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Counterparty Risk Modeling: Beyond Immersion

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Abstract Counterparty risk reduced-form models typically hinge on an immersion property of a reference filtration into the full model filtration enlarged by the default times of the counterparties, as well as on a continuity assumption on some of the data at default time. This is too restrictive for cases of strong wrong-way risk, i.e. adverse dependence between the exposure and the credit riskiness of the counterparty. In this paper we generalize the approach by switching from the class of pseudo-stopping times, which is classically used to model the defaults of the counterparties, to the much more flexible class of invariant times. For instance, these can be marked default times, where the role of the mark is to convey some additional information about the defaults, in order to account for various possible wrong-way and gap risk scenarios and features. Additional tools are introduced to analyze the cure period (time interval between the default and the liquidation) and the ensuing gap risk of diverging evolutions of the portfolio and of its collateral. In particular, the liquidation time is predictable (as announced by the default), which modifies the nature of the pricing problem. We illustrate our approach in two dynamic copula models of portfolio credit risk.

Keywords: Counterparty risk, funding, reduced-form credit modeling, BSDE, immersion, invariant times, wrong-way risk, gap risk, collateral, credit derivatives, dynamic copulas.

Mathematics Subject Classification: 91G40, 60H10, 60G07.

1 Introduction

Counterparty risk is the risk of default of a party in an OTC derivative transaction, a topical issue since the global financial crisis. As banks themselves have become risky, counterparty risk must be understood in a bilateral perspective (not only CVA, i.e. credit valuation adjustment but also DVA, i.e. debit valuation adjustment), which raises the companion issue of a proper accounting of funding costs (FVA, i.e. funding valuation adjustment). See Brigo, Morini, and Pallavicini (2013) for a general reference. To mitigate counterparty risk, a margining procedure is set up according to the specifications of a CSA (credit support annex). However, accounting for various frictions and delays, notably the cure period (time interval between default and liquidation), there is gap risk, i.e. risk of a residual gap between the collateral and the debt of the defaulted counterparty. This is why another layer of collateralization, called initial margins (as opposed to the variation margin that accounts for market risk), is now maintained in both centrally cleared transactions and bilateral transactions under a sCSA (standard CSA). Gap risk is magnified in the presence of wrong-way risk, i.e. adverse dependence between the underlying exposure and the credit risk of the counterparty. This is a special case of

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concern regarding counterparty risk on credit derivatives, given the strong dependence (contagion) effects between the credit risks of the two parties and the ones of the underlying credit names.

In a bilateral counterparty risk framework also accounting for the nonlinear funding costs involved, the pricing equations for the corresponding valuation adjustment (TVA equations, where TVA stands for total valuation adjustment) are implicit and nonlinear (see Crépey (2012), Pallavicini, Perini, and Brigo (2012)). Moreover, they are posed over random time intervals determined by the first default time of a party. To deal with such equations, a first reduced-form counterparty risk modeling approach was proposed in Crépey (2012, Part II), under a standard immersion hypothesis between the reference (or market) filtration and the full model filtration progressively enlarged by the default times of the two parties. But this basic immersion setup, with the related continuity assumptions on some of the data at default time, is too restrictive for wrong-way risk applications, like counterparty risk of credit derivatives (an issue that also poses specific dependence modeling and dimensionality challenges). Moreover, additional tools are required to analyze the cure period. In particular, the liquidation time is predictable (as announced by the default), which modifies the nature of the pricing problem.

In the first part of this paper (Sect. 2 through 4), we generalize, resorting to the notion of invariant times in Crépey and Song (2014), the basic reduced-form approach of Crépey (2012, Part II), in view of a proper wrong-way and gap risk modeling in applications. In a second part (Sect. 5 through 7), the general approach is implemented through marked default times in two dynamic copula models of portfolio credit risk. The role of the mark is to convey some information about the default, in order to account for various possible wrong-way and gap risk scenarios and features. The detailed outline of the paper is as follows. Sect. 2 fixes the setup and provides the full TVA BSDE with respect to the full model filtration $\mathbb{G}$. Sect. 3 develops an extended reduced-form approach with τ modeled as an invariant time. Sect. 4 deals with the cure period. In the marked default time specification of Sect. 5, we derive a CVA, DVA, FVA decomposition of the all-inclusive TVA. Sect. 6 and Sect. 7 illustrate our approach in two dynamic copula models of counterparty risk for credit derivatives. Sect. 8 concludes with numerical perspectives.

Any "deterministic function" of real arguments is measurable with respect to the corresponding Borel σ field, denoted in the case of $\mathbb{R}$ by $\mathcal{B}(\mathbb{R})$. We write $\mathcal{P}(\mathbb{F})$, $\mathcal{O}(\mathbb{F})$ and $\mathcal{R}(\mathbb{F})$ for the predictable, optional and progressive σ fields with respect to a filtration $\mathbb{F}$. When a process f_t can be represented in terms of a function of some factor process X, we typically write $f(t, X_t)$ (i.e. the function is denoted by the same letter as the related process, if no confusion arises). Order relationships between random variables (respectively processes) are meant almost surely (respectively in the indistinguishable sense).

2 Counterparty Risk Setup

We consider a netted portfolio of OTC derivatives with maturity T between two defaultable counterparties, generically referred to as the "contract between the bank and its counterparty". After having bought the contract from its counterparty at time 0, the bank sets-up a hedging, collateralization (i.e. margining) and funding portfolio. We call "funder" of the bank a third party (possibly composed in practice of several entities or devices) insuring funding of the bank's strategy. The funder, assumed default-free for simplicity, plays the role of lender/borrower of last resort after exhaustion of the internal sources of funding provided to the bank through its hedge and its collateral. For reasons explained in Sect. 1 of Crépey (2012, Part II), the price of the contract is computed as the difference between a reference price (ignoring counterparty risk and assuming a risk-free funding rate) and a counterparty risk and funding adjustment. Let $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, with $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ satisfying the usual conditions and $\mathbb{Q}$ expectation denoted by $\mathbb{E}$, represent a prevailing pricing stochastic basis, such that all our processes are $\mathbb{G}$ adapted and all random times are $\mathbb{G}$ stopping times. The meaning of a risk-neutral pricing measure in our setup, with different funding rates in particular, will be specified by martingale conditions introduced below in the form of pricing BSDEs, i.e. backward stochastic differential equations. In the first place (cf. Assumption 1.1 in Crépey (2012, Part I)), a pricing measure must be such that the gain processes related to the trading of the hedging assets (assumed done via repo and swap markets) are local martingales. We denote by r_t a progressive OIS rate process (overnight indexed swap rate, the best market proxy for a risk-free rate), by D_t a finite variation cumulative promised dividend process of the contract (contractual cash-flows ignoring counterparty risk) and by P_t the corresponding mark-to-market (risk-neutral conditional expectation of future promised cash flows discounted at the OIS rate r_t). But the two parties are defaultable, with first-default-time modeled as a stopping time τ with intensity γ_t (so that any event $\{\tau = t\}$ for a fixed time t has zero probability and can be ignored in the analysis). This results in an effective time horizon of the problem $\bar{\tau} = \tau \wedge T$ (there are no cash-flows beyond $\bar{\tau}$) and an effective dividend stream $J_t dD_t$ on $[0, \bar{\tau}]$, where $J_t = \mathbb{1}_{\{t<\tau\}}$. The position of the bank is supposed to be closed at τ (if $\tau < T$), with a terminal cash-flow given as a $\mathcal{G}_\tau \otimes \mathcal{B}(\mathbb{R})$ measurable function $\xi(\pi)$ (exposure at default) of the form

$$\xi(\pi) = \xi_c - \Lambda(\pi - \mathfrak{C}_\star)^+, \tag{2.1}$$

for $\mathcal{G}_\tau$ measurable random variables ξ_c, Λ and $\mathfrak{C}_\star$ respectively corresponding to the CVA/DVA exposure (of the bank to the default of its counterparty/its own default), to the fractional loss of the funder (in case of default of the bank) and to the collateral funded by the bank; see Sect. 5 for a concrete specification. By Lemma 2.2 in Crépey and Song (2014), for any $\mathcal{G}_\tau \otimes \mathcal{B}(\mathbb{R})$ measurable function $\zeta(\pi)$ (e.g. $\zeta = \xi$), if ζ is nonnegative, then there exists a $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$ measurable nonnegative function $\widehat{\zeta}_t(\omega, x)$ such that, for any real valued $\mathcal{G}_{\tau-}$ measurable random variable η,

$$\mathbb{1}_{\{\tau<\infty\}}\mathbb{E}[\zeta(\eta)|\mathcal{G}_{\tau-}] = \mathbb{1}_{\{\tau<\infty\}}\widehat{\zeta}_\tau(\eta), \tag{2.2}$$

and the (respectively $\mathcal{G}_{\tau-}$ local) integrability of $\zeta(\eta)$ implies that of $\widehat{\zeta}_\tau(\eta)$. For a general (nonnecessarily ≥ 0) $\zeta(\eta)$, we define $\widehat{\zeta} = \widehat{\zeta^+} - \widehat{\zeta^-}$ whenever well defined as an $\bar{\mathbb{R}}$ valued function, which we assume henceforth regarding $\zeta = \xi$ (see Lemma 5.1 for a concrete specification), so that it also holds for $\mathbb{1}_{\tau<T}\xi$, with $\widehat{\mathbb{1}_{\tau<T}\xi} = \mathbb{1}_{[0,T)}\widehat{\xi}$.

Lemma 2.1 *For every $(\mathbb{G}, \mathbb{Q})$ semimartingale Y such that $\mathbb{1}_{\tau<T}\xi(Y_{\tau-})J$ has locally integrable total variation, the process $\xi(Y_{t-})dJ_t + \gamma_t\widehat{\xi}_t(Y_t)dt$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, \bar{\tau}]$.*

Proof. Without loss of generality, we suppose that $\mathbb{1}_{\tau<T}\xi(Y_{\tau-})J$ has integrable total variation, so that $\mathbb{1}_{\tau<T}\xi(Y_{\tau-})$ is integrable. Then, for every stopping time υ, we have in $\mathbb{R}$:

$$\begin{aligned}\mathbb{E}\int_0^\upsilon \mathbb{1}_{t<T}\xi(Y_{t-})dJ_t &= -\mathbb{E}[\mathbb{1}_{\{\tau\leq\upsilon\}}\mathbb{1}_{\tau<T}\xi(Y_{\tau-})] = -\mathbb{E}\mathbb{E}\left[\mathbb{1}_{\{\tau\leq\upsilon\}}\mathbb{1}_{\tau<T}\xi(Y_{\tau-})\,\middle|\,\mathcal{G}_{\tau-}\right]\\ &= -\mathbb{E}[\mathbb{1}_{\{\tau\leq\upsilon\}}\mathbb{1}_{\tau<T}\widehat{\xi}_\tau(Y_{\tau-})] = \mathbb{E}\int_0^\upsilon \mathbb{1}_{t<T}\widehat{\xi}_t(Y_{t-})dJ_t,\end{aligned}$$

where the $\mathcal{G}_{\tau-}$ measurability of $\mathbb{1}_{\{\tau\leq\upsilon\}}$ was used to pass to the second line. Therefore, Theorem 4.40 in He, Wang, and Yan (1992) shows that the process $\mathbb{1}_{\tau<T}(\xi(Y_{t-}) - \widehat{\xi}_t(Y_{t-}))dJ_t$ is a martingale on $\mathbb{R}_+$, hence $(\xi(Y_{t-}) - \widehat{\xi}_t(Y_{t-}))dJ_t$ is a martingale on $[0, \bar{\tau}]$, as is in turn

$$\xi(Y_{t-})dJ_t + \gamma_t\widehat{\xi}_t(Y_t)dt = (\xi(Y_{t-}) - \widehat{\xi}_t(Y_{t-}))dJ_t + \widehat{\xi}_t(Y_{t-})\,(dJ_t + \gamma_t dt)$$

(since $\widehat{\xi}_t(Y_{t-})\,(dJ_t + \gamma_t dt)$ also is, as a well defined stochastic integral against a local martingale). ∎

Moreover, the bank needs to fund its position (contract, hedge and collateral) before $\bar{\tau}$. We denote by $g_t = g_t(\pi)$ an $\mathcal{R}(\mathbb{G})\otimes\mathcal{B}(\mathbb{R})$ measurable funding coefficient such that $(r_t\pi + g_t(\pi))dt$ represents the bank's funding cost over $(t, t+dt)$, depending on the contract's value represented by π. Concrete specifications for ξ and g_t will be studied in Sect. 5. Writing $f_t(\vartheta) = g_t(P_t - \vartheta) - r_t\vartheta$, $\vartheta \in \mathbb{R}$, a TVA process Θ on $[0, \bar{\tau}]$ is implicitly defined as a solution to the following BSDE over $[0, \bar{\tau}]$:

$$\begin{aligned}&\Theta_{\bar{\tau}} = \mathbb{1}_{\{\tau<T\}}\xi(P_{\tau-} - \Theta_{\tau-}),\\ &d\mu_t := d\Theta_t + f_t(\Theta_t)dt \text{ is a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale on } [0, \bar{\tau}]\end{aligned} \tag{2.3}$$

(which includes the time integrability of $f_t(\Theta_t)$ on $[0, \bar{\tau}]$; similar convention is in force and will not be repeated regarding the various BSDEs introduced below). The reader is referred to Proposition 2.1 of Crépey (2012, Part II) for more details and for the derivation of the TVA BSDE (2.3) as the output of a computation based on a TVA primarily defined as

$$\Theta_t = P_t - \Pi_t + \int_{[\tau,t]} e^{\int_s^t r_u du} dD_s, \tag{2.4}$$

where Π represents the overall price process of the contract (cost of the hedge inclusive of counterparty risk and funding costs) and where the integral includes the jump $\Delta_\tau = D_\tau - D_{\tau-}$ of D at τ. Instead, for simplicity of presentation in this paper, we take (2.3) as a definition. Of course, this approach assumes the existence (at the very least) of a solution to (2.3), a quite nonstandard BSDE. The object of the next section is to develop a reduced-form approach for (2.3), beyond the basic immersion setup of Crépey (2012, Part II).

3 Reduced Form Approach

For $t \in [0, \bar{\tau}]$ and $\vartheta \in \mathbb{R}$, we write

$$\widehat{f}_t(\vartheta) := f_t(\vartheta) + \gamma_t\big(\widehat{\xi}_t(P_t - \vartheta) - \vartheta\big) = g_t(P_t - \vartheta) + \gamma_t\widehat{\xi}_t(P_t - \vartheta) - \widetilde{r}_t\vartheta, \tag{3.1}$$

where $\widetilde{r}_t = r_t + \gamma_t$. By a (left-limited) process Y stopped at $\tau-$, we mean the process $Y^{\tau-} = JY + (1 - J)Y_-$.

Condition (C). There exist:

(C.1) a subfiltration $\mathbb{F}$ of $\mathbb{G}$ satisfying the usual conditions and such that $\mathbb{F}$ semimartingales stopped at τ are $\mathbb{G}$ semimartingales,

(C.2) a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ and such that any $(\mathbb{F}, \mathbb{P})$ local martingale stopped at $\tau-$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$,

(C.3) an $\mathbb{F}$ representative $\widetilde{f}_t(\vartheta)$ of $\widehat{f}_t(\vartheta)$, i.e. an $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ function $\widetilde{f}_t(\vartheta)$ such that $\int_0^{\cdot} \widehat{f}_t(\vartheta)dt = \int_0^{\cdot} \widetilde{f}_t(\vartheta)dt$ on $[0, \bar{\tau}]$.

The condition (C.1) relates to the $(\mathcal{H}')$ hypothesis between $\mathbb{F}$ and $\mathbb{G}$, i.e. $\mathbb{F}$ semimartingales are $\mathbb{G}$ semimartingales (see Bielecki, Jeanblanc, and Rutkowski (2009)). The condition (C.3) is a mild technical condition, which holds in particular under the condition (B) in Crépey and Song (2014) that any $\mathbb{G}$ predictable process Y admits an $\mathbb{F}$ predictable process $\widetilde{Y}$ coinciding with Y until τ; see Crépey and Song (2014), where it is shown that the condition (B) also implies (C.1). The condition (C.2) is the condition (A) of Crépey and Song (2014), where it is characterized in terms of the Azéma supermartingale $S_t = \mathbb{Q}(\tau > t|\mathcal{F}_t)$ of τ. This condition obviously holds if $(\mathbb{F}, \mathbb{P})$ local martingales don't jump at τ (then "stopped at $\tau-$" reduces to "stopped at τ" in (C.2)) and $(\mathbb{F}, \mathbb{P})$ local martingales stopped at τ are $(\mathbb{G}, \mathbb{Q})$ local martingales. In the case where $\mathbb{P} = \mathbb{Q}$, these properties are related to the notions of immersion of $\mathbb{F}$ into $\mathbb{G}$, i.e. $\mathbb{F}$ local martingales are $\mathbb{G}$ local martingales (see Bielecki et al. (2009)) and of an $\mathbb{F}$ pseudo-stopping time τ, i.e. $\mathbb{F}$ local martingales stopped at τ are $\mathbb{G}$ local martingales (see Nikeghbali and Yor (2005)). However, even in this "immersion" case where $\mathbb{P} = \mathbb{Q}$, the condition (C) offers a richer setup than a standard reduced-form intensity model of credit risk, where the full model filtration $\mathbb{G}$ is given as the reference filtration $\mathbb{F}$ progressively enlarged by τ, i.e. in a standard notation: "$\mathbb{G} = \mathbb{F} \vee \mathbb{H}$" (see Bielecki et al. (2009)). Indeed, under (C.1-2-3), the full filtration $\mathbb{G}$ can be bigger than $\mathbb{F} \vee \mathbb{H}$ (to some extent limited, as discussed in Crépey and Song (2014), by the condition (B) that is latent in (C.3)). In particular, the conditions (C.1-2-3) (even for $\mathbb{P} = \mathbb{Q}$) do not exclude a jump of an $\mathbb{F}$ adapted càdlàg process at time τ, which happens for instance with a nonvanishing random variable $\Delta_\tau = D_\tau - D_{\tau-}$ in the DMO model of Sect. 7 (e.g. to render the case, actually the key feature in Bielecki, Crépey, Jeanblanc, and Zargari (2012), of a joint default of the counterparty and a reference firm in a CDS). By contrast, a jump of an $\mathbb{F}$ adapted càdlàg process at time τ cannot happen in a standard reduced-form setup of credit risk (see Lemma 2.1(ii) in Crépey (2012, Part II)). But all these comments should not hide the main feature, namely, the great flexibility of the condition (C) comes from the possibility to choose $(\mathbb{F}, \mathbb{P})$ ensuring (C.1-2-3): see Sect. 6 and Sect. 7 for concrete examples, with $\mathbb{P} \neq \mathbb{Q}$ (and $\Delta_\tau = 0$) in the first case and $\mathbb{P} = \mathbb{Q}$ (but $\Delta_\tau \neq 0$) in the second case. In this paper we work under the "minimal condition" (C). However, since the condition (B) of Crépey and Song (2014) is in fact latent in (C.1) and (C.3) and since (C.2) is exactly the condition (A) of Crépey and Song (2014), our random time τ is essentially an invariant time in the sense of Crépey and Song (2014).

Theorem 3.1 (Reduced-form TVA modeling) *Assume that an $(\mathbb{F}, \mathbb{P})$ semimartingale $\widetilde{\Theta}$ satisfies the following pre-default TVA BSDE on $[0, T]$:*

$$\widetilde{\Theta}_T = 0 \text{ and } d\widetilde{\mu}_t := d\widetilde{\Theta}_t + \widetilde{f}_t(\widetilde{\Theta}_t)dt \text{ is an } (\mathbb{F}, \mathbb{P}) \text{ local martingale on } [0, T]. \tag{3.2}$$

Let $\Theta = \widetilde{\Theta}$ on $[0,\bar\tau)$ and $\Theta_{\bar\tau} = \mathbb{1}_{\{\tau<T\}}\xi(P_{\tau-} - \widetilde{\Theta}_{\tau-})$. We write $\xi_\star = \xi(P_{\tau-} - \widetilde{\Theta}_{\tau-})$ and $\widehat{\xi}^\star_t = \widehat{\xi}_t(P_t - \widetilde{\Theta}_t)$. If $\mathbb{1}_{\tau<T}\xi_\star J$ has locally integrable total variation, then Θ satisfies the full TVA equation (2.3) on $[0,\bar\tau]$ and

$$d\mu_t = d\widetilde{\mu}^{\tau-}_t - \Big((\xi_\star - \widetilde{\Theta}_{t-})dJ_t + \gamma_t(\widehat{\xi}^\star_t - \widetilde{\Theta}_t)dt\Big)\,,\ \ t\in[0,\bar\tau]. \tag{3.3}$$

Proof. By definition of Θ here, a $(\mathbb{G},\mathbb{Q})$ semimartingale by (C.1), we have, for $t\in[0,\bar\tau]$:

$$d\Theta_t = d(J_t\widetilde{\Theta}_t) - \xi_\star dJ_t = d\widetilde{\Theta}^{\tau-}_t + \widetilde{\Theta}_{t-}\,dJ_t - \xi_\star dJ_t. \tag{3.4}$$

Then by (3.2), for $t\in[0,\bar\tau]$:

$$\begin{aligned}-d\Theta_t &= \widetilde{f}_t(\widetilde{\Theta}_t)dt - d\widetilde{\mu}^{\tau-}_t + (\xi_\star - \widetilde{\Theta}_{t-})dJ_t\\ &= f_t(\Theta_t)dt - d\widetilde{\mu}^{\tau-}_t + \Big((\xi_\star - \widetilde{\Theta}_{t-})dJ_t + \gamma_t(\widehat{\xi}^\star_t - \widetilde{\Theta}_t)dt\Big)\,,\end{aligned}$$

by (C.3). By (C.2), $\widetilde{\mu}^{\tau-}_t$ is a $(\mathbb{G},\mathbb{Q})$ local martingale, as is also on $[0,\bar\tau]$

$$(\xi_\star - \widetilde{\Theta}_{t-})dJ_t + \gamma_t(\widehat{\xi}^\star_t - \widetilde{\Theta}_t)dt,$$

by Lemma 2.1. This yields the decomposition (3.3) of the $(\mathbb{G},\mathbb{Q})$ local martingale part μ of Θ, which implies (2.3). ∎

3.1 Markov Case

Assume that the pre-default TVA BSDE (3.2) is Markov, in the sense that there exists an $(\mathbb{F},\mathbb{P})$ Markov process $\widetilde{X}$, called $(\mathbb{F},\mathbb{P})$ (or pre-default) factor process, along with:

- a deterministic function $\widetilde{f}(t,\widetilde{x},\vartheta)$ such that

$$\widetilde{f}_t(\vartheta) = \widetilde{f}(t,\widetilde{X}_t,\vartheta),\ \ t\in[0,T], \tag{3.5}$$

- a stochastic or random measure stochastic integral "$Z_t\cdot d\widetilde{\mathbf{M}}_t$" and a linear operator $\widetilde{\mathcal{B}}$ such that, for any sufficiently regular function $\varphi=\varphi(t,\widetilde{x})$, the $(\mathbb{F},\mathbb{P})$ local martingale part of $\varphi(t,\widetilde{X}_t)$ is given in Itô-Markov form as

$$\widetilde{\mathcal{B}}\varphi(t,\widetilde{X}_{t-})\cdot d\widetilde{\mathbf{M}}_t. \tag{3.6}$$

Then the pre-default TVA BSDE (3.2) reduces to the following Markov BSDE in $\widetilde{\Theta}_t = \widetilde{\Theta}(t,\widetilde{X}_t)$:

$$\begin{cases}\widetilde{\Theta}(T,\widetilde{X}_T) = 0 \text{ and, for } t\in[0,T],\\ -\,d\widetilde{\Theta}(t,\widetilde{X}_t) = \widetilde{f}\left(t,\widetilde{X}_t,\widetilde{\Theta}(t,\widetilde{X}_t)\right)dt - \widetilde{\mathcal{B}}\widetilde{\Theta}(t,\widetilde{X}_{t-})\cdot d\widetilde{\mathbf{M}}_t,\end{cases} \tag{3.7}$$

for which an equivalent semilinear PIDE could be written in terms of the generator of $\widetilde{X}$ (a similar statement applies and will not be repeated regarding the various Markov BSDEs introduced below). Accordingly, Theorem 3.1 admits the following Markov counterpart. Note that under mild regularity and growth conditions on the data, the BSDE (3.7) is well-posed in the space of square integrable processes $\widetilde{\Theta}$ (assuming an $(\mathbb{F},\mathbb{P})$ martingale representation in $\widetilde{\mathbf{M}}$).

Proposition 3.1 *If the BSDE (3.7) has a solution $\widetilde{\Theta}_t = \widetilde{\Theta}(t,\widetilde{X}_t)$ such that $\mathbb{1}_{\tau<T}\xi_\star J$ has locally integrable total variation, where $\xi_\star = \xi(P_{\tau-} - \widetilde{\Theta}_{\tau-})$, then we obtain a solution Θ to the full TVA equation (2.3) by $\Theta = \widetilde{\Theta}$ on $[0,\bar\tau)$ and $\Theta_{\bar\tau} = \mathbb{1}_{\tau<T}\xi$. Writing $\widehat{\xi}^\star_t = \widehat{\xi}_t(P_t - \widetilde{\Theta}_t)$,*

$$d\mu_t = \widetilde{\mathcal{B}}\widetilde{\Theta}(t,\widetilde{X}_{t-})\cdot d\widetilde{\mathbf{M}}^{\tau-}_t - \Big((\xi_\star - \widetilde{\Theta}_{t-})dJ_t + \gamma_t(\widehat{\xi}^\star_t - \widetilde{\Theta}_t)dt\Big)\,,\ \ t\in[0,\bar\tau].\ ∎ \tag{3.8}$$

4 Cure Period

An additional feature is a time lag $\delta > 0$, called the cure period, typically taken as ten (respectively five) days in the case of bilateral (respectively cleared) transactions, between default (at time τ) and liquidation (delivery of the close-out cash-flow). We assume that the liquidator takes in charge the funding of a defaulted party during the cure period (for possibly modified funding conditions, which would correspond to a funding coefficient $g_t(\pi)$ presenting a "change of regime" at the default time of a party). For every time u, we write

$$\bar{u} = u \wedge T,\ \ u^\delta = u + \delta,\ \ \bar{u}^\delta = \mathbb{1}_{u<T} u^\delta + \mathbb{1}_{\{u \geq T\}} T,$$

and we define

$$P_t^\delta = P_t + \int_{[\tau,t]} e^{\int_s^t r_u du} dD_s,\ \ f_t^\delta(\vartheta) = g_t(P_t^\delta - \vartheta) - r_t \vartheta \tag{4.1}$$

(in particular, $P_t^\delta = P_t$ and $f_t^\delta = f_t$ for $t \leq \tau$). With a positive cure period δ, the effective time horizon of the problem becomes $\bar{\tau}^\delta$. If $\tau < T$, the position is liquidated at τ^δ, for a modified exposure $\xi^\delta(\pi)$ of the same general structure as (2.1), i.e.

$$\xi^\delta(\pi) = \xi_c - \Lambda(\pi - \mathfrak{C}_\star)^+, \tag{4.2}$$

except that the random variables ξ_c, Λ and $\mathfrak{C}_\star$ are now $\mathcal{G}_{\tau^\delta}$ measurable (and in fact, typically, $\mathcal{G}_{\tau^\delta-}$ measurable; see Lemma 5.2). Following the approach of Crépey (2012, Part II), the full TVA BSDE with cure period is written as:

$$\begin{aligned} &\Theta^\delta_{\bar{\tau}^\delta} = \mathbb{1}_{\{\tau<T\}} \xi^\delta(P^\delta_{\tau^\delta -} - \Theta^\delta_{\tau^\delta -}) \text{ and} \\ &d\mu_t^\delta := d\Theta_t^\delta + f_t^\delta(\Theta_t^\delta)dt \text{ is a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } [0,\bar{\tau}^\delta]. \end{aligned} \tag{4.3}$$

This BSDE is structurally different from (2.3) in that τ^δ in (4.3) is predictable, as announced by τ (see Theorem 3.27 in He et al. (1992)), whereas τ in (2.3) is totally inaccessible, as endowed with an intensity (see the third assertion of Corollary 5.23 in He et al. (1992)). In particular, as will become clear after Lemmas 4.1-4.2, the terminal condition in (4.3) really plays the role of an equation for $\Theta^\delta_{\bar{\tau}^\delta-}$, instead of the role of a recovery that is played by the terminal condition in (2.3).

Lemma 4.1 *If ξ_c, Λ and $\mathfrak{C}_\star$ are $\mathcal{G}_{\tau^\delta-}$ measurable and $\Lambda < 1$, then the equation*

$$\theta = \mathbb{1}_{\{\tau<T\}} \xi^\delta(P^\delta_{\tau^\delta-} - \theta) \tag{4.4}$$

has a unique $\mathcal{G}_{\bar{\tau}^\delta-}$ measurable solution $\theta = \mathbb{1}_{\{\tau<T\}} \xi_\star^\delta$, where

$$\xi_\star^\delta = \begin{cases} (1-\Lambda)^{-1}\xi_c + (1-(1-\Lambda)^{-1})(P^\delta_{\tau^\delta-} - \mathfrak{C}_\star) \ \textit{if}\ \{\xi_c \leq P^\delta_{\tau^\delta-} - \mathfrak{C}_\star\} \\ \xi_c \ \textit{otherwise.} \end{cases} \tag{4.5}$$

Proof. In view of (4.2), (4.4) is equivalent to

$$\big(\mathbb{1}_{\{\tau<T\}}\xi_c - \theta\big)^+ - \big(\mathbb{1}_{\{\tau<T\}}\xi_c - \theta\big)^- = \mathbb{1}_{\{\tau<T\}} \Lambda(P^\delta_{\tau^\delta-} - \mathfrak{C}_\star - \theta)^+, \tag{4.6}$$

i.e.

$$\begin{cases} \theta \leq \mathbb{1}_{\{\tau<T\}}\xi_c \\ \mathbb{1}_{\{\tau<T\}}\xi_c = \theta + \mathbb{1}_{\{\tau<T\}} \Lambda(P^\delta_{\tau^\delta-} - \mathfrak{C}_\star - \theta)^+. \end{cases} \tag{4.7}$$

Let us assume that a $\in \mathcal{G}_{\bar{\tau}^\delta-}$ measurable random variable θ solves (4.4), or equivalently (4.7). We write $A = \{\tau < T\} \cap \{\xi_c \leq P^\delta_{\tau^\delta-} - \mathfrak{C}_\star\}$. Then $\mathbb{1}_{\{\tau<T\}}\xi_c = \theta$ outside A. Indeed this follows from the second line in (4.7) if $\tau \geq T$, whereas on the set $\{\tau < T\}$ but for $\xi_c > P^\delta_{\tau^\delta-} - \mathfrak{C}_\star$, we have $\mathbb{1}_{\{\tau<T\}}\xi_c - \theta > P^\delta_{\tau^\delta-} - \mathfrak{C}_\star - \theta$, only compatible with the second line in (4.7) for $P^\delta_{\tau^\delta-} - \mathfrak{C}_\star - \theta \leq 0$, in which case this second line yields $\theta = \mathbb{1}_{\{\tau<T\}}\xi_c = \mathbb{1}_{\{\tau<T\}}\xi_\star^\delta$. Moreover, on A, we have by the first line in (4.7) $\theta \leq \xi_c \leq P^\delta_{\tau^\delta-} - \mathfrak{C}_\star$ by definition of A. Therefore on A the second line in (4.7) is rewritten as

$\xi_c = \theta + \Lambda(P^{\delta}_{\tau^{\delta}-} - \mathfrak{C}_{\star} - \theta)$, i.e. $\theta = (1-\Lambda)^{-1}\xi_c + (1-(1-\Lambda)^{-1})(P^{\delta}_{\tau^{\delta}-} - \mathfrak{C}_{\star}) = \xi^{\delta}_{\star} = \mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star}$ (on A). In conclusion, $\theta = \mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star}$.

Conversely, let us prove that $\mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star}$ solves (4.7). Noticing that $\xi^{\delta}_{\star} \le \xi_c$ (as $\Lambda < 1$), therefore $\mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star}$ satisfies the first line in (4.7). If $\tau \ge T$, then $\mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star}$ trivially satisfies the second line in (4.7). On the set $\{\tau < T\}$, either $P^{\delta}_{\tau^{\delta}-} - \mathfrak{C}_{\star} < \xi_c$ and $\mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star} = \xi^{\delta}_{\star} = \xi_c$, $(P^{\delta}_{\tau^{\delta}-} - \mathfrak{C}_{\star} - \mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star})^{+} = 0$, thus $\mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star}$ satisfies the second line in (4.7); or $\xi_c \le P^{\delta}_{\tau^{\delta}-} - \mathfrak{C}_{\star}$ and $\xi^{\delta}_{\star} \le \xi_c \le P^{\delta}_{\tau^{\delta}-} - \mathfrak{C}_{\star}$, $(P^{\delta}_{\tau^{\delta}-} - \mathfrak{C}_{\star} - \mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star})^{+} = P^{\delta}_{\tau^{\delta}-} - \mathfrak{C}_{\star} - \mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star}$, so that $\mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star} = (1-(1-\Lambda)^{-1})\xi_c + (1-(1-(1-\Lambda)^{-1}))(P^{\delta}_{\tau^{\delta}-} - \mathfrak{C}_{\star})$ again satisfies the second line in (4.7). ∎

Henceforth in this section, we assume the conditions of Lemma 4.1.

Lemma 4.2 *The full TVA BSDE with cure period* (4.3) *is equivalent to*

$$\Theta_{\bar\tau^{\delta}} = \mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star} \text{ and } d\mu^{\delta}_t := d\Theta^{\delta}_t + f^{\delta}_t(\Theta^{\delta}_t)dt \text{ is a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } [0,\bar\tau^{\delta}]. \tag{4.8}$$

***Proof*.** Since

$$\bar\tau^{\delta} = \mathbb{1}_{\{\tau<T\}}\tau^{\delta} + \mathbb{1}_{\{T\le\tau\}}T = \tau^{\delta}_{\{\tau<T\}} \wedge T_{\{T\le\tau\}},$$

where T and τ^{δ} (by Theorem 3.27 in He et al. (1992)) and in turn $\tau^{\delta}_{\{\tau<T\}}$ and $T_{\{T\le\tau\}}$ (by Theorem 3.29 in He et al. (1992)) are predictable stopping times, so is therefore $\bar\tau^{\delta}$. If Θ^{δ} solves (4.8), then the $(\mathbb{G},\mathbb{Q})$ local martingale μ^{δ}, hence Θ^{δ}, cannot jump at the predictable time $\bar\tau^{\delta}$. That's because $\mathbb{E}\left(\Delta\mu^{\delta}_{\bar\tau^{\delta}} \middle| \mathcal{G}_{\bar\tau^{\delta}-}\right) = 0$ ($\bar\tau^{\delta}$ being predictable), joint to the $\mathcal{G}_{\bar\tau^{\delta}-}$ measurability of

$$\mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star} - \Theta^{\delta}_{\bar\tau^{\delta}-} = \Delta\Theta^{\delta}_{\bar\tau^{\delta}} = \Delta\mu_{\bar\tau^{\delta}}.$$

Consequently,

$$\mathbb{1}_{\{\tau<T\}}\xi^{\delta}(P^{\delta}_{\tau^{\delta}-} - \Theta^{\delta}_{\tau^{\delta}-}) = \mathbb{1}_{\{\tau<T\}}\xi^{\delta}(P^{\delta}_{\tau^{\delta}-} - \Theta^{\delta}_{\bar\tau^{\delta}}) = \Theta^{\delta}_{\bar\tau^{\delta}},$$

by (4.4), so that Θ^{δ} also solves (4.3). Conversely, if Θ^{δ} solves (4.3), then, likewise, Θ^{δ} cannot jump at $\bar\tau^{\delta}$ and

$$\Theta^{\delta}_{\bar\tau^{\delta}} = \mathbb{1}_{\{\tau<T\}}\xi^{\delta}(P^{\delta}_{\tau^{\delta}-} - \Theta^{\delta}_{\tau^{\delta}-}) = \mathbb{1}_{\{\tau<T\}}\xi^{\delta}(P^{\delta}_{\tau^{\delta}-} - \Theta^{\delta}_{\bar\tau^{\delta}})$$

solves (4.4), so that by the uniqueness in Lemma 4.1:

$$\Theta^{\delta}_{\bar\tau^{\delta}} = \mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star},$$

hence Θ^{δ} solves (4.8). ∎

4.1 Reduced-Form Approach

Obviously, we can rewrite (4.8) in two parts, after and until $\bar\tau$:

Lemma 4.3 *If*

$$\begin{cases} \check\Theta_{\bar\tau^{\delta}} = \mathbb{1}_{\{\tau<T\}}\xi^{\delta}_{\star} \text{ and } \big(d\check\Theta_t + f^{\delta}_t(\check\Theta_t)dt\big) \text{ is a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } [\bar\tau,\bar\tau^{\delta}] & (4.9)\\ \bar\Theta_{\bar\tau} = \check\Theta_{\bar\tau} \text{ and } \big(d\bar\Theta_t + f_t(\bar\Theta_t)dt\big) \text{ is a } (\mathbb{G},\mathbb{Q}) \text{ local martingale on } [0,\bar\tau], & (4.10)\end{cases}$$

then $\Theta^{\delta} = \mathbb{1}_{[0,\bar\tau)}\bar\Theta + \mathbb{1}_{[\bar\tau,\bar\tau^{\delta}]}\check\Theta$ *solves* (4.8). ∎

Given $\check\Theta$ solving (4.9), (4.10) will be solved in Theorem 4.1 much like (2.3) in Theorem 3.1 above. But first, in order to solve (4.9), we need to be able to view it as a "classical" BSDE on a deterministic time interval. Since it is posed on $[\bar\tau,\bar\tau^{\delta}]$, which is random but $\mathcal{G}_{\bar\tau}$ measurable, this is possible through the concept of regular conditional probability, assuming $\mathbb{G}$ given as the usual augmentation of the natural filtration $\mathbb{Z} = (\mathcal{Z}_t)_{t\ge0}$ of some Polish space valued càdlàg process Z. Note that $\mathcal{Z}_{\infty}$ is Borel as generated by the paths of Z. We identify Ω with the path space set of Z equipped with the Skorohod topology (cf. Billingsley (1968)). By Theorem 13.6 in Koralov and Sinai (2007), there exists a $(\mathbb{Q},\mathcal{G}_{\bar\tau})$

regular conditional probability of the identity map from $(\Omega, \mathcal{G})$ into $(\Omega, \mathcal{Z}_\infty)$.

Condition (G). *$\mathcal{G}_{\bar\tau}$ is generated by $\mathrm{Z}_{\cdot\wedge\bar\tau}$ and the $\mathbb{Q}$ negligible sets.*

Under this assumption, the above regular conditional probability can be taken as a function of $\overline{w} = w_{|[0,\bar\tau(w)]}$ and we write $\mathcal{Q}_{\bar w}$, with related expectation denoted by $\mathbb{E}^{\bar w}$, for every $w \in \Omega$. In particular, for $\mathbb{Q}$ almost every w, $\bar\omega = \bar w$ holds $\mathcal{Q}_{\bar w}(d\omega)$ a.s., so that under $\mathcal{Q}_{\bar w}$ any dependence on ω reduces to a dependence on $\omega_{|[\bar\tau(\omega), \bar\tau^\delta(\omega)]}$. Note that $\mathcal{Q}_{\bar w}$ is only defined on $\mathcal{Z}_\infty$, for $\mathbb{Q}$ almost every w.

Lemma 4.4 *For any measurable space $(E, \mathcal{E})$ and $\mathcal{Z}_\infty \otimes \mathcal{E}$ measurable function $\zeta(\omega, e)$, for any E valued $\mathcal{G}_{\bar\tau}$ measurable random variable η, if $\zeta(\eta)$ is $\mathcal{G}_{\bar\tau}$ locally integrable, then*

$$\mathbb{E}[\zeta(\eta)|\mathcal{G}_{\bar\tau}] = \left(\mathbb{E}^{\bar\cdot}[\zeta(e)]\right)_{|e=\eta}, \mathbb{Q} \ a.s.. \tag{4.11}$$

***Proof*.** The identities (4.11) without "parameter" η characterize $\mathcal{Q}_{\bar\cdot}$. The extension with η can be proven by a standard monotone class argument. ∎

We consider the following family of $(\mathbb{Z}, \mathcal{Q}_{\bar w})$ BSDEs with solutions, assumed to exist, $\Theta^{\bar w}$ (one per $w \in \Omega$):

$$\begin{aligned} &\Theta^{\bar w}_{\bar\tau^\delta} = \mathbb{1}_{\{\tau<T\}}\xi^\delta_\star \\ &d\mu^{\bar w}_t := d\Theta^{\bar w}_t + f^\delta_t(\Theta^{\bar w}_t)dt \text{ is a } (\mathbb{G}, \mathcal{Q}_{\bar w}) \text{ local martingale on } [\bar\tau(w), \bar\tau^\delta(w)], \end{aligned} \tag{4.12}$$

and we write $\check\Theta : (t,\omega) \mapsto \Theta^{\bar\omega}_t(\omega)$, $\check\mu : (t,\omega) \mapsto \mu^{\bar\omega}_t(\omega)$ (or $\Theta^{\bar\cdot}_t, \mu^{\bar\cdot}_t$ for short).

Lemma 4.5 *If $(t,\omega,w) \mapsto \Theta^{\bar w}_t(\omega)$ is $\mathcal{O}(\mathbb{Z}) \otimes \mathcal{G}_\infty$ measurable and if there exists a sequence $(\upsilon_n)_{n\geq 0}$ of $\mathbb{Z}$ stopping times increasing to ∞, reducing $\check\Theta_\cdot - \check\Theta_{\cdot\wedge\bar\tau}$ and $\int_{\bar\tau}^{\cdot} f^\delta_t(\check\Theta_{t-})dt$ to $(\mathbb{G}, \mathbb{Q})$ uniformly integrable processes and localizing every $(\mathbb{Z}, \mathcal{Q}_{\bar w})$ local martingale $d\Theta^{\bar w}_t + f^\delta_t(\Theta^{\bar w}_t)dt$ on $[\bar\tau, \bar\tau^\delta]$, then the process $\mathbb{1}_{\tau<T}\check\Theta$ restricted to $[\bar\tau, \bar\tau^\delta]$ solves (4.9).*

***Proof*.** By a standard class monotone argument, $\check\Theta_\cdot - \check\Theta_{\cdot\wedge\bar\tau}$ can be shown to be a $\mathbb{G}$ optional process, hence so is in turn $\check\mu_\cdot - \check\mu_{\cdot\wedge\bar\tau}$. Let υ be a $\mathbb{Z}$ stopping time such that $\bar\tau \leq \upsilon \leq \upsilon_n \wedge \bar\tau^\delta$ (assuming without loss of generality that $\bar\tau \leq \upsilon_n$). For any reals $a < b$ and $A \in \mathcal{Z}_a$, we compute by Lemma 4.4 with $e \equiv w$, $\eta \equiv \bar\cdot$ and $\zeta(e) \equiv \mathbb{1}_{\{\bar\tau(w)\leq\upsilon\}}\mathbb{1}_A\mathbb{1}_{\{\bar\tau(w)\leq a\}}\left(\mu^{\bar w}_{b\wedge\upsilon} - \mu^{\bar w}_{\bar\tau(w)}\right)$:

$$\begin{aligned} &\mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{\bar\tau\leq a\}}\left(\check\mu_{b\wedge\upsilon} - \check\mu_{\bar\tau\wedge b\wedge\upsilon}\right)] = \mathbb{E}\mathbb{E}[\mathbb{1}_{\{\bar\tau\leq\upsilon\}}\mathbb{1}_A\mathbb{1}_{\{\bar\tau\leq a\}}\left(\check\Theta_{b\wedge\upsilon} - \check\Theta_{\bar\tau} + \int_{\bar\tau}^{b\wedge\upsilon} f^\delta_s(\check\Theta_{s-})ds\right)|\mathcal{G}_{\bar\tau}] \\ &= \mathbb{E}\mathbb{E}^{\bar\cdot}[\mathbb{1}_{\{\bar\tau\leq\upsilon\}}\mathbb{1}_A\mathbb{1}_{\{\bar\tau\leq a\}}\left(\Theta^{\bar\cdot}_{b\wedge\upsilon} - \Theta^{\bar\cdot}_{\bar\tau} + \int_{\bar\tau}^{b\wedge\upsilon} f^\delta_s(\Theta^{\bar\cdot}_{s-})ds\right)] = \mathbb{E}\mathbb{E}^{\bar\cdot}[\mathbb{1}_{\{\bar\tau\leq\upsilon\}}\mathbb{1}_A\mathbb{1}_{\{\bar\tau\leq a\}}\left(\mu^{\bar\cdot}_{b\wedge\upsilon} - \mu^{\bar\cdot}_{\bar\tau}\right)] \\ &= \mathbb{E}\mathbb{E}^{\bar\cdot}[\mathbb{1}_{\{\bar\tau\leq\upsilon\}}\mathbb{1}_A\mathbb{1}_{\{\bar\tau\leq a\}}\left(\mu^{\bar\cdot}_{a\wedge\upsilon} - \mu^{\bar\cdot}_{\bar\tau}\right)] = \mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{\bar\tau\leq a\}}\left(\check\mu_{a\wedge\upsilon} - \check\mu_{\bar\tau\wedge a\wedge\upsilon}\right)]. \end{aligned}$$

In the same way,

$$\begin{aligned} &\mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{a<\bar\tau\leq b\}}\left(\check\mu_{b\wedge\upsilon} - \check\mu_{\bar\tau\wedge b\wedge\upsilon}\right)] = \mathbb{E}\mathbb{E}[\mathbb{1}_{\{\bar\tau\leq\upsilon\}}\mathbb{1}_A\mathbb{1}_{\{a<\bar\tau\leq b\}}\left(\check\Theta_{b\wedge\upsilon} - \check\Theta_{\bar\tau} + \int_{\bar\tau}^{b\wedge\upsilon} f^\delta_s(\check\Theta_{s-})ds\right)|\mathcal{G}_{\bar\tau}] \\ &= \mathbb{E}\mathbb{E}^{\bar\cdot}[\mathbb{1}_{\{\bar\tau\leq\upsilon\}}\mathbb{1}_A\mathbb{1}_{\{a<\bar\tau\leq b\}}\left(\Theta^{\bar\cdot}_{b\wedge\upsilon} - \Theta^{\bar\cdot}_{\bar\tau} + \int_{\bar\tau}^{b\wedge\upsilon} f^\delta_s(\Theta^{\bar\cdot}_{s-})ds\right)] \\ &= \mathbb{E}\mathbb{E}^{\bar\cdot}[\mathbb{1}_{\{\bar\tau\leq\upsilon\}}\mathbb{1}_A\mathbb{1}_{\{a<\bar\tau\leq b\}}(\mu^{\bar\cdot}_{b\wedge\upsilon} - \mu^{\bar\cdot}_{\bar\tau})] = 0 = \mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{a<\bar\tau\leq b\}}\left(\check\mu_{a\wedge\upsilon} - \check\mu_{\bar\tau\wedge a\wedge\upsilon}\right)] \end{aligned}$$

and of course $\mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{b<\bar\tau\}}\left(\check\mu_{b\wedge\upsilon} - \check\mu_{\bar\tau\wedge b\wedge\upsilon}\right)] = 0 = \mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{b<\bar\tau\}}\left(\check\mu_{a\wedge\upsilon} - \check\mu_{\bar\tau\wedge a\wedge\upsilon}\right)]$. Adding them up and sending n, hence υ_n, to ∞, these identities for every $A \in \mathcal{Z}_a$ show that

$$\mathbb{E}[(\check\mu_b - \check\mu_{\bar\tau\wedge b}) \mid \mathcal{Z}_a] = \mathbb{E}[(\check\mu_a - \check\mu_{\bar\tau\wedge a}) \mid \mathcal{Z}_a].$$

But these $\mathcal{Z}_a$ conditional expectations $\mathbb{Q}$ almost surely coincide with their $\mathcal{G}_a$ analogs, so that

$$\mathbb{E}[(\check\mu_b - \check\mu_{\bar\tau\wedge b}) \mid \mathcal{G}_a] = \mathbb{E}[(\check\mu_a - \check\mu_{\bar\tau\wedge a}) \mid \mathcal{G}_a] = \check\mu_a - \check\mu_{\bar\tau\wedge a},$$

by $\mathbb{G}$ adaptedness of $\check\mu_\cdot - \check\mu_{\cdot\wedge\bar a}$. Consequently, Theorem 4.40 in He et al. (1992) shows that the process $\check\mu$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[\bar\tau, \bar\tau^\delta]$. ∎

Theorem 4.1 (TVA modeling with cure period) *Assuming the conditions of Lemma 4.5 and $\mathbb{1}_{\tau<T}\check{\Theta}_\tau J$ of locally integrable total variation, let $\widehat{\Theta}^\delta$ be a $\mathbb{G}$ predictable process such that*

$$\mathbb{1}_{\tau<T}\mathbb{E}(\check{\Theta}_\tau \,|\, \mathcal{G}_{\tau-}) = \mathbb{1}_{\tau<T}\widehat{\Theta}^\delta_\tau. \tag{4.13}$$

We denote by $\widetilde{f}^\delta_t(\vartheta)$ an $\mathbb{F}$ representative (cf. (C.3)), assumed to exist, of $\widehat{f}^\delta_t(\vartheta)$, where

$$\widehat{f}^\delta_t(\vartheta) = f_t(\vartheta) + \gamma_t\big(\widehat{\Theta}^\delta_t - \vartheta\big) \tag{4.14}$$

(i.e. $\widehat{f}_t(\vartheta)$ of (3.1) with $\widehat{\xi}_t(P_t - \vartheta)$ replaced by $\widehat{\Theta}^\delta_t$). If an $(\mathbb{F},\mathbb{P})$ semimartingale $\widetilde{\Theta}^\delta$ satisfies the BSDE

$$\widetilde{\Theta}^\delta_T = 0 \text{ and } d\widetilde{\mu}^\delta_t := d\widetilde{\Theta}^\delta_t + \widetilde{f}^\delta_t(\widetilde{\Theta}^\delta_t)dt \text{ is an } (\mathbb{F},\mathbb{P}) \text{ local martingale on } [0,T], \tag{4.15}$$

then the process $\Theta^\delta = \mathbb{1}_{[0,\bar\tau)}\widetilde{\Theta}^\delta + \mathbb{1}_{[\bar\tau,\bar\tau^\delta]}\mathbb{1}_{\tau<T}\check{\Theta}$ satisfies the full TVA BSDE with cure period (4.3), and

$$d\mu^\delta_t = d((\widetilde{\mu}^\delta)^{\tau-})_t - \Big((\check{\Theta}_\tau - \widetilde{\Theta}^\delta_{t-})dJ_t + \gamma_t(\widehat{\Theta}^\delta_t - \widetilde{\Theta}^\delta_t)dt\Big),\ \ 0 \le t \le \bar\tau. \tag{4.16}$$

Proof. The existence of $\widehat{\Theta}^\delta$ in (4.13) is ensured by Corollary 3.23 2) in He et al. (1992). By Lemma 4.5, $\mathbb{1}_{\tau<T}\check{\Theta}$ satisfies the BSDE (4.9) over $[\bar\tau,\bar\tau^\delta]$. An application of Theorem 3.1 with $\widetilde{f}_t(\vartheta)$ replaced by $\widetilde{f}^\delta_t(\vartheta)$ shows that $\mathbb{1}_{[0,\bar\tau)}\widetilde{\Theta}^\delta + \mathbb{1}_{\{\bar\tau\}}\mathbb{1}_{\tau<T}\check{\Theta}$ satisfies the BSDE (4.10) over $[0,\bar\tau]$. We conclude that $\Theta^\delta = \mathbb{1}_{[0,\bar\tau)}\widetilde{\Theta}^\delta + \mathbb{1}_{[\bar\tau,\bar\tau^\delta]}\mathbb{1}_{\tau<T}\check{\Theta}$ satisfies (4.9)–(4.10), i.e. (4.8) by Lemma 4.3, or equivalently (4.3) by Lemma 4.2. ∎

4.2 Markov Case

We want to solve (4.3) or, as sufficient by Theorem 4.1, (4.12) (one BSDE (4.12) per $\bar w$) and (4.15). For solving (4.12), we assume, on top of the condition (G), that there exist:

- a $(\mathbb{G},\mathbb{Q})$ jump-diffusion X^δ with a (3.6)-like $(\mathbb{G},\mathbb{Q})$ Itô-Markov local martingale part formula $\mathcal{B}^\delta\varphi(t,X^\delta_{t-})\cdot d\mathbf{M}^\delta_t$, for some $\mathbb{Z}$ optional $\mathbf{M}^\delta$; hence, Lemma 4.6 below shows that $X^\delta_{|[\bar\tau(w),+\infty)}$ has the analogous properties relative to $(\mathbb{G},\mathcal{Q}_{\bar w})$, for $\mathbb{Q}$ a.e. w;
- functions $\phi^{\bar w}(t,x_\delta)$ and $f^{\bar w}(t,x_\delta,\vartheta)$ such that

$$\mathbb{1}_{\tau<T}\xi^\delta_\star = \phi^{\bar\cdot}(\tau^\delta, X^\delta_{\tau^\delta}) \quad \text{and} \quad f^\delta_t(\vartheta) = f^{\bar\cdot}(t, X^\delta_t, \vartheta),\ \ t \in [\bar\tau,\bar\tau^\delta]. \tag{4.17}$$

Lemma 4.6 *Let M be a $\mathbb{Z}$ optional $(\mathbb{G},\mathbb{Q})$ local martingale. The restriction of M to $[\overline{\tau}(w),\infty)$ is a $(\mathbb{Z},\mathcal{Q}_{\overline{w}})$ local martingale, for $\mathbb{Q}$ almost every w.*

Proof. By stopping, we can assume that $\sup_{t>0}|M_t|$ is $\mathbb{Q}$ integrable, hence $\mathbb{E}\big(\sup_{t>0}|M_t| \,\big|\, \mathcal{G}_{\bar\tau}\big)$ is $\mathbb{Q}$ almost surely finite, i.e. $\sup_{t>0}|M_t|$ is $\mathcal{Q}_{\overline{w}}$ integrable for $\mathbb{Q}$ almost every w. For $B \in \mathcal{Z}_{(b+\epsilon)-}$ $(b,\epsilon > 0)$,

$$\mathbb{E}[\mathbb{1}_A\mathbb{1}_B(M_\infty - M_{\bar\tau(w)})] = \mathbb{E}[\mathbb{1}_A\mathbb{1}_B(M_{(b+\epsilon)} - M_{(b+\epsilon)\wedge\bar\tau(w)})],\ \ A \in \mathcal{G}_{\bar\tau(w)},$$

hence $\mathbb{E}^{\bar w}[\mathbb{1}_B(M_\infty - M_{\bar\tau(w)})] = \mathbb{E}^{\bar w}[\mathbb{1}_B(M_{(b+\epsilon)} - M_{(b+\epsilon)\wedge\bar\tau(w)})]$, $\mathbb{Q}(dw)$ a.s.. But $\mathcal{Z}_{(b+\epsilon)-}$ is countably generated, so that: $\mathbb{Q}(dw)$ almost surely,

$$\mathbb{E}^{\bar w}[\mathbb{1}_B(M_\infty - M_{\bar\tau(w)})] = \mathbb{E}^{\bar w}[\mathbb{1}_B(M_{(b+\epsilon)} - M_{(b+\epsilon)\wedge\bar\tau(w)})],\ \ B \in \mathcal{Z}_{(b+\epsilon)-}$$

and therefore by dominated convergence for $B \in \mathcal{Z}_b \subseteq \mathcal{Z}_{(b+\epsilon)-}$

$$\mathbb{E}^{\bar w}[\mathbb{1}_B(M_\infty - M_{\bar\tau(w)})] = \mathbb{E}^{\bar w}[\mathbb{1}_B(M_b - M_{b\wedge\bar\tau(w)})],\ \ B \in \mathcal{Z}_b,$$

hence the restriction of M to $[\bar\tau(w),\infty)$ is a $(\mathbb{Z},\mathcal{Q}_{\overline{w}})$ local martingale, by Theorem 4.40 in He et al. (1992). ∎

In this setup the BSDE (4.12) is equivalent to the following $(\mathbb{G},\mathcal{Q}_{\bar w})$ Markov BSDE in $\Theta^{\bar w}_t = \Theta^{\bar w}(t,X^\delta_t)$:

$$\begin{cases} \Theta^{\bar w}(\tau^\delta(w), X^\delta_{\tau^\delta(w)}) = \phi^{\bar w}(\tau^\delta(w), X^\delta_{\tau^\delta(w)}) \text{ and, for } t \in [\tau^\delta(w),\tau^\delta(w)], \\ -\,d\Theta^{\bar w}(t,X^\delta_t) = f^{\bar w}\Big(t, X^\delta_t, \Theta^{\bar w}(t,X^\delta_t)\Big)\,dt - \mathcal{B}^\delta\Theta^{\bar w}(t,X^\delta_{t-})\cdot d\mathbf{M}^\delta_t. \end{cases} \tag{4.18}$$

Therefore $\breve{\Theta}_t = \bar{\Theta}(t, X_t^\delta)$ and, in view of (4.13), $\widehat{\Theta}_t^\delta$ in $\widehat{f}_t^\delta(\vartheta)$ must satisfy

$$\mathbb{1}_{\tau<T}\widehat{\Theta}_\tau^\delta = \mathbb{1}_{\tau<T}\mathbb{E}(\bar{\Theta}(\tau, X_\tau^\delta) \,|\, \mathcal{G}_{\tau-}). \tag{4.19}$$

Next, in order to solve the pre-default $(\mathbb{F},\mathbb{P})$ TVA BSDE (4.15), we assume, regarding $\widetilde{f}_t^\delta(\vartheta)$ there, that

$$\widetilde{f}_t^\delta(\vartheta) = \widetilde{f}^\delta(t, \widetilde{X}_t^\delta, \vartheta), \tag{4.20}$$

where $\widetilde{X}^\delta$ is a pre-default $(\mathbb{F},\mathbb{P})$ factor process in the sense of $\widetilde{X}$ in Sect. 3.1, with corresponding Itô-Markov local martingale part formula $\widetilde{\mathcal{B}}^\delta\varphi(t, \widetilde{X}_{t-}^\delta) \cdot d\widetilde{\mathbf{M}}_t^\delta$. This results in the following Markov form, similar to (3.7), of the BSDE (4.15) for $\widetilde{\Theta}_t^\delta = \widetilde{\Theta}^\delta(t, \widetilde{X}_t^\delta)$:

$$\begin{cases} \widetilde{\Theta}^\delta(T, \widetilde{X}_T^\delta) = 0 \text{ and, for } t \in [0,T], \\ -\,d\widetilde{\Theta}^\delta(t, \widetilde{X}_t^\delta) = \widetilde{f}^\delta\left(t, \widetilde{X}_t^\delta, \widetilde{\Theta}^\delta(t, \widetilde{X}_t^\delta)\right)dt - \widetilde{\mathcal{B}}^\delta\widetilde{\Theta}^\delta(t, \widetilde{X}_{t-}^\delta) \cdot d\widetilde{\mathbf{M}}_t^\delta. \end{cases} \tag{4.21}$$

These observations imply the following Markov counterpart of Theorem 4.1. Note that under mild regularity and growth conditions on the data, the BSDEs (4.18) (one per $\bar{w}$) and (4.21) are well-posed in the related spaces of square integrable processes (assuming a $(\mathbb{G},\mathbb{Q})$ martingale representation in $\mathbf{M}^\delta$ and an $(\mathbb{F},\mathbb{P})$ martingale representation in $\widetilde{\mathbf{M}}^\delta$, respectively).

Proposition 4.1 *Assuming solutions* $\Theta_t^{\bar{w}} = \Theta^{\bar{w}}(t, X_t^\delta)$ *to* (4.18) *(one per* $\bar{w}$*) satisfying the conditions of Lemma 4.5 and such that* $\mathbb{1}_{\tau<T}\bar{\Theta}(\tau, X_\tau^\delta)J$ *has locally integrable total variation, assuming in turn a solution* $\widetilde{\Theta}_t^\delta = \widetilde{\Theta}^\delta(t, \widetilde{X}_t^\delta)$ *to* (4.21)*, then* $\Theta_t^\delta = \mathbb{1}_{t<\bar{\tau}}\widetilde{\Theta}_t^\delta + \mathbb{1}_{t\geq\bar{\tau}}\mathbb{1}_{\tau<T}\bar{\Theta}(t, X_t^\delta)$ *solves the full TVA equation with positive cure period* (4.3) *on* $[\bar{\tau}, \bar{\tau}^\delta]$*, and:*

- *on* $[0, \bar{\tau}]$,

$$d\mu_t^\delta = \widetilde{\mathcal{B}}_t^\delta\widetilde{\Theta}^\delta(t, \widetilde{X}_{t-}^\delta) \cdot d\widetilde{\mathbf{M}}_t^{\tau-} - \left((\bar{\Theta}(\tau, X_\tau^\delta) - \widetilde{\Theta}_{t-}^\delta)dJ_t + \gamma_t(\widehat{\Theta}_t^\delta - \widetilde{\Theta}_t^\delta)dt\right),$$

- *on* $[\bar{\tau}, \bar{\tau}^\delta]$,

$$d\mu_t^\delta = \mathcal{B}^\delta\bar{\Theta}(t, X_{t-}^\delta) \cdot d\mathbf{M}_t^\delta. \blacksquare$$

Summarizing so far, assuming the main condition (C) of an invariant default time τ (a mild assumption), based on Theorems 3.1 and 4.1, we can design a TVA process in terms of a solution to the pre-default BSDE (3.2) or, in case of a positive cure period δ, to the BSDEs (4.12) (one per $\bar{w}$) and (4.15). Note that this approach is not arbitrary since, using the results of Crépey and Song (2014) (see in particular Lemma 4.4 there), one can establish a converse to these results. In jump diffusion setups all these equations are well-posed under mild technical conditions on the data. The next sections implement this program based on invariant times obtained as times with a mark, where the role of the mark is to convey some additional information about the default, e.g. to encode wrong-way and gap risk features that would be out-of-reach in a basic immersion setup.

5 Marked Default Times Specification

For concrete applications, we need to specify the exposure at default $\xi(\pi)$ and the funding coefficient $g_t(\pi)$. At time τ (if $< T$), a terminal cash-flow $\mathfrak{R}$ paid to the bank closes out its position, where $\mathfrak{R} = \mathfrak{R}(\pi)$ can also depend on the value, represented by the real number π, of the contract right before time τ (see Crépey (2012, Part I)). In case $\delta = 0$ (no cure period), consistent with (2.4) and a terminal condition $\Pi_{\bar{\tau}} = \mathbb{1}_{\tau<T}\mathfrak{R}(\Pi_{\bar{\tau}-})$ for Π at $\bar{\tau}$, the exposure of the bank is

$$\xi(\pi) = P_\tau + \Delta_\tau - \mathfrak{R}(\pi), \tag{5.1}$$

where we need to specify $\mathfrak{R}(\pi)$ conformly with usual CSA specifications (see Sect. 1), in particular a CSA close-out valuation scheme Q_t and a CSA collateralization scheme. Let τ_b and τ_c stand for the default times of the bank and the counterparty, so that $\tau = \tau_b \wedge \tau_c$. The bank's close-out cash-flow $\mathfrak{R} = \mathfrak{R}(\pi)$ decomposes into a close-out cash-flow $\mathfrak{R}_c$ from the counterparty to the bank plus, in case

$\tau = \tau_b$ (default of the bank), a cash-flow from the funder to the bank, $\mathfrak{R}_f(\pi)$, depending on the wealth π of the bank right before time τ. These two cash-flows are respectively derived from the debt of the counterparty to the bank, the debt of the bank to the counterparty and the debt of the bank to its funder, respectively modeled at time τ as

$$\chi = (Q_\tau + \Delta_\tau - C_\tau)^+, \;\; \mathfrak{X} = (Q_\tau + \Delta_\tau - \mathfrak{C}_\tau)^-, \;\; \bar{\mathfrak{X}}(\pi) = (\pi - \mathfrak{C}_{\tau-})^+, \tag{5.2}$$

where

$$C = M + N \;\; \text{and} \;\; \mathfrak{C} = M + \mathfrak{N} \tag{5.3}$$

respectively represent the collateral funded by the counterparty and by the bank: re-hypothecable variation margin M plus segregated initial margin $N \geq 0$, respectively $\mathfrak{N} \leq 0$ (so that $\chi \times \mathfrak{X} \equiv 0$). Specifically, the close-out cash-flow is given as $\mathfrak{R}(\pi) = \mathfrak{R}_c + \mathbb{1}_{\tau=\tau_b}\mathfrak{R}_f(\pi)$, where $\mathfrak{R}_f(\pi) = (1 - \bar{R}_b)\bar{\mathfrak{X}}(\pi)$ and

$$\mathfrak{R}_c = \begin{cases} C_\tau + R_c\chi \text{ if } \chi > 0 \text{ and } \tau = \tau_c, \\ \mathfrak{C}_\tau - R_b\mathfrak{X} \text{ if } \mathfrak{X} > 0 \text{ and } \tau = \tau_b, \\ Q_\tau + \Delta_\tau \text{ otherwise.} \end{cases} \tag{5.4}$$

Here R_c and R_b stand for the recovery rates between the two parties and $\bar{R}_b$ for the recovery rate of the bank to its funder (all assumed constant). The ensuing exposure at default results from (5.1) as

$$\begin{aligned} \xi(\pi) &= P_\tau + \Delta_\tau - \mathfrak{R}(\pi) \\ &= P_\tau - Q_\tau + \mathbb{1}_{\tau=\tau_c}(1 - R_c)\chi - \mathbb{1}_{\tau=\tau_b}\big((1 - R_b)\mathfrak{X} + (1 - \bar{R}_b)\bar{\mathfrak{X}}(\pi)\big), \end{aligned} \tag{5.5}$$

consistent with the general form postulated in (2.1). Moreover, given spreads c_t for the remuneration of the collateral and λ_t and $\bar{\lambda}_t$ for the external lending and borrowing costs of the bank, the funding coefficient of the bank is defined by

$$\begin{aligned} g_t(\pi) &= c_t(M_t + \mathfrak{N}_t + N_t) + \bar{\lambda}_t\,(\pi - \mathfrak{C}_t)^+ - \lambda_t\,(\pi - \mathfrak{C}_t)^- \\ &= c_t(\mathfrak{C}_t + N_t) + \bar{\lambda}_t\,(\pi - \mathfrak{C}_t)^+ - \lambda_t\,(\pi - \mathfrak{C}_t)^- \end{aligned} \tag{5.6}$$

(see Crépey (2012, Part I) and Brigo and Pallavicini (2014)).

In the sequel we assume that τ is endowed with a mark e in a finite set E, in the sense that

$$\tau = \min_{e \in E} \tau_e, \tag{5.7}$$

where each τ_e is a stopping time with intensity γ^e_t, such that $\mathbb{Q}(\tau^e \neq \tau^{e'}) = 1$, $e \neq e'$. We denote by $\mathcal{E}$ the powerset of E.

Lemma 5.1 *Given a $\mathcal{G}_\tau \otimes \mathcal{B}(\mathbb{R})$ measurable function $\zeta(\pi)$, if there exists a $\mathcal{P}(\mathbb{G}) \otimes \mathcal{E} \otimes \mathcal{B}(\mathbb{R})$ measurable function $\widetilde{\zeta}^e_t(\pi)$ such that*

$$\mathbb{1}_{\{\tau=\tau_e\}}\zeta(\pi) = \mathbb{1}_{\{\tau=\tau_e\}}\widetilde{\zeta}^e_\tau(\pi), \;\; e \in E, \tag{5.8}$$

then $\widehat{\zeta}$ exists and a version of it is given by $\widehat{\zeta} = J_- \sum_E \frac{\gamma^e}{\gamma}\widetilde{\zeta}^e$ (with $\frac{0}{0} = 0$). In particular, $\gamma = \mathbb{1}_{[0,\tau]}\sum_{e \in E}\gamma^e$.

Proof. For any $\mathcal{G}_{\tau-}$ measurable random variable η, the $\widetilde{\zeta}^e_\tau(\eta)$ are $\mathcal{G}_{\tau-}$ locally integrable, by predictability of the $\widetilde{\zeta}^e_t(\pi)$, so that, by localization, one can assume the $\widetilde{\zeta}^e_\tau(\eta)$ integrable. Then, on $\{\tau < \infty\}$,

$$\mathbb{E}[\zeta(\eta)|\mathcal{G}_{\tau-}] = \mathbb{E}[\textstyle\sum_{e \in E}\mathbb{1}_{\{\tau=\tau^e\}}\widetilde{\zeta}^e_\tau(\eta)|\mathcal{G}_{\tau-}] = \sum_{e \in E}\widetilde{\zeta}^e_\tau(\eta)\mathbb{E}[\mathbb{1}_{\{\tau=\tau^e\}}|\mathcal{G}_{\tau-}].$$

Let $q^e \in \mathcal{P}(\mathbb{G}):\; q^e_\tau\mathbb{1}_{\{\tau<\infty\}} = \mathbb{E}[\mathbb{1}_{\{\tau=\tau^e\}}|\mathcal{G}_{\tau-}]\mathbb{1}_{\{\tau<\infty\}}$, which exists by Corollary 3.23 2) in He et al. (1992). For bounded $Z \in \mathcal{P}(\mathbb{G})$, we compute $\mathbb{E}[Z_\tau\mathbb{1}_{\{\tau=\tau^e<\infty\}}]$ in two ways:

$$\mathbb{E}[Z_\tau\mathbb{1}_{\{\tau=\tau^e<\infty\}}] = \mathbb{E}[Z_\tau q^e_\tau\mathbb{1}_{\{\tau<\infty\}}] = \mathbb{E}[\int_0^\infty Z_s q^e_s\gamma_s ds],$$

and

$$\mathbb{E}[Z_\tau\mathbb{1}_{\{\tau=\tau^e<\infty\}}] = \mathbb{E}[Z_{\tau^e}\mathbb{1}_{\{\tau=\tau^e<\infty\}}] = \mathbb{E}[Z_{\tau^e}\mathbb{1}_{\{\tau^e\leq\tau<\infty\}}] = \mathbb{E}[\int_0^\infty Z_s\mathbb{1}_{\{s\leq\tau\}}\gamma^e_s ds].$$

Hence, $\mathbb{Q}$ almost surely: $q^e_t\gamma_t = \mathbb{1}_{\{t\leq\tau\}}\gamma^e_t$, dt almost surely, so that

$$\mathbb{Q}[q^e_\tau\gamma_\tau \neq \gamma^e_\tau,\ \tau<\infty] = \mathbb{E}[\mathbb{1}_{q^e_\tau\gamma_\tau\neq\gamma^e_\tau,\ \tau<\infty}] = \mathbb{E}[\int_0^\infty \mathbb{1}_{q^e_t\gamma_t\neq\gamma^e_t}\gamma_t dt] = 0.$$

We conclude that on $\{\tau<\infty\}$

$$\gamma_\tau\widehat{\zeta}_\tau(\eta) = \gamma_\tau\mathbb{E}[\zeta(\eta)|\mathcal{G}_{\tau-}] = \sum_{e\in E}\widetilde{\zeta}^e_\tau(\eta)\gamma_\tau q^e_\tau = \sum_{e\in E}\widetilde{\zeta}^e_\tau(\eta)\gamma^e_\tau. \blacksquare$$

Consistent with (5.7), we assume

$$\tau_b = \min_{e\in E_b}\tau_e,\ \ \tau_c = \min_{e\in E_c}\tau_e, \tag{5.9}$$

for some finite sets E_b and E_c such that $E = E_b\cup E_c$ (not necessarily a disjoint union, as will be illustrated in Sect. 7). Let us postulate that

$$\begin{aligned}&\text{For every process } U = P, \mathit{\Delta}, Q, C \text{ and } \mathfrak{C}, \text{there exists a } \mathcal{P}(\mathbb{G})\otimes\mathcal{E} \text{ measurable function}\\ &\widetilde{U}^e_t \text{ such that } U_\tau = \widetilde{U}^e_\tau \text{ on the event } \{\tau=\tau_e\}.\end{aligned} \tag{5.10}$$

Then, in view of (5.5) and (5.2), ξ satisfies (5.8), for

$$\begin{aligned}\widetilde{\xi}^e_t(\pi) = \widetilde{P}^e_t - \widetilde{Q}^e_t + \mathbb{1}_{e\in E_c}(1-R_c)(\widetilde{Q}^e_t + \widetilde{\mathit{\Delta}}^e_t - \widetilde{C}^e_t)^+\\ - \mathbb{1}_{e\in E_b}\big((1-R_b)(\widetilde{Q}^e_t + \widetilde{\mathit{\Delta}}^e_t - \widetilde{\mathfrak{C}}^e_t)^- + (1-\bar{R}_b)(\pi - \mathfrak{C}_{t-})^+\big).\end{aligned} \tag{5.11}$$

Thus, by Lemma 5.1, $\widehat{f}_t(\vartheta)$ in (3.1) can be taken as (on $[0,\bar{\tau}]$)

$$\widehat{f}_t(\vartheta) = g_t(P_t-\vartheta) + \sum_E \gamma^e_t\widetilde{\xi}^e_t(P_t-\vartheta) - \widetilde{r}_t\vartheta, \tag{5.12}$$

where $\widetilde{r}_t = r_t + \gamma_t$, so that

$$\begin{aligned}\widehat{f}_t(\vartheta) + r_t\vartheta = \quad &(1-R_c)\sum_{e\in E_c}\gamma^e_t\left(\widetilde{Q}^e_t + \widetilde{\mathit{\Delta}}^e_t - \widetilde{C}^e_t\right)^+\\ &-(1-R_b)\sum_{e\in E_b}\gamma^e_t\left(\widetilde{Q}^e_t + \widetilde{\mathit{\Delta}}^e_t - \widetilde{\mathfrak{C}}^e_t\right)^-\\ &+ c_t(\mathfrak{C}_t + N_t) + \widetilde{\lambda}_t\,(P_t-\vartheta-\mathfrak{C}_t)^+ - \lambda_t\,(P_t-\vartheta-\mathfrak{C}_t)^-\\ &+\sum_{e\in E}\gamma^e_t\left(\widetilde{P}^e_t - \vartheta - \widetilde{Q}^e_t\right),\end{aligned} \tag{5.13}$$

where $\widetilde{\lambda}_t = \bar{\lambda}_t - (1-\bar{R}_b)\sum_{e\in E_b}\gamma^e_t$ can be interpreted as a liquidity borrowing spread for the bank, net of its credit spread. From the perspective of the bank, the four terms (lines) in the decomposition (5.13) of the TVA coefficient $\widehat{f}_t(\vartheta)$ (up to the $r_t\vartheta$ discount term at the OIS rate r_t) can respectively be interpreted as a costly credit valuation adjustment (CVA coefficient), a beneficial debit valuation adjustment (DVA coefficient), a funding liquidity valuation adjustment (LVA coefficient) and a replacement cost/benefit (RC coefficient). The positive (respectively negative) TVA terms can be considered as deal adverse (respectively deal friendly) as they increase (respectively decrease) the TVA $\mathit{\Theta}$ and therefore decrease (respectively increase) the cost of the hedge $\mathit{\Pi} = P - \mathit{\Theta}$ for the bank—with, depending on the sign of $\mathit{\Pi}$, a “less positive” $\mathit{\Pi}$ interpreted as a lower buyer price by the bank or a “more negative” $\mathit{\Pi}$ interpreted as a higher seller price by the bank.

Remark 5.1 The materiality of a debit benefit at own default (DVA proportional to $(1-R_b)$) or of a funding benefit at own default (proportional to $(1-\bar{R}_b)$) is clearly subject to caution, unless a corresponding hedge allows the bank to monetize these before its default. Otherwise, the bank should better set the recovery rates R_b and $\bar{R}_b$ equal to one in the equations, in order to avoid reckoning such “fake benefits” (or, at least, benefits to senior bondholders only, whereas a sound management should only consider the interest of the shareholders; see Albanese, Brigo, and Oertel (2013) and Albanese and Iabichino (2013)).

5.1 Gap Risk

As illustrated by the bailout of AIG on 16 September 2008, largely triggered by increasing margin calls on sell-protection CDS positions (on the distressed Lehman, in particular), it is important to use an accurate model of the collateral (processes M, N and $\mathfrak{N}$). An extreme form of variation margining would correspond to $M = Q$ at all times. However, accounting for various frictions and delays regarding formation (Sect. 5.1.1) and delivery (Sect. 5.1.2) of the collateral, there is gap risk, i.e. risk of a residual gap between the variation margin M and the debt of a defaulting party at the time of liquidation, which is the motivation for the initial margins.

5.1.1 Collateral Slippage, Thresholds and Minimal Transfer Amounts

In practice, variation margin calls are executed according to a discrete schedule (t_l), based on the following CSA data:

- the thresholds (free credit lines) of the bank and the counterparty : $\varepsilon^b \le 0$ and $\varepsilon^c \ge 0$
- the minimum transfer amounts (MTA) of the bank and the counterparty: $\epsilon^b \le 0$ and $\epsilon^c \ge 0$.

In a realistic variation margin scheme conform to ISDA requirements, M_t tracks the thresholded exposure

$$Q_t^{\varepsilon} = (Q_t - \varepsilon^c)^+ - (Q_t - \varepsilon^b)^- \tag{5.14}$$

through a càdlàg and piecewise-constant process reset at every $t_l < \tau$, unless the corresponding margin adjustment is less than the MTA of the concerned party. Namely, at every $t_l < \tau$,

$$M_{t_l} - M_{t_l-} = \mathbb{1}_{Q^{\varepsilon}_{t_l-} - M_{t_l-} \notin [\epsilon^b, \epsilon^c]} (Q^{\varepsilon}_{t_l-} - M_{t_l-}), \tag{5.15}$$

so that, in particular,

$$M_{t_l} \in [Q_{t_l-} - (\varepsilon^c + \epsilon^c), Q_{t_l-} - (\varepsilon^b + \epsilon^b)]. \tag{5.16}$$

The initial margins N and $\mathfrak{N}$ are also updated at discrete times (of the same grid (t_l) as above for notational simplicity below), based on risk measures (see Brigo and Pallavicini (2014) for concrete specifications) of the profit-and-loss of the position at the horizon of the cure period δ that is dealt with below. Note that discrete-time collateralization induces a path dependence, which, from a computational perspective, implies to add M_t, N_t and $\mathfrak{N}_t$ as extra dimensions to the factor process (see Sect. 7.3).

5.1.2 Cure Period

With a positive cure period δ as in Sect. 4, we need to modify the definition of the debts χ and $\mathfrak{X}$ in (5.2) into (assuming $Q = P$ for simplicity)

$$\chi^\delta = (P^\delta_{\tau^\delta} - C_{\widehat{\tau}})^+, \;\; \mathfrak{X}^\delta = (P^\delta_{\tau^\delta} - \mathfrak{C}_{\widehat{\tau}})^- \tag{5.17}$$

(so that $\chi^\delta \times \mathfrak{X}^\delta \equiv 0$ by nonnegativity of the initial margins), where, for every t, $\widehat{t}$ denotes the greatest t_l less or equal than t. The $\cdot_{\widehat{\tau}}$ in (5.17) reflect the fact that the collateral is frozen during the margin period of risk $(\widehat{\tau}, \bar{\tau}]$. Note that

$$P^\delta_{\tau^\delta} - C_{\widehat{\tau}} = (P_{\widehat{\tau}-} - M_{\widehat{\tau}}) + \int_{[\tau,\tau^\delta]} e^{\int_s^{\tau^\delta} r_u du} dD_s + (P_{\tau^\delta} - P_{\widehat{\tau}-}) - N_{\widehat{\tau}} \tag{5.18}$$

(and likewise for $P^\delta_{\tau^\delta} - \mathfrak{C}_{\widehat{\tau}}$), where $P_{\widehat{\tau}-} - M_{\widehat{\tau}} \in [\varepsilon^b + \epsilon^b, \varepsilon^c + \epsilon^c]$ (by (5.16)) is controlled by the thresholds and the MTAs, but where the "gaps" $\beta^{-1}_{\tau^\delta} \int_{[\tau,\tau^\delta]} \beta_s dD_s$ (including $\Delta_\tau = D_\tau - D_{\tau-}$) and $(P_{\tau^\delta} - P_{\widehat{\tau}-})$ can be quite substantial, hence the need for the initial margin N. This results in the following expressions for the terminal cash-flow $\mathfrak{R}^\delta(\pi)$ and the exposure $\xi^\delta(\pi) = P^\delta_{\tau^\delta} - \mathfrak{R}^\delta(\pi)$ (cf. (5.1) through (5.4)):

$$\mathfrak{R}^\delta(\pi) = \mathfrak{R}^\delta_c + \mathbb{1}_{\tau_b \le \tau_c^\delta} \mathfrak{R}_f(\pi),$$

where

$$\mathfrak{R}_c = \begin{cases} C_{\widehat{\tau}} + R_c \chi^\delta \text{ if } \chi^\delta > 0 \text{ and } \tau_c \le \tau_b^\delta, \\ \mathfrak{C}_{\widehat{\tau}} - R_b \mathfrak{X}^\delta \text{ if } \mathfrak{X}^\delta > 0 \text{ and } \tau_b \le \tau_c^\delta, \\ \quad P^\delta_\tau + \Delta_\tau \text{ otherwise} \end{cases}$$

and therefore, consistent with (4.2):

$$\xi^\delta(\pi) = P^\delta_{\tau^\delta} - \mathfrak{R}^\delta(\pi) = \xi_c - \Lambda(\pi - \mathfrak{C}_\star)^+,$$

for the random variables ξ_c, Λ and $\mathfrak{C}_\star$ given as

$$\xi_c = \mathbb{1}_{\tau_c \le \tau_b^\delta}(1 - R_c)\chi^\delta - \mathbb{1}_{\tau_b \le \tau_c^\delta}(1 - R_b)\mathfrak{X}^\delta, \;\; \Lambda = \mathbb{1}_{\tau_b \le \tau_c^\delta}(1 - \bar{R}_b), \;\; \mathfrak{C}_\star = \mathfrak{C}_{\widehat{\tau}}. \tag{5.19}$$

Remark 5.2 The resulting expression of ξ_c is consistent with the equation (14) in Brigo and Pallavicini (2014) (note that they don't consider the possibility of a funding benefit at own default, i.e. $\bar{R}_b = 1$ and thus $\Lambda = 0$ in their setup). According to them, to render the case of a cleared transaction where one of the parties is a clearing member of a clearinghouse (here or already in (5.4) in case $\delta = 0$), it is sufficient to set the corresponding recoveries and initial margin equal to one and zero, respectively (as clearing members of a clearinghouse are backed-up by others if they default and don't post initial margins).

Lemma 5.2 *If P^δ cannot jump at τ^δ (as for typical contracts in the models of Sect. 6 and 7; see before Lemma 7.4), then ξ_c, Λ and $\mathfrak{C}_\star$ are $\mathcal{G}_{\tau^\delta-}$ measurable. Assuming $\bar{R}_b \neq 0$, the conditions of Lemma* (4.1) *hold and $\xi^\delta_\star$ in* (4.5) *can be rewritten as*

$$\xi^\delta_\star = \begin{cases} \bar{R}_b^{-1}\xi_c + (1 - \bar{R}_b^{-1})(P^\delta_{\tau^\delta-} - \mathfrak{C}_{\widehat{\tau}}) \text{ on } \{\tau_b \le \tau_c^\delta\} \cap \{P^\delta_{\tau^\delta-} \ge \mathfrak{C}_{\widehat{\tau}}\} \\ \xi_c \text{ otherwise.} \end{cases} \tag{5.20}$$

Proof. (5.20) follows from (4.5) and (5.19). Regarding the $\mathcal{G}_{\tau^\delta-}$ measurabilities, it suffices to check that $\{\tau_c \le \tau_b^\delta\}$ (and, likewise, $\{\tau_b \le \tau_c^\delta\}$) is $\mathcal{G}_{\tau^\delta-}$ measurable, which holds since

$$\{\tau_c \le \tau_b^\delta\} \cap \{\tau_c \le \tau_b\} = \{\tau_c \le \tau_b\} \in \mathcal{G}_\tau \subseteq \mathcal{G}_{\tau^\delta-}$$

and

$$\{\tau_c \le \tau_b^\delta\} \cap \{\tau_c > \tau_b\} = \{\tau_c \le \tau^\delta\} \cap \{\tau_c > \tau\} = \{\tau_c < \tau^\delta\} \cap \{\tau_c > \tau\} \in \mathcal{G}_{\tau^\delta-},$$

where the last equality follows by avoidance between predictable and totally inaccessible stopping times. ∎

Remark 5.3 The above collateralization scheme finely renders the path dependence of the margins. It does not include other gap risk features, such as a possible jump of the collateral at the default of a party (e.g. in case of a collateral posted in a currency strongly dependent on the credit of a party; cf. Ehlers and Schönbucher (2006)). However, the general framework (5.10) does allow for such features if wanted, through $\widetilde{C}^e$ and/or $\widetilde{\mathfrak{C}}^e$ in (5.10) that would effectively depend on e—whereas they do not in the above specification.

In the remaining sections, we apply the above approach to deal with the counterparty risk of credit derivatives in the dynamic copula models of Crépey, Jeanblanc, and Wu (2013) (DGC for dynamic Gaussian copula) or Bielecki, Cousin, Crépey, and Herbertsson (2014b,2014a)) (common-shock model or DMO for dynamic Marshall-Olkin copula). In each case, we dynamize a copula model of portfolio credit risk by the introduction of a suitable filtration.

For any Euclidean vector $\mathbf{k} = (k_e)$ indexed by marks e, we denote by $\mathbf{k}^e$ (respectively $\mathbf{k}^{e,t}$) the vector obtained from $\mathbf{k}$ by replacing the component with index e by 1 (respectively t). MRP refers to a local martingale predictable representation property. The optional splitting formula is Theorem 6.9 in Song (2013b), which holds in any density or immersion model of portfolio credit risk (such as the DGC and the DMO model, respectively, as will be seen below).

6 Dynamic Gaussian Copula TVA Model

6.1 Model of Default Times

Let $N = \{-1, 0, 1, \dots, n\}$, $N^\star = \{1, \dots, n\}$. We consider a multivariate Brownian motion $\mathbf{B} = (B^i)_{i \in N}$ with pairwise correlation $\varrho \geq 0$ in its own completed filtration $\mathbb{B} = (\mathcal{B})_{t \geq 0}$. For any $i \in N$, let h_i be a continuously differentiable increasing function from $\mathbb{R}_+^*$ to $\mathbb{R}$, with $\lim_0 h_i(s) = -\infty$ and $\lim_{+\infty} h_i(s) = +\infty$, and let

$$\tau_i = h_i^{-1}\Big(\int_0^{+\infty} \varsigma(u) dB_u^i\Big), \tag{6.1}$$

where $\varsigma(\cdot)$ is a square integrable function with unit L^2 norm. We set $\tau_b = \tau_{-1}$ $\tau_c = \tau_0$, whilst the $(\tau_i)_{i \in N^\star}$ represent the default times of n reference credit names. Thus the $(\tau_i)_{i \in N}$ follow the standard Gaussian copula model of Li (2000), with correlation parameter ϱ and with marginal survival function $\Phi \circ h_i$ of τ_i, where Φ is the standard normal survival function. In order to make the model dynamic as required by counterparty risk applications, we introduce the model filtration $\mathbb{G}$ given as the Brownian filtration $\mathbb{B}$ progressively enlarged by the τ_i (augmented so as to satisfy the usual conditions, as is standard). We write, for every $i \in N$,

$$m_t^i = \int_0^t \varsigma(u) dB_u^i, \;\; k_t^i = \tau_i \mathbb{1}_{\{\tau_i \leq t\}}$$

and $\mathbf{m}_t = (m_t^i)_{i \in N}, \;\; \mathbf{k}_t = (k_t^i)_{i \in N}, \;\; X_t = (\mathbf{m}_t, \mathbf{k}_t)$.

Lemma 6.1 *For every $t \geq 0$,*

$$\mathcal{G}_t = \mathcal{B}_t \vee \bigvee_{i \in N} \big(\sigma(\tau_i \wedge t) \vee \sigma(\{\tau_i > t\})\big). \tag{6.2}$$

There exist processes β_t^i and γ_t^i of the form

$$\beta_t^i, \gamma_t^i = \beta_i, \gamma_i\,(t, \mathbf{m}_t, \mathbf{k}_t), \tag{6.3}$$

such that $\mathbb{G}$ Brownian motions $dW_t^i = dB_t^i - \beta_t^i dt$ and $(\mathbb{G}, \mathbb{Q})$ compensated default indicator processes $dM_t^i = d\mathbb{1}_{\tau_i \leq t} - \gamma_t^i dt$, $i \in N$, have the $(\mathbb{G}, \mathbb{Q})$ MRP. The process $X = (\mathbf{m}, \mathbf{k})$ is a $(\mathbb{G}, \mathbb{Q})$ jump-diffusion.

Proof. For the Markov and martingale (other than representation) properties, see Crépey et al. (2013). The expression of the γ^i is given in (6.16), whilst the β^i can be computed by making use of the probability measure $\mathbb{Q}^\star$ such that $\frac{d\mathbb{Q}^\star}{d\mathbb{Q}} \propto p_T(\tau_i, \, i \in N)$, where p is the conditional density of the τ_i in (6.14). More precisely, all the τ_i are $\mathbb{Q}^\star$ independent between them and from $\mathcal{B}_T$ (cf. Theorem 4.7 in Song (2013a)), so that the B^i are $(\mathbb{G}, \mathbb{Q}^\star)$ Brownian motions. Hence, their $(\mathbb{G}, \mathbb{Q})$ drifts β^i can be obtained by application of a Girsanov formula from $(\mathbb{G}, \mathbb{Q}^\star)$ to $(\mathbb{G}, \mathbb{Q})$, which reveals the functional form claimed in (6.3). The MRP and (6.2) are proved by induction over the cardinality of N as follows. We write $\mathbb{G} = \mathbb{G}^N$. If N is reduced to a singleton, then the density property of τ given $\mathcal{B}_t$ implies the results, by the optional splitting formula for (6.2) and by Theorem 6.4 in Jeanblanc and Song (2013) for the MRP. If one adds a new name, say $(n+1)$, to N, then the density properties of τ_{n+1} and of $(\tau_i)_{i \in N}$ given $\mathcal{B}_t$ imply the density property of τ_{n+1} given $\mathcal{B}_t \bigvee \vee_{i \in N} \big(\sigma(\tau_i \wedge t) \vee \sigma(\{\tau_i > t\})\big)$. Hence, the results for $\mathbb{G}^{N \cup \{n+1\}}$ follow as above from those, if assumed, for $\mathbb{G}^N$. ∎

6.2 TVA Model

A DGC setup can be used as a TVA model for credit derivatives, with mark i and $E_b = \{-1\}, \;\; E_c = \{0\}$ in (5.9). Since there are no joint defaults in this model, it is harmless to assume that the contract promises no cash-flow at τ, $\Delta_\tau = 0$, so that (cf. (5.2))

$$\chi = (Q_\tau - C_\tau)^+, \;\; \mathfrak{X} = (Q_\tau - \mathfrak{C}_\tau)^-.$$

We assume that for every process $U = P$, Q, C and $\mathfrak{C}$, there exists a continuous function $\widetilde{U}_i$ such that

$$U_\tau = \widetilde{U}_i(\tau, \mathbf{m}_\tau, \mathbf{k}_{\tau-}) \tag{6.4}$$

($=\widetilde{U}^i_\tau$ for brevity) on every event of the form $\{\tau=\tau_i\}$, $i=-1,0$. The results of Crépey et al. (2013) show that the condition (6.4) holds regarding $U=P$ (for every $i\in N$) for vanilla credit derivatives, including CDS contracts and CDO tranches, with semi-explicit formulas for P. The conditions (6.4) regarding $U=Q$, C and $\mathfrak{C}$ may be satisfied or not depending on the CSA (see e.g. Sect. 5.1 regarding C and $\mathfrak{C}$). By (5.11) and (5.13),

$$\begin{aligned}
\widetilde{\xi}^i_t(\pi) &= \widetilde{P}^i_t - \widetilde{Q}^i_t + \mathbb{1}_{i=0}(1-R_c)(\widetilde{Q}^i_t - \widetilde{C}_t)^+ \\
&\quad - \mathbb{1}_{i=-1}\big((1-R_b)(\widetilde{Q}^i_t - \widetilde{\mathfrak{C}}_t)^- + (1-\bar{R}_b)(\pi - \mathfrak{C}_{t-})^+\big),\;\; i=-1,0,\\
\widehat{f}_t(\vartheta) + r_t\vartheta &= \quad (1-R_c)\gamma^0_t\big(\widetilde{Q}^0_t - \widetilde{C}^0_t\big)^+ - (1-R_b)\gamma^{-1}_t\big(\widetilde{Q}^{-1}_t - \widetilde{\mathfrak{C}}^{-1}_t\big)^- \\
&\quad + c_t(\mathfrak{C}_t + N_t) + \widetilde{\lambda}_t\,(P_t - \vartheta - \mathfrak{C}_t)^+ - \lambda_t\,(P_t-\vartheta-\mathfrak{C}_t)^- + \sum_{i=-1,0}\gamma^i_t\left(\widetilde{P}^i_t - \vartheta - \widetilde{Q}^i_t\right),\;\; t\in[0,\bar{\tau}],
\end{aligned}$$

where $\widetilde{\lambda}_t = \bar{\lambda}_t - (1-\bar{R}_b)\gamma^{-1}_t$. We assume that the processes $r, c, \lambda, \bar{\lambda}, P$, $\mathfrak{C}$ and N are given before τ as continuous functions of $(t,\widetilde{X}_t)$, where $\widetilde{X}_t = (\mathbf{m}_t, \widetilde{\mathbf{k}}_t)$ with $\widetilde{\mathbf{k}}_t = (k^i_t)_{i\in N^\star}$.

Lemma 6.2 *The condition (C) holds with, writing $\mathbf{k}=(0,0,\widetilde{\mathbf{k}})$ for every $\widetilde{\mathbf{k}}\in\mathbb{R}^n_+$:*

(DGC.1) *a reference filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ in (C.1) given as $\mathbb{B}$ progressively enlarged by the default times of the reference names, which satisfies*

$$\mathcal{F}_t = \mathcal{B}_t \vee \bigvee_{i\in N^\star}\big(\sigma(\tau_i\wedge t)\vee\sigma(\{\tau_i>t\})\big),\;\; t\geq 0, \tag{6.5}$$

(DGC.2) *a changed measure $\mathbb{P}$ ($\neq\mathbb{Q}$, so that this is a case of "no immersion" in the sense of the comments following the statement of (C)) equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ and such that a family of $(\mathbb{F},\mathbb{P})$ martingales with the MRP is given by the*

$$d\widetilde{W}^i_t = dB^i_t - \widetilde{\beta}^i_t dt,\; i\in N \text{ and } d\widetilde{M}^i_t = d\mathbb{1}_{\tau_i\leq t} - \widetilde{\gamma}^i_t dt,\; i\in N^\star, \tag{6.6}$$

where

$$\widetilde{\beta}^i_t = \widetilde{\beta}_i(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t) := \beta_i(t,\mathbf{m}_t,\mathbf{k}_t),\;\; \widetilde{\gamma}^i_t = \widetilde{\gamma}_i(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t) := \gamma_i(t,\mathbf{m}_t,\mathbf{k}_t), \tag{6.7}$$

(DGC.3) *a Markov specification $\widetilde{f}_t(\vartheta) = \widetilde{f}(t,\widetilde{X}_t,\vartheta)$ in (C.3), for the function $\widetilde{f}=\widetilde{f}(t,\widetilde{x},\vartheta)$ given, writing $\widetilde{x}=(\mathbf{m},\widetilde{\mathbf{k}})$ and $x=(\mathbf{m},\mathbf{k})$, by*

$$\begin{aligned}
\widetilde{f}(t,\widetilde{x},\vartheta) + r(t,x)\vartheta &= (1-R_c)\gamma_0\left(\widetilde{Q}_0 - \widetilde{C}_0\right)^+(t,x) - (1-R_b)\gamma_{-1}\left(\widetilde{Q}_{-1} - \widetilde{\mathfrak{C}}_{-1}\right)^-(t,x)\\
&\quad + \left(c(\mathfrak{C}+N) + \widetilde{\lambda}\,(P-\vartheta-\mathfrak{C})^+ - \lambda\,(P-\vartheta-\mathfrak{C})^-\right)(t,\widetilde{x}) + \sum_{i=-1,0}\gamma_i\left(\widetilde{P}_i - \vartheta - \widetilde{Q}_i\right)(t,x),
\end{aligned} \tag{6.8}$$

where $\widetilde{\lambda} = \bar{\lambda} - (1-\bar{R}_b)\gamma_{-1}$, and for the $(\mathbb{F},\mathbb{P})$ jump-diffusion $\widetilde{X}_t = (\mathbf{m}_t,\widetilde{\mathbf{k}}_t)$.

Proof. (DGC.1) and (DGC.3) can be proven as Lemma 6.1 above. (DGC.2) is proven in Sect. 6.3 (see Theorem 6.1). Note that (DGC.2) implies (C.2) (so the condition (C) holds) via the included $(\mathbb{F},\mathbb{P})$ MRP, where each of the $(\mathbb{F},\mathbb{P})$ fundamental martingales in (6.6) stopped at $\tau-$ (or, equivalently by avoidance in this model, at τ) is a $(\mathbb{G},\mathbb{Q})$ local martingale. ∎

Since (C) is satisfied, we can state the following specification of Proposition 3.1, where the shape (6.10) of $\widetilde{\mu}$ reflects the Itô-Markov local martingale part formula (3.6) that applies to the $(\mathbb{F},\mathbb{P})$ jump-diffusion $\widetilde{X}_t = (\mathbf{m}_t,\widetilde{\mathbf{k}}_t)$, and where there is no need to stop $\widetilde{\mu}$ at $\tau-$ in (6.11) since $\widetilde{\mu}$ does not jump at τ.

Proposition 6.1 *We obtain a solution Θ to the full TVA BSDE (2.3) by setting $\Theta=\widetilde{\Theta}$ on $[0,\bar{\tau})$ and $\Theta_{\bar{\tau}} = \mathbb{1}_{\tau<T}\xi$, provided $\widetilde{\Theta}_t = \widetilde{\Theta}(t,\widetilde{X}_t)$ is such that $\mathbb{1}_{\tau<T}\xi_\star J$ (with $\xi_\star = \xi(P_{\tau-}-\widetilde{\Theta}_{\tau-})$) has locally integrable total variation, $\widetilde{\Theta}_T=0$ and, for $t\in[0,T]$,*

$$-d\widetilde{\Theta}_t = \widetilde{f}(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t,\widetilde{\Theta}_t)dt - d\widetilde{\mu}_t, \tag{6.9}$$

where

$$d\widetilde{\mu}_t = \varsigma(t)\sum_{i\in N}\partial_{m_i}\widetilde{\Theta}(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_t)d\widetilde{W}^i_t + \sum_{i\in N^\star}\delta_i\widetilde{\Theta}(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_{t-})d\widetilde{M}^i_t. \tag{6.10}$$

On $[0,\bar{\tau}]$, *writing* $\widetilde{\xi}^{\star,i}_t = \widetilde{\xi}^i_t(P_t-\widetilde{\Theta}_t)$,

$$d\mu_t = d\widetilde{\mu}_t - \left((\xi_\star - \widetilde{\Theta}_{t-})dJ_t + \sum_{i=-1,0}\gamma^i_t(\widetilde{\xi}^{\star,i}_t - \widetilde{\Theta}_t)dt\right). \blacksquare \tag{6.11}$$

6.3 Proof of (DGC.2)

We use the notation $\centerdot$ for stochastic integration and $\boldsymbol{\lambda}$ for the Lebesgue measure. For $i\in N^\star$, let $\overline{\gamma}^i$ be the $(\mathbb{F},\mathbb{Q})$ intensity of τ_i and let $\overline{M}^i = H^i - \overline{\gamma}^i \centerdot \boldsymbol{\lambda}$. Let $S_t = \mathbb{Q}(\tau > t\,|\,\mathcal{F}_t)$ denote the Azéma supermartingale of τ, with continuous local martingale part Q^c. We consider the $(\mathbb{F},\mathbb{Q})$ exponential martingale $\mathcal{E}(\nu)$, where $\nu = \mathbb{1}_{(0,T]}\frac{1}{S}\centerdot Q^c + \sum_{i\in N^\star}\mathbb{1}_{(0,T]}(\frac{\widetilde{\gamma}^i}{\overline{\gamma}^i}-1)\centerdot\overline{M}^i$, in which $\widetilde{\gamma}^i$, defined through γ^i by (6.6), is positive until τ_i, by (6.16). Hence

$$\mathcal{E}(\nu) = \mathcal{E}\left(\mathbb{1}_{(0,T]}\nu^c\right)\prod_{i\in N^\star}\left(1+(\frac{\widetilde{\gamma}^i_{\tau_i}}{\overline{\gamma}^i_{\tau_i}}-1)H^i_{\cdot\wedge T}\right)\exp\int_0^{\cdot\wedge\tau_i\wedge T}\left(\overline{\gamma}^i_s - \widetilde{\gamma}^i_s\right)ds > 0, \tag{6.12}$$

where $\nu^c = \frac{1}{S}\centerdot Q^c$. Also, $S>0$ by (6.15), so that:

Lemma 6.3 *Two* $\mathbb{F}$ *predictable processes* α *and* $\widetilde{\alpha}$ *undistinguishable until* τ *are undistinguishable on* $\mathbb{R}_+$.

Proof. Otherwise, the optional section theorem would imply the existence of an $\mathbb{F}$ stopping time κ with indicator process K such that $\mathbb{E}[\mathbb{1}_{\alpha_\kappa\neq\widetilde{\alpha}_\kappa}S_\kappa]\neq 0$ (since also $S>0$), in contradiction with

$$\mathbb{E}\left[\mathbb{1}_{\alpha_\kappa\neq\widetilde{\alpha}_\kappa}S_\kappa\right] = \mathbb{E}\left[\mathbb{1}_{\alpha\neq\widetilde{\alpha}}S\centerdot K\right] = \mathbb{E}\left[\mathbb{1}_{\alpha\neq\widetilde{\alpha}}J\centerdot K\right] = 0,$$

where Theorems 5.4 and 5.16 1) in He et al. (1992) were used in the next-to-last equality. $\blacksquare$

Theorem 6.1 $\mathcal{E}(\nu)$ *is an* $(\mathbb{F},\mathbb{Q})$ *martingale. The probability measure* $\mathbb{P}$ *with* $\mathbb{Q}$ *density process* $\mathcal{E}(\nu)$ *satisfies the condition (DGC.2).*

Proof. We claim the following property, which is proved after (cf. Lemma 6.4):

$$\text{There exists } \epsilon>0 \text{ such that, for any } s\in[0,T]\text{: } \mathbb{E}\big[\mathcal{E}\big(\mathbb{1}_{(s,t]}\nu\big)\,\big|\,\mathcal{F}_s\big]=1,\ \ t\in[s,s+\epsilon]. \tag{6.13}$$

If $T\le\epsilon$, the first part of the theorem follows from the above claim. Otherwise, we write

$$\mathbb{E}\mathcal{E}(\nu) = \mathbb{E}\left[\mathcal{E}\left(\mathbb{1}_{(0,T-\epsilon]}\nu\right)\mathbb{E}\left[\mathcal{E}\left(\mathbb{1}_{(T-\epsilon,T]}\nu\right)\,\big|\,\mathcal{F}_{T-\epsilon}\right]\right] = \mathbb{E}\left[\mathcal{E}\left(\mathbb{1}_{(0,T-\epsilon]}\nu\right)\right],$$

so that the first part of the theorem follows by induction. The equivalence of $\mathbb{P}$ and $\mathbb{Q}$ on $\mathcal{F}_T$ follows by positivity of $\mathcal{E}(\nu)$ in (6.12). By Girsanov's theorem, the $(\mathbb{F},\mathbb{P})$ intensity of τ_i is $\widetilde{\gamma}^i$, $i\in N^\star$, and, denoting by $\overline{W}^i = B^i - \overline{\beta}^i\centerdot\boldsymbol{\lambda}$ the $(\mathbb{F},\mathbb{Q})$ local martingale part (Brownian motion) of B^i, $i\in N$, the process

$$\widehat{W}^i = \overline{W}^i - \langle\nu^c,B^i\rangle = B^i - (\overline{\beta}^i\centerdot\boldsymbol{\lambda} + \langle\nu^c,B^i\rangle)$$

is an $(\mathbb{F},\mathbb{P})$ Brownian motion. Moreover, by the Jeulin formula (see e.g. no 77 Remarques b) in Dellacherie, Maisonneuve, and Meyer (1992)), $\widehat{W}^i$ is a $(\mathbb{G},\mathbb{Q})$ Brownian motion until time τ, as is also $\widetilde{W}^i = B^i - \widetilde{\beta}^i\centerdot\boldsymbol{\lambda}$ in (6.6) (since $\widetilde{W}^i_{\cdot\wedge\tau} = W^i_{\cdot\wedge\tau}$). Therefore, the $\mathbb{F}$ predictable processes $\overline{\beta}^i\centerdot\boldsymbol{\lambda}+\langle\nu^c,B^i\rangle$ and $\widetilde{\beta}^i\centerdot\boldsymbol{\lambda}$ coincide until τ, hence on $[0,T]$ by Lemma 6.3. In conclusion, $\widetilde{W}^i = \widehat{W}^i$, an $(\mathbb{F},\mathbb{P})$ Brownian motion. Finally, the $(\mathbb{F},\mathbb{P})$ MRP of the $\widetilde{W}^i, i\in N$ and $\widetilde{M}^i, i\in N^\star$ follows by equivalent change of measure from the $(\mathbb{F},\mathbb{Q})$ MRP of the $\overline{W}^i, i\in N$ and $\overline{M}^i, i\in N^\star$, which can be proven as the $(\mathbb{G},\mathbb{Q})$ MRP of the $W^i, i\in N$ and $M^i, i\in N$ in Lemma 6.1. $\blacksquare$

From now on in this section we prove the claim used in the proof of Theorem 6.1. For notational simplicity, we only prove it for $s=0$, i.e.:

Lemma 6.4 *For* t *small enough,* $\mathcal{E}(\nu)$ *is an* $(\mathbb{F},\mathbb{Q})$ *martingale on* $[0,t]$.

6.3.1 Explicit Formulas

Let $f^2(t) = \int_t^{+\infty} \varsigma^2(v)dv$, assumed positive for all t for simplicity of presentation. Denoting by I and J generic subsets of N (representing sets of defaulted and alive obligors in the financial interpretation), we write:

$$
\begin{aligned}
&\mathbb{B}^I = (\mathcal{B}_t^I)_{t\geq 0} \text{ with } \mathcal{B}_t^I = \mathcal{B}_t \vee \bigvee_{i\in I} \tau_i,\\
&\rho^I = \frac{\varrho}{|I|\varrho + 1}, \quad (\sigma^I)^2 = \frac{(|I|-1)\varrho + 1 - \varrho^2|I|}{(|I|-1)\varrho+1}, \quad \alpha^I = \frac{\varrho}{(|I|-1)\varrho+1},\\
&Z_t^{j,I}(u) = \frac{h_j(u) - m_t^j}{f(t)} - \alpha^I \sum_{i\in I} \frac{h_i(\tau_i) - m_t^i}{f(t)},\\
&d\zeta_t^{j,I}, \text{ the } (\mathbb{F},\mathbb{Q}) \text{ local martingale part of } \left(-\frac{1}{f(t)}dm_t^j + \alpha^I \sum_{i\in I} \frac{1}{f(t)} dm_t^i\right),\\
&\Phi_{\rho,\sigma}(\mathbf{z}) = \mathbb{Q}(Z_j > z_j, j\in J), \quad \psi^j_{\rho,\sigma}(\mathbf{z}) = -\frac{\partial_{z_j}\Phi_{\rho,\sigma}}{\Phi_{\rho,\sigma}}(\mathbf{z}),
\end{aligned}
$$

where $\mathbf{z} = (z_j)_{j\in J}$, whilst $\mathbf{Z} = (Z_j)_{j\in J}$ follows a $|J|$-dimensional centered Gaussian vector with homogenous marginal variances σ^2 and pairwise correlations ρ. In addition, we define

$$
\begin{aligned}
&\rho_t, \sigma_t, \kappa_t = \rho^I, \sigma^I, \alpha^I, \ Z_t^j = Z_t^{j,I}(t) \text{ on } \{I = \mathrm{supp}(\mathbf{k}_t)\},\\
&\mathcal{I}_t^\star = \mathcal{B}_t^I, \ \rho_t^\star, \sigma_t^\star, \alpha_t^\star = \rho^I, \sigma^I, \alpha^I, \ Z_t^{\star,j} = Z_t^{j,I}(t), \ d\zeta_t^j = d\zeta_t^{j,I} \text{ on } \{I = \mathrm{supp}(\widetilde{\mathbf{k}}_t)\},
\end{aligned}
$$

where $\mathrm{supp}(\mathbf{k}_t)$ and $\mathrm{supp}(\widetilde{\mathbf{k}}_t)$ respectively correspond to the defaulted obligors in N and in $N^\star$. Let also

$$
\begin{aligned}
&\mathfrak{J}_t = N \setminus \mathrm{supp}(\mathbf{k}_t), \ \mathfrak{J}_t^\star = N^\star \setminus \mathrm{supp}(\widetilde{\mathbf{k}}_t), \ \widetilde{\mathfrak{J}}_t = \{-1, 0\} \cup \mathfrak{J}_t^\star,\\
&\mathcal{Z}_t = \big(Z_t^{\star,j}, \, j\in\mathfrak{J}_t\big), \ \mathcal{Z}_t^\star = \big(Z_t^{\star,j}, \, j\in\mathfrak{J}_t^\star\big), \ \widetilde{\mathcal{Z}}_t = \big(Z_t^{\star,-1}, Z_t^{\star,0}; Z_t^{\star,j}, \, j\in\mathfrak{J}_t^*\big)
\end{aligned}
$$

and $p_t(t_i, i\in N) = \partial_{t_1}\ldots\partial_{t_n}\mathbb{Q}(\tau_i < t_i, \, i\in N \,|\, \mathcal{B}_t)$, the conditional Lebesgue density of the τ_i, $i\in N$.

Lemma 6.5 *For any nonnegative t and t_i, $i\in N$,*

$$
p_t(t_i, \, i\in N) = \int_{\mathbb{R}} \phi(y) \prod_{i\in N} \phi\Big(\frac{h_i(t_i) - m_t^i + f(t)\sqrt{\varrho}y}{f(t)\sqrt{1-\varrho}}\Big) \frac{h_i'(t_i)}{f(s)\sqrt{1-\varrho}} dy, \tag{6.14}
$$

$$
S_t = \frac{\Phi_{\rho_t^\star,\sigma_t^\star}\big(\widetilde{\mathcal{Z}}_t\big)}{\Phi_{\rho_t^\star,\sigma_t^\star}\big(\mathcal{Z}_t^\star\big)} > 0\,, \ \ d\nu_t^c = \sum_{j\in\widetilde{\mathfrak{J}}_{t-}} \psi^j_{\rho_t^\star,\sigma_t^\star}\big(\widetilde{\mathcal{Z}}_t\big) d\zeta_t^j - \sum_{j\in\mathfrak{J}^\star_{t-}} \psi^j_{\rho_t^\star,\sigma_t^\star}\big(\mathcal{Z}_t^\star\big) d\zeta_t^j. \tag{6.15}
$$

For any nonnegative t and $j\in N$, respectively $N^\star$,

$$
\gamma_t^j = \mathbb{1}_{\{\tau_j\geq t\}}\Big(\frac{h_j'}{f}\Big)(t)\psi^j_{\rho_t,\sigma_t}\big(\mathcal{Z}_t\big) = \gamma_j(t, \mathbf{m}_t, \mathbf{k}_t), \tag{6.16}
$$

respectively

$$
\overline{\gamma}_t^j = \mathbb{1}_{\{\tau_j\geq t\}}\Big(\frac{h_j'}{f}\Big)(t)\psi^j_{\rho_t^\star,\sigma_t^\star}\big(\mathcal{Z}_t^\star\big) = \overline{\gamma}_j(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_t). \tag{6.17}
$$

Proof. The expression for the conditional density p of the τ_i is obtained by differentiation of their conditional survival function given by the last line page 3 in Crépey et al. (2013). The expression for S_t, which implies the one for $d\nu_t^c$, results from the following "multiname key lemma formula" valid in any density model of default times (as can be established by optional splitting):

$$
S_t = \frac{\mathbb{Q}(\tau > t; \tau_j > t, \ j\in\mathfrak{J}_t^\star \,|\, \mathcal{I}_t^\star)}{\mathbb{Q}(\tau_j > t, \ j\in\mathfrak{J}_t^\star \,|\, \mathcal{I}_t^\star)}.
$$

Finally, (6.16) and (6.17) result from the following Laplace formulas that can also be established by optional splitting:

$$\gamma_t^j = -\mathbb{1}_{\{\tau_j \geq t\}} \frac{\partial_u \Phi_{\rho_t,\sigma_t}\Big(Z_t^{j,I}(u); Z_t^l,\, l \in \mathfrak{J}_t \setminus \{j\}\Big) \,|_{u=t, I=\mathrm{supp}(\mathbf{k}_t)}}{\Phi_{\rho_t,\sigma_t}\Big(Z_t^j; Z_t^l,\, l \in \mathfrak{J}_t \setminus \{j\}\Big)}$$

$$\overline{\gamma}_t^j = -\mathbb{1}_{\{\tau_j \geq t\}} \frac{\partial_u \Phi_{\rho_t^\star,\sigma_t^\star}\Big(Z_t^{j,I}(u); Z_t^{\star,l},\, l \in \mathfrak{J}_t^\star \setminus \{j\}\Big) \,|_{u=t, I=\mathrm{supp}(\widetilde{\mathbf{k}}_t)}}{\Phi_{\rho_t^\star,\sigma_t^\star}\Big(Z_t^{\star,j}; Z_t^{\star,l},\, l \in \mathfrak{J}_t^\star \setminus \{j\}\Big)}. \blacksquare$$

6.3.2 Estimates

Let $g(y) = -\frac{\Gamma'(y)}{\Gamma(y)}$ and $G(y) = \int_y^\infty t^k \Gamma(t)dt$, where $\Gamma(y)$ a positive decreasing C^1 function on $\mathbb{R}_+$ such that $\int_{\mathbb{R}_+} t^k \Gamma(t)dt < \infty$ and $\lim_{t\uparrow\infty} t^{k-1}\Gamma(t) \to 0$, for some integer $k \geq 0$.

Lemma 6.6 *Let $\overline{y} \geq 0$ and $\alpha, \epsilon > 0$.*
(i) *If $g(y) \geq \alpha y$ for $y > \overline{y}$, then*

$$G(y) \leq (\frac{1}{\alpha} + \epsilon) y^{k-1}\Gamma(y) \quad \textit{for} \quad y > \overline{y} \vee \sqrt{|k-1|(\frac{1}{\epsilon\alpha^2} + \frac{1}{\alpha})}.$$

(ii) *If $g(y) \leq \beta y$ for $y > \overline{y}$, then*

$$G(y) \geq (\frac{1}{\beta} - \epsilon) y^{k-1}\Gamma(y) \quad \textit{for} \quad y > \overline{y} \vee \sqrt{|k-1|(\frac{1}{\epsilon\alpha^2} - \frac{1}{\alpha})}.$$

Proof. We only prove (i) (the proof of (ii) is similar). For every positive C^1 function φ on $\mathbb{R}_+^*$,

$$\begin{aligned}(G(y) - \varphi(y)\Gamma(y))' &= -y^k\Gamma(y) - \varphi'(y)\Gamma(y) + \varphi(y)g(y)\Gamma(y) \\ &= (\varphi(y)g(y) - y^k - \varphi'(y))\Gamma(y) \geq (\alpha y\varphi(y) - y^k - \varphi'(y))\Gamma(y)\end{aligned}$$

for $y \geq \overline{y}$. For $\varphi(y) = (\frac{1}{\alpha} + \epsilon)y^{k-1}$,

$$\begin{aligned}\alpha y\varphi(y) - y^k - \varphi'(y) = (1+\epsilon\alpha)y^k - y^k - (\frac{1}{\alpha} + \epsilon)(k-1)y^{k-2} = \\ \epsilon\alpha y^k - (\frac{1}{\alpha} + \epsilon)(k-1)y^{k-2} = (\epsilon\alpha y^2 + -(\frac{1}{\alpha} + \epsilon)(k-1))y^{k-2}.\end{aligned}$$

Therefore, if $y > \overline{y} \vee \sqrt{|k-1|(\frac{1}{\epsilon\alpha^2} + \frac{1}{\alpha})}$, then $(G(y) - \varphi(y)\Gamma(y))' \geq \alpha y\varphi(y) - y^k - \varphi'(y) \geq 0$. But $\lim_{y\uparrow\infty}(G(y) - \varphi(y)\Gamma(y)) = 0$, hence $G(y) - \varphi(y)\Gamma(y) \leq 0$. $\blacksquare$

By a first application of Lemma 6.6, to the standard normal density $\Gamma(y) = \phi$, we recover the following classical inequalities on $\psi = \frac{\phi}{\Phi}$ (recall Φ is the standard normal survival function): for any constant $c > 1$,

$$c^{-1}y \leq \psi(y) \leq cy, \; y > y_0, \tag{6.18}$$

for some $y_0 > 0$ depending on c. The following estimate, where c and y_0 are as in (6.18), can be seen as a multivariate extension of the right hand side in (6.18).

Lemma 6.7 *There exist constants a and b such that, for every $j \in J$,*

$$0 \leq \psi_{\rho,\sigma}^j(\mathbf{z}) \leq a + b||\mathbf{z}||_\infty.$$

Proof. By conditional independence in the Gaussian copula model, we have $\Phi_{\rho,\sigma}\big(\mathbf{z}\big) = \int_{\mathbb{R}} \Gamma(y)dy$, where $\Gamma(y) = \prod_{l\in J} \Phi\left(\frac{z_l+\sigma\sqrt{\rho}\,y}{\sigma\sqrt{1-\rho}}\right)\phi(y)$. Hence

$$\psi^j_{\rho,\sigma}\big(\mathbf{z}\big) = \frac{1}{\sigma\sqrt{1-\rho}}\int_{\mathbb{R}} w_{\rho,\sigma}(\mathbf{z},y)\psi\left(\frac{z_j+\sigma\sqrt{\rho}\,y}{\sigma\sqrt{1-\rho}}\right)dy, \tag{6.19}$$

where $w_{\rho,\sigma}(\mathbf{z},y) = \frac{\Gamma(y)}{\Phi_{\rho,\sigma}\big(\mathbf{z}\big)}$. Straightforward computations yield

$$g(t) = -\frac{\Gamma'(t)}{\Gamma(t)} = \sum_{l\in J}\psi(\frac{z_l+\sigma\sqrt{\rho}t}{\sigma\sqrt{1-\rho}})\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} + t \geq t,$$

whereas for $t > \max_{l\in J}\frac{1}{\sigma\sqrt{\rho}}(\sigma\sqrt{1-\rho}y_0 - z_l)$ and $t > \frac{1}{\sigma\sqrt{\rho}}\max_{l\in J} z_l$, we have

$$g(t) \leq \textstyle\sum_{l\in J} c\frac{z_l+\sigma\sqrt{\rho}t}{\sigma\sqrt{1-\rho}}\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} + t \leq \bar{\alpha}t,$$

with $\bar{\alpha} := \sum_{l\in J} 2c\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} + 1 \geq 1$. Applying Lemma 6.6 (i) with $k = 1, \alpha = 1$ and $\epsilon = 1$, respectively (ii) with $k = 0$, $\alpha = \bar{\alpha}$ and $\epsilon = \frac{1}{2\bar{\alpha}}$, yields

$$\int_y^\infty t\Gamma(t)dt \leq 2\Gamma(y),\;\; y > 0, \quad \text{respectively} \int_y^\infty \Gamma(t)dt \geq \frac{1}{2\bar{\alpha}y}\Gamma(y),\;\; y > \overline{y} \vee \frac{1}{\sqrt{\bar{\alpha}}},$$

where $\overline{y} = \frac{1}{\sigma\sqrt{\rho}}\max_{l\in J}|z_l| + \frac{1}{\sigma\sqrt{\rho}}\sigma\sqrt{1-\rho}y_0$. Thus, setting $y_1 = \overline{y}+1 = \frac{1}{\sigma\sqrt{\rho}}\max_{l\in J}|z_l| + \frac{1}{\sigma\sqrt{\rho}}\sigma\sqrt{1-\rho}y_0 + 1$,

$$\begin{aligned}\int_0^\infty t\Gamma(t)dt &= \int_0^{y_1} t\Gamma(t)dt + \int_{y_1}^\infty t\Gamma(t)dt \leq y_1\int_0^{y_1}\Gamma(t)dt + 2\Gamma(y_1)\\ &\leq y_1\int_0^{y_1}\Gamma(t)dt + 4\bar{\alpha}y_1\int_{y_1}^\infty \Gamma(t)dt \leq (1+4\bar{\alpha})\int_0^\infty \Gamma(t)dt,\end{aligned}$$

i.e.

$$\int_0^\infty t w_{\rho,\sigma}(\mathbf{z},t)dt \leq (1+4\bar{\alpha})y_1. \tag{6.20}$$

Now, by (6.19) and the right-hand side inequality in (6.18),

$$\begin{aligned}0 &\leq \sigma\sqrt{1-\rho}\,\psi^j_{\rho,\sigma}\big(\mathbf{z}\big)\\ &\leq \int_{\mathbb{R}}\left(\frac{1}{\Phi(y_0)}\mathbb{1}_{\{\frac{z_j+\sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}\leq y_0\}} + c\frac{z_j+\sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}\mathbb{1}_{\{\frac{z_j+\sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}>y_0\}}\right)w_{\rho,\sigma}(\mathbf{z},y)dy\\ &= \left(\frac{1}{\Phi(y_0)} + \frac{cz_j}{\sigma\sqrt{1-\rho}}\right) + \frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}\int_{\mathbb{R}}\mathbb{1}_{\{\sigma\sqrt{\rho}y>\sigma\sqrt{1-\rho}y_0-z_j\}}y w_{\rho,\sigma}(\mathbf{z},y)dy\\ &\leq \left(\frac{1}{\Phi(y_0)} + \frac{cz_j}{\sigma\sqrt{1-\rho}}\right) + \frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}\int_0^\infty y w_{\rho,\sigma}(\mathbf{z},y)dy,\end{aligned} \tag{6.21}$$

so that by substitution of (6.20) into (6.21)

$$0 \leq \sigma\sqrt{1-\rho}\,\psi^j_{\rho,\sigma}\big(\mathbf{z}\big) \leq \left(\frac{1}{\Phi(y_0)} + \frac{cz_j}{\sigma\sqrt{1-\rho}}\right) + \frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}(1+4\bar{\alpha})y_1. \blacksquare$$

Corollary 6.1 **(i)** *There exists a constant $C > 0$ such that, for $0 \leq r \leq t$ and $j \in N^\star$,*

$$\langle \nu^c\rangle_t \leq C(\sum_{i\in N}\sup_{0<s\leq t}|m^i_s|^2 + 1)t, \tag{6.22}$$

$$\widetilde{\gamma}^j_r \vee \bar{\gamma}^j_r \leq C(\sum_{i\in N}\sup_{0<s\leq t}|m^i_s| + 1),\;\; \widetilde{\gamma}^j_r\ln(\widetilde{\gamma}^j_r \vee \bar{\gamma}^j_r) \leq C\sum_{i\in N}\sup_{0<s\leq t}(|m^i_s|+1)\ln(|m^i_s|+1). \tag{6.23}$$

Proof. Applying Lemma 6.7 to the formulas derived in Lemma 6.5 (and $\widetilde{\gamma}^j_t = \gamma_j(t, \mathbf{m}_t, (0, 0, \widetilde{\mathbf{k}}_t)))$, we obtain, for constants C changing from line to line

$$\langle \nu^c \rangle_t \leq C \int_0^t (\sum_{I \subseteq N} \sum_{j \in N \setminus I} |Z^{j,I}_s(s)| + 1)^2 ds \leq C(\sum_{I \subseteq N} \sum_{j \in N \setminus I} \sup_{0 < s \leq t} |Z^{j,I}_s(s)| + 1)^2 t,$$

as well as the left-hand side inequality in (6.23), whence the right-hand side follows by

$$\begin{aligned} \widetilde{\gamma}^j_r \ln(\widetilde{\gamma}^j_r \vee \bar{\gamma}^j_r) &\leq C(\max_{i \in N} \sup_{0 < s \leq t} |m^i_s| + 1) \ln\big(C(\max_{i \in N} \sup_{0 < s \leq t} |m^i_s| + 1)\big) \\ &= \max_{i \in N} \sup_{0 < s \leq t} C(|m^i_s| + 1) \ln(C|m^i_s| + 1). \blacksquare \end{aligned}$$

Lemma 6.8 *For any constant $q > 0$, for sufficiently small t, $e^{q \sup_{0 \leq s \leq t} (m^i_s)^2}$ is $\mathbb{Q}$ integrable, $i \in N$.*

Proof. The process $(m^i_t)_{t \geq 0}$ is equal in law to a time changed Brownian motion $(B_{\bar{t}})_{t \geq 0}$, where B is a a univariate standard Brownian motion and $\bar{t} = \int_0^t \varsigma^2(s) ds$ goes to 0 with t. Thus, it suffices to show the result with m^i replaced by B. Let r_t be the density function of the law of $\sup_{0 \leq s \leq t} |B_s|$ and let $R_t(y) = \int_y^\infty r_t(x) dx, y > 0$, so that

$$\mathbb{E}[e^{q \sup_{0 \leq s \leq t} B^2_s}] = \int_0^\infty e^{qy^2} r_t(y) dy = -[R_t(y) e^{qy^2}]_0^\infty + 2q \int_0^\infty y R_t(y) e^{qy^2} dy \tag{6.24}$$

and

$$\begin{aligned} R_t(y) &= \mathbb{P}[\sup_{0 \leq s \leq t} (B^+_s + B^-_s) > y] \leq \mathbb{P}[\sup_{0 \leq s \leq t} B^+_s > \tfrac{y}{2}] + \mathbb{P}[\sup_{0 \leq s \leq t} B^-_s > \tfrac{y}{2}] \\ &= 2\mathbb{P}[\sup_{0 \leq s \leq t} B_s > \tfrac{y}{2}] = 2\mathbb{P}[|B_t| > \tfrac{y}{2}] = 2\mathbb{P}[|B_1| > \tfrac{y}{2\sqrt{t}}] = 4\Phi(\tfrac{y}{2\sqrt{t}}), \end{aligned}$$

where by the left-hand side in (6.18)

$$\Phi(\frac{y}{2\sqrt{t}}) \frac{y}{2\sqrt{t}} \leq c\phi(\frac{y}{2\sqrt{t}}) = \frac{c}{\sqrt{2\pi}} e^{-\frac{y^2}{8t}}, \quad \frac{y}{2\sqrt{t}} > y_0.$$

Therefore, for $\frac{1}{8t} > q$, both terms are finite in the right-hand side of (6.24). $\blacksquare$

By Corollary 6.1, multivariate Hölder inequality and Lemma 6.8,

$$\exp\Big(\langle \nu^c \rangle_t + \sum_{i \in N^\star} \int_0^t (\widetilde{\gamma}^i_s \ln(\widetilde{\gamma}^i_s) - \widetilde{\gamma}^i_s \ln(\bar{\gamma}^i_s) - \widetilde{\gamma}^i + \bar{\gamma}^i_s) ds\Big)$$

is $\mathbb{Q}$ integrable for sufficiently small t. Hence Lemma 6.4 follows by an application of Theorem IV.3 in Lepingle and Mémin (1978).

7 Dynamic Marshall-Olkin Copula TVA Model

The above dynamic Gaussian copula (DGC) model can suffice to deal with TVA on portfolios of CDS contracts. If CDO tranches are also present in the portfolio, a Gaussian copula dependence structure is not rich enough. Instead one can use the following dynamic Marshall-Olkin (DMO) copula model.

7.1 Model of Default Times

We define a family $\mathcal{Y}$ of "shocks", i.e. subsets $Y \subseteq N$ of obligors, including the singletons $\{-1\}, \{0\}, \{1\}, \ldots, \{n\}$ and a (typically small) number of "common shocks" representing simultaneous defaults. The shock intensities are given in the form of extended CIR processes as, for every $Y \in \mathcal{Y}$,

$$dX^Y_t = a(b_Y(t) - X^Y_t) dt + c\sqrt{X^Y_t} dW^Y_t, \tag{7.1}$$

for nonnegative constants a and c, functions $b_Y(t)$ and for independent Brownian motions W^Y in their own completed filtration $\mathbb{W} = (\mathcal{W}_t)_{t \geq 0}$, under the pricing measure $\mathbb{Q}$ (the case of deterministic intensities $X^Y_t = b_Y(t)$ can be embedded in this framework as the limiting case of an "infinite mean-reversion speed" a). For $Y \in \mathcal{Y}$, we define

$$\tau_Y = \inf\{t > 0; \int_0^t X^Y_s ds > \epsilon_Y\}, \;\; H^Y_t = \mathbb{1}_{\tau_Y \le t}, \tag{7.2}$$

where the ϵ_Y are i.i.d. standard exponential random variables. We consider the filtration $\mathbb{G}$ given as $\mathbb{W}$ progressively enlarged by the random times τ_Y, so that $\mathbb{G}$ is the usual $\mathbb{Q}$ augmentation of the natural filtration $\mathbb{Z}$ of $\mathrm{Z} = (\mathbf{W}, \mathbf{H})$. We denote by M^Y the compensation of H^Y, i.e. $dM^Y_t = dH^Y - (1 - H^Y_t)X^Y_t dt, \;\; t \ge 0$.

Lemma 7.1 *For $t \ge 0$,*

$$\mathcal{G}_t = \mathcal{W}_t \vee \bigvee_{Y \in \mathcal{Y}} \big(\sigma(\tau_Y \wedge t) \vee \sigma(\{\tau_Y > t\})\big). \tag{7.3}$$

The W^Y and the M^Y, $Y \in \mathcal{Y}$, have the $(\mathbb{G}, \mathbb{Q})$ MRP. In addition, $X = (\mathbf{X}, \mathbf{H}) = ((X^Y)_{Y\in\mathcal{Y}}, (H^Y)_{Y\in\mathcal{Y}})$ is a $(\mathbb{G}, \mathbb{Q})$ jump-diffusion.

Proof. The MRP and (7.3) are proved by induction as follows. We write $\mathbb{G} = \mathbb{G}^{\mathcal{Y}}$. If $\mathcal{Y}$ is a singleton (case of a Cox time in view of (7.2)), then the immersion of $\mathbb{W}$ into $\mathbb{G}^{\mathcal{Y}}$ implies the results, by the optional splitting formula for (7.3) and by Theorem 6.4 in Jeanblanc and Song (2013) for the MRP. Moreover, if $\mathcal{Z}$ is obtained by addition of a new $Z \subseteq N$ to $\mathcal{Y}$, then the independence between the ϵ_Y implies that τ_Z is in turn a Cox time with intensity in $\mathbb{G}^{\mathcal{Y}}$, hence immersion of $\mathbb{G}^{\mathcal{Y}}$ into $\mathbb{G}^{\mathcal{Z}}$ follows and the results for $\mathbb{G}^{\mathcal{Z}}$ are implied as above from those, if assumed, for $\mathbb{G}^{\mathcal{Y}}$. Regarding the $(\mathbb{G}, \mathbb{Q})$ jump-diffusion feature of $X = (\mathbf{X}, \mathbf{H})$, see Bielecki et al. (2014a, Part I). ■

7.2 TVA Model

A DMO setup can be used as a TVA model for credit derivatives, with

$$E_b = \mathcal{Y}_b := \{Y \in \mathcal{Y}; -1 \in Y\}, \;\; E_c = \mathcal{Y}_c := \{Y \in \mathcal{Y}; 0 \in Y\}, \;\; E = \mathcal{Y}_\bullet := \mathcal{Y}_b \cup \mathcal{Y}_c.$$

We assume that for every process $U = P$, Δ, Q, C and $\mathfrak{C}$, there exists a continuous function $\widetilde{U}$ such that

$$U_\tau = \widetilde{U}_Y(\tau, \mathbf{X}_\tau, \mathbf{H}_{\tau-}) \tag{7.4}$$

($= \widetilde{U}^Y_\tau$ for brevity) on every event of the form $\{\tau = \tau_Y\}$, $Y \in \mathcal{Y}_\bullet$. The results of Bielecki et al. (2014a, Part II) show that the condition (7.4) holds on $U = P$ and Δ for vanilla credit derivatives, including CDS contracts and CDO tranches, with semi-explicit formulas for P. The conditions (7.4) on $U = Q$, C and $\mathfrak{C}$ may be satisfied or not depending on the CSA (regarding C and $\mathfrak{C}$, see Sect. 5.1 and 7.3). By (7.4) assumed for every $U = P$, Δ, Q, C and $\mathfrak{C}$, the coefficient $\widetilde{\xi}$ in (5.11) is given as

$$\begin{aligned} \widetilde{\xi}^Y_t(\pi) = \widetilde{P}^Y_t - \widetilde{Q}^Y_t + \mathbb{1}_{Y\in\mathcal{Y}_c}(1 - R_c)(\widetilde{Q}^Y_t + \widetilde{\Delta}^Y_t - \widetilde{C}^Y_t)^+ \\ - \mathbb{1}_{Y\in\mathcal{Y}_b}\big((1 - R_b)(\widetilde{Q}^Y_t + \widetilde{\Delta}^Y_t - \widetilde{\mathfrak{C}}^Y_t)^- + (1 - \bar{R}_b)(\pi - \mathfrak{C}_{t-})^+\big), \;\; Y \in \mathcal{Y}_\bullet. \end{aligned} \tag{7.5}$$

The coefficient $\widehat{f}_t(\vartheta)$ in (5.13) is given by

$$\begin{aligned} \widehat{f}_t(\vartheta) + r_t\vartheta = \;\; & (1 - R_c) \sum_{Y\in\mathcal{Y}_c} X^Y_t \big(\widetilde{Q}^Y_t + \widetilde{\Delta}^Y_t - \widetilde{C}^Y_t\big)^+ - (1 - R_b) \sum_{Y\in\mathcal{Y}_b} X^Y_t \big(\widetilde{Q}^Y_t + \widetilde{\Delta}^Y_t - \widetilde{\mathfrak{C}}^Y_t\big)^- \\ & + c_t(\mathfrak{C}_t + N_t) + \widetilde{\lambda}_t \,(P_t - \vartheta - \mathfrak{C}_t)^+ \big) - \lambda_t\,(P_t - \vartheta - \mathfrak{C}_t)^- + \sum_{Y\in\mathcal{Y}_\bullet} X^Y_t \left(\widetilde{P}^Y_t - \vartheta - \widetilde{Q}^Y_t\right), \;\; t \in [0, \bar{\tau}], \end{aligned} \tag{7.6}$$

where $\widetilde{\lambda}_t = \bar{\lambda}_t - (1 - \bar{R}_b)\sum_{Y\in\mathcal{Y}_b} X^Y_t$. Let $\mathcal{Y}_\circ = \mathcal{Y} \setminus \mathcal{Y}_\bullet$ (collection of the $Y \in \mathcal{Y}$ that don't intersect $\{-1, 0\}$) and $\widetilde{X}_t = (\mathbf{X}_t, \widetilde{\mathbf{H}}_t)$, where $\widetilde{\mathbf{H}} = (H^Y)_{Y\in\mathcal{Y}_\circ}$. We assume that the processes $r, c, \lambda, \bar{\lambda}, P$, $\mathfrak{C}$ and N are given before τ as continuous functions of $(t, \widetilde{X}_t)$. The next result is the DMO analog of Lemma 6.2 in the DGC setup, where the main difficulty, related to (DGC.2), was due to the fact that $\mathbb{P} \neq \mathbb{Q}$ there, so we give no proof here.

Lemma 7.2 *The condition (C) holds, with:*

(DMO.1) *a reference filtration $\mathbb{F}=(\mathcal{F}_t)$ in (C.1) given as $\mathbb{W}$ progressively enlarged by the $\tau_Y, Y\in\mathcal{Y}_\circ$, which satisfies*

$$\mathcal{F}_t=\mathcal{W}_t\vee\bigvee_{Y\in\mathcal{Y}_\circ}\big(\sigma(\tau_Y\wedge t)\vee\sigma(\{\tau_Y>t\})\big),\ \ t\geq 0,$$

(DMO.2) *$\mathbb{P}=\mathbb{Q}$ in (C.2) (so that "immersion holds" in the sense of the comments following the statement of the condition (C)), where the $W^Y, Y\in\mathcal{Y}$, and the $M^Y, Y\in\mathcal{Y}_\circ$, have the $(\mathbb{F},\mathbb{Q})$ MRP,*

(DMO.3) *a Markov specification $\widetilde{f}_t(\vartheta)=\widetilde{f}(t,\widetilde{X}_t,\vartheta)$ in (C.3), for the function $\widetilde{f}=\widetilde{f}(t,\mathbf{x},\widetilde{\mathbf{k}},\vartheta)$ given, writing $\mathbf{k}=(\mathbf{1}_{Y\in\mathcal{Y}_\circ}\widetilde{k}_Y)_{Y\in\mathcal{Y}}$ and $\widetilde{x}=(t,\mathbf{x},\widetilde{\mathbf{k}}), x=(t,\mathbf{x},\mathbf{k})$ for every $\mathbf{x}=(x_Y)_{Y\in\mathcal{Y}},\widetilde{\mathbf{k}}=(\widetilde{k}_Y)_{Y\in\mathcal{Y}_\circ}$, by:*

$$\begin{aligned}&\widetilde{f}(t,\widetilde{x},\vartheta)+r(t,x)\vartheta=\\&\quad(1-R_c)\sum_{Y\in\mathcal{Y}_c}x_Y\big(\widetilde{Q}_Y+\widetilde{\Delta}_Y-\widetilde{C}_Y\big)^+(t,x)-(1-R_b)\sum_{Y\in\mathcal{Y}_b}x_Y\big(\widetilde{Q}_Y+\widetilde{\Delta}_Y-\widetilde{\mathfrak{C}}_Y\big)^-(t,x)\\&\qquad+\big(c(\mathfrak{C}+N)+\widetilde{\lambda}\big(P-\vartheta-\mathfrak{C}\big)^+-\lambda\big(P-\vartheta-\mathfrak{C}\big)^-\big)(t,\widetilde{x})+\sum_{Y\in\mathcal{Y}_\bullet}x_Y\big(\widetilde{P}_Y-\vartheta-\widetilde{Q}_Y\big)(t,x),\end{aligned}\tag{7.7}$$

with $\widetilde{\lambda}=\bar{\lambda}-(1-\bar{R}_b)\sum_{Y\in\mathcal{Y}_b}x_Y$, and for the $(\mathbb{F},\mathbb{Q})$ jump-diffusion $\widetilde{X}_t=(\mathbf{X}_t,\widetilde{\mathbf{H}}_t)$.

Lemma 7.2 implies the following specification of Proposition 3.1, where the shape (7.9) of $\widetilde{\mu}$ reflects the Itô-Markov local martingale part formula (3.6) that applies to the $(\mathbb{F},\mathbb{P})$ jump-diffusion $\widetilde{X}_t=(\mathbf{X}_t,\widetilde{\mathbf{H}}_t)$, and where there is no need to stop $\widetilde{\mu}$ at $\tau-$ in (7.10) since $\widetilde{\mu}$ does not jump at τ.

Proposition 7.1 *We obtain a solution Θ to the full TVA equation (2.3) by setting $\Theta=\widetilde{\Theta}$ on $[0,\bar{\tau})$ and $\Theta_{\bar{\tau}}=\mathbb{1}_{\tau<T}\xi$, provided $\widetilde{\Theta}_t=\widetilde{\Theta}(t,\widetilde{X}_t)$ is such that $\mathbb{1}_{\tau<T}\xi_\star J$ (with $\xi_\star=\xi(P_{\tau-}-\widetilde{\Theta}_{\tau-})$) has locally integrable total variation, $\widetilde{\Theta}_T=0$ and, for $t\in[0,T]$,*

$$-d\widetilde{\Theta}_t=\widetilde{f}(t,\mathbf{X}_t,\widetilde{\mathbf{H}}_t,\widetilde{\Theta}_t)dt-d\widetilde{\mu}_t,\tag{7.8}$$

where

$$d\widetilde{\mu}_t=c\sum_{Y\in\mathcal{Y}}\sqrt{x_Y}\partial_{x_Y}\widetilde{\Theta}(t,\mathbf{X}_t,\widetilde{\mathbf{H}}_t)dW^Y_t\ +\sum_{Y\in\mathcal{Y}_\circ}\delta_Y\widetilde{\Theta}(t,\mathbf{X}_t,\widetilde{\mathbf{H}}_{t-})dM^Y_t.\tag{7.9}$$

On $[0,\bar{\tau}]$, writing $\widetilde{\xi}^{\star,Y}_t=\widetilde{\xi}^Y_t(P_t-\widetilde{\Theta}_t)$,

$$d\mu_t=d\widetilde{\mu}_t-\left((\xi_\star-\widetilde{\Theta}_{t-})dJ_t+\sum_{Y\in\mathcal{Y}_\bullet}X^Y_t(\widetilde{\xi}^{\star,Y}_t-\widetilde{\Theta}_t)dt\right).\ \blacksquare\tag{7.10}$$

This proposition can be used for any TVA valuation and hedging purposes in a DMO setup. One can for instance extend to the present TVA setup the unilateral CVA dynamic hedging developments of Bielecki and Crépey (2013), the extension being straightforward in an asymmetrical (but still bilateral) CVA approach with $R_b=\bar{R}_b=1$ (see the remark 5.1).

7.3 Gap Risk

In the DGC and DMO examples of Sect. 6 and 7, we postulated a "vanilla" collateralization scheme, without path dependence of $\mathbf{\Gamma}:=(M,N,\mathfrak{N})$ nor cure period. In both models, the only modification required to deal with a path-dependent $\mathbf{\Gamma}$ as in Sect. 5.1.1 (but without cure period) would be to augment the corresponding pre-default factor process $\widetilde{X}_t$ into $\widetilde{X}^{\mathbf{\Gamma}}_t=(\widetilde{X}_t,\mathbf{\Gamma}_t)$.

Remark 7.1 Under the margining scheme of Sect. 5.1.1, $\mathbf{\Gamma}$ can only jump at the constant times t_l, hence it cannot jump at τ (which is totally inaccessible). In particular, for $U=C$ or $\mathfrak{C}$, $\widetilde{U}^Y_\tau$ does in fact not depend on Y in (7.4). As explained in the remark 5.3, the flexibility given by a potential dependence in Y could be used to deal with other gap risk features, such as a jump of $U=C$ and/or $\mathfrak{C}$ at τ.

In addition, let's assume a positive cure period δ as in Sect. 5.1.2, with for simplicity $Q = P$ (as already done there) and deterministic interest rates, in the above DMO setup (very similar considerations would apply in a DGC setup). Recall that $\mathbb{G}$ is the usual $\mathbb{Q}$ augmentation of the natural filtration of $\mathrm{Z} = (\mathbf{W}, \mathbf{H})$.

Lemma 7.3 *The condition (G) holds.*

Proof. By σ-algebra order relationships here, we mean relationships between the completed σ-algebras. Since the $\tau_Y, Y \in \mathcal{Y}_\bullet$, don't intercept each other nor $\mathbb{F}$ stopping times, we have by Lemma 2.6 in Crépey and Song (2014):

$$\mathcal{G}_\tau = \mathcal{G}_{\tau-} \vee \sigma(\{\tau = \tau_Y\}, Y \in \mathcal{Y}_\bullet),$$

where $\mathcal{G}_{\tau-} \subseteq \sigma(\mathrm{Z}_{\cdot\wedge\tau})$ and where, for $Y \in \mathcal{Y}_\bullet$,

$$\tau = \tau_Y \Longleftrightarrow H^Y_\tau \prod_{Z \in \mathcal{Y}_\bullet \setminus Y} (1 - H^Z_\tau) = 1,$$

so that $\{\tau = \tau_Y\} \in \mathcal{Z}_\tau \subseteq \sigma(\mathrm{Z}_{\cdot\wedge\tau})$. Thus, $\mathcal{G}_\tau = \sigma(\mathrm{Z}_{\cdot\wedge\tau})$. In addition, by the optional splitting formula, $\mathcal{G}_T = \mathcal{F}_T \vee \sigma(\mathbf{H}_{\cdot\wedge T})$, where $\mathcal{F}_T = \sigma(\mathbf{W}_{\cdot\wedge T})$ by Markov, hence $\mathcal{G}_T = \sigma(Z_{\cdot\wedge T})$. Finally, denoting by $\mathcal{S} + \mathcal{T}$ the collection of (disjoint) unions of a set in a collection $\mathcal{S}$ and a set in a collection $\mathcal{T}$, we have

$$\mathcal{G}_{\bar{\tau}} = \{\tau \le T\} \cap \mathcal{G}_\tau + \{T < \tau\} \cap \mathcal{G}_T = \{\tau \le T\} \cap \sigma(Z_{\cdot\wedge\tau}) + \{T < \tau\} \cap \sigma(Z_{\cdot\wedge T}) = \sigma(Z_{\cdot\wedge\bar{\tau}}),$$

since $\{\tau \le T\} \in \sigma(Z_{\cdot\wedge\bar{\tau}})$. ∎

The probability of a default at time τ^δ is zero because τ^δ is predictable whereas default times are totally inaccessible in a DMO setup. Therefore, the conditions of Lemma 5.2 hold. In the case of credit derivatives with deterministic interest rates, the process $\int_{[\tau,t]} e^{\int_s^t r_u du} dD_s$ in P^δ_t is a function of the default times in $[\tau, t]$, which we assume henceforth. We write $K^Y_t = \tau_Y \mathbb{1}_{\tau_Y \le t}$, $\mathbf{K} = (K^Y)_{Y \in \mathcal{Y}}$ and we consider a cure period $(\mathbb{G}, \mathbb{Q})$ factor process $X^\delta_t = (\mathbf{X}_t, \mathbf{K}_t)$. For concreteness we assume the definition (5.6) for g, but it could be any other definition respecting the Markov structure of the setup (cf. the first paragraph in Sect. 4).

Lemma 7.4 *There exist functions $P^{\bar{w}}(t, x_\delta)$ and $f^{\bar{w}}(t, x_\delta, \vartheta)$, $x_\delta = (\mathbf{x}, \mathbf{k})$, such that*

$$P^\delta_t = P^{\bar{\cdot}}(t, X^\delta_t), \;\; f^\delta_t(\vartheta) = f^{\bar{\cdot}}(t, X^\delta_t, \vartheta), \;\; t \in [\bar{\tau}, \bar{\tau}^\delta]. \tag{7.11}$$

We have

$$\mathbb{1}_{\tau<T}\xi^\delta_\star = \phi^{\bar{\cdot}}(\tau^\delta, X^\delta_{\tau^\delta}) \tag{7.12}$$

where, writing, for $a = b$ or c, $k_a = \min\{k_Y; Y \in \mathcal{Y}_a, k_Y > 0\}$ or 0 if there are no such k_Y,

$$\begin{aligned}
&\phi^{\bar{w}}(v, x_\delta) = \mathbb{1}_{\tau(w)<T} \times \\
&\begin{cases} \bar{R}_b^{-1}\psi^{\bar{w}}(v, x_\delta) + (1 - \bar{R}_b^{-1})(P^{\bar{w}}(v, x_\delta) - \mathfrak{C}_{\widehat{\tau}(w)}(w)) \text{ if} \\ \qquad (k_b > 0 \wedge (k_c = 0 \vee (k_c > 0 \wedge k_b \le k_c^\delta)) \text{ and } P^{\bar{w}}(v, x_\delta) \ge \mathfrak{C}_{\widehat{\tau}(w)}(w) \\ \psi^{\bar{w}}(v, x_\delta) \text{ otherwise,} \end{cases}
\end{aligned} \tag{7.13}$$

with

$$\begin{aligned}
\psi^{\bar{w}}(v, x_\delta) = &\mathbb{1}_{(k_c>0 \wedge (k_b=0 \vee (k_b>0 \wedge k_c \le k_b^\delta))}(1 - R_c)(P^{\bar{w}}(v, x_\delta) - \mathfrak{C}_{\widehat{\tau}(w)}(w))^+ \\
&- \mathbb{1}_{(k_b>0 \wedge (k_c=0 \vee (k_c>0 \wedge k_b \le k_c^\delta))}(1 - R_b)(P^{\bar{w}}(v, x_\delta) - \mathfrak{C}_{\widehat{\tau}(w)}(w))^-.
\end{aligned} \tag{7.14}$$

Moreover, $P^{\bar{w}}$, $f^{\bar{w}}$ and $\phi^{\bar{w}}$ only depend on $\bar{w}$ through $\tau(w)$ and $\mathfrak{C}_{\widehat{\tau}(w)}(w)$.

Proof. Since $P_t = P(t, X^\delta_t)$ and $\int_{[\tau,t]} e^{\int_s^t r_u du} dD_s$ in P^δ_t is a function of the default times in $[\tau, t]$, (7.11) just rephrases (4.1). Likewise, (7.12) through (7.14) are simply the Markov counterparts of (5.20) and (5.19). ∎

We consider the pre-default $(\mathbb{F}, \mathbb{Q})$ factor process $\widetilde{X}^\delta_t = (\mathbf{X}_t, \widetilde{\mathbf{K}}_t, \mathbf{\Gamma}_t)$, $\widetilde{\mathbf{K}} = (K^Y)_{Y\in\mathcal{Y}_\circ}$, where we recall that $\mathbb{Q} = \mathbb{P}$ in a DMO setup. We postulate that the process $g_t(P_t - \vartheta)$ is given before τ as a continuous function $\widetilde{g}(t, \widetilde{X}^\delta_t, \vartheta)$. In (7.18) below, we write, with $\widetilde{x}_\delta = (\mathbf{x}, \widetilde{\mathbf{k}}, \boldsymbol{\gamma})$,

$$\widetilde{f}^\delta(t, \widetilde{x}_\delta, \vartheta) + r(t)\vartheta = \widetilde{g}(t, \widetilde{x}_\delta, \vartheta) + (\sum_{Y\in\mathcal{Y}_\bullet} x_Y)\left(\widehat{\Theta}^\delta(t, \widetilde{x}_\delta) - \vartheta\right), \tag{7.15}$$

where $\widehat{\Theta}^\delta(t, \widetilde{x}_\delta)$ is the function such that (observing that, by the final statement in Lemma 7.4, all the data of (7.17) and hence its solution $\Theta^{\bar{w}}$ only depend on $\bar{w}$ through $\tau(w)$ and $\mathfrak{C}_{\widehat{\tau}(w)}(w)$), on $\{\tau < T\}$,

$$\widehat{\Theta}^\delta(\tau, \widetilde{X}^\delta_\tau) := \sum_{Y\in\mathcal{Y}_\bullet} \frac{X^Y_\tau}{\sum_{Z\in\mathcal{Y}_\bullet} X^Z_\tau} \Theta^{\bar{\cdot}}\left(\tau, \mathbf{X}_\tau, ((\mathbf{1}_{Z\in\mathcal{Y}_\circ}\widetilde{K}^Z_\tau)_{Z\in\mathcal{Y}})^{Y,\tau}\right) = \mathbb{E}(\Theta^{\bar{\cdot}}(\tau, X^\delta_\tau) \,|\, \mathcal{G}_{\tau-}), \tag{7.16}$$

by Lemma 5.1 applied with

$$\zeta(\pi) \equiv \Theta^{\bar{\cdot}}(\tau, X^\delta_\tau),\ \ \widetilde{\zeta}^Y_\tau(\pi) \equiv \Theta^{\bar{\cdot}}\left(\tau, \mathbf{X}_\tau, ((\mathbf{1}_{Z\in\mathcal{Y}_\circ}\widetilde{K}^Z_\tau)_{Z\in\mathcal{Y}})^{Y,\tau}\right) = \Theta^{\bar{\cdot}}\left(\tau, \mathbf{X}_\tau, ((\mathbf{1}_{Z\in\mathcal{Y}_\circ}\widetilde{K}^Z_{\tau-})_{Z\in\mathcal{Y}})^{Y,\tau}\right)$$

(independent of π).

Since (C) and (G) hold by Lemmas 7.2 and 7.3, comparing (7.11)-(7.12) with (4.17) as well as (7.15)–(7.16) with (4.19)–(4.20), it comes by application of Proposition 4.1:

Proposition 7.2 *We obtain a solution Θ to the full TVA equation with positive cure period* (4.3) *by setting $\Theta^\delta_t = \mathbb{1}_{t<\bar{\tau}}\widetilde{\Theta}^\delta_t + \mathbb{1}_{t\geq\bar{\tau}}\mathbb{1}_{\tau<T}\Theta^{\bar{\cdot}}(t, X^\delta_t)$, provided:*

- *for every w, $\Theta^{\bar{w}}_t = \Theta^{\bar{w}}(t, X^\delta_t)$ satisfies $\Theta^{\bar{w}}_{\bar{\tau}^\delta(w)} = \phi^{\bar{w}}(\tau^\delta(w), X^\delta_{\tau^\delta(w)})$ and, for $t \in [\bar{\tau}(w), \bar{\tau}^\delta(w)]$,*

$$-d\Theta^{\bar{w}}_t = f^{\bar{w}}(t, \mathbf{X}_t, \mathbf{K}_t, \Theta^{\bar{w}}_t)dt - d\mu^{\bar{w}}_t \tag{7.17}$$

with

$$d\mu^{\bar{w}}_t = c\sum_{Y\in\mathcal{Y}} \sqrt{X^\delta_Y}\partial_{X^\delta_Y}\Theta^{\bar{w}}(t, \mathbf{X}_t, \mathbf{K}_t)dW^Y_t + \sum_{Y\in\mathcal{Y}} \delta_Y\Theta^{\bar{w}}(t, \mathbf{X}_t, \mathbf{K}_{t-})dM^Y_t,$$

- *$\mathbb{1}_{\tau<T}\Theta^{\bar{\cdot}}(\tau, X^\delta_\tau)J$ has locally integrable total variation and $\widetilde{\Theta}^\delta_t = \widetilde{\Theta}^\delta(t, \widetilde{X}^\delta_t)$ satisfies $\widetilde{\Theta}^\delta_T = 0$ and, for $t \in [0, T]$,*

$$-d\widetilde{\Theta}^\delta_t = \widetilde{f}^\delta(t, \mathbf{X}_t, \widetilde{\mathbf{K}}_t, \mathbf{\Gamma}_t, \widetilde{\Theta}^\delta_t)dt - d\widetilde{\mu}^\delta_t \tag{7.18}$$

with

$$d\widetilde{\mu}^\delta_t = c\sum_{Y\in\mathcal{Y}} \sqrt{X^\delta_Y}\partial_{X^\delta_Y}\widetilde{\Theta}^\delta(t, \mathbf{X}_t, \widetilde{\mathbf{K}}_t, \mathbf{\Gamma}_t)dW^Y_t + \sum_{Y\in\mathcal{Y}_\circ} \delta_Z\widetilde{\Theta}^\delta(t, \mathbf{X}_t, \widetilde{\mathbf{K}}_{t-}, \mathbf{\Gamma}_t)dM^Y_t.$$

On $[0, \bar{\tau})$,

$$d\mu^\delta_t = d\widetilde{\mu}^\delta_t - \left(\left(\Theta^{\bar{\cdot}}(\tau, X^\delta_\tau) - \widetilde{\Theta}^\delta_{t-}\right)dJ_t + (\sum_{Y\in\mathcal{Y}_\bullet} X^Y_t)\left(\widehat{\Theta}^\delta(t, \mathbf{X}_t, \widetilde{\mathbf{K}}_t, \mathbf{\Gamma}_t) - \widetilde{\Theta}^\delta_t\right)dt\right),$$

and $d\mu^\delta_t = d\mu^{\bar{\cdot}}(t, X^\delta_t)$ on $[\bar{\tau}, \bar{\tau}^\delta]$. ∎

8 Perspectives

Accounting for funding costs, the TVA equations are nonlinear. In the case of credit derivatives, the problem is also very high-dimensional. For nonlinear and very high-dimensional problems, any numerical scheme based, even to some extent, on dynamic programming, such as purely backward deterministic PDE schemes, but also forward/backward simulation/regression BSDE schemes, are ruled out by the curse of dimensionality. The only feasible TVA schemes are purely forward simulation schemes, such as the linear expansions of Fujii and Takahashi (2012) in vanilla cases with explicit formulas for P_t, or the CVA branching particles of Henry-Labordère (2012) in more exotic situations. To conclude this paper, Fig. 1[1] shows TVAs (CVA and FVA, no DVA to ease the interpretation of the results, and no collateral nor cure period) computed by expansions a la Fujii and Takahashi (2012) of increasing order, for different levels of nonlinearity (the unsecured borrowing spread $\bar{\lambda}$ in (6.8) and (7.7)). These numerical aspects will be developed in a follow-up paper.

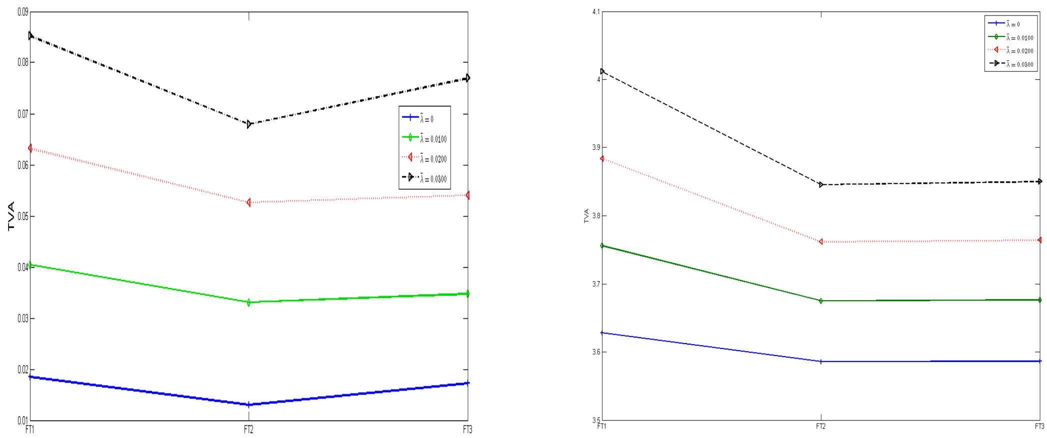

Fig. 1 TVA computed by linear Monte Carlo expansions a la Fujii and Takahashi (2012) ("FT schemes") of order 1 to 3. *Left*: TVA on a netted portfolio of ten CDS on ten different names in a DGC model (equations (6.9)-(6.10)) with 10^5 paths (at time 0 the protection legs of the CDS have a cumulative mark-to-market of 45.12). *Right*: TVA on a junior-mezzanine CDO tranche in a DMO model (equations (7.8)-(7.9)) with 120 names and 2×10^4 paths (at time 0 the protection leg of the tranche has a mark-to-market of 5.69).

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