UP AND DOWN CREDIT RISK

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Abstract

This paper discusses the main modeling approaches that have been developed so far for handling portfolio credit derivatives, with a focus on the question of hedging. In particular the so called top, top down and bottom up approaches are considered. We give some mathematical insights regarding the fact that information, namely, the choice of a relevant model filtration, is the major modeling issue. In this regard, we examine the notion of thinning that was recently advocated for the purpose of hedging a multi-name derivative by single-name derivatives. We then illustrate by means of numerical simulations (semi-static hedging experiments) why and when the portfolio loss process may not be a "sufficient statistic" for the purpose of valuation and hedging of portfolio credit risk.

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1 Introduction

Presently, most if not all credit portfolio derivatives have cash flows that are determined solely by the evolution of the cumulative loss process generated by the underlying portfolio. Thus, as of today, credit portfolio derivatives can be considered as derivatives of the cumulative loss process L. The consequence of this is that most of the models of portfolio credit risk, and related derivatives, focus on modeling of the dynamics of the process L, or, directly on modeling of the dynamics of the related conditional probabilities, such as

$$\text{Prob}(L \text{ takes some values at future time(s)} \mid \text{given present information}).$$

In this paper we shall study various methodologies that have been developed for this purpose, particularly the so called *top*, *top down* and *bottom up* approaches. In addition, we shall discuss the issue of *hedging* of loss process derivatives, and we shall argue that loss process may not provide a sufficient basis for this, in the sense described later in the paper. In fact, we engage in some in depth study of the *role of information* with regard to valuation and hedging of derivatives written on the loss process.

The paper is organized as follows. In Section 2 we provide an overview of the main modeling approaches that have been developed so far for handling portfolio credit derivatives. In Section 3 we revisit the notion of *thinning* that was recently advocated for the purpose of hedging a multi-name credit derivative by single-name credit derivatives, such as CDS contracts. In Section 4 we illustrate by means of numerical simulations why and when the portfolio loss process may not be a *"sufficient statistic"* for the purpose of valuation and hedging of portfolio credit risk. Conclusions and perspectives are drawn in Section 5. Finally, an Appendix gathers definitions and results from the theory of processes that we use repeatedly in this paper, such as, for instance, the definition of the *compensator* of a non-decreasing adapted process.

2 Top, Top-Down and Bottom-Up Approaches: an Overview

This section provides an overview and a discussion about the so called top, top-down and bottom-up approaches in portfolio credit risk modeling. Some related discussion can also be found in Inglis et al. [23].

Let us first introduce some standing notation:
• If X is a given process, we denote by $\mathbb{F}^X$ its natural filtration satisfying usual conditions (perhaps after completion and augmentation);
• By the $\mathbb{F}$-compensator of an $\mathbb{F}$-stopping time τ, where $\mathbb{F}$ is a given filtration, we mean the $\mathbb{F}$-compensator of the (non-decreasing) one point process $\mathbb{1}_{\tau \le t}$ (see section A.2);
• For every $d, k \in \mathbb{N}$, we denote $\mathbb{N}_k = \{0, \cdots, k\}$, $\mathbb{N}_k^* = \{1, \cdots, k\}$ and $\mathbb{N}_k^d = \{0, \cdots, k\}^d$.

From now on, t will denote the present time, and $T > t$ will denote some future time. Suppose that ξ represents a future payment at time T, which will be derived from the evolution of the loss process L on a credit portfolio, and representing a specific (stylized[1]) credit portfolio derivative claim. There may be two tasks at hand:
• to compute the time-t *price* of the claim, given the information that we may have available and we are willing to use at time t;
• to *hedge* the claim at time t. By this, we mean computing hedging sensitivities of the claim with respect to hedging instruments that are available and that we may want to use.

For simplicity we shall assume that we use spot martingale measure, say $\mathbb{P}$, for pricing, and that the interest rate is zero. Thus, denoting by $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ a filtration that represents flow of information we use for pricing, and by $\mathbb{E}$ the expectation relative to $\mathbb{P}$, the pricing task amounts to computation of the conditional expectation $\mathbb{E}(\xi \mid \mathcal{F}_t)$ (ξ being assumed $\mathcal{F}_T$-measurable and $\mathbb{P}$-integrable).

More specifically, on a standard stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we consider a (strictly) increasing sequence of stopping times t_i, for $i \in \mathbb{N}_n^*$, representing the *ordered default times* of the names in the credit pool, and

[1] Of course most credit products are swapped and involve therefore coupon streams, so in general we need to consider a *cumulative ex-dividend cash flow* ξ^t on the time interval $(t, T]$.

we define the ($\mathbb{F}$-adapted) portfolio loss process L by, for $t \geq 0$:

$$L_t = \sum_{i=1}^{n} \mathbb{1}_{t_i \leq t} \tag{1}$$

(assuming for simplicity zero recoveries). So L is a non-decreasing càdlàg process stopped at time t_n, taking its values in $\mathbb{N}_n$, with jumps of size one (L is in particular a *point process*, see, e.g., Brémaud [6], Last and Brandt [25]).

We shall then consider (stylized) portfolio loss derivatives with payoff $\xi = \pi(L_T)$, where $\pi(\cdot)$ is appropriately integrable function.

In all the paper we work under the ***standing assumption*** that the t_i's are *totally inaccessible* $\mathbb{F}$-stopping times, which is tantamount to assuming that their compensators Λ^i's are continuous processes (and are therefore stopped at the t_i's, cf. Appendix A.2). The compensator $\Lambda = \sum_{i=1}^{n} \Lambda_i$ of L is therefore in turn continuous and stopped at t_n.

Let $\tau_i,\ i \in \mathbb{N}_n^*,$ denote an arbitrary collection of (mutually avoiding) random (*not necessarily stopping*) times on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\tau_{(i)},\ i \in \mathbb{N}_n^*,$ denote the corresponding ordered sequence, that is $\tau_{(1)} < \tau_{(2)} < \cdots < \tau_{(n)}$. We denote $H_t^i = \mathbb{1}_{\tau_i \leq t}$. Accordingly, we set $H_t^{(i)} = \mathbb{1}_{\tau_{(i)} \leq t}$. So, obviously, $\sum_{i=1}^{n} H^i = \sum_{i=1}^{n} H^{(i)}$, and the representation

$$L = \sum_{i=1}^{n} H^i \tag{2}$$

holds if and only if

$$t_i = \tau_{(i)},\ i \in \mathbb{N}_n^* \tag{3}$$

(in which case the $\tau_{(i)}$'s are $\mathbb{F}$-stopping times).

From now on we assume that (2) is satisfied. The random times τ_i can thus be interpreted as the default times of the pool names, and H^i as the *default indicator process* of name i. We stress that for any i the random time τ_i may or may not be an $\mathbb{F}$-stopping time, and thus the process H^i may or may not be $\mathbb{F}$-adapted, though all the $\tau_{(i)}$'s are $\mathbb{F}$-stopping times in this case.

We denote $\mathbb{H}^i = \mathbb{F}^{H^i}$, $\mathbb{H} = \bigvee_{i \in \mathbb{N}_n^*} \mathbb{H}^i$.

2.1 Information is it!

Various approaches to valuation of derivatives written on credit portfolios differ between themselves depending on what is the content of the model filtration $\mathbb{F}$. Thus, loosely speaking, these approaches differ between themselves depending on what they presume to be sufficient information so to price, and consequently to hedge, credit portfolio derivatives.

The choice of a filtration is of course a crucial modeling issue. In particular the *compensator* Λ of an adapted non-decreasing (and bounded, say) process K, defined as the *predictable non-decreasing Doob-Meyer component* of K (see section A.2), is an information- (i.e. filtration-) dependent quantity. So is, therefore, the *intensity* process (time-derivative of Λ, assumed to exist) of K.

Let thus K denote an $\mathbb{F}$-adapted non-decreasing process and $\mathbb{G}$ be a filtration larger than $\mathbb{F}$ (so K is of course $\mathbb{G}$-adapted). Let Λ and Γ denote the $\mathbb{F}$-compensator and the $\mathbb{G}$-compensator of K, respectively. The following general result, which is proved in section A.3 (see also sections A.1 and A.2 for the various notions of projections involved), establishes the relation between Λ and Γ and the related $\mathbb{F}$- and $\mathbb{G}$- intensity processes λ and γ (whenever they exist, for the latter).

Proposition 2.1 **(i)** *Λ is the dual predictable projection of Γ on $\mathbb{F}$.*
(ii) *Moreover, in case Λ and Γ are time-differentiable with related $\mathbb{F}$- and $\mathbb{G}$- intensity processes λ and γ, then λ is the optional projection of γ on $\mathbb{F}$.*

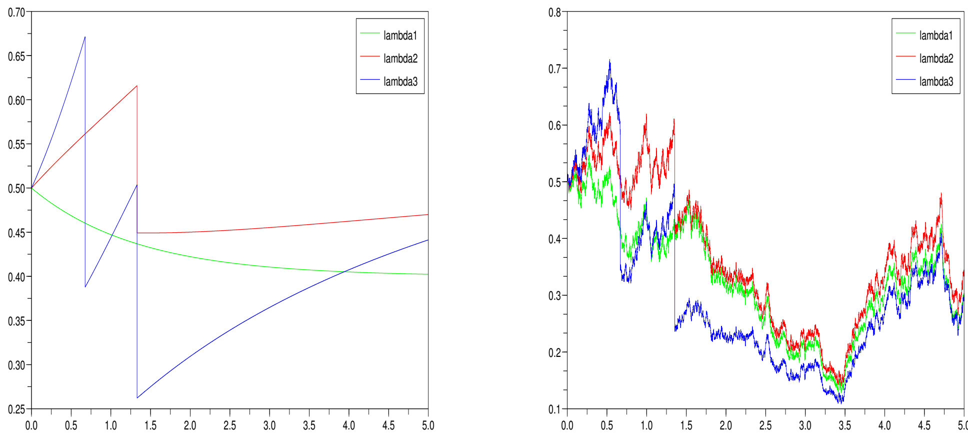

Figure 1: *Simulated sample path of the pre-default intensity of τ_1 with respect to $\mathbb{H}^1 \vee \mathbb{F}^Z$, $\mathbb{H}^1 \vee \mathbb{H}^2 \vee \mathbb{F}^Z$ or $\mathbb{H}^1 \vee \mathbb{H}^2 \vee \mathbb{H}^3 \vee \mathbb{F}^Z$, with Z trivial (constant) on the left side versus $Z = W$ on the right side.*

Figure 1 provides an illustration of the dependence of intensities on information in a simple model with $n = 3$ stopping times[2]. The figure shows a trajectory over the time interval $[0, 5]yr$ of the pre-default intensity of τ_1 with respect to $\mathbb{H}^1 \vee \mathbb{F}^Z$, $\mathbb{H}^1 \vee \mathbb{H}^2 \vee \mathbb{F}^Z$ and $\mathbb{H}^1 \vee \mathbb{H}^2 \vee \mathbb{H}^3 \vee \mathbb{F}^Z$ (curves respectively labelled `lambda1`, `lambda2` and `lambda3` in Figure 1), where the $\mathbb{H}^i$'s correspond to the filtrations generated by the H^i's, and:

- the reference filtration $\mathbb{F}^Z$ is trivial on the left side,
- it is given as a (scalar) Brownian filtration $\mathbb{F}^Z = \mathbb{F}^W$ on the right side.

We refer the reader to Zargari [30] for more details about these simulations. In this example we have $\tau_2 = 1.354$, $\tau_3 = 0.669$ in the case where $\mathbb{F}^Z$ is trivial and $\tau_2 = 1.3305$, $\tau_3 = 0.676$ in the case where $\mathbb{F}^Z = \mathbb{F}^W$ (the same random numbers were used in the two experiments). Observe that:

- `lambda2` and `lambda3` jump at τ_2 ($= 1.354$ in the left graph and 1.3305 in the right one),
- only `lambda3` jumps at τ_3 ($= 0.669$ in the left graph and 0.676 in the right one), and
- `lambda1` does not jump at all.

The facts that `lambda2` does not jump at τ_3 and `lambda1` does not jump at all are of course consistent with the definitions of `lambda1` and `lambda2` as the pre-default intensities of τ_1 with respect to $\mathbb{H}^1 \vee \mathbb{F}^Z$ and $\mathbb{H}^1 \vee \mathbb{H}^2 \vee \mathbb{F}^Z$, respectively. Also note the effect of adding a reference filtration (noisy pre-default intensities on the right side, versus pre-default intensities 'deterministic between default times' on the left side).

2.2 Top and Top-Down Approaches

The approach, that we dub the *pure top* approach takes as $\mathbb{F}$ the filtration generated by the loss process alone. Thus, in the pure top approach we have that $\mathbb{F} = \mathbb{F}^L$. Examples of this approach are Laurent, Cousin and Fermanian [26], Cont and Minca [8], or (most of) Herbertsson [22].

The approach that we dub the *top* approach takes as $\mathbb{F}$ the filtration generated by the loss process and by some additional relevant (preferably low dimensional) *auxiliary factor process*, say Y. Thus, in this case, $\mathbb{F} = \mathbb{F}^L \vee \mathbb{F}^Y$. Examples of this approach are Bennani [2], Schönbucher [28], Sidenius, Piterbarg and Andersen [29], Arnsdorf and Halperin [1] or Ehlers and Schönbucher [13].

The so-called *top-down* approach starts from *top*, that is, it starts with modeling of evolution of the

[2] We thank Behnaz Zargari from the Mathematics Departments at University of Evry, France, and Sharif University of Technology, Tehran, Iran, for these simulations.

portfolio loss process subject to information structure $\mathbb{F}$. Then, it attempts to decompose the dynamics of the portfolio loss process *down* to the individual constituent names of the portfolio, so to deduce the dynamics of processes H^i (for the purpose typically of hedging of credit portfolio derivatives by vanilla individual contracts such as default swaps). This decomposition is done by a method of *random thinning* formalized in Giesecke and Goldberg [19] (see also Halperin and Tomecek [20]), and which will be discussed in detail in Section 3.

2.3 Bottom-Up Approaches

The approach that we dub the *pure bottom-up* approach takes as $\mathbb{F}$ the filtration generated by the state of the pool process $H = (H^1, \dots, H^n)$, i.e. $\mathbb{F} = \mathbb{F}^H = \mathbb{H}$. (see, for instance, Herbertsson [21]).

The approach that we dub the *bottom-up* approach takes as $\mathbb{F}$ the filtration generated by process H and by an *auxiliary factor process* Z. Thus, in this case, $\mathbb{F} = \mathbb{F}^H \vee \mathbb{F}^Z$. Examples of this approach are Duffie and Garleanu [12], Frey and Backhaus [15, 16], Bielecki, Crépey, Jeanblanc and Rutkowski [3], or Bielecki, Vidozzi and Vidozzi [4].

2.4 Discussion

The *pure top* approach is undoubtedly the best suited for *fast valuation* of portfolio loss derivatives, as it only refers to a single driver – the loss process itself. However, this approach may produce incorrect pricing results, as it is rather unlikely that financial market evaluates derivatives of the loss process based only on the history of evolution of the loss process alone. Note in particular that loss process is not a traded instrument.

Thus, it seems to be advisable to work with a larger amount of information than the one carried by filtration $\mathbb{F}^L$ alone. This is quite likely the reason why several versions of the top approach have been developed. Enlarging filtration from $\mathbb{F}^L$ to $\mathbb{F}^L \vee \mathbb{F}^Y$ may lead to increased computational complexity, but at the same time it is quite likely to increase accuracy in calculation of important quantities, such as CDO tranche spreads and/or CDO prices.

From the *hedging* perspective both the pure top approach and the top approach may not be adequate. Indeed, operating on the top level prohibits computing sensitivities of a loss process derivative with respect to constituents of the credit portfolio. So, for example, when operating just on top level one cannot compute sensitivities of CDO tranche prices with respect to prices of the CDS contracts underlying the portfolio. In these approaches, it is only possible to hedge one loss derivative by another (e.g., hedging a CDO tranche using iTraxx). However, as we shall see in section 4, in certain circumstances this kind of hedging may not be quite precise, or even not possible at all.

This is of course the problem that led to the idea of the *top-down approach*, that is the idea of *thinning*. But, as we shall now see, it seems to us that thinning cannot really help in developing a consistent approach to hedging credit loss derivatives by single-name credit derivatives.

3 Thinning Revisited

Note that processes H^i and $H^{(i)}$ are sub-martingales, and can therefore be compensated, with respect to any filtration for which they are adapted, as non-decreasing processes (see section A.2). *Thinning* refers to the recovery of individual compensators of $H^{(i)}$ and H^i, starting from the loss compensator Λ as input data. Since the compensator is an information- (filtration-) dependent quantity, thinning of course depends on the filtration under consideration.

A preliminary question regarding thinning is why would one wish to know the individual compensators.

Suppose that all one wants to do is pricing, in other words computing the expectation $\mathbb{E}(\xi \,|\, \mathcal{F}_t)$ for $0 \leq t < T$, where the integrable random variable $\xi = \pi(L_T)$ represents the stylized payoff of a portfolio loss derivative. Under Markovian assumptions, or conditionally Markovian assumptions (assuming further factors Y), about process L with respect to the filtration $\mathbb{F}$, then, in principle, the expectation $\mathbb{E}(\xi \,|\, \mathcal{F}_t)$ can be computed (at least numerically). For computation of $\mathbb{E}(\xi \,|\, \mathcal{F}_t)$, one does not really need to know the individual compensators of the τ_i's (which do not even need to be assumed to be $\mathbb{F}$-stopping times in this regard). So, with regard to the problem of pricing of derivatives of the loss process, a top model may be fairly adequate. In particular, the filtration $\mathbb{F}$ may not necessarily contain the pool filtration $\mathbb{H}$. Also, the representation $L = \sum_{i=1}^{n} H^i$ (cf. (2)) need not be considered at all in this context.

But computing the price $\mathbb{E}(\xi \,|\, \mathcal{F}_t)$ is just one task of interest. Another key task is *hedging*. From the mathematical point of view hedging relies on the derivation of a *martingale representation* of $\mathbb{E}(\xi \,|\, \mathcal{F}_t)$, which is useful in the context of computing sensitivities of the price of ξ with respect to changes in prices of liquid instruments, such as credit indices and/or CDS contracts, corresponding to the credit names composing the credit pool underlying the loss process L. Typically, *assuming here that the τ_i's are $\mathbb{F}$ stopping times*, one will seek a martingale representation in the form

$$\mathbb{E}\left(\xi \,|\, \mathcal{F}_t\right) = \mathbb{E}\xi + \sum_{i=1}^{n} \int_0^t \zeta_s^i dM_s^i + \sum_{j=1}^{m} \int_0^t \eta_s^j dN_s^j, \tag{4}$$

where the M^i's are some fundamental martingales associated with the non-decreasing processes H^i's, and the N^j's are some fundamental martingales associated with all relevant auxiliary factors included in the model. The coefficients ζ^i's and η^j's can, in principle, be computed given a particular model specification; now, for the practical computation of the ζ^i's and η^j's, but also for the very definition of the M^i's and N^j's, one will typically need to know the compensators Λ^i's.

3.1 Thinning of the Ordered Default Times

Let $\Lambda^{(i)}$ denote the $\mathbb{F}$-compensator of $\tau_{(i)}$ (recall that the $\tau_{(i)}$ are $\mathbb{F}$-stopping times).

Proposition 3.1 *We have, for $t \geq 0$,*

$$\Lambda_t^{(i)} = \Lambda_{t\wedge\tau_{(i)}} - \Lambda_{t\wedge\tau_{(i-1)}}. \tag{5}$$

So in particular $\Lambda^{(i)} = 0$ on the set $t \leq \tau_{(i-1)}$.

Proof. Note first that

$$L_{t\wedge\tau_{(i)}} - \Lambda_{t\wedge\tau_{(i)}} \tag{6}$$

is an $\mathbb{F}$-martingale, as it is equal to the $\mathbb{F}$-martingale $L - \Lambda$ (cf. equation (21) in the Appendix) stopped at the $\mathbb{F}$-stopping time $\tau_{(i)}$. Taking the difference between expression in (6) for i and $i-1$ yields that $H_t^{(i)} - \bar{\Lambda}_t^{(i)}$, with $\bar{\Lambda}_t^{(i)}$ defined as the RHS of (5), is an $\mathbb{F}$-martingale (starting at $\tau_{(i-1)}$ and stopped at $\tau_{(i)}$). Hence (5) follows, due to uniqueness of compensators (recall Λ is continuous, so $\bar{\Lambda}^{(i)}$ is continuous, hence predictable). □

Formula (5) represents the 'ordered thinning' of Λ. Note that Proposition 3.1 is true regardless of whether the τ_i's are $\mathbb{F}$-stopping times or not. This reflects the fact that modeling the loss process L is the same as modeling the ordered sequence of the $\tau_{(i)}$'s, no matter what is the informational context of the model otherwise.

3.2 Thinning of the Default Times

Let us first denote by Λ^i the $\mathbb{F}$-compensator of τ_i, assumed to be an $\mathbb{F}$-stopping time. We of course have that

$$\Lambda = \sum_{i=1}^{n} \Lambda^i. \tag{7}$$

Moreover, the following is true.

Proposition 3.2 *There exists $\mathbb{F}$-predictable non-negative processes Z^i, $i \in \mathbb{N}_n^*$, such that $Z^1 + Z^2 + \cdots + Z^n = 1$ and*

$$\Lambda^i = \int_0^{\cdot} Z_t^i d\Lambda_t, \; i \in \mathbb{N}_n^*. \tag{8}$$

Proof. In view of (7), existence of $Z^i = \frac{d\,\Lambda^i}{d\,\Lambda}$ follows from Theorem VI 68, page 130, in Dellacherie and Meyer [11] (see also Giesecke and Goldberg [19]). □

In the special case where the random times τ_i's constitute an ordered sequence, so $\tau_i = \tau_{(i)}$, then the ordered thinnning formula (5) yields that $Z_t^i = \mathbb{1}_{\tau_{i-1} < t \leq \tau_i}$.

Proposition 3.2 tells us that, if one starts building a model from top, that is, if one starts building the model by first modeling the $\mathbb{F}$-compensator Λ of the loss process L, then the only way to go down relative to the information carried by $\mathbb{F}$, i.e., to obtain $\mathbb{F}$-compensators Λ^i, is to do thinning in the sense of equation (8). We shall refer to this as to $\mathbb{F}$-*thinning* of Λ.

3.2.1 Thinning with respect to a Sub-filtration

Now, suppose that $\mathbb{F}^i$ is some sub-filtration of $\mathbb{F}$ and that τ_i is an $\mathbb{F}^i$-stopping time. We want to compute the $\mathbb{F}^i$-compensator $\widehat{\Lambda}^i$ of τ_i, starting with Λ.

The first step is to do the $\mathbb{F}$-thinning of Λ, that is, to obtain the $\mathbb{F}$-compensator Λ^i of τ_i (cf. (8)). The second step is to obtain the $\mathbb{F}^i$-compensator $\widehat{\Lambda}^i$ of τ_i from Λ^i. The following results follows by application of Proposition 2.1.

Proposition 3.3 *$\widehat{\Lambda}^i$ is the dual predictable projection of Λ^i on $\mathbb{F}^i$. Moreover, in case $\widehat{\Lambda}^i$ and Λ^i are time-differentiable with related $\mathbb{F}^i$- and $\mathbb{F}$- intensity processes $\widehat{\lambda}^i$ and λ^i, then $\widehat{\lambda}^i$ is the optional projection of λ^i on $\mathbb{F}^i$.*

Remark 3.1 Note that $\widehat{\Lambda}^i$ is also the dual predictable projection of H^i on $\mathbb{F}^i$ (see section A.2).

Proposition 3.3 is important regarding the issue of *calibration* of a portfolio credit model to marginal data, one of the key issues in relation with hedging a credit loss derivative by single-name credit instruments. For example, one may want to calibrate the credit portfolio model to spreads on individual CDS contracts. If the spread on the i^{th} CDS contract is computed using conditioning with respect to $\mathbb{F}^i$, then the $\mathbb{F}^i$-intensity $\widehat{\lambda}^i$ of τ_i will typically be used as an input data in the calibration (for determining an $\mathbb{F}$-adapted process λ^i with $\mathbb{F}^i$-optional projection $\widehat{\lambda}^i$ given in the market).

3.3 The case when τ_i's are not stopping times

In case τ_i is not an $\mathbb{F}$-stopping time, Giesecke and Goldberg [19] introduce a notion of (we call it *top-down*) intensity of τ_i, defined as the time-derivative, assumed to exist, of the dual predictable projection of H^i on $\mathbb{F}$. In view of Remark 3.1, this is indeed a generalization of the usual notion of intensity to the case where τ_i is not an $\mathbb{F}$-stopping time.

However, our opinion is that such a top-down intensity does not make much sense. Indeed the *market* intensity of name i (intensity of name i as extracted from the marginal market data on name i, typically the CDS curve on i) corresponds to an intensity in a filtration adapted to τ_i, which in particular vanishes after τ_i (contrarily to a top-down intensity, unless τ_i is an $\mathbb{F}$-stopping time). A top-down intensity is thus not represented in the market, and it can therefore not be calibrated (unless, again, τ_i is an $\mathbb{F}$-stopping time).

3.4 Limitations of Thinning

In view of the above observations, one must, in our opinion, restrict consideration of thinning to the case where all τ_i's are $\mathbb{F}$-stopping times, that is, to the case thinning in the sense of section 3.2. Observe though that thinning in this sense is equivalent to building the model from the bottom up. This is because modeling of processes Λ and Z^i's, that show in Proposition 3.2, is equivalent to modeling the processes Λ^i's. The relevance of top-down construction of a model by thinning with respect to a filtration containing all the $\mathbb{H}^i$'s (so, a bottom-up model, ultimately) thus seems questionable.

In a defense of such an approach one might say that, since this approach starts from top, then it gives the modeler a better control over designing the dynamics of the portfolio loss process L so to tailor-design this process through a 'nice' and simple dynamics. But the point is precisely that in a model with a 'nice' and simple top portfolio loss process L, there is no need of use of single-name instruments for hedging. In fact, typically a small number of other loss derivatives will be able to do the hedging job (see, for instance, Laurent, Cousin and Fermanian [26]). Models with a 'too simple' loss process L are actually not a good family for considering the issue of hedging credit loss derivatives by single-name instruments, because single-name instruments are, in principle, not required for hedging in such model.

4 Sufficient Statistics

For credit derivatives with stylized payoff given as $\xi = \pi(L_T)$ at maturity time T, it is tempting to adopt a Black–Scholes like approach, modeling L as a Markov process and performing factor hedging of one derivative by another, balancing the related sensitivities computed by the Itô-Markov formula (see, for instance, Laurent, Cousin and Fermanian [26]). However, since the loss process L may be far from Markovian in the market, there may be circumstances under which L is not a "sufficient statistic" for the purpose of valuation and hedging of portfolio credit risk. In other words, ignoring the potentially non-Markovian dynamics of L for pricing and/or hedging may cause significant model risk, even though the payoffs of the products at hand are given as functions of L_T.

In this section we want to illustrate this point by means of numerical hedging simulations (see also Cont and Kan [7] for an extensive empirical study of the real-life hedging performances of a variety of top models on pre- as well as post-crisis data sets). For these numerical experiments we introduce a non-zero recovery R, taken as a constant $R = 40\%$. We thus need to distinguish the *cumulative default process* $N_t = \sum_{i=1}^n H_t^i$ and the *cumulative loss process* $L_t = (1 - R)N_t$.

We shall consider the benchmark problem of pricing and hedging a stylized loss derivative. Specifically, for simplicity, we only consider protection legs of *of equity tranches*, resp. *super-senior tranches* (i.e. detachment of 100%), with stylized payoffs

$$\pi(N_T) = \frac{L_T}{n} \wedge k \text{ , resp. } \left(\frac{L_T}{n} - k\right)^+$$

at a maturity time T. The 'strike' (detachment, resp. attachment point) k belongs to $[0, 1]$. In this formalism the stylized *credit index* corresponds to the equity tranche with $k = 100\%$ (or senior tranche with $k = 0$). With a slight abuse of terminology, we shall refer to our stylized loss derivatives as to *tranches* and *index*, respectively.

We shall now consider the problem of hedging the tranches with the index, using a simplified market model of credit risk.

4.1 Homogeneous Groups Model

We consider a Markov chain model of credit risk as of Frey and Backhaus [16] (see also Bielecki et al. [3]). Namely, the n names of a pool are grouped in d classes of $\nu - 1 = \frac{n}{d}$ homogeneous obligors (assuming $\frac{n}{d}$

integer). The cumulative default processes $N^l, l \in \mathbb{N}_d^*$ of different groups are jointly modeled as a d-variate Markov point process $\mathcal{N}$, with $\mathbb{F}^{\mathcal{N}}$-intensity of N^l given as

$$\lambda_t^l = (\nu - 1 - N_t^l)\widetilde{\lambda}^l(t, \mathcal{N}_t) , \tag{9}$$

for some *pre-default individual intensity functions* $\widetilde{\lambda}^l = \widetilde{\lambda}^l(t, \imath)$, where $\imath = (i_1, \cdots, i_d) \in \mathbb{N}_{\nu-1}^d$. The related infinitesimal generator at time t may then be written in the form of a ν^d-dimensional (very sparse) matrix, say $\mathcal{A}_t$. Also note that $N = \sum N^l$.

For $d = 1$, we recover the well-known *local intensity model* (N modeled as a Markov birth point process stopped at level n) of Laurent, Cousin and Fermanian [26] or Cont and Minca [8]. At the other extreme, for $d = n$, we are in effect modeling the vector of the default indicator processes of the pool names. As d varies between 1 and n, we thus get a variety of models of credit risk, ranging from pure top models for $d = 1$ to pure bottom-up models for $d = n$.

Remark 4.1 Observe that in the *homogeneous case* where $\widetilde{\lambda}^l(t, \imath) = \widehat{\lambda}(t, \sum_j i_j)$ for some function $\widehat{\lambda} = \widehat{\lambda}(t, i)$ (independent of l), the model effectively reduces to a local intensity model (with $d = 1$ and pre-default individual intensity $\widehat{\lambda}(t, i)$ therein).
Further specifying the model to $\widehat{\lambda}$ independent of i corresponds to the situation of homogeneous and independent obligors.
In general, introducing parsimonious parameterizations of the intensities allows one to account for inhomogeneity between groups and/or defaults contagion. It is also possible to extend this set-up to more general credit migrations models, or to generic bottom-up models of credit migrations influenced by macro-economic factors (see Bielecki et al. [3, 4] or Frey and Backhaus [17]).

4.1.1 Pricing in the Homogeneous Groups Model

Since $\mathcal{N}$ is a Markov process and N_t is a function of $\mathcal{N}_t$, the related tranche price process writes, for $t \in [0, T]$ (assuming $\pi(N_T)$ integrable):

$$\Pi_t = \mathbb{E}(\pi(N_T) \,|\, \mathcal{F}_t^{\mathcal{N}}) = u(t, \mathcal{N}_t) , \tag{10}$$

where $u(t, \imath)$ or $u_\imath(t)$ for $t \in [0, T]$ and $\imath \in \mathbb{N}_{\nu-1}^d$, is the *pricing function* (system of time-functions $u_\imath$). Using the Itô formula in conjunction with the martingale property of Π, the pricing function can then be characterized as the solution to the following *pricing equation* (system of ODEs):

$$(\partial_t + \mathcal{A}_t)u = 0 \text{ on } [0, T) \tag{11}$$

with terminal condition $u_\imath(T) = \pi(\imath)$, for $\imath \in \mathbb{N}_{\nu-1}^d$. In particular, in the case of a time-homogeneous generator $\mathcal{A}$ (independent of t), one has the semi-closed matrix exponentiation formula,

$$u(t) = e^{(T-t)\mathcal{A}}\pi . \tag{12}$$

Pricing in this model can be achieved by various means, like numerical resolution of the ODE system (11), numerical matrix exponentiation based on (12) (in the time-homogeneous case) or Monte Carlo simulation. However resolution of (11) or computation of (12) by deterministic numerical schemes is typically precluded by the curse of dimensionality for d greater than a few units (depending on ν). So for high d simulation methods appear to be the only viable computational alternative. Appropriate variance reduction methods may help in this regard (see, for instance, Carmona and Crépey [9]).

The distribution of the vector of time-t losses (for each group), that is, $q_\imath(t) = \mathbb{P}(\mathcal{N}_t = \imath)$ for $t \in [0, T]$ and $\imath \in \mathbb{N}_{\nu-1}^d$, and the portfolio cumulative loss distribution, $p = p_i(t) = \mathbb{P}(N_t = i)$ for $t \in [0, T]$ and $i \in \mathbb{N}_n$, can be computed likewise by numerical solution of the associated forward Kolmogorov equations (for more detail, see, e.g., [9]).

4.1.2 Hedging in the Homogeneous Groups Model

In general, in the Markovian model described above, it is possible to replicate dynamically in continuous time any payoff provided d non-redundant hedging instruments are available (see Frey and Backhaus [15] or Bielecki, Vidozzi and Vidozzi [4]; see also Laurent, Cousin and Fermanian [26] for results in the special case where $d = 1$). From the mathematical side this corresponds to the fact that in general this model is of *multiplicity* d (model with d fundamental martingales, see, e.g., Davis and Varaiya [10]). So, in general, it is not possible to replicate a payoff, such as tranche, *by the index alone* in this model, unless the model dimension d is equal to one (or reducible to one, cf. Remark 4.1). Now our point is that this potential lack of replicability is not purely speculative, but can be very significant in practice.

Since delta-hedging in continuous time is expensive in terms of transaction costs, and because main changes occur at default times in this model (in fact, default times are the only events in this model, if not for time flow and the induced time-decay effects), we shall focus on *semi-static hedging* in what follows, only updating at default times the composition of the hedging portfolio. More specifically, denoting by t_1 the first default time of a reference obligor, we shall examine the result at t_1 of a static hedging strategy on the random time interval $[0, t_1]$.
Let Π and P denote the tranche and index model price processes, respectively. Using a constant hedge ratio $\widehat{\delta}_0$ over the time interval $[0, t_1]$, the *tracking error* or *profit-and-loss* of a delta-hedged tranche at t_1 writes:

$$e_{t_1} = (\Pi_{t_1} - \Pi_0) - \widehat{\delta}_0 (P_{t_1} - P_0) . \tag{13}$$

The question we want to consider is whether it is possible to make this quantity 'small', in terms, say, of variance, relative to the variance of $\Pi_{t_1} - \Pi_0$ (which corresponds to the risk without hedging), by a suitable choice of $\widehat{\delta}_0$. It is expected that this should depend:
• First, on the characteristics of the tranche, and in particular on the value of the strike k: A high strike equity tranche or low strike senior tranche is quite close to the index in terms of cash flows, and should therefore exhibit a higher degree of correlation and be easier to hedge with the index, than a low strike equity tranche or high strike senior tranche;
• Second, on the 'degree of Markovianity' of the loss process L, which in the case of the homogeneous groups model depends both on the model nominal dimension d and on the specification of the intensities (see, e.g., Remark 4.1).

Moreover, it is intuitively clear that for too large values of t_1 time-decay effects matter and the hedge should be rebalanced at some intermediate points of the time interval $[0, t_1]$ (even though no default occurred yet). To keep it as simple as possible we shall merely apply a cutoff and restrict our attention to the random set $\{\omega : t_1(\omega) < T_1\}$ for some fixed $T_1 \in [0, T]$.

4.2 Numerical Results

We work with the above model for $d = 2$ and $\nu = 5$. We thus consider a two-dimensional model of a stylized credit portfolio of $n = 8$ obligors. The model generator is a $\nu^d \otimes \nu^d$ – (sparse) matrix with $\nu^{2d} = 5^4 = 625$. Recall that the computation time for exact pricing using matrix exponentiation based on (12) in such model grows as ν^{2d}, which motivated the previous modest choices for d and ν.

Moreover we take the $\widetilde{\lambda}^l$'s given by (cf. (9)),

$$\widetilde{\lambda}^1(t, \imath) = \frac{2(1 + i_1)}{9n} , \ \widetilde{\lambda}^2(t, \imath) = \frac{16(1 + i_2)}{9n} . \tag{14}$$

So in this case, which is an admittedly extreme case of inhomogeneity between two independent groups of obligors, the individual intensities of the obligors of group 1 and 2 are given as $\frac{1+i_1}{36}$ and $\frac{8(1+i_2)}{36}$, where i_1 and i_2 represent the number of currently defaulted obligors in groups 1 and 2, respectively.
For instance, at time 0 with $\mathcal{N}_0 = (0, 0)$, the individual intensities of obligors of group 1 and 2 are equal to $1/36$ and $8/36$, respectively; the average individual intensity at time 0 is thus equal to $1/8 = 0.125 = 1/n$.

We set the maturity T equal to 5 years and the cutoff T_1 equal to 1 year. We thus make a focus on the random set of trajectories for which $t_1 < 1$, meaning that a default occurred during the first year of hedging.

In this toy model the simulation takes the following very simple form:
Compute Π_0 for the tranche and P_0 for the index by numerical matrix exponentiation based on (12). Then, for every $j = 1, \cdots, m$:
• Draw a pair $(\widetilde{t}_1^j, \widehat{t}_1^j)$ of independent exponential random variables with parameter (cf. (9), (14))

$$(\lambda_0^1, \lambda_0^2) = 4 \times (\frac{1}{36}, \frac{8}{36}) = (\frac{1}{9}, \frac{8}{9}) \; ; \tag{15}$$

• Set $t_1^j = \min(\widetilde{t}_1^j, \widehat{t}_1^j)$ and $\mathcal{N}_{t_1} = (1,0)$ or $(0,1)$ depending on whether $t_1^j = \widetilde{t}_1^j$ or $\widehat{t}_1^j$;
• Compute $\Pi_{t_1^j}$ for the tranche and $P_{t_1^j}$ for the index by (12).

Doing this for $m = 10^4$, we got 9930 draws with $t_1 < T = 5yr$, among which 6299 ones with $t_1 < T_1 = 1yr$, subdividing themselves into 699 defaults in the first group of obligors and 5600 defaults in the second one.

4.2.1 Pricing Results

We consider two $T = 5yr$-tranches in the above model: an 'equity tranche' with $k = 30\%$, corresponding to a payoff $\frac{(1-R)N_T}{n} \wedge k = (\frac{60N_T}{8} \wedge 30)\%$ (of a unit nominal amount), and a 'senior tranche' defined simply as the complement of the equity tranche to the index, thus with payoff $(\frac{(1-R)N_T}{n} - k)^+ = (\frac{60N_T}{8} - 30)^+\%$.

We computed the portfolio loss distribution at maturity by numerical matrix exponentiation corresponding to explicit solution of the associated forward Kolmogorov equations (see, e.g., [9]).

Note that there is virtually no error involved in the previous computations, in the sense that our simulation is exact (without simulation bias), and the prices and loss probabilities are computed by numerical quasi-exact matrix exponentiation.

The left side of Figure 2 represents the histogram of the loss distribution at the time horizon T; we indicate by a vertical line the loss level x beyond which the equity tranche is wiped out, and the senior tranche starts being hit (so $\frac{(1-R)x}{n} = k$, i.e. $x = 4$).
The right side of Figure 2 displays the equity (labeled by $+$), senior ($\times$) and index ($\circ$) tranche prices at t_1 (in ordinate) versus t_1 (in abscissa), for all the points in the simulated data with $t_1 < 5$ (9930 points). Blue and red points correspond to defaults in the first ($\mathcal{N}_{t_1} = (1,0)$) and in the second ($\mathcal{N}_{t_1} = (0,1)$) group of obligors, respectively. We also represented in black the points $(0, \Pi_0)$ (for the equity tranche and the senior tranche) and $(0, P_0)$ (for the index).

Note that in the case of the senior tranche and of the index, there is a clear difference between prices at t_1 depending on whether t_1 corresponds to a default in the first or in the second group of obligors, whereas in the case of the equity tranche there seems to be little difference in this regard.
In view of the portfolio loss distribution in the left side, this can be explained by the fact that in the case of the equity tranche, the probability conditional on t_1 that the tranche will be wiped out at maturity is important unless t_1 is rather large. Therefore the equity tranche price at t_1 is close to $k = 30\%$ for t_1 close to 0. Moreover for t_1 close to T the intrinsic value of the tranche at t_1 constitutes the major part of the equity tranche price at t_1, for the tranche has low time-value close to maturity. In conclusion the state of $\mathcal{N}$ at t_1 has a low impact on Π_{t_1}, unless t_1 is in the middle of the time-domain.
On the other hand, in the case of the senior tranche or in case of the index, the state of $\mathcal{N}$ at t_1 has a high impact on the corresponding price, unless t_1 is close to T (in which case intrinsic value effects are dominant). This explains the 'two-track' pictures seen for the senior tranche and for the index in the right side of Figure 2, except close to T (whereas the two-tracks are superimposed close to 0 and T in the case of the equity tranche).

Looking at these results in terms of price changes $\Pi_0 - \Pi_{t_1}$ of a tranche versus the corresponding index price changes $P_0 - P_{t_1}$, we obtain the graphs of Figure 3 for the equity tranche and 4 for the senior tranche. We consider all points with $t_1 < T$ in the left sides and focus on the points with $t_1 < T_1$ in the right sides.

We use the same blue/red color code as above, and we further highlight in green in the left sides the points with $t_1 < 1$, which are focused upon in the right sides.
Figure 3 gives a further graphical illustration of the low level of correlation between price changes of the equity tranche and of the index. Indeed the cloud of points on the right side is obviously "far from a straight line", due to the partitioning of points between blue points / defaults in group one on one segment versus red points / defaults in group two on a different segment.
On the opposite (Figure 4), at least for t_1 not too far from 0 (see the zoom on the points for which $t_1 < 1$ in the right side), there is an evidence of linear correlation between price changes of the senior tranche and of the index, since in this case the blue and the red segments are not far from being on a common line.

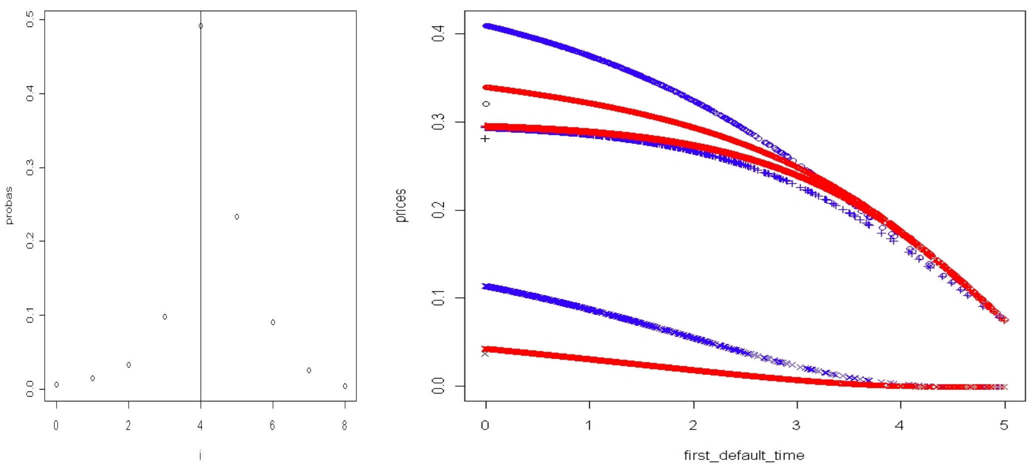

Figure 2: *(Left) Portfolio loss distribution at maturity $T = 5yr$; (Right) Tranche prices at t_1 for $t_1 < T = 5$ (equity tranche $(+)$, senior tranche $(\times)$ and index $(\circ)$).). On this and the following figures, blue and red points correspond to defaults in the first and in the second group of obligors, respectively.*

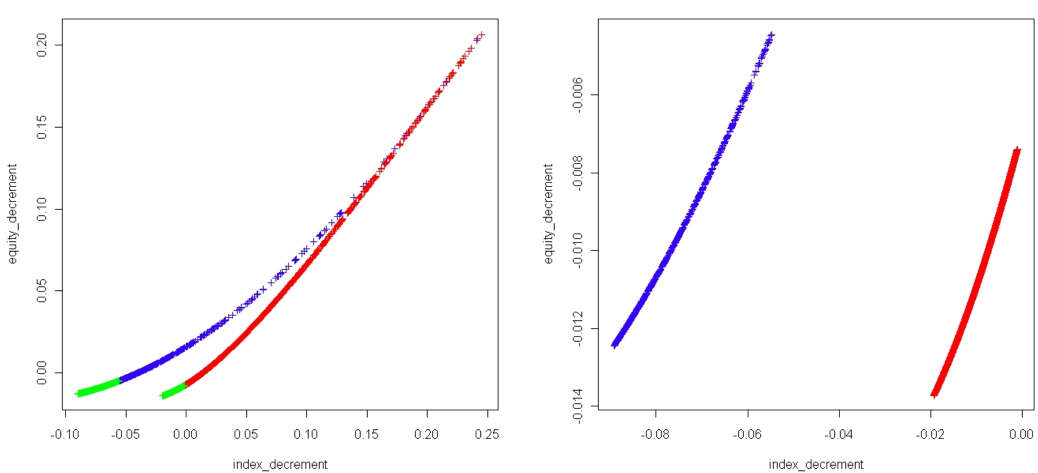

Figure 3: *Equity tranche vs Index Price Changes between 0 and t_1 (Left: $t_1 < T = 5$; Right: zoom on $t_1 < T_1 = 1$).*

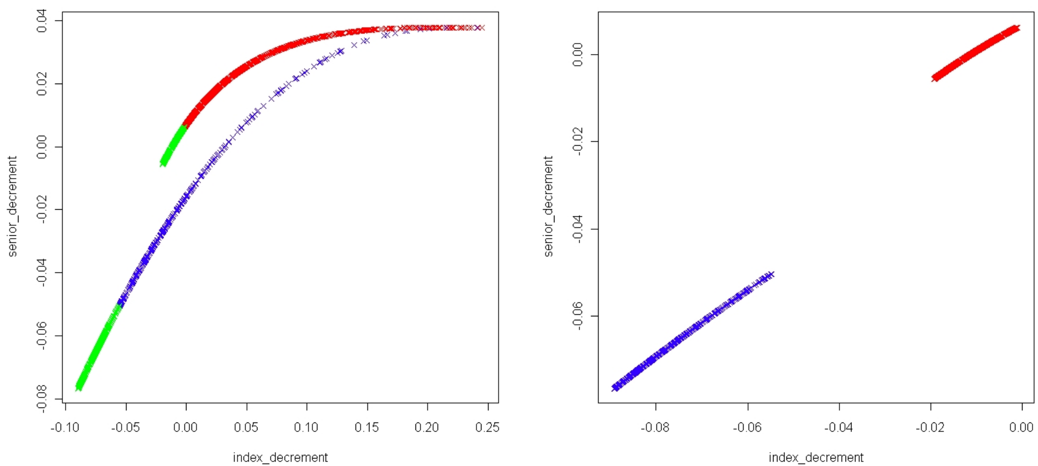

Figure 4: *Senior tranche vs Index Price Changes between 0 and t_1 (Left: $t_1 < T = 5$; Right: zoom on $t_1 < T_1 = 1$).*

4.2.2 Hedging Results

We then computed the empirical variance of $\Pi_{t_1} - \Pi_0$ and of the profit-and-loss e_{t_1} in (13) on the subset $t_1 < T_1 = 1$ of the trajectories, using for $\widehat{\delta}_0$ the empirical regression delta of the tranche with respect to the index at time 0, so

$$\widehat{\delta}_0 = \frac{\widehat{\mathbb{C}\mathrm{ov}}(\Pi_{t_1} - \Pi_0, P_{t_1} - P_0)}{\widehat{\mathbb{V}\mathrm{ar}}(P_{t_1} - P_0)} . \tag{16}$$

Moreover, we also did these computations restricting further attention to the subsets of $t_1 < 1$ corresponding to defaults in the first and in the second group of obligors (blue and red points on the figures), respectively. The latter results are to be understood as giving proxies of the situation that would prevail in a one-dimensional complete model of credit risk ('local intensity model' for N, see section 4.2.3).

The results are displayed in Tables 1 and 2.

In Table 1 we denote by:

- $\Sigma_0 = \frac{10^4}{kT}\Pi_0$ or $\frac{10^4}{(1-R-k)T}\Pi_0$ (for the equity or senior tranche) or $S_0 = \frac{10^4}{(1-R)T}P_0$ (for the index), stylized 'bp spreads' corresponding to the time zero prices Π_0 and P_0 of the equity or senior tranche and of the index;
- δ_0^1, δ_0^2 and δ_0, the functions $\frac{\delta^1 u}{\delta^1 v}$, $\frac{\delta^2 u}{\delta^2 v}$ and the *continuous time min-variance delta function* (as it follows easily by application of a bilinear regression formula)

$$\frac{\lambda^1(\delta^1 u)(\delta^1 v) + \lambda^2(\delta^2 u)(\delta^2 v)}{\lambda^1(\delta^1 v)^2 + \lambda^2(\delta^2 v)^2} = \frac{\lambda^1(\delta^1 v)^2}{\lambda^1(\delta^1 v)^2 + \lambda^2(\delta^2 v)^2}(\frac{\delta^1 u}{\delta^1 v}) + \frac{\lambda^2(\delta^2 v)^2}{\lambda^1(\delta^1 v)^2 + \lambda^2(\delta^2 v)^2}(\frac{\delta^2 u}{\delta^2 v})$$

evaluated at $t = 0$ and $\imath = \mathcal{N}_{0-} = (0,0)$, so

$$\delta_0^1 = \frac{u_{1,0} - u_{0,0}}{v_{1,0} - v_{0,0}}(0) \; , \; \delta_0^2 = \frac{u_{0,1} - u_{0,0}}{v_{0,1} - v_{0,0}}(0) \tag{17}$$

$$\delta_0 = \frac{\lambda_0^1(u_{1,0} - u_{0,0})(v_{1,0} - v_{0,0}) + \lambda_0^2(u_{0,1} - u_{0,0})(v_{0,1} - v_{0,0})}{\lambda_0^1(v_{1,0} - v_{0,0})^2 + \lambda_0^2(v_{0,1} - v_{0,0})^2} \tag{18}$$

where we recall from (15) that $(\lambda_0^1, \lambda_0^2) = (\frac{1}{9}, \frac{8}{9})$.
The three deltas δ_0^1, δ_0^2 and δ_0 were thus computed by matrix exponentiation based on (12) for the various terms $u, v_\imath(0)$ involved in formulas (17), (18). Note that the prices and deltas of the equity and senior tranche of same strike k respectively sum up to P and to one, by construction (see also Table 2 for $\widehat{\delta}_0$). So the results

for the senior tranche could be deduced from those for the equity tranche and conversely. However we present detailed results for the equity and senior tranche, for the reader's convenience.

Remark 4.2 The instantaneous min-variance delta δ_0 (which is a suitably weighted average of δ_0^1 and δ_0^2) can be considered as a measure of the *distance to the index* of a tranche: *far-from-the-index* low strike equity tranche or high strike senior tranche with δ_0 less than 0.5, versus *close-to-the-index* high strike equity tranche or low strike senior tranche with δ_0 greater than 0.5. The further from the index a tranche and/or 'the less Markovian' a porfolio loss process L, the poorer the hedge by the index (cf. end of section 4.1.2).

	Π_0 or P_0	Σ_0 or S_0	δ_0^1	δ_0^2	δ_0
Eq	0.2821814	1881.209	0.1396623	0.7157741	0.2951399
Sen	0.03817907	254.5271	0.8603377	0.2842259	0.7048601

Table 1: *Time $t = 0$ – Prices, Spreads and Instantaneous Deltas in the Semi-Homogeneous Model.*

In Table 2 (cf. also (16)):

- ρ in column two is the empirical correlation of the tranche price increments $\Pi_{t_1} - \Pi_0$ versus the index price increments $P_{t_1} - P_0$,
- $R2 = 1 - \frac{\widehat{\mathbb{V}\mathrm{ar}}(e_{t_1})}{\widehat{\mathbb{V}\mathrm{ar}}(\Pi_{t_1} - \Pi_0)}$ in column three is the *coefficient of determination* of the regression, which, in the present set-up of a simple linear regression, coincides with ρ^2,
- Dev in column 4 stands for $\widehat{\mathrm{Stdev}}(\Pi_{t_1} - \Pi_0)/\Pi_0$,
- The *hedging variance reduction factor* RedVar $= \frac{\widehat{\mathbb{V}\mathrm{ar}}(\Pi_{t_1} - \Pi_0)}{\widehat{\mathbb{V}\mathrm{ar}}(e_{t_1})}$ in the last column is equal to $\frac{1}{1-R2} = \frac{1}{1-\rho^2}$.

Remark 4.3 It is expected that $\widehat{\delta}_0$ should converge to δ_0 in the limit where the cutoff T_1 would tend to zero, provided the number of simulations m jointly goes to infinity. For $T_1 = 1yr$ and $m = 10^4$ simulations however, we shall see below that there is a clear discrepancy between δ_0 and $\widehat{\delta}_0$, and all the more so that we are in a non-homogeneous model with low correlation between the tranche and index price changes between times 0 and t_1. The reason is that the *coefficient of determination* of the linear regression with slope $\widehat{\delta}_0$ is given by $R2 = \rho^2$. In case ρ is small, $R2$ is even smaller, and the significance of the estimator (for low T_1's) $\widehat{\delta}_0$ of δ_0 is low too. In other words, in case ρ is small, we recover mainly noise through $\widehat{\delta}_0$. This however does not weaken our statements below regarding the ability or not to hedge the tranche by the index, since the variance reduction factor RedVar $= \frac{\widehat{\mathbb{V}\mathrm{ar}}(\Pi_{t_1} - \Pi_0)}{\widehat{\mathbb{V}\mathrm{ar}}(e_{t_1})}$ is equal to $\frac{1}{1-\rho^2}$, which for ρ small is close to one, whatever the noisy value of $\widehat{\delta}_0$ may be.

	$\widehat{\delta}_0$	ρ	$R2$	Dev	RedVar
Eq	-0.00275974	-0.03099014	0.0009603885	0.006612626	**1.000961**
Eq1	0.2269367	0.9980242	0.9960522	0.007576104	253.306
Eq2	0.3391563	0.997375	0.994757	0.006134385	190.7276
Sen	1.002760	0.9960836	0.9921825	0.07475331	**127.9176**
Sen1	0.7730633	0.9998293	0.9996586	0.02576152	2928.847
Sen2	0.6608437	0.9993066	0.9986137	0.01192970	721.3244

Table 2: *Hedging Tranches by the Index in the Semi-Homogeneous Model.*

Recall that qualitatively the senior tranche's dynamics is rather close to that of the index (at least for t_1 close to 0, see Section 4.2.1, right side of Figure 4). Accordingly, we find that hedging the senior tranche with the index is possible (variance reduction factor of about 128 in bold blue in the last column). This case thus seems to be supportive of the claim according to which one could use the index for hedging a loss derivative, even in a non Markovian model of portfolio loss process L.

But in the case of the equity tranche we get the opposite message: the index is useless for hedging the equity tranche (variance reduction factor essentially equal to 1 in bold red in the table, so *no variance reduction* in this case).
Moreover, the equity tranche variance reduction factors conditional on defaults in the first and in the second group of obligors (in purple in the table) amount to 253 and 190, respectively. This supports the interpretation that the unhedgeability of the equity tranche by the index really comes from the fact that the full model dynamics is not represented in the loss process L.

Incidentally this also means that hedging the senior tranche by the equity tranche, or vice versa, is not possible either.

We conclude that in general, at least for certain ranges of the model parameters and tranche characteristics (strongly non-Markovian loss process L and/or far-from-the-index tranche), hedging tranches with the index may not be possible.

Since the equity and the senior tranche sum-up to the index, therefore a perfect static replication of the equity tranche is provided by a long position in the index and a short position in the senior tranche. As a reality-check of this statement, we performed a bilinear regression of the equity price increments versus the index and the senior tranche price increments, in order to minimize over $(\widehat{\delta}_0^{ind}, \widehat{\delta}_0^{sen})$ the (risk-neutral) variance of

$$\widetilde{e}_{t_1} = (\Pi_{t_1}^{eq} - \Pi_0^{eq}) - \widehat{\delta}_0^{ind}(P_{t_1} - P_0) - \widehat{\delta}_0^{sen}(\Pi_{t_1}^{sen} - \Pi_0^{sen}) \ . \tag{19}$$

The results are displayed in Tables 3. We recover numerically the perfect two-instruments replication strategy which was expected theoretically, whereas a single-instrument hedge using only the index was essentially useless in this case (cf. bold red entry in Table 2).

$\widehat{\delta}_0^{ind}$	$\widehat{\delta}_0^{sen}$	RedVar
1	-1	2.56e+29

Table 3: *Replicating the equity tranche by the index and the senior tranche in the Semi-Homogeneous Model.*

4.2.3 Fully Homogeneous Case

For confirmation of the previous analysis and interpretation of the results, we redid the computations using the same values as before for all the model, products and simulation parameters, except for the fact that the following pre-default individual intensities were used, for $l = 1, 2$:

$$\widetilde{\lambda}^l(\imath) = \frac{1}{n} + \frac{\sum_{1 \le \ell \le d} i_\ell}{nd} =: \widehat{\lambda}(\sum_{1 \le \ell \le d} i_\ell) \ . \tag{20}$$

For instance, at time 0 with $\mathcal{N}_0 = 0$, the individual intensities of the obligors are all equal to $1/8 = 0.125 = 1/n$.

We are thus in a case of homogeneous obligors, reducible to a local intensity model (with $d = 1$ and pre-default individual intensity $\widehat{\lambda}(i)$ therein, see Remark 4.1). So in this case we expect that hedging tranches by the index should work, including in the case of the far-from-the-index equity tranche.

This is what happens numerically, as it is evident from the following Figures and Tables (which are the analogs of those in previous sections, using the same notation everywhere). Note that all red and blue curves are now superimposed, which is consistent with the fact that the group of a defaulted name has no bearing in this case, given the present specification of the identities.

Out of new 10^4 draws using the intensities given in (20), we got 9922 draws with $t_1 < 5$, among which 6267 ones with $t_1 < 1$, subdividing themselves into 3186 defaults in the first group of obligors and 3081 defaults in the second one.

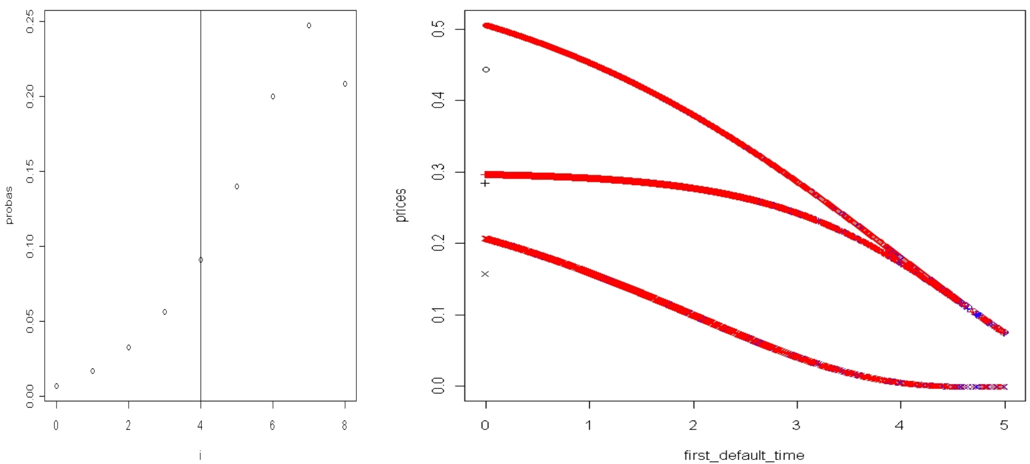

Figure 5: *(Left) Portfolio loss distribution at maturity $T = 5y$ (Right) Tranche Prices at t_1 (for $t_1 < T$).*

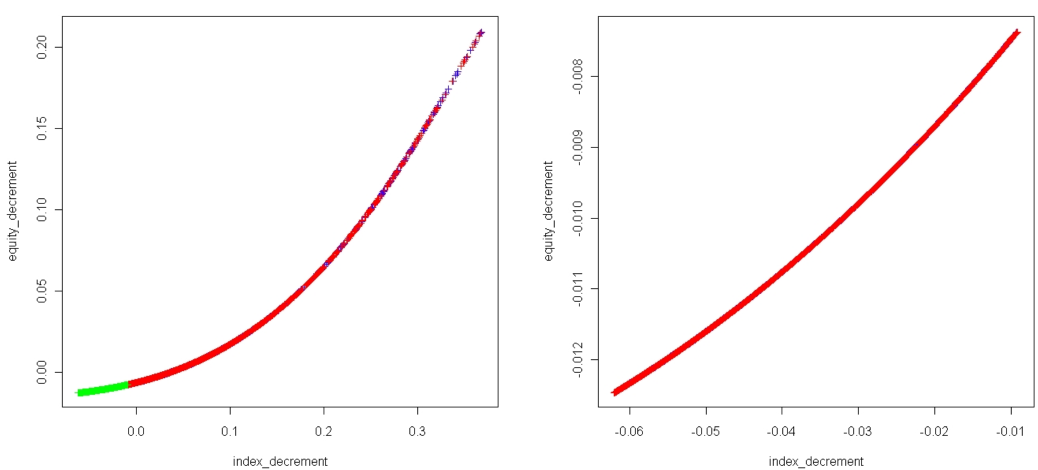

Figure 6: *Equity tranche vs Index Price Decrements between 0 and t_1 (Left: $t_1 < T = 5$; Right: zoom on $t_1 < T_1 = 1$).*

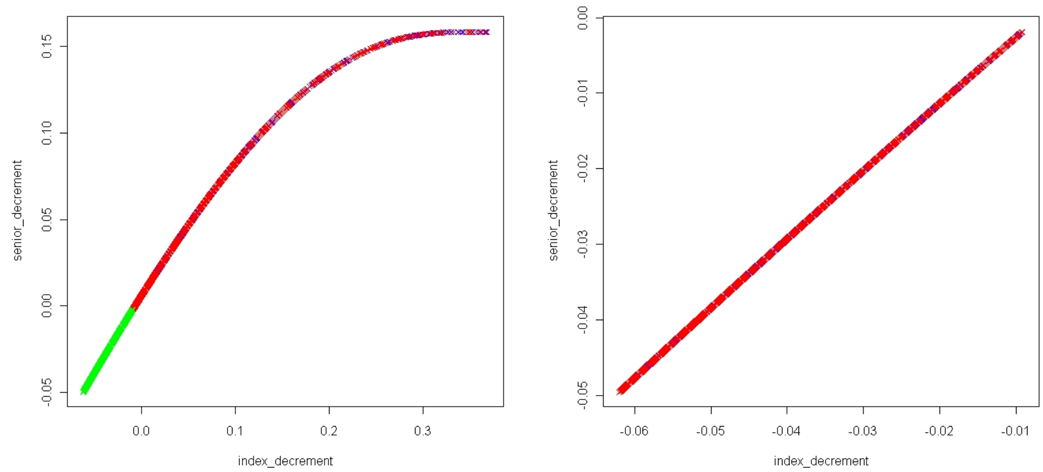

Figure 7: *Senior tranche vs Index Price Decrements between 0 and t_1 (Left: $t_1 < T = 5$; Right: zoom on $t_1 < T_1 = 1$).*

Looking at Table 5, we find as in the semi-homogeneous case that hedging the senior tranche with the index works very well, and even better than before: variance reduction factor of 11645 in bold blue in the last column. Yet these even better results may be partly due to an effect of distance to the index and not only to the fact that we are now in a fully homogeneous case: for the senior tranche is now closer to the index than before, with a senior tranche δ_0 of about 0.7 in Table 1 versus 0.8 in Table 4.
But as opposed to the situation in the semi-homogeneous case, hedging the equity tranche with the index now also works very well (variance reduction factor of about 123 in bold purple in the last column), and this holds even though the equity tranche is further from the index now than it was before, with an equity tranche δ_0 of about 0.3 in Table 1 versus 0.2 in Table 4 (cf. Remark 4.2). So the degradation of the hedge when we pass from the homogeneous model to the semi-homogeneous model is really due to the non-Markovianity of L, and not to an effect of distance to the index (cf. end of section 4.1.2).
Moreover the unconditional variance reduction factor and variance reduction factor conditional on defaults in the first and in the second group of obligors are now essentially the same (for the equity tranche as for the senior tranche).

This also means that hedging the equity tranche by the senior tranche, or vice versa, is quite effective in this case.

These results support our previous analysis that the impossibility of hedging the equity tranche by the index in the semi-hompogeneous model was due to the non-Markovianity of the loss process L.

Note incidentally that $\widehat{\delta}_0$ and δ_0 are closer now (in Tables 4–5) than they were previously (in Tables 1–2). This is consistent with the fact that $R2$ is now larger than before ($\widehat{\delta}_0$ and δ_0 would be even closer if the cutoff T_1 was less than $1yr$, provided of course the number of simulations m is large enough; see Remark 4.3).

	Π_0 or P_0	Σ_0 or S_0	δ_0^1	δ_0^2	δ_0
Eq	0.2850154	1900.103	0.2011043	0.2011043	0.2011043
Sen	0.1587075	1058.050	0.7988957	0.7988957	0.7988957

Table 4: *Time $t = 0$ – Prices, Spreads and Instantaneous Deltas in the Fully-Homogeneous Model.*

	$\widehat{\delta}_0$	ρ	$R2$	Dev	RedVar
Eq	0.0929529	0.9959361	0.9918887	0.004754811	**123.2852**
Eq1	0.09307564	0.995929	0.9918745	0.004794916	123.0695
Eq2	0.09282067	0.995946	0.9919084	0.004713333	123.5853
Sen	0.9070471	0.999957	0.9999141	0.04621152	**11645.15**
Sen1	0.9069244	0.9999569	0.9999137	0.04653322	11590.83
Sen2	0.9071793	0.9999573	0.9999146	0.0458808	11710.42

Table 5: *Hedging Tranches by the Index in the Fully-Homogeneous Model.*

5 Conclusions and Perspectives

In case of a non-Markovian portfolio process L, factor hedging of a loss derivative by another one may not work, and hedging by single name credit derivatives, such as CDS contracts, may then be necessary. Models with filtration that is at least as large as the filtration $\mathbb{H}$, that is the filtration of the indicator processes of all the default times in the pool, are the only ones which are able to deal with this issue in a theoretically sound way. Such models can, arguably, be constructed in a top-down way by thinning, starting from a top model with 'nice' dynamics for the portfolio process L. But focusing on having a model with a 'nice' dynamics for the top process L is misguided for dealing with a situation in which the index does not do the job in terms of hedging, for such a situation precisely means that the market dynamics of the top process L is *not* nice, and, as illustrated in section 4, insisting on using a simplistic model for L in a complex world may lead to a highly ineffective hedge.

It is thus our opinion that *bottom-up models* are the only ones, which are really suited to deal in a self-consistent way with the issue of hedging credit loss derivatives by single-name derivatives.

5.1 A Tractable Bottom Up Model of Portfolio Credit Risk

A common objection to the use of a bottom-up model is made with regard the issue of the so called *curse of dimensionality*. In this regard we wish to stress that suitable developments in the bottom up modeling enable one to efficiently cope with this curse of dimensionality: See, for instance, Elouerkhaoui [14], or Bielecki, Vidozzi and Vidozzi [4]. It is thus possible to specify bottom-up Markovian models of portfolio credit risk *with automatically calibrated model marginals* (to the individual CDS curves, say). Much like in the standard static copula models, but in a dynamized set-up, this effectively reduces the main computational effort, i.e. the effort related to model calibration, to calibration of only a few *dependence parameters* in the model at hand. Thus, model calibration can be achieved in a very reasonable time, also by pure simulation procedures if need be (without using any closed pricing formulae, if there aren't any available for the model under consideration).

To illustrate the previous statements let us briefly present a simple model (see [4] for more general theory and models). We postulate that for $i \in \mathbb{N}_n^*$ the individual default indicator process H^i is a Markov process admitting the following generator, for $u = u_e(t)$ with $e \in \{0, 1\}$,

$$\mathcal{A}_t^i u_e(t) = \eta_i(t)\big(u_1(t) - u_e(t)\big),$$

for a *pre-default intensity function* η_i of name i given by $\eta_i(t) = a_i + b_i t$. For constant and given interest rate $r \geq 0$ and recovery rate $R_i \in [0, 1]$, the individual time-$t = 0$ spread $\kappa_0^i(T)$ is then given by a standard explicit formula in terms of r, R_i, a_i and b_i (see [4] for the detail). The (non-negative) parameters a_i's and b_i's are then fitted so as to match the 5 and 10 year (say) spreads of the related credit index constituents.

Next, in order to couple together the model marginals H^i's, we define a certain number of groups of obligors susceptible to default simultaneously. Setting $n = 125$, for $l \in L = \{10, 20, 40, 125\}$, we thus define I_l as the set containing the indices of the l riskiest obligors, as measured by the spread of the corresponding five year CDS. In particular, we have

$$I_{10} \subseteq I_{20} \subseteq I_{40} \subseteq I_{125} = \mathbb{N}_n^* = \{1, 2, \ldots, 125\} .$$

Let $\mathcal{I} = \{I_l\}_{l \in L}$. We then construct the generator of process $H = (H^1, \cdots, H^n)$ as, for $u = u_\epsilon(t)$ with $\epsilon = (e_1, \cdots, e_n) \in \{0,1\}^n$:

$$\begin{aligned} \mathcal{A}_t u_\epsilon(t) &= \sum_{i=1}^{125} \Big(\eta_i(t) - \sum_{I \in \mathcal{I};\, I \ni i} \lambda_l(t)\Big)\Big(u_{\epsilon^i}(t) - u_\epsilon(t)\Big) \\ &+ \sum_{I \in \mathcal{I}} \lambda_I(t)\Big(u_{\epsilon^I}(t) - u_\epsilon(t)\Big) , \end{aligned}$$

where, for $i \in \mathbb{N}_n^*$ and $I \in \mathcal{I}$:
• ϵ^i, resp. ϵ^I, denotes the vectors obtained from ϵ by replacing the component e_i, resp. the components e_j for $j \in I$, by number one,
• $\lambda_I(t) = \bar{a}_I + \bar{b}_I t$, with $\bar{a}_I = \alpha_I \min_{\{i \in I\}} a_i$, $\bar{b}_I = \alpha_I \min_{\{i \in I\}} b_i$, for some $[0,1]$-valued *model dependence parameters* α_i.

In words, the form of the above generator implies that, at every time instant, either each alive obligor can default individually, or all the surviving names whose indices are in one of the sets $I \in \mathcal{I}$ can default simultaneously.

Observe that the *martingale dimension* (or *multiplicity*, cf. section 4.1.2) of the model is $125 + 4 = 129$. This makes the simulation of process H very fast, as we essentially need to draw $125 + 4$ IID exponential random variables in order to recover a set of default times $(\tau_i)_{i \in \mathbb{N}_n^*}$. Pricing CDO tranches in this model can thus be effectively done by simulation.

Moreover, we only need to calibrate four parameters, namely α_I with $I \in \mathcal{I}$ (since the marginal model parameters a_I's and b_I's were calibrated in a previous stage).

Finally, since this is a bottom-up Markov model, dynamic delta hedging by multi- and single-name derivatives can be considered in a model-consistent way, in the sense of replication if there are enough hedging instruments at hand, or in a min-variance sense in any case.

Of course the grouping of the names in the above model is rather arbitrary. Also, it would be better to use exact or approximate analytics for the CDO tranches, rather than relying only on simulation as proposed above. There is thus much room for improvement. However we refer the reader to [4] for numerical results on real market data demonstrating that this simple approach already does a very good job in practice in terms of calibration to CDS and CDO data.

A A glimpse at the General Theory

For the convenience of the reader, in this Appendix we recall definitions and results from the theory of processes, that we used in this paper. We refer to, e.g., Dellacherie and Meyer [11] for a comprehensive exposition.

Let us be given a standard stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The probability measure $\mathbb{P}$ will be fixed throughout. By default all the filtration-dependent notions like *adapted*, *stopping time*, *compensator*, *intensity*, *(local) martingale*, etc., implicitly refer to the filtration $\mathbb{F}$ (as opposed to, for instance, the larger filtration $\mathbb{G}$ which appears in section A.3).

A.1 Optional Projections

Let X be an integrable process, *not necessarily ($\mathbb{F}$-)adapted*. Then there exists a unique adapted process $({}^oX_t)_{t \geq 0}$, called the *optional projection* of X on $\mathbb{F}$, such that, for any stopping time τ,

$$\mathbb{E}\left(X_\tau \mathbb{1}_{\{\tau < +\infty\}} \,|\, \mathcal{F}_\tau\right) = {}^oX_\tau \mathbb{1}_{\{\tau < +\infty\}} .$$

In case X is non-decreasing, then ${}^oX$ a submartingale.

A.2 Dual Predictable Projections and Compensators

Let K be a non-decreasing and bounded process, *not necessarily adapted* (typically in the context of this paper, K corresponds to marginal or portfolio loss processes H^i or L). Then there exists a unique predictable non-decreasing process $(K^p_t)_{t\geq 0}$, called the *dual predictable projection* of K on $\mathbb{F}$, such that, for any positive predictable process H:

$$\mathbb{E}\left(\int_0^\infty H_s dK_s\right) = \mathbb{E}\left(\int_0^\infty H_s dK^p_s\right) .$$

In case K is adapted, it is a sub-martingale, and it admits as such a unique Doob-Meyer decomposition

$$K_t = M_t + \Lambda_t \tag{21}$$

where M is a uniformly integrable martingale (since K is bounded) and the *compensator* Λ of K is a predictable finite variation process. So $K^p = \Lambda$, by identification in the Doob-Meyer decomposition.
If, moreover, K is stopped at some stopping time τ, and if $K^p = \Lambda$ is continuous, then $K^p = \Lambda$ is also stopped at τ, by uniqueness of the Doob-Meyer decomposition of $K = K_{\cdot\wedge\tau}$. In case Λ is time-differentiable, so $\Lambda = \int_0^\cdot \lambda_t dt$ for some *intensity process* λ of K (also called *intensity of* τ, when $K = \mathbb{1}_{\tau\geq t}$ for some stopping time τ), the intensity process λ vanishes after τ.

Remark A.1 If K is a point-process (like a marginal or cumulative default process H^i or L in this paper), the continuity of Λ is equivalent to the ordered jump times of K being *totally inaccessible* stopping times (see, e.g., Dellacherie and Meyer [11]).

A.3 Proof of Proposition 2.1

We recall that, in the context of Proposition 2.1, K denotes an $\mathbb{F}$-adapted non-decreasing process with $\mathbb{F} \subseteq \mathbb{G}$, while Λ and Γ denote the $\mathbb{F}$-compensator and the $\mathbb{G}$-compensator of K, respectively.

Let $\bar\Gamma$ denote the $\mathbb{F}$-predictable non-decreasing component of the $\mathbb{F}$-submartingale ${}^o\Gamma$, the optional projection of Γ on $\mathbb{F}$ (see section A.1). The tower property of iterated conditional expectations yields,

$$\begin{aligned}\mathbb{E}\Big(\int_t^T dK_u - d\bar\Gamma_u \,|\, \mathcal{F}_t\Big) &= \mathbb{E}\Big(\int_t^T dK_u - d({}^o\Gamma)_u \,|\, \mathcal{F}_t\Big)\\ &= \mathbb{E}\Big(\int_t^T dK_u - d\Gamma_u \,|\, \mathcal{F}_t\Big) = \mathbb{E}\Big(\mathbb{E}\Big(\int_t^T dK_u - d\Gamma_u \,|\, \mathcal{G}_t\Big) \,|\, \mathcal{F}_t\Big) = 0\,,\end{aligned}$$

since $K - \Gamma$ is an $\mathbb{G}$-martingale. This proves that

$$\bar\Gamma = \Lambda\ . \tag{22}$$

Moreover, one has $\bar\Gamma = \Gamma^p$, the dual predictable projection of Γ on $\mathbb{F}$ (see, e.g., Proposition 3 of Brémaud and Yor [5]), hence

$$\Lambda = \Gamma^p\ ,$$

as stated in words in Proposition 2.1(i).

Now, in case Λ and Γ are time-differentiable with related intensity processes λ and γ, (22) means that

$$\int_0^t \lambda_s ds - \mathbb{E}(\int_0^t \gamma_s ds \,|\, \mathcal{F}_t) \tag{23}$$

is an $\mathbb{F}$-martingale. Moreover it is immediate to check, using the tower property of iterated conditional expectations, that

$$\mathbb{E}(\int_0^t \gamma_s ds \,|\, \mathcal{F}_t) - \int_0^t \mathbb{E}\Big(\gamma_s \,|\, \mathcal{F}_s\Big) ds \tag{24}$$

is an $\mathbb{F}$-martingale as well. By addition between (23) and (24),

$$\int_0^t \lambda_s ds - \int_0^t \mathbb{E}(\gamma_s \,|\, \mathcal{F}_s) ds$$

is in turn an $\mathbb{F}$-martingale. Since it is also a predictable (as continuous) finite variation process, it is thus in fact identically equal to 0, so for $t \geq 0$,

$$\lambda_t = \mathbb{E}(\gamma_t \,|\, \mathcal{F}_t) ,$$

and therefore

$$\lambda = {}^o\gamma ,$$

which is the statement of Proposition 2.1(ii).

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