaults. Otherwise, the bank should better set $R_b = 1$ and $\Lambda = 0$ in the equations, a best (at least, the most conservative) practice that we refer to as the asymmetrical TVA approach. This avoids reckoning "fake benefits" or benefits to bondholders only, whereas a sound management should only consider the interest of the shareholders (see Albanese, Brigo, and Oertel (2013) and Albanese and Iabichino (2013)).

The assumptions in the following reduced BSDE well-posedness result, even though not minimal, are sufficient for our later purposes in this paper. In particular, the boundedness assumption on P, from which the one on the margins V, I and $\mathfrak{I}$ follows quite naturally, is satisfied in the case of vanilla credit derivatives such as CDS contracts and CDO tranches (cf. (5.12)). Note that, on $[0, \bar{\tau}]$,

$$\begin{aligned}\widehat{f}_t(\vartheta) &= \gamma_t\widehat{\xi}_t + c_t\mathcal{C}_t + \widetilde{\lambda}_t\,(P_t - \mathcal{C}_t - \vartheta)^+ - \lambda_t\,(P_t - \mathcal{C}_t - \vartheta)^- - (r_t + \gamma_t)\vartheta \\ &= \gamma_t\widehat{\xi}_t + c_t\mathcal{C}_t + \bar{\lambda}_t\,(P_t - \mathcal{C}_t - \vartheta)^+ - \lambda_t\,(P_t - \mathcal{C}_t - \vartheta)^- - \gamma_t\,(P_t - \mathcal{C}) \\ &\quad + (P_t - \mathcal{C}_t - \vartheta)^+\,(\gamma_t - \gamma_t^b\Lambda) - \gamma_t\,(P_t - \mathcal{C} - \vartheta)^- - r_t\vartheta.\end{aligned} \tag{4.7}$$

Assuming the condition (B) and denoting by U' the $\mathbb{F}$ predictable reduction of U_-, for any $\mathbb{G}$ semimartingale U, we may and do henceforth choose for $\widetilde{f}_t(\vartheta)$ the process obtained from $\widehat{f}_t(\vartheta)$ by replacing each process U involved in (4.7) by the corresponding process U'. Let $\|U\|^p_{\widetilde{\mathcal{H}}_p} = \widetilde{\mathbb{E}}\int_0^T U_t^p dt$ $(p > 0)$, where $\widetilde{\mathbb{E}}$ means $\mathbb{P}$ expectation.

Theorem 4.1 *Under the condition (B), assuming λ', $\bar{\lambda}'$ and r' bounded from below on $[0, T]$, P', V', I' and $\mathfrak{I}'$ bounded on $[0, T]$ and c', λ', $\bar{\lambda}'$, r', γ' in $\widetilde{\mathcal{H}}_2$, then the resulting reduced BSDE (3.5) is well-posed in $\widetilde{\mathcal{H}}_2$, where well-posedness includes existence, uniqueness, comparison and the standard BSDE a priori bound and error estimates. The $\widetilde{\mathcal{H}}_2$ solution $\widetilde{\Theta}$ to (3.5) satisfies*

$$\widetilde{\Theta}_t = \widetilde{\mathbb{E}}\left[\int_t^T \widetilde{f}_s(\widetilde{\Theta}_s)ds \,\Big|\, \mathcal{F}_t\right] = \widetilde{\mathbb{E}}\left[\int_t^T e^{-\int_t^s \gamma'_r dr}\bar{f}_s(\widetilde{\Theta}_s)ds \,\Big|\, \mathcal{F}_t\right],\ \ t \in [0, T], \tag{4.8}$$

where we set $\bar{f}_t(\vartheta) = \widetilde{f}_t(\vartheta) + \gamma'_t\vartheta$.

Proof. Assuming λ', $\bar{\lambda}'$ and r' bounded from below, it follows from (4.7), where $\gamma_t \geq \gamma_t^b\Lambda \geq 0$, that $\widetilde{f}_t(\vartheta)$ satisfies the classical BSDE monotonicity assumption

$$\big(\widetilde{f}_t(\vartheta) - \widetilde{f}_t(\vartheta')\big)(\vartheta - \vartheta') \leq C(\vartheta - \vartheta')^2, \tag{4.9}$$

for some constant C. Hence, by application of the results in Kruse and Popier (2014), the reduced BSDE (3.5) with coefficient $\widetilde{f}_t(\vartheta)$ is well-posed in $\widetilde{\mathcal{H}}_2$ (which includes existence, uniqueness, comparison and the standard BSDE bound and error estimates) as soon as the following auxiliary integrability conditions hold:

$$\sup_{|\vartheta|\le\bar{\vartheta}} |\widetilde{f}_\cdot(\vartheta) - \widetilde{f}_\cdot(0)| \in \widetilde{\mathcal{H}}_1 \ (\bar{\vartheta} > 0), \ \ \widetilde{f}_\cdot(0) \in \widetilde{\mathcal{H}}_2. \tag{4.10}$$

Since

$$|\widetilde{f}(\vartheta) - \widetilde{f}(0)| \le \big(|\bar{\lambda}'| + |\lambda'| + |r'| + 3\gamma'\big)|\vartheta|,$$

(4.10) holds in particular for P', V', I' and $\mathfrak{I}'$ bounded and c', λ', $\bar{\lambda}'$, r', γ' in $\widetilde{\mathcal{H}}_2$. The left hand side identity in (4.8) is just the usual integral representation for an $\widetilde{\mathcal{H}}_2$ solution $\widetilde{\Theta}$ to the reduced BSDE (3.5), whereas the right hand side identity is obtained by treating the $(-\gamma'\vartheta)$ term in $\widetilde{f}$ as a discount factor at rate γ'. ∎

5 Marked Default Times Framework

A residual issue left open by the above is the specification of a concrete but general enough framework where $cdva = \gamma\widehat{\xi}$ in (4.6) can be computed in practice. Toward this aim, this section implements the extended reduced-form approach of the last section based on an invariant time τ obtained as a $\mathbb{G}$ stopping time with a mark. The role of the mark is to convey some additional information about the default, e.g. to encode wrong-way and gap risk features that would be out-of-reach in the basic immersion setup of Crépey (2015, Part II).

We assume in the sequel that τ is endowed with a mark e in a finite set E, i.e.

$$\tau = \min_{e\in E} \tau_e, \tag{5.1}$$

where each τ_e is a stopping time with intensity γ^e_t such that $\mathbb{Q}(\tau^e \neq \tau^{e'}) = 1$, $e \neq e'$. We assume that

$$\mathcal{G}_\tau = \mathcal{G}_{\tau-} \vee \sigma(\epsilon), \tag{5.2}$$

where the random variable $\epsilon = \mathrm{argmin}_{e\in E}\tau_e$ yields the “identity” of the mark of τ, and we denote by $\mathcal{E}$ the powerset of E.

Lemma 5.1 *For any $\mathcal{G}_\tau$ measurable random variable ζ, there exists a $\mathcal{P}(\mathbb{G})\otimes\mathcal{E}$ measurable function $\widetilde{\zeta}^e_t$ such that*

$$\mathbb{1}_{\{\tau=\tau_e\}}\zeta = \mathbb{1}_{\{\tau=\tau_e\}}\widetilde{\zeta}^e_\tau, \ \ e \in E. \tag{5.3}$$

For any such function $\widetilde{\zeta}^e_t$, a $\mathbb{Q}\times\boldsymbol{\lambda}$ a.e. version of $\gamma\widehat{\zeta}$ is given by $J_- \sum_E \gamma^e\widetilde{\zeta}^e$. In particular, the intensity of τ satisfies $\gamma = J_- \sum_{e\in E}\gamma^e$, $\mathbb{Q}\times\boldsymbol{\lambda}$ a.e., and we have

$$cdva_t = J_- \sum_{e\in E} \gamma^e\widetilde{\xi}^e, \ \ \mathbb{Q}\times\boldsymbol{\lambda} \ a.e., \tag{5.4}$$

for any $\mathcal{P}(\mathbb{G})\otimes\mathcal{E}$ measurable function $\widetilde{\xi}^e_t$, which exists, such that, for each $e \in E$,

$$\bar{\xi}_\tau = \widetilde{\xi}^e_\tau \text{ on the event } \{\tau = \tau_e\}. \tag{5.5}$$

Proof. The $\widetilde{\zeta}^e_\tau$ are $\mathcal{G}_{\tau-}$ locally integrable, by $\mathbb{G}$ predictability of the processes $\widetilde{\zeta}^e(\pi)$, processes which exist due to (5.2). Hence, by localization, one can assume the $\widetilde{\zeta}^e_\tau$ integrable. Then, on $\{\tau < \infty\}$,

$$\mathbb{E}[\zeta|\mathcal{G}_{\tau-}] = \mathbb{E}[\textstyle\sum_{e\in E}\mathbb{1}_{\{\tau=\tau^e\}}\widetilde{\zeta}^e_\tau|\mathcal{G}_{\tau-}] = \sum_{e\in E}\widetilde{\zeta}^e_\tau\mathbb{E}[\mathbb{1}_{\{\tau=\tau^e\}}|\mathcal{G}_{\tau-}].$$

Let q^e_t denote a $\mathcal{P}(\mathbb{G})\otimes\mathcal{E}$ measurable function, which exists by Corollary 3.23 2) in He et al. (1992), such that $q^e_\tau\mathbb{1}_{\{\tau<\infty\}} = \mathbb{E}[\mathbb{1}_{\{\tau=\tau^e\}}|\mathcal{G}_{\tau-}]\mathbb{1}_{\{\tau<\infty\}}$ ($e \in E$). For bounded $Z \in \mathcal{P}(\mathbb{G})$, we compute $\mathbb{E}[Z_\tau\mathbb{1}_{\{\tau=\tau^e<\infty\}}]$ in two ways:

$$\mathbb{E}[Z_\tau\mathbb{1}_{\{\tau=\tau^e<\infty\}}] = \mathbb{E}[Z_\tau q^e_\tau\mathbb{1}_{\{\tau<\infty\}}] = \mathbb{E}[\int_0^\infty Z_s q^e_s\gamma_s ds],$$

and

$$\mathbb{E}[Z_\tau \mathbb{1}_{\{\tau=\tau^e<\infty\}}] = \mathbb{E}[Z_{\tau^e} \mathbb{1}_{\{\tau=\tau^e<\infty\}}] = \mathbb{E}[Z_{\tau^e} \mathbb{1}_{\{\tau^e\le\tau<\infty\}}] = \mathbb{E}[\int_0^\infty Z_s \mathbb{1}_{\{s\le\tau\}}\gamma_s^e ds].$$

Hence, $\mathbb{Q}$ almost surely: $q_t^e\gamma_t = \mathbb{1}_{\{t\le\tau\}}\gamma_t^e$, dt almost everywhere, so that

$$\mathbb{Q}[q_\tau^e\gamma_\tau \neq \gamma_\tau^e,\ \tau<\infty] = \mathbb{E}[\mathbb{1}_{\{q_\tau^e\gamma_\tau\neq\gamma_\tau^e,\ \tau<\infty\}}] = \mathbb{E}[\int_0^\infty \mathbb{1}_{\{q_t^e\gamma_t\neq\gamma_t^e\}}\gamma_t dt] = 0.$$

Therefore, on $\{\tau<\infty\}$,

$$\gamma_\tau\widehat{\zeta}_\tau = \gamma_\tau\mathbb{E}[\zeta|\mathcal{G}_{\tau-}] = \sum_{e\in E}\widetilde{\zeta}_\tau^e\gamma_\tau q_\tau^e = \sum_{e\in E}\widetilde{\zeta}_\tau^e\gamma_\tau^e.$$

This implies that

$$\gamma\widehat{\zeta} = J_-\sum_E\gamma^e\widetilde{\zeta}^e,\ \ \mathbb{Q}\times\boldsymbol{\lambda}\text{-a.e.}$$

In particular, (5.4) follows by definition (4.6) of *cdva*. ∎

We now give a concrete specification ensuring (5.2), in case where $\mathbb{G}$ is the progressive enlargement of a reference filtration $\mathbb{F}$ by n mutually avoiding random times $\eta_1,\dots,\eta_n$ ($n\ge1$). Let the $\eta_{(i)}$ be the increasing ordering of the η_i, with also $\eta_{(0)}=0$ and $\eta_{(n+1)}=\infty$. In any multivariate density or recursively immersed model of default times (see the introductory paragraph to Part II as well as Theorems 6.1 and 7.1), the optional splitting formula of Song (2013b) holds. Namely, for any $\mathbb{G}$ optional process Y, there exist $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}([0,\infty]^n)$-measurable functions $Y^{(0)},Y^{(1)},\dots,Y^{(n)}$ such that

$$Y = \sum_{i=0}^n Y^{(i)}(\eta_1\nmid\eta_{(i)},\dots,\eta_n\nmid\eta_{(i)})\mathbb{1}_{[\eta_{(i)},\eta_{(i+1)})}, \tag{5.6}$$

where $a\nmid b$ denotes a if $a\le b$ and ∞ if $a>b$, for $a,b\in[0,\infty]$.

Lemma 5.2 *Assuming the optional splitting formula in force, e.g. in any multivariate density or recursively immersed model of default times, let $\eta=\eta_1\wedge\eta_2$. If the η_i avoid each other and $\mathbb{F}$ stopping times, then*

$$\mathcal{G}_\eta = \mathcal{G}_{\eta-}\vee\sigma(\{\eta=\eta_1\},\{\eta=\eta_2\}).$$

Proof. Let $N=\{1,2,3,\dots,n\}$. By the optional splitting formula (5.6), for any $\mathbb{G}$ optional process Y and $i\in N$,

$$\begin{aligned}
Y_\eta\mathbb{1}_{\{\eta=\eta_{(i)}\}} &= Y_\eta^{(i)}(\eta_1\nmid\eta,\dots,\eta_n\nmid\eta)\mathbb{1}_{\{\eta=\eta_{(i)}\}}\\
&= \textstyle\sum_{I\subseteq N;|I|=i-1} Y_\eta^{(i)}(\eta_1\nmid\eta,\eta_2\nmid\eta,\dots,\eta_n\nmid\eta)\mathbb{1}_{\{\forall j\in I,\eta_j<\eta\}}\mathbb{1}_{\{\forall j\in N\setminus I,\eta\le\eta_j\}}\\
&= \textstyle\sum_{k=1}^2\mathbb{1}_{\{\eta=\eta_k\}}\sum_{I\subseteq N;|I|=i-1} Y_\eta^{(i,I,k)}(\eta;\eta_j,j\in I)\mathbb{1}_{\{\forall j\in I,\eta_j<\eta\}}\mathbb{1}_{\{\forall j\in N\setminus I,\eta\le\eta_j\}},
\end{aligned}$$

where $Y_t^{(i,I,k)}(\omega,y;y_j,j\in I)$ is $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}([0,\infty])\otimes\mathcal{B}([0,\infty]^{i-1})$ measurable. Moreover, as η avoids $\mathbb{F}$ stopping times, He et al. (1992, Theorem 3.20) and the monotone class theorem imply that $Y_\eta^{(i,I,k)}(\eta;\eta_j,j\in I)$ is $\mathcal{G}_{\eta-}$ measurable. So, on each event $\{\eta=\eta_k\}$, $Y_\eta\mathbb{1}_{\{\eta=\eta_{(i)}\}}$ is $\mathcal{G}_{\eta-}$ measurable. As $\mathcal{G}_\eta$ is generated by the Y_η for the $\mathbb{G}$ optional processes Y, this proves the result. ∎

5.1 No Cure Period

If $\delta=0$, then $\bar{\xi}=\xi$, for which the expression in (4.4) reduces to

$$\xi = \mathbb{1}_{\{\tau=\tau_c\}}(1-R_c)\big(P_\tau+\Delta_\tau-C_\tau\big)^+ - \mathbb{1}_{\{\tau=\tau_b\}}(1-R_b)\big(P_\tau+\Delta_\tau-\mathfrak{C}_\tau\big)^-, \tag{5.7}$$

where $\Delta_\tau=D_\tau-D_{\tau-}$. Moreover, for every process $U=P,(D-D_-)$, V, I and $\mathfrak{I}$, hence also C and $\mathfrak{C}$, there exists by (5.2) a $\mathcal{P}(\mathbb{G})\otimes\mathcal{E}$ measurable function $\widetilde{U}_t^e$, abusively renoted $\widetilde{\Delta}_t^e$ in the case of $(D-D_-)$, such that $U_\tau=\widetilde{U}_\tau^e$ on the event $\{\tau=\tau_e\}$.

Remark 5.1 As will be illustrated in the DGC and DMO models (see Sect. 8, Figure 7 in particular, and the explanations following (6.6) and (7.6)), the $\widetilde{P}^e$ and $\widetilde{\Delta}^e$ may include respective wrong way and gap risk effects. The dependence of $\widetilde{C}^e$ and $\widetilde{\mathfrak{C}}^e$ on e can be used to render further gap risk features such as a possible jump of the collateral at the default of a counterparty, e.g. in case of a collateral posted in a currency strongly dependent on this counterparty (cf. Ehlers and Schönbucher (2006)).

Consistent with (5.1), let's assume $\tau_b = \min_{e\in E_b} \tau_e$ and $\tau_c = \min_{e\in E_c} \tau_e$, where $E = E_b \cup E_c$ (not necessarily a disjoint union, as will be exploited in Sect. 7). We may then take in (5.5) (where $\bar{\xi} = \xi$ when $\delta = 0$)

$$\widetilde{\xi}_t^e = \mathbb{1}_{e\in E_c}(1-R_c)(\widetilde{P}_t^e + \widetilde{\Delta}_t^e - \widetilde{C}_t^e)^+ - \mathbb{1}_{\{e\in E_b\}}(1-R_b)(\widetilde{P}_t^e + \widetilde{\Delta}_t^e - \widetilde{\mathfrak{C}}_t^e)^-, \tag{5.8}$$

so that by (5.4), we have on $[0,\bar{\tau}]$:

$$cdva_t = (1-R_c)\sum_{e\in E_c}\gamma_t^e\left(\widetilde{P}_t^e + \widetilde{\Delta}_t^e - \widetilde{C}_t^e\right)^+ - (1-R_b)\sum_{e\in E_b}\gamma_t^e\left(\widetilde{P}_t^e + \widetilde{\Delta}_t^e - \widetilde{\mathfrak{C}}_t^e\right)^-, \tag{5.9}$$

where the two terms have clear respective CVA and DVA interpretation. Hence, in the no cure period $\delta = 0$ case, (4.6) is rewritten, on $[0,\bar{\tau}]$, as

$$\begin{aligned}\widehat{f}_t(\vartheta) + (r_t+\gamma_t)\vartheta = {} & \underbrace{(1-R_c)\sum_{e\in E_c}\gamma_t^e\left(\widetilde{P}_t^e + \widetilde{\Delta}_t^e - \widetilde{C}_t^e\right)^+}_{cva_t} - \underbrace{(1-R_b)\sum_{e\in E_b}\gamma_t^e\left(\widetilde{P}_t^e + \widetilde{\Delta}_t^e - \widetilde{\mathfrak{C}}_t^e\right)^-}_{dva_t} \\ & + \underbrace{c_t\mathcal{C}_t + \widetilde{\lambda}_t\left(P_t - \mathcal{C}_t - \vartheta\right)^+ - \lambda_t\left(P_t - \mathcal{C}_t - \vartheta\right)^-}_{lva_t(\vartheta)},\end{aligned} \tag{5.10}$$

where we set $\widetilde{\lambda}_t = \bar{\lambda}_t - \Lambda\sum_{e\in E_b}\gamma_t^e$. We can then choose the coefficient $\widetilde{f}$ of the reduced TVA BSDE (3.5) based on (5.10) in the manner described before Theorem 4.1. Once stated in a Markov setup where

$$\widetilde{f}_t(\vartheta) = \widetilde{f}(t, \widetilde{X}_t, \vartheta), \;\; t\in[0,T], \tag{5.11}$$

for some $(\mathbb{F},\mathbb{P})$ jump diffusion $\widetilde{X}$, all the ingredients in the coefficient $\widetilde{f}$ can be computed and the reduced BSDE (3.5) can be solved numerically. This is only for $\delta = 0$ here but we'll study in Sect. 7.3 a case with $\delta > 0$.

Part II

Application to Credit Derivatives

We write $N = \{-1, 0, 1, \ldots, n\}$ and $N^\star = \{1, \ldots, n\}$, for some nonnegative integer n. In the second part of this paper, we apply the above approach to counterparty risk on credit derivatives traded between the bank and the counterparty respectively labeled as -1 and 0, i.e. for $\tau_b = \tau_{-1}$ and $\tau_c = \tau_0$, and referencing the names in N (or rather $N^\star$ for one does not trade credit protection on oneself in practice, but this makes no difference mathematically). The examples in this part are important, not only to provide some insights regarding the condition (C) introduced abstractly in the first part of this paper, but also as concrete models addressing challenge posed by counterparty risk and funding costs on credit derivatives. This will be demonstrated in two different setups, the dynamic Gaussian copula (DGC) model of Crépey, Jeanblanc, and Wu (2013) and the dynamic Marshall-Olkin (DMO) copula or common-shock model of Bielecki, Cousin, Crépey, and Herbertsson (2014b,2014a). These two models, reviewed in Chapters 7 and 8 of Crépey et al. (2014), are dynamic extensions of the perhaps best known copula models, namely the Gaussian copula and the exponential (or Marshall-Olkin) copula. We chose them as prototypes of multivariate density and Markov copula (i.e. recursively immersed Markov) models of portfolio credit risk, respectively. See Pham (2010) (cf. the condition (DH) in page 1800), extending to the multivariate setup Jeanblanc and Le Cam (2009) or El Karoui, Jeanblanc, and Jiao (2010), regarding the former, and see the survey in Crépey et al. (2014, Chapitre 14) for the latter. Other respectively related models include the one-period Merton model of Fermanian and Vigneron (2013, Section 6) and the multivariate Poisson model of Brigo, Pallavicini, and Torresetti (2007). In these models, we shall consider stylized CDS contracts and protection legs of CDO tranches corresponding to dividend processes D of the respective form, for $0 \le t \le T$:

$$\begin{aligned}
D_t = D^i_t &= \big((1-R_i)\mathbb{1}_{\{t\ge\tau_i\}} - S_i(t\wedge\tau_i)\big)Nom_i \\
D_t = D^\star_t &= \big(((1-R_\star)\sum_{j\in N}\mathbb{1}_{\{t\ge\tau_j\}} - (n+2)a)^+ \wedge (n+2)(b-a)\big)Nom_\star,
\end{aligned} \tag{5.12}$$

where all the recoveries R_i and $R_\star$ (resp. nominals Nom_i and $Nom_\star$) will be set in the numerics of Sect. 8 to 40% (resp. to 100). The contractual spreads S_i of the CDS contracts will be set such that the corresponding prices are equal to 0 at time 0. Protection legs of CDO tranches, where the attachment and detachment points a and b are such that $0 \le a \le b \le 100\%$, can also be seen as CDO tranches with upfront payment. Note that credit derivatives traded as swaps or with upfront payment coexist since the crisis.

Until Sect. 7.3, it is assumed that $\delta = 0$.

6 Dynamic Gaussian Copula TVA Model

6.1 Model of Default Times

We consider a multivariate Brownian motion $\mathbf{B} = (B^i)_{i\in N}$ with pairwise correlation $\varrho \ge 0$ in its own completed filtration $\mathbb{B} = (\mathcal{B})_{t\ge 0}$. Specifically, for $i \in N$, we assume

$$B^i_t = \sqrt{\varrho}Z_t + \sqrt{1-\varrho}Z^i_t, \tag{6.1}$$

where Z and the Z^i are independent Brownian motions. For any $i \in N$, let h_i be a continuously differentiable increasing function from $\mathbb{R}^*_+$ to $\mathbb{R}$, with $\lim_0 h_i(s) = -\infty$ and $\lim_{+\infty} h_i(s) = +\infty$, and let

$$\tau_i = h_i^{-1}\Big(\int_0^{+\infty} \varsigma(u)dB^i_u\Big), \tag{6.2}$$

where $\varsigma(\cdot)$ is a square integrable function with unit L^2 norm. As a consequence, the $(\tau_i)_{i\in N}$ follow the standard Gaussian copula model of Li (2000), with correlation parameter ϱ and with marginal survival function $\Phi\circ h_i$ of τ_i, where Φ is the standard normal survival function. In particular, the τ_i avoid each other. In order to make the model dynamic as required by counterparty risk applications, we introduce the model filtration $\mathbb{G} = (\mathcal{G}_t)_{t\ge 0}$ given as the Brownian filtration $\mathbb{B}$ progressively enlarged by the τ_i, augmented so as to satisfy the usual conditions. Note that the τ_i are $\mathcal{B}_\infty$ measurable, totally unpredictable $\mathbb{G}$ stopping times. Let

$$m^i_t = \int_0^t \varsigma(u)dB^i_u,\;\; k^i_t = (\mathbb{1}_{\{\tau_i\le t\}}, \tau_i\mathbb{1}_{\{\tau_i\le t\}}),\;\; \mathbf{m}_t = (m^i_t)_{i\in N},\;\; \mathbf{k}_t = (k^i_t)_{i\in N}.$$

The reason why we consider $\tau_i\mathbb{1}_{\{\tau_i\le t\}}$ on top of $\mathbb{1}_{\{\tau_i\le t\}}$ in k^i_t is because of a dependence of future prices on past default times in the DGC model. Augmenting the factor process in this way allows to take care of this path-dependence. Conversely, only having $\tau_i\mathbb{1}_{\{\tau_i\le t\}}$ in k^i_t would lead to time discontinuous functions in the representations (6.4), which would be nonstandard from the point of view of PDE (continuous) viscosity solutions in Corollaries 6.1, 7.1 and 7.2. Note that $\tau_i\mathbb{1}_{\{\tau_i\le t\}} \le t$, hence the process $\mathbf{k}$ is bounded on $[0, T]$.

Theorem 6.1 *The dynamic Gaussian copula model is a multivariate density model of default times. For every $t \ge 0$, we have*

$$\mathcal{G}_t = \mathcal{B}_t \vee \bigvee_{i\in N}\big(\sigma(\tau_i\wedge t)\vee\sigma(\{\tau_i > t\})\big). \tag{6.3}$$

There exist processes β^i_t and γ^i_t of the form

$$\beta^i_t,\ \ \gamma^i_t = \beta_i, \gamma_i\,(t, \mathbf{m}_t, \mathbf{k}_t), \tag{6.4}$$

for continuous functions β_i and γ_i with linear growth in the m_j, $j \in N$, such that the $dW^i_t = dB^i_t - \beta^i_t dt$ are $(\mathbb{G}, \mathbb{Q})$ Brownian motions and the $(\mathbb{G}, \mathbb{Q})$ compensated default indicator processes are written as $dM^i_t = d\mathbb{1}_{\tau_i \le t} - \gamma^i_t dt$, $i \in N$. The W^i and the M^i, $i \in N$, have the $(\mathbb{G}, \mathbb{Q})$ martingale representation property.

Proof. The τ_i have the joint conditional density p_t given $\mathcal{B}_t$ derived in the formula (B.1). Hence, the dynamic Gaussian copula model is a multivariate density model of default times. The expression for γ^i is given in (B.3), where the stated continuity is apparent and the linear growth in the m_j follows from the left inequality in (B.11). The β^i can be computed by making use of the probability measure $\mathbb{Q}^T$, classically used in density models, such that $\frac{d\mathbb{Q}^T}{d\mathbb{Q}} \propto p_T(\tau_j,\, j \in N)$, for every $T > 0$. More precisely, the τ_i are $\mathbb{Q}^T$ independent between them and from $\mathcal{B}_T$ (cf. Theorem 4.7 in Song (2013a)), so that the B^i are $(\mathbb{G}, \mathbb{Q}^T)$ Brownian motions on $[0, T]$, for every $T > 0$. Hence, their $(\mathbb{G}, \mathbb{Q})$ drifts β^i can be obtained by application of a Girsanov formula from $(\mathbb{G}, \mathbb{Q}^T)$ to $(\mathbb{G}, \mathbb{Q})$, $T > 0$, which reveals the functional dependence in (6.4) with the claimed properties. The martingale representation property and (6.3) are proved by induction over the cardinality of N as follows. We write $\mathbb{G} = \mathbb{G}^N$. If N is reduced to a singleton, then the density property of τ given $\mathcal{B}_t$ implies the results, by the optional splitting formula (5.6) for (6.3) and by Jeanblanc and Song (2013, Theorem 6.4) for the martingale representation property. If N' is obtained by adding a new name, say $(n+1)$, to N, then the density properties of τ_{n+1} and of $(\tau_i)_{i \in N}$ given $\mathcal{B}_t$ imply the density property of τ_{n+1} given $\mathcal{B}_t \vee \bigvee_{i \in N} \big(\sigma(\tau_i \wedge t) \vee \sigma(\{\tau_i > t\})\big)$. Hence, the results for $\mathbb{G}^{N'}$ follow likewise from those, if assumed, for $\mathbb{G}^N$. ■

6.2 TVA Model

A DGC setup can be used as a TVA model for credit derivatives, with mark $i = -1, 0$ and $E_b = \{-1\}$, $E_c = \{0\}$. Since there are no joint defaults in this model, it is harmless to assume that the contract promises no cash-flow at τ, i.e. $\Delta_\tau = 0$, so that (cf. (4.2) with currently $\delta = 0$)

$$\chi = (P_\tau - C_\tau)^+, \ \ \mathfrak{X} = (P_\tau - \mathfrak{C}_\tau)^-.$$

The results of Crépey et al. (2014, Corollaries 7.3.1 page 178 and 7.3.3 page 181)[5] show that in the case of a portfolio of vanilla credit derivatives on names in N, e.g. CDS contracts and CDO tranches as of (5.12), one has a semi-explicit formula for P of the form

$$P_t = P(t, \mathbf{m}_t, \mathbf{k}_t), \tag{6.5}$$

for a continuous function P.

Remark 6.1 The "true domain of definition" of the variables t_i corresponding to the second components of the $k_i \in \{0,1\} \times \mathbb{R}_+$, $i \in N$, is given as the union of all the strict order sets of t_i for $i \in I$, i.e. the sets of the form $\{(t_i)_{i \in I}; t_{\pi(1)} < \cdots < t_{\pi(|I|)}\}$, for all the subsets $I \subseteq N$ and all the bijections π of $\{1, \cdots, |I|\}$ to I. In fact, the τ_i, which correspond to the second (positive) components of the k_i in the probabilistic interpretation, don't take values outside this union. Here and below any continuity statement with respect to the t_i has to be understood in the sense of the corresponding domain and topology.

We assume that for every process $U = P$, V, I and $\mathfrak{I}$, hence also C and $\mathfrak{C}$, there exists a continuous function $\widetilde{U}_i$ such that

$$U_\tau = \widetilde{U}_i(\tau, \mathbf{m}_\tau, \mathbf{k}_{\tau-}), \tag{6.6}$$

or $\widetilde{U}^i_\tau$ in a shorthand notation, on the event $\{\tau = \tau_i\}$, $i = -1, 0$. In view of (6.5), this always holds regarding $U = P$ for vanilla credit derivatives on names in N, since

$$P_\tau = P(\tau, \mathbf{m}_\tau, \mathbf{k}_\tau) = P(\tau, \mathbf{m}_\tau, \mathbf{k}^{i,\tau}_{\tau-}) \text{ on } \{\tau = \tau_i\}, \tag{6.7}$$

[5] Or Crépey et al. (2013, Corollaries 3.1 and 3.2) in the journal version.

where $\mathbf{k}^{i,t}$ denotes the vector obtained from $\mathbf{k}$ by replacing the component with index i by $(1,t)$. Note that, since the i^{th} component of $\mathbf{k}^{i,\tau}_{\tau-}$ equals $(1,\tau)$ and not $(0,0)$, the functions $\widetilde{P}^i$ incorporate a wrong-way risk effect through the spike of intensities of surviving names at other names' defaults in the DGC model (this will be illustrated by the left graph in Figure 7). The conditions (6.6) regarding $U=C$ and $\mathfrak{C}$ may be satisfied or not depending on the CSA. By (5.8) and (5.10),

$$\begin{aligned}&\widetilde{\xi}^i_t = \mathbb{1}_{i=0}(1-R_c)(\widetilde{P}^i_t - \widetilde{C}_t)^+ - \mathbb{1}_{i=-1}(1-R_b)(\widetilde{P}^i_t - \widetilde{\mathfrak{C}}_t)^-,\ \ i=-1,0,\\ &\widetilde{f}_t(\vartheta) + (r_t+\gamma_t)\vartheta = (1-R_c)\gamma^0_t\big(\widetilde{P}^0_t - \widetilde{C}^0_t\big)^+ - (1-R_b)\gamma^{-1}_t\big(\widetilde{P}^{-1}_t - \widetilde{\mathfrak{C}}^{-1}_t\big)^-\\ &\qquad + c_t\mathcal{C}_t + \widetilde{\lambda}_t\,(P_t - \mathcal{C}_t - \vartheta)^+ - \lambda_t\,(P_t-\mathcal{C}_t-\vartheta)^-,\ \ t\in[0,\bar{\tau}],\end{aligned} \tag{6.8}$$

where $\gamma_t = \gamma^0_t + \gamma^1_t$ and $\widetilde{\lambda}_t = \bar{\lambda}_t - \Lambda\gamma^{-1}_t$. We assume that the processes $r, c, \lambda, \bar{\lambda}, P$ and $\mathcal{C}$ are given before τ as continuous functions of $(t, \widetilde{X}_t)$, where $\widetilde{X}_t = (\mathbf{m}_t, \widetilde{\mathbf{k}}_t)$ with $\widetilde{\mathbf{k}}_t = (\mathbb{1}_{i\in N^\star} k^i_t)_{i\in N}$. Regarding P, (6.5) shows that this property always holds in the case of vanilla credit derivatives on names in N. In view of (6.4), this property is also verified by the process γ.

Remark 6.2 In the DGC or in the DMO model without cure period, the only modification required to deal with path-dependent margins tracking the mark-to-market P at discrete grid times (cf. the remark 4.1) would be to augment the pre-default factor process $\widetilde{X}_t$ by additional margin components, as already done for reasons pertaining to the cure period in the DMO setup of Sect. 7.3.

Note that $\mathbb{P}\neq\mathbb{Q}$ in (DGC.2), hence the DGC model is a case of "no immersion" in the sense of the comments following the statement of the condition (C).

Theorem 6.2 *The condition (C) holds, for:*

(DGC.1) *a reference filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ in (C.1) given as $\mathbb{B}$ progressively enlarged by the default times for $i\in N^\star$, which satisfies*

$$\mathcal{F}_t = \mathcal{B}_t \vee \bigvee_{i\in N^\star}\big(\sigma(\tau_i\wedge t)\vee\sigma(\{\tau_i>t\})\big),\ \ t\geq 0, \tag{6.9}$$

(DGC.2) *a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that a family of $(\mathbb{F},\mathbb{P})$ martingales with the $(\mathbb{F},\mathbb{P})$ martingale representation property is given by the*

$$d\widetilde{W}^i_t = dB^i_t - \widetilde{\beta}^i_t dt,\ i\in N \text{ and } d\widetilde{M}^i_t = d\mathbb{1}_{\tau_i\leq t} - \widetilde{\gamma}^i_t dt,\ i\in N^\star, \tag{6.10}$$

where

$$\widetilde{\beta}^i_t := \beta_i(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t) = \beta_i(t,\widetilde{X}_t),\ \ \widetilde{\gamma}^i_t := \gamma_i(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t) = \gamma_i(t,\widetilde{X}_t), \tag{6.11}$$

(DGC.3) *a Markov specification*

$$\widetilde{f}_t(\vartheta) = \widetilde{f}(t,\widetilde{X}_t,\vartheta) \tag{6.12}$$

in (C.3), for the $(\mathbb{F},\mathbb{P})$ jump diffusion $\widetilde{X}_t = (\mathbf{m}_t,\widetilde{\mathbf{k}}_t)$ and for the function $\widetilde{f}=\widetilde{f}(t,\widetilde{x},\vartheta)$ given, writing $\widetilde{x}=(\mathbf{m},\widetilde{\mathbf{k}})$ for every $\mathbf{m}=(m_i)_{i\in N}\in\mathbb{R}^N$ and $\widetilde{\mathbf{k}}=(k_i)_{i\in N}\in\{(0,0)\}\times\{(0,0)\}\times(\{(0,0)\}\cup(\{1\}\times\mathbb{R}_+))^{N^\star}$, by:

$$\begin{aligned}\widetilde{f}(t,\widetilde{x},\vartheta) + (r(t,\widetilde{x})+\gamma(t,\widetilde{x}))\vartheta &= (1-R_c)\gamma_0\left(\widetilde{P}_0 - \widetilde{C}_0\right)^+(t,\widetilde{x}) - (1-R_b)\gamma_{-1}\left(\widetilde{P}_{-1}-\widetilde{\mathfrak{C}}_{-1}\right)^-(t,\widetilde{x})\\ &+\Big(c\mathcal{C} + \widetilde{\lambda}\,(P-\mathcal{C}-\vartheta)^+ - \lambda\,(P-\mathcal{C}-\vartheta)^-\Big)(t,\widetilde{x}),\end{aligned} \tag{6.13}$$

where $\gamma(t,\widetilde{x}) = \gamma_0(t,\widetilde{x})+\gamma_{-1}(t,\widetilde{x})$ and $\widetilde{\lambda}(t,\widetilde{x}) = \bar{\lambda}(t,\widetilde{x}) - \gamma_{-1}(t,\widetilde{x})\Lambda$.

In addition, $(\mathbb{F},\mathbb{P})$ local martingales don't jump at τ, τ avoids $\mathbb{F}$ stopping times, the condition (B) is satisfied and the Azéma supermartingale S of τ is positive. In particular, the DGC model is a marked default times setup, satisfying (5.2), where the full and reduced BSDEs are equivalent.

Proof. (DGC.1) and (DGC.3) can be proven as Theorem 6.1 above, whereas (DGC.2) is proven in Theorem 6.3 below. Moreover, (DGC.1) and (DGC.3) obviously imply (C.1) and (C.3). In view of (6.11), each $(\mathbb{F},\mathbb{P})$ martingale in (6.10), stopped at $(\tau-)$ or, equivalently by avoidance between the τ_i, at τ, is a $(\mathbb{G},\mathbb{Q})$ local martingale. Hence, (DGC.2) implies (C.2) via the $(\mathbb{F},\mathbb{P})$ martingale representation property that is included in (DGC.2). This also shows that $(\mathbb{F},\mathbb{P})$ local martingales don't jump at τ. By He et al. (1992, Theorem 5.27 1)) applied to the indicator process $\mathbb{1}_{\{\tau=\nu\}}$, where ν is any $\mathbb{F}$ predictable stopping time, we have $\mathbb{Q}(\tau=\nu)=0$ as soon as the $(\mathbb{F},\mathbb{Q})$ drift of S is continuous, as it is in the DGC model in view of the formula (B.2) for S. Besides, by the $(\mathbb{F},\mathbb{Q})$ martingale representation property in this model (see the end of the proof of Theorem 6.3), the $(\mathbb{F},\mathbb{Q})$ compensated martingale of the default indicator process of an $\mathbb{F}$ totally inaccessible stopping time ν only jumps at the τ_i, $i\in N^\star$. But, by the same argument as for $(\mathbb{F},\mathbb{P})$ above, $(\mathbb{F},\mathbb{Q})$ local martingales don't jump at τ. Hence, $\mathbb{Q}(\tau=\nu)=0$. We conclude that τ avoids all $\mathbb{F}$ stopping times. To check the condition (B), by the monotone class theorem, we only need consider the elementary $\mathbb{G}$ predictable processes of the form $Y=\nu f(\tau_c\wedge s,\tau_b\wedge s)\mathbb{1}_{(s,t]}$, for an $\mathcal{F}_s$ measurable random variable ν and a Borel function f. Since $Y\mathbb{1}_{(0,\tau]}=\nu f(s,s)\mathbb{1}_{(s,t]}\mathbb{1}_{(0,\tau]}$, we may take $\widetilde{Y}=\nu f(s,s)\mathbb{1}_{(s,t]}$. The positivity of S is apparent on the explicit formula in (B.2). The final statements follows from the other results by Lemma 5.2 and in view of the explanations given in the last paragraph of Sect. 3. ∎

Note that for any $\mathbb{P}$ ensuring the martingale representation property in (DGC.2), the corresponding argument in the above proof shows that (C.2) is satisfied. This reduces the problem of finding $\mathbb{P}$ satisfying (C.2) to finding $\mathbb{P}$ for which the martingale representation property stated in (DGC.2) holds. As we will see in Sect. 6.3, the tentative stochastic exponential $\mathbb{Q}$ to $\mathbb{P}$ density (6.17) is precisely defined so that, on the one hand, the drift in the $(\mathbb{F},\mathbb{Q})$ to $(\mathbb{F},\mathbb{P})$ Girsanov measure change compensates the drift in the $(\mathbb{F},\mathbb{Q})$ to $(\mathbb{G},\mathbb{Q})$ Jeulin progressive enlargement of filtration formula for the $(\mathbb{F},\mathbb{Q})$ Brownian motions $\bar{W}^i$ and, on the other hand, the default intensities, given by $\bar{\gamma}^i$ as of (B.4) under $(\mathbb{F},\mathbb{Q})$, become $\widetilde{\gamma}^i$ under $(\mathbb{F},\mathbb{P})$, where by construction $\widetilde{\gamma}^i$ coincides with the $(\mathbb{G},\mathbb{Q})$ intensity γ^i before τ. Then the proof of (DGC.2) becomes a matter of checking that the tentative measure change density defined in this way is a valid measure change, i.e. a positive $(\mathbb{F},\mathbb{Q})$ martingale, which is done in Theorem 6.3.

Corollary 6.1 *In the DGC model we have $\gamma'=\gamma(\cdot,\widetilde{X}_{\cdot-})\in\widetilde{\mathcal{H}}_2$. Assuming all the other conditions in Theorem 4.1 and no cure period, so for $\delta=0$ and $\widetilde{f}=\widetilde{f}(t,\widetilde{X},\vartheta)=\widetilde{f}(t,\mathbf{m},\widetilde{\mathbf{k}},\vartheta)$ as of (6.13), then the corresponding reduced TVA BSDE (3.5) admits a unique square integrable solution $\widetilde{\Theta}_t=\widetilde{\Theta}(t,\widetilde{X}_t)$, where the function $\widetilde{\Theta}(t,\widetilde{x})$ is a continuous viscosity solution to the corresponding semilinear PIDE (not written as not directly used in the paper). A solution Θ to the full TVA BSDE (2.8) is obtained by setting $\Theta=\widetilde{\Theta}$ on $[0,\bar{\tau})$ and*

$$\Theta_{\bar{\tau}}=\mathbb{1}_{\{\tau<T\}}\left(\widetilde{\xi}^{\imath}_{\tau}-(P_{\tau-}-\mathcal{C}_{\tau-}-\widetilde{\Theta}_{\tau-})^{+}\mathbb{1}_{\{\tau=\tau_{-1}\}}\Lambda\right),$$

where $\imath=0$ or -1 denotes the identity of the defaulting counterparty (cf. (6.8)). The $(\mathbb{G},\mathbb{Q})$ local martingale component μ of Θ satisfies, for $t\in[0,\bar{\tau}]$:

$$\begin{aligned}d\mu_t=d\widetilde{\mu}_t&-\left(\widetilde{\xi}^{\imath}_{\tau}-(P_{\tau-}-\mathcal{C}_{\tau-}-\widetilde{\Theta}_{\tau-})^{+}\mathbb{1}_{\{\tau=\tau_{-1}\}}\Lambda-\widetilde{\Theta}_{\tau-}\right)dJ_t\\&-\Big(\sum_{i=-1,0}(\widetilde{\xi}^{i}_{t}-\widetilde{\Theta}_t)\gamma^i_t-(P_t-\mathcal{C}_t-\widetilde{\Theta}_t)^{+}\gamma^{-1}_t\Lambda\Big)dt.\end{aligned}\tag{6.14}$$

Proof. By (DGC.2) in Theorem 6.2, $\widetilde{W}^i=B^i-\widetilde{\beta}^i\boldsymbol{\cdot}\boldsymbol{\lambda}$ is an $(\mathbb{F},\mathbb{P})$ Brownian motion ($i\in N^\star$). Set $\widetilde{m}^i=\varsigma\boldsymbol{\cdot}\widetilde{W}^i$, hence $m^i=\widetilde{m}^i+\varsigma\widetilde{\beta}^i\boldsymbol{\cdot}\boldsymbol{\lambda}$. For fixed $t\leq T$, for any $p\geq 1$, there exist constants, all denoted by the same symbol C, such that

$$\begin{aligned}\widetilde{\mathbb{E}}[\textstyle\sum_i\sup_{s\leq t}|m^i_s|^p]&\leq C\widetilde{\mathbb{E}}\Big[\textstyle\sum_i\sup_{s\leq t}|\widetilde{m}^i_s|^p+\sum_i(|\varsigma\widetilde{\beta}^i|\boldsymbol{\cdot}\boldsymbol{\lambda}_t)^p\Big]\\&\leq C\widetilde{\mathbb{E}}\Big[\textstyle\sum_i\sup_{s\leq t}|\widetilde{m}^i_s|^p\Big]+C\widetilde{\mathbb{E}}\Big[(1+\sum_j\sup_{s\leq\cdot}|m^j_s|^p)\boldsymbol{\cdot}\boldsymbol{\lambda}_t\Big]\\&=C\widetilde{\mathbb{E}}\Big[\textstyle\sum_i\sup_{s\leq t}|\widetilde{m}^i_s|^p\Big]+C+C\widetilde{\mathbb{E}}\Big[\textstyle\sum_j\sup_{s\leq\cdot}|m^j_s|^p\Big]\boldsymbol{\cdot}\boldsymbol{\lambda}_t,\end{aligned}$$

where the definition of the process $\widetilde{\beta}^i$ in (6.11) was used in conjunction with the linear growth of the function β_i in (6.4) to pass to the second line. Hence, by virtue of the Gronwall inequality,

$$\widetilde{\mathbb{E}}[\sum_i\sup_{s\leq t}|m^i_s|^p]\leq\left(1+\widetilde{\mathbb{E}}\left[\sum_i\sup_{s\leq t}|\widetilde{m}^i_s|^p\right]\right)Ce^{Ct}<\infty,$$

from which $\gamma' = \gamma(\cdot, \widetilde{X}_{\cdot-}) = \gamma_0(\cdot, \widetilde{X}_{\cdot-}) + \gamma_{-1}(\cdot, \widetilde{X}_{\cdot-}) \in \widetilde{\mathcal{H}}_2$ follows by linear growth in the m_j of the functions γ_i in (6.4). As a consequence, well-posedness in $\widetilde{\mathcal{H}}_2$ of the corresponding reduced TVA BSDE (3.5) follows from Theorem 4.1 (assuming all the other conditions there). Since well-posedness in the sense of Theorem 4.1 includes comparison and the usual a priori bound and error BSDE estimates, the representation of the solution $\widetilde{\Theta}_\cdot$ to the BSDE in terms of a continuous viscosity solution to the corresponding semilinear PIDE (with continuous coefficients) follows from standard arguments (see e.g. Delong (2013) or Crépey (2013, Chapter 13)). By Theorem 6.2, the reduced stochastic basis $(\mathbb{F}, \mathbb{P})$ satisfies the condition (C), so that the remaining statements follow from Theorem 3.1. ∎

Presumably, reinforcing as needed the assumptions on $\widetilde{f}$, one could show that the function $\widetilde{\Theta}(t, \widetilde{x})$ in Corollary 6.1 is a classical (not only viscosity) solution to the corresponding PIDE (see e.g. Becherer and Schweizer (2005)). In this case, an application of the Itô formula related to the $(\mathbb{F}, \mathbb{P})$ jump diffusion $\widetilde{X}_t = (\mathbf{m}_t, \widetilde{\mathbf{k}}_t)$ yields the following functional representation of $\widetilde{\mu}$ in (6.14):

$$d\widetilde{\mu}_t = \varsigma(t) \sum_{i \in N} \partial_{m_i} \widetilde{\Theta}(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_t) d\widetilde{W}^i_t + \sum_{i \in N^\star} \delta_i \widetilde{\Theta}(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_{t-}) d\widetilde{M}^i_t, \tag{6.15}$$

where $\partial_{m_i} \widetilde{\Theta}$ denotes the partial derivative of the function $\widetilde{\Theta}$ with respect to m_i and where

$$\delta_i \widetilde{\Theta}(t, \mathbf{m}, \widetilde{\mathbf{k}}) = \widetilde{\Theta}(t, \mathbf{m}, \mathbf{k}^{i,t}) - \widetilde{\Theta}(t, \mathbf{m}, \widetilde{\mathbf{k}}). \tag{6.16}$$

This formula shows the nature of the TVA Greeks in the DGC model, namely the integrands $\partial_{m_i} \widetilde{\Theta}(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_t)$ and $\delta_i \widetilde{\Theta}(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_{t-})$ in (6.15). For length care, we do not conduct the above-mentioned regularity analysis. Similar comments apply and will not be repeated regarding Corollaries 7.1 and 7.2.

6.3 Proof of (DGC.2)

Let Q^c denote the continuous martingale component of S. For $i \in N^\star$, let $\overline{\gamma}^i$ be the $(\mathbb{F}, \mathbb{Q})$ intensity of τ_i and let $\overline{M}^i = \mathbb{1}_{[\tau_i, +\infty)} - \overline{\gamma}^i \centerdot \boldsymbol{\lambda}$. We consider the $(\mathbb{F}, \mathbb{Q})$ local martingale given as the Doléans-Dade exponential $\mathcal{E}(\nu)$, where $\nu = \mathbb{1}_{(0,T]} \frac{1}{S} \centerdot Q^c + \sum_{i \in N^\star} \mathbb{1}_{(0,T]} (\frac{\widetilde{\gamma}^i}{\overline{\gamma}^i} - 1) \centerdot \overline{M}^i$, i.e.

$$\mathcal{E}(\nu) = \mathcal{E}\left(\mathbb{1}_{(0,T]} \nu^c\right) \prod_{i \in N^\star} \left(1 + (\frac{\widetilde{\gamma}^i_{\tau_i}}{\overline{\gamma}^i_{\tau_i}} - 1) \mathbb{1}_{\{\tau_i \le T\}} \mathbb{1}_{[\tau_i, +\infty)}\right) \exp \int_0^{\cdot \wedge \tau_i \wedge T} \left(\overline{\gamma}^i_s - \widetilde{\gamma}^i_s\right) ds, \tag{6.17}$$

where $\nu^c = \frac{1}{S} \centerdot Q^c$.

Theorem 6.3 *$\mathcal{E}(\nu)$ is a positive $(\mathbb{F}, \mathbb{Q})$ martingale and the probability measure $\mathbb{P}$ with $\mathbb{Q}$ density process $\mathcal{E}(\nu)$ satisfies the condition (DGC.2).*

Proof. Note that $\widetilde{\gamma}^i$, defined through γ^i by (6.10), is positive in view of (B.3), for any $i \in N^\star$. Hence, $\mathcal{E}(\nu) > 0$. The following property will be proven later (cf. Lemma 6.1):

$$\text{There exists } \epsilon > 0 \text{ such that, for any } s \in [0, T]\text{: } \mathbb{E}\big[\mathcal{E}\big(\mathbb{1}_{(s,t]} \nu\big) \,\big|\, \mathcal{F}_s\big] = 1, \ \ t \in [s, s + \epsilon]. \tag{6.18}$$

If $T \le \epsilon$, then the first part of the theorem follows. Otherwise, we write

$$\mathbb{E}[\mathcal{E}(\nu)] = \mathbb{E}\left[\mathcal{E}\left(\mathbb{1}_{(0,T-\epsilon]} \nu\right) \mathbb{E}\left[\mathcal{E}\left(\mathbb{1}_{(T-\epsilon,T]} \nu\right) \,\big|\, \mathcal{F}_{T-\epsilon}\right]\right] = \mathbb{E}\left[\mathcal{E}\left(\mathbb{1}_{(0,T-\epsilon]} \nu\right)\right],$$

by (6.18) applied with $s = T - \epsilon$ and $t = T$, so that the first part of the theorem follows by induction. Since $\mathcal{E}(\nu)$ is a positive $(\mathbb{F}, \mathbb{Q})$ martingale, one can define a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ by the $\mathbb{Q}$ density process $\mathcal{E}(\nu)$. By the Girsanov theorem, the $(\mathbb{F}, \mathbb{P})$ intensity of τ_i is $\widetilde{\gamma}^i$, $i \in N^\star$, and, denoting by $\overline{W}^i = B^i - \overline{\beta}^i \centerdot \boldsymbol{\lambda}$ the $(\mathbb{F}, \mathbb{Q})$ Brownian motion obtained as the $(\mathbb{F}, \mathbb{Q})$ martingale component of B^i, the process

$$\widehat{W}^i = \overline{W}^i - \langle \nu^c, B^i \rangle = B^i - (\overline{\beta}^i \centerdot \boldsymbol{\lambda} + \langle \nu^c, B^i \rangle)$$

is an $(\mathbb{F}, \mathbb{P})$ Brownian motion, for each $i \in N$. Moreover, by the Jeulin formula (see e.g. Dellacherie, Maisonneuve, and Meyer (1992, no 77 Remarques b))), $\widehat{W}^i$ is a $(\mathbb{G}, \mathbb{Q})$ Brownian motion until time τ, as is also $\widetilde{W}^i = B^i - \widetilde{\beta}^i \centerdot \boldsymbol{\lambda}$ in (6.10), for $\widetilde{\beta}^i_{\cdot \wedge \tau} = \beta^i_{\cdot \wedge \tau}$. Hence, $\widetilde{W}^i_{\cdot \wedge \tau} = W^i_{\cdot \wedge \tau}$. Therefore, the $\mathbb{F}$ predictable processes $\bar{\beta}^i \centerdot \boldsymbol{\lambda} + \langle \nu^c, B^i \rangle$ and $\widetilde{\beta}^i \centerdot \boldsymbol{\lambda}$ coincide until τ, hence on $[0,T]$ by a classical optional section argument (valid for S positive, as so in our case in view of (B.2); see e.g. Crépey and Song (2014c, Lemma 2.3)). In conclusion, $\widetilde{W}^i = \widehat{W}^i$, an $(\mathbb{F},\mathbb{P})$ Brownian motion, for any $i \in N^\star$. The $(\mathbb{F},\mathbb{P})$ martingale representation property of the $\widetilde{W}^i, i \in N$ and $\widetilde{M}^i, i \in N^\star$ follows by equivalent change of measure from the $(\mathbb{F},\mathbb{Q})$ martingale representation property of the $\overline{W}^i, i \in N$ and $\overline{M}^i, i \in N^\star$, which can be proven as the $(\mathbb{G},\mathbb{Q})$ martingale representation property of the $W^i, i \in N$ and $M^i, i \in N$ in Theorem 6.1. Thus, all the conditions hold in (DGC.2). ∎

Remark 6.3 $\overline{W}^i$ yields an $(\mathbb{F},\mathbb{Q})$ local martingale that, stopped at τ, or equivalently $(\tau-)$ by continuity of $\overline{W}^i$, fails to be a $(\mathbb{G},\mathbb{Q})$ local martingale. In fact, by the above computations:

$$\overline{W}^i_{\cdot\wedge\tau} = \widetilde{W}^i_{\cdot\wedge\tau} + J_-(\widetilde{\beta}^i - \overline{\beta}^i) \centerdot \boldsymbol{\lambda},$$

where $(\widetilde{\beta}^i - \overline{\beta}^i) \centerdot \boldsymbol{\lambda} = \langle \nu^c, B^i \rangle$ is not null on $[0,\tau]$, in view of the expression for $d\nu^c_t$ in (B.2). This justifies the "no immersion" statement before Theorem 6.2.

It remains to establish (6.18). For notational simplicity, we only prove it for $s = 0$, i.e.:

Lemma 6.1 *For t small enough, $\mathcal{E}(\nu)$ is an $(\mathbb{F},\mathbb{Q})$ martingale on $[0,t]$.*

Proof. This relies on explicit formulas in the dynamic Gaussian copula model, which then need be controlled by means of Gaussian estimates, so that Lepingle and Mémin (1978, Theorem IV.3) can eventually be applied to establish the true martingality of $\mathcal{E}(\nu)$. This is done in Sect. B. ∎

Note that the controls of Sect. B, even if technical, are not optimal and don't need be so for showing the desired integrability. The practical conclusion is that once the "right" candidate $\mathbb{P}$ has been guessed based on martingale representation considerations (see the comments following Theorem 6.2), checking the condition (C) reduces to checking a rather mild integrability condition.

7 Dynamic Marshall-Olkin Copula TVA Model

The above dynamic Gaussian copula model can suffice to deal with TVA on CDS contracts. But a Gaussian copula dependence structure is not rich enough for permitting a joint calibration to CDS and CDO data. If CDO tranches are also present in a portfolio, a possible alternative is the following dynamic Marshall-Olkin (DMO) copula model.

7.1 Model of Default Times

We define a family $\mathcal{Y}$ of "shocks", i.e. subsets $Y \subseteq N$ of obligors, usually consisting of the singletons $\{-1\}$, $\{0\}$, $\{1\}$, $\dots$, $\{n\}$, and a few "common shocks" representing simultaneous defaults. The shock intensities are given in the form of extended CIR processes as, for every $Y \in \mathcal{Y}$,

$$d\gamma^Y_t = a(b_Y(t) - \gamma^Y_t)dt + c\sqrt{\gamma^Y_t}\, dW^Y_t\,, \tag{7.1}$$

for nonnegative constants a and c, continuous functions $b_Y(t)$ and independent Brownian motions W^Y in their own completed filtration $\mathbb{W} = (\mathcal{W}_t)_{t\geq 0}$, under the pricing measure $\mathbb{Q}$. The case of deterministic intensities $\gamma^Y_t = b_Y(t)$ can be embedded in this framework as the limiting case of an "infinite speed of mean-reversion" a. In fact, one could use any independent Markov processes γ^Y with semi-analytic formulas for $\mathbb{E}e^{-\int_0^t \gamma^Y_s\, ds}$, for calibration purposes, and square integrable, for the sake of (4.10). We emphasize that, even if we don't engage into any calibration exercise in this paper, the empirical study in Bielecki, Cousin, Crépey, and Herbertsson (2014a, Part II) shows that this model is efficiently calibratable to CDS and CDO market data, including at the peak of the credit crisis. For $Y \in \mathcal{Y}$, we define mutually avoiding random times and their indicator processes by

$$\eta_Y = \inf\{t > 0; \int_0^t \gamma^Y_s\, ds > \epsilon_Y\}, \quad H^Y_t = \mathbb{1}_{\{\eta_Y \leq t\}}, \tag{7.2}$$

where the ϵ_Y are i.i.d. standard exponential random variables, $Y \in \mathcal{Y}$. The full model filtration $\mathbb{G}$ is given as $\mathbb{W}$ progressively enlarged by the random times η_Y, $Y \in \mathcal{Y}$. Let M^Y denote the compensated martingale $dM_t^Y = dH^Y - (1 - H_t^Y)\gamma_t^Y dt,\ t \geq 0$. We define $\boldsymbol{\Gamma} = (\gamma^Y)_{Y\in\mathcal{Y}}, \mathbf{H} = (H^Y)_{Y\in\mathcal{Y}}$ and $\tau_i = \min_{\{Y\in\mathcal{Y};i\in Y\}} \eta_Y$, $i \in N$.

Theorem 7.1 *The common-shock model is a recursively immersed model of default times. For $t \geq 0$,we have*

$$\mathcal{G}_t = \mathcal{W}_t \vee \bigvee_{Y\in\mathcal{Y}} \left(\sigma(\eta_Y \wedge t) \vee \sigma(\{\eta_Y > t\})\right). \tag{7.3}$$

The W^Y and the M^Y, $Y \in \mathcal{Y}$, have the $(\mathbb{G}, \mathbb{Q})$ martingale representation property.

Proof. We prove the martingale representation property and (7.3) by induction as follows. We write $\mathbb{G} = \mathbb{G}^{\mathcal{Y}}$. If $\mathcal{Y}$ is a singleton (case of a Cox time in view of (7.2)), then the immersion of $\mathbb{W}$ into $\mathbb{G}^{\mathcal{Y}}$ implies the results, by the optional splitting formula (5.6) for (7.3) and by Jeanblanc and Song (2013, Theorem 6.4) for the martingale representation property. Moreover, if $\mathcal{Z}$ is obtained by addition of a new $Z \subseteq N$ to $\mathcal{Y}$, then the independence of the ϵ_Y implies that η_Z is a Cox time with intensity in $\mathbb{G}^{\mathcal{Y}}$, hence immersion of $\mathbb{G}^{\mathcal{Y}}$ into $\mathbb{G}^{\mathcal{Z}}$ follows (this is the recursively immersed feature stated in the lemma) and the results for $\mathbb{G}^{\mathcal{Z}}$ are implied likewise from those, if assumed, for $\mathbb{G}^{\mathcal{Y}}$. ∎

7.2 TVA Model

A DMO setup can be used as a TVA model for credit derivatives, with

$$E_b = \mathcal{Y}_b := \{Y \in \mathcal{Y};\, -1 \in Y\},\ \ E_c = \mathcal{Y}_c := \{Y \in \mathcal{Y};\, 0 \in Y\},\ \ E = \mathcal{Y}_\bullet := \mathcal{Y}_b \cup \mathcal{Y}_c.$$

In particular,

$$\tau_b = \tau_{-1} = \min_{Y\in\mathcal{Y}_b} \eta_Y,\ \ \tau_c = \tau_0 = \min_{Y\in\mathcal{Y}_c} \eta_Y,\ \text{hence}\ \tau = \min_{Y\in\mathcal{Y}_\bullet} \eta_Y,\ \ \gamma = J_- \sum_{Y\in\mathcal{Y}_\bullet} \gamma^Y. \tag{7.4}$$

The results of Crépey et al. (2014, Corollary 8.3.1 page 205)[6] show that in the case of a portfolio of vanilla credit derivatives on names in N, e.g. CDS contracts and CDO tranches as of (5.12), one has a semi-explicit formula for P of the form

$$P_t = P(t, \boldsymbol{\Gamma}_t, \mathbf{H}_t), \tag{7.5}$$

for a continuous function P. We assume that for every process $U = P$, $(D - D_-)$, V, I and $\mathfrak{I}$, hence also C and $\mathfrak{C}$, there exists a continuous function $\widetilde{U}$, abusively renoted $\widetilde{\Delta}$ in the case of $(D - D_-)$, such that

$$U_\tau = \widetilde{U}_Y(\tau, \boldsymbol{\Gamma}_\tau, \mathbf{H}_{\tau-}), \tag{7.6}$$

or $\widetilde{U}_\tau^Y$ in a shorthand notation, on every event of the form $\{\tau = \eta_Y\}$, $Y \in \mathcal{Y}_\bullet$. For vanilla credit derivatives on names in N, e.g. CDS contracts and CDO tranches as of (5.12), this always holds regarding $U = P$, by (7.5) and the DMO analog of the DGC identity (6.7), and in view of (5.12) it also holds for $U = D - D_-$. As will be illustrated by the right graph in our concluding figure 7, the $\widetilde{\Delta}^Y$ incorporate the gap risk effect in the DMO model. The conditions (7.6) on $U = V$, I and $\mathfrak{I}$ may be satisfied or not depending on the CSA (see the remark 6.2). In view of (5.8), the coefficient $\widetilde{\xi}$ (in the present no cure period case where $\delta = 0$) is given as

$$\widetilde{\xi}_t^Y = \mathbb{1}_{Y\in\mathcal{Y}_c}(1 - R_c)\big(\widetilde{P}_t^Y + \widetilde{\Delta}_t^Y - \widetilde{C}_t^Y\big)^+ - \mathbb{1}_{Y\in\mathcal{Y}_b}(1 - R_b)\big(\widetilde{P}_t^Y + \widetilde{\Delta}_t^Y - \widetilde{\mathfrak{C}}_t^Y\big)^-,\ \ Y \in \mathcal{Y}_\bullet. \tag{7.7}$$

The coefficient $\widehat{f}_t(\vartheta)$ in (5.10) is given, on $[0, \bar{\tau}]$, by

$$\begin{aligned} \widehat{f}_t(\vartheta) + (r_t + \gamma_t)\vartheta = \quad & (1 - R_c) \sum_{Y\in\mathcal{Y}_c} \gamma_t^Y \big(\widetilde{P}_t^Y + \widetilde{\Delta}_t^Y - \widetilde{C}_t^Y\big)^+ - (1 - R_b) \sum_{Y\in\mathcal{Y}_b} \gamma_t^Y \big(\widetilde{P}_t^Y + \widetilde{\Delta}_t^Y - \widetilde{\mathfrak{C}}_t^Y\big)^- \\ & + c_t\mathcal{C}_t + \widetilde{\lambda}_t \left(P_t - \mathcal{C}_t - \vartheta\right)^+ - \lambda_t \left(P_t - \mathcal{C}_t - \vartheta\right)^-, \end{aligned} \tag{7.8}$$

[6] Or Bielecki et al. (2014a, Part II, Corollary 3.1) in the journal version.

where $\widetilde{\lambda}_t = \bar{\lambda}_t - \Lambda \sum_{Y \in \mathcal{Y}_b} \gamma_t^Y$. Let $\mathcal{Y}_\circ = \mathcal{Y} \setminus \mathcal{Y}_\bullet$ and let $\widetilde{X}_t = (\boldsymbol{\Gamma}_t, \widetilde{\mathbf{H}}_t)$, where $\widetilde{\mathbf{H}} = (\mathbb{1}_{Y \in \mathcal{Y}_\circ} H^Y)_{Y \in \mathcal{Y}}$. We assume that the processes $r, c, \lambda, \bar{\lambda}, P$ and $\mathcal{C}$ are given before τ as continuous functions of $(t, \widetilde{X}_t)$. Regarding P, (7.5) shows that this property always holds in the case of vanilla credit derivatives on names in N. In view of the last identity in (7.4), this property is also verified by the process γ. The next result, stated without proof, is the DMO analog of the DGC Theorem 6.2, where the main difficulty, related to (DGC.2), came from the fact that we had to use $\mathbb{P} \neq \mathbb{Q}$ there, as opposed to $\mathbb{P} = \mathbb{Q}$ simply here. This is an easier case of immersion in the sense of the comments following the statement of the condition (C), hence we state the result without proof.

Theorem 7.2 *The condition (C) holds, for:*

(DMO.1) *a reference filtration $\mathbb{F} = (\mathcal{F}_t)$ in (C.1) given as $\mathbb{W}$ progressively enlarged by the $\eta_Y, Y \in \mathcal{Y}_\circ$, which satisfies*

$$\mathcal{F}_t = \mathcal{W}_t \vee \bigvee_{Y \in \mathcal{Y}_\circ} \big(\sigma(\eta_Y \wedge t) \vee \sigma(\{\eta_Y > t\})\big), \ \ t \geq 0,$$

(DMO.2) *$\mathbb{P} = \mathbb{Q}$ in (C.2), where the $W^Y, Y \in \mathcal{Y}$, and the $M^Y, Y \in \mathcal{Y}_\circ$, have the $(\mathbb{F}, \mathbb{P} = \mathbb{Q})$ martingale representation property,*

(DMO.3) *a Markov specification $\widetilde{f}_t(\vartheta) = \widetilde{f}(t, \widetilde{X}_t, \vartheta)$ of (C.3), for the $(\mathbb{F}, \mathbb{P} = \mathbb{Q})$ jump diffusion $\widetilde{X}_t = (\boldsymbol{\Gamma}_t, \widetilde{\mathbf{H}}_t)$ and the function $\widetilde{f} = \widetilde{f}(t, \widetilde{x}, \vartheta)$ given, writing $\widetilde{x} = (t, \boldsymbol{\gamma}, \widetilde{\mathbf{k}})$ for $\boldsymbol{\gamma} = (\gamma_Y)_{Y \in \mathcal{Y}} \in \mathbb{R}_+^{\mathcal{Y}}$ and $\widetilde{\mathbf{k}} = (k_Y)_{Y \in \mathcal{Y}} \in \{0,1\}^{\mathcal{Y}}$ with $k_Y = 0$ if $Y \in \mathcal{Y}_\bullet$, by:*

$$\begin{aligned} &\widetilde{f}(t, \widetilde{x}, \vartheta) + (r(t, \widetilde{x}) + \gamma(t, \widetilde{x}))\vartheta = \\ &\quad (1 - R_c) \sum_{Y \in \mathcal{Y}_c} \gamma_Y \big(\widetilde{P}_Y + \widetilde{\Delta}_Y - \widetilde{C}_Y\big)^+(t, \widetilde{x}) - (1 - R_b) \sum_{Y \in \mathcal{Y}_b} \gamma_Y \big(\widetilde{P}_Y + \widetilde{\Delta}_Y - \widetilde{\mathfrak{C}}_Y\big)^-(t, \widetilde{x}) \\ &\qquad + \big(c\mathcal{C} + \widetilde{\lambda}\big(P - \vartheta - \mathcal{C}\big)^+ - \lambda\big(P - \vartheta - \mathcal{C}\big)^-\big)(t, \widetilde{x}), \end{aligned} \tag{7.9}$$

where $\gamma(t, \widetilde{x}) = \sum_{Y \in \mathcal{Y}_\bullet} \gamma_Y$, $\widetilde{\lambda} = \bar{\lambda} - \Lambda \sum_{Y \in \mathcal{Y}_b} \gamma_Y$.

In addition, $(\mathbb{F}, \mathbb{P} = \mathbb{Q})$ local martingales don't jump at τ, τ avoids $\mathbb{F}$ stopping times, the condition (B) is satisfied and the Azéma supermartingale S of τ is given, for $t \in [0, T]$, by

$$S_t = e^{-\sum_{Y \in \mathcal{Y}_\bullet} \int_0^t \gamma_s^Y ds} > 0. \tag{7.10}$$

In particular, the DMO model is a marked default times setup, satisfying (5.2)*, where the full and reduced DGC BSDEs are equivalent.*

Similar comments as those formulated after the DGC Theorem 6.2 are applicable regarding the role of the martingale representation property here. Also note that the Azéma supermartingale (7.10) is continuous and nonincreasing, consistent with the immersion property of the setup. Since the condition (C) is satisfied, we can derive the following DMO analog, stated without proof, of the DGC corollary 6.1.

Corollary 7.1 *In the DMO model, we have $\gamma' = \gamma(\cdot, \widetilde{X}_\cdot) = \sum_{Y \in \mathcal{Y}_\bullet} \gamma^Y \in \widetilde{\mathcal{H}}_2$. Assuming all the other conditions in Theorem 4.1 and without cure period, so for $\delta = 0$ and $\widetilde{f} = \widetilde{f}(t, \boldsymbol{\gamma}, \widetilde{\mathbf{k}}, \vartheta)$ as of* (7.9)*, the corresponding reduced TVA BSDE* (3.5) *admits a unique square integrable solution $\widetilde{\Theta}_t = \widetilde{\Theta}(t, \widetilde{X}_t)$, where the function $\widetilde{\Theta}(t, \widetilde{x})$ is a continuous viscosity solution to the corresponding semilinear PIDE. A solution Θ to the full TVA BSDE* (2.8) *is obtained by setting $\Theta = \widetilde{\Theta}$ on $[0, \bar{\tau})$ and*

$$\Theta_{\bar{\tau}} = \mathbb{1}_{\{\tau < T\}} \Big(\widetilde{\xi}_\tau^\imath - (P_{\tau-} - \mathcal{C}_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{-1 \in \imath\}} \Lambda\Big),$$

where $\imath \in \mathcal{Y}_\bullet$ is the identity of the shock triggering the first default of a party. The $(\mathbb{G}, \mathbb{Q})$ local martingale component μ of Θ satisfies, for $t \in [0, \bar{\tau}]$:

$$\begin{aligned} d\mu_t = d\widetilde{\mu}_t &- \Big(\widetilde{\xi}_\tau^\imath - \widetilde{\Theta}_{\tau-} - (P_{\tau-} - \mathcal{C}_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{-1 \in \imath\}} \Lambda\Big) dJ_t \\ &- \Big(\sum_{Y \in \mathcal{Y}_\bullet} (\widetilde{\xi}_t^Y - \widetilde{\Theta}_t)\gamma_t^Y - (P_t - \mathcal{C}_t - \widetilde{\Theta}_t)^+ \Lambda \sum_{Y \in \mathcal{Y}_b} \gamma_t^Y\Big) dt. \ \blacksquare \end{aligned} \tag{7.11}$$

7.3 Cure Period

In our DGC and DMO examples so far, we postulated no cure period, i.e. $\delta = 0$. Let's now assume, in a DMO setup, a positive cure period δ, with a $\mathcal{P}(\mathbb{G})$ measurable $\mathbf{C} = (V, I, \mathfrak{I})$ and deterministic interest rates r_t for simplicity. Similar considerations would apply in a DGC setup. In the case of vanilla credit derivatives (CDS contracts and CDO tranches with dividends given by (5.12)), in a deterministic interest rate environment, the process $\Delta_t = \int_{[\tau,t]} e^{\int_s^t r_u du} dD_s$ is a function of the default times in $[\tau, t]$. We write $K_t^Y = (\mathbb{1}_{\{\eta_Y \le t\}}, \eta_Y \mathbb{1}_{\{\eta_Y \le t\}}) = (H_t^Y, \eta_Y H_t^Y)$, $\mathbf{K} = (K^Y)_{Y \in \mathcal{Y}}$, $\Delta_t^\star = \int_{[0,t]} e^{-\int_t^s r_u du} dD_s$, so that $\Delta_t = \Delta_t^\star - \Delta_{\tau-}^\star$, and we consider a cure period $(\mathbb{G}, \mathbb{Q})$ factor process $X_t = (t, \boldsymbol{\Gamma}_t, \mathbf{K}_t)$. By application of the results of Bielecki, Jakubowski, and Nieweglowski (2012) (or by a direct proof based on Heath and Schweizer (2000, Theorem 1) and Becherer and Schweizer (2005, Corollary 2.3)), X is a $(\mathbb{G}, \mathbb{Q})$ homogenous strong Markov process. Recall from (4.4) and (4.2) that

$$\xi = \mathbb{1}_{\{\tau_c \le \tau_b^\delta\}} (1 - R_c)(Q_{\tau^\delta} - C_\tau)^+ - \mathbb{1}_{\{\tau_b \le \tau_c^\delta\}} (1 - R_b)(Q_{\tau^\delta} - \mathfrak{C}_\tau)^+,$$

where

$$\mathbb{1}_{\{\tau_c \le \tau_b^\delta\}} = \mathbb{1}_{\{\tau_c \le \tau^\delta\}} = 1 - \prod_{Y \in \mathcal{Y}_c} (1 - H_{\tau^\delta}^Y), \;\; \mathbb{1}_{\{\tau_b \le \tau_c^\delta\}} = \mathbb{1}_{\{\tau_b \le \tau^\delta\}} = 1 - \prod_{Y \in \mathcal{Y}_b} (1 - H_{\tau^\delta}^Y).$$

Moreover, in the case of vanilla credit derivatives such as CDS and CDO contracts as of (5.12), we have in the DMO model:

$$Q_{\tau^\delta} = P_{\tau^\delta} + \Delta_{\tau^\delta} = P_{\tau^\delta} + \Delta_{\tau^\delta}^\star - \Delta_{\tau-}^\star = P(\tau^\delta, \boldsymbol{\Gamma}_{\tau^\delta}, \mathbf{K}_{\tau^\delta}) + \Delta_\star(\tau^\delta, \boldsymbol{\Gamma}_{\tau^\delta}, \mathbf{K}_{\tau^\delta}) - \Delta_{\tau-}^\star,$$

for continuous functions P and $\Delta_\star$.

Remark 7.1 Similar as in the DGC case (see the remark 7.1), any continuity statement with respect to the second components t_Y of the k_Y values of the K_t^Y, $Y \in \mathcal{Y}$, has to be understood in the sense of the corresponding "order sets" domain and topology.

Hence, ξ can be written in functional form as

$$\xi = \xi_\star(\tau^\delta, \boldsymbol{\Gamma}_{\tau^\delta}, \mathbf{K}_{\tau^\delta}, \mathbf{C}_\tau, \Delta_{\tau-}^\star) = \xi_\star(X_{\tau^\delta}, \mathbf{C}_\tau, \Delta_{\tau-}^\star), \tag{7.12}$$

for some function $\xi_\star$ continuous in the values x of X and where $\mathbf{C}_\tau$ and $\Delta_{\tau-}^\star$ are considered as $\mathcal{G}_\tau$ measurable parameters. We consider the $(\mathbb{F}, \mathbb{P} = \mathbb{Q})$ reduced factor process $\widetilde{X}_t = (\boldsymbol{\Gamma}_t, \widetilde{\mathbf{K}}_t, \widetilde{\mathbf{C}}_t, \Delta_t')$, where $\widetilde{\mathbf{K}} = (\mathbb{1}_{Y \in \mathcal{Y}_\circ} K^Y)_{Y \in \mathcal{Y}}$ and where $\widetilde{\mathbf{C}}$ (resp. Δ') is an $\mathbb{F}$ predictable process that coincides with $\mathbf{C}$ (resp. $\Delta_-^\star$) on $(0, \tau]$. Such processes $\widetilde{\mathbf{C}}$ (having assumed a $\mathbb{G}$ predictable process $\mathbf{C}$) and Δ' exist by virtue of the DMO condition (B) established in Theorem 6.2. We denote by $\mathbf{k}$ and $\widetilde{x}$ respective values of $\mathbf{K}$ (or $\widetilde{\mathbf{K}}$) and $\widetilde{X}$ and we write $\mathbf{k}^{Y,t}$ for the vector obtained from $\mathbf{k}$ by replacing the component with index Y by $(1, t)$.

Lemma 7.1 *We have $cdva_t = J_{t-} cdva(t, \widetilde{X}_t)$, $\mathbb{Q} \times \boldsymbol{\lambda}$ a.e., where*

$$cdva(t, \widetilde{X}_t) = \sum_{Y \in \mathcal{Y}_\bullet} \gamma_t^Y \bar{\xi}\big(t, \boldsymbol{\Gamma}_t, \widetilde{\mathbf{K}}_t^{Y,t}, \widetilde{\mathbf{C}}_t, \Delta_t'\big), \tag{7.13}$$

for a continuous function $\bar{\xi}(t, \widetilde{x})$ such that

$$\bar{\xi}_\tau = \bar{\xi}(\tau, \widetilde{X}_\tau). \tag{7.14}$$

Proof. In view of (7.12), the $(\mathbb{G}, \mathbb{Q})$ Markov property of X yields

$$\bar{\xi}_\tau = \bar{\xi}(X_\tau, \mathbf{C}_\tau, \Delta_{\tau-}^\star), \tag{7.15}$$

for some continuous function $\bar{\xi}(x, \mathbf{c}, d)$. On $\{\tau = \eta_Y < T\}$, we have

$$\mathbf{K}_\tau = (\mathbf{K}_{\tau-})^{Y,\tau}$$

and hence

$$\bar{\xi}_\tau = \bar{\xi}\big(X_\tau, \mathbf{C}_\tau, \Delta^\star_{\tau-}\big) = \bar{\xi}\big(\tau, \boldsymbol{\Gamma}_\tau, \mathbf{K}_\tau, \mathbf{C}_\tau, \Delta^\star_{\tau-}\big)$$
$$= \bar{\xi}\big(\tau, \boldsymbol{\Gamma}_\tau, (\mathbf{K}_{\tau-})^{Y,\tau}, \mathbf{C}_\tau, \Delta^\star_{\tau-}\big) = \bar{\xi}\big(\tau, \boldsymbol{\Gamma}_\tau, (\widetilde{\mathbf{K}}_{\tau-})^{Y,\tau}, \widetilde{\mathbf{C}}_\tau, \Delta'_\tau\big), \tag{7.16}$$

from which the results follows by an application of Lemma 5.1. ∎

As a consequence of Lemma 7.1, postulating that $lva_t(\vartheta)$ in (4.6) is given before τ as a continuous function $lva(t, \widetilde{X}_t, \vartheta)$, the condition (5.11) holds for the function $\widetilde{f}(t, \widetilde{x}, \vartheta)$ such that, with $\widetilde{x} = (\boldsymbol{\gamma}, \widetilde{\mathbf{k}}, \mathbf{c}, d)$ and $\gamma(t, \widetilde{x}) = \sum_{Y\in\mathcal{Y}_\bullet} \gamma_Y$:

$$\widetilde{f}(t, \widetilde{x}, \vartheta) + (r(t) + \gamma(t, \widetilde{x}))\vartheta = cdva(t, \widetilde{x}) + lva(t, \widetilde{x}, \vartheta). \tag{7.17}$$

Therefore, Theorem 7.2 still holds for $\delta > 0$ with $\widetilde{f}$ as of (7.17) instead of (7.9) in (DMO.3). Hence, the condition (C) is satisfied and we can derive the following cure period analog of Corollary 7.1.

Corollary 7.2 *In the DMO model, we have $\gamma' = \gamma(\cdot, \widetilde{X}_\cdot) = \sum_{Y\in\mathcal{Y}_\bullet} \gamma^Y \in \widetilde{\mathcal{H}}_2$. Assuming all the other conditions in Theorem 4.1 and for a cure period $\delta > 0$, hence for $\widetilde{f} = \widetilde{f}(t, \boldsymbol{\gamma}, \widetilde{\mathbf{k}}, \mathbf{c}, d, \vartheta)$ as of (7.17), the corresponding reduced TVA BSDE (3.5) admits a unique square integrable solution $\widetilde{\Theta}_t = \widetilde{\Theta}(t, \widetilde{X}_t)$, where the function $\widetilde{\Theta}(t, \widetilde{x})$ is a continuous viscosity solution to the corresponding semilinear PIDE. A solution Θ to the full TVA BSDE (2.8) is obtained by setting $\Theta = \widetilde{\Theta}$ on $[0, \bar{\tau})$ and*

$$\Theta_{\bar{\tau}} = \mathbb{1}_{\{\tau<T\}} \Big(\bar{\xi}\big(\tau, \boldsymbol{\Gamma}_\tau, (\widetilde{\mathbf{K}}_{\tau-})^{\imath,\tau}, \widetilde{\mathbf{C}}_\tau, \Delta'_\tau\big) - (P_{\tau-} - \mathcal{C}_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{-1\in\imath\}} \Lambda\Big),$$

where $\imath \in \mathcal{Y}_\bullet$ is the identity of the shock triggering the first default of a party. The $(\mathbb{G}, \mathbb{Q})$ local martingale component μ of Θ satisfies, for $t \in [0, \bar{\tau}]$:

$$d\mu_t = d\widetilde{\mu}_t - \Big(\bar{\xi}\big(\tau, \boldsymbol{\Gamma}_\tau, (\widetilde{\mathbf{K}}_{\tau-})^{\imath,\tau}, \widetilde{\mathbf{C}}_\tau, \Delta'_\tau\big) - \widetilde{\Theta}_{t-} - (P_{\tau-} - \mathcal{C}_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{\tau=\tau_b\}} \Lambda\Big) dJ_t$$
$$- \Big(\sum_{Y\in\mathcal{Y}_\bullet} (\bar{\xi}\big(t, \boldsymbol{\Gamma}_t, (\widetilde{\mathbf{K}}_t)^{Y,t}, \widetilde{\mathbf{C}}_t, \Delta'_t\big) - \widetilde{\Theta}_t)\gamma^Y_t - (P_t - \mathcal{C}_t - \widetilde{\Theta}_t)^+ \Lambda \sum_{Y\in\mathcal{Y}_b} \gamma^Y_t\Big) dt. \ ∎$$

8 Numerical Implementation and Results

Due to funding costs, the TVA equations are nonlinear BSDEs. In the case of credit derivatives, they are also very high-dimensional. For nonlinear and very high-dimensional problems, any numerical scheme based, even to some extent, on dynamic programming, such as purely backward deterministic PDE schemes, but also forward/backward simulation/regression BSDE schemes, are ruled out by the curse of dimensionality (see e.g. Crépey (2013, Part IV)). Now, in any bottom-up credit portfolio model such as the DGC or the DMO model, the dimension is at least the number of credit names. Hence, for n greater than a few units, the only feasible TVA schemes are purely forward simulation schemes, such as Monte Carlo simulation with m runs based on the linear expansion of Fujii and Takahashi (2012a,2012b) or the branching particles scheme of Henry-Labordère (2012), respectively dubbed "FT scheme" and "PHL scheme" below. In our setup, the PHL scheme involves a nontrivial and rather sensitive fine-tuning for finding a polynom in ϑ that approximates the terms $(P_t - \mathcal{C}_t - \vartheta)^\pm$ in $lva_t(\vartheta)$ in a suitable range for ϑ. Ideally, such a polynom should be adaptive and depend on $(P_t - \mathcal{C}_t)$, but in a PHL scheme the approximating polynom has to be fixed once for all in the simulation. The only way we were able to achieve a good fine-tuning is by using a preliminary knowledge on the solution obtained by running the FT scheme in the first place. Since a numerical scheme that can be run automatically and does not involve any fine-tuning is preferable, we focus on the FT scheme in the sequel.

8.1 Fujii and Takahashi's TVA Linear Expansion

The FT scheme is based on an expansion of the coefficient, hence of the solution $\widetilde{\Theta}$, to a Markovian BSDE on $[0,T]$ such as (3.5)/(5.11), as a series $\widetilde{\Theta} \approx \widetilde{\Theta}^{(0)} + \widetilde{\Theta}^{(1)} + \widetilde{\Theta}^{(2)} + \widetilde{\Theta}^{(3)} + \cdots$ of solutions to linear BSDEs, where the next BSDE in the series uses the solution to the previous one as input data. Let

$$\bar{f}(t,\widetilde{x},\vartheta) := \widetilde{f}(t,\widetilde{x},\vartheta) + \gamma(t,\widetilde{x})\vartheta = cdva(t,\widetilde{x}) + lva(t,\widetilde{x},\vartheta) - r(t,\widetilde{x})\vartheta \tag{8.1}$$

(cf. (4.6)). In order to exploit some cancellation between related discount factors in Lemma 8.1, it is preferable to use an FT expansion of the coefficient $\bar{f}$ rather than $\widetilde{f}$, treating the $\gamma(t,\widetilde{x})\vartheta$ term in $\bar{f}$ as a discount factor (cf. the right identity in (4.8)). In terms of $\bar{f}$, the reduced BSDE (3.5) is written as: $\widetilde{\Theta}_T = 0$ and, for $t \in [0,T]$,

$$-d\widetilde{\Theta}_t = \widetilde{f}(t,\widetilde{X}_t,\widetilde{\Theta}_t)dt - d\widetilde{\mu}_t = \big(\bar{f}(t,\widetilde{X}_t,\widetilde{\Theta}_t) - \gamma(t,\widetilde{X}_t)\widetilde{\Theta}_t\big)dt - d\widetilde{\mu}_t.$$

The corresponding FT expansion reads (cf. Fujii and Takahashi (2012a, Equations (2.4), (2.6) and (2.7) in the arxiv version)): $\widetilde{\Theta}^{\epsilon}_T = 0$ and, for $t \in [0,T]$,

$$\begin{aligned}
&- d\widetilde{\Theta}^{\epsilon}_t = \big(\epsilon \bar{f}(t,\widetilde{X}_t,\widetilde{\Theta}^{\epsilon}_t) - \widetilde{\gamma}(t,\widetilde{X}_t)\widetilde{\Theta}^{\epsilon}_t\big)dt - d\widetilde{\mu}^{\epsilon}_t\\
&\widetilde{\Theta}^{\epsilon}_t = \widetilde{\Theta}^{(0)}_t + \epsilon\widetilde{\Theta}^{(1)}_t + \epsilon^2\widetilde{\Theta}^{(2)}_t + \epsilon^3\widetilde{\Theta}^{(3)}_t + \cdots\\
&\bar{f}(t,\widetilde{X}_t,\widetilde{\Theta}^{\epsilon}_t) = \bar{f}(t,\widetilde{X}_t,\widetilde{\Theta}^{0}_t) + \big(\epsilon\widetilde{\Theta}^{(1)} + \epsilon^2\widetilde{\Theta}^{(2)} + \epsilon^3\widetilde{\Theta}^{(3)} + \cdots\big)\partial_\vartheta \bar{f}(t,\widetilde{X}_t,\widetilde{\Theta}^{(0)}_t) + \ldots.
\end{aligned}$$

Quoting Fujii and Takahashi (2012a, page 4 in the arxiv version), "by putting $\epsilon = 1$, $\widetilde{\Theta}^{(0)} + \widetilde{\Theta}^{(1)} + \widetilde{\Theta}^{(2)} + \widetilde{\Theta}^{(3)} + \cdots$ is expected to provide a reasonable approximation for the original $\widetilde{\Theta}$ as long as the residual term is small enough to allow the perturbative treatment." This is studied mathematically in a diffusive setup in Takahashi and Yamada (2014). Collecting all terms in ϵ^i, the resulting first $\widetilde{\Theta}^{(i)}$ terms are written as $\widetilde{\Theta}^{(0)} = 0$, due to the null terminal condition of the reduced TVA BSDE (3.5), and

$$\begin{aligned}
\widetilde{\Theta}^{(1)}_t &= \widetilde{\mathbb{E}}\Big[\int_t^T e^{-\int_t^s \gamma(r,\widetilde{X}_r)dr}\,\bar{f}\Big(s,\widetilde{X}_s,\widetilde{\Theta}^{(0)}_s = 0\Big)ds \,\big|\, \mathcal{F}_t\Big],\\
\widetilde{\Theta}^{(2)}_t &= \widetilde{\mathbb{E}}\Big[\int_t^T e^{-\int_t^s \gamma(r,\widetilde{X}_r)dr}\partial_\vartheta \bar{f}\Big(s,\widetilde{X}_s,\widetilde{\Theta}^{(0)}_s = 0\Big)\widetilde{\Theta}^{(1)}_s ds \,\big|\, \mathcal{F}_t\Big],\\
\widetilde{\Theta}^{(3)}_t &= \widetilde{\mathbb{E}}\Big[\int_t^T e^{-\int_t^s \gamma(r,\widetilde{X}_r)dr}\partial_\vartheta \bar{f}\Big(s,\widetilde{X}_s,\widetilde{\Theta}^{(0)}_s = 0\Big)\widetilde{\Theta}^{(2)}_s ds \,\big|\, \mathcal{F}_t\Big].
\end{aligned} \tag{8.2}$$

The first two lines correspond to the identities (2.19) and (2.22) in the arxiv version of Fujii and Takahashi (2012a). Compared with the third line, the complete third order term comprises another component based on $\partial^2_{\vartheta^2}\widetilde{f}$. In our case, $\partial^2_{\vartheta^2}\widetilde{f}$ involves a Dirac measure via the terms $(P_t - \mathcal{C}_t - \vartheta)^{\pm}$ in $lva_t(\vartheta)$ (cf. (4.6)), so that we truncate the expansion to the term $\widetilde{\Theta}^{(3)}_t$ as above. Moreover, we use the interacting particles implementation of these formulas in Fujii and Takahashi (2012c) and Fujii, Sato, and Takahashi (2014). Namely, we estimate by randomization, based on independent exponential draws ζ_j with parameters μ_j, each time integral that intervenes in (8.2) either explicitly or implicitly through the terms $\widetilde{\Theta}^{(1)}_s$ and $\widetilde{\Theta}^{(2)}_s$. Specifically, the following identities follow from (8.2) by the tower rule :

$$\begin{aligned}
\widetilde{\Theta}^{(1)}_0 &= \widetilde{\mathbb{E}}\Big[\mathbb{1}_{\zeta_1<T}\frac{e^{\mu_1\zeta_1}}{\mu_1}e^{-\int_0^{\zeta_1}\gamma(r,\widetilde{X}_r)dr}\,\bar{f}\Big(\zeta_1,\widetilde{X}_{\zeta_1},0\Big)\Big],\\
\widetilde{\Theta}^{(2)}_0 &= \widetilde{\mathbb{E}}\Big[\mathbb{1}_{\zeta_1+\zeta_2<T}\frac{e^{\mu_1\zeta_1+\mu_2\zeta_2}}{\mu_1\mu_2}e^{-\int_0^{\zeta_1+\zeta_2}\gamma(r,\widetilde{X}_r)dr}\partial_\vartheta\bar{f}\Big(\zeta_1,\widetilde{X}_{\zeta_1},0\Big)\bar{f}\Big(\zeta_1+\zeta_2,\widetilde{X}_{\zeta_1+\zeta_2},0\Big)\Big],\\
\widetilde{\Theta}^{(3)}_0 &= \widetilde{\mathbb{E}}\Big[\mathbb{1}_{\zeta_1+\zeta_2+\zeta_3<T}\frac{e^{\mu_1\zeta_1+\mu_2\zeta_2+\mu_3\zeta_3}}{\mu_1\mu_2\mu_3}e^{-\int_0^{\zeta_1+\zeta_2+\zeta_3}\gamma(r,\widetilde{X}_r)dr}\times\\
&\qquad\partial_\vartheta\bar{f}\Big(\zeta_1,\widetilde{X}_{\zeta_1},0\Big)\partial_\vartheta\bar{f}\Big(\zeta_1+\zeta_2,\widetilde{X}_{\zeta_1+\zeta_2},0\Big)\bar{f}\Big(\zeta_1+\zeta_2+\zeta_3,\widetilde{X}_{\zeta_1+\zeta_2+\zeta_3},0\Big)\Big].
\end{aligned} \tag{8.3}$$

In the spirit of reduction of filtration of this work, a model is originally defined and can be simulated with respect to the full stochastic basis $(\mathbb{G},\mathbb{Q})$. Even in the case where there exists a stochastic basis $(\mathbb{F},\mathbb{P})$ satisfying the condition (C), $(\mathbb{F},\mathbb{P})$ simulation may be nontrivial. That would for instance be an issue in the DGC model (in the DMO model $(\mathbb{F},\mathbb{P}=\mathbb{Q})$ simulation is equally easy as $(\mathbb{G},\mathbb{P}=\mathbb{Q})$ simulation due to the immersion properties of the setup). But, as explained after the next result, it is always possible to reformulate the $\mathbb{P}$ expectations in (8.3) as $\mathbb{Q}$ expectations, by a direct formula not involving any Girsanov weights, which allows estimating the $\widetilde{\Theta}_0^{(i)}$ simply by $(\mathbb{G},\mathbb{Q})$ simulation.

Lemma 8.1 *Assume the conditions (B) and (C). If the Azema supermartingale S of τ is positive, then, for any $\mathbb{F}$ progressively measurable process h and independent nonnegative random variable ζ,*

$$\widetilde{\mathbb{E}}\left[\mathbb{1}_{\{\zeta<T\}}e^{-\int_0^\zeta \gamma'_s ds}h_\zeta\right]=\mathbb{E}\left[\mathbb{1}_{\{\zeta<\bar\tau\}}h_\zeta\right]. \tag{8.4}$$

Proof. By Fubini's theorem, we only need to prove the lemma for $\zeta=s$ constant positive. Let $H_s=\mathbb{1}_{\{s<T\}}h_s$. Since h is $\mathbb{F}$ adapted, $\mathbb{E}[H_s\mathbb{1}_{\{s<\tau\}}]=\mathbb{E}[H_sS_s]$. We recall from Crépey and Song (2014c, Lemma 2.2 5)) the following multiplicative stochastic exponential decomposition of S on $\mathbb{R}_+$ (for S positive):

$$S=S_0\mathcal{E}(-\frac{1}{S_-}\centerdot\mathcal{D})\mathcal{E}(\frac{1}{pS}\centerdot\mathcal{Q}),$$

where $S=\mathcal{Q}-\mathcal{D}$ is the $(\mathbb{F},\mathbb{Q})$ canonical Doob-Meyer decomposition of S. By Crépey and Song (2014c, Theorem 3.1 2)), the $(\mathbb{F},\mathbb{Q})$ density process of $\mathbb{P}$ is given by $\mathcal{E}(\frac{1}{pS}\centerdot\mathcal{Q})$ on $[0,T]$. Therefore,

$$\mathbb{E}[H_s\mathbb{1}_{\{s<\tau\}}]=\mathbb{E}[H_sS_s]=\mathbb{E}[H_sS_0\mathcal{E}(-\frac{1}{S_-}\centerdot\mathcal{D})_s\mathcal{E}(\frac{1}{pS}\centerdot\mathcal{Q})_s]=\widetilde{\mathbb{E}}[H_sS_0\mathcal{E}(-\frac{1}{S_-}\centerdot\mathcal{D})_s]. \tag{8.5}$$

Note that $S_0=1$ and $\mathcal{D}$ is the drift of the Azema supermartingale of τ in $(\mathbb{F},\mathbb{Q})$, so that $\mathcal{D}$ is absolutely continuous, i.e. $\mathcal{D}=\nu\centerdot\boldsymbol{\lambda}$ for some density process ν. Hence, the stochastic exponential $\mathcal{E}(-\frac{1}{S_-}\centerdot\mathcal{D})$ is a usual exponential. By Jeulin (1980, Remark 4.5), $\mathbb{1}_{(0,\tau]}\frac{\nu}{S_-}\centerdot\boldsymbol{\lambda}$ is the $(\mathbb{G},\mathbb{Q})$ predictable dual projection of $\mathbb{1}_{[\tau,\infty)}$. Therefore,

$$\mathbb{1}_{(0,\tau]}\frac{\nu}{S_-}=\gamma_-\mathbb{1}_{(0,\tau]},$$

so that $\frac{\nu}{S_-}$ is the $\mathbb{F}$ predictable reduction of γ_-, i.e. $\frac{\nu}{S_-}=\gamma'$. As a consequence, after substitution of ζ for s, the identity (8.5) is rewritten as (8.4). ■

Applications of Lemma 8.1 to (8.3) yield

$$\begin{aligned}
\widetilde{\Theta}_0^{(1)}&=\mathbb{E}\Big[\mathbb{1}_{\zeta_1<\bar\tau}\frac{e^{\mu_1\zeta_1}}{\mu_1}\bar f\Big(\zeta_1,\widetilde X_{\zeta_1},0\Big)\Big],\\
\widetilde{\Theta}_0^{(2)}&=\mathbb{E}\Big[\mathbb{1}_{\zeta_1+\zeta_2<\bar\tau}\frac{e^{\mu_1\zeta_1+\mu_2\zeta_2}}{\mu_1\mu_2}\partial_\vartheta\bar f\Big(\zeta_1,\widetilde X_{\zeta_1},0\Big)\bar f\Big(\zeta_1+\zeta_2,\widetilde X_{\zeta_1+\zeta_2},0\Big)\Big],\\
\widetilde{\Theta}_0^{(3)}&=\mathbb{E}\Big[\mathbb{1}_{\zeta_1+\zeta_2+\zeta_3<\bar\tau}\frac{e^{\mu_1\zeta_1+\mu_2\zeta_2+\mu_3\zeta_3}}{\mu_1\mu_2\mu_3}\\
&\qquad\partial_\vartheta\bar f\Big(\zeta_1,\widetilde X_{\zeta_1},0\Big)\partial_\vartheta\bar f\Big(\zeta_1+\zeta_2,\widetilde X_{\zeta_1+\zeta_2},0\Big)\bar f\Big(\zeta_1+\zeta_2+\zeta_3,\widetilde X_{\zeta_1+\zeta_2+\zeta_3},0\Big)\Big].
\end{aligned} \tag{8.6}$$

In the two models considered with $\delta=0$ in this paper, the reduced factor process $\widetilde X$ consists of some components of a full factor process X, namely $X_t=(\mathbf{m}_t,\mathbf{k}_t)$ in the DGC setup without cure period of Sect. 6.2 and $X_t=(\boldsymbol{\Gamma}_t,\mathbf{H}_t)$ in the DMO setup without cure period of Sect. 7.2. Hence, for $\delta=0$, we can compute the $\widetilde{\Theta}_0^{(i)}$ based on (8.6) by $(\mathbb{G},\mathbb{Q})$ simulation of X, which is well understood in both cases (cf. Crépey et al. (2014, Part IV)). In the DMO setup with positive cure period $\delta>0$ of Sect. 7.3, one residual difficulty with the formulas (8.6) is that $\bar f(t,\widetilde x,\vartheta)$ in (8.1) involves nontrivial $cdva(t,\widetilde x)$ terms as of (7.13). But the computation of these can be avoided by resorting to the following add-on to Lemma 8.1, where (8.7) implies that the $\widetilde{\Theta}_0^{(i)}$ in (8.6) can be computed by $(\mathbb{G},\mathbb{Q})$ simulation of X, $\mathbf{C}$ and $\Delta^\star$.

Lemma 8.2 *In the DMO setup with positive cure period $\delta>0$ of Sect. 7.3, the notation of which we use here, for any $\mathbb{F}$ predictable process h and for any independent random variable ζ with density p, we have:*

$$\mathbb{E}[\mathbb{1}_{\{\zeta<\bar\tau\}}h_\zeta cdva(\zeta,\widetilde X_\zeta)]=\mathbb{E}\big[\mathbb{1}_{\{\zeta<\bar\tau\}}h_\zeta\textstyle\sum_{Y\in\mathcal{Y}_\bullet}\gamma_\zeta^Y e^{-\int_\zeta^{\zeta+\delta}r(s)ds}\xi_\star\big(X_{\zeta^\delta}^{Y,\zeta},\mathbf{C}_\zeta,\Delta_\zeta^\star\big)\big], \tag{8.7}$$

where we write $X_t^{Y,s}=(t,\boldsymbol{\Gamma}_t,(\mathbf{K}_t)^{Y,s})$, for any $0\le s\le t$.

Proof. On $\{\tau=\eta_Y\}$, we have $\mathbf{K}_{\tau^\delta}=(\mathbf{K}_{\tau^\delta})^{Y,\tau}$, hence

$$\xi=\xi_\star(\tau^\delta,\boldsymbol{\Gamma}_{\tau^\delta},(\mathbf{K}_{\tau^\delta})^{Y,\tau},\mathbf{C}_\tau,\Delta^\star_{\tau-})=\xi_Y(X^{-Y}_{\tau^\delta},\tau,\mathbf{C}_\tau,\Delta^\star_{\tau-}),$$

for some function ξ_Y. By the recursively immersed feature (Markov copula properties) of the DMO model (see Bielecki et al. (2014a, Part I) and Bielecki, Jakubowski, and Nieweglowski (2012)), the process X^{-Y} obtained as X deprived from the Y^{th} component of $\mathbf{K}$ is a $(\mathbb{G},\mathbb{Q})$ homogenous strong Markov process, for any $Y\in\mathcal{Y}$. We denote by $\mathcal{T}^Y_\epsilon$ the transition function of the process X^{-Y} killed at the rate r over a time horizon ϵ, i.e., $\mathbf{k}^{-Y}$ representing a value of $\mathbf{K}$ deprived from its Y^{th} component,

$$\begin{aligned}(\varphi,(t,\boldsymbol{\gamma},\mathbf{k}^{-Y}))\to\mathcal{T}^Y_\epsilon[\varphi](t,\boldsymbol{\gamma},\mathbf{k}^{-Y})&=\mathbb{E}\big[e^{-\int_t^{t+\epsilon}r(s)ds}\varphi(X^{-Y}_{t+\epsilon})|X^{-Y}_t=(t,\boldsymbol{\gamma},\mathbf{k}^{-Y})\big]\\&=\mathbb{E}\big[e^{-\int_t^{t+\epsilon}r(s)ds}\varphi(X^{-Y}_{t+\epsilon})|\mathcal{G}_t\big],\end{aligned}$$

we compute:

$$\begin{aligned}\bar{\xi}&=\mathbb{E}[e^{-\int_\tau^{\tau+\delta}r(s)ds}\xi|\mathcal{G}_\tau]=\sum_{Y\in\mathcal{Y}_\bullet}\mathbb{E}[e^{-\int_\tau^{\tau+\delta}r(s)ds}\xi_Y(X^{-Y}_{\tau^\delta},\tau,\mathbf{C}_\tau,\Delta^\star_{\tau-})\mathbb{1}_{\{\tau=\eta_Y\}}|\mathcal{G}_\tau]\\&=\sum_{Y\in\mathcal{Y}_\bullet}\mathcal{T}^Y_\delta[\xi_Y(\cdot,\tau,\mathbf{C}_\tau,\Delta^\star_{\tau-})](X^{-Y}_\tau)\mathbb{1}_{\{\tau=\eta_Y\}}=\sum_{Y\in\mathcal{Y}_\bullet}\mathcal{T}^Y_\delta[\xi_Y(\cdot,\tau,\mathbf{C}_\tau,\Delta^\star_{\tau-})](X^{-Y}_{\tau-})\mathbb{1}_{\{\tau=\eta_Y\}},\end{aligned}$$

where the fact that $\{\tau=\eta_Y\}=\{\tau\geq\eta_Y\}\in\mathcal{G}_\tau$ (resp. X^{-Y} doesn't jump at τ on $\{\tau=\eta_Y\}$) was used to pass to the second line (resp. in the last identity). Hence, Lemma 5.1 yields

$$\gamma_t\widehat{\xi}_t=J_{t-}\sum_{Y\in\mathcal{Y}_\bullet}\gamma^Y_t\mathcal{T}^Y_\delta[\xi_Y(\cdot,t,\mathbf{C}_t,\Delta^\star_{t-})](X^{-Y}_{t-}),\ \ \mathbb{Q}\times\boldsymbol{\lambda}\text{ a.e.}\tag{8.8}$$

As a consequence, given an independent random variable ζ with density p, we can write, using respectively the formula (8.8) and the definition of $\mathcal{T}^Y_\delta$ to pass to the second and third line:

$$\begin{aligned}\mathbb{E}[h_\zeta\ \mathbb{1}_{\{\zeta\leq\bar{\tau}\}}cdva(\zeta,\widetilde{X}_\zeta)]&=\textstyle\int_0^T\mathbb{E}\big[h_t\mathbb{1}_{\{t<\bar{\tau}\}}cdva(t,\widetilde{X}_t)\big]\,p(t)dt\\&=\textstyle\int_0^T\mathbb{E}\Big[h_t\mathbb{1}_{\{t\leq\tau\}}\sum_{Y\in\mathcal{Y}_\bullet}\gamma^Y_t\mathcal{T}^Y_\delta[\xi_Y(\cdot,t,\mathbf{C}_t,\Delta^\star_{t-})](X^{-Y}_t)\Big]p(t)dt\\&=\textstyle\int_0^T\ \mathbb{E}\Big[h_t\mathbb{1}_{\{t\leq\tau\}}\sum_{Y\in\mathcal{Y}_\bullet}\gamma^Y_t\mathbb{E}\big[e^{-\int_t^{t+\delta}r(s)ds}\xi_Y(X^{-Y}_{t^\delta},t,\mathbf{C}_t,\Delta^\star_{t-})|\mathcal{G}_t\big]\Big]\ p(t)dt\\&=\textstyle\int_0^T\ \mathbb{E}\big[h_t\mathbb{1}_{\{t\leq\tau\}}\sum_{Y\in\mathcal{Y}_\bullet}\gamma^Y_t e^{-\int_t^{t+\delta}r(s)ds}\xi_\star(t^\delta,\boldsymbol{\Gamma}_{t^\delta},(\mathbf{K}_{t^\delta})^{Y,t},\mathbf{C}_t,\Delta^\star_{t-})\big]\ p(t)dt\\&=\mathbb{E}\big[\mathbb{1}_{\{\zeta\leq T\}}h(\zeta)\mathbb{1}_{\{\zeta\leq\tau\}}\textstyle\sum_{Y\in\mathcal{Y}_\bullet}\gamma^Y_\zeta e^{-\int_\zeta^{\zeta+\delta}r(s)ds}\xi_\star(X^{Y,\zeta}_{\zeta^\delta},\mathbf{C}_\zeta,\Delta^\star_\zeta)\big].\ \blacksquare\end{aligned}$$

Note that this proof exploits the specific immersion and Markov copula properties of the DMO model, through which each process X^{-Y} is Markov in the full model filtration $\mathbb{G}$.

Summarizing this section, in our dynamic copula specifications of the generic marked default times setup of Sect. 5, the reduced factor process $\widetilde{X}$ is made of certain components of a full factor process X that one knows how to simulate under $(\mathbb{G},\mathbb{Q})$, and the successive terms of the FT expansion (8.6) can be computed by $(\mathbb{G},\mathbb{Q})$ simulation of the related components of X. By contrast with methods such as importance sampling, which requires simulation under a modified probability measure and involves a related reweighting of all paths and payoffs, in our case, once the existence of $(\mathbb{F},\mathbb{P})$ satisfying the condition (C) has been checked mathematically, algorithmically, all one needs is $(\mathbb{G},\mathbb{Q})$ simulation and use of the appropriate reduced coefficient $\widetilde{f}$ (for which, admittedly, the explicit expression of the $(\mathbb{G},\mathbb{Q})$ intensity of the marked default times τ_e, $e\in E_b\cup E_c$, is required). One doesn't even need remember what $\mathbb{F}$ and $\mathbb{P}$ really are. As soon as the existence of a reduced stochastic basis $(\mathbb{F},\mathbb{P})$ satisfying the condition (C) is known and there is an explicit formula for the $(\mathbb{G},\mathbb{Q})$ intensity of the τ_e, the method can be implemented by any standard user.

To conclude this paper we present[7] TVA computations in the above DGC and DMO models, on CDS contracts and protection legs of CDO tranches corresponding to dividend processes of the respective forms D^i and $D^\star$ in (5.12). For these computations we follow the most conservative TVA

[7] We thank Hai Nam Nguyen, Tuyet Mai Nguyen, Abdallah Rahal and Dong Li Wu, current and former PhD students of the University of Evry, for their Matlab implementation of the DGC and DMO models.

approach of ignoring windfall benefits at own default, setting $R_b = 1$ and $\Lambda = 0$ (see the remark 4.4), which allows numerical validation of the results based on the reduced BSDE by results based on the full BSDE. In fact, also setting $c = \lambda = 0$, we have the following linear approximation formula for the time-0 value of the solution to the full TVA BSDE (2.8):

$$\begin{aligned}\Theta_0 &\approx \mathbb{E}\Big[\mathbb{1}_{\{\tau<T\}}\beta_{\tau^\delta}\xi + \int_t^{\bar{\tau}} \beta_s g_s(P_s)ds\Big] \\ &= \mathbb{E}\Big[\mathbb{1}_{\{\tau<T\}}\beta_{\tau^\delta}\mathbb{1}_{\{\tau_c\le\tau_b^\delta\}}(1-R_c)(P_{\tau^\delta}+\Delta_{\tau^\delta}-C_\tau)^+ + \int_t^{\bar{\tau}} \beta_s\bar{\lambda}_s\,(P_s-\mathcal{C}_s)^+\,ds\Big].\end{aligned} \tag{8.9}$$

On the one hand, for $\bar{\lambda} = 0$, this approximation is exact and a Monte Carlo loop based on the second line in (8.9) yields an unbiased estimate for $\Theta_0 = \widetilde{\Theta}_0$ alternative to Monte Carlo estimates for $\widetilde{\Theta}_0^{(1)} + \widetilde{\Theta}_0^{(2)} + \widetilde{\Theta}_0^{(3)}$ in (8.3) or (8.6). But, on the other hand, for $\bar{\lambda} \neq 0$, (8.9) analogs of higher order terms such as the second or third lines in (8.3) or (8.6), even if they could possibly be devised based on the full BSDE (2.8), would not be practical, due to the nonzero terminal condition in the full TVA BSDE (2.8). The tractability of (8.3) or (8.6) results from the null terminal condition in the reduced TVA BSDE (3.5). Otherwise, nested simulations would be required for estimating the $\Theta_s^{(0)}$ (whereas $\widetilde{\Theta}_s^{(0)} = 0$ everywhere in (8.2), (8.3) and (8.6)). Based on the full TVA BSDE (2.8), it might be possible to devise a PHL scheme, but at the cost of the already mentioned fine-tuning issue and for a variance bounded below by the variance of the standard Monte Carlo scheme based on (8.9), already found significatively greater below than the variance of the FT scheme based on the reduced BSDE.

Unless stated otherwise, the following numerical values are used (on top of $R_b = 1$ and $\Lambda = c = \lambda = 0$ for consistency with (8.9)):

$$r = 0,\ \ R_c = 40\%,\ \ \delta = 0,\ \ V = I = \mathfrak{I} = 0,\ \ \bar{\lambda} = 100 \text{ bp } = 0.01,\ \ \mu_j = \frac{2}{T},\ \ m = 10^4 \tag{8.10}$$

(see also after (5.12)). In particular, Θ is nonnegative.

8.2 Numerical Results in the DGC Model

We start by TVA computations on CDS contracts with maturity $T = 10$ years in a DGC model with $\varsigma = \frac{\mathbb{1}_{[0,T+1]}}{\sqrt{T+1}}$ and $\varrho = 0.6$ unless otherwise stated. Table 1 displays the contractual spreads of the CDS contracts used in these experiments. In Figure 2, the left graph shows the TVA on a CDS on name 1,

i	-1	0	1
S_i	36	41	47

i	-1	0	1	2	3	4	5	6	7	8	9	10
S_i	39	40	47	36	41	48	54	54	27	30	36	50

Table 1 Time-0 bp CDS spreads of names -1 (the bank), 0 (the counterparty) and of the reference names 1 to n used when $n = 1$ (*left*) and $n = 10$ (*right*).

computed in a DGC model with $n = 1$ by FT scheme of order 1 to 3, for different levels of nonlinearity represented by the value of the unsecured borrowing spread $\bar{\lambda}$. The right graph shows similar results regarding a portfolio of one CDS contract on each name $i = 1, \ldots, 10$. The time-0 clean value of the default leg of the CDS in case $n = 1$, respectively the sum of the ten default legs in case $n = 10$, is 4.52, respectively 40.78 (of course $P_0 = 0$ in both cases by definition of fair contractual spreads). Hence, in relative terms, the TVA numbers visible in Figure 2 are quite high, much greater for instance than in the cases of counterparty risk on interest rate derivatives considered in Crépey, Gerboud, Grbac, and Ngor (2013). This is explained by the wrong-way risk feature of the DGC model, namely, the default intensities of the surviving names and the value of the CDS protection spike at defaults in this model (cf. the left graph in our concluding figure 7). When $\bar{\lambda}$ increases (for $\bar{\lambda} = 0$ that's a case of linear TVA where FT higher order terms equal 0), the second (resp. third) FT term may represent in each case up to 5% to 10% of the first (resp. second) FT term, from which we conclude that the first FT term can be used as a first order linear estimate of the TVA, with a nonlinear correction that can be estimated by the second FT term.

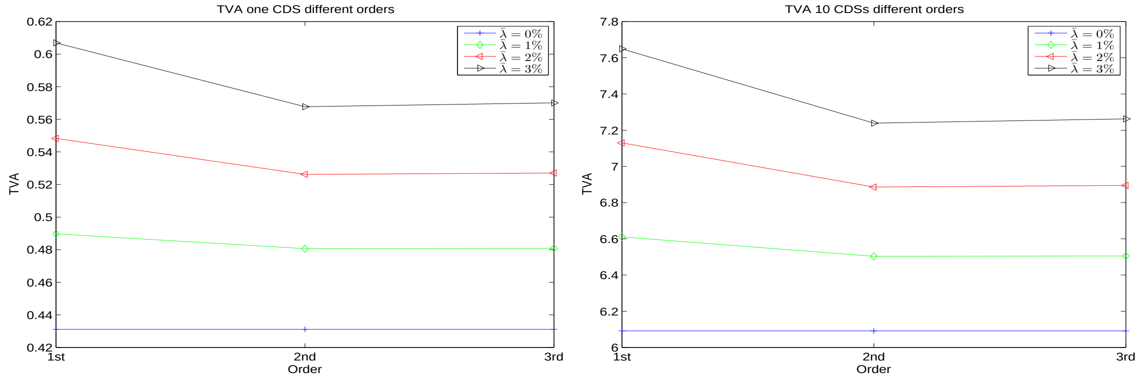

Fig. 2 *Left*: DGC TVA on one CDS computed by FT scheme of order 1 to 3, for different levels of nonlinearity (unsecured borrowing spread $\bar{\lambda}$). *Right*: Similar results regarding the portfolio of CDS contracts on ten names.

In Figure 3, the left graph shows the TVA on one CDS computed by FT scheme of order 3 as a function of the DGC correlation parameter ϱ, with other parameters set as before. The right graph shows the analogous results regarding the portfolio of ten CDS contracts. In both cases, the TVA numbers increase (roughly linearly) with ϱ, including for high values of ϱ, as desirable from the financial interpretation point of view, whereas it has been noted in Brigo and Chourdakis (2008) (see the blue curve in Figure 1 of the ssrn version of the paper) that for high levels of the correlation between names, other models may show some pathological behaviours.

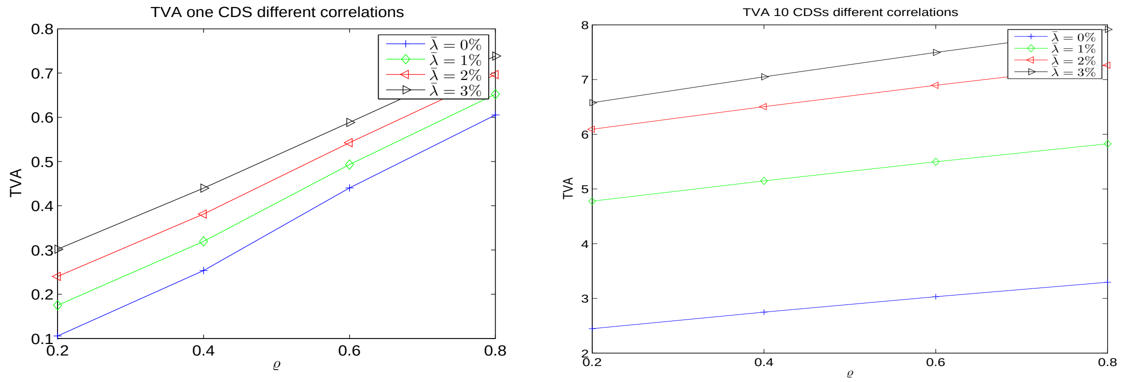

Fig. 3 *Left*: TVA on one CDS computed by FT scheme of order 3 as a function of the DGC correlation parameter ϱ. *Right*: Similar results regarding a portfolio of CDS contracts on ten different names.

In Figure 4, the left graph shows that the errors, in the sense of the % relative standard errors (% rel. SE), of the different orders of the FT scheme don't explode with the dimension (number of credit names that underlie the CDS contracts). The middle graph, produced with $n = 1$, shows that the errors don't explode with the level of nonlinearity represented by the unsecured borrowing spread $\bar{\lambda}$. Consistent with the fact that the successive FT terms are computed by purely forward Monte Carlo schemes, their computation times are essentially linear in the number of names, as visible in the right graph.

Table 2 illustrates the statements made after (8.9). Namely, for $\bar{\lambda} = 0$, the 95% confidence interval of the FT scheme based on (8.6) is included into (in particular, fully consistent with) the 95% confidence interval of the Monte Carlo based on the formula (8.9), formula which is unbiased for $\bar{\lambda} = 0$. But, as $\bar{\lambda}$ increases (i.e. from bottom to top in the table), the approximation (8.9) reveals a significant upward bias increasing with $\bar{\lambda}$, for (8.9) uses $(P_s - \mathcal{C}_s)^+$ instead of what should be $(P_s - \mathcal{C}_s - \Theta_s)^+$ under the time integral, with $\Theta \geq 0$ here. On top of this bias, one can see in Table 2 that the confidence intervals of the Monte Carlo estimates based on (8.9) are substantially broader than the FT ones. In addition,

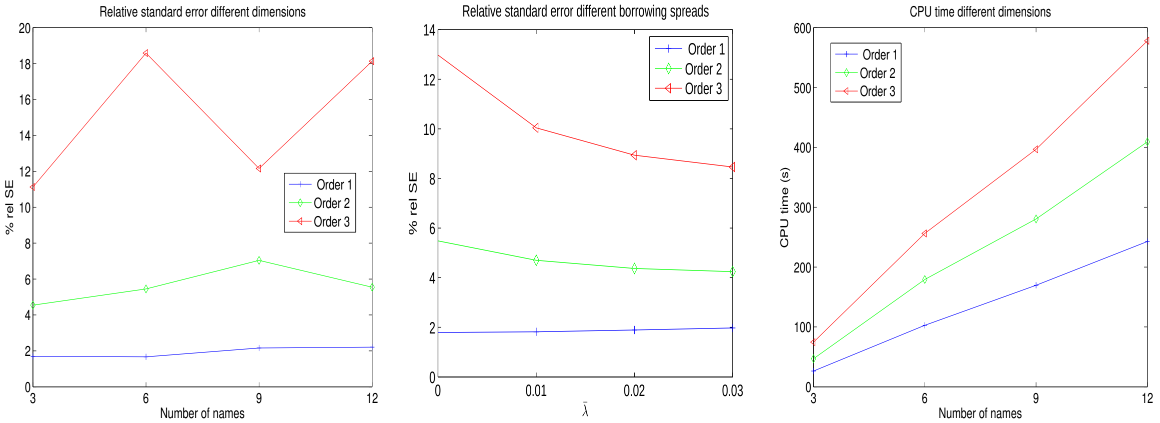

Fig. 4 *Left*: The % relative standard errors of the different orders of the expansions don't explode with the number of names ($\bar{\lambda} = 100$ bp). *Middle*: The % relative standard errors of the different orders of the expansions don't explode with the level of nonlinearity represented by the unsecured borrowing spread $\bar{\lambda}$ ($n = 1$). *Right*: Since FT terms are computed by purely forward Monte Carlo schemes, their computation times are linear in the number of names ($\bar{\lambda} = 100$ bp).

to evaluate the time integral in (8.9) with the required accuracy, we used time discretisation with 500 time points (i.e. a time mesh of $T/500 \approx$ one week for $T = 10y$ here), so that the computation times with (8.9) are much larger than the FT ones, typically a few minutes for FT versus a few hours with (8.9) (all of course proportional to the number of CDS contracts that are used). Note that regarding a PHL scheme that one could possibly devise for solving the full TVA BSDE (2.8), on top of the fine-tuning issue discussed after (8.9), such a scheme would use randomization to evaluate the time integral in (8.9), resulting in an even greater variance than for (8.9) in Table 2 (for computation times similar to the FT scheme). Summarizing, in the DGC model, TVA numbers based on the full BSDE take significantly more time and/or are significantly less accurate than TVA numbers based on the reduced BSDE.

λ (bps)	FT3	95% CI	(8.9)	95% CI
300	0.60	$[0.58, 0.63]$	1.17	$[1.11, 1.22]$
200	0.54	$[0.52, 0.56]$	0.91	$[0.85, 0.96]$
100	0.48	$[0.47, 0.50]$	0.65	$[0.59, 0.70]$
0	0.43	$[0.41, 0.44]$	0.40	$[0.34, 0.45]$

Table 2 TVA computations on one CDS ($\varrho = 0.6$). *Columns 2 and 3:* by the FT scheme based on (8.6)). *Columns 4 and 5:* by Monte Carlo based on the formula (8.9).

8.3 Numerical Results in the DMO Model

We consider a DMO model with constant shock intensities and $n = 120$ credit names (unless stated otherwise). Note that the dependence between names is all in the common shocks in this model. The stochasticity of the intensities is not crucial for the gap risk feature that we want to investigate here. Using deterministic (constant in this case) intensities allows speeding up the simulations. We use individual shock intensities $\gamma^{\{i\}} = 10^{-4} \times (100 + i)$, which increases from 101 bp to 220 bp as i increases from 1 to 120. We use four nested groups of joint defaults, respectively consisting of the riskiest 3%, 9%, 21% and 100% (i.e. all) names, with respective shock intensities of 20, 10, 6.67 and 5 bp. The counterparty (resp. the bank) is taken as the eleventh (resp. tenth) less riskiest name (name with median risk) in the portfolio. In this model, we consider CDO tranches with upfront payment (dividend process $D^\star$ as of (5.12)), for a maturity $T = 2$ years and attachment (resp. detachment) points 0%, 3% and 14% (resp. 3%, 14% and 100%). Figure 5 shows the corresponding TVA computed by FT scheme of order 1 to 3 for different levels of nonlinearity (unsecured borrowing spread $\bar{\lambda}$). The respective values of P_0 (upfront payment) for the equity, mezzanine and senior tranche are 229.65, 5.68 and 2.99. Compared with these, the TVA numbers of Figure 5 are very high, especially for the higher tranches, considerably greater again (cf. Figure 2 and related comments) than the TVA numbers computed on interest rate derivatives in Crépey et al. (2013). This is explained by the gap risk feature of the DMO model, namely, the joint default dividend $\Delta_\tau \neq 0$. By comparison, in the DGC model we have $\Delta_\tau = 0$ but the default intensities of surviving names and the cost of credit protection spike at defaults (whereas default intensities of surviving names are not affected by defaults in the DMO model). It is in this sense that we view the DGC and the DMO model as respective wrong-way and gap risk setups (see Figure 7).

The second (resp. third) FT term never exceeds in each case, depending on $\bar{\lambda}$ increasing from 0 to 300 bp (for $\bar{\lambda} = 0$ that's a case of linear TVA with higher order FT terms all equal to 0), more than 5% of the first (resp. second) FT term in Figure 5, from which we conclude that the first FT term can be used as a first order linear estimate of the TVA, with a nonlinear correction that can be estimated by the second FT term. Figure 6 is the analog of the DGC CDS Figure 4, but for the DMO CDO tranches of Figure 5, with similar conclusions. Table 3 compared with Figures 5 and 6 shows that

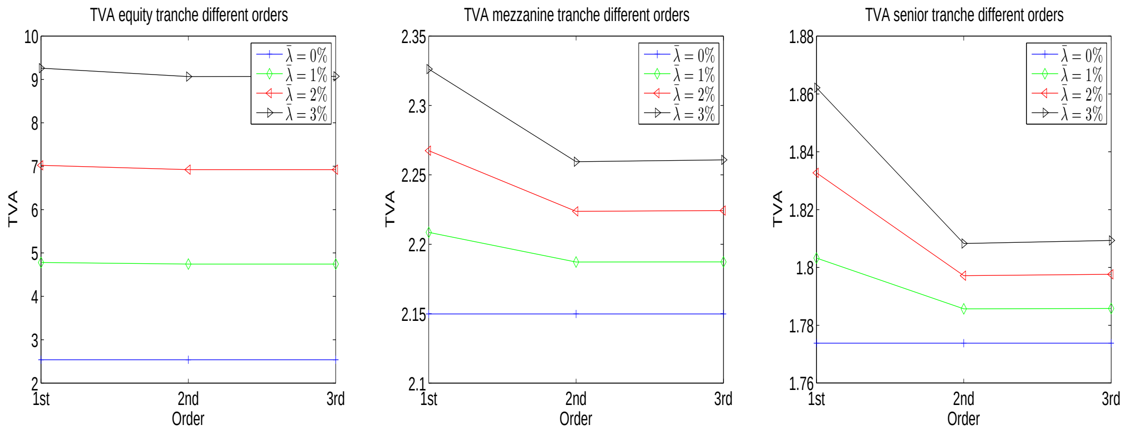

Fig. 5 TVA on CDO tranches with 120 underlying names computed by FT scheme of order 1 to 3, for different levels of nonlinearity (unsecured borrowing spread $\bar{\lambda}$). *Left*: Equity tranche. *Middle*: Mezzanine tranche. *Right*: Senior Tranche.

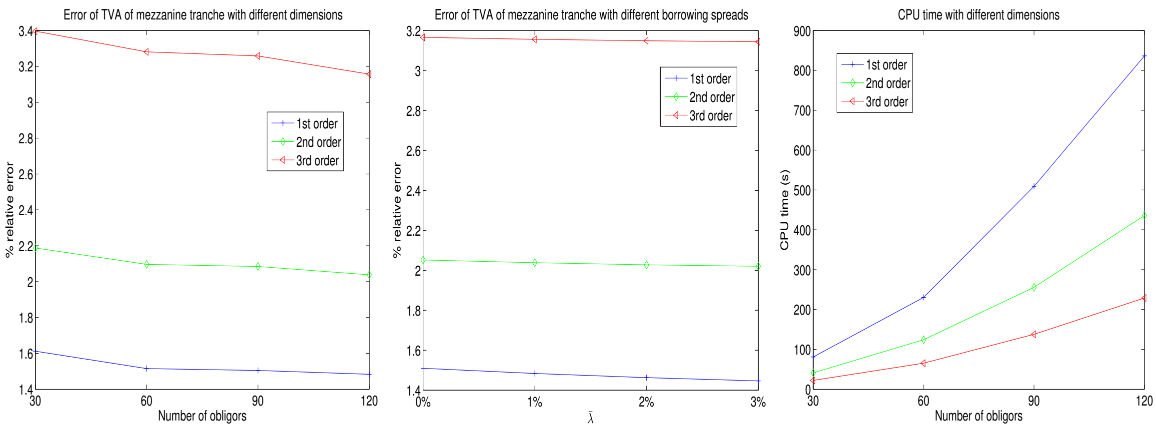

Fig. 6 Analog of Figure 4 for the CDO tranche of Figure 5 in the DMO model.

on top of being biased (depending on $\bar{\lambda}$, equal to 100 bp in Table 3), a Monte Carlo estimate based on the linear approximation formula (8.9) has a large variance, especially for higher tranches. In fact, for higher tranches, nonzero payoffs become quite rare events, so that exploiting the knowledge of the explicit formulas for the intensities in an FT scheme greatly improves the variance by comparison with a crude simulation based on (8.9). In addition, the simulations for (8.9) take considerably more time,

Tranche	TVA	Rel. SE	95% CI	Tranche	TVA	Rel. SE	95% CI
Eq.	5.00	7.43%	[4.63 , 5.37]	Eq.	4.94	2.40%	[4.82 , 5.06]
Mezz.	2.05	63.25%	[0.75 , 3.34]	Mezz.	2.14	19.60%	[1.72 , 2.55]
Sen.	1.67	64.59 %	[0.59 , 2.75]	Sen.	1.74	20.02%	[1.39 , 2.09]

Table 3 Linearized DMO TVA on CDO tranches computed by a Monte Carlo based on the formula (8.9) ($\bar{\lambda} = 100$ bp). *Left*: $m = 10^4$. *Right*: $m = 10^5$.

due to the discretisation that is used for valuing the time integral (or, if the integral is randomized as in the FT scheme, then this increases the variance further).

Table 4 compares the performance of the FT scheme based on the reduced BSDE and of Monte Carlo simulations based on the formula (8.9) to compute the TVA in the continuous variation-margining case where $V_\tau = P_{\tau-}$ (cf. the remark 4.1). As already repeatedly found above, the FT scheme has significantly less variance, crucially so for higher tranches. The FT scheme also takes considerably less computation time, due to the need of computing the time integral in the case of (8.9) (unless this integral is randomized but this increases the variance).

Tranche	TVA	Rel. SE	95% CI	Tranche	TVA	Rel. err.	95% CI
Eq.	0.99	5.02%	[0.96 , 0.99]	Eq.	1.02	17.02%	[0.84 , 1.19]
Mezz.	2.12	4.94%	[2.09 , 2.15]	Mezz.	1.95	66.51%	[0.65 , 3.24]
Sen.	1.76	4.94%	[1.74 , 1.79]	Sen.	1.62	66.64%	[0.54 , 2.70]

Table 4 TVA computations in the continuous variation-margining case $V_\tau = P_{\tau-}$ ($\bar{\lambda} = 100$ bp, $\delta = 0$, $m = 10^4$). *Left*: FT scheme based on (8.3). *Right* : Monte Carlo based on the formula (8.9).

Last, we show some results computed in the DMO setup with positive cure period δ or initial margin I, in the continuous variation-margining case where $V_\tau = P_{\tau-}$ (cf. Table 4). Even in this continuous variation-margining case, we have

$$Q_{\tau^\delta} - C_\tau = (P_{\tau^\delta} - P_{\tau-}) + \Delta_{\tau^\delta} - I_\tau, \tag{8.11}$$

where the wrong-way and gap terms $(P_{\tau^\delta} - P_{\tau-})$ and $\Delta_{\tau^\delta} = \beta_{\tau^\delta}^{-1} \int_{[\tau,\tau^\delta]} \beta_s dD_s$, which includes the "joint default dividend" $D_\tau - D_{\tau-} = \Delta_\tau$, can be quite substantial. Accordingly, observe from the left panel in Table 4 that the TVA numbers are still important relatively to the corresponding values of P_0, even for $\delta = 0$, especially for higher tranches. This motivates the need for the initial margins I. First, still for $I = 0$, the left panel in Table 5 shows the impact of the cure period δ, using Lemma 8.2 to compute the $\widetilde{\Theta}_0^{(i)}$ by $(\mathbb{G}, \mathbb{Q})$ simulation based on the formula (8.7). Regarding long default protection positions of the bank corresponding to payoffs such as $D^\star$ in (5.12), most of the CVA is due to the joint default dividend Δ_τ in the common-shock model, especially for higher tranches (cf. the right graph in Figure 7). This gap risk is already there for $\delta = 0$, instantaneously realized in the joint default dividend $\Delta_\tau = D_\tau - D_{\tau-}$, rather than developing progressively through $\delta > 0$. Accordingly, the left panel in Table 5 shows that the impact of δ is very limited, even if a bit less for the equity tranche. In the DGC model, we would have a similar effect through a large term $(P_{\tau^\delta} - P_{\tau-})$ in (8.11), already large for $\delta = 0$ as revealed by the comparison between the left graph in Figure 3 and the right graph in Figure 7. However, this limited impact of $\delta > 0$ (or instantaneous impact of τ, already present for $\delta = 0$) is likely to be rather restricted to this particular situation of counterparty risk on credit derivatives that we consider here. For counterparty risk on other kinds of derivatives, the methodology of the first part of this paper (Sect. 2 through 5) is equally relevant and the relative impact of δ would typically be much bigger. Next, the right panel of Table 5, computed with $\delta = 0$ based on (8.3), shows that the amount of initial margins, assumed a constant proportional to Θ_0 for simplicity here, that is required to balance the DMO gap risk term (joint default dividend $\Delta_\tau = D_\tau - D_{\tau-}$) is huge. This reflects the extreme tail event feature of CVA on long protection (especially higher) CDO tranches. Namely, most of the CVA comes from the few joint default scenarios giving rise to the joint default dividends $\Delta_\tau = D_\tau - D_{\tau-}$. To compete with these, initial margins must be of the same level of magnitude, i.e. very large, and this at every point in time of every possible scenario, as DMO default times are totally unpredictable. Such levels of initial margins would represent a huge funding charge for the counterparty. We only consider the perspective of the bank here, but a huge funding charge for the counterparty means that the bank could hardly claim such levels of initial margins.

Tranche/ δ	0	2 weeks	1 year	Tranche/I	Θ_0	$10\Theta_0$	$10^2\Theta_0$	$10^3\Theta_0$	$10^4\Theta_0$
Eq.	0.99	0.99	1.49	Eq.	0.97	0.85	0.16	0.00	0.00
Mezz.	2.12	2.12	2.14	Mezz.	2.12	2.11	1.99	0.87	0.00
Sen.	1.76	1.76	1.76	Sen.	1.77	1.76	1.66	0.73	0.00

Table 5 Impact of the cure period $\delta \geq 0$ *(left panel with $I = 0$)* and of the initial margin I posted by the counterparty *(right panel with $\delta = 0$)* on Θ_0, in the continuous variation-margining case where $V_\tau = P_{\tau-}$.

8.4 Conclusions

To put a final point on the respective wrong-way and gap risk features of the DGC and DMO models, our concluding figure 7 shows the analogs of the left graph in Figure 3 and of the middle graph in Figure 5 using flawed simulations where we replace $\widetilde{P}^e_t + \widetilde{\Delta}^e_t$ by P_{t-} in all the coefficients $\widetilde{f}$ (cf. (5.10), (6.13) and (7.9)), thus artificially removing the respective wrong-way and gap risk from the DGC and DMO models. We can see from the figure that the corresponding fake TVA numbers are up to five (resp. ten) times smaller than the "true" TVA levels that can be seen in Figure 3 (resp. 5).

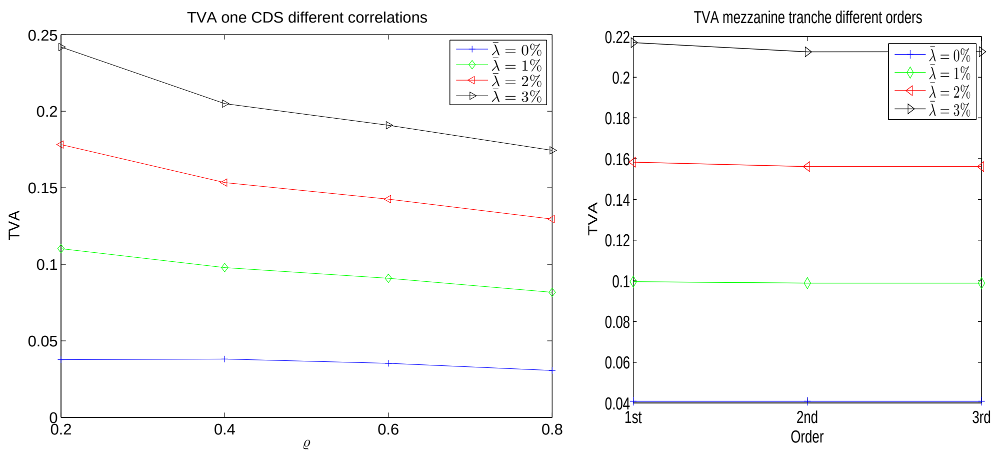

Fig. 7 *Left*: Analog of the left graph of Figure 3 in a fake DGC model without wrong-way-risk. *Right*: Analog of the middle graph of Figure 5 in a fake DMO model without gap risk.

From a broader numerical perspective, independent of the particular models that are used in the second part of this paper (Sect. 6 through 8), let's recap the advantages of the reduced TVA BSDE (3.5) with respect to the full TVA BSDE (2.8) from a numerical point of view. First, in case $\Lambda > 0$, no direct simulation approach for the full TVA BSDE (2.8) seems possible. Second, when $\Lambda = 0$ and a direct simulation approach for the full BSDE (2.8) could be possible, in the case of high dimensional credit portfolio applications where only purely forward simulation schemes are feasible, it's only a first order linear approximation (8.9) that can be estimated directly based on (2.8) (in our setup a tentative PHL scheme would involve the fine-tuning of a polynomial approximation hard to automate in practice). By contrast, successive orders of approximation and a nonlinear correction can be computed for (3.5) based on the FT expansion (8.2) and its particle implementation (8.3) or (8.6). The high tractability of the FT schemes (8.3) or (8.6) is due to the null terminal condition $\widetilde{\Theta}_T = 0$ in (2.8), implying that $\widetilde{\Theta}^{(0)}_s = 0$ in (8.2), (8.3) or (8.6), which would not be the case in a tentative adaptation of the FT scheme to the full BSDE (2.8). Third, even in cases where one can neglect the nonlinearity in (2.8) and solve it by standard Monte Carlo, using the reduced BSDE (3.5) significantly improves the variance of the simulations.

The method of this paper is applicable in any marked default times intensity setup satisfying a suitable integrability condition. The integrability condition expresses that no mass is lost in some tentative measure change. The changed probability measure is not needed algorithmically. All one needs in practice is an explicit expression for the intensities.

A Derivation of the Exact TVA Equation (2.7)

As introduced in Sect. 2, $\mathcal{M}$ is the vector $(\mathbb{G}, \mathbb{Q})$ local martingale of the gain processes related to the trading of unit positions in the hedging assets, assumed securely funded. We assume that the bank, having obtained the contract from the counterparty at time 0 in exchange of an upfront payment of a certain premium Π_0, sets up a hedge $(-\zeta)$, which is a left-continuous row-vector process of the same dimension as $\mathcal{M}$. The "short" negative sign notation in $(-\zeta)$ is only for consistency with the idea, just to fix the mindset, that the contract is "bought" by the bank at time 0. Let $\mathcal{W}^{\Pi_0,\zeta} = \mathcal{W}$ denote the value of the collateralization, hedging and funding portfolio, supposed held by the bank itself before $\bar{\tau}$ and, if $\tau < T$, taken over by a risk-free liquidator on $[\bar{\tau}, \bar{\tau}^\delta]$. We write $J'_t = \mathbb{1}_{\{t<\tau'\}}$, where we recall that $\tau' = \tau_b \wedge \tau_c^\delta$.

Lemma A.1 *Ignoring the terminal cashflow $\mathfrak{R}$ at $\bar{\tau}^\delta$ if $\tau < T$, which will be added separately in Lemma A.2, we have $\mathcal{W}_0 = -\Pi_0$ and, for $0 < t \le \bar{\tau}^\delta$,*

$$d\mathcal{W}_t = r_t\mathcal{W}_t dt + J_t dD_t - J'_t g_t(-\mathcal{W}_t)dt - (-\mathcal{W}_{\tau_b-} - \mathcal{C}_{\tau_b-})^+ \Lambda \mathbb{1}_{\{\tau'=\tau_b<T\}} dJ'_t - \zeta_t d\mathcal{M}_t. \tag{A.1}$$

Proof. Collecting all terms in the collateralization, hedging and funding scheme described in Sect. 2, we obtain $\mathcal{W}_0 = -\Pi_0$ and, for $0 < t \le \bar{\tau}^\delta$:

$$\begin{aligned}
d\mathcal{W}_t = &\underbrace{J_t dD_t}_{\text{bank gets dividends}} - \underbrace{J'_{t-}\zeta_t d\mathcal{M}_t}_{\text{bank pays on its hedge}} \\
+ &\underbrace{J'_t(r_t\mathcal{W}_t - g_t(-\mathcal{W}_t))dt}_{\text{funding benefits / costs to bank}} \\
- &\underbrace{(-\mathcal{W}_{\tau_b-} - \mathcal{C}_{\tau_b-})^+ \Lambda \mathbb{1}_{\{\tau'=\tau_b<T\}} dJ'_t}_{\text{windfall funding benefit to bank at own default}} \\
- &\underbrace{(1 - J'_{t-})\zeta_t d\mathcal{M}_t}_{\text{liquidator pays on the hedge of the bank during the cure period}} \\
+ &\underbrace{(1 - J'_t)r_t\mathcal{W}_t dt,}_{\text{risk-free funding benefits/costs of the liquidator during the cure period}}
\end{aligned}$$

which yields (A.1). ∎

In particular, our assumption of a securely funded hedge is reflected by the fact that the hedge ζ doesn't enter the g and $(-\mathcal{W}_{\tau_b-} - \mathcal{C}_{\tau_b-})^+$ terms in (A.1). This can be compared with Crépey et al. (2014, Example 4.4.3 page 97)[8], where a more general funding policy for the hedge is considered

[8] Or the equation (3.4) in the journal version Crépey (2015, Part I).

A.1 Price

Definition A.1 A price of the contract for the bank is a $(\mathbb{G},\mathbb{Q})$ semimartingale Π that satisfies the following price BSDE on $[0,\bar{\tau}^{\delta}]$:

$$\begin{aligned}&\Pi_{\bar{\tau}^{\delta}} = \mathbb{1}_{\{\tau<T\}}\mathfrak{R},\\ &d\nu_t := d\Pi_t - r_t\Pi_t dt + (\Pi_{\tau_b-} - \mathcal{C}_{\tau_b-})^{+}\Lambda\mathbb{1}_{\{\tau'=\tau_b<T\}}dJ'_t + J_t dD_t - J'_t g_t(\Pi_t)dt \\ &\text{defines a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } [0,\bar{\tau}^{\delta}].\end{aligned} \tag{A.2}$$

The justification for this definition is provided by the following result.

Lemma A.2 *Assume $g_t(\pi)$ given by (4.5). If a price Π can be found with $d\nu_t = \zeta_t d\mathcal{M}_t$ for some hedge ζ, then (Π_0,ζ) yields an exact replication price and hedge for the bank, i.e. $\mathcal{W} = \mathcal{W}^{\Pi_0,\zeta} = -\Pi$ on $[0,\bar{\tau}^{\delta}]$. In particular,*

$$\mathcal{W}_{\bar{\tau}^{\delta}} = -\Pi_{\bar{\tau}^{\delta}} = -\mathbb{1}_{\{\tau<T\}}\mathfrak{R},$$

so that after the terminal cash flow $\mathbb{1}_{\{\tau<T\}}\mathfrak{R}$ at $\bar{\tau}^{\delta}$, the bank's position is closed break-even.

Proof. If $d\nu_t = \zeta_t d\mathcal{M}_t$, then $\mathcal{W} = -\Pi$ on $[0,\bar{\tau}^{\delta}]$, which shows the result. In fact, by construction, $Z = \beta\Pi + \beta\mathcal{W}$ satisfies $Z_0 = 1$ and $dZ_t = \alpha_t Z_t dt$ on $[0,\bar{\tau})$, where $\alpha_t := \mathbb{1}_{\{\Pi_t\neq\mathcal{W}_t\}}\frac{g_t(\Pi_t)+g_t(-\mathcal{W}_t)}{\Pi_t+\mathcal{W}_t}$ is Lebesgue integrable over $[0,T]$ (for $g_t(\pi)$ given by (4.5)). Hence,

$$d\big(e^{-\int_0^t\alpha_s ds}Z_t\big) = e^{-\int_0^t\alpha_s ds}\big(dZ_t - \alpha_t Z_t dt\big) = 0,$$

i.e. $e^{-\int_0^t\alpha_s ds}Z_t$ is constant on $[0,\bar{\tau})$, equal to 0 in view of the initial condition for Z, i.e. $\mathcal{W} = -\Pi$ on $[0,\bar{\tau})$. This is followed by a jump of the two processes by the same amount

$$(-\mathcal{W}_{\tau_b-} - \mathcal{C}_{\tau_b-})^{+}\Lambda = (\Pi_{\tau_b-} - \mathcal{C}_{\tau_b-})^{+}\Lambda$$

at $\bar{\tau}'$ (if $=\tau_b<T$), after which $\mathcal{W}$ and $(-\Pi)$ coincide again on $[\bar{\tau}',\bar{\tau}^{\delta}]$ by the same argument as above. Hence, $\mathcal{W} = -\Pi$ on $[0,\bar{\tau}^{\delta}]$. ∎

More broadly, if a price can be found with $d\nu_t = \zeta_t d\mathcal{M}_t + d\varepsilon_t$ for some hedge ζ and a "small" cost martingale ε, then the hedging error $\rho = \mathcal{W}+\Pi$, which starts from 0 at time 0, remains "small" all the way through. In particular,

$$\mathcal{W}_{\bar{\tau}^{\delta}} \approx -\Pi_{\bar{\tau}^{\delta}} = -\mathbb{1}_{\{\tau<T\}}\mathfrak{R},$$

so that after the close-out cash flow $\mathbb{1}_{\{\tau<T\}}\mathfrak{R}$ at $\bar{\tau}^{\delta}$, the bank's position is closed with a "small" hedging error.

A.2 TVA

In this section we define and derive a representation of the bilateral counterparty risk and funding valuation adjustment (TVA).

Definition A.2 *Given a price Π, the corresponding TVA is the process defined on $[0,\bar{\tau}^{\delta}]$ as $\Theta = Q - \Pi$.*

Lemma A.3 *Let there be given $\mathbb{G}$ semimartingales Π and Θ such that $\Theta = Q - \Pi$ on $[0,\bar{\tau}^{\delta}]$. The process Π is a price of the contract for the bank if and only if the process Θ satisfies the following $(\mathbb{G},\mathbb{Q})$ TVA BSDE on $[0,\bar{\tau}^{\delta}]$:*

$$\begin{aligned}&\Theta_{\bar{\tau}^{\delta}} = \mathbb{1}_{\{\tau<T\}}\xi,\\ &d\mu_t = d\Theta_t - r_t\Theta_t dt + J'_t g_t(Q_t-\Theta_t)dt + (Q_{\tau_b-} - \Theta_{\tau_b-} - \mathcal{C}_{\tau_b-})^{+}\Lambda\mathbb{1}_{\{\tau'=\tau_b<T\}}dJ'_t \\ &\textit{defines a } (\mathbb{G},\mathbb{Q}) \textit{ local martingale on } [0,\bar{\tau}^{\delta}],\end{aligned} \tag{A.3}$$

which is equivalent to (2.7) on $[0,\bar{\tau}']$ and $\Theta = \mathbb{1}_{\{\tau<T\}}\bar{\xi}$ on $(\bar{\tau}',\bar{\tau}^{\delta}]$. This equivalence is modulo the value of Θ at $\tau_b=\tau'$ in case $\tau_b=\tau'<T$, with a windfall funding benefit at default $(Q_{\tau_b-}-\mathcal{C}_{\tau_b-}-\Theta_{\tau_b-})^{+}\Lambda\mathbb{1}_{\{\tau=\tau_b<T\}}$ considered as a dividend at the intermediate time τ_b in (A.3) and as part of the valuation adjustment at the terminal time $\tau_b=\tau'$ in (2.7), but this difference is immaterial in practice for the bank.

Proof. Assuming Θ defined in terms of a price Π as $(Q-\Pi)$ on $[0,\bar\tau^\delta]$, the terminal condition for Θ in (A.3) results from the terminal condition for Π in (A.2) and the left hand side in (2.6). Moreover, for $t\in[0,\bar\tau^\delta]$,

$$-\beta_t\Theta_t=-\beta_tQ_t+\beta_t\Pi_t=-\beta_tP_t-\int_0^t\beta_sdD_s+(\beta_t\Pi_t+\int_0^t\beta_sJ_sdD_s).$$

Hence, by the martingale equation in (A.2),

$$\begin{aligned}&-\beta_t\Theta_t-\int_0^t\beta_sJ'_sg_s(Q_s-\Theta_s)ds\\&\quad+\int_0^t(Q_{\tau_b-}-\Theta_{\tau_b-}-\mathcal{C}_{\tau_b-})^+\Lambda\mathbb{1}_{\{\tau'=\tau_b<T\}}dJ'_s=-\Big(\beta_tP_t+\int_0^t\beta_sdD_s\Big)+\int_0^t\beta_sd\nu_s,\end{aligned}$$

a $(\mathbb{G},\mathbb{Q})$ local martingale (cf. (2.1)). This establishes the martingale condition in (A.3). Hence, (A.2) implies (A.3). The converse implication is proven similarly. Next, (A.3) is obviously equivalent to $\Theta=\mathbb{1}_{\{\tau<T\}}\bar\xi$ on $[\bar\tau,\bar\tau^\delta]$ and (2.7) on $[0,\bar\tau]$, up to the value of Θ at τ' if $=\tau_b<T$, but this is immaterial for the bank, which only risk manages Θ before τ_b. ∎

B Proof of Lemma 6.1

B.1 Explicit Formulas in the DGC Model

Let $f^2(t)=\int_t^{+\infty}\varsigma^2(v)dv$, assumed positive for all t for simplicity. Denoting by I and J generic subsets of N, meant to represent sets of defaulted and alive obligors in the financial interpretation, and for $j\in N$, we write:

$$\begin{aligned}&\rho^I=\frac{\varrho}{|I|\varrho+1},\quad(\sigma^I)^2=\frac{(|I|-1)\varrho+1-\varrho^2|I|}{(|I|-1)\varrho+1},\quad\alpha^I=\frac{\varrho}{(|I|-1)\varrho+1},\\&Z_t^{j,I}(u)=\frac{h_j(u)-m_t^j}{f(t)}-\alpha^I\sum_{i\in I}\frac{h_i(\tau_i)-m_t^i}{f(t)},\\&d\zeta_t^{j,I},\ \text{the }(\mathbb{F},\mathbb{Q})\text{ martingale component of }\left(-\frac{1}{f(t)}dm_t^j+\alpha^I\sum_{i\in I}\frac{1}{f(t)}dm_t^i\right),\end{aligned}$$

and for any $\sigma>0$ and $\rho\in[-1,1]$:

$$\Phi_{\rho,\sigma}(\mathbf{z})=\mathbb{Q}(Z_j>z_j,j\in J),\ \ \psi^j_{\rho,\sigma}(\mathbf{z})=-\frac{\partial_{z_j}\Phi_{\rho,\sigma}}{\Phi_{\rho,\sigma}}(\mathbf{z}),$$

where $\mathbf{z}=(z_j)_{j\in J}$ is a real vector and $\mathbf{Z}=(Z_j)_{j\in J}$ is a $|J|$-dimensional centered Gaussian vector under $\mathbb{Q}$, with homogenous marginal variances σ^2 and pairwise correlations ρ. The standard normal survival and density functions are respectively denoted by Φ and ϕ. In addition, we define

$$\begin{aligned}\rho_t,\sigma_t,\alpha_t&=\rho^I,\sigma^I,\alpha^I,\ \ Z_t^j=Z_t^{j,I}(t)\ \text{ on }\ \{I=\mathcal{I}_t\},\\\rho_t^\star,\sigma_t^\star,\alpha_t^\star&=\rho^I,\sigma^I,\alpha^I,\ \ Z_t^{\star,j}=Z_t^{j,I}(t),\ d\zeta_t^j=d\zeta_t^{j,I}\ \text{ on }\ \{I=\mathcal{I}_t^\star\},\end{aligned}$$

where $\mathcal{I}_t$ (resp. $\mathcal{I}_t^\star$) represents the set of obligors in N (resp. $N^\star$) that are in default at time t. Let also

$$\begin{aligned}&\mathcal{J}_t=N\setminus\mathcal{I}_t,\ \ \mathcal{J}_t^\star=N^\star\setminus\mathcal{I}_t^\star,\ \ \widetilde{\mathcal{J}}_t=\{-1,0\}\cup\mathcal{J}_t^\star,\\&\mathcal{Z}_t=\big(Z_t^j,\,j\in\mathcal{J}_t\big),\ \ \mathcal{Z}_t^\star=\big(Z_t^{\star,j},\,j\in\mathcal{J}_t^\star\big),\ \ \widetilde{\mathcal{Z}}_t=\big(Z_t^{\star,-1},Z_t^{\star,0};Z_t^{\star,j},\,j\in\mathcal{J}_t^\star\big).\end{aligned}$$

Let $p_t(t_i,i\in N)=\partial_{t_1}\ldots\partial_{t_n}\mathbb{Q}(\tau_i<t_i,\,i\in N\,|\,\mathcal{B}_t)$ denote the conditional Lebesgue density of the τ_i, $i\in N$, and recall ν^c from (6.17) .

Lemma B.1 *For any nonnegative* t_i, $i \in N$, *we have, for* $t \in \mathbb{R}_+$,

$$p_t(t_i,\, i\in N) = \int_{\mathbb{R}} \phi(y) \prod_{i\in N} \phi\left(\frac{h_i(t_i) - m_t^i + f(t)\sqrt{\varrho}y}{f(t)\sqrt{1-\varrho}}\right) \frac{h_i'(t_i)}{f(s)\sqrt{1-\varrho}} dy, \tag{B.1}$$

$$S_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t) = \frac{\Phi_{\rho_t^\star,\sigma_t^\star}\big(\widetilde{\mathcal{Z}}_t\big)}{\Phi_{\rho_t^\star,\sigma_t^\star}\big(\mathcal{Z}_t^\star\big)} > 0\,,\ \ d\nu_t^c = \sum_{j\in\widetilde{\mathcal{J}}_{t-}} \psi^j_{\rho_t^\star,\sigma_t^\star}\big(\widetilde{\mathcal{Z}}_t\big)d\zeta_t^j - \sum_{j\in\mathcal{J}_{t-}^\star} \psi^j_{\rho_t^\star,\sigma_t^\star}\big(\mathcal{Z}_t^\star\big)d\zeta_t^j. \tag{B.2}$$

For any $j \in N$, *respectively* $N^\star$, *we have, for* $t \in \mathbb{R}_+$,

$$\gamma_t^j = \mathbb{1}_{\{\tau_j \ge t\}}\big(\frac{h_j'}{f}\big)(t)\psi^j_{\rho_t,\sigma_t}\big(\mathcal{Z}_t\big) = \gamma_j(t, \mathbf{m}_t, \mathbf{k}_t), \tag{B.3}$$

respectively

$$\overline{\gamma}_t^j = \mathbb{1}_{\{\tau_j \ge t\}}\big(\frac{h_j'}{f}\big)(t)\psi^j_{\rho_t^\star,\sigma_t^\star}\big(\mathcal{Z}_t^\star\big) = \overline{\gamma}_j(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_t). \tag{B.4}$$

Proof. The expression for the conditional density p of the τ_i given $\mathcal{B}_t$ is obtained by differentiation of their conditional survival function, given in Crépey et al. (2014, page 172)[9]. The expression for the Azéma supermartingale S of τ, which implies the one for $d\nu_t^c$, results from the following "multiname key lemma formula" (cf. Crépey et al. (2014, Lemma 13.7.6 page 333)[10]), which can be established in any density model of default times thanks to the optional splitting formula (5.6):

$$S_t = \frac{\mathbb{Q}\big(\tau > t; \tau_j > t,\ j\in\mathcal{J}_t^\star \,|\, \mathcal{B}_t \vee \bigvee_{i\in\mathcal{I}_t^\star}\sigma(\tau_i)\big)}{\mathbb{Q}\big(\tau_j > t,\ j\in\mathcal{J}_t^\star \,|\, \mathcal{B}_t \vee \bigvee_{i\in\mathcal{I}_t^\star}\sigma(\tau_i)\big)}.$$

Finally, (B.3) and (B.4) result from the following Laplace formulas that can also be rigorously established using (5.6):

$$\gamma_t^j = -\mathbb{1}_{\{\tau_j\ge t\}} \frac{\partial_u \Phi_{\rho_t,\sigma_t}\Big(Z_t^{j,I}(u); Z_t^l,\, l\in\mathcal{J}_t\setminus\{j\}\Big)\,|_{u=t, I=\mathcal{I}_t}}{\Phi_{\rho_t,\sigma_t}\Big(Z_t^j; Z_t^l,\, l\in\mathcal{J}_t\setminus\{j\}\Big)}$$

$$\overline{\gamma}_t^j = -\mathbb{1}_{\{\tau_j\ge t\}} \frac{\partial_u \Phi_{\rho_t^\star,\sigma_t^\star}\Big(Z_t^{j,I}(u); Z_t^{\star,l},\, l\in\mathcal{J}_t^\star\setminus\{j\}\Big)\,|_{u=t, I=\mathcal{I}_t^\star}}{\Phi_{\rho_t^\star,\sigma_t^\star}\Big(Z_t^{\star,j}; Z_t^{\star,l},\, l\in\mathcal{J}_t^\star\setminus\{j\}\Big)}. \ \blacksquare$$

B.2 Gaussian Estimates

Lemma B.2 *Given a positive decreasing continuously differentiable function* Γ *on* $\mathbb{R}_+$ *such that*

$$\int_{\mathbb{R}_+} t^d \Gamma(t)dt < \infty \quad \text{and} \quad \lim_{t\uparrow\infty} t^{d-1}\Gamma(t) \to 0,$$

for some integer $d \ge 0$, *let*

$$g(y) = -\frac{\Gamma'(y)}{\Gamma(y)},\ \ G(y) = \int_y^\infty t^d \Gamma(t)dt.$$

Let $\overline{y} \ge 0$ *and* $\alpha, \epsilon > 0$.
(i) *If* $g(y) \ge \alpha y$ *for* $y > \overline{y}$, *then*

$$G(y) \le \big(\frac{1}{\alpha} + \epsilon\big)y^{d-1}\Gamma(y) \quad \text{for} \quad y > \overline{y} \vee \sqrt{|d-1|(\frac{1}{\epsilon\alpha^2} + \frac{1}{\alpha})}.$$

(ii) *If* $g(y) \le \alpha y$ *for* $y > \overline{y}$, *then*

$$G(y) \ge \big(\frac{1}{\alpha} - \epsilon\big)y^{d-1}\Gamma(y) \quad \text{for} \quad y > \overline{y} \vee \sqrt{|d-1|(\frac{1}{\epsilon\alpha^2} - \frac{1}{\alpha})}.$$

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[9] Or Crépey et al. (2013, page 3) in the journal version.

[10] Or Crépey et al. (2013, Lemma 2.5) in the journal version.

Proof. We only prove (i), for (ii) is similar. For every positive continuously differentiable function φ on $(0,+\infty)$,

$$\begin{aligned}(G(y)-\varphi(y)\Gamma(y))' &= -y^d\Gamma(y)-\varphi'(y)\Gamma(y)+\varphi(y)g(y)\Gamma(y)\\ &=(\varphi(y)g(y)-y^d-\varphi'(y))\Gamma(y)\geq(\alpha y\varphi(y)-y^d-\varphi'(y))\Gamma(y)\end{aligned}$$

for $y\geq\overline{y}$. For $\varphi(y)=(\frac{1}{\alpha}+\epsilon)y^{d-1}$,

$$\begin{aligned}\alpha y\varphi(y)-y^d-\varphi'(y)&=(1+\epsilon\alpha)y^d-y^d-(\frac{1}{\alpha}+\epsilon)(d-1)y^{d-2}=\\ \epsilon\alpha y^d-(\frac{1}{\alpha}+\epsilon)(d-1)y^{d-2}&=(\epsilon\alpha y^2-(\frac{1}{\alpha}+\epsilon)(d-1))y^{d-2}.\end{aligned}$$

Therefore, if $y>\overline{y}\vee\sqrt{|d-1|(\frac{1}{\epsilon\alpha^2}+\frac{1}{\alpha})}$, then $(G(y)-\varphi(y)\Gamma(y))'\geq\alpha y\varphi(y)-y^d-\varphi'(y)\geq0$. But $\lim_{y\uparrow\infty}(G(y)-\varphi(y)\Gamma(y))=0$, hence $G(y)-\varphi(y)\Gamma(y)\leq0$. ∎

By a first application of Lemma B.2, to the standard normal density $\Gamma=\phi$, we recover the following classical inequalities on $\psi=\frac{\phi}{\Phi}$: for any constant $c>1$,

$$c^{-1}y\leq\psi(y)\leq cy,\ \ y>y_0, \tag{B.5}$$

for some $y_0>0$ depending on c. The following estimate, where c and y_0 are as here, can be seen as a multivariate extension of the right hand side inequality in (B.5).

Lemma B.3 *There exist constants a and b such that, for every $j\in J$,*

$$0\leq\psi^j_{\rho,\sigma}(\mathbf{z})\leq a+b||\mathbf{z}||_\infty. \tag{B.6}$$

Proof. By conditional independence, we have $\Phi_{\rho,\sigma}(\mathbf{z})=\int_{\mathbb{R}}\Gamma(y)dy$, where $\Gamma(y)=\prod_{l\in J}\Phi\left(\frac{z_l+\sigma\sqrt{\rho}\,y}{\sigma\sqrt{1-\rho}}\right)\phi(y)$. Hence

$$\psi^j_{\rho,\sigma}(\mathbf{z})=\frac{1}{\sigma\sqrt{1-\rho}}\int_{\mathbb{R}}w_{\rho,\sigma}(\mathbf{z},y)\psi\left(\frac{z_j+\sigma\sqrt{\rho}\,y}{\sigma\sqrt{1-\rho}}\right)dy, \tag{B.7}$$

where $w_{\rho,\sigma}(\mathbf{z},y)=\frac{\Gamma(y)}{\Phi_{\rho,\sigma}(\mathbf{z})}$. Straightforward computations yield

$$g(t)=-\frac{\Gamma'(t)}{\Gamma(t)}=\sum_{l\in J}\psi(\frac{z_l+\sigma\sqrt{\rho}t}{\sigma\sqrt{1-\rho}})\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}+t\geq t,$$

whereas for $t>\max_{l\in J}\frac{1}{\sigma\sqrt{\rho}}(\sigma\sqrt{1-\rho}y_0-z_l)$ and $t>\frac{1}{\sigma\sqrt{\rho}}\max_{l\in J}z_l$, we have

$$g(t)\leq\textstyle\sum_{l\in J}c\frac{z_l+\sigma\sqrt{\rho}t}{\sigma\sqrt{1-\rho}}\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}+t\leq\bar{\alpha}t,$$

with $\bar{\alpha}:=\sum_{l\in J}2c\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}+1\geq1$. Applying Lemma B.2(i) with $d=1,\alpha=1$ and $\epsilon=1$, respectively (ii) with $d=0$, $\alpha=\bar{\alpha}$ and $\epsilon=\frac{1}{2\bar{\alpha}}$, yields

$$\int_y^\infty t\Gamma(t)dt\leq2\Gamma(y),\ \ y>0,\quad\text{respectively}\int_y^\infty\Gamma(t)dt\geq\frac{1}{2\bar{\alpha}y}\Gamma(y),\ \ y>\overline{y}\vee\frac{1}{\sqrt{\bar{\alpha}}},$$

where $\overline{y}=\frac{1}{\sigma\sqrt{\rho}}\max_{l\in J}|z_l|+\frac{1}{\sigma\sqrt{\rho}}\sigma\sqrt{1-\rho}y_0$. Thus, setting $y_1=\overline{y}+1=\frac{1}{\sigma\sqrt{\rho}}\max_{l\in J}|z_l|+\frac{1}{\sigma\sqrt{\rho}}\sigma\sqrt{1-\rho}y_0+1$,

$$\begin{aligned}\int_0^\infty t\Gamma(t)dt&=\int_0^{y_1}t\Gamma(t)dt+\int_{y_1}^\infty t\Gamma(t)dt\leq y_1\int_0^{y_1}\Gamma(t)dt+2\Gamma(y_1)\\ &\leq y_1\int_0^{y_1}\Gamma(t)dt+4\bar{\alpha}y_1\int_{y_1}^\infty\Gamma(t)dt\leq(1+4\bar{\alpha})\int_0^\infty\Gamma(t)dt,\end{aligned}$$

i.e.

$$\int_0^\infty t w_{\rho,\sigma}(\mathbf{z},t)dt \le (1+4\bar{\alpha})y_1. \tag{B.8}$$

Now, by (B.7) and the right hand side inequality in (B.5),

$$\begin{aligned}
0 &\le \sigma\sqrt{1-\rho}\,\psi^j_{\rho,\sigma}(\mathbf{z}) \\
&\le \int_{\mathbb{R}} \left(\frac{1}{\Phi(y_0)} \mathbb{1}_{\{\frac{z_j+\sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}} \le y_0\}} + c\frac{z_j+\sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}} \mathbb{1}_{\{\frac{z_j+\sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}} > y_0\}} \right) w_{\rho,\sigma}(\mathbf{z},y)dy \\
&= \left(\frac{1}{\Phi(y_0)} + \frac{cz_j}{\sigma\sqrt{1-\rho}} \right) + \frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} \int_{\mathbb{R}} \mathbb{1}_{\{\sigma\sqrt{\rho}y > \sigma\sqrt{1-\rho}y_0 - z_j\}} y w_{\rho,\sigma}(\mathbf{z},y)dy \\
&\le \left(\frac{1}{\Phi(y_0)} + \frac{cz_j}{\sigma\sqrt{1-\rho}} \right) + \frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} \int_0^\infty y w_{\rho,\sigma}(\mathbf{z},y)dy,
\end{aligned} \tag{B.9}$$

so that by substitution of (B.8) into (B.9)

$$0 \le \sigma\sqrt{1-\rho}\,\psi^j_{\rho,\sigma}(\mathbf{z}) \le \left(\frac{1}{\Phi(y_0)} + \frac{cz_j}{\sigma\sqrt{1-\rho}} \right) + \frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}(1+4\bar{\alpha})y_1. \ \blacksquare$$

Corollary B.1 **(i)** *There exists a constant $C > 0$ such that, for $0 \le r \le t$ and $j \in N^\star$,*

$$\langle \nu^c \rangle_t \le C\Big(\sum_{i\in N} \sup_{0<s\le t} |m^i_s|^2 + 1\Big)t, \tag{B.10}$$

$$\gamma^j_r \vee \widetilde{\gamma}^j_r \vee \bar{\gamma}^j_r \le C\Big(\sum_{i\in N} \sup_{0<s\le t} |m^i_s| + 1\Big),\ \ \widetilde{\gamma}^j_r \ln(\widetilde{\gamma}^j_r \vee \bar{\gamma}^j_r) \le C \sum_{i\in N} \sup_{0<s\le t} (|m^i_s|+1)\ln(|m^i_s|+1). \tag{B.11}$$

Proof. Applying Lemma B.3 to the formulas derived in Lemma B.1, we obtain, for constants denoted by the same symbol C,

$$\langle \nu^c \rangle_t \le C\int_0^t \Big(\sum_{I\subseteq N} \sum_{j\in N\setminus I} |Z^{j,I}_s(s)| + 1\Big)^2 ds \le C\Big(\sum_{I\subseteq N} \sum_{j\in N\setminus I} \sup_{0<s\le t} |Z^{j,I}_s(s)| + 1\Big)^2 t,$$

as well as the left hand side inequality in (B.11) (recalling also $\widetilde{\gamma}^j_t = \gamma_j(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t)$), whence the right hand side inequality follows from

$$\begin{aligned}
\widetilde{\gamma}^j_r \ln(\widetilde{\gamma}^j_r \vee \bar{\gamma}^j_r) &\le C\big(\max_{i\in N} \sup_{0<s\le t} |m^i_s| + 1\big) \ln\Big(C\big(\max_{i\in N} \sup_{0<s\le t} |m^i_s| + 1\big)\Big) \\
&= \max_{i\in N} \sup_{0<s\le t} C(|m^i_s|+1)\ln(C|m^i_s|+1). \ \blacksquare
\end{aligned}$$

Lemma B.4 *For any constant $q > 0$ and $i \in N$, $e^{q\sup_{0\le s\le t}(m^i_s)^2}$ is $\mathbb{Q}$ integrable for sufficiently small t.*

Proof. The process $(m^i_t)_{t\ge0}$ is equal in law to a time changed Brownian motion $(B_{\bar{t}})_{t\ge0}$, where B is a a univariate standard Brownian motion and $\bar{t} = \int_0^t \varsigma^2(s)ds$ goes to 0 with t. Thus, it suffices to show the result with m^i replaced by B. Let r_t be the density function of the law of $\sup_{0\le s\le t}|B_s|$ and let $R_t(y) = \int_y^\infty r_t(x)dx, y > 0$, so that

$$\mathbb{E}[e^{q\sup_{0\le s\le t} B^2_s}] = \int_0^\infty e^{qy^2} r_t(y)dy = -[R_t(y)e^{qy^2}]^\infty_0 + 2q\int_0^\infty yR_t(y)e^{qy^2}dy \tag{B.12}$$

and

$$\begin{aligned}
R_t(y) &= \mathbb{Q}[\sup_{0\le s\le t}(B^+_s + B^-_s) > y] \le \mathbb{Q}[\sup_{0\le s\le t} B^+_s > \tfrac{y}{2}] + \mathbb{Q}[\sup_{0\le s\le t} B^-_s > \tfrac{y}{2}] \\
&= 2\mathbb{Q}[\sup_{0\le s\le t} B_s > \tfrac{y}{2}] = 2\mathbb{Q}[|B_t| > \tfrac{y}{2}] = 2\mathbb{Q}[|B_1| > \tfrac{y}{2\sqrt{t}}] = 4\Phi(\tfrac{y}{2\sqrt{t}}),
\end{aligned}$$

where by the left hand side in (B.5)

$$\Phi\Big(\frac{y}{2\sqrt{t}}\Big)\frac{y}{2\sqrt{t}} \le c\phi\Big(\frac{y}{2\sqrt{t}}\Big) = \frac{c}{\sqrt{2\pi}}e^{-\frac{y^2}{8t}},\ \ \frac{y}{2\sqrt{t}} > y_0.$$

Therefore, for $\frac{1}{8t} > q$, both terms are finite in the right hand side of (B.12), which shows the result. ∎

By Corollary B.1, multivariate Hölder inequality and Lemma B.4,

$$\exp\Big(\langle \nu^c \rangle_t + \sum_{i\in N^\star} \int_0^t (\widetilde{\gamma}^i_s \ln(\widetilde{\gamma}^i_s) - \widetilde{\gamma}^i_s \ln(\overline{\gamma}^i_s) - \widetilde{\gamma}^i + \overline{\gamma}^i_s) ds\Big)$$

is $\mathbb{Q}$ integrable for sufficiently small t. Hence Lemma 6.1 follows by an application of Lepingle and Mémin (1978, Theorem IV.3).

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