JUMP PROCESSES
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Introduction

These notes will be an introduction to jump processes and application to finance. There are many books, articles, and thesis devoted to that subject, and we give here only some references.

The general theory of stochastic processes is presented in Dellacherie and Meyer [12, 13], He et al. [15], Protter, version 2.1. [27].

Jacod and Shiryaev [16], Bichteler [7], Prigent [26] study processes with discontinuous path in a semi-martingale framework.

For general jump processes with finance in view, one can consult Jeanblanc, Yor and Chesney [17], and Shiryaev [33].

Excellent surveys papers are Bass [2, 1], Kunita [21], Runggaldier [28].

Bertoin [5, 6, 4, 3], Kyprianou [22], Sato [29, 30] contain the theory of Lévy processes. Applications to Finance can be found in Boyarchenko and Levendorskii [9], Cont and Tankov [11], Overhaus et al. [25], Schoutens [32].

Many thanks to all the participants to the school.

Chapter 1

Poisson Processes

1.1 Some Particular Laws of Random Variables

We recall some important results on exponential and Poisson laws. We give only some proofs.

1.1.1 Exponential Law

Definition 1.1.1 *The exponential law with parameter $\lambda > 0$ has a density $f_\lambda(x) = \lambda e^{-\lambda x} \mathbb{1}_{x>0}$ with respect to Lebesgue measure. If X is a random variable with exponential law, its cumulative distribution function is, for $t \in \mathbb{R}^+$, given by $\mathbb{P}(X \le t) = 1 - e^{-\lambda t}$.*

The characteristic property of the exponential law is the lack of memory that we recall now

Proposition 1.1.1 *(i) The exponential law is the unique density on $\mathbb{R}^+$ such that*

$$\forall t, s > 0, \quad \mathbb{P}(X > t + s | X > s) = \mathbb{P}(X > t)$$

(ii) The exponential law is the unique density such that there exists a constant λ such that

$$\forall t > 0, \quad \lim_{s \to 0} s^{-1} \mathbb{P}(X > t + s | X > s) = \lambda$$

PROOF: From definition of conditional probability

$$\mathbb{P}(X > t + s | X > s) = \frac{\mathbb{P}(X > t + s)}{\mathbb{P}(X > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$$

and the result follows. For the converse, let $G(s) = \mathbb{P}(X > s)$ be the survival probability. Then, from hypothesis $G(t + s) = G(t)G(s)$. This implies, using that G is continuous that $G(s) = e^{\theta s}$ for some θ. Since G is a survival probability, $G(s) < 1$, hence $\theta < 0$. We leave to the reader the proof of (ii).$\triangleleft$

The property (ii) can be written

$$\forall t > 0, \quad \mathbb{P}(X \in dt | X > t) = \lambda dt$$

We now recall some properties of the exponential law and sum of i.i.d. exponential random variables.

Proposition 1.1.2 *(i) If X has an exponential law with parameter λ, then $\mathbb{E}(X) = \lambda^{-1}$ and $\mathrm{Var}(X) = \lambda^{-2}$.*
(ii) The Laplace transform $\mathbb{E}[e^{-\mu X}]$ of an exponential law of parameter λ is defined for $\mu > -\lambda$ and is given by

$$\phi(\mu) = \mathbb{E}[e^{-\mu X}] = \frac{\lambda}{\lambda + \mu}$$

(iii) The sum X_n of n independent exponential r.v's of parameter λ has a Gamma law with parameters (n, λ):

$$\mathbb{P}(X_n \in dt) = \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t} \, \mathbb{1}_{\{t>0\}} dt,$$

The cumulative distribution function is

$$\mathbb{P}(X_n \leq x) = 1 - e^{-\lambda x}(1 + \frac{\lambda x}{1!} + \cdots + \frac{(\lambda x)^{n-1}}{(n-1)!})$$

and its Laplace transform is given, for $\mu > -\lambda$ by

$$\mathbb{E}(e^{-\mu X_n}) = \left(\frac{\lambda}{\lambda + \mu}\right)^n .$$

(iv) The law of $(X_1, \cdots, X_n)$ has density $\lambda^n e^{-\lambda x_n} 1\!\!1_{0<x_1<\cdots<x_n}$
(v) The law of $(X_1, \cdots, X_n)$ given $X_n = x$ is

$$\mathbb{P}(X_1 \in dx_1, \cdots, X_n \in dx_n | X_n = x) = \frac{(n-1)!}{x^n} 1\!\!1_{0<x_1<\cdots<x_n<x} dx_1 \cdots dx_n$$

Proof: The proof of (i) and (ii) is standard. The proof of (iii) is done in a recursive way and is based on the computation of the law of a sum of two independent random variables. The proof of (iv) and (v) follows from simple computations. See Feller [14] for details. $\lhd$

Exercise 1.1.1 Let X and Y be two independent exponential random variables. Prove that

$$\begin{aligned} \mathbb{P}(X - Y \in A | X > Y) &= \mathbb{P}(X \in A) \\ \mathbb{P}(X - Y \in A | X < Y) &= \mathbb{P}(-Y \in A) \end{aligned}$$

1.1.2 Poisson Law

Definition 1.1.2 *A random variable X with integer values has a Poisson law with parameter $\theta > 0$ if*

$$\mathbb{P}(X = k) = e^{-\theta}\frac{\theta^k}{k!}.$$

Proposition 1.1.3 *If X has a Poisson law with parameter $\theta > 0$, then*
(i) for any $s \in \mathbb{R}$, $\mathbb{E}[s^X] = e^{\theta(s-1)}$.
(ii) $\mathbb{E}[X] = \theta$, $\mathrm{Var}\,(X) = \theta$.
(iii) for any $u \in \mathbb{R}$, $\mathbb{E}(e^{iuX}) = \exp(\theta(e^{iu} - 1))$
(iv) for any $\alpha \in \mathbb{R}$, $\mathbb{E}(e^{\alpha X}) = \exp(\theta(e^{\alpha} - 1))$

Proof: If X has a Poisson law with parameter $\theta > 0$, for any $s \in \mathbb{R}$,

$$f(s) := \mathbb{E}[s^X] = \sum_{k=0}^{\infty} e^{-\theta}\frac{\theta^k}{k!}s^k = e^{-\theta}\sum_{k=0}^{\infty}\frac{(s\theta)^k}{k!}s^k = e^{-\theta}e^{s\theta} .$$

Then, using the usual relation giving the expectation and the second moment in terms of the generating function f: $\mathbb{E}(X) = f'(0)$ and $\mathbb{E}(X^2) = f''(0) + f'(0)$, we obtain the assertion (ii).
The proof of assertion (iii) is left to the reader, we give only the proof of (iv):

$$\mathbb{E}(e^{\alpha X}) = \sum_{k=0}^{\infty} e^{-\theta}\frac{\theta^k}{k!}e^{\alpha k} = e^{-\theta}\sum_{k=0}^{\infty}\frac{(e^{\alpha}\theta)^k}{k!} = e^{-\theta}e^{\theta e^{\alpha}}$$

and the result follows. $\lhd$

Conversely, if X is a r.v. such that one of the property (i), (iii), (iv) holds, then X has a Poisson law with parameter θ. The proof follows from the characterization of a probability law by the generating function (resp. the characteristic function, the Laplace transform)

In particular, the form of the characteristic function implies the infinitely divisible property of the Poisson law: if X has a Poisson law, for any n there exists n i.i.d. random variables $X_i^n, i = 1, \cdots, n$ such that $X \overset{law}{=} X_1^n + \cdots + X_n^n$. Indeed, for any n, the obvious equality

$$\mathbb{E}(e^{iuX}) = \exp(\theta(e^{iu} - 1)) = \left(\exp((\theta/n)(e^{iu} - 1))\right)^n$$

and the fact that $\exp((\theta/n)(e^{iu} - 1))$ is the characteristic function of a Poisson law with parameter θ/n yields the result.

Corollary 1.1.1 *If X and Y are two independent random variables, with a Poisson law of parameter θ and μ, then $X+Y$ has a Poisson law with parameter $\lambda+\mu$.*

PROOF: If X and Y are two independent random variables, with a Poisson law of parameter θ and μ, the generating function of $X+Y$ is the product of the generating function of X and of the generating function of Y, hence

$$\mathbb{E}[s^{X+Y}] = e^{\theta(s-1)} e^{\mu(s-1)} = e^{(\theta+\mu)(s-1)}$$

which gives the result. $\triangleleft$

Exercise 1.1.2 Let X be a random variable with a Poisson law with parameter θ and $(U_i)_{i\geq 1}$ a sequence of i.i.d. Bernoulli random variables, with parameter p, and independent of X. Prove that $X_1 = \sum_{k=1}^{X} U_k$ and $X_2 = X - X_1$ are independent random variables, with a Poisson law of parameters $p\theta$ and $(1-p)\theta$.

1.1.3 Poisson mixture model

We now assume that $\Lambda := (\Lambda_1, \cdots, \Lambda_n)$ is an $\mathbb{R}_+^n$ valued random variable with cumulative distribution function

$$F(\lambda_1, \cdots, \lambda_n) = \mathbb{P}(\Lambda_1 \leq \lambda_1, \cdots, \Lambda_n \leq \lambda_n)$$

and that $(X_1, \cdots, X_n)$ are random variables, valued in the set of non negative integers such that

$$\mathbb{P}(X_1 = i_1, \cdots, X_n = i_n | \Lambda) = \prod_{k=1}^{n} \mathbb{P}(X_k = i_k | \Lambda) = \prod_{k=1}^{n} e^{-\Lambda_k} \frac{\Lambda_k}{i_k!}$$

so that, the r.v's X_k are independent conditionally w.r.t. Λ. In particular,

$$\mathbb{P}(X_k = i_k | \Lambda) = \mathbb{P}(X_k = i_k | \Lambda_k) = e^{-\Lambda_k} \frac{\Lambda_k}{i_k!}$$

and

$$\mathbb{P}(X_1 = i_1, \cdots, X_n = i_n) = \int_{\mathbb{R}_+^n} e^{-(\lambda_1 + \cdots + \lambda_n)} \prod_{k=1}^{n} \frac{\lambda_k^{i_k}}{i_k!} F(d\lambda_1, \cdots, d\lambda_n)$$

Using results on Poisson law, we obtain

$$\begin{aligned} \mathbb{E}(X_k | \Lambda_k) &= \Lambda_k, \mathrm{Var}(X_k | \Lambda_k) = \Lambda_k \\ \mathbb{E}(X_k X_m) &= \mathbb{E}(\mathbb{E}(X_k X_m | \Lambda) = \mathbb{E}(\mathbb{E}(X_k | \Lambda_k) \mathbb{E}(X_m | \Lambda_m)) = \mathbb{E}(\Lambda_k \Lambda_m) \end{aligned}$$

Then, $\mathrm{Cov}(X_k X_m) = \mathrm{Cov}(\Lambda_k \Lambda_m)$ and $\mathrm{Var}(X_k) = \mathbb{E}\mathrm{Var}(X_k | \Lambda_k) + \mathrm{Var}\mathbb{E}(X_k | \Lambda_k) = \mathbb{E}(\Lambda_k) + \mathrm{Var}(\Lambda_k)$

Proposition 1.1.4 *In the Poisson mixture framework, for $X = \sum_{k=1}^{n} X_k$*

1. $\mathbb{E}(X) = \sum_{k=1}^{n} \mathbb{E}(\Lambda_k)$
2. $\mathrm{Var}X = \sum_{k=1}^{n} \mathrm{Var}X_k + \sum_{k\neq m} \mathrm{Cov}(X_k X_m)$
3. $\mathbb{E}(s^X) = \mathbb{E}(e^{(s-1)(\Lambda_1 + \cdots + \Lambda_m)})$

See Bluhm et al. [8] and Schmock [31] application to credit risk.

1.2 Standard Poisson Processes

1.2.1 Counting processes

A **counting process** is a process that increases in unit steps at isolated times and is constant between these times. Let $(T_n, n \geq 0)$ be a strictly increasing sequence of random variables $T_0 = 0 < T_1 < \cdots < T_n$. The (increasing and right-continuous) counting process associated with this sequence is defined as

$$N_t = \begin{cases} n & \text{if } t \in [T_n, T_{n+1}[\\ +\infty & \text{otherwise} \end{cases}$$

or, equivalently

$$N_t = \sum_{n \geq 1} \mathbb{1}_{\{T_n \leq t\}} = \sum_{n \geq 0} n \mathbb{1}_{\{T_n \leq t < T_{n+1}\}},\ N_0 = 0\,.$$

We shall assume that $\lim T_n = +\infty$ to avoid explosion at finite time. For $t > s$, the increment $N_t - N_s$ is the number of random times T_n that occurs between s and t.

Let $\mathbf{F}$ be a given filtration. Then, the counting process N is $\mathbf{F}$-adapted if and only if $(T_n, n \geq 1)$ are $\mathbf{F}$-stopping times. We denote by $\mathbf{F}^N$ the natural filtration of the process N.

1.2.2 Definition and first Properties of Poisson Processes

Definition 1.2.1 *Let $(\tau_i)_{i \geq 1}$ be a sequence of i.i.d. random variable with exponential law of parameter λ and let $T_n = \sum_{i=1}^n \tau_i$, $n \geq 1$. The associated counting process is called a Poisson process.*

Proposition 1.2.1 *The Poisson process has independent and stationary increments.*

PROOF: Let us prove that

$$\mathbb{P}(N_{t+s} - N_t = k \mid \mathcal{F}_t^N) \;=\; \mathbb{P}(N_s = k) \;=\; e^{-\theta s} \frac{(\theta s)^k}{k!}$$

In a first step, one prove that $\mathbb{P}(N_s = k) \;=\; e^{-\theta s} \frac{(\theta s)^k}{k!}$. From the definition of the process N, the event $N_s = k$ is equal to the event $T_k \leq s < T_{k+1}$. Then

$$\mathbb{P}(N_s = k) = \mathbb{P}(T_k \leq s < T_{k+1}) = \mathbb{P}(T_k \leq s) - \mathbb{P}(T_{k+1} \leq s)$$

and the result follows from the form of the cumulative distribution function of T_k.

Let $t_0 = 0 < t_1 < \cdots < t_k$, and

$$A = \mathbb{P}(N_{t_1} = n_1, \cdots N_{t_j} - N_{t_{j-1}} = n_j, \cdots N_{t_k} - N_{t_{k-1}} = n_k)$$

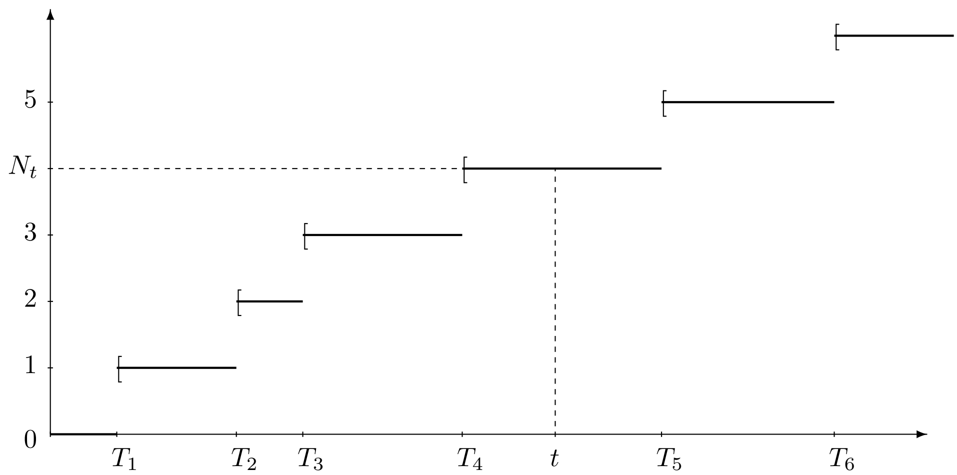
Hence, if $i_m = \sum_{j \leq m} n_j$

$$A = \mathbb{P}(T_{i_1} \leq t_1 < T_{i_1+1}, \cdots T_{i_m} \leq t_m < T_{i_m+1}, \cdots T_{i_k} \leq t_k < T_{i_k+1})$$

The law of the vector T_i is known, however, it is more efficient to work with conditional laws.

$$A = \mathbb{P}(T_{i_1} \leq t_1 < T_{i_1+1}, \cdots T_{i_m} \leq t_m < T_{i_m+1}, \cdots | T_{i_k} \leq t_k < T_{i_k+1}) \mathbb{P}(T_{i_k} \leq t_k < T_{i_k+1})$$

The result follows in a recursive way. See Cont and Tankov [11] for details.◁



Exercise 1.2.1 Let t fixed. Give the law of the r.v. $T_{N_t+1} - t$.
HINT: For any $s \geq 0$,

$$\mathbb{P}(T_{N_t+1} - t > s) \;=\; \mathbb{P}(T_{N_t+1} > t + s) \;=\; \mathbb{P}(N_{t+s} = N_t) \;=\; \mathbb{P}(N_s = 0) \;=\; e^{-\theta s}$$

hence $T_{N_t+1} - t \overset{law}{=} T_1$.

Theorem 1.2.1 *For any stopping time T the process $N_{T+t} - N_T, t \geq 0$ is independent of $\mathcal{F}_T^N$ and has the same law as N.*

PROOF: This property is the Markov property for process with independent increments (see Appendix).$\lhd$

Note that, in particular, if f is a bounded Borel function, and $s < t$,

$$\mathbb{E}(f(N_t)|\mathcal{F}_s^N) = \mathbb{E}(f(N_t)|N_s) = \mathbb{E}(f(N_t - N_s + N_s)|N_s) = F(N_s)$$

with $F(x) = E(f(N_{t-s}+x))$. Indeed, if (X, Y) are independent random variables $E(f(X, Y)|Y) = \Psi(Y)$ with $\Psi(y) = E(f(X, y))$.

Theorem 1.2.2 *The Poisson process with parameter λ is the unique counting process with independent stationary increments.*

PROOF: Let N be a counting process with independent stationary increments and $T_1 = \inf\{t : N_t = 1\}$. Let us prove that T_1 has an exponential law. Using the property of independent stationary increments of the counting process N, we obtain

$$\begin{aligned}
\mathbb{P}(T_1 > t + s) &= \mathbb{P}(N_{t+s} = 0) = \mathbb{P}(N_{t+s} - N_t = 0, N_t = 0) \\
&= \mathbb{P}(N_{t+s} - N_t = 0)\, \mathbb{P}(N_t = 0) = \mathbb{P}(T_1 > t)\, \mathbb{P}(T_1 > s)
\end{aligned}$$

hence the result from Proposition 1.1.1. The independence and the stationarity of the sequence $T_1, \cdots, T_n - T_{n-1}$ follows from the strong Markov property.

Exercise 1.2.2 Let N be a Poisson process. Then

$$\mathbb{P}(T_1, \cdots, T_n \in A|N_t = n) = \mathbb{P}(U_{\sigma_1}, \cdots U_{\sigma_n} \in A)$$

where $(U_i, i = 1, \cdots, n)$ are i.i.d. random variable with uniform law on $[0, t]$ and $U_{\sigma_1} < U_{\sigma_2} < \cdots < U_{\sigma_n}$

Exercise 1.2.3 [Law of Large numbers]Let N be a Poisson process with parameter λ. Prove that $\lim_{t\to\infty} N_t = \infty$ a.s. and

$$\lim_{t\to\infty} \frac{N_t}{t} = \lambda \text{ a.s..}$$

Exercise 1.2.4 Let $N_1, \cdots, N_k$ be independent Poisson processes with parameter $\theta_1, \cdots, \theta_k$. Prove that the processes N_i have no common jumps and $N_1 + \cdots + N_k$ is a Poisson process with parameter $\theta_1 + \cdots + \theta_k$.

1.2.3 Martingale Properties

From the independence of the increments of the Poisson process, we derive martingale properties of various processes.

Proposition 1.2.2 *Let N be a Poisson process with intensity λ.*
(i) The process $M_t = N_t - \lambda t$ is a martingale.
(ii) The process $M_t^2 - \lambda t = (N_t - \lambda t)^2 - \lambda t$ is a martingale.
(iii) The process $M_t^2 - N_t$ is a martingale.
(iv) For any α, the process $\exp(\alpha N_t - \lambda t(e^\alpha - 1))$ is a martingale.
(v) For any β, the process $(1 + \beta)^{N_t} e^{-\lambda\beta t}$ is a martingale.

PROOF: The properties (i) (ii) and (iv) are an application of the more general following Proposition 2.2.5. Now, adding the two $\mathbf{F}^N$-martingales M_t and $M_t^2 - \lambda t$ proves that $M_t^2 - N_t$ is a martingale. Property (v) follows from the independence of the increments and the computation of the generating function.$\lhd$

Definition 1.2.2 *The martingale $(M_t := N_t - \lambda t,\, t \geq 0)$ is called the* ***compensated process*** *of N, and λ is the* ***intensity*** *of the process N.*

It follows from the martingale property of $M_t^2 - \lambda t$ that λt is the predictable variation process of M. The process $M_t^2 - N_t$ is a martingale, N is increasing and $\Delta N_s = (\Delta M_s)^2$. Hence, the quadratic variation process of M is N.

Remark 1.2.1 *One can think that, since $N_t = N_{t-}$, a.s. the process $N_t - N_{t-}$ is null, hence is a martingale, hence the predictable variation process is N_{t-}. This is obviously false. Firstly, the predictable variation is unique, up to indistinguability. The process N_{t-} is not equal to λt. Furthermore, the process $N_t - N_{t-}$ is not a null process, since $\mathbb{P}(\forall t, X_t = X_{t-})$ is not equal to 1.*

The previous Proposition 1.2.2 admits an extension:

Proposition 1.2.3 *Let N be an $\mathbf{F}$-Poisson process. For each bounded Borel function h, for any $\beta > -1$, and any bounded Borel function φ valued in $]-1, \infty[$, the following processes are $\mathbf{F}$-martingales:*

$$\exp[\ln(1+\beta)N_t - \lambda\beta t]$$

$$\exp\Big[\int_0^t h(s)dN_s - \lambda\int_0^t (e^{h(s)} - 1)ds\Big] = \exp\Big[\int_0^t h(s)dM_s - \lambda\int_0^t (e^{h(s)} - h(s) - 1)ds\Big]\,,$$

$$\exp\Big[\int_0^t \ln(1+\varphi(s))dN_s - \lambda\int_0^t \varphi(s)ds\Big] = \exp\Big[\int_0^t \ln(1+\varphi(s))dM_s + \lambda\int_0^t (\ln(1+\varphi(s)) - \varphi(s))ds\Big]\,,$$

Proof: The study of $\exp[\ln(1+\beta)N_t - \lambda\beta t] = (1+\beta)^{N_t}e^{-\lambda\beta t}$ was done in Proposition 1.2.2. In the specific case $\beta > -1$, it can also be done setting $\ln(1+\beta) = \alpha$ in (iv) of the same Proposition. For the other processes, proceed using Monotone class Theorem.$\lhd$

We have chosen to write $\exp\big[\int_0^t h(s)dN_s - \lambda\int_0^t (e^{h(s)} - 1)ds$ instead of $\exp\big[\int_0^t h(s)dN_s + \lambda\int_0^t (1 - e^{h(s)})ds$ to help the reader to memorize the formule: for h non-negative, the process $\int_0^t h(s)dN_s$ is increasing, so we have to subtract a non negative a quantity to obtain a martingale.

Proposition 1.2.4 *Let N be an $\mathbf{F}$-Poisson process and H be an $\mathbf{F}$-predictable bounded process, then the following processes are martingales*

$$\begin{aligned}
&(H\star M)_t := \int_0^t H_s dM_s = \int_0^t H_s dN_s - \lambda\int_0^t H_s ds \\
&((H\star M)_t)^2 - \lambda\int_0^t H_s^2 ds \qquad (1.1)\\
&\exp\left(\int_0^t H_s dN_s - \lambda\int_0^t (e^{H_s} - 1)ds\right)
\end{aligned}$$

Proof: In a first step, the proof is done for elementary predictable processes H

Exercise 1.2.5 Assuming that $\int_0^t N_{s-}dM_s$ is a martingale as stated in Proposition 1.2.4, prove that $\int_0^t N_s dM_s$ is not a martingale.

1.2.4 Change of Probability

Theorem 1.2.3 *Let N be a Poisson process with intensity λ, and $\mathbb{Q}$ be the probability defined as $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = (1+\beta)^{N_t}e^{-\lambda\beta t}$.*
Then, the process N is a $\mathbb{Q}$-Poisson process with intensity equal to $(1+\beta)\lambda$.

Proof: In a first step, one note that, from Proposition 1.2.3, for $\beta > -1$, the process L defined as

$$L_t = (1+\beta)^{N_t}e^{-\lambda\beta t}$$

is a strictly positive martingale with expectation equal to 1. Then, from the definition of $\mathbb{Q}$, for any sequence $0 = t_1 < t_2 < \cdots < t_{n+1} = t$,

$$\mathbb{E}_Q\left(\prod_{i=1}^n x_i^{N_{t_{i+1}} - N_{t_i}}\right) = \mathbb{E}_P\left(e^{-\lambda\beta t}\prod_{i=1}^n ((1+\beta)x_i)^{N_{t_{i+1}} - N_{t_i}}\right)$$

The right-hand side is computed using that, under $\mathbb{P}$, the process N is a Poisson process (hence with independent increments) and is equal to

$$e^{-\lambda\beta t}\prod_{i=1}^{n}\mathbb{E}_P\left(((1+\beta)x_i)^{N_{t_{i+1}-t_i}}\right)=e^{-\lambda\beta t}\prod_{i=1}^{n}e^{-\lambda(t_{i+1}-t_i)}\,e^{\lambda(t_{i+1}-t_i)(1+\beta)x_i}=\prod_{i=1}^{n}e^{(1+\beta)\lambda(t_{i+1}-t_i)(x_i-1)}\,.$$

It follows that, for any j (take all the x_i's, except the jth one, equal to 1)

$$\mathbb{E}_Q\left(x_j^{N_{t_{j+1}}-N_{t_j}}\right)=e^{(1+\beta)\lambda(t_{j+1}-t_j)(x_j-1)}\,,$$

which establishes that, under $\mathbb{Q}$, the r.v. $N_{t_{j+1}}-N_{t_j}$ has a Poisson law with parameter $(1+\beta)\lambda$, then that

$$\mathbb{E}_Q\left(\prod_{i=1}^{n}x_i^{N_{t_{i+1}}-N_{t_i}}\right)=\prod_{i=1}^{n}\mathbb{E}_Q\left(x_i^{N_{t_{i+1}}-N_{t_i}}\right)$$

which is equivalent to the independence of the increments.$\triangleleft$

1.3 Inhomogeneous Poisson Processes

1.3.1 Definition

Instead of considering a constant intensity λ as before, now $(\lambda(t), t\geq 0)$ is an $\mathbb{R}^+$-valued function satisfying $\Lambda(t):=\int_0^t\lambda(u)du<\infty, \forall t$. An **inhomogeneous Poisson process** N with intensity λ is a counting process with independent increments which satisfies for $t>s$

$$\mathbb{P}(N_t-N_s=n)=e^{-\Lambda(s,t)}\frac{(\Lambda(s,t))^n}{n!}\tag{1.2}$$

where $\Lambda(s,t)=\Lambda(t)-\Lambda(s)=\int_s^t\lambda(u)du$, and $\Lambda(t)=\int_0^t\lambda(u)du$.
If $(T_n, n\geq 1)$ is the sequence of successive jump times associated with N, the law of T_n is:

$$\mathbb{P}(T_n\leq t)=\frac{1}{n!}\int_0^t\exp(-\Lambda(s))\,(\Lambda(s))^{n-1}\,d\Lambda(s)\,.$$

It can easily be shown that an inhomogeneous Poisson process with deterministic intensity is an inhomogeneous Markov process. Moreover, $\mathbb{E}(N_t)=\Lambda(t), \mathrm{Var}(N_t)=\Lambda(t)$.
An inhomogeneous Poisson process can be constructed as a deterministic changed time Poisson process: if $\widetilde{N}$ is a standard Poisson process with intensity 1, the process N defined as $N_t=\widetilde{N}_{\Lambda(t)}$ is an inhomogeneous Poisson process with intensity λ. An inhomogeneous Poisson process can also be constructed using a change of probability (see below).

1.3.2 Martingale Properties

Proposition 1.3.1 *Let N be an inhomogeneous Poisson process with deterministic intensity λ and $\mathbf{F}^N$ its natural filtration. Define $\Lambda(t)=\int_0^t\lambda(s)ds$. The process*

$$(M_t=N_t-\Lambda(t),\, t\geq 0)$$

is an $\mathbf{F}^N$-martingale, and the increasing function Λ is called the (deterministic) **compensator** *of N.*

Let ϕ be an $\mathbf{F}^N$-predictable process such that $\mathbb{E}(\int_0^t|\phi_s|\lambda(s)ds)<\infty$ for every t. Then, the process $(\int_0^t\phi_s dM_s, t\geq 0)$ is an $\mathbf{F}^N$-martingale. In particular,

$$\mathbb{E}\left(\int_0^t\phi_s\,dN_s\right)=\mathbb{E}\left(\int_0^t\phi_s\lambda(s)ds\right)\,.\tag{1.3}$$

As an immediate extension of results obtained in the constant intensity case, for any bounded $\mathbf{F}^N$-predictable process H, the following processes are martingales

$$\begin{array}{ll} a) & (H\star M)_t=\int_0^t H_s dM_s=\int_0^t H_s dN_s-\int_0^t\lambda(s)H_s ds\\ b) & ((H\star M)_t)^2-\int_0^t\lambda(s)H_s^2 ds\\ c) & \exp\left(\int_0^t H_s dN_s-\int_0^t\lambda(s)(e^{H_s}-1)ds\right)\,. \end{array}$$

1.3.3 Stochastic Intensity

Let $(\Omega, \mathbf{F}, \mathbb{P})$ be a filtered probability space and λ a non-negative $\mathbf{F}$-adapted process such that $\int_0^t \lambda_s ds < \infty$.

Definition 1.3.1 *A counting process N is said to be an inhomogeneous Poisson process with stochastic intensity λ if the process*

$$(M_t = N_t - \int_0^t \lambda_s ds,\, t \geq 0)$$

is a martingale, called the compensated martingale.

1.3.4 Stochastic Calculus

In this section, M is the compensated martingale of an inhomogeneous Poisson process N with intensity $(\lambda_s, s \geq 0)$. From now on, we restrict our attention to integrals of predictable processes, even if the stochastic integrals are defined in a more general setting.

Integration by parts formula

We start with an elementary case.

Let $(x_t, t \geq 0)$ and $(y_t, t \geq 0)$ be two predictable processes and define two processes X and Y as $X_t = x + \int_0^t x_s dN_s$ and $Y_t = y + \int_0^t y_s dN_s$. The jumps of X (resp. of Y) occur at the same times as the jumps of N and $\Delta X_s = x_s \Delta N_s, \Delta Y_s = y_s \Delta N_s$. The processes X and Y are of finite variation and are constant between two jumps. Then

$$X_t Y_t = xy + \sum_{s \leq t} \Delta(XY)_s = xy + \sum_{s \leq t} X_{s-} \Delta Y_s + \sum_{s \leq t} Y_{s-} \Delta X_s + \sum_{s \leq t} \Delta X_s \, \Delta Y_s$$

The first equality is obvious, the second one is easy to check. Hence, from the definition of stochastic integrals (see Section 4.2)

$$X_t Y_t = xy + \int_0^t Y_{s-} dX_s + \int_0^t X_{s-} dY_s + [X, Y]_t$$

where (note that $(\Delta N_t)^2 = \Delta N_t$)

$$[X, Y]_t := \sum_{s \leq t} \Delta X_s \, \Delta Y_s = \sum_{s \leq t} x_s y_s \Delta N_s = \int_0^t x_s \, y_s \, dN_s \, .$$

More generally, if $dX_t = \mu_t dt + x_t dN_t$ with $X_0 = x$ and $dY_t = \nu_t dt + y_t dN_t$ with $Y_0 = y$, one gets

$$X_t Y_t = xy + \int_0^t Y_{s-} dX_s + \int_0^t X_{s-} dY_s + [X, Y]_t$$

where $[X, Y]_t = \int_0^t x_s \, y_s \, dN_s$. In particular, if $dX_t = x_t dM_t$ and $dY_t = y_t dM_t$, the process $X_t Y_t - [X, Y]_t$ is a local martingale.

Itô's Formula

For inhomogeneous Poisson processes, Itô's formula is obvious as we now explain.
Let N be a Poisson process and f a bounded Borel function. The decomposition

$$f(N_t) = f(N_0) + \sum_{0 < s \leq t} [f(N_s) - f(N_{s-})] \tag{1.4}$$

is trivial and is the main step to obtain Itô's formula for a Poisson process.
We can write the right-hand side of (1.4) as a stochastic integral:

$$\begin{aligned} \sum_{0 < s \leq t} [f(N_s) - f(N_{s-})] &= \sum_{0 < s \leq t} [f(N_{s-} + 1) - f(N_{s-})] \Delta N_s \\ &= \int_0^t [f(N_{s-} + 1) - f(N_{s-})] dN_s \, , \end{aligned}$$

hence, the canonical decomposition of $f(N_t)$ as the sum of a martingale and an absolute continuous adapted process is

$$f(N_t) = f(N_0) + \int_0^t [f(N_{s^-}+1) - f(N_{s^-})]dM_s + \int_0^t [f(N_{s^-}+1) - f(N_{s^-})]\lambda_s ds\,.$$

More generally, let h be an adapted process and g a predictable process such that $\int_0^t |h_s|ds < \infty$, $\int_0^t |g_s|\lambda_s ds < \infty$.

Proposition 1.3.2 *Let* $dX_t = h_t dt + g_t dM_t = (h_t - g_t\lambda_t)dt + g_t dN_t$ *and* $F \in C^{1,1}(\mathbb{R}^+ \times \mathbb{R})$*. Then*

$$\begin{aligned}
F(t, X_t) &= F(0, X_0) + \int_0^t \partial_t F(s, X_s)ds + \int_0^t \partial_x F(s, X_{s-})(h_s - g_s\lambda_s)ds + \sum_{s\le t} F(s, X_s) - F(s, X_{s-}) \\
&= F(0, X_0) + \int_0^t \partial_t F(s, X_s)ds + \int_0^t \partial_x F(s, X_{s-})dX_s \\
&\qquad + \sum_{s\le t} [F(s, X_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-})g_s\Delta N_s]\,. \qquad (1.5) \\
&= \int_0^t \partial_t F(s, X_s)ds + \int_0^t \partial_x F(s, X_s)(h_s - g_s\lambda_s)ds + \int_0^t [F(s, X_s) - F(s, X_{s-})]dN_s
\end{aligned}$$

Proof: Indeed, between two jumps, $dX_t = (h_t - \lambda_t g_t)dt$, and for $T_n < s < t < T_{n+1}$,

$$F(t, X_t) = F(s, X_s) + \int_s^t \partial_t F(u, X_u)du + \int_s^t \partial_x F(u, X_u)(h_u - g_u\lambda_u)du\,.$$

At jump times, $F(T_n, X_{T_n}) = F(T_n, X_{T_n-}) + \Delta F(\cdot, X)_{T_n}.\triangleleft$

Remark that, in the "ds" integrals, we can write X_{s-} or X_s, since, for any bounded Borel function f,

$$\int_0^t f(X_{s-})ds = \int_0^t f(X_s)ds\,.$$

Note that since dN_s a.s. $N_s = N_{s-} + 1$, one has

$$\int_0^t f(N_{s-})dN_s = \int_0^t f(N_s + 1)dN_s\,.$$

We shall use systematically use the form $\int_0^t f(N_{s-})dN_s$, even if the $\int_0^t f(N_s+1)dN_s$ has a meaning. The reason is that $\int_0^t f(N_{s-})dM_s = \int_0^t f(N_{s-})dN_s + \lambda \int_0^t f(N_{s-})ds$ is a martingale, whereas $\int_0^t f(N_s+1)dM_s$ is not.

Exercise 1.3.1 Check that the formula (1.5) can be written as

$$\begin{aligned}
&F(t, X_t) - F(0, X_0) \\
= &\int_0^t \partial_t F(s, X_s)ds + \int_0^t \partial_x F(s, X_s)(h_s - g_s\lambda(s))ds + \int_0^t [F(s, X_s) - F(s, X_{s-})]dN_s \\
= &\int_0^t \partial_t F(s, X_s)ds + \int_0^t \partial_x F(s, X_{s-})dX_s + \int_0^t [F(s, X_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-})g_s]dN_s \\
= &\int_0^t \partial_t F(s, X_s)ds + \int_0^t \partial_x F(s, X_{s-})dX_s + \int_0^t [F(s, X_{s-} + g_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-})g_s]dN_s \\
= &\int_0^t (\partial_t F(s, X_s) + [F(s, X_{s-} + g_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-})g_s]\lambda)\, ds \\
&\qquad + \int_0^t [F(s, X_{s-} + g_s) - F(s, X_{s-})]dM_s
\end{aligned}$$

1.3.5 Change of Probability

Doléans-Dade exponential

Let φ be a predictable process such that $\int_0^t |\varphi_s|\lambda_s ds < \infty$. The process $Z = \mathcal{E}(\varphi\star M)$ is the unique solution to the SDE (called the Doléans Dade exponential)

$$Z_t = 1 + \int_{]0,t]} Z_{u-}\, \varphi_u dM_u .$$

Let

$$X_t = \int_0^t \varphi_s dM_s = \sum_{s\le t} \varphi_s \Delta N_s - \int_0^t \varphi_s \lambda_s ds\,.$$

Then, $dZ_t = Z_{t^-} dX_t$. It is easy to prove that

$$Z_t \quad = \quad \exp\big(-\int_0^t \varphi_s\lambda_s ds\big)\prod_{s\le t}(1+\varphi_s\Delta N_s) = \exp\big(X_t\big)\prod_{s\le t}(1+\Delta X_s)e^{-\Delta X_s}$$

Indeed, for $t\in[T_n, T_{n+1}[$, the equation $Z_t = 1 + \int_{]0,t]} Z_{u-}\,\varphi_u dM_u$ writes

$$dZ_t = Z_{t-}\varphi_t dM_t = -Z_{t-}\lambda_t\varphi_t dt\,,$$

hence $Z_t = Z_{T_n}\exp - \int_{T_n}^t \lambda_s\varphi_s ds$. At time T_n, $Z_{T_n} = Z_{T_n^-}(1+\varphi_{T_n})$.

In the case where $\varphi > -1$,

$$\prod_{s\le t}(1+\varphi_s\Delta N_s) = e^{\sum_{s\le t}\ln(1+\varphi_s\Delta N_s)} = \exp\int_0^t \ln(1+\varphi_s)dN_s$$

hence

$$Z_t = \exp\big(\int_0^t \ln(1+\varphi_s)dN_s - \int_0^t \varphi_s\lambda_s ds\big)\,.$$

Note that in Section 1.3.2, we have already obtained this kind of martingales (see c), with $H = \ln(1+\varphi)$.

Radon-Nykodỳm densities

If $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probabilities, there exists a predictable process γ, with $\gamma > -1$ such that the Radon-Nikodym density $L = d\mathbb{Q}/d\mathbb{P}$ is of the form

$$dL_t = L_{t-}\gamma_t dM_t\,.$$

Girsanov's theorem

Let $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probabilities, with RN density L with $dL_t = L_{t-}\gamma_t dM_t$. Let N be a $\mathbb{P}$ inhomogeneous Poisson process with intensity λ, and M its compensated martingale. Then, the process

$$\widetilde{M}_t = M_t - \int_0^t \lambda_s\gamma_s ds$$

is a $\mathbb{Q}$-martingale. In particular, the process N is a $\mathbb{Q}$-Poisson process with (stochastic) intensity $\lambda_s(1+\gamma_s)$. This gives a way to construct Poisson processes with a given intensity.

1.4 Compound Poisson process

Definition 1.4.1 *Let N be a standard Poisson process with parameter λ and $(Y_i)_{i\ge 1}$, be i.i.d. random variables, independent of the process N. The process $X_t := \sum_{i=1}^{N(t)} Y_i$ is called a compound Poisson process.*

We denote by F the cumulative distribution function for the Y's: $F(y) = \mathbb{P}(Y_1 \leq y)$. We shall say that X is a (λ, F)-compound Poisson process

Theorem 1.4.1 *Let X be a (λ, F)-compound Poisson process*
(i) The process X has stationary and independent increments
(ii) The characteristic function of the r.v. X_t is

$$\mathbb{E}[e^{-iuX_t}] = e^{\lambda t(\mathbb{E}[e^{-iuY_1}]-1)} = \exp\left(\lambda t \int_{\mathbb{R}} (e^{-iuy} - 1)F(dy)\right).$$

Assume that $\mathbb{E}[e^{\alpha Y_1}] < \infty$. Then, the Laplace transform of the r.v.is

$$\mathbb{E}[e^{\alpha X_t}] = e^{\lambda t(\mathbb{E}[e^{\alpha Y_1}]-1)} = \exp\left(\lambda t \int_{\mathbb{R}} (e^{\alpha y} - 1)F(dy)\right).$$

(iii) Assume that $\mathbb{E}(|Y_1|) < \infty$. Then, the process $(Z_t = X_t - t\lambda\mathbb{E}(Y_1), t \geq 0)$ is a martingale and in particular, $\mathbb{E}(X_t) = \lambda t\mathbb{E}(Y_1) = \lambda t \int_{-\infty}^{\infty} yF(dy)$.
(iii) If $\mathbb{E}(Y_1^2) < \infty$, the process $(Z_t^2 - t\lambda\mathbb{E}(Y_1^2), t \geq 0)$ is a martingale and Var $(X_t) = \lambda t\mathbb{E}(Y_1^2)$.

PROOF: (i) We leave the proof to the reader.
(ii) Let us compute the characteristic function of X_t:

$$\begin{aligned}
\mathbb{E}(e^{iuX_t}) &= \sum_{n=0}^{\infty} \mathbb{E}(\mathbb{1}_{N_t=n} e^{iu\sum_{k=1}^{n} Y_k}) = \sum_{n=0}^{\infty} \mathbb{P}(N_t = n)(\mathbb{E}(e^{iuY_1}))^n \\
&= \exp\left(-\lambda t + \lambda t \int e^{iuy} F(dy)\right)
\end{aligned}$$

Properties (iii), (iv) follow from (i)

Proposition 1.4.1 *The predictable process $\lambda t\mathbb{E}(Y_1)$ is the predictable quadratic variation of the martingale Z. The quadratic variation of Z is $[Z]_t = \sum_{n=1}^{N_t} Y_n^2$.*

PROOF: Let us give the proof in the case $\mathbb{E}(Y^2) < \infty$. The increasing process $\sum_{n=1}^{N_t} Y_n^2$ is a compound Poisson process, hence $\sum_{n=1}^{N_t} Y_n^2 - \lambda t\mathbb{E}(Y_1^2)$ is a martingale. Therefore

$$Z_t^2 - t\lambda\mathbb{E}(Y_1^2) - (\sum_{n=1}^{N_t} Y_n^2 - \lambda t\mathbb{E}(Y_1^2)) = Z_t^2 - \sum_{n=1}^{N_t} Y_n^2$$

is a martingale. Moreover, the jumps of Z_t^2 are the jumps of $\sum_{n=1}^{N_t} Y_n^2$. $\triangleleft$

We now denote by ν the positive measure $\nu(dy) = \lambda F(dy)$. Using that notation, a (λ, F)-compound Poisson process will be called a **ν-compound Poisson** process. Conversely, to any positive finite measure ν on $\mathbb{R}$, we can associate a cumulative distribution function setting $\lambda = \nu(\mathbb{R})$ and $F(dy) = \nu(dy)/\lambda$ and construct a ν-compound Poisson process. The measure ν satisfies

$$\nu(A) = \mathbb{E}\left(\sum_{n=1}^{N_1} \mathbb{1}_{Y_n \in A}\right) = \int_A \nu(dx).$$

By application of results on processes with independent increments, we obtain that, for any $\alpha \in \mathbb{R}$ such that $\int_{-\infty}^{\infty} |e^{\alpha x} - 1|\nu(dx) < \infty$ the process

$$\exp\left(\alpha X_t - t\int_{-\infty}^{\infty} (e^{\alpha x} - 1)\nu(dx)\right)$$

is a martingale.

1.4.1 Random Measure

Let (Y_n, T_n) be a sequence of random variables, with $0 < T_1 < \cdots < T_n < \cdots$. We now introduce the random measure $\mathbf{N} = \sum_n \delta_{T_n, Y_n}$ on $\mathbb{R}^+ \times \mathbb{R}$, i.e. $\mathbf{N}(\omega, [0,t] \times A) = \sum_{n=1}^{N_t(\omega)} 1\!\!1_{Y_n(\omega) \in A}$. As usual, we shall omit the ω, and write only $\mathbf{N}([0,t] \times A)$ or $\mathbf{N}(dt, dx)$. We shall also write $N_t(dx) = \mathbf{N}([0,t], dx)$. We denote by $(f * \mathbf{N})_t$ the integral

$$\int_0^t \int_{\mathbb{R}} f(x) \mathbf{N}(ds, dx) = \int_{\mathbb{R}} f(x) N_t(dx) = \sum_{k=1}^{N_t} f(Y_k) \,.$$

In particular

$$X_t = \int_0^t \int_{\mathbb{R}} x \mathbf{N}(ds, dx)$$

Proposition 1.4.2 *If $\nu(|f|) < \infty$, the process*

$$M_t^f = (f * \mathbf{N})_t - t\nu(f) = \int_0^t \int_{\mathbb{R}} f(x)(\mathbf{N}(ds, dx) - ds\nu(dx))$$

is a martingale.

Proof: Indeed, the process $Z_t = \sum_{k=1}^{N_t} f(Y_k)$ is a $\widehat{\nu}$ compound Poisson process, where $\widehat{\nu}$, defined as

$$\widehat{\nu}(A) = \mathbb{E}(\sum_{n=1}^{N_1} 1\!\!1_{f(Y_n) \in A}) = \int_A f(y)\nu(dy)$$

is the image of ν by f. Hence, if $\mathbb{E}(f(Y_1)) < \infty$, the process $Z_t - t\lambda\mathbb{E}(f(Y_1)) = Z_t - t\int f(x)\nu(dx)$ is a martingale. $\triangleleft$

Using again that Z is a compound Poisson process, it follows that the process

$$\exp\left(\sum_{k=1}^{N_t} f(Y_k) - t\int_{-\infty}^{\infty} (e^{f(x)} - 1)\nu(dx)\right) = \exp\left(\int_0^t \int_{\mathbb{R}} f(x)\mathbf{N}(ds, dx) - t\int_{-\infty}^{\infty} (e^{f(x)} - 1)\nu(dx)\right) \tag{1.6}$$

is a martingale

1.4.2 Change of Measure

Let X be a ν-compound Poisson process under $\mathbb{P}$, we present some particular probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that, under $\mathbb{Q}$, X is still a compound Poisson process.

Let $\widetilde{\nu}$ a positive finite measure on $\mathbb{R}$ absolutely continuous w.r.t. ν, and $\widetilde{\lambda} = \widetilde{\nu}(\mathbb{R}) > 0$. Let

$$L_t = \exp\left(t(\lambda - \widetilde{\lambda}) + \sum_{s \le t} \ln\left(\frac{d\widetilde{\nu}}{d\nu}\right)(\Delta X_s)\right) \,.$$

Applying the martingale property (1.6) for $f = \ln\left(\frac{d\widetilde{\nu}}{d\nu}\right)$, the process L is a martingale. Set $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$.

Proposition 1.4.3 *Under $\mathbb{Q}$, the process X is a $\widetilde{\nu}$-compound Poisson process.*

Proof: First we find the law of the r.v. X_t under $\mathbb{Q}$. From the definition of $\mathbb{Q}$

$$\begin{aligned}
\mathbb{E}_Q(e^{iuX_t}) &= \mathbb{E}_P(e^{iuX_t} \exp\left(t(\lambda - \widehat{\lambda}) + \sum_{k=1}^{N_t} f(Y_k)\right) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} e^{t(\lambda - \widehat{\lambda})} \left(\mathbb{E}_P(e^{iuY_1 + f(Y_1)})\right)^n \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} e^{t(\lambda - \widehat{\lambda})} \left(\mathbb{E}_P(\frac{d\widehat{\nu}}{d\nu}(Y_1) e^{iuY_1})\right)^n \\
&= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-t\widehat{\lambda}} \left(\frac{1}{\lambda}\int e^{iuy} d\widehat{\nu}(y)\right)^n = \exp t\int (e^{iuy} - 1) d\widehat{\nu}(y)
\end{aligned}$$

It remains to check that X is with independent and stationary increments under $\mathbb{Q}$. By Bayes formula, for $t > s$

$$\begin{aligned} \mathbb{E}_Q(e^{iu(X_t - X_s)}|\mathcal{F}_s) &= \frac{1}{L_s}\mathbb{E}_P(L_t e^{iu(X_t-X_s)}|\mathcal{F}_s) \\ &= \exp\left((t-s)\int(e^{iux}-1)\widetilde{\nu}(dx)\right) . \end{aligned}$$

$\triangleleft$

We can also write this theorem in terms of the random measure $\mathbf{N}$. Let

$$\begin{aligned} L_t &= \exp\left(\int_{\mathbb{R}} f(x)N_t(dx) - t\int_{-\infty}^{\infty}(e^{f(x)}-1)\nu(dx)\right) \\ &= \exp\left(\int_0^t \int_{\mathbb{R}} f(x)\mathbf{N}(ds,dx) - t\int_{-\infty}^{\infty}(e^{f(x)}-1)\nu(dx)\right) \end{aligned}$$

be a martingale. Define $d\mathbb{Q}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$. Then

$$\int_0^t \int_{\mathbb{R}} (\mathbf{N}(ds,dx) - ds\, e^{f(x)}\nu(dx))$$

is a $\mathbb{Q}$-martingale.

1.5 A Specific Example: Processes with a single jump

Processes with a single jump are of great interest in Credit risk. We give some examples and we state some easy rules of computation. The reader can refer to the lecture of M. Rutkowski in this volume for more informations.

1.5.1 Elementary example

Let N be a Poisson process with deterministic intensity λ and $H_t = N_{t\wedge T_1}$. The process $M_t := N_t - \int_0^t \lambda(s)ds$ being a martingale, the stopped process $M^d_t = H_t - \int_0^{t\wedge T_1}\lambda(s)ds$ is a martingale. The quadratic variation process of M^d is equal to H_t.

1.5.2 Cox Processes

Let $\mathbf{F}$ be a given filtration and λ an $\mathbf{F}$-adapted non-negative process. Let Θ be a random variable, independent of $\mathbf{F}$ with an unit exponential law. We define

$$\tau = \inf\{t \,:\, \int_0^t \lambda_s ds > \Theta\}$$

Then $\{\tau > t\} := \{\int_0^t \lambda_s ds < \Theta\}$, hence

$$\mathbb{P}(\tau > t) = \mathbb{P}(\int_0^t \lambda_s ds < \Theta) = \mathbb{E}\left(\exp\left(-\int_0^t \lambda_s ds\right)\right)$$

(We have used that if X and Y are independent r.v.'s, then $\mathbb{P}(X < Y) = \mathbb{E}(\Phi(X))$ with $\Phi(x) = \mathbb{P}(x < Y)$).

We also obtain

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\int_0^t \lambda_s ds < \Theta|\mathcal{F}_t) = \exp\left(-\int_0^t \lambda_s ds\right)$$

(we use that if Y is independent of $\mathcal{G}$ and if X is $\mathcal{G}$ measurable, $\mathbb{P}(X < Y|\mathcal{G}) = \Psi(X)$ with $\Psi(x) = \mathbb{P}(x < Y)$.

Proposition 1.5.1 $\mathbb{1}_{\tau\leq t} - \int_0^{t\wedge\tau}\lambda_s ds$ *is a* $\mathcal{F}_t \vee \sigma(\tau\wedge t)$ *martingale.*

Proof: See Rutkowski's lecture.$\triangleleft$

1.5.3 Quadratic variation

In the so-called intensity approach in a credit risk setting, one works under the following hypothesis: there exists a non-negative process λ such that

$$M_t = H_t - \int_0^{t\wedge\tau} \lambda_s ds = H_t - \Lambda_{t\wedge\tau} = H_t - \int_0^t (1-H_s)\lambda_s ds$$

is a martingale. The quadratic variation process of M is H: indeed

$$\begin{aligned} M_t^2 - H_t &= H_t - 2H_t\Lambda_{t\wedge\tau} + \Lambda_{t\wedge\tau}^2 - H_t \\ &= \Lambda_{t\wedge\tau}(\Lambda_{t\wedge\tau} - 2H_t) := X_t \end{aligned}$$

Now, if $Z_t = \Lambda_{t\wedge\tau}$, then $dZ_t = (1-H_t)\lambda_t dt$. Integration by parts formula leads to, using that Z is a bounded variation continuous process

$$\begin{aligned} dX_t &= (Z_t - 2H_t)dZ_t + Z_t(dZ_t - 2dH_t) = -2Z_t(dH_t - dZ_t) - 2H_t dZ_t \\ &= -2Z_t(dH_t - dZ_t) = -2Z_t dM_t \end{aligned}$$

since

$$H_t dZ_t = H_t(1-H_t)\lambda_t dt = 0\,.$$

Hence, X is a martingale, and $\Delta H_t = \Delta M_t^2$.

Chapter 2

Lévy Processes

2.1 Infinitely Divisible Random Variables

2.1.1 Definition

A random variable X taking values in $\mathbb{R}$ is infinitely divisible if its characteristic function $\widehat{\mu}$ satisfies

$$\forall u,\ \widehat{\mu}(u) := \mathbb{E}(e^{iuX}) = (\widehat{\mu}_n(u))^n$$

where $\widehat{\mu}_n$ is a characteristic function.
In other terms, a random variable X is infinitely divisible if, for any n, X has the same law as $\sum_{i=1}^n X_{i,n}$ where $X_{i,n}, i = 1, \cdots n$ are i.i.d. random variables.

Example 2.1.1 A Gaussian variable and a Poisson variable are examples of infinitely divisible random variables. Indeed, for a Gaussian variable

$$\widehat{\mu}(u) = \exp(ium - \frac{1}{2}\sigma^2 u^2) = \left(\exp(iu\frac{m}{n} - \frac{1}{2}\frac{\sigma^2}{n}u^2)\right)^n = (\widehat{\mu}_n(u))^n$$

where $\widehat{\mu}_n(u) = \exp(iu\frac{m}{n} - \frac{1}{2}\frac{\sigma^2}{n}u^2)$ is the c.f. of a $\mathcal{N}(\frac{m}{n}, \frac{\sigma^2}{n})$ r.v.
Cauchy laws and Gamma laws are also infinitely divisible (see below).

Definition 2.1.1 *A Lévy measure is a positive measure ν on $\mathbb{R} \setminus \{0\}$ such that*

$$\int_{\mathbb{R}\setminus\{0\}} \min(1, \|x\|^2)\nu(dx) < \infty\,.$$

In what follows, we shall assume that ν does not charge the set $\{0\}$, so that we shall write $\int_{\mathbb{R}} f(x)\nu(dx)$ instead of $\int_{\mathbb{R}\setminus\{0\}} f(x)\nu(dx)$ for suitable functions f.

Proposition 2.1.1 (Lévy-Khintchine representation.)
If X is an infinitely divisible random variable with characteristic function $\widetilde{\mu}$, there exists a triple (m, σ^2, ν) where $m \in \mathbb{R}$ and ν is a Lévy measure such that

$$\widehat{\mu}(u) = \exp\left(ium - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}})\nu(dx)\right).$$

Note that the integral $\int_{\mathbb{R}}(e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}})\nu(dx)$ converges, due to the assumptions on ν. If the integral $\int |x|\mathbb{1}_{\{|x|\leq 1\}}\nu(dx)$ converges,

$$\widehat{\mu}(u) = \exp\left(ium_0 - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx)\right)$$

with $m_0 = m - \int x\mathbb{1}_{\{|x|\leq 1\}}\nu(dx)$. In that case, we shall say that the LK representation is written in a reduced form

Example 2.1.2 a) Gaussian law. The characteristic triple of the Gaussian law $\mathcal{N}(m, \sigma^2)$ is $(m, \sigma, 0)$.
b) Poisson law The characteristic triple of a Poisson law of parameter λ is $(0, 0, \lambda\delta_1(dx))$.
c) Cauchy law. The standard Cauchy law has the characteristic function

$$\exp(-c|u|) = \exp\left(\frac{c}{\pi}\int_{-\infty}^{\infty}(e^{iux}-1)x^{-2}dx\right).$$

Its reduced form characteristic triple is $(0, 0, \pi^{-1}x^{-2}dx)$.
d) Gamma law. The Gamma law $\Gamma(a, \nu)$ has the characteristic function

$$(1-iu/\nu)^{-a} = \exp\left(a\int_{0}^{\infty}(e^{iux}-1)e^{-\nu x}\frac{dx}{x}\right).$$

Its reduced form characteristic triple is $(0, 0, \mathbb{1}_{\{x>0\}}\, ax^{-1}e^{-\nu x}dx)$.

2.1.2 Stable Random Variables

A random variable is stable if for any $a > 0$, there exist $b > 0$ and $c \in \mathbb{R}$ such that $[\hat{\mu}(u)]^a = \hat{\mu}(bu)\, e^{icu}$. A stable r.v. is infinitely divisible.

Proposition 2.1.2 *The characteristic function of a stable law can be written*

$$\hat{\mu}(u) = \begin{cases} \exp(ibu - \frac{1}{2}\sigma^2u^2), & \text{for } \alpha = 2 \\ \exp\left(-\gamma|u|^\alpha[1 - i\beta\,\mathrm{sgn}(u)\tan(\pi\alpha/2)]\right), & \text{for } \alpha \neq 1, \neq 2 \\ \exp\left(\gamma|u|(1 - i\beta v \ln|u|)\right), & \alpha = 1 \end{cases},$$

where $\beta \in [-1, 1]$. For $\alpha \neq 2$, the Lévy measure of a stable law is absolutely continuous with respect to the Lebesgue measure, with density

$$\nu(dx) = \begin{cases} c^+x^{-\alpha-1}dx & \text{if } x > 0 \\ c^-|x|^{-\alpha-1}dx & \text{if } x < 0\,. \end{cases}$$

Here $c^{\pm}$ are non-negative real numbers, such that $\beta = (c^+ - c^-)/(c^+ + c^-)$.
More precisely,

$$\begin{aligned} c^+ &= \frac{1}{2}(1+\beta)\frac{\alpha\gamma}{\Gamma(1-\alpha)\cos(\alpha\pi/2)}, \\ c^- &= \frac{1}{2}(1-\beta)\frac{\alpha\gamma}{\Gamma(1-\alpha)\cos(\alpha\pi/2)}. \end{aligned}$$

Example 2.1.3 A Gaussian variable is stable with $\alpha = 2$. The Cauchy law is stable with $\alpha = 1$.

2.2 Definition and Main Properties of Lévy Processes

2.2.1 Definition

An real-valued process X such that $X_0 = 0$ is a Lévy process if
a- for every $s, t, 0 \leq s \leq t < \infty$, the r.v. $X_t - X_s$ is independent of $\mathcal{F}_s^X$
b- for every s, t the r.v's $X_{t+s} - X_t$ and X_s have the same law.
c- X is continuous in probability, i.e., $P(|X_t - X_s| > \epsilon) \to 0$ when $s \to t$ for every $\epsilon > 0$.

Brownian motion, Poisson process and compound Poisson processes are examples of Lévy processes.

2.2.2 Poisson Point Process, Lévy Measure

For every Borel set $\Lambda \in \mathbb{R}$, such that $0 \notin \bar{\Lambda}$, where $\bar{\Lambda}$ is the closure of Λ, we define

$$N_t^\Lambda = \sum_{0<s\leq t} \mathbb{1}_\Lambda(\Delta X_s),$$

to be the number of jumps before time t which take values in Λ.

Definition 2.2.1 *The σ-additive measure ν defined on $\mathbb{R} \setminus \{0\}$ by*

$$\nu(\Lambda) = \mathbb{E}(N_1^\Lambda)$$

is called the Lévy measure of the process X.

• If $\nu(\mathbb{R} \setminus \{0\}) < \infty$, the process X has a finite number of jumps in any finite time interval. In finance, when $\nu(\mathbb{R} \setminus \{0\}) < \infty$, one refers to **finite activity**.

• If $\nu(\mathbb{R} \setminus \{0\}) = \infty$, the process corresponds to **infinite activity**.

Proposition 2.2.1 *Let Λ be a Borel set and assume $\nu(\Lambda) < \infty$.*
a) The process N^Λ defined as

$$N_t^\Lambda = \sum_{0<s\leq t} \mathbb{1}_\Lambda(\Delta X_s)$$

is a standard Poisson process with constant intensity $\nu(\Lambda)$.
b) Let Γ be another Borel set with $\nu(\Gamma) < \infty$. The processes N^Λ and N^Γ are independent if $\nu(\Gamma \cap \Lambda) = 0$, in particular if Λ and Γ are disjoint.

The map $\Lambda \to N_t^\Lambda(\omega)$ defines a σ-finite measure on $\mathbb{R}^d$ denoted by $N_t(\omega, dx)$. Let Λ be a Borel set of $\mathbb{R}$ with $0 \notin \bar{\Lambda}$, and f a Borel function defined on Λ. We have

$$\int_\Lambda f(x) N_t(\omega, dx) = \sum_{0<s\leq t} f(\Delta X_s(\omega)) \mathbb{1}_\Lambda(\Delta X_s(\omega)) \,.$$

As usual, we shall omit the ω and write $N_t(dx)$ for $N_t(\omega, dx)$. The process $\int_\Lambda f(x) N_t(dx)$ is a Lévy process.
If $f \mathbb{1}_\Lambda \in L^1(d\nu)$, then

$$\mathbb{E}\left(\int_\Lambda f(x) N_t(dx)\right) = t \int_\Lambda f(x) \nu(dx)$$

and, if $f \mathbb{1}_\Lambda \in L^1(d\nu) \cap L^2(d\nu)$,

$$\mathbb{E}\left[\left(\int_\Lambda f(x) N_t(dx) - t \int_\Lambda f(x) \nu(dx)\right)^2\right] = t \int_\Lambda f^2(x) \nu(dx)$$

If f is bounded and vanishes in a neighborhood of 0,

$$\mathbb{E}(\sum_{0<s\leq t} f(\Delta X_s)) = t \int_{\mathbb{R}} f(x) \nu(dx) \,,$$

and, for any bounded predictable process H

$$\mathbb{E}\left[\sum_{s\leq t} H_s f(\Delta X_s)\right] = \mathbb{E}\left[\int_0^t ds H_s \int_{\mathbb{R}} f(x) d\nu(x)\right] .$$

More generally, if H is a predictable function (i.e. $H \ : \Omega \times \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{P} \times \mathcal{B}$ measurable) such that $\mathbb{E}\left[\int_0^t ds \int d\nu(x) |H_s(\cdot, x)|\right] < \infty$, then

$$\mathbb{E}\left[\sum_{s\leq t} H_s(\cdot, \Delta X_s)\right] = \mathbb{E}\left[\int_0^t ds \int d\nu(x) H_s(\cdot, x)\right] .$$

Let X be a Lévy process with jumps bounded by 1. Then, $\mathbb{E}(|X_t|^n) < \infty$ for any $n = 1, 2, .$ The process $Z_t = X_t - \mathbb{E}(X_t)$ is a martingale with decomposition $Z_t = Z_t^c + Z_t^d$ where Z^c is a martingale with continuous path (in fact a Brownian motion up to a multiplicative constant) and $Z_t^d = \int x N_t(dx) - t\nu(dx)$

Proposition 2.2.2 (Lévy-Itô's decomposition.) *If X is a Lévy process, it can be decomposed into*

$$X_t = \alpha t + \sigma B_t + \int_{|x|<1} x\,(N_t(dx) - t\nu(dx)) + \int_{|x|\geq 1} x N_t(dx)$$

Proposition 2.2.3 (Exponential formula.)

Let X be a Lévy process and ν its Lévy measure. For all t and all Borel function f defined on $\mathbb{R}^+ \times \mathbb{R}^d$ such that $\int_0^t ds \int_{\mathbb{R}} |1 - e^{f(s,x)}|\nu(dx) < \infty$, one has

$$\mathbb{E}\left[\exp\left(\sum_{s\leq t} f(s,\Delta X_s)\mathbb{1}_{\{\Delta X_s \neq 0\}}\right)\right] = \exp\left(\int_0^t ds \int (e^{f(s,x)} - 1)\nu(dx)\right).$$

Warning 1 The above property does not extend to predictable functions.

2.2.3 Lévy-Khintchine Representation

If X is a Lévy process, then, for any t, the r.v. X_t is infinitely divisible.

Proposition 2.2.4 *Let X be a Lévy process. There exists $m, \sigma \in \mathbb{R}$, a Lévy measure ν such that for $u \in \mathbb{R}$*

$$\mathbb{E}(\exp(iuX_1)) = \exp\left(ium - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1})\nu(dx)\right) \tag{2.1}$$

It can be proved that the Lévy measure is indeed the one defined in Definition (2.2.1)

Definition 2.2.2 *The complex valued continuous function Φ such that*

$$\mathbb{E}\left[\exp(iuX_1)\right] = \exp(-\Phi(u))$$

*is called the **characteristic exponent** (sometimes the Lévy exponent) of the Lévy process X.*
If $\mathbb{E}\left[e^{\lambda X_1}\right] < \infty$ for any $\lambda > 0$, the function Ψ defined on $[0, \infty[$, such that

$$\mathbb{E}\left[\exp(\lambda X_1)\right] = \exp(\Psi(\lambda))$$

*is called the **Laplace exponent** of the Lévy process X.*
It follows that, if $\Psi(\lambda)$ exists,

$$\mathbb{E}\left[\exp(iuX_t)\right] = \exp(-t\Phi(u)), \qquad \mathbb{E}\left[\exp(\lambda X_t)\right] = \exp(t\Psi(\lambda))$$

and

$$\Psi(\lambda) = -\Phi(-i\lambda).$$

From LK formula, the characteristic exponent and the Laplace exponent can be computed as follows:

$$\begin{aligned}
\Phi(u) &= -ium + \frac{1}{2}\sigma^2 u^2 - \int (e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1})\nu(dx) \\
\Psi(\lambda) &= \lambda m + \frac{1}{2}\sigma^2 \lambda^2 + \int (e^{\lambda x} - 1 - \lambda x\mathbb{1}_{|x|\leq 1})\nu(dx).
\end{aligned}$$

• If $\sigma = 0$ and $\nu(\mathbb{R}) < \infty$, the process X is a compound Poisson process with "drift".
• If $\sigma = 0$, $\nu(\mathbb{R}) = \infty$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$, the paths of X are of bounded variation on any finite time interval.
• If $\sigma = 0$, $\nu(\mathbb{R}) = \infty$ and $\int_{|x|\leq 1} |x|\nu(dx) = \infty$, the paths of X are no longer of bounded variation on any finite time interval.

2.2.4 Martingales

Proposition 2.2.5 *Let X be a Lévy process.*

(i) If $\mathbb{E}(|X_t|) < \infty$, then the process $X_t - \mathbb{E}(X_t)$ is a martingale.

(ii) For any u, the process $Z_t(u) := \frac{e^{iuX_t}}{\mathbb{E}(e^{iuX_t})}$ is a martingale.

(iii) If $\mathbb{E}(e^{\lambda X_t}) < \infty$, the process $\frac{e^{\lambda X_t}}{\mathbb{E}(e^{\lambda X_t})}$ is a martingale

PROOF: (i)From independence properties $\mathbb{E}(X_t - X_s|\mathcal{F}_t) = \mathbb{E}(X_t - X_s)$.
(ii) Using independence of the increments,

$$E(Z_t(u)|\mathcal{F}_s) = \frac{e^{iuX_s}}{\mathbb{E}(e^{iuX_t})}\mathbb{E}(e^{iu(X_t-X_s)}) = e^{iuX_s}\frac{\mathbb{E}(e^{iu(X_t-X_s)})}{\mathbb{E}(e^{iu(X_t-X_s)})\,\mathbb{E}(e^{iuX_s})} = Z_s(u)$$

and the result follows.

2.2.5 Itô's formula

Let X be a Lévy process with decomposition

$$dX_t = \alpha dt + \sigma dB_t + \int_{|x|<1} x\,(\mathbf{N}(dt,dx) - \nu(dx)dt) + \int_{|x|\geq 1} x\mathbf{N}(dt,dx)$$

Let $Y_t = f(t, X_t)$ where f is a $C^{1,2}$ function. Then, Y is a semi-martingale

$$\begin{aligned} dY_t &= \partial_t f(t,X_t)dt + \partial f_x(t,X_t)\,(\alpha dt + \sigma dB_t) + \frac{1}{2}\sigma^2\partial_{xx}f(t,X_t)\,dt \\ &+ \int_{|x|<1} (f(t,X_{t^-}+x) - f(t,X_{t^-}) - x\partial_x f(t,X_{t^-}))\,\nu(dx)dt \\ &+ \int_{|x|<1} (f(t,X_{t^-}+x) - f(t,X_{t^-}))\,(\mathbf{N}(dt,dx) - \nu(dx)dt) \\ &+ \int_{|x|\geq 1} (f(t,X_{t^-}+x) - f(t,X_{t^-}))\,\mathbf{N}(dt,dx) \end{aligned}$$

Comments 2.2.1 As a consequence of the semi-martingale property, if F is a C^2 function , then, the series

$$\sum_{s\leq t} |F(\Delta X_s) - F(0) - F'(0)\Delta X_s|$$

converges.

2.2.6 Representation Theorem

Proposition 2.2.6 *Let X be a Lévy process and $\mathbf{F}^X$ its natural filtration. Let M be a locally square integrable martingale with $M_0 = m$. Then, there exists a family (φ, ψ) of predictable processes such that*

$$\int_0^t |\varphi_s|^2 ds < \infty, \text{ a.s.}, \quad \int_0^t \int_{\mathbb{R}} |\psi_s(x)|^2 ds\,\nu(dx) < \infty, \text{ a.s.}$$

and

$$M_t = m + \int_0^t \varphi_s dW_s + \int_0^t \int_{\mathbb{R}} \psi_s(x)(N(ds,dx) - ds\,\nu(dx))\,.$$

2.3 Change of measure

2.3.1 Esscher transform

Assume that $\mathbb{E}(e^{\lambda X_t}) < \infty$. The process $L_t = \frac{e^{\lambda X_t}}{\mathbb{E}(e^{\lambda X_t})}$ is then a strictly positive martingale with expectation equal to 1. We define a probability Q, equivalent to P by the formula

$$Q|_{\mathcal{F}_t} = \frac{e^{\lambda X_t}}{\mathbb{E}(e^{\lambda X_t})}\,P|_{\mathcal{F}_t}\,. \tag{2.2}$$

This particular choice of measure transformation, (called an Esscher transform) preserves the Lévy process property, as we prove now.

Proposition 2.3.1 *Let X be a P-Lévy process with parameters (m, σ, ν). Let λ be such that $\mathbb{E}(e^{\lambda X_t}) < \infty$ and suppose Q is defined by (2.2). Then X is a Lévy process under Q, and if the Lévy-Khintchine decomposition of X under P is (2.1), then the Lévy-Khintchine decomposition of X under Q is*

$$\mathbb{E}_Q(\exp(iuX_1)) = \exp\left(ium^{(\lambda)} - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux 1\!\!1_{|x|\leq 1})\nu^{(\lambda)}(dx)\right)$$

with

$$\begin{aligned} m^{(\lambda)} &= m + \sigma^2\lambda + \int_{|x|\leq 1} x(e^{\lambda x} - 1)\nu(dx) \\ \nu^{(\lambda)}(dx) &= e^{\lambda x}\nu(dx)\,. \end{aligned}$$

Proof: The characteristic exponent of X under Q is obtained from

$$e^{-t\Phi^{(\lambda)}(u)} = \mathbb{E}_Q(e^{iuX_t}) = \frac{\mathbb{E}_P(e^{i(u-i\lambda)X_t})}{\mathbb{E}_P(e^{\lambda X_t})} = e^{-t\Phi(u-i\lambda)}e^{t\Phi(-i\lambda)}$$

hence,

$$\Phi^{(\lambda)}(u) = \Phi(u - i\lambda) - \Phi(-i\lambda)\,.$$

If $\mathbb{E}(e^{(\lambda+\gamma)X_1}) < \infty$, the Laplace exponent of X under Q is obtained from

$$e^{t\Psi^{(\lambda)}(\gamma)} = \mathbb{E}_Q(e^{\gamma X_t}) = \frac{\mathbb{E}_P(e^{(\gamma+\lambda)X_t})}{\mathbb{E}_P(e^{\lambda X_t})}$$

hence, $\Psi^{(\lambda)}(\gamma) = \Psi(\gamma + \lambda) - \Psi(\lambda)\,.$

2.3.2 General case

More generally, any density $(L_t, t \geq 0)$ which is a positive martingale can be used to define an equivalent change of probability. From representation martingales property, any martingale can be written as

$$dL_t = \widetilde{\varphi}_t dW_t + \int_{\mathbb{R}} \widetilde{\psi}_t(x)[\mathbf{N}(dt, dx) - dt\nu(dx)]\,.$$

From the strict positivity of L, there exists φ, ψ such that $\widetilde{\varphi}_t = L_{t-}\varphi_t$, $\widetilde{\psi}_t = L_{t-}(e^{\psi(t,x)} - 1)$, hence the process L satisfies

$$dL_t = L_{t-}\left(\varphi_t dW_t + \int (e^{\psi(t,x)} - 1)[\mathbf{N}(dt, dx) - dt\nu(dx)]\right) \tag{2.3}$$

Proposition 2.3.2 *Let $Q|_{\mathcal{F}_t} = L_t\, P|_{\mathcal{F}_t}$ where L is defined in (2.3). With respect to Q,*

(i) $W_t^{\varphi} \stackrel{def}{=} W_t - \int_0^t \varphi_s ds$ is a Brownian motion

(ii) The process N is compensated by $e^{\psi(s,x)}ds\nu(dx)$ meaning that for any Borel function h such that

$$\int_0^T \int_{\mathbb{R}} |h(s,x)|e^{\psi(s,x)}ds\nu(dx) < \infty\,,$$

the process

$$\int_0^t \int_{\mathbb{R}} h(s,x)\left(\mathbf{N}(ds, dx) - e^{\psi(s,x)}ds\nu(dx)\right)$$

is a local martingale.

In this general setting the Lévy property is lost

2.4 Exponential Lévy Processes as Stock Price Processes

In a Black and Scholes model, prices can be written as an exponential of a drifted Brownian motion, or as a Doléans-Dade martingale of a drifted Brownian motion. We prove here that this property extends to Lévy processes.

Proposition 2.4.1 *Let X be a (m, σ^2, ν) Lévy process.*

(i) Let $S_t = e^{X_t}$ be the ordinary exponential of the process X. The stochastic logarithm of S (i.e., the process Y which satisfies $S_t = \mathcal{E}(Y)_t$) is a Lévy process and is given by

$$Y_t = \mathcal{L}(S)_t = X_t + \frac{1}{2}\sigma^2 t - \sum_{0<s\le t}\left(1 + \Delta X_s - e^{\Delta X_s}\right) .$$

The Lévy characteristics of Y are

$$\begin{aligned} m_Y &= m + \frac{1}{2}\sigma^2 + \int \left((e^x - 1)\mathbb{1}_{\{|e^x-1|\le 1\}} - x\mathbb{1}_{\{|x|\le 1\}}\right) \nu(dx) \\ \sigma_Y^2 &= \sigma^2 \\ \nu_Y(A) &= \nu(\{x : e^x - 1 \in A\}) = \int \mathbb{1}_A(e^x - 1)\, \nu(dx) . \end{aligned}$$

(ii) Let $Z_t = \mathcal{E}(X)_t$ the Doléans-Dade exponential of X. If $Z > 0$, the ordinary logarithm of Z is a Lévy process L given by

$$L_t = \ln(Z_t) = X_t - \frac{1}{2}\sigma^2 t + \sum_{0<s\le t}\left(\ln(1 + \Delta X_s) - \Delta X_s\right) .$$

Its Lévy characteristics are

$$\begin{aligned} m_L &= m - \frac{1}{2}\sigma^2 + \int \left(\ln(1+x)\mathbb{1}_{\{|\ln(1+x)|\le 1\}} - x\mathbb{1}_{\{|x|\le 1\}}\right) \nu(dx) \\ \sigma_L^2 &= \sigma^2 \\ \nu_L(A) &= \nu(\{x : \ln(1+x) \in A\}) = \int \mathbb{1}_A(\ln(1+x))\, \nu(dx) \end{aligned}$$

PROOF: We only prove part (i) and leave part (ii) to the reader. Note that the series $\sum_{0<s\le t}(1+\Delta X_s - e^{\Delta X_s})$ is absolutely convergent by the result stated in Comment 2.2.1.
The process $Y_t = X_t + \frac{1}{2}\sigma^2 t - \sum_{0<s\le t}\left(1 + \Delta X_s - e^{\Delta X_s}\right)$ is a Lévy process, $\sigma_Y^2 = \sigma^2$, and $\Delta Y_t = e^{\Delta X_t} - 1$. Using the equality

$$\sum_{s\le t}\left(1 + \Delta X_s - e^{\Delta X_s}\right) = \int_0^t \int_{\mathbb{R}} (1 + x - e^x)\mathbf{N}(ds, dx)$$

we obtain that the Lévy-Itô decomposition of Y is (where m_Y is defined in Proposition 2.4.1)

$$\begin{aligned} Y_t &= mt + \sigma B_t + \int_0^t \int_{\{|x|<1\}} x\widetilde{\mathbf{N}}(ds, dx) + \int_0^t \int_{\{|x|>1\}} x\mathbf{N}(ds, dx) \\ &\quad + \frac{1}{2}\sigma^2 t - \int_0^t \int (1 + x - e^x)\mathbf{N}(ds, dx) \\ &= m_Y t + \sigma B_t + \int_0^t \int (e^x - 1)\mathbb{1}_{\{|e^x-1|<1\}}\widetilde{\mathbf{N}}(ds, dx) \\ &\quad + \int_0^t \int (e^x - 1)\mathbb{1}_{\{|e^x-1|>1\}}\mathbf{N}(ds, dx) \\ &= m_Y t + \sigma B_t + \int_0^t \int y\mathbb{1}_{\{|y|<1\}}\widetilde{\mathbf{N}}_Y(ds, dy) + \int_0^t \int y\mathbb{1}_{\{|y|>1\}}\mathbf{N}_Y(ds, dy) . \end{aligned}$$

The result follows. $\lhd$

2.4.1 Option pricing with Esscher Transform

Let $S_t = S_0 e^{rt+X_t}$ where X is a Lévy process under the historical probability P.

Proposition 2.4.2 *We assume that $\mathbb{E}(e^{\alpha X_1}) < \infty$ on some open interval (a,b) with $b-a>1$ and that there exists a real number θ such that $\Psi(\theta) = \Psi(\theta+1)$. The process $e^{-rt}S_t = S_0 e^{X_t}$ is a martingale under the probability Q defined as $Q|_{\mathcal{F}_t} = Z_t P|_{\mathcal{F}_t}$ where $Z_t = \frac{e^{\theta X_t}}{\mathbb{E}(e^{\theta X_t})}$*

PROOF: The process X is a $\mathbb{Q}$-Lévy process, hence $e^{X_t}/\mathbb{E}_Q(e^{X_t})$ is a Q- martingale. Now,

$$\mathbb{E}_Q(e^{X_t}) = \mathbb{E}_P(e^{(\theta+1)X_t})\frac{1}{\mathbb{E}(e^{\theta X_t})} = e^{\Psi(\theta+1)\,t}\frac{1}{e^{\Psi(\theta)\,t}} = 1$$

The martingale property follows.$\triangleleft$

Exercise 2.4.1 Check that, if $S_t = S_0 e^{\mu t+\sigma B_t}$, the previous Proposition gives the well know result of change of probability in a Black Scholes model.

Hence, the value of a contingent claim $h(S_T)$ can be obtained, assuming that the emm chosen by the market is Q as

$$V_t = e^{-r(T-t)}\mathbb{E}_Q(h(S_T)|\mathcal{F}_t) = e^{-r(T-t)}\frac{1}{\mathbb{E}(e^{\theta X_t})}\mathbb{E}_P(h(ye^{r(T-t)+X_{T-t}}e^{\theta X_{T-t}})\big|_{y=S_t}$$

Note that the dynamics of S are

$$dS_t = S_{t^-}\left(rdt + \sigma dW_t + \int_{\mathbb{R}}(e^x-1)\widetilde{\mathbf{N}}_X(dt,dx)\right)$$

2.4.2 A Differential Equation for Option Pricing

Let $S_t = S_0 e^{rt+X_t}$ where X is a (m,σ^2,ν)-Lévy process under the risk-neutral probability Q. Assume that

$$V(t,S) = e^{-r(T-t)}\mathbb{E}_Q(H(S_T)|S_t = S)$$

belongs to $C^{1,2}$. Then

$$rV = \frac{1}{2}\sigma^2\partial_{SS}V + \partial_t V + rS\partial_S V + \int (V(t,Se^y) - V(t,S) - S(e^y-1)\partial_S V(t,S))\,\nu(dy)\,.$$

Introducing the change of variables $\tau = T-t$, $x = \ln(S/K) + r\tau$, and the function $h(x) = H(e^x K)/K$, then $u(\tau,x) = e^{r(T-t)}V(t,S)/K = \mathbb{E}_Q(h(x+X_\tau))$ satisfies

$$\partial_\tau u = m\partial_x u + \frac{1}{2}\sigma^2\partial_{xx}u + \int \big(u(\tau,x+y) - u(\tau,x) - y1\!\!1_{|y|<1}\partial_x u(\tau,x)\big)\,\nu(dy)\,.$$

2.4.3 Put-call Symmetry

Let us study a financial market with a riskless asset with constant interest rate r and dividend yield δ, and a price process $S_t = S_0 e^{X_t}$ where X is a Lévy process such that $e^{-(r-\delta)t}S_t$ is a martingale. In terms of characteristic exponent, this condition means that $\psi(1) = r-\delta$, and the characteristic triple of X is such that

$$m = r - \delta - \sigma^2/2 - \int (e^y - 1 - y1\!\!1_{\{|y|\le 1\}})\nu(dy)\,.$$

Then, the following symmetry between call and put prices holds:

$$C_E(S_0,K,r,\delta,T,\psi) = P_E(K,S_0,\delta,r,T,\widetilde{\psi})\,.$$

2.5 Subordinators

A Lévy process which takes values in $[0,\infty[$ (i.e. with increasing paths) is a subordinator. In this case, the parameters in the Lévy-Khintchine decomposition are $m \geq 0, \sigma = 0$ and the Lévy measure ν is a measure on $]0,\infty[$ with $\int_{]0,\infty[}(1 \wedge x)\nu(dx) < \infty$. The Laplace exponent can be expressed as

$$\Phi(u) = \delta u + \int_{]0,\infty[}(1 - e^{-ux})\nu(dx)$$

where $\delta \geq 0$.

Definition 2.5.1 *Let Z be a subordinator and X an independent Lévy process. The process $\widetilde{X}_t = X_{Z_t}$ is a Lévy process, called subordinated Lévy process.*

Example 2.5.1 Compound Poisson process. A compound Poisson process with $Y_k \geq 0$ is a subordinator.

Example 2.5.2 Gamma process. The Gamma process is an increasing Lévy process, hence a subordinator, with one sided Lévy measure

$$\frac{1}{x}\exp(-\frac{x}{\gamma})1\!\!1_{x>0}\,.$$

Example 2.5.3 Let W be a BM, and

$$T_r = \inf\{t \geq 0 \,:\, W_t \geq r\}\,.$$

The process $(T_r, r \geq 0)$ is a stable (1/2) subordinator, its Lévy measure is $\frac{1}{\sqrt{2\pi}\,x^{3/2}}1\!\!1_{x>0}dx$. Let B be a BM independent of W. The process B_{T_t} is a Cauchy process, its Lévy measure is $dx/(\pi x^2)$.

Proposition 2.5.1 (Changes of Lévy characteristics under subordination.) *Let X be a (m^X, σ^X, ν^X) Lévy process and Z be a subordinator with drift β and Lévy measure ν^Z, independent of X. The process $\widetilde{X}_t = X_{Z_t}$ is a Lévy process with characteristic exponent*

$$\Phi(u) = i\widetilde{a}u + \frac{1}{2}\widetilde{A}(u) - \int_{\mathbb{R}}(e^{iux} - 1 - iux1\!\!1_{|x|\leq 1})\widetilde{\nu}(dx)$$

with

$$\begin{aligned}
\widetilde{a} &= \beta a^X + \int_{\mathbb{R}^+}\int_{\mathbb{R}}\nu^Z(ds)1\!\!1_{|x|\leq 1}xP(X_s \in dx) \\
\widetilde{A} &= \beta A^X \\
\widetilde{\nu}(dx) &= \beta\nu^X dx + \int_{\mathbb{R}^+}\nu^Z(ds)P(X_s \in dx)\,.
\end{aligned}$$

Example 2.5.4 Normal Inverse Gaussian. The NIG Lévy process is a subordinated process with Lévy measure $\frac{\delta\alpha}{\pi}\frac{e^{\beta x}}{|x|}K_1(\alpha|x|)dx$.

2.6 Examples of Lévy Processes

2.6.1 Variance-Gamma Model

The variance Gamma process is a Lévy process where X_t has a Variance Gamma law VG(σ,ν,θ). Its characteristic function is

$$\mathbb{E}(\exp(iuX_t)) = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-t/\nu}.$$

The Variance Gamma process can be characterized as a time changed BM with drift as follows: let W be a BM, $\gamma(t)$ a $G(1/\nu, 1/\nu)$ process. Then

$$X_t = \theta\gamma(t) + \sigma W_{\gamma(t)}$$

is a VG(σ, ν, θ) process. The variance Gamma process is a finite variation process. Hence it is the difference of two increasing processes. Madan et al. [24, 23] showed that it is the difference of two independent Gamma processes

$$X_t = G(t; \mu_1, \gamma_1) - G(t; \mu_2, \gamma_2)\,.$$

Indeed, the characteristic function can be factorized

$$\mathbb{E}(\exp(iuX_t)) = \left(1 - \frac{iu}{\nu_1}\right)^{-t/\gamma} \left(1 + \frac{iu}{\nu_2}\right)^{-t/\gamma}$$

with

$$\nu_1^{-1} = \frac{1}{2}\left(\theta\nu + \sqrt{\theta^2\nu^2 + 2\nu\sigma^2}\right)$$
$$\nu_2^{-1} = \frac{1}{2}\left(\theta\nu - \sqrt{\theta^2\nu^2 + 2\nu\sigma^2}\right)$$

The Lévy density of X is

$$\frac{1}{\gamma}\frac{1}{|x|}\exp(-\nu_1|x|) \qquad \text{for } x < 0$$
$$\frac{1}{\gamma}\frac{1}{x}\exp(-\nu_2 x) \qquad \text{for } x > 0\,.$$

The density of X_1 is

$$\frac{2e^{\frac{\theta x}{\sigma^2}}}{\gamma^{1/\gamma}\sqrt{2\pi}\sigma\Gamma(1/2)}\left(\frac{x^2}{\theta^2 + 2\sigma^2/\gamma}\right)^{\frac{1}{2\gamma}-\frac{1}{4}} K_{\frac{1}{\gamma}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x^2(\theta^2 + 2\sigma^2/\gamma)}\right)$$

where K_α is the modified Bessel function.

Stock prices driven by a Variance-Gamma process have dynamics

$$S_t = S_0 \exp\left(rt + X(t; \sigma, \nu, \theta) + \frac{t}{\nu}\ln(1 - \theta\nu - \frac{\sigma^2\nu}{2})\right)$$

From $\mathbb{E}(e^{X_t}) = \exp\left(-\frac{t}{\nu}\ln(1 - \theta\nu - \frac{\sigma^2\nu}{2})\right)$, we get that $S_t e^{-rt}$ is a martingale. The parameters ν and θ give control on skewness and kurtosis. See Madan et al. for more comments.

The CGMY model, introduced by Carr et al. [10] is an extension of the Variance-Gamma model. The Lévy density is

$$\begin{cases} \frac{C}{x^{Y+1}}e^{-Mx} & x > 0 \\ \frac{C}{|x|^{Y+1}}e^{Gx} & x < 0 \end{cases}$$

with $C > 0, M \geq 0, G \geq 0$ and $Y < 2, Y \notin \mathbb{Z}$.
If $Y < 0$, there is a finite number of jumps in any finite interval, if not, the process has infinite activity.
If $Y \in [1, 2[$, the process is with infinite variation.

2.6.2 Double Exponential Model

The Model

A particular Lévy model is the double exponential jumps model, introduced by Kou [18] and Kou and Wang [19, 20]. In this model

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\,,$$

where W is a Brownian motion independent of N and $\sum_{i=1}^{N_t} Y_i$ is a compound Poisson process. The r.v's Y_i are i.i.d., independent of N and W and the density of the law of Y_1 is

$$f(x) = p\eta_1 e^{-\eta_1 x}\mathbb{1}_{\{x>0\}} + (1-p)\eta_2 e^{\eta_2 x}\mathbb{1}_{\{x<0\}}\,.$$

The Lévy measure of X is $\nu(dx) = \lambda f(x)dx$.

Here, η_i are positive real numbers, and $p \in [0,1]$. With probability p (resp. $(1-p)$), the jump size is positive (resp. negative) with exponential law with parameter η_1 (resp η_2).

It is easy to prove that

$$\mathbb{E}(Y_1) = \frac{p}{\eta_1} - \frac{1-p}{\eta_2},\ \text{var}\ (Y_1) = \frac{p}{\eta_1^2} + \frac{1-p}{\eta_2^2} + p(1-p)\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)^2$$

and that, for $\eta_1 > 1$, $\mathbb{E}(e^{Y_1}) = p\frac{\eta_1}{\eta_1 - 1} + (1-p)\frac{\eta_2}{1+\eta_2}$. Moreover

$$\mathbb{E}(e^{iuX_t}) = \exp\left(t\left\{-\frac{1}{2}\sigma^2 u^2 + ibu + \lambda\left(\frac{p\eta_1}{\eta_1 - iu} + \frac{(1-p)\eta_2}{\eta_2 + iu} - 1\right)\right\}\right),$$

where $b = \mu + \lambda\mathbb{E}(Y_1 \mathbb{1}_{|Y_1|\leq 1}) = \mu + \lambda p\left(\frac{1-e^{-\eta_1}}{\eta_1} - e^{-\eta_1}\right) - \lambda(1-p)\left(\frac{1-e^{-\eta_2}}{\eta_2} - e^{-\eta_2}\right)$. The Laplace exponent of X, i.e., the function Ψ such that $\mathbb{E}(e^{\beta X_t}) = \exp(\Psi(\beta)t)$ is defined for $-\eta_2 < \beta < \eta_1$ as

$$\Psi(\beta) = \beta\mu + \frac{1}{2}\beta^2\sigma^2 + \lambda(\frac{p\eta_1}{\eta_1 - \beta} + \frac{(1-p)\eta_2}{\beta + \eta_2} - 1)\,.$$

Change of probability

Let $S_t = S_0 e^{rt + X_t}$ where $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$. Then, setting $V_i = e^{Y_i}$, using an Escher transform, the process $S_t e^{-rt}$ will be a $\mathbb{Q}$ martingale with $\mathbb{Q}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$ and $L_t = \frac{e^{\alpha X_t}}{\mathbb{E}(e^{\alpha X_t})}$, for α such that $\Psi(\alpha) = \Psi(\alpha + 1)$. Under $\mathbb{Q}$, the Lévy measure of X is $\widehat{\nu}(dx) = e^{\alpha x}\nu(dx) = e^{\alpha x}\lambda f(x)dx = \widehat{\lambda}\widehat{f}(x)dx$ where, after some standard computations

$$\begin{aligned}
\widehat{f}(x) &= \left(\widehat{p}\,\widehat{\eta}_1 e^{-\widehat{\eta}_1 x}\mathbb{1}_{\{x>0\}} + (1-\widehat{p})\widehat{\eta}_2 e^{\widehat{\eta}_2 x}\mathbb{1}_{\{x<0\}}\right).\\
\widehat{\eta}_1 &= \eta_1 - \alpha\,, \qquad \widehat{\eta}_2 = \eta_2 + \alpha\\
\widehat{\lambda} &= \lambda\left(\frac{p\eta_1}{\eta_1 - \alpha} + \frac{(1-p)\eta_2}{\eta_2 + \alpha}\right)\\
\widehat{p} &= p\eta_1 \frac{\eta_2 + \alpha}{\alpha p\eta_1 + \eta_2(\eta_1 - \alpha + \alpha p\eta_1)}
\end{aligned}$$

In particular, the process X is a double exponential process under $\mathbb{Q}$.

Hitting times

Proposition 2.6.1 *For any $x > 0$*

$$\mathbb{P}(\tau_b \leq t,\, X_{\tau_b} - b \geq x) = e^{-\eta_1 x}\,\mathbb{P}(\tau_b \leq t, X_{\tau_b} - b \geq 0)$$

PROOF:

The infinitesimal generator of X is

$$\mathcal{L}f = \frac{1}{2}\sigma^2\partial_{xx}f + \mu\partial_x f + \lambda\int_{\mathbb{R}}(f(x+y) - f(x))\nu(dx)$$

Let $T_x = \inf\{t : X_t \geq x\}$. Then Kou and Wang [19] establish that, for $r > 0$ and $x > 0$,

$$\begin{aligned}
\mathbb{E}(e^{-rT_x}) &= \frac{\eta_1 - \beta_1}{\eta_1}\frac{\beta_2}{\beta_2 - \beta_1}e^{-x\beta_1} + \frac{\beta_2 - \eta_1}{\eta_1}\frac{\beta_1}{\beta_2 - \beta_1}e^{-x\beta_2}\\
\mathbb{E}(e^{-rT_x}\mathbb{1}_{X_{T_x} - x > y}) &= e^{\eta_1 y}\frac{\eta_1 - \beta_1}{\eta_1}\frac{\beta_2 - \eta_1}{\beta_2 - \beta_1}\left(e^{-x\beta_1} - e^{-x\beta_2}\right)\\
\mathbb{E}(e^{\theta X_{T_x} - rT_x}) &= e^{\theta x}\left(\frac{\eta_1 - \beta_1}{\beta_2 - \beta_1}\frac{\beta_2 - \theta}{\eta_1 - \theta}e^{-x\beta_1} + (\frac{\beta_2 - \eta_1}{\beta_2 - \beta_1}\frac{\beta_1 - \theta}{\eta_1 - \theta}e^{-x\beta_2}\right)
\end{aligned}$$

where $0 < \beta_1 < \eta_1 < \beta_2$ are roots of $G(\beta) = r$. The method is based on finding an explicit solution of $\mathcal{L}u = ru$ where $\mathcal{L}$ is the infinitesimal generator of the process X.

Chapter 3

Mixed Processes

3.1 Definition

A mixed process is a process X with dynamics

$$X_t = X_0 + \int_0^t a_s\, ds + \int_0^t \sigma_s dW_s + \int_0^t \varphi_s dM_s \,,$$

where W is a standard Brownian motion and M is the compensated martingale of an inhomogeneous Poisson process N, i.e., $M_t = N_t - \int_0^t \lambda_s ds$. Here the processes W and M are independent and adapted with respect to a filtration $\mathbf{F}$. The coefficients a, σ, φ are assumed to be $\mathbf{F}$-predictable processes, satisfying integrability conditions, i.e.,

$$\int_0^t |a_s| ds < \infty, \int_0^t \sigma_s^2 ds < \infty, \int_0^t \varphi_s^2 \lambda_s ds < \infty, \int_0^t |\varphi_s| \lambda_s ds < \infty \,.$$

The process X is a special semi-martingale, its continuous martingale part is $X_t^c = \int_0^t \sigma_s dW_s$. The jump times of the process X are those of N, the jump of X is $\Delta X_t = X_t - X_{t-} = \varphi_t \Delta N_t$.
The predictable bracket of X is

$$\langle X \rangle_t = \int_0^t \sigma_s^2 ds + \int_0^t \lambda_s \varphi_s^2 ds$$

The quadratic variation process is $[X]_t = \int_0^t \sigma_s^2 ds + \int_0^t \varphi_s^2 dN_s$. If X and Y be two mixed processes

$$\begin{aligned} dX_t &= a_t dt + \sigma_t dW_t + \varphi_t dM_t \,, \\ dY_t &= \tilde{a}_t dt + \tilde{\sigma}_t dW_t + \tilde{\varphi}_t dM_t \,. \end{aligned}$$

then, the covariation process is $d[X, Y]_t = \sigma_t \tilde{\sigma}_t dt + \varphi_t \tilde{\varphi}_t dN_t$ and the predictable bracket is $d\langle X, Y\rangle_t = (\sigma_t \tilde{\sigma}_t + \lambda \varphi_t \tilde{\varphi}_t) dt$.

Remark 3.1.1 One can extend this definition to the case where M is the compensated martingale of a compound Poisson process. However, in that case, one has to introduce the random measure associated with the compound process.

3.2 Itô's Formula

3.2.1 One Dimensional Case

Let F be a $C^{1,2}$ function, and

$$dX_t = a_t dt + \sigma_t dW_t + \varphi_t dM_t \,.$$

Then,

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \partial_s F(s, X_s)\, ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\ &\quad + \frac{1}{2} \int_0^t \partial_{xx} F(s, X_s) \sigma_s^2\, ds + \sum_{s \le t} [F(s, X_s) - F(s, X_{s^-}) - \partial_x F(s, X_{s-}) \Delta X_s] \,. \end{aligned} \tag{3.1}$$

This formula can be written in different forms. An important form is the following, where the canonical decomposition of the semi-martingale Y is given

$$F(t,X_t)-F(0,X_0)=\int_0^t \partial_x F(s,X_s)\sigma_s dW_s+\int_0^t [F(s,X_{s^-}+\varphi_s)-F(s,X_{s^-})]\,dM_s$$
$$+\int_0^t \left[(\partial_t F+a_s\partial_x F+\frac{1}{2}\sigma_s^2\partial_{xx}F)(s,X_s)+\lambda_s[F(s,X_s+\varphi_s)-F(s,X_s)-\partial_x F(s,X_s)\varphi_s]\right]ds\,.$$

In the particular case of deterministic t coefficients a,σ,φ and λ, it follows that if F solves the following IPDE

$$\begin{aligned}\partial_t F(t,x) \quad &+ \quad a(t)\partial_x F(t,x)+\frac{1}{2}\sigma^2(t)\partial_{xx}F(t,x)\\ &+ \quad \lambda(t)[F(t,x+\varphi(t))-F(t,x)-\partial_x F(t,x)\varphi(t)]=0,\ dt\,a.s.\end{aligned}$$

the process $F(t,X_t)$ is a local martingale

3.3 Predictable Representation Theorem

Let Z be a square integrable $\mathbf{F}$-martingale. There exist two predictable processes (ψ,γ) such that $Z=z+\psi\cdot W+\gamma\cdot M$, with

$$\int_0^t \psi_s^2 ds<\infty\,,\int_0^t \gamma_s^2\lambda(s)ds<\infty,\ a.s.$$

3.4 Change of probability

3.4.1 Exponential Martingales

Let γ and ψ be two predictable processes. The solution of

$$dL_t=L_{t^-}(\psi_t dW_t+\gamma_t dM_t)$$

is the strictly positive exponential local martingale

$$L_t=L_0\prod_{s\le t}(1+\gamma_s\Delta N_s)\,e^{-\int_0^t\gamma_s\lambda(s)ds}\ \exp\left(\int_0^t \psi_s dW_s-\frac{1}{2}\int_0^t\psi_s^2 ds\right)$$

If $\gamma_t>-1$, the process L is sticly positive and can be written as

$$L_t=L_0\exp\left(\int_0^t \ln(1+\gamma_s)dN_s-\int_0^t\lambda(s)\gamma_s ds+\int_0^t\psi_s dW_s-\frac{1}{2}\int_0^t\psi_s^2 ds\right)\,.$$

3.4.2 Girsanov's Theorem

If P and Q are equivalent probabilities, there exist two predictable processes ψ and γ, with $\gamma>-1$ such that the Radon-Nikodym density L is of the form

$$dL_t=L_{t-}(\psi_t dW_t+\gamma_t dM_t)\,.$$

Then, the processes

$$\widetilde{W}_t=W_t-\int_0^t\psi_s ds,\ \widetilde{M}_t=M_t-\int_0^t\lambda(s)\gamma_s ds$$

are Q-martingales. The process $\widetilde{W}$ is a Q-Brownian motion. Note that $\widetilde{W}$ and $\widetilde{M}$ can fail to be independent.

3.5 Hitting Times

Here, N is a standard Poisson process, with compensated martingale M. Let

$$dX_t = adt + \sigma dW_t + \varphi dM_t,\ X_0 = 0\,.$$

Let us denote by $T_\ell(X)$, the first passage time of the process X at level ℓ, for $\ell > 0$ as $T_\ell(X) = \inf\{t \geq 0 \,:\, X_t \geq \ell\}$.

The process $Z_t = \exp(kX_t - tg(k))$ where

$$g(k) = ak + \frac{1}{2}\sigma^2 k(k-1) + \lambda((1+\varphi)^k - 1 - k\varphi)\,. \tag{3.2}$$

is a martingale When there are no positive jumps, i.e., $\varphi < 0$, $X_{T_\ell} = \ell$, hence

$$\mathbb{E}[\exp(-g(k)T_\ell)] = \exp(-k\ell)\,.$$

Inverting the Lévy exponent $g(k)$ we obtain

$$\mathbb{E}(\exp(-uT_\ell)) = \exp(-g^{-1}(u)\,\ell),$$

where $g^{-1}(u)$ is the positive root of $g(k) = u$.

If the jump size is positive there is a non zero probability that X_{T_ℓ} is strictly greater than ℓ. In this case, we introduce the so-called overshoot $K(\ell)$

$$K(\ell) = X_{T_\ell} - \ell\,. \tag{3.3}$$

The difficulty is to obtain the law of the overshoot.

3.6 Mixed Processes in Finance

The dynamics of the price are supposed to be given by

$$dS_t = S_{t^-}(b_t dt + \sigma_t dW_t + \phi_t dM_t) \tag{3.4}$$

or in closed form

$$S_t = S_0 \exp\left(\int_0^t b_s ds\right) \mathcal{E}(\phi\star M)_t\, \mathcal{E}(\sigma\star W)_t\,.$$

In order that the price remains positive, one assumes that $\phi_t > -1$.

3.6.1 Symmetry Formula

We now restrict our attention to the case of constant coefficients $r, \delta, \sigma, \phi, \lambda)$, and we establish the symmetry formula, which gives the price of an European Call in terms of the price of an European Put. We assume that, under Q,

$$dS_t = S_{t^-}((r-\delta)dt + \sigma dW_t + \phi dM_t)$$

where δ is a dividend (or, in case of currency, the foreign interest rate). In that case, the process $Z_t = S_t e^{(\delta - r)t}/S_0$ is a strictly positive Q-martingale with expectation equal to 1. Here $M_t = N_t - \lambda t$ is a Q-martingale. We can write

$$\mathbb{E}(e^{-rt}(K-S_t)^+) = \mathbb{E}\left(e^{-\delta t} Z_t(\frac{KS_0}{S_t} - S_0)^+\right) = \widehat{\mathbb{E}}\left(e^{-\delta t}(\frac{KS_0}{S_t} - S_0)^+\right),$$

where $d\widehat{Q}|_{\mathcal{F}_t} = Z_t dQ|_{\mathcal{F}_t}$. The process $\widehat{S} = 1/S$ follows

$$d\widehat{S}_t = \widehat{S}_{t-}((\delta - r)dt - \sigma d\widehat{W}_t - \frac{\phi}{1+\phi} d\widehat{M}_t)$$

where $\widehat{W}_t = W_t - \sigma t$ is a $\widehat{Q}$-BM and $\widehat{M}_t = N_t - \lambda(1+\phi)t$ is a $\widehat{Q}$-martingale. Hence, denoting by C_E (resp. P_E) the price of a European call (resp. put)

$$P_E(x, K, r, \delta; \sigma, \phi, \lambda) = C_E(K, x, \delta, r; \sigma, -\frac{\phi}{1+\phi}, \lambda(1+\phi))\,.$$

3.6.2 Incompleteness

A market in which a riskless asset and a risky asset, with a mixed process dynamics is incomplete. We determine here the set of e.m.m. and we determine the range of prices for a European call. We denote by $R(t) = \exp - \int_0^t r(s)ds$ the discount factor, where the interest rate r is assumed to be deterministic. Assume that

$$d(RS)_t = R(t)S_{t-}([b(t) - r(t)]dt + \sigma(t)dW_t + \phi(t)dM_t) \tag{3.5}$$

The set of probability measures equivalent to $\mathbb{P}$ is the set of measures $\mathbb{P}^{\psi,\gamma}$ such that $\left.\frac{dP^{\psi,\gamma}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t^{\psi,\gamma}$ where $L_t^{\psi,\gamma} \stackrel{def}{=} L_t^{\psi,W} L_t^{\gamma,M}$

$$\begin{cases} L_t^{\psi,W} &= \mathcal{E}(\psi\star W)_t &= \exp\left[\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right] \\ \\ L_t^{\gamma,M} &= \mathcal{E}(\gamma\star M)_t &= \exp\left[\int_0^t \ln(1+\gamma_s)dN_s - \int_0^t \lambda(s)\gamma_s ds\right] . \end{cases}$$

In order that $P^{\psi,\gamma}$ is an e.m.m., one has to impose conditions on the parameters such that the discounted price process SR is a $P^{\psi,\gamma}$-martingale, or that $L^{\psi,\gamma}SR$ is a $\mathbb{P}$ martingale. Some Itô calculus yields to the relation

$$b(t) - r(t) + \sigma(t)\psi_t + \lambda(t)\phi(t)\gamma_t = 0 \;\; , \;\; dP \otimes dtp.s. \tag{3.6}$$

We study now the range of viable prices associated with a European call option, that is, the interval $]\inf_{\gamma\in\Gamma} V_t^\gamma \, , \sup_{\gamma\in\Gamma} V_t^\gamma[$, for $V_t^\gamma = e^{-r(T-t)}\mathbb{E}^{\psi,\gamma}((S_T-K)^+|\mathcal{F}_t)$.
We denote by $\mathcal{BS}$ the Black-Scholes function, that is, the function such that

$$R(t)\mathcal{BS}(x,t) = \mathbb{E}(R(T)(X_T-K)^+ \,|X_t = x) \;, \;\; \mathcal{BS}(x,T) = (x-K)^+$$

when

$$dX_t = X_t(r(t)dt + \sigma(t)\, dW_t)\,. \tag{3.7}$$

Proposition 3.6.1 *Let $P^\gamma \in \mathcal{Q}$. Then, the associated viable price is bounded below by the Black-Scholes function, evaluated at the underlying asset value, and bounded above by the underlying asset value, i.e.,*

$$R(t)\mathcal{BS}(S_t,t) \le \mathbb{E}^\gamma(R(T)\,(S_T-K)^+|\mathcal{F}_t) \le R(t)\, S_t$$

The range of viable prices $V_t^\gamma = \frac{R(T)}{R(t)}\mathbb{E}^\gamma((S_T-K)^+|\mathcal{F}_t)$ is exactly the interval $]\mathcal{BS}(S_t,t),\, S_t[$.

3.7 Other models

3.7.1 Affine Jump Diffusion Models

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t + dX_t$$

where X is a (λ,ν) compound Poisson process. The infinitesimal generator of S is

$$\mathcal{L}f = \partial_t f + \mu(x)\partial_x f + \frac{1}{2}\mathrm{Tr}(\partial_{xx} f\sigma\sigma^T) + \lambda\int (f(x+z,t) - f(x,t))d\nu(z)$$

for $f \in C_b^2$.

Proposition 3.7.1 *Suppose that $\mu(x) = \mu_0 + \mu_1 x\,; \sigma^2(x) = \sigma_0 + \sigma_1 x$ are affine functions, and that $\int e^{zy}\nu(dy) < \infty, \forall z$. Then, for any affine function $\psi(x) = \psi_0 + \psi_1 x$, there exist two functions α and β such that*

$$\mathbb{E}(e^{\theta S_T}\exp\left(-\int_t^T \psi(S_s)ds\right)|\mathcal{F}_t) = e^{\alpha(t)+\beta(t)S_t}\,.$$

3.7.2 Mixed Processes involving Compound Poisson Processes

Proposition 3.7.2 *Let W be a Brownian motion and X be a (λ, F) compound Poisson process independent of W. Let*

$$dS_t = S_{t-}(\mu dt + \sigma dW_t + dX_t)\,.$$

The process $(S_t e^{-rt}, t \geq 0)$ is a martingale if and only if $\mu + \lambda \mathbb{E}(Y_1) = r$.

3.7.3 General Jump-Diffusion Processes

Let W be a Brownian motion and $p(ds, dz)$ a marked point process. Let $\mathcal{F}_t = \sigma(W_s, p([0,s], A), A \in \mathcal{E};\ s \leq t)$. The solution of

$$dS_t = S_{t-}(\mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}} \varphi(t,x) p(dt, dx))$$

can be written in an exponential form as

$$S_t = S_0 \exp\left(\int_0^t \left[\mu_s - \frac{1}{2}\sigma_s^2\right] ds + \int_0^t \sigma_s dW_s\right) \prod_{n=1}^{N_t} (1 + \varphi(T_n, Z_n))$$

where $N_t = p((0,t], \mathbb{R})$ is the total number of jumps.

3.8 Lévy-Itô processes

One can consider the more general class of **Lévy-Itô processes**

$$dX_t = a_t dt + \sigma_t dW_t + \int \mathbb{1}_{\{|x|<1\}} \gamma_t(x) \widetilde{N}(dt, dx) + \int \mathbb{1}_{\{|x|\geq 1\}} \gamma_t(x) N(dt, dx)\,,$$

. These processes are semi-martingales.

Chapter 4

Appendix

4.1 Processes and Filtrations

A continuous time process is a family of random variables $(X_t, t \geq 0)$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The process is said to be measurable if the map

$$\begin{array}{rcl} (\Omega \times \mathbb{R}^+, \mathcal{F} \otimes \mathcal{B}) & \rightarrow & \mathbb{R} \\ (\omega, t) & \rightarrow & X_t(\omega) \end{array}$$

is measurable. In all this text we say process for continuous time measurable process.

If a filtration $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$ is given, the process X is said to be $\mathbf{F}$-adapted if, for any t, the r.v. X_t is $\mathcal{F}_t$-mesurable. The natural filtration of a (measurable) process is the filtration $\mathbf{F}^X$ defined as $\mathcal{F}^X_t = \sigma(X_s, s \leq t)$.

A process Y is said to be a modification of X if, for any t, $\mathbb{P}(X_t = Y_t) = 1$. A process is continuous, (resp. continuous on right (càd), continuous on right with limits on left (càdlàg[1])) if the map $t \rightarrow X_t(\omega)$ is continuous a.s. (resp. continuous on right, continuous on right with limits on left). If X is càdlàg, one denotes by $X_{t-} = \lim_{s\uparrow t} X_s$ the left limit of X and by $\Delta X_t = X_t - X_{t-}$ the jump of X at time t.

A filtration $\mathbf{F}$ satisfies the usual hypotheses if $\mathcal{F}_0$ contains all the negligeable sets (i.e., all the sets A such that there exists $A_i, i = 1, 2; A_i \in \mathcal{F}_\infty, A_1 \subseteq A \subseteq A_2, \mathbb{P}(A_2 \setminus A_1) = 0$) and is right-continuous, i.e., $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$. If a process X is continuous on right, its natural filtration is continuous on right. In the following, we shall assume that the filtrations satisfy the usual hypothesis (i.e., if it is not the case, we complete the filtration and take its right-continuous regularization).

4.2 Integration w.r.t. a Finite Variation Process

4.2.1 Some Definitions

An increasing process is a process $(A_t, t \geq 0)$ such that $A_0 = 0, A_s \leq A_t\, a.s.$, for $s < t$. The process is said to be integrable if $\mathbb{E}(A_\infty) < \infty$. Increasing processes admit a right-continuous modification with limit on left (we shall always take this version).

Any increasing process can be written as $A_t = A^c_t + A^d_t$ where A^c is an increasing continuous process and A^d is an increasing pure jump process, i.e., $A^d_t = \sum_{s\leq t} \Delta A_s$. The summation $\sum_{s\leq t} \Delta A_s$ is in fact a summation over a denumerable number of times s, i.e., the times where A admits a jump.

Finite variation processes are the difference between two increasing process. We consider always their right-continuous modification with limit on left.

4.2.2 Stieltjes Integral

Let U be a càdlàg process with bounded variation (i.e., the difference between two increasing processes). The **Stieltjes integral** $\int_0^\infty \theta_s dU_s$ is defined for elementary processes θ of the form $\theta_s = \vartheta_a \mathbb{1}_{]a,b]}(s)$,

[1] we use the french acronym for **c**ontinu **à** **d**roite, pourvu de **l**imites **à** **g**auche

with ϑ_a a r.v. as $\int_0^\infty \theta_s dU_s = \vartheta_a\,(U(b) - U(a))$ and for θ such that $\int_0^\infty |\theta_s||dU(s)| < \infty$ by linearity and passage to the limit. (Hence, the integral is defined path-by-path.) Then, one defines the integral

$$\int_0^t \theta_s dU_s = \int_{]0,t]} \theta_s dU_s = \int_0^\infty 1\!\!1_{\{]0,t]\}} \theta_s dU_s\,.$$

Note that if U has a jump at time t_0, then $(\Theta_t := \int_0^t \theta_s dU_s,\ t \geq 0)$ has also a jump at time t_0 given as $\Delta\Theta_{t_0} = \Theta_{t_0} - \Theta_{t_0^-} = \theta_{t_0}\Delta U_{t_0}$.

4.2.3 Integration by Parts

If U and V are two finite variation processes, Stieltjes' integration by parts formula can be written as follows

$$\begin{aligned} U_t V_t &= U_0 V_0 + \int_{]0,t]} V_s dU_s + \int_{]0,t]} U_{s^-} dV_s \\ &= U_0 V_0 + \int_{]0,t]} V_{s^-} dU_s + \int_{]0,t]} U_{s^-} dV_s + \sum_{s\leq t} \Delta U_s\, \Delta V_s\,. \end{aligned} \tag{4.1}$$

The summation $\sum_{s\leq t} \Delta U_s\, \Delta V_s$ is in fact a summation over a denumerable number of times s, i.e., the times where U and V admit a common jump. As a partial check, one can verify that the jumps of the left-hand side, i.e., $U_t V_t - U_{t^-} V_{t^-}$, are equal to the jumps of the right hand side $V_{t^-}\Delta U_t + U_{t^-}\Delta V_t + \Delta U_t\, \Delta V_t$.

4.2.4 Chain Rule

Let $F \in C^1$ and A a finite variation process. Then,

$$F(A_t) = F(A_0) + \int_0^t F'(A_{s^-}) dA_s + \sum_{s\leq t} (F(A_s) - F(A_{s^-}) - F'(A_{s^-}))\Delta A_s$$

or,

$$F(A_t) = F(A_0) + \int_0^t F'(A_{s^-}) dA_s^c + \sum_{s\leq t} F(A_s) - F(A_{s^-})$$

where A^c is the continuous part of A.

4.2.5 Exponential Process

Proposition 4.2.1 *Let A be a given finite variation process. The unique solution of $dZ_t = Z_{t^-} dA_t,\ Z_0 = 1$ is*

$$Z_t = \exp(A_t^c) \prod_{s\leq t} (1 + \Delta A_s)$$

This process is non-negative iff $\Delta A_s \geq -1$.

PROOF: It suffices to solve the SDE between two jumps (i.e., $dZ_t = Z_{t^-} dA_t^c$) , and to take care about the jumps. From definition, the solution satisfies $\Delta Z_t = Z_{t^-}\Delta A_t$, i.e., $Z_t = Z_{t^-}(1 + \Delta A_t)$. $\lhd$

4.3 General Theory of Stochastic Processes

Let $(\Omega, \mathbf{F}, \mathbb{P})$ be a filtered probability space. The process X is **indistinguishable** from Y if $\{\omega : X_t(\omega) = Y_t(\omega), \forall t\}$ is a mesurable set and $P(X_t = Y_t, \forall t) = 1$. If X and Y are modifications of each other and are a.s. continuous, they are indistinguishable.

4.3.1 Stopping Times

A random variable τ, valued in $\mathbb{R}^+ \cup \{+\infty\}$ is an **F-stopping time** if, for any t,

$$\{\tau \le t\} \in \mathcal{F}_t$$

When no confusion is possible, we shall say only stopping time. If $\mathbf{F} \subseteq \mathbf{G}$, any $\mathbf{F}$-stopping time is a $\mathbf{G}$-stopping time.
If τ is an $\mathbf{F}$-stopping time, the σ-algebra $\mathcal{F}_\tau$ of events prior to τ is defined as:

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty \,:\, A \cap \{\tau \le t\} \in \mathcal{F}_t, \,\forall t\}.$$

The σ-algebra $\mathcal{F}_{\tau-}$ of the events strictly prior to τ is the smallest σ-algebra which contains $\mathcal{F}_0$ and all the sets of the form $A \cap \{t < \tau\}$ for $A \in \mathcal{F}_t, t > 0$.

Proposition 4.3.1 *Any stopping time τ is a decreasing limit of stopping times τ_n where τ_n are valued in a denumerable set*

PROOF: Write $\tau_n = \sum_{k=0}^{\infty} \frac{k+1}{2^n} \mathbb{1}_{\frac{k}{2^n} \le \tau < \frac{k+1}{2n}} + (+\infty)\mathbb{1}_{\tau=+\infty}$ $\triangleleft$

Proposition 4.3.2 *If X is a càd process, and τ an $\mathbf{F}^X$-stopping time, then X_τ is $\mathcal{F}_\tau$ measurable.*

An important example of stopping time is the following: if A is a closed set, and X is a continuous process, then $D_A = \inf\{t \ge 0 \,:\, X_t \in A\}$ is an $\mathbf{F}$-stopping time. (We recall that $\inf \emptyset = +\infty$.) The **predictable** σ-algebra $\mathcal{P}$ is the σ-algebra on $\Omega \times \mathbb{R}^+$ generated generated by the stochastic intervals $]\!]S, T]\!]$ where S and T are two $\mathbf{F}$-stopping times such that $S \le T$. It is also generated by the $\mathbf{F}$-adapted càg (or continuous) processes. A process is said to be **predictable** if it is measurable with respect to the predictable σ-field. If X is a càdlàg process, then $(X_{t-}, t \ge 0)$ is a predictable process.

Definition 4.3.1 *A stopping time T is **predictable** if there exists an increasing sequence (T_n) of stopping times such that almost surely*

i) $\lim_n T_n = T$

ii) $T_n < T$ *for every n on the set $\{T > 0\}$.*

*A stopping time T is **totally inaccessible** if $\mathbb{P}(T = S < \infty) = 0$ for any predictable stopping time S. An equivalent definition is: for any increasing sequence of stopping times T_n, $\mathbb{P}(\{\lim T_n = T\} \cap A) = 0$ where $A = \cap\{T_n < T\}$.*

4.3.2 Local Martingales and Semi-martingales

Local martingale

Definition 4.3.2 *An adapted, right-continuous process M is an **F-local martingale** if there exists a sequence of stopping times (T_n) such that*

(i) The sequence T_n is increasing and $\lim_n T_n = \infty$, a.s.

(ii) For every n, the stopped process $M^{T_n} \mathbb{1}_{\{T_n > 0\}}$ is an $\mathbf{F}$-martingale.

A sequence of stopping times such that (i) holds is called a localizing or reducing sequence.

Semi-martingale
An **F-semi-martingale** is a càdlàg process X which can be written as $X_t = X_0 + M_t + A_t$ where M is an $\mathbf{F}$-local martingale with value 0 at time 0 and where A is an $\mathbf{F}$-adapted càdlàg process with finite variation. In general, this decomposition is not unique (see the Poisson process), and it is necessary to add some conditions on the finite variation process to get the uniqueness.

Definition 4.3.3 *A **special semi-martingale** is a semi-martingale with a predictable finite variation part. Such a decomposition $X = X_0 + M + A$ with A **predictable**, is unique. Call it the* **canonical decomposition** *of X, if it exists.*

The martingale part of the semi martingale X can be written as a sum of a continuous martingale and a discontinuous martingale: $X = M^c + M^d + A$. The continuous martingale M is often called the continuous martingale part of X and is denoted by X^c (This notation is the one in use, one should avoid the confusion with the continuous part of X)

A continuous semi-martingale is a special semi-martingale, and A is continuous

4.3.3 Covariation of Local Martingales

• **Continuous Local martingales:** Let X be a continuous local martingale. The predictable quadratic variation process of X is the continuous increasing process $\langle X \rangle$ such that $X^2 - \langle X \rangle$ is a local martingale. Let X and Y be two continuous local martingales. The predictable covariation process is the continuous finite variation process $\langle X, Y \rangle$ such that $XY - \langle X, Y \rangle$ is a local martingale. Note that $\langle X \rangle = \langle X, X \rangle$ and

$$\langle X + Y \rangle = \langle X \rangle + \langle Y \rangle + 2\langle X, Y \rangle \,.$$

In the particular case where $X_t = x + \int_0^t x_s dW_s$ and $Y_t = y + \int_0^t y_s dW_s$, where W is a BM and x, y two adapted processes such that $\int_0^t x_s^2 ds < \infty$, $\int_0^t y_s^2 ds < \infty$, then $\langle X, Y \rangle_t = \int_0^t x_s y_s ds$.

• **General local martingales:** Let X and Y be two local martingales.
The covariation process is the finite variation process $[X, Y]$ such that
$XY - [X, Y]$ is a local martingale
$\Delta[X, Y]_t = \Delta X_t \Delta Y_t$
The process $[X] = [X, X]$ is non-decreasing. If the martingales X and Y are continuous, $[X, Y] = \langle X, Y \rangle$. This covariation process is the limit in probability of $\sum_{i=1}^{p(n)} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$, for $0 < t_1 < \cdots < t_{p(n)} \leq t$ when $\sup_{i \leq p(n)}(t_i - t_{i-1})$ goes to 0.
If X and Y are continuous, $\langle X, Y \rangle = [X, Y]$.
If $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures, the quadratic covariation process $[X, Y]$ under $\mathbb{P}$ and under $\mathbb{Q}$ are the same. The covariation $[X, Y]$ of both processes X and Y can be also defined by polarisation

$$[X + Y] = [X] + [Y] + 2[X, Y]$$

Let us recall that, if W is a Brownian motion $\langle W \rangle_t = [W]_t = t$. If M is the compensated martingale of a Poisson process, $[M] = N$.

The predictable covariation process is the continuous finite variation process $\langle X, Y \rangle$ such that $XY - \langle X, Y \rangle$ is a local martingale. The existence of such a process may fail for discontinuous martingales.

4.3.4 Covariation of Semi-martingales

If X and Y are semi-martingales and if X^c, Y^c are their continuous martingale parts, their quadratic covariation is

$$[X, Y]_t = \langle X^c, Y^c \rangle_t + \sum_{s \leq t} (\Delta X_s)(\Delta Y_s) \,.$$

The integration by parts formula is

$$d(X_t Y_t) = X_{t^-} dY_t + Y_{t^-} dX_t + d[X, Y]_t$$

4.4 Change of probability

4.4.1 Doléans-Dade exponential

If X is a semi-martingale, then the process $Z = \mathcal{E}(X)$ is the unique solution to the SDE (called the Doléans Dade exponential)

$$Z_t = 1 + \int_{]0,t]} Z_{u-} \, dX_u.$$

It is known that

$$\mathcal{E}_t(X) = \exp\left(X_t - X_0 - \frac{1}{2}\langle X^c \rangle_t\right) \prod_{u \leq t} (1 + \Delta X_u) e^{-\Delta X_u},$$

where X^c is the continuous martingale component of X.
If X is a local martingale, Z is also a local martingale.

4.4.2 Girsanov's Theorem

Theorem 4.4.1 *Let X be a local martingale with respect to $\mathbb{P}$ and $\mathbb{Q}|_{\mathcal{F}_t} = L_t \, \mathbb{P}|_{\mathcal{F}_t}$. Then,*

$$X_t - \int_0^t \frac{d[X,L]_s}{L_s}$$

is a $\mathbb{Q}$-local martingale.

PROOF: In a first step, one notes that a process Z is a $\mathbb{Q}$-local martingale iff LZ is a $\mathbb{P}$-local martingale. Then, the proof relies on stochastic calculus. $\triangleleft$

4.5 Processes with Stationary and Independent Increments

4.5.1 Definition

Let X be a càdlàg process. We denote by $\mathbf{F}^X$ its natural filtration. The process is said to have

- Independent increments if, for any pair (s,t) of positive numbers $X_{t+s} - X_t$ is independent of $\mathcal{F}_t^X$
- Stationary increments if for any pair (s,t) of positive numbers, $X_{t+s} - X_t \overset{law}{=} X_s$

4.5.2 Strong Markov Property

Theorem 4.5.1 *Let X be a process with stationary and independent increments. For any $\mathbf{F}^X$-stopping time τ, the process Y defined on the set $\tau < \infty$ as $Y_t = X_{t+\tau} - X_\tau$ has the same law as the process X and is independent of $\mathcal{F}_\tau$.*

PROOF: Let us set $\varphi(t;u) = \mathbb{E}(e^{iuX_t})$. Let us assume that the stopping time τ is bounded and let $A \in \mathcal{F}_\tau$. Then, applying several time the optional sampling theorem

$$\begin{aligned}
\mathbb{E}\left(\mathbb{1}_A \exp\left(i\sum_{j=1}^n u_j(Y_{t_j} - Y_{t_{j-1}})\right)\right) &= \mathbb{E}\left(\mathbb{1}_A \prod_j \frac{Z_{\tau+t_j}(u_j)}{Z_{\tau+t_{j-1}}(u_j)}\, \varphi(t_j - t_{j-1}, u_j)\right) \\
&= \mathbb{P}(A) \prod_j \varphi(t_j - t_{j-1}, u_j)
\end{aligned}$$

$\triangleleft$

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