PRICING AND TRADING CREDIT DEFAULT SWAPS

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Dynamics of Prices of Defaultable claims

The default time is a strictly positive random variable $\tau$, defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. The filtration generated by the jump process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ is denoted by $\mathcal{H}$. We assume that some auxiliary filtration $\mathcal{F}$ is given, and we write $\mathcal{G} = \mathcal{H} \vee \mathcal{F}$. The filtration $\mathcal{G}$ is referred to as to the full filtration. We assume that any $\mathcal{F}$-martingale is a continuous process.
Survival Process

$G_t = \mathbb{Q}(\tau > t \mid \mathcal{F}_t)$ is the survival process assumed to satisfy $G_0 = 1$ and $G_t > 0$ for every $t \in \mathbb{R}_+$. We assume that $G$ is continuous.

Then

$$M_t = H_t - \int_0^t (1 - H_u)\lambda_u \, du,$$

is a $\mathbb{G}$-martingale, where $\lambda$ is defined via the Doob-Meyer decomposition of the sub-martingale $G$. 
Defaultable Claims

By a defaultable claim maturing at $T$ we mean a quadruple $(X, A, Z, \tau)$, where

- $X$ is an $\mathcal{F}_T$-measurable random variable,
- $A = (A_t)_{t \in [0,T]}$ is an $\mathbb{F}$-adapted process of finite variation with $A_0 = 0$,
- $Z = (Z_t)_{t \in [0,T]}$ is an $\mathbb{F}$-predictable process,
- and $\tau$ is the default time.
The total dividend process $D^X = (D^X_t)_{t \in \mathbb{R}_+}$ of a defaultable claim maturing at $T$ equals, for every $t \in \mathbb{R}_+$,

$$D^X_t = X 1_{\{\tau > T\}} 1_{[T, \infty[}(t) + \int_{]0, t \wedge T]} (1 - H_u) dA_u + \int_{]0, t \wedge T]} Z_u dH_u.$$ 

The reduced total dividend process $D$ of a defaultable claim maturing at $T$ equals, for every $t \in \mathbb{R}_+$,

$$D_t = \int_{]0, t \wedge T]} (1 - H_u) dA_u + \int_{]0, t \wedge T]} Z_u dH_u.$$
Price Dynamics of a Defaultable Claim

The **ex-dividend price** process $S$ associated with the dividend process $D^X$ equals, for every $t \in [0, T]$, 

$$S_t = B_t \mathbb{E}_{Q^*} \left( B_T^{-1} X \mathbf{1}_{\{\tau > T\}} + \int_{[t,T]} B_u^{-1} dD_u \Big| \mathcal{G}_t \right).$$

The **ex-dividend pre-default price** of a defaultable claim is the unique $\mathbb{F}$-adapted process $\tilde{S}$ such that

$$S_t = \mathbf{1}_{\{t < \tau\}} \tilde{S}_t$$

The **cumulative price** process $S^{cum}$ associated with the dividend process $D^X$ is

$$S^{cum}_t = B_t \mathbb{E}_{Q^*} \left( \int_{[0,T]} B_u^{-1} dD^X_u \Big| \mathcal{G}_t \right) = S_t + B_t \int_{[0,t]} B_u^{-1} dD_u.$$

The discounted cumulative price $B^{-1} S^{cum}$ is a $\mathcal{G}$-martingale under $Q^*$. 
Let $n$ be any $\mathbb{F}$-martingale. Then the process $\hat{n}$ given by

$$
\hat{n}_t = n_{t \wedge \tau} - \int_0^{t \wedge \tau} G_u^{-1} d\langle n, \mu \rangle_u
$$

is a continuous $\mathcal{G}$-martingale.
• For any $Q^*$-integrable and $\mathcal{F}_T$-measurable random variable $Y$ we have

$$E_{Q^*}(1_{\{T<\tau\}} Y \mid \mathcal{G}_t) = 1_{\{t<\tau\}} G_t^{-1} E_{Q^*}(G_T Y \mid \mathcal{F}_t).$$

• For any $\mathbb{F}$-predictable process $R$ such that $E_{Q^*}|R_\tau| < \infty$

$$E_{Q^*}(1_{\{t<\tau \leq T\}} R_\tau \mid \mathcal{G}_t) = -1_{\{t<\tau\}} \frac{1}{G_t} E_{Q^*} \left( \int_t^T R_u \, dG_u \mid \mathcal{F}_t \right).$$
The dynamics of the cumulative price $S^{cum}$ on $[0, T]$ are
\[ dS^\text{cum}_t = r_t S^\text{cum}_t \, dt + (Z_t - S_{t-}) \, dM_t + G_t^{-1} (B_t \, d\hat{m}_t - S_t \, d\hat{\mu}_t) \]
where
\[ m_t = \mathbb{E}_{Q^*} \left( B_T^{-1} G_T X + \int_0^T B_u^{-1} G_u Z_u \lambda_u \, du - \int_0^T B_u^{-1} G_u \, dA_u \bigg| \mathcal{F}_t \right). \]
and $\mu$ is the martingale part of the submartingale $G$. 
Proof. We derive the dynamics of the pre-default ex-dividend price $\tilde{S}$.
The price $S$ can be represented as follows

$$S_t = 1_{\{t<\tau\}} \tilde{S}_t = 1_{\{t<\tau\}} B_t G_t^{-1} Y_t,$$

where $Y$ is defined as follows

$$Y_t = m_t - \int_0^t B_u^{-1} G_u Z_u \lambda_u \, du + \int_0^t B_u^{-1} G_u \, dA_u,$$

An application of Itô’s formula leads to the result $\Box$
Price Dynamics of a CDS

A credit default swap (CDS) with a constant rate $\kappa$ and recovery at default is a defaultable claim $(0, A, Z, \tau)$ where $Z_t = \delta_t$ and $A_t = -\kappa t$ for every $t \in [0, T]$.

The process $\delta : [0, T] \to \mathbb{R}$ represents the default protection, and $\kappa$ is the CDS rate (also termed the spread, premium or annuity of a CDS).
The ex-dividend price of a CDS equals, for any \( t \in [0, T] \),

\[
S_t(\kappa) = \mathbb{1}_{\{t < \tau\}} \frac{B_t}{G_t} \mathbb{E}_{Q^*}\left( \int_t^T B_u^{-1} G_u \delta_u \lambda_u \, du - \kappa \int_t^T B_u^{-1} G_u \, du \bigg| \mathcal{F}_t \right).
\]

Define the \( F \)-martingale \( n \) by the formula

\[
n_t = \mathbb{E}_{Q^*}\left( \int_0^T B_u^{-1} G_u \delta_u \lambda_u \, du - \kappa \int_0^T B_u^{-1} G_u \, du \bigg| \mathcal{F}_t \right).
\]

Then, the dynamics of the cumulative price \( S^{\text{cum}}(\kappa) \) are

\[
dS^{\text{cum}}_t(\kappa) = r_t S^{\text{cum}}_t(\kappa) \, dt + (\delta_t - S_{t-}(\kappa)) \, dM_t + G_t^{-1}(B_t \, d\hat{n}_t - S_t(\kappa) \, d\hat{\mu}_t).
\]
Replication of a Defaultable Claim

We now assume that $k$ credit default swaps with maturities $T_i \geq T$, spreads $\kappa_i$ and protection payments $\delta^i$ for $i = 1, \ldots, k$ are traded over the time interval $[0, T]$. The 0th traded asset is the savings account $B$. Our goal is to examine hedging strategies for a defaultable claim $(X, A, Z, \tau)$.

Here, we assume that immersion property holds: any $\mathcal{F}$-martingale is a $\mathcal{G}$-martingale. In that case, $G$ is a non-increasing process.
We consider a trading strategy \( \phi = (\phi^0, \ldots, \phi^k) \) where \( \phi^0 \) is \( \mathbb{G} \)-adapted and \( \phi^1, \ldots, \phi^k \) are \( \mathbb{G} \)-predictable processes. The associated wealth process \( V(\phi) \) equals, for \( t \in [0, T] \),

\[
V_t(\phi) = \phi^0_t B_t + \sum_{i=1}^{k} \phi^i_t S^i_t(\kappa_i)
\]
A strategy \( \phi \) is said to be **self-financing** if

\[
dV_t(\phi) = \phi_t^0 dB_t + \sum_{i=1}^{k} \phi_t^i dS_{t}^{c,i}(\kappa_i)
\]

where \( S_{t}^{c,i}(\kappa_i) \) is the cumulative price process of the \( i \)th traded CDS.
We consider a defaultable claim \((X, A, Z, \tau)\) such that the price process \(S\) for this claim is well defined:

\[
S_{t}^{\text{cum}} = 1_{\{t < \tau\}} \frac{B_t}{G_t} \left( - \int_0^t B_u^{-1} G_u Z_u \lambda_u \, du + \int_0^t B_u^{-1} G_u \, dA_u + m_t \right)
\]

with

\[
m_t = \mathbb{E}_{Q^*} \left( B_T^{-1} G_T X + \int_0^T B_u^{-1} G_u Z_u \lambda_u \, du - \int_0^T B_u^{-1} G_u \, dA_u \mid \mathcal{F}_t \right).
\]

We say that a self-financing strategy \(\phi = (\phi^0, \ldots, \phi^k)\) replicates a defaultable claim \((X, A, Z, \tau)\) if its wealth process \(V(\phi)\) is equal to the price \(S\) of the claim \(t \in [0, T]\).

In particular, the equality \(V_{t \wedge \tau}(\phi) = S_{t \wedge \tau}\) holds for every \(t \in [0, T]\).
For any $t \in [0, T]$,

$$dV_t(\phi) = r_t V_t(\phi) \, dt + \sum_{i=1}^{k} \phi_t^i \left( (\delta_t^i - \tilde{S}_t^i(\kappa_i)) \, dM_t + (1 - H_t) B_t G_t^{-1} \, dn_t^i \right)$$

where

$$n_t^i = \mathbb{E}_{Q^*} \left( \left( \int_0^{T_i} B_u^{-1} G_u \delta_u \lambda_u \, du - \kappa_i \int_0^{T_i} B_u^{-1} G_u \, du \right) \mid \mathcal{F}_t \right) .$$
We assume from now on that the filtration $\mathbb{F}$ is generated by a (possibly multi-dimensional) Brownian motion $W$ under $\mathbb{Q}^*$ and Hypothesis (H) holds (so that $W$ is also a Brownian motion with respect to $\mathbb{G}$).

In view of the predictable representation property of a Brownian motion, there exist $\mathbb{F}$-predictable processes $\zeta$ and $\zeta^i$, $i = 1, \ldots, k$ such that $dm_t = \zeta_t \, dW_t$ and $dn_t^i = \zeta^i_t \, dW_t$.
Assume that there exist \( \mathbb{F} \)-predictable processes \( \phi^1, \ldots, \phi^k \) such that, for any \( t \in [0, T] \),

\[
\sum_{i=1}^{k} \phi^i_t (\delta^i_t - \tilde{S}^i_t(\kappa_i)) = Z_t - \tilde{S}_t, \quad \sum_{i=1}^{k} \phi^i_t \zeta^i_t = \zeta_t.
\]

Let

\[
dV_t(\phi) = r_t V_t(\phi) \, dt + \sum_{i=1}^{k} \phi^i_t \left( (\delta^i_t - \tilde{S}^i_t(\kappa_i)) \, dM_t + (1 - H_t) B_t G_t^{-1} \, dn^i_t \right)
\]

with the initial condition \( V_0(\phi) = S_{0}^{cum} \). Then the self-financing trading strategy \( (B^{-1}(V(\phi) - \phi \cdot S^i), \phi^1, \ldots, \phi^k) \) replicates the defaultable claim \( (X, A, Z, \tau) \).
**Enlargement of filtration formula**

Let $\tau$ be a unique random time. In general, it is not an honest time. However, it is possible to prove that any $\mathbb{F}$-martingale is a $\mathbb{G} = \mathbb{F} \lor \mathbb{F}$-semi-martingale. If $X$ is a $\mathbb{F}$ martingale, if

$$
\mathbb{P}(\tau > u | \mathcal{F}_t) = \int_u^{\infty} f(v; t)dv
$$

then

$$
X_t = \tilde{X}_t + \int_0^{t \land \tau} \frac{d\langle X, M^\tau \rangle_s}{S_{s-}} + \int_{t \land \tau}^{t} \varphi(\tau, ds),
$$

$$
= \tilde{X}_t + \int_0^{t \land \tau} \int_s^{\infty} \eta(dv) \frac{d\langle X, f^v \rangle_s}{S_{s-}} + \int_{t \land \tau}^{t} \varphi(\tau, ds)
$$

where $\tilde{X}$ is a $\mathbb{G}$-martingale and

$$
\varphi(u, ds) = \frac{d\langle f(u; \cdot), X \rangle_s}{f(u; s)}
$$
In this section, we assume that $\mathbb{F}$ is a Brownian filtration and that the interest rate is null. Our aim is to obtain the dynamics of a CDS in the simple case where two different credit names are considered.
Joint Survival Process

Hence we assume that we are given two strictly positive random times $\tau_1$ and $\tau_2$. We introduce the conditional joint survival process $G(u, v; t)$

$$G(u, v; t) = \mathbb{Q}^*(\tau_1 > u, \tau_2 > v \mid \mathcal{F}_t).$$

We write

$$\partial_1 G(u, v; t) = \frac{\partial}{\partial u} G(u, v; t), \quad \partial_{12} G(u, v; t) = \frac{\partial^2}{\partial u \partial v} G(u, v; t).$$
We assume that the density \( f(u, v; t) = \partial_{12} G(u, v; t) \) with respect to \( u \) and \( v \) exists, so that

\[
G(u, v; t) = \int_{u}^{\infty} \left( \int_{v}^{\infty} f(x, y; t) \, dy \right) \, dx.
\]

For any fixed \((u, v) \in \mathbb{R}_+^2\), the \( \mathbb{F} \)-martingale

\( G(u, v; t) = \mathbb{Q}^*(\tau_1 > u, \tau_2 > v \mid \mathcal{F}_t) \) admits the integral representation

\[
G(u, v; t) = \mathbb{Q}^*(\tau_1 > u, \tau_2 > v) + \int_0^t g(u, v; s) \, dW_s
\]

where \( g(u, v; s) \) is some \( \mathbb{F} \)-predictable process (in fact an \( \mathbb{F} \)-martingale under \( \mathbb{Q}^* \)).
Let us introduce the filtrations $\mathcal{H}^i, \mathcal{H}, \mathcal{G}^i$ and $\mathcal{G}$ associated with default times by setting

$$
\mathcal{H}^i_t = \sigma(H^i_s; s \in [0, t]), \quad \mathcal{H}_t = \mathcal{H}^1_t \vee \mathcal{H}^2_t, \quad \mathcal{G}^i_t = \mathcal{F}_t \vee \mathcal{H}^i_t, \quad \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t,
$$

where $H^i_t = 1_{\{\tau_i \leq t\}}$. We assume that the usual conditions of completeness and right-continuity are satisfied by these filtrations.
We assume that immersion hypothesis holds between $\mathbb{F}$ and $\mathbb{G}$. In particular, any $\mathbb{F}$-martingale is also a $\mathbb{G}^i$-martingale for $i = 1, 2$. 
In general, there is no reason that any $\mathbb{G}^i$-martingale is a $\mathbb{G}$-martingale. Indeed, when $\mathbb{F}$ is a trivial filtration, denoting by $G_{t}^{1|2} = \mathbb{Q}^*(\tau_1 > t \mid \mathcal{H}_{t}^{2})$ and $G(u, v) = \mathbb{Q}(\tau_1 > u, \tau_2 > v)$,

$$dG_{t}^{1|2} = \left(\frac{\partial_2 G(t, t)}{\partial_2 G(0, t)} - \frac{G(t, t)}{G(0, t)}\right) dM_{t}^2 + \left(H_{t}^{2} \partial_1 h(t, \tau_2) + (1 - H_{t}^{2}) \frac{\partial_1 G(t, t)}{G(0, t)}\right) dt$$

where $M^2$ is the $\mathbb{H}^2$-martingale given by

$$M_{t}^{2} = H_{t}^{2} + \int_{0}^{t \wedge \tau_2} \frac{\partial_2 G(0, s)}{G(0, s)} ds$$

and $h(t, s) = \frac{\partial_2 G(t, s)}{\partial_2 G(0, s)}$. Hence immersion hypothesis is not always valid between $\mathbb{H}^2$ and $\mathbb{H}^1 \vee \mathbb{H}^2$, since $\frac{\partial_2 G(t, t)}{\partial_2 G(0, t)} - \frac{G(t, t)}{G(0, t)}$ is not always null.
Valuation of Single-Name CDSs

Let us now examine the valuation of single-name CDSs under the assumption that the interest rate equals zero.

We consider the CDS

- with the constant spread $\kappa_1$,
- which delivers $\delta_1(\tau_1)$ at time $\tau_1$ if $\tau_1 < T_1$, where $\delta_1$ is a deterministic function.

The value $S_1^{1}(\kappa_1)$ of this CDS, computed in the filtration $\mathcal{G}$, i.e., taking care on the information on the second default contained in that filtration, is computed in two successive steps.
On the set $t < \tau_{(1)} = \tau_1 \land \tau_2$, the ex-dividend price of the CDS equals, on the event \{ $t < \tau_{(1)}$ \},

$$S_t^1(\kappa_1) = \tilde{S}_t^1(\kappa_1) = \frac{1}{G(t, t; t)} \left( - \int_t^{T_1} \delta_1(u) \partial_1 G(u, t; t) \, du - \kappa_1 \int_t^{T_1} G(u, t; t) \, du \right).$$
On the event $\{\tau_2 \leq t < \tau_1\}$, we have that

$$S^1_t(\kappa_1) = \frac{1}{\partial_2 G(t, \tau_2; t)} \left( - \int_t^{T_1} \delta_1(u) f(u, \tau_2; t) \, du - \kappa_1 \int_t^{T_1} \partial_2 G(u, \tau_2; t) \, du \right).$$
Price Dynamics of Single-Name CDSs

Let us return to the study of a general case. By applying the Itô-Wentzell theorem, we get

\[ G(u, t; t) = G(u, 0; 0) + \int_{0}^{t} g(u, s; s) \, dW_s + \int_{0}^{t} \partial_2 G(u, s; s) \, ds \]

\[ G(t, t; t) = G(0, 0; 0) + \int_{0}^{t} g(s, s; s) \, dW_s + \int_{0}^{t} (\partial_1 G(s, s; s) + \partial_2 G(s, s; s)) \, ds. \]
The cumulative price $S^{c,1}(\kappa_1)$ satisfies, on $[0, T \wedge \tau(1)]$,

$$
\begin{align*}
    dS^{c,1}_t(\kappa_1) &= (\delta_1(t) - \tilde{S}^1_t(\kappa_1)) \, d\hat{M}^1_t + (S^1_{t|2}(\kappa_1) - \tilde{S}^1_t(\kappa_1)) \, d\hat{M}^2_t \\
    &\quad - \frac{1}{G(t, t; t)} \left( \int_t^{T_1} \delta_1(u) \partial_1 g(u, t; t) \, du + \kappa_1 \int_t^{T_1} g(u, t; t) \, du \right) dW_t.
\end{align*}
$$

Here

$$
\hat{M}^i_t = H^i_{t\wedge \tau(1)} - \int_0^{t\wedge \tau(1)} \tilde{\lambda}^i_u \, du,
$$

is a $\mathcal{G}$-martingale, where $\tilde{\lambda}^i_t = -\frac{\partial_i G(t, t; t)}{G(t, t; t)}$ is the pre-default intensity for the $i$th name.
Replication of a First-to-Default Claim

A **first-to-default claim** with maturity $T$ is a defaultable claim $(X, A, Z, \tau_{(1)})$ where $X$ is an $\mathcal{F}_T$-measurable amount payable at maturity if no default occurs, a continuous process of finite variation $A : [0, T] \rightarrow \mathbb{R}$ with $A_0 = 0$ represents the dividend stream up to $\tau_{(1)}$, and $Z = (Z^1, Z^2, \ldots, Z^n)$ is the vector of $\mathbb{F}$-predictable, real-valued processes, where $Z^i_{\tau_{(1)}}$ specifies the recovery received at time $\tau_{(1)}$ if the $i$th name is the first defaulted name, that is, on the event \{\tau_i = \tau_{(1)} \leq T\}. 
The cumulative price $S^{cum}$ is given by

$$dS^{cum}_t = r_t S^{cum}_t \, dt + \sum_{i=1}^{n} (Z^i_t - S_{t-}) \, d\hat{M}^i_t + (1 - H_t^{(1)}) B_t (G_{(1)}(t; t))^{-1} dm_t,$$

where the $\mathbb{F}$-martingale $m$ is given by the formula

$$m_t = \mathbb{E}_{Q^*} \left( B_{T}^{-1} G_{(1)}(T; T) X + \sum_{i=1}^{n} \int_{0}^{T} B_{u}^{-1} G_{(1)}(u; u) Z^i_u \tilde{\lambda}^i_u \, du \
- \int_{0}^{T} B_{u}^{-1} G_{(1)}(u; u) \, dA_u \bigg\lvert \mathcal{F}_t \right).$$
The pre-default ex-dividend price satisfies

\[ d\tilde{S}_t = (r_t + \tilde{\lambda}_t)\tilde{S}_t dt - \sum_{i=1}^n \tilde{\lambda}_t^i Z_t^i dt + dA_t + B_t(G(t, t; t))^{-1} dm_t. \]

Since \( \mathbb{F} \) is generated by a Brownian motion, there exists an \( \mathbb{F} \)-predictable process \( \zeta \) such that

\[ d\tilde{S}_t = (r_t + \tilde{\lambda}_t)\tilde{S}_t dt - \sum_{i=1}^n \tilde{\lambda}_t^i Z_t^i dt + dA_t + B_t(G(t, t; t))^{-1} \zeta_t dW_t. \]
Hedging of Credit Spreads and Default Correlations

We say that a self-financing strategy \( \phi = (\phi^0, \phi^1, \ldots, \phi^k) \) replicates a first-to-default claim \((X, A, Z, \tau(1))\) if its wealth process \(V(\phi)\) satisfies the equality \(V_{t \wedge \tau(1)}(\phi) = S_{t \wedge \tau(1)}\) for any \(t \in [0, T]\).

We have, for any \(t \in [0, T]\),

\[
dV_t(\phi) = r_t V_t(\phi) \, dt + \sum_{i=1}^{n} \phi_t^i \left( (\delta_t^i - \tilde{S}_t^i(\kappa_i)) \, dM_t^i + \sum_{j=1, j \neq i}^{n} (S_{t|j}^i - \tilde{S}_t^i(\kappa_i)) \, dM_t^j \right) \\
+ (1 - H_t) B_t (G(t, t; t))^{-1} dn_t^i
\]

where

\[
n_t^i = \mathbb{E}_{Q^*} \left( \int_0^{T_i} G(u, u; u) \frac{\delta_u \tilde{\lambda}_u^i + \sum_{j=1, j \neq i}^{n} S_{u|j}^i \tilde{\lambda}_u^j}{B_u} \, du - \kappa_i \int_0^{T_i} G(u, u; u) \frac{1}{B_u} \, du \bigg| \mathcal{F}_t \right).
\]
Let $\tilde{\phi}_t = (\tilde{\phi}_1^t, \tilde{\phi}_2^t, \ldots, \tilde{\phi}_n^t)$ be a solution to the following equations

$$\tilde{\phi}_i^t (\delta_t^i - \tilde{S}_t^i (\kappa_i)) + \sum_{j=1, j \neq i}^n \tilde{\phi}_j^t (S^j_{t|i} (\kappa_j) - \tilde{S}_t^j (\kappa_j)) = Z^i_t - \tilde{S}_t$$

and $\sum_{i=1}^n \tilde{\phi}_i^t \zeta_t^i = \zeta_t$. Let us set $\phi^i_t = \tilde{\phi}^i_t (\tau(1) \wedge t)$ for $i = 1, 2, \ldots, n$ and $t \in [0, T]$. Then the self-financing trading strategy

$\phi = (B^{-1} (V(\phi) - \phi \cdot S), \ldots, \phi^k)$ replicates the first-to-default claim $(X, A, Z, \tau(1))$.
References


REFERENCES


REFERENCES

Hedging of Credit Spreads and Default Correlations

Working paper.

