A BSDE Approach to Counterparty Risk under Funding Constraints

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Outline

1. Counterparty Risk under Funding Constraints
2. Application to Counterparty Risk on Interest Rate Derivatives
3. Conclusions
What is it about?

Setup

- A generic “contract” between two defaultable counterparties
  - “The bank” and the “investor”
  - Contract ≡ CSA portfolio of OTC derivatives
    - **CSA** Credit Support Annex prescribing collateral and close-out cash-flows covenants
- **Funding Costs**
  - Higher than the risk-free rate for a credit risky party
  - Also liquidity funding costs
Valuation and Hedging Issues

- Bilateral counterparty risk perspective necessary to agree (at least at first order, if not for funding costs) on the level of the Credit Valuation Adjustment (CVA)
-Benefiting from one’s own default?
  - Contract only partially settled to the other party
  - Funding debt only partially reimbursed to an external funder
-Risk-neutral valuation of this benefit??
  - Hedging one’s own default hardly possible
    - Legal and practical constraints
    - Selling CDS protection on oneself forbidden and repurchasing one’s own bond not always practical
A reduced-form CVA BSDE approach to valuation and hedging of bilateral counterparty risk under funding constraints

CVA price-and-hedge correction for counterparty risk under funding constraints

- Introduced in banks for organizational reasons
  - Industry trading desks lack the global view, and specifically the aggregated data, needed to properly value the CSA cash-flows
  - Industry trading desks in charge of clean valuation and hedging
  - Central CVA desk in charge of CVA valuation and hedging
- Also useful mathematically

- Problem of valuation and hedging of counterparty risk under funding constraints reduced to solving pre-default CVA BSDEs in which defaultability of the two parties only shows up through their default intensities
  - Or equivalent semilinear PDEs in the Markov setup

- Immersion hypothesis and the case of credit derivatives
References

Based on


Related Papers


- Many CVA papers by Brigo and co-authors.


...and a huge (often unpublished) ‘quants’ literature.
Cash-Flows and Strategies

\[
\begin{align*}
\text{Hedge } S_t &= 0 \\
\text{Hedge } \mathcal{P}_t \\
\zeta_t^s (d\mathcal{P}_t^s + dC_t^s) &\quad (r_t + \lambda_t)X_t^- (\mathcal{W}_t, \zeta_t^-)dt - (r_t + \bar{\lambda}_t)X_t^+ (\mathcal{W}_t, \zeta_t)dt \\
\zeta_t^s (d\mathcal{P}_t^s - (r_t\mathcal{P}_t^s + c_t^s)dt + dC_t^s) &\quad (1 - r)X_t^+ (\mathcal{W}_{\tau -}, \zeta_{\tau -})\delta_\theta(dt) = \tilde{R}^f \delta_\theta(dt) \\
\tau &= \theta \land \bar{\theta} \\
\bar{\tau} &= \tau \land T \\
(r_t + b_t)\Gamma_t^+ - (r_t + \bar{b}_t)\Gamma_t^- dt &\quad \Gamma_t \\
W_t - \Gamma_t - \zeta_t^s \mathcal{P}_t^s = -X_t (\mathcal{W}_t, \zeta_t) \\
\text{Investor } \bar{\theta} \\
dC_t = 1_{t<\tau} dD_t &\quad \tilde{R}^f = (1 - r)X_{\theta -}^+ (\mathcal{W}_{\theta -}, \zeta_{\theta -}) \\
R^f\delta_\tau(dt) &\quad \mathcal{W}_t = \tilde{W}_t - 1_{t=\theta} \tilde{R}^f \\
\mathcal{W}_0 = \tilde{W}_0
\end{align*}
\]
Price-and-hedge

Standard stochastic basis \((\Omega, \mathcal{G}_T, \mathcal{G} = (\mathcal{G}_t)_{t \in [0, T]}, \mathbb{P})\) used for modeling the evolution of a financial market model

- “Martingale pricing measure” \(\mathbb{P}\) equivalent to the historical probability measure \(\widehat{\mathbb{P}}\) over \((\Omega, \mathcal{G}_T)\)

**Definition (Price-and-hedge \((\Pi, \zeta)\) over \([0, \bar{t}]\))**

- Semimartingale price process \(\Pi\) such that \(\Pi_{\bar{t}} = 1_{\bar{t} < T} R^i\)
- Hedge \(\zeta\) in primary risky (infinite variation) assets
- Wealth \(\tilde{W}\) of the collateralization, hedging and funding portfolio
- Algebraic debt \(\bar{x}_t (\tilde{W}_t, \zeta_t)\) of the bank towards its “external funder”
- Hedging error \(\varrho = \Pi - \tilde{W} = \Pi^* - \mathcal{W}\) where
  \[
  \Pi_t^* = \Pi_t - 1_{t=\theta} \bar{R}^f, \quad \mathcal{W}_t = \tilde{W}_t - 1_{t=\theta} \bar{R}^f
  \]

in which \(\bar{R}^f := (1 - r) \bar{x}^+_t (\tilde{W}_{\theta-}, \zeta_{\theta-}) = (1 - r) \bar{x}^+_t (\mathcal{W}_{\theta-}, \zeta_{\theta-})\)
Notation

For every real number $\pi$ and $\mathbb{R}^d$-valued row-vector $\zeta$

$$f_t(\pi, \zeta) = b_t \Gamma^+_t - \bar{b}_t \Gamma^-_t + \lambda_t (\pi - \Gamma_t - \zeta^s \mathcal{P}_t^s)^+ - \bar{\lambda}_t (\pi - \Gamma_t - \zeta^s \mathcal{P}_t^s)^- - \zeta^s \mathcal{C}_t^s$$

Self-Financing Property (Assumed throughout)

$\mathcal{W}_0 = \Pi_0$ and for $t \in [0, \bar{t}]$

$$d\mathcal{W}_t = -dC_t + (r_t \mathcal{W}_t + f_t(\mathcal{W}_t, \zeta_t)) dt + \zeta_t (d\mathcal{P}_t - r_t \mathcal{P}_t dt + dC_t)$$

where $f_t(\mathcal{W}_t, \zeta_t)$ is the $dt$-excess-funding-benefit of the bank
Assumption (Primary risky assets gain martingale)

\[ M_0 = 0 \text{ and for } t \in [0, \bar{t}] \]

\[ dM_t = dP_t - (r_t P_t + c_t) dt + dC_t \]

- Precludes arbitrage opportunities in the primary market of hedging assets (traded in swapped form)

→ Self-financing property rewritten “in martingale form”

\[ d\mathcal{W}_t = -dC_t + (r_t \mathcal{W}_t + g_t(\mathcal{W}_t, \zeta_t)) dt + \zeta_t dM_t \]

where \( g_t(\pi, \zeta) = f_t(\pi, \zeta) + \zeta c_t \)
Given a hedge $\zeta$ and a semimartingale $\Pi$, we denote $R = R^i - 1_{\tau=\theta}R^f$ in which

$$R^f := (1 - r)\mathcal{X}^+_{\tau-}(\Pi_{\tau-}, \zeta_{\tau-})$$

**Definition ($\mathbb{P}$-Price-and-Hedge)**

- Solution $(\Pi, \zeta)$ to the following Backward Stochastic Differential Equation (BSDE) on $[0, \bar{\tau}]$:
  \[ \Pi_{\bar{\tau}} = 1_{\tau<T}R \] and for $t \in [0, \bar{\tau}]$:
  \[
  d\Pi_t + dC_t - (r_t \Pi_t + g_t(\Pi_t, \zeta_t))dt = \zeta_t dM_t + d\varepsilon_t
  \]
  for a cost martingale $\varepsilon$ null at time 0

- $\mathbb{P}$-price-and-hedge $(\tilde{\Pi}, \zeta)$ where for $t \in [0, \bar{\tau}]$
  \[
  \tilde{\Pi}_t := \Pi_t + 1_{t=\theta}R^f
  \]
The $\mathbb{P}$-price-and-hedge BSDE made non-standard by:

- the random terminal time $\tau$
- the dependence of the terminal condition $R$ in $(\Pi_\tau, \zeta_\tau)$
- the contract effective dividend term $dC_t$
- the fact that it is not driven by an explicit set of fundamental martingales like Brownian motions and/or compensated jump measures
Lemma

In case of a \( \mathbb{P} \)-price-and-hedge \((\tilde{\Pi}, \zeta)\):

- The wealth process \( \mathcal{W} \) of the hedging portfolio satisfies for \( t \in [0, \tilde{\tau}] \):
  \[
  \left( \Pi_t - \int_0^t (r_s \Pi_s + g_s(\Pi_s, \zeta_s)) \, ds \right) - \left( \mathcal{W}_t - \int_0^t (r_s \mathcal{W}_s + g_s(\mathcal{W}_s, \zeta_s)) \, ds \right) = \varepsilon_t
  \]

- The hedging error \( \varrho \) of \((\tilde{\Pi}, \zeta)\) is such that \( \varrho_0 = \varepsilon_0 = 0 \) and for \( t \in [0, \tilde{\tau}] \):
  \[
  d\varrho_t = d\varepsilon_t + \left( r_t \varrho_t + g(\Pi_t, \zeta_t) - g(\mathcal{W}_t, \zeta_t) \right) dt
  - 1_{\tau=\theta}(1 - r)(\tilde{R}^f - R^f)\delta\tau(dt)
  \]

where \( \tilde{R}^f - R^f = \mathcal{X}_{\tau}^+(\mathcal{W}_{\tau^-}, \zeta_{\tau^-}) - \mathcal{X}_{\tau}^+(\Pi_{\tau^-}, \zeta_{\tau^-}) \).

- \( \Pi = \mathcal{W} \) and \( \varrho = 0 \) ("Complete Market Case") if \( \varepsilon = 0 \)
  - \( \Pi \approx \mathcal{W} \) and \( \varrho \) "small" if \( \varepsilon \) "small"

- Theoretically arbitrable \( \mathbb{P} \)-price-and-hedge if the market is incomplete and \( r < 1 \)
  - The corresponding "free lunch" seems difficult to lock in however
CVA Valuation and Hedging Approach (Sketched)

- “Clean” (of counterparty risk and funding costs) price-and-hedge $(P, \phi)$
- CVA price-and-hedge correction $(\Theta, \eta)$
  - Counterparty risk exposure $= \text{Positive part of the mark-to-market } P_\tau$
    - Cannot be simply handled by the application of a credit spread, has an intrinsically optional and dynamic flavor
  - CVA $\sim \text{Option ("CCDS") on the counterparty clean price process } P$
- Overall price-and-hedge $(\Pi = P - \Theta, \zeta = \phi - \eta)$
• Markov pre-default factor process $X$ with infinitesimal generator $\mathcal{X}$
  • (Pre-default) funding costs $\tilde{g}(t, x, \theta, \eta)$
• Relating a suitable notion of orthogonal solution of a pre-default CVA BSDE to a corresponding min-variance hedging objective of the bank
• CVA Greeking task of the bank reduced to the (numerical) solution of a classical Markovian BSDE, or an equivalent semilinear parabolic PDE
Approach applicable to the risk-management of either the overall contract, or of its CVA component

- Pre-default CVA BSDE key to the mathematical analysis in both cases

Three possible hedging objectives of the bank considered

- Min-variance hedging of market risk ignoring jump-to-default risk
- Min-variance hedging (of market risk) constrained to perfect hedging of jump-to-default risk
- Min-variance hedging of market risk constrained to perfect hedging of an isolated default of the investor
Case of a Fully Swapped Hedge $\tilde{g}(t, X_t, \theta, \eta) = \tilde{g}(t, X_t, \theta)$

- **Pre-default CVA PDE**
  $$
  \begin{cases}
  \tilde{\Theta}(T, x) = 0 \\
  (\partial_t + X) \tilde{\Theta}(t, x) + \tilde{g}(t, x, \tilde{\Theta}(t, x)) = 0, \quad t < T
  \end{cases}
  $$

- **Deterministic PDE schemes** only practical provided the dimension of $X$ is less than 3 or 4
- Otherwise **simulation schemes** are the only viable computational alternative
  - **BSDE simulation schemes** unless linear funding costs $\tilde{g}$
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Multiple Curve Issue

Left: Eonia swaps - Libor bases 2005-10, Right: Discount functions bootstrapped from 2 Sept 2010 data (We thank Jeroen Kerkhof, from Jefferies bank, for these graphs)

- Credit risk and liquidity risk fundamentals
- Special concern for interest rate derivatives (this part)
Counterparty Clean Valuation

- Multiple curve modeling of $P$
  - Using different yield curves for discounting and fixing
- Defaultable HJM methodology
  - Introducing credit risk to explain the divergence in the yield curves
  - Low-dimensional Markovian short-term specifications $X$
    - $P_t = P(t, X_t)$ needed for all vanilla interest rate derivatives needed for the CVA application
    - Vector factor process $X$ made of the risk-free short rate process $r$, a stylized short credit spread process $\lambda$ of the LIBOR banks, and auxiliary processes if need be for the Markov sake
Extended CIR Model

- **Given Brownian Motions** $W^r$ and $W^\lambda$,
  \[
  dr_t = (\rho(t) - k(t)r(t))dt + \zeta(t)\sqrt{r_t}dW_t^r \\
  d\lambda_t = (\rho^*(t) - k^*(t)\lambda(t))dt + \zeta^*(t)\sqrt{\lambda_t}dW_t^\lambda \\
  d\nu_t = ((\zeta(t))^2r_t - 2k(t)\nu_t)dt \\
  d\nu_t = (((\zeta^*(t))^2\lambda_t - 2k^*(t)\nu_t)dt
\]
  in which the model coefficients are defined in connection with a
defaultable HJM setup.

- **Generator** $\mathcal{X}$ of process $X_t = (r_t, \lambda_t, \nu_t, \mu_t)$ with $x = (r, \lambda, \nu, \mu)$,
  \[
  \mathcal{X}\tilde{\Theta}(t, x) = (\rho(t) - k(t)r)\partial_r\tilde{\Theta} + (\rho^*(t) - k^*(t)\lambda)\partial_\lambda\tilde{\Theta} \\
  + ((\zeta(t))^2r - 2k(t)\nu)\partial_\nu\tilde{\Theta} + (((\zeta^*(t))^2\lambda - 2k^*(t)\nu)\partial_\nu\tilde{\Theta} \\
  + \frac{1}{2}((\zeta(t))^2r + \frac{1}{2}((\zeta^*(t))^2\lambda - 2k^*(t)\nu)\partial_\nu\tilde{\Theta},
\]
  where $\varrho$ is the correlation between $W^r$ and $W^\lambda$. 
Assuming $\tilde{g}(t, X_t, \theta)$, the corresponding pre-default CVA Markovian BSDE writes: $\tilde{\Theta}(T, X_T) = 0$, and for every $t \in [0, T]$

$$
- d\tilde{\Theta}(t, X_t) = \tilde{g}(t, X_t, \tilde{\Theta}(t, X_t))dt
- \partial_r \tilde{\Theta}(t, X_t) \zeta(t) \sqrt{r_t} dW_t^r
- \partial_{\lambda} \tilde{\Theta}(t, X_t) \zeta^*(t) \sqrt{\lambda_t} dW_t^\lambda.
$$
Lévy Hull–White Model

- **Lévy subordinators** (non-decreasing Lévy processes) $L^r$ and $L^\lambda$
  \[
  \begin{align*}
  dr_t &= a(\rho(t) - r_t)dt + \sigma dL_t^r \\
  d\lambda_t &= a^*(\rho^*(t) - \lambda_t)dt + \sigma^* dL_t^\lambda.
  \end{align*}
  \]

- **Generator** $\mathcal{X}$ of $X_t = (r_t, \lambda_t)$, with $x = (r, \lambda)$,
  \[
  \mathcal{X}\tilde{\Theta}(t, x) = (a(\rho(t) - r))\partial_r\tilde{\Theta} + (a^*(\rho^*(t) - \lambda))\partial_\lambda\tilde{\Theta} + \\
  \int_{\delta,\varepsilon > 0} \left( \tilde{\Theta}(t, r + \sigma\delta, \lambda + \sigma^*\varepsilon) - \tilde{\Theta}(t, r, \lambda) \right) F(d(\delta, \varepsilon)),
  \]

  where $F$ stands for the Lévy measure of $(L^r, L^\lambda)$. 
Assuming \( \tilde{g}(t, X_t, \theta) \), the corresponding pre-default CVA Markovian BSDE writes: \( \tilde{\Theta}(T, X_T) = 0 \), and for every \( t \in [0, T] \):

\[
-d\tilde{\Theta}(t, X_t) = \tilde{g}(t, X_t, \tilde{\Theta}(t, X_t))dt \\
-\int_{\delta, \varepsilon > 0} \left( \tilde{\Theta}(t, r_{t-} + \sigma \delta, \lambda_{t-} + \sigma^* \varepsilon) - \tilde{\Theta}(t, X_{t-}) \right) N(dt, d(\delta, \varepsilon))
\]

where \( N \) stands for the compensated jump measure of \( (L^r, L^\lambda) \).
Possible Specifications of the Subordinators $L^r, L^\lambda$

- The jumps of a subordinator can only be positive
  - Compound Poisson process with positive jumps
  - Inverse Gaussian (IG) process and generalized inverse Gaussian (GIG) processes
  - Gamma process
  - Some degenerate CGMY processes
- Common factor specifications such as $L^r = Z^1 + Z^3$, $L^\lambda = Z^2 + Z^3$, for independent Lévy subordinators $Z^1$, $Z^2$ and $Z^3$
- Two Lévy subordinators $L^r, L^\lambda$ with a given Lévy copula
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Take-Away Messages

- Even though considering bilateral counterparty risk on a contract between two parties, focus on a party of interest, “the bank” in these slides
  - Symmetrical considerations apply to “the investor” but with non-symmetrical hedging positions and funding conditions
    - Different prices/CVAs

- Consider explicitly the three pillars of the position consisting of the contract itself, its hedging portfolio and its funding portfolio
  - As opposed to getting rid of funding costs through discounting at the risk-free rate in a classical one-curve setup

- Concrete recipes for risk-managing the contract as a whole or its CVA component, according to various possible objectives of the bank
  - E.g., minimizing the (risk-neutral) variance of the cost process (~hedging error) of the contract or of its CVA component, whilst achieving a perfect hedge of the related jump-to-default exposure

- Practical notion of CVA also useful mathematically
Devising a Cross – Asset Classes Model of Counterparty Risk

- Above reduced-form approach possible on most markets
- But $\Delta D_T \neq 0$ needed for counterparty credit risk

Facing the simulation computational challenge of CCR on real-life portfolios with tens of thousands of contracts

- More intensive than (Credit-)VaR or other risk measure computations
  - Value the portfolio at every time point of every simulated trajectory
- Devise appropriate variance reduction techniques
  - Importance Sampling exploiting the Markovian structure of the model
  - Particles
- Approximate simulation/regression pricing schemes for ‘exotic’ and/or path-dependently collateralized portfolios.