Liquidation in Limit Order Books with Controlled Intensity

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Liquidation via Limit Orders

- An investor wishes to liquidate a large position.
- To have a guaranteed execution price, use limit orders.
- No guaranteed execution time: trading frequency depends on the spread between limit price and bid price.
- Objective: maximize total revenue by a fixed liquidation date \( T \).
- Use a queue representation of limit order book.
- Study a stochastic control problem where the investor controls the frequency of trading.
- Leads to nonlinear first-order ordinary differential equations.
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Price Model

- \((P_t)\): bid price process. Assume \(e^{-rt}P_t\) is a martingale.
- \((N_t)\): counting process of order fills and \(\tau_k\) the corresponding arrival times, \(N_t = \sum_k 1\{\tau_k \leq t\}\). Each order is unit size.
- \(\Lambda_t\): (controlled) intensity of order fill.
- \(s_t \geq 0\): spread between the bid price and the limit order of the investor.
- \(N_t - \int_0^t \Lambda_s \, ds\) is a martingale and expected revenue is

\[
\mathbb{E} \left[ \sum_{i=1}^n e^{-r\tau_i} (P_{\tau_i} + s_{\tau_i} 1\{\tau_i \leq T\}) \right].
\]
Background

- Unaffected price is a martingale.
- *No price impact*, however, trade intensity depends on the strategy.
- Inspired by the model of Stoikov and Avellaneda (2009).
- Our view of the LOB as a **system of Poisson processes** is similar to recent work by Cont and co-authors on multi-queue formulations.
- In previous work (BL11) considered a similar case but without control over the spreads.
Optimization Problem

- Start with $n$ shares to sell.
- Maximize expected revenue until $T$: $\mathbb{E} \left[ \sum_{i=1}^{n} e^{-r \tau_i} (P_{\tau_i} + s_{\tau_i} 1_{\{\tau_i \leq T\}}) \right]$.
- Assume intensity of order fills is $\Lambda(s_t)$.
- Remaining shares at $T$ are liquidated at the bid price.
- Since $e^{-rt} P_t$ is a martingale, ignore the baseline revenue $nP_0$.
- Maximize expected profit due to limit orders:

$$V(n, T) := \sup_{(s_t) \in S_T} \mathbb{E} \left[ \sum_{i=1}^{n} e^{-r \tau_i} s_{\tau_i} 1_{\{\tau_i \leq T\}} \right].$$

- $S_T$ is the collection of $\mathcal{F}$-adapted controls, $s_t \geq 0$ with $\mathcal{F}_t := \sigma(N_s : s \leq t)$. 
Inventory Process

- $X_t$: remaining inventory at time $t$ with $X_0 = n$.
- $X_t := X_0 - N_t$ is a “death" process with intensity $\Lambda(s_t)$.
- Re-write

$$V(n, T) = \sup_{(s_t) \in S_T} \mathbb{E} \left[ \int_0^{T \wedge \tau(X)} e^{-rt} s_t \, dN_t \right] = \sup_{(s_t) \in S_T} \mathbb{E} \left[ \int_0^{T \wedge \tau(X)} e^{-rt} s_t \Lambda(s_t) \, dt \right].$$

- $\tau(X) := \inf\{t \geq 0 : X_t = 0\}$ is the time of liquidation.
- Boundary conditions are $V(n, 0) = 0 \ \forall n$ (terminal condition in time)
- $V(0, T) = 0 \ \forall T$ (exhaustion).
Nonlinear ODE

- Using standard methods, value function is the \textit{viscosity solution} of

\[
- V_T + \sup_{s \geq 0} \{ \Lambda(s) \cdot (V(n - 1, T) - V(n, T) + s) \} - rV(n, T) = 0,
\]

with boundary conditions \( V(0, T) = V(n, 0) = 0 \) and \( V_T \) denoting partial derivative wrt time-to-expiration.

- Optimal control is of Markov feedback type, \( s_t^* = s(X_t^*, T - t) \).

- Focus on:
  - Special functional forms of \( \Lambda(s) \) that admit closed-form solutions.
  - The \textit{fluid limit} where order size \( \Delta \to 0 \) and trade intensity is \( \Lambda(s)/\Delta \to \infty \).
Power-Law LOB: Explicit Solution

Proposition

Assume that \( \Lambda(s) = \lambda s^{-\alpha}, \alpha > 1 \). Then

\[
V(n, T) = c_n \left( 1 - e^{-r\alpha T} \right)^{1/\alpha}, \quad s^*(n, T) = \left( \frac{\lambda}{\alpha r c_n} \right)^{1/(\alpha-1)} \cdot (1 - e^{-r\alpha T})^{1/\alpha},
\]

with \( c_n \) satisfying the recursion

\[
rc_n = A_{\alpha} \lambda (c_n - c_{n-1})^{1-\alpha}, \quad n \geq 1, \quad c_0 = 0,
\]

where \( A_{\alpha} := \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha} \).
Solution Structure

- Empirical tests suggest that $\alpha \in [1.5, 3]$.
- $V(n, T)$ is concave in $n$.
- $n \mapsto s^*(n, T)$ is decreasing.
- $\Lambda(s^*(n, T)) = \frac{C(n)}{1 - e^{-r\alpha T}}$ so $\mathbb{P}(\sigma_i \leq T - \tau_{i-1}) = 1$ for all $i \leq n$. Liquidate everything by $T$.
- On infinite horizon, $\lim_{T \to \infty} V(n, T) = c_n$.
- Also have an explicit solution when $r = 0$. 
Continuous Selling Limit

- Consider the problem where shares are sold at $\Delta$ increments and $\Lambda^\Delta(s) := \Lambda(s)/\Delta$.
- The corresponding value function $V^\Delta(x, T)$ with $x \in \{0, \Delta, 2\Delta, \cdots \}$, $T \in \mathbb{R}_+$ is the viscosity solution of

$$
-V_T^\Delta + \sup_{s \geq 0} \frac{\lambda}{s^\alpha \Delta} (V^\Delta(x - \Delta, T) - V^\Delta(x, T) + s\Delta) - rV^\Delta = 0. \quad (1)
$$

- As $\Delta \to 0$, the limiting PDE is $-V_T + \sup_{s \geq 0} \frac{\lambda}{s^\alpha} s^{\frac{s - V_x}{s^\alpha}} - rv = 0$, which has explicit solution

$$
V(x, T) = \left(\frac{\lambda}{r\alpha}\right)^{1/\alpha} x^{(\alpha - 1)/\alpha} (1 - e^{-r\alpha T})^{1/\alpha}.
$$

- The optimizer above is $s^{(0)}(x, T) = \left(\frac{\lambda}{\alpha r}\right)^{1/\alpha} \frac{1}{x^{1/\alpha}} (1 - e^{-r\alpha T})^{1/\alpha}$. 
Relationship with Fluid Limit

**Theorem**

As $\Delta \to 0$, $V^\Delta \to v$ uniformly on compact sets.

- Use viscosity arguments of Barles-Souganides.
- Idea: the pre-limit is a space-discretization of the pde. Immediately get convergence to the viscosity solution and related convergence of the controls.
- Note: our control set and payoffs are unbounded.
- Extends results on fluid limit of queues due to Bäuerle, Piunovskiy, Day, etc.

**Proposition**

For any sequence $(\Delta_k)$ with $\Delta_k = \delta 2^{-k}$, we have $V^{\Delta_k} \uparrow v$ as $k \to \infty$. 
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For any sequence $(\Delta_k)$ with $\Delta_k = \delta 2^{-k}$, we have $V^{\Delta_k} \uparrow v$ as $k \to \infty$. 
Implications for the Original Model

- Theorem tells us that \( c_n \sim \left( \frac{\lambda}{r \alpha} \right)^{1/\alpha} n^{(\alpha-1)/\alpha} \) as \( n \to \infty \).

**Corollary**

Let us denote by \( s^{(\Delta)} \) the pointwise optimizer in (1). Then we have that \( s^{(\Delta)} \to s^{(0)} \) uniformly on compacts.

\[ \Rightarrow \] Clearly, the marginal spread will go to zero as \( n \to \infty \). The corollary gives the rate of convergence: \( s^*(n, T) \sim \left( \frac{\lambda}{\alpha r} \right)^{1/\alpha} \frac{1}{n^{1/\alpha}} \) as \( n \to \infty \).
Numerical Illustration of the Convergence

Figure: Convergence to the fluid limit for $\Lambda(s) = s^{-2}$. Left panel: the ratio between discrete and continuous $V^\Delta(x)/v(x)$ for $\Delta = 0.05$ and $\Delta = 0.01$. Right panel: ratio of the fluid limit optimal control $s^{(0)}(x)$ to the discrete $s^{(\Delta)}(x)$ for $\Delta \in \{0.01, 0.05\}$. 
Exponential Decay Order Books

- In power-law LOB, can trade \textit{instantaneously} at the bid price.
- Realistically, order fill intensity might be finite even at zero spread.
- Also, the LOB \textit{tail} for large spreads might be thinner.

⇒ Consider \( \Lambda(s) = \lambda e^{-\kappa s} \).
Explicit Solution II: Finite Horizon

Proposition

Consider order size $\Delta \in [0, 1]$ and $\Lambda(s) = \lambda e^{-\kappa s}$. Then for $x = n\Delta$, the value function $V^\Delta(x, T)$ with boundary condition $V^\Delta(x, 0) = V^\Delta(0, T) = 0$ is

$$V^\Delta(x, T) = \frac{\Delta}{\kappa} \log \left( \sum_{j=0}^{n} \frac{1}{j!} \left( \frac{\lambda T}{\Delta} \right)^j \right), \quad s^*(n\Delta, T) = \frac{1}{\kappa} + \frac{V^\Delta(n\Delta) - V^\Delta((n - 1)\Delta)}{\Delta}.$$

As $\Delta \downarrow 0$, $V^\Delta(n\Delta, T) \rightarrow v(x, T)$ uniformly on compacts, where $v(x, T)$ solves the nonlinear first order PDE

$$v_T(x, T) = \frac{\lambda}{\kappa} e^{-1-\kappa v_x(x, T)}, \quad v(0, T) = v(x, 0) = 0.$$
Solution Structure

- $s^*(n\Delta, T) \geq \frac{1}{\kappa}$. Never trade close to the bid.
- Have the bounds
  \[
  \frac{x}{\kappa} \log \left( \frac{\lambda}{x} T \right) \leq v(x, T) \leq \frac{\lambda}{\kappa e} T.
  \]
- Cannot guarantee liquidation by $T$. Impossible to do so when $x > \lambda e^{-1} T$. 
Explicit Solution III: Infinite Horizon

Proposition

For exponential-decay LOB with $T = +\infty$ and discounting rate $r > 0$ we have

$$V^\Delta(x) = \frac{\Delta}{\kappa} W \left( \lambda r^{-1} \Delta^{-1} \exp \left( \kappa \frac{V^\Delta(x - \Delta)}{\Delta} - 1 \right) \right), \quad x \in \{0, \Delta, \cdots\}.$$  

where $W$ is the Lambert-$W$ function.

As $\Delta \downarrow 0$, $V^\Delta(x) \to v(x)$ uniformly on compacts where

$$\text{li} \left( \frac{e^{\kappa r} v(x)}{\lambda} \right) := -\frac{e^{rx}}{\lambda}, \quad \text{li}(y) := \int_0^y \frac{1}{\log t} \, dt.$$
Power Law vs. Exponential LOB Fluid Limit Optimizer

Figure: Optimal controls for power-law and exponential-decay order books. We take $r = 0.1$ and depth functions $\Lambda(s) \in \{s^{-2}, s^{-3}, e^{1-s}\}$, which have been normalized such that $\Lambda(1) = 1$ in all three cases. The plot shows the resulting fluid limit spreads $s^{(0)}(x)$. 
General Limit Order Book Shapes

Our fluid limit **Convergence Theorem** holds for general shapes of limit order books.

Furthermore if $s \mapsto \Lambda(s)$ is decreasing and

$$\frac{\Lambda(s)\Lambda''(s)}{(\Lambda'(s))^2} < 2, \quad \forall s \in \mathbb{R}_+,$$

then:

- both $V^\Delta$ and $v$ are concave;
- $s^{(\Delta)}$ and $s^{(0)}$ are decreasing;
- $s^{(\Delta)} \to s^{(0)}$ uniformly on compacts.
Other Extensions

- Liquidity might be stochastic.

- As a first approximation, have closed-form solution to a power-law LOB with regime-switching $\Lambda(s, t) = \lambda M_t / s^\alpha$ where $(M_t)$ is a two-state indep. Markov chain.

- Can also consider multiple trading venues. For instance, continuous trading on exchange C, discrete orders of $\Delta$ on exchange L.

- Obtain a nonlinear delay ODE.

$$A_\alpha \lambda_0 v'(x)^{1-\alpha} + A_\alpha \lambda_1 (x \wedge \delta)^\alpha (v(x) - v((x - \delta)_+))^{1-\alpha} - rv = 0,$$

- Can do asymptotic expansions in $\lambda_1$. 
Still To Do

- Price impact.
- Combine market orders + limit orders.
- THANK YOU!
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References

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