Market Micro Structure knowledge needed to control an intra-day trading process

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Based on joint works with B. Bouchard and M. Dang (Impulse control); G. Pagès and S. Laruelle (stoch. algo); O. Guéant and J. Fernandez Tapia (HJB for MM); O. Guéant and J. Razafinimanana (MFG)

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Optimal trading?

Financial mathematics evolve to control (hedge / replicate) the first order risk of the major actors:

- Asset management is about controlling the risk of holding a static portfolio,
- Derivative pricing is about replicating the risk of a non linear inventory,
- now it is the turn of the risk of modifying an inventory: liquidity risk highly related to market micro-structure

This talk is a journey across the key elements of liquidity risk control using financial mathematics: optimal / quantitative trading.
Outline

Introduction

Usual formal tools for optimal execution
  Market micro-structure
  Execution and trading
  Trade scheduling

Practical and new elements
  Statistical invariants
  Algorithmic trading
  Stochastic algorithms
  Market making
  Back and stress-testing: our “greeks”

Conclusion
Market microstructure recent changes

- Automated trading takes place on electronic markets
- In statistics and modelling, the more knowledge you have on the phenomena, the less data you will need to achieve a given accuracy
- First understand (and model) the environment of your algorithms
- Micro-structure, order-books definition, auction mechanisms, market-impact...
Previous organization of Equity market microstructure

- Investors
- Intermediaries
- OTC
- Visible book
- Primary
Actual market organisation of Equity markets

US: Reg NMS (National Market System) / EU: MiFID (Market in Financial Instruments Directive)
Electronic trading

Before:
# Electronic trading

Now:

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FLP | 500000 | 13.3800 / 13.4300 | 500000 | FLP

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Market Micro Structure knowledge needed to control an intra-day trading process
Continuous auctions

For continuous auctions, once the bid-ask spread is created at the end of the open fixing, the match takes place continuously:

The universe is discrete and event driven
### Continuous auctions Actors

#### Liquidity provider
- Wait to be executed (not executed for sure)
- Execution price is better than the last quoted one
- (sometimes) pays less fees

#### Liquidity consumer
- Executed for sure
- Pays around half a bid-ask spread more than the last quoted price
- Pays *Price Impact* for large quantities
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  Back and stress-testing: our “greeks”

Conclusion
Market impact (volume driven) demands to trade slowly
A good way to reduce your Market Impact is to trade on a larger time interval $\Delta T$.

$$v \equiv \text{cst}$$

$$V_{\Delta T} \uparrow \text{ with } \Delta T \uparrow \quad \Rightarrow \quad \alpha \psi + F \left( \sigma, \frac{v}{V_{\Delta T}} \right) \downarrow \text{ with } \Delta T \uparrow$$

$\Rightarrow$ trade as slowly as possible!
Market risk (volatility driven) demands to trade fast

The more you wait and the more you take market risk.
For an arithmetic Brownian diffusion, the amplitude of the risk is:

\[ a \cdot \sigma \sqrt{\Delta T} \]

⇒ trade as fast as possible!
Those two effects are mixed
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Conclusion
The Almgren-Chriss framework (simplified)

I have to buy $V^*$ shares between time 0 and $T$; I assume a regular temporal grid with a time step of $\delta t$ ($n$ goes from 1 to $N = \lceil T/\delta t \rceil$). My volume is split into $N$ slices $v_n$ such that $\sum_n v_n = V^*$ ([Almgren and Chriss, 2000]). The price follows a Brownian motion:

$$S_{n+1} = S_n + \alpha \delta t + \sigma_{n+1} \sqrt{\delta t} \xi_{n+1}$$

The additive, temporary only, market impact function is $\eta_n(v_n)$. Then the total cost is:

$$W = \sum_{n=1}^{N} v_n \cdot (S_n + \eta_n(v_n))$$

Of course, far more sophisticated models have been studied: [Almgren and Lorenz, 2007], [Almgren, 2009], [Bouchard et al., 2011], [Gatheral et al., 2010], [Alfonsi et al., 2010],
Classical solution

When the market impact is linear with respect to the participation rate $v_n/V_n$ and proportional to the volatility (change of variable $x_n = \sum_{k=n}^{N} v_n$).

$$W = V^* S_0 + \sum_{n=1}^{N} x_n \sigma_n \xi_n + \sum_{n=1}^{N} \eta \sigma_n \frac{V_n^2}{V_n}$$

For a broker algo, we want to minimize a mean-variance criteria (no "more" than Markowitz through time):

$$J_\lambda = \mathbb{E}(W|V_1, \ldots, V_N, \sigma_1, \ldots, \sigma_N) + \lambda \mathbb{V}(W|V_1, \ldots, V_N, \sigma_1, \ldots, \sigma_N)$$

We obtain a recurrence equation in $x_n$:

$$x_{n+1} = \left(1 + \frac{\sigma_{n-1}}{\sigma_n} \frac{V_n}{V_{n-1}} + \frac{\lambda}{\eta} \sigma_n^2 \right) x_n - \frac{\sigma_{n-1}}{\sigma_n} \frac{V_n}{V_{n-1}} x_{n-1}$$
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Conclusion
We need to "capture statistical invariants"

- No risk-neutral measure available,
- so we will work under historical measure,
- any statistical knowledge is welcome!
Intra day seasonality - what is stationary?

Usual phases (in Europe):

- Open: uncertainty on prices and unwind of the overnight positions
- Macro economic news
- NY opens

Volatility

Volumes
Measure or estimate: the case of Intra day volatility

- Proportion of volatility seems to be more stable by volatility itself
- How could we use proportion of volatility?
- But measuring volatility itself is difficult

Our prices are discretized (rounded?) on a price grid

A lot of interesting papers have been written on this subject: Jacod, Delattre, Aït-Sahalia, Zhang, Mykland, Shepard, Rosenbaum, Yoshida (covariance); [Aït-Sahalia and Jacod, 2007], [Zhang et al., 2005], [Hayashi and Yoshida, 2005], [Robert and Rosenbaum, 2010]
A "simple" model:

\[ X_{n+1} = X_n + \sigma \sqrt{\delta t} \xi_n, \quad S_n = X_n + \varepsilon \]

\[ \sum_n (S_{n+1} - S_n)^2 = 2n \mathbb{E}(\varepsilon^2) + O(\sqrt{n}) \]

Courtesy of Mathieu Rosenbaum
Limit order book matching mechanism

- Buy orders
- Sell orders
- Best Bid
- Mid price
- Best Ask
- Bid-Ask spread
- $f^+(s-S_a)$
- Prices

Market Micro Structure knowledge needed to control an intra-day trading process
Dependencies: spread - volatility

“Spread effect” on three stocks

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Market Micro Structure knowledge needed to control an intra-day trading process
Dependencies: spread - volatility - traded volumes

Given one of the three curves, you can always derive the missing one...
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Optimal liquidation under estimation assumptions: a practical point of view

Volumes and volatility change from one day to another (here two years of volume and its log for a French large cap between 11:45 and 12:30):

You have to consider that $V_n$ is a random variable/process, or that you have only access to $\hat{V}_n$, an estimator of $V_n$, with a variance.
A new set of equations

When the $V_n$ and $\sigma_n$ are i.i.d. random variables, and when $V_n$ and $\xi_n$ are correlated:

\[
\mathbb{E}(W) = \nu^* S_0 + \sum_{n=1}^{N} \eta \mathbb{E}\left(\frac{\sigma_n}{V_n}\right) (x_n - x_{n+1})^2
\]

\[
\text{Var}(W) = \sum_{n=1}^{N} x_n^2 \sigma_n^2 + \sum_{n=1}^{N} \eta x_n (x_n - x_{n+1}) \mathbb{E}(\sigma_n) \mathbb{E}\left(\frac{\xi_n}{V_n}\right) + \sum_{n=1}^{N} \eta^2 \text{Var}\left(\frac{\sigma_n}{V_n}\right) (x_n - x_{n+1})^4
\]

It is easy to add a lot of other effects like: noise on “expected” market impact, auto-correlations, volume-volatility coupled-dynamic. No more closed-form formula.
Results

The obtained results are obviously different, but what does it mean?

When the “noise” is higher than the estimated market impact, the optimal trajectory focus on market risk (no use to try to optimise on a variable not accurate enough)
A real execution: organisation of an algo in two layers

Now that you know your instantaneous optimal trading rate (if the market context is the expected one), how can you really capture this flow at the best possible price?

A “tactical” layer has to follow the targeted trading rate.
Why tactics are so important?

- robustness of the whole algorithmic cinematics (a global optimisation could suffer from polluted HF data)
- an unique place dedicated to address the specificities of each trading pool
- a need to react quickly as possible to behavioural changes
- trade scheduling is about backward solving (stochastic control) with an horizon of several hours (very long) relying of statistical invariants through days, tactical trading is about forward solving (stochastic algorithms): filtering immediate past to optimise the present
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Stochastic algorithms

- The stationary solutions of the ODE: $\dot{x} = h(x)$ contains the extremal values of $F(x) = \int_0^x h(x) \, dx$

- A discretized version of the ODE is ($\gamma$ is a step):

  \begin{equation}
  x_{n+1} = x_n + \gamma_{n+1} h(x_n)
  \end{equation}

- A stochastic version of this being ($\xi_n$ are i.i.d. realizations of a random variable, $h(X) = \mathbb{E}(H(X, \xi_1))$):

  \begin{equation}
  X_{n+1} = X_n + \gamma_{n+1} H(X_n, \xi_{n+1})
  \end{equation}

- The stochastic algorithms theory is a set of results describing the relationship between these 3 formula and the nature of $\gamma$, $H$, $h$ and $\xi$ ([Hirsch and Smith, 2005], [Kushner and Yin, 2003], [Doukhan, 1994]

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Stochastic algorithms theory can be used when you only have a sequential access to a functional you need to minimize:

- to minimize a criteria $\mathbb{E}(F(X, \xi_1))$ of a state variable $X$
- if it is possible to compute:

$$H(X_n, \xi_{n+1}) := \frac{\partial F}{\partial X}(X_n, \xi_{n+1})$$

- the results of the stochastic algorithms theory (like the Robbins-Monro theorem [Pagès et al., 1990]) can be used to study the properties of the long term solutions of the recurrence equation:

$$X_{n+1} = X_n + \gamma_{n+1} H(X_n, \xi_{n+1})$$
Optimal flow capture in the dark: can be extended to any routing problem

- at high frequency, historical statistics are not so useful
- the limit price $S$ and the quantity $V$ are random variables,
- the executed quantity on dark pools has to be maximized (it is *market impact free*) and sometimes fees are different; this effect is modelled by a *discount factor* $\theta_i \in (0, 1)$ (normalized with respect to a “reference” Lit pool)
- the quantity $V$ is split into $N$ parts (one for each DP): $r_i \times V$ is sent to the $i$th DP ($\sum_{i=1}^{N} r_i = 1$)

(see [Pagès et al., 2009])
The cost $C$ of the whole executed order is given by

$$C = S \sum_{i=1}^{N} \theta_i \min (r_i V, D_i) + S \left( V - \sum_{i=1}^{N} \min (r_i V, D_i) \right)$$

$$= S \left( V - \sum_{i=1}^{N} \rho_i \min (r_i V, D_i) \right)$$

where

$$\rho_i = 1 - \theta_i \in (0, 1), i = 1, \ldots, N.$$
Mean Execution Cost

Minimizing the mean execution cost, *given the price* $S$, amounts to:

Maximization problem to solve

\[
\begin{align*}
(6) \quad \max \left\{ \sum_{i=1}^{N} \rho_i \mathbb{E} \left( S \min \left( r_i V, D_i \right) \right), \ r \in \mathcal{P}_N \right\}
\end{align*}
\]

where $\mathcal{P}_N := \left\{ r = (r_i)_{1 \leq i \leq N} \in \mathbb{R}_+^N \mid \sum_{i=1}^{N} r_i = 1 \right\}$.

It is then convenient to *include the price* $S$ *into both random variables* $V$ and $D_i$ by considering $\tilde{V} := V S$ and $\tilde{D}_i := D_i S$ instead of $V$ and $D_i$. Assume that the distribution function of $D/V$ is continuous on $\mathbb{R}_+$. Let $\varphi(r) = \rho \mathbb{E} \left( \min \left( rV, D \right) \right)$ be the mean execution function of a single dark pool ($\Phi = \sum_i \varphi(r_i)$), and assume that $V > 0 \ \mathbb{P} - \text{a.s.}$ and $\mathbb{P}(D > 0) > 0$.
The Lagrangian Approach

We aim at solving the following maximization problem

\[
\max_{r \in \mathcal{P}_N} \Phi(r)
\]

The Lagrangian associated to the sole affine constraint is

\[
L(r, \lambda) = \Phi(r) - \lambda \left( \sum_{i=1}^{N} r_i - 1 \right)
\]

So,

\[
\forall i \in \mathcal{I}_N, \quad \frac{\partial L}{\partial r_i} = \varphi'_i(r_i) - \lambda.
\]

This suggests that any \( r^* \in \arg \max_{\mathcal{P}_N} \Phi \) iff \( \varphi'_i(r_i^*) \) is constant when \( i \) runs over \( \mathcal{I}_N \) or equivalently if

\[
\forall i \in \mathcal{I}_N, \quad \varphi'_i(r_i^*) = \frac{1}{N} \sum_{j=1}^{N} \varphi'_j(r_j^*).
\]
Design of the stochastic algorithm

Using the representation of the derivatives \( \varphi'_i \) yields that, if Assumption (C) is satisfied, then

Characterization of the solution

\[
r^* \in \arg \max_{\mathcal{P}_N} \Phi \iff \forall i \in \{1, \ldots, N\},
\]

\[
E \left( V \left( \rho_i \mathbb{1}_{\{r^*_i V < D_i\}} - \frac{1}{N} \sum_{j=1}^{N} \rho_j \mathbb{1}_{\{r^*_j V < D_j\}} \right) \right) = 0.
\]

Consequently, this leads to the following recursive zero search procedure

\[
(10) \quad r_{i}^{n+1} = r_{i}^{n} + \gamma_{n+1} H_i(r^{n}, Y^{n+1}), \quad r^0 \in \mathcal{P}_N, \quad i \in \mathcal{I}_N,
\]
where for \( i \in \mathcal{I}_N \), every \( r \in \mathcal{P}_N \), every \( V > 0 \) and every \( D_1, \ldots, D_N \geq 0 \),

\[
H_i(r, Y) = V \left( \rho_i 1 \{ r_i V < D_i \} - \frac{1}{N} \sum_{j=1}^{N} \rho_j 1 \{ r_j V < D_j \} \right)
\]

with \( (Y^n)_{n \geq 1} \) a sequence of random vectors with non-negative components such that, for every \( n \geq 1 \),

\( (V^n, D^n_i, i = 1, \ldots, N) \overset{d}{=} (V, D_i, i = 1, \ldots, N) \).

The underlying idea of the algorithm

is to reward the dark pools which outperform the mean of the \( N \) dark pools by increasing the allocated volume sent at the next step (and conversely).
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Market making in equations

Another style of trading is to be massively liquidity provider (see [Menkveld, 2010] for an in depth study of the behaviour of HF MM): maintain an inventory contributing to the bid $S^b$ and ask $S^a$. Provided that the way your orders are consumed is a Poisson process which intensity being a function of your distance to the “fair” (i.e. mid) price: introduce a Bellman function $u$ defined as:

$$u(t, x, q, s) = \sup_{S^a, S^b} \mathbb{E} \left[ -\exp (-\gamma(X_T + q_T S_T)) \right] | x, s, q$$
The Hamilton-Jacobi-Bellman equation for $u$ is (see [Avellaneda and Stoikov, 2008]):

\[(HJB.1)\quad 0 = \frac{\partial_t u(t, x, q, s)}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 u(t, x, q, s)}{\partial s^2} + \sup_{s^b} \lambda^b(s^b, s) \left[ u(t, x - s^b, q + 1, s) - u(t, x, q, s) \right] \\
+ \sup_{s^a} \lambda^a(s^a, s) \left[ u(t, x + s^a, q - 1, s) - u(t, x, q, s) \right] \]

with the final condition:

\[u(T, x, q, s) = -\exp(-\gamma(x + qs))\]
With the change of variable

\[ u(t, x, q, s) = -\exp(-\gamma(x + qs))v_q(t)^{-\frac{\gamma}{k}} \]

it is enough (see [Guéant et al., 2011]) to consider a family of strictly positive functions \((v_q)_{q\in\mathbb{Z}}\) solution of the linear system of ODEs that follows:

\[ \forall q \in \mathbb{Z}, \, \dot{v}_q(t) = \alpha q^2 v_q(t) - \eta (v_{q-1}(t) + v_{q+1}(t)) \]

with \(v_q(T) = 1\), where \(\alpha = \frac{k}{2}\gamma\sigma^2\) and \(\eta = A(1 + \frac{k}{\gamma})^{-\left(1 + \frac{k}{\gamma}\right)}\).

Moreover, you have asymptotics:

\[ \delta_{\infty}^{b^*}(q) \approx \frac{1}{\gamma} \ln \left(1 + \frac{\gamma}{k}\right) + \frac{2q + 1}{2} \sqrt{\frac{\sigma^2 \gamma}{2kA \left(1 + \frac{\gamma}{k}\right)^{1 + \frac{k}{\gamma}}}} \]

\[ \delta_{\infty}^{a^*}(q) \approx \frac{1}{\gamma} \ln \left(1 + \frac{\gamma}{k}\right) - \frac{2q - 1}{2} \sqrt{\frac{\sigma^2 \gamma}{2kA \left(1 + \frac{\gamma}{k}\right)^{1 + \frac{k}{\gamma}}}} \]
Application

Details for the quotes and trades when the strategy is used on AXA, 02/11/2010 with $\gamma = 0.05$. Thin lines: the market; bold lines: quotes of the market maker. Dotted lines: bid side; plain lines: ask side. Black points: obtained trades.
The associated inventory
Extension: passive brokerage

The Bellman function $u$ is now defined as:

\[ u(t, x, q, s) = \sup_{S^a} \mathbb{E} \left[ - \exp (-\gamma X_T) 1_{q_T=0} - \infty 1_{q_T>0} \right | x, s, q \]

The Hamilton-Jacobi-Bellman equation for $u$ is:

(HJB.2) \[ 0 = \partial_t u(t, x, q, s) + \mu \partial_s u(t, x, q, s) + \frac{1}{2} \sigma^2 \partial_{ss} u(t, x, q, s) \]

\[ + \sup_{s^a} \lambda^a(s^a, s) \left[ u(t, x + s^a, q - 1, s) - u(t, x, q, s) \right] \]

with the final condition:

\[ u(T, x, q, s) = - \exp (-\gamma x) 1_{q=0} - \infty 1_{q>0} \]

and the boundary condition:

\[ u(t, x, 0, s) = - \exp (-\gamma x) \]
Again a change of variables:

$$u(t, x, q, s) = -A^{-\frac{\gamma q}{k}} \exp(-\gamma(x + qs))w_q(t)^{-\frac{\gamma}{k}}$$

to obtain the linear system of ODEs ($S$) that follows:

$$\forall q \in \mathbb{Z}, \dot{w}_q(t) = (\alpha q^2 - \beta q)w_q(t) - \eta w_{q-1}(t)$$

with $w_q(T) = 1_{q=0}$ and $w_0 = 1$, where $\alpha = \frac{k}{2} \gamma \sigma^2$, $\beta = k \mu$ and $\eta = (1 + \frac{\gamma}{k})^{-(1 + \frac{k}{\gamma})}$ and the optimal ask quote can be expressed as:

$$s^{a^*}(t, q, s) = s + \left(\frac{1}{k} \ln \left(\frac{w_q(t)}{w_{q-1}(t)}\right) + \frac{1}{k} \ln(A) + \frac{1}{\gamma} \ln \left(1 + \frac{\gamma}{k}\right)\right)$$
Example of a passive sell order

Trading example. The strategy is used with $\gamma = 1 \, (\text{euro}^{-1})$ to get rid of a quantity of shares equal to 20 times the ATS within 2 hours. quotes of the trader (bold line), market best bid and ask quotes (thin lines).
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Formalisation of back and stress testing I

A trading algorithm lives in a state space made of three kinds of variables:

- The **instantaneous market data** $X_t$: it is usually the state of the Limit Order Book (LOB) on several trading venues, trades when they occur, news, etc.

- a trading algorithm has to filter the instantaneous market data $X_t$ to build its **view on the state of the market**: $Y_t$. It takes the market data into account via an updating rule $H_\theta$:

  $$ Y_t = H_\theta(Y_{t-\delta_t}, X_t) $$

where $\theta$ is a set of static parameters.
Formalisation of back and stress testing II

- And the **internal state of the algorithm** \(Z_t\) containing for instance its inventory, its pending orders in the books, its risk constraints, etc.

The trading algorithm has to take decisions based on available information:

\[
D_t = F_\theta(Y_t, Z_t)
\]

Trading between \(t = 0\) and \(t = T\), an algorithm tries to maximize:

\[
V_T(H, F, \theta) = \mathbb{E}\left(\int_{t=0}^{T} g(Z_t) \, dt + G(Z_T)\right)
\]
What will be the real outcome of an algo

▶ Its actions will change the behaviour of the market: you cannot be sure that an identified pattern in the order-books will be there once you identified it statistically

▶ So it is not “usual” backtesting... What can be done?

▶ Anyway, the use of a “market replayer” will guarantee that even having no influence on the market, your behaviour will have a positive outcome

▶ You also need to know what to expect if the market behaviour changes
How to control the impact of unexpected changes?

- ⇒ you need a market simulator to inject assumptions on the influence of your algo on the market and measure what can appen (for instance you can base it on a Mean Filed Game-based model [Lehalle et al., 2010])

- It is reasonnable to think that the real outcome will be close to:

$$V_T(H, F, \theta) + \sum_i \alpha_i \frac{\partial V_T}{\partial X_i} dX_i$$

($X_i$ span the state space of the market, it is hidden inside $V_T$ via $F_\theta \circ H_\theta$) but you do not know the $\alpha_i$

- at least you can measure the $\frac{\partial V_T}{\partial X_i}$. 
Back and stress-testing: our “greeks”

- **Vega** - sensitivity to intra day volatility:
  \[ V := \frac{\partial V_T(H, F, \theta)}{\partial \sigma} \]

- **Psi** - sensitivity to bid-ask spread:
  \[ \psi := \frac{\partial V_T(H, F, \theta)}{\partial \text{(bid-ask spread)}} \]

- **Phi** - sensitivity to trading frequency:
  \[ \phi := \frac{\partial V_T(H, F, \theta)}{\partial \text{(trading frequency)}} \]

- **Iota** - sensitivity to imbalance between sell and buy orders:
  \[ \iota := \frac{\partial V_T(H, F, \theta)}{\partial \text{(sell vs buy imbalance)}} \]
Focus on back tests

“Naive” backtesting

Introduction
Usual formal tools for optimal execution
Practical and new elements
Conclusion

Back and stress-testing: our “greeks”

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QR CA Cheuvreux

Market Micro Structure knowledge needed to control an intra-day trading process
Focus on back tests

“Really” backtesting

- Strategy
- Matching Engine
- Market Model
- History
- LOB
- Trades

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Conclusion

- Two layers: a “risk control” one (trade scheduling) and a “tactical one” (interactions with order books, adding and removing liquidity)
- Formal models are important, but they must not hide that you work under historical measure (estimates)
- We cannot throw away the uncertainty / inaccuracy of parameters and conduct optimisation as if they were deterministic
- There is intra-day seasonality almost everywhere (use of statistics)
- We cannot afford to miss a temporary behaviour change of the market (use of on-line estimates / optimisation, i.e. stochastic algo)
- We need to confront our ideas to data flows reproducing market behaviour (use of order books dynamics modelling, for instance MFG-based)


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