Dynamics copula

Tomasz R. Bielecki, Illinois Institute of Technology, USA
Nicole El Karoui, Ying Jiao, Ecole polytechnique, France
Yann Le Cam, Monique Jeanblanc, Université d'Évry Val d'Essonne, France
Marek Rutkowski, University of New South Wales, Australia

Innovations in Mathematical Finance

AMAMEF, Loen 26-30 June 2007

Initial times

Let $(\Omega, \mathbf{F}, \mathbb{P})$ be a given filtered probability space, τ a random time and

$$H_t = 1\!\!1_{\tau \leq t}$$

Let $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$ where $\mathcal{H}_t = \sigma(H_s, s \leq t)$. For any random time τ, we write

$$G_t^T(\omega) = \mathbb{P}(\tau > T | \mathcal{F}_t)(\omega)$$

the conditional survival process.

The positive random time τ is called an ***initial time*** *if there exists a measure η on $\mathcal{B}(\mathbb{R}^+)$ such that* $\mathbb{P}(\tau \in ds|\mathcal{F}_t) \ll \eta(ds)$. Then,

$$G_t^T = \mathbb{P}(\tau > T|\mathcal{F}_t) = \int_T^\infty f(u;t)\eta(du)\,.$$

From $G_s^T = \mathbb{E}(G_t^T|\mathcal{F}_s)$ for any $s \leq t$, it follows that for any $u \geq 0$, $(f(u;t))_t$ is a non negative $\mathbb{F}$-martingale.

• Under the condition that the initial time τ avoids the $\mathbb{F}$-stopping times, there is equivalence between $\mathbb{F}$ is immersed in $\mathbb{G}$ and for any $u \geq 0$, the martingale $f(u;\cdot)$ is constant after u.

• Under the condition that the initial time τ avoids the $\mathbb{F}$-stopping times, there is equivalence between $\mathbb{F}$ is immersed in $\mathbb{G}$ and for any $u \geq 0$, the martingale $f(u;\cdot)$ is constant after u.

• Let $(K_t^u)_{t\geq 0}$ be a family of $\mathbb{F}$-predictable processes indexed by $u \geq 0$. Then

$$\mathbb{E}\left(K_t^\tau \,|\, \mathcal{F}_t \right) = \int_0^\infty K_t^u f(u;t) \eta\,(du)$$

• Under the condition that the initial time τ avoids the $\mathbb{F}$-stopping times, there is equivalence between $\mathbb{F}$ is immersed in $\mathbb{G}$ and for any $u \geq 0$, the martingale $f(u;\cdot)$ is constant after u.

• Let $(K_t^u)_{t\geq 0}$ be a family of $\mathbb{F}$-predictable processes indexed by $u \geq 0$. Then

$$\mathbb{E}\left(K_t^\tau \,|\, \mathcal{F}_t\right) = \int_0^\infty K_t^u f(u;t)\eta\,(du)$$

• If X is an $\mathbb{F}$-martingale, assuming that $G_t = G_t^t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ is continuous

$$Y_t = X_t - \int_0^{t\wedge\tau} \frac{d\left\langle X, G\right\rangle_s}{G_s} - \int_{t\wedge\tau}^t \frac{d\left\langle X, f(\theta;\cdot)\right\rangle_s}{f(\theta;s)}\bigg|_{\theta=\tau} \in \mathcal{M}(\mathbb{G}).$$

Example: "Cox-like" construction. Here

- λ is a non-negative $\mathbb{F}$-adapted process, $\Lambda_t = \int_0^t \lambda_s ds$
- Θ is a given r.v. independent of $\mathcal{F}_\infty$ with unit exponential law
- V is a $\mathcal{F}_\infty$ -measurable non-negative random variable
- $\tau = \inf\{t : \Lambda_t \geq \Theta V\}$.

Example: "Cox-like" construction. Here

- λ is a non-negative $\mathbb{F}$-adapted process, $\Lambda_t = \int_0^t \lambda_s ds$
- Θ is a given r.v. independent of $\mathcal{F}_\infty$ with unit exponential law
- V is a $\mathcal{F}_\infty$ -measurable non-negative random variable
- $\tau = \inf\{t : \Lambda_t \geq \Theta V\}$.

For any T and t,

$$G_t(T) = \mathbb{P}(\tau \geq T|\mathcal{F}_t) = \mathbb{P}(\Lambda_T \leq \Theta V|\mathcal{F}_t) = \mathbb{P}\left(\exp -\frac{\Lambda_T}{V} \geq e^{-\Theta}\middle| \mathcal{F}_t\right).$$

Let us denote $\exp(-\Lambda_t/V) = 1 - \int_0^t \psi_s ds$, with

$$\psi_s = (\lambda_s/V)\,\exp -\int_0^s (\lambda_u/V)\,du,$$

and define $\gamma(s;t) = \mathbb{E}\left(\psi_s|\,\mathcal{F}_t\right)$. Then, $f(s;t) = \gamma(s;t)/\gamma(s;0)$.

HJM model

Assume that for any $T > 0$, the process $(G_t(T), 0 \le t)$ satisfies

$$\frac{dG_t(T)}{G_t(T)} = \Psi(t,T)dW_t$$

where $\Psi(t,T)$ is an $\mathbb{F}$-adapted process which is differentiable with respect to T. Similar as in the interest rate modelling, we define the forward rate $\gamma_t(T) = -\frac{\partial}{\partial T}\ln G_t(T)$. If, in addition, $\psi(t,T) = \dfrac{\partial}{\partial T}\Psi(t,T)$ is bounded, then we have

1. $G_t(T) = G_0(T)\exp\left(\int_0^t \Psi(s,T)dW_s - \frac{1}{2}\int_0^t |\Psi(s,T)|^2 ds\right)$
2. $\gamma_t(T) = \gamma_0(T) - \int_0^t \psi(s,T)dW_s + \int_0^t \psi(s,T)\Psi(s,T)^* ds.$
3. $G_t = \exp\left(-\int_0^t \gamma_s(s)ds + \int_0^t \Psi(u,u)dW_u - \frac{1}{2}\int_0^t |\Psi(u,u)|^2 du\right).$

A general model of density process

Our aim is to give a large class of examples of density process. We start with an explicit construction of a family $(f(u) = f(u;t), t \geq 0)$ which satisfy:

(a) $f(u;\cdot)$ is a family of martingales

(b) $f(u;t) > 0, \forall t \geq 0, \forall u \geq 0$

(c) $\int_0^\infty f(u;t)du = 1$, $\forall t \geq 0$.

A general model of density process

Our aim is to give a large class of examples of density process. We start with an explicit construction of a family $(f(u) = f(u;t), t \geq 0)$ which satisfy:

(a) $f(u;\cdot)$ is a family of martingales

(b) $f(u;t) > 0, \forall t \geq 0, \forall u \geq 0$

(c) $\int_0^\infty f(u;t)du = 1, \ \forall t \geq 0.$

Define $f(u;t) = \lambda_t(u) \exp\left(-\int_0^u \lambda_t(v)dv\right)$ where $d\lambda_t(u) = m_t(u)dt + \sigma_t(u)dW_t$ with

$$m_t(u) = \sigma_t(u) \int_0^u \sigma_t(v)dv, \qquad \forall t, u \geq 0.$$

Then, for any u, the process $f(u;\cdot)$ is a martingale. Condition b) is satisfied if $\lambda_t(u) \geq 0$

We have $\lambda_t(u) = \lambda_0(u) + \int_0^t m_s(u)ds + \int_0^t \sigma_s(u)dW_s$. The non negativity of λ will be satisfied if

- m is non negative
- for any u, there exists a positive constant $C(u)$ such that

$\lambda_0(u) \geq C(u)$

- the quantity $X_t = C(u) + \int_0^t \sigma_s(u)dW_s$ is a positive martingale.

We have $\lambda_t(u) = \lambda_0(u) + \int_0^t m_s(u)ds + \int_0^t \sigma_s(u)dW_s$. The non negativity of λ will be satisfied if

- m is non negative
- for any u, there exists a positive constant $C(u)$ such that

$\lambda_0(u) \geq C(u)$

- the quantity $X_t = C(u) + \int_0^t \sigma_s(u)dW_s$ is a positive martingale.

This is the case if X is a Doléans Dade exponential: there exists a family $b(u)$ of adapted processes

$$X_t = C(u) \exp\left(\int_0^t b_s(u)dW_s - \frac{1}{2}\int_0^t b_s^2(u)ds\right)$$

In that case, $\int_0^t \sigma_s(u)dW_t = \int_0^t b_s(u)X_s dW_s$. Hence, we make the choice of

$$\sigma_t(u) = b_t(u)X_t = b_t(u)C(u) \exp\left(\int_0^t b_s(u)dW_s - \frac{1}{2}\int_0^t b_s^2(u)ds\right)$$

and propose a family of density processes.

Let $\lambda_0(u)$ *be a family of probability densities on* $\mathbb{R}^+$*. Assume that* $b(u)$ *is a given family of non-negative adapted processes, and* $C(u)$ *a family of constants such that* $\lambda_0(u) \geq C(u) \geq 0$*. Define*

$$\sigma_t(u) = b_t(u)C(u)\exp\left(\int_0^t b_s(u)dW_s - \frac{1}{2}\int_0^t b_s^2(u)ds\right)$$

and

$$\begin{aligned} f_t(u) &= \lambda_t(u)\exp\left(-\int_0^t \lambda_t(v)dv\right) \\ \lambda_t(u) &= \lambda_0(u) + \int_0^t m_s(u)ds + \int_0^t \sigma_s(u)dW_s \\ m_t(u) &= \sigma_t(u)\int_0^u \sigma_t(v)dv\,. \end{aligned}$$

Then the family $f(u)$ *satisfies the above conditions.*

Hedging of Credit Spreads and Default Correlations

In this section, we assume that $\mathbf{F}$ is a Brownian filtration and that the interest rate is null.

Our aim is to obtain the dynamics of a CDS in the simple case where two different credit names are considered.

We assume that we are given two strictly positive random times τ_1 and τ_2.

Joint Survival Process

We introduce the **conditional joint survival process** $G(u, v; t)$

$$G(u, v; t) = \mathbb{Q}(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t).$$

We write

$$\partial_1 G(u, v; t) = \frac{\partial}{\partial u} G(u, v; t), \quad \partial_{12} G(u, v; t) = \frac{\partial^2}{\partial u \partial v} G(u, v; t) = f(u, v; t)$$

so that

$$G(u, v; t) = \int_u^\infty \Big(\int_v^\infty f(x, y; t)\, dy \Big)\, dx$$

where $f(u, v; s)$ is some $\mathbf{F}$-predictable process (in fact an $(\mathbf{F}, \mathbb{Q})$-martingale).

For any fixed $(u, v) \in \mathbb{R}^2_+$, the $\mathbf{F}$-martingale $G(u, v; t) = \mathbb{Q}(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t)$ admits the integral representation

$$G(u, v; t) = \mathbb{Q}(\tau_1 > u, \tau_2 > v) + \int_0^t g(u, v; s)\, dW_s$$

Let us introduce the filtrations $\mathbf{H}^i, \mathbf{H}, \mathbf{G}^i$ and $\mathbf{G}$ associated with default times by setting

$$\mathcal{H}_t^i = \sigma(H_s^i;\, s \in [0,t]), \quad \mathcal{H}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2, \quad \mathcal{G}_t^i = \mathcal{F}_t \vee \mathcal{H}_t^i, \quad \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t,$$

where $H_t^i = 1\!\!1_{\{\tau_i \leq t\}}$. **In this talk, we assume that immersion hypothesis holds between F and G**. In particular, any $\mathbf{F}$-martingale is also a $\mathbf{G}^i$-martingale for $i = 1, 2$.

In general, there is no reason that any $\mathbf{G}^i$-martingale is a $\mathbf{G}$-martingale. (see a specific case in Ehlers and Schonbucher)

Let $\tau_{(1)} = \tau_1 \wedge \tau_2$. We denote $\widetilde{\lambda}^i_t = -\frac{\partial_i G(t,t;t)}{G(t,t;t)}$ the **pre-default intensity** for the ith name:

$$\widehat{M}^i_t = H^i_{t\wedge\tau_{(1)}} - \int_0^{t\wedge\tau_{(1)}} \widetilde{\lambda}^i_u \, du,$$

is a $\mathbf{G}$-martingale, and $\widetilde{\lambda} = \widetilde{\lambda}^1 + \widetilde{\lambda}^2$ the intensity of $\tau_{(1)}$.

Valuation of Single-Name CDSs

Let us now examine the valuation of single-name CDSs under the assumption that the interest rate equals zero.

We consider the CDS

- with the constant spread κ_1,
- which delivers $\delta_1(\tau_1)$ at time τ_1 if $\tau_1 < T_1$, where δ_1 is a

deterministic function.

The value $S^1(\kappa_1)$ of this CDS, computed in the filtration $\mathbf{G}$, i.e., taking care on the information on the second default contained in that filtration, is computed in two successive steps.

On the set $t < \tau_{(1)}$, the ex-dividend price of the CDS equals

$$S_t^1(\kappa_1) = \widetilde{S}_t^1(\kappa_1) = \frac{1}{G(t,t;t)} \left(- \int_t^{T_1} \delta_1(u) \partial_1 G(u,t;t)\, du - \kappa_1 \int_t^{T_1} G(u,t;t)\, du \right).$$

On the set $t < \tau_{(1)}$, the ex-dividend price of the CDS equals

$$S_t^1(\kappa_1) = \widetilde{S}_t^1(\kappa_1) = \frac{1}{G(t,t;t)} \left(- \int_t^{T_1} \delta_1(u) \partial_1 G(u,t;t)\, du - \kappa_1 \int_t^{T_1} G(u,t;t)\, du \right).$$

On the event $\{\tau_2 \leq t < \tau_1\}$, we have that

$$S_t^1(\kappa_1) = \widehat{S}_t^1(\kappa_1) = \frac{1}{\partial_2 G(t,\tau_2;t)} \left(- \int_t^{T_1} \delta_1(u) f(u,\tau_2;t)\, du - \kappa_1 \int_t^{T_1} \partial_2 G(u,\tau_2;t)\, du$$

Price Dynamics of Single-Name CDSs

By applying the Itô-Wentzell theorem, we get

$$\begin{aligned} G(u,t;t) &= G(u,0;0) + \int_0^t g(u,s;s)\,dW_s + \int_0^t \partial_2 G(u,s;s)\,ds \\ G(t,t;t) &= G(0,0;0) + \int_0^t g(s,s;s)\,dW_s + \int_0^t (\partial_1 G(s,s;s) + \partial_2 G(s,s;s))\,ds. \end{aligned}$$

The dynamics of the process $\widetilde{S}^1(\kappa_1)$ are

$$d\widetilde{S}^1_t(\kappa_1) = \Big(- \widetilde{\lambda}^1_t \delta_1(t) + \kappa_1 + \widetilde{\lambda}_t \widetilde{S}^1_t(\kappa_1) - \widetilde{\lambda}^2_t S^1_{t|2}(\kappa_1) \Big) dt + \sigma^1(t, T_1)\, dW_t$$

where

$$\begin{aligned}
\sigma^1(t, T_1) &= -\frac{1}{G_{(1)}(t;t)} \left(\int_t^{T_1} \big(\delta_1(u)\, \partial_1 g(u, t; t) + \kappa_1 g(u, t; t)\big)\, du \right) \\
S^1_{t|2}(\kappa_1) &= \frac{1}{\partial_2 G(t, t; t)} \left(- \int_t^{T_1} \delta_1(u) f(u, t; t)\, du - \kappa_1 \int_t^{T_1} \partial_2 G(u, t; t)\, du \right).
\end{aligned}$$

The cumulative price

$$S^{c,1}(\kappa_1) = S_t^1(\kappa_1) + B_t \int_{]0,t]} B_u^{-1}\, dD_u$$

where

$$D_t = D_t(\kappa_1, \delta_1, T_1, \tau_1) = \delta_1(\tau_1) 1\!\!1_{\{\tau_1 \le t\}} - \kappa_1(t \wedge (T_1 \wedge \tau_1))$$

satisfies, on $[0, T_1 \wedge \tau_{(1)}]$,

$$dS_t^{c,1}(\kappa_1) \quad = \quad (\delta_1(t) - \widetilde{S}_t^1(\kappa_1))\, d\widehat{M}_t^1 + (S_{t|2}^1(\kappa_1) - \widetilde{S}_t^1(\kappa_1))\, d\widehat{M}_t^2 + \sigma^1(t, T_1) dW_t \,.$$

On $\tau_1 > t > \tau_2$

$$dS_t^1 = d\widehat{S}_t^1 = \sigma_{1|2}(t, T_1)dW_t + (\delta_1(t)\lambda_t^{1|2}(\tau_2) - \kappa_1 + \widehat{S}_t^1 \lambda_t^{1|2}(\tau_2))dt$$

where

$$\sigma_{1|2}(t, T^1) = -\int_t^T \delta_1(u)\partial_1\partial_2 g(u, \tau_2; t)du - \kappa_1 \int_t^{T_1} \partial_2 g(u, \tau_2; t)du$$

$$\lambda^{1|2}(t, s) = -\frac{f(t, s; t)}{\partial_2 G(t, s; t)}$$

Replication of a First-to-Default Claim

A **first-to-default claim** with maturity T is a claim $(X, A, Z, \tau_{(1)})$ where

- X is an $\mathcal{F}_T$-measurable amount payable at maturity if no default occurs
- $A : [0, T] \to \mathbb{R}$ with $A_0 = 0$ represents the dividend stream up to $\tau_{(1)}$,
- $Z = (Z^1, Z^2, \dots, Z^n)$ is the vector of $\mathbf{F}$-predictable, real-valued processes, where $Z^i_{\tau_{(1)}}$ specifies the recovery received at time $\tau_{(1)}$ if the ith name is the first defaulted name, that is, on the event $\{\tau_i = \tau_{(1)} \leq T\}$.
- We denote by $G_{(1)}(t;t) = G(t, \cdots, t; t)$

The cumulative price S^{cum} is given by

$$dS_t^{cum} = \sum_{i=1}^{n} (Z_t^i - S_{t-})\, d\widehat{M}_t^i + (1 - H_t^{(1)})(G_{(1)}(t;t))^{-1}\, dm_t,$$

where the **F**-martingale m is given by the formula

$$m_t \quad = \quad \mathbb{E}_{Q^*}\left(G_{(1)}(T;T)X + \sum_{i=1}^{n} \int_0^T G_{(1)}(u;u) Z_u^i \widetilde{\lambda}_u^i \, du - \int_0^T G_{(1)}(u;u)\, dA_u \,\middle|\, \mathcal{F}_t \right)$$

The pre-default ex-dividend price satisfies

$$d\widetilde{S}_t = \widetilde{\lambda}_t \widetilde{S}_t \, dt - \sum_{i=1}^{n} \widetilde{\lambda}_t^i Z_t^i \, dt + dA_t + (G(t,t;t))^{-1} \, dm_t.$$

Since $\mathbf{F}$ is generated by a Brownian motion, there exists an $\mathbf{F}$-predictable process ζ such that

$$d\widetilde{S}_t = \widetilde{\lambda}_t \widetilde{S}_t \, dt - \sum_{i=1}^{n} \widetilde{\lambda}_t^i Z_t^i \, dt + dA_t + (G(t,t;t))^{-1} \zeta_t \, dW_t.$$

We say that a self-financing strategy $\phi = (\phi^0, \phi^1, \dots, \phi^k)$ **replicates** a first-to-default claim $(X, A, Z, \tau_{(1)})$ if its wealth process $V(\phi)$ satisfies the equality $V_{t\wedge\tau_{(1)}}(\phi) = S_{t\wedge\tau_{(1)}}$ for any $t \in [0, T]$.

We have, for any $t \in [0, T]$,

$$dV_t(\phi) = \sum_{\ell=1}^{k} \phi_t^i \Big(\big(\delta_t^\ell - \widetilde{S}_t^\ell(\kappa_\ell)\big)\, dM_t^\ell + \sum_{j=1\,,j\neq\ell}^{k} \big(S_{t|j}^\ell - \widetilde{S}_t^\ell(\kappa_\ell)\big)\, dM_t^j + (1 - H_t)(G(t,t;t))^{-1}\, dn_t^\ell \Big)$$

where

$$n_t^\ell = \mathbb{E}_{Q^*}\left(\int_0^{T_\ell} G(u,u;u)\Big(\delta_u^\ell \widetilde{\lambda}_u^i + \sum_{j=1,j\neq\ell}^{n} S_{u|j}^\ell \widetilde{\lambda}_u^j\Big)\, du - \kappa_\ell \int_0^{T_\ell} G(u,u;u)\, du \,\Big|\, \mathcal{F}_t \right).$$

Let $\widetilde{\phi}_t = (\widetilde{\phi}^1_t, \widetilde{\phi}^2_t, \ldots, \widetilde{\phi}^k_t)$ be a solution to the following equations

$$\widetilde{\phi}^\ell_t \big(\delta^\ell_t - \widetilde{S}^\ell_t(\kappa_\ell)\big) + \sum_{j=1,\, j\neq\ell}^{n} \widetilde{\phi}^j_t \big(S^j_{t|\ell}(\kappa_j) - \widetilde{S}^j_t(\kappa_j)\big) = Z^\ell_t - \widetilde{S}_t$$

and $\sum_{\ell=1}^k \widetilde{\phi}^\ell_t \zeta^\ell_t = \zeta_t$.

Let us set $\phi^\ell_t = \widetilde{\phi}^\ell(\tau_{(1)} \wedge t)$ for $\ell = 1, 2, \ldots, k$ and $t \in [0, T]$.

Then the self-financing trading strategy $\phi = ((V(\phi) - \phi \cdot S), \ldots, \phi^k)$ replicates the first-to-default claim $(X, A, Z, \tau_{(1)})$.

Replication with Market CDSs

When considering trading strategies involving CDSs issued in the past, one encounters a practical difficulty regarding their liquidity.

Recall that for each maturity T_i by the CDS issued at time t we mean the CDS over $[t, T]$ with the spread $\kappa(t, T_i) = \kappa_i$.

We now define a **market CDS** — which at any time t has similar features as the T_i-maturity CDS issued at this date t, in particular, it has the ex-dividend price equal to zero.

A T_i**-maturity market CDS** has the dividend process equal to

$$ {}^*D_t^i = \int_{]0,t]} B_u \, d(B_u^{-1} S_u^i(\kappa_i)) + D_t^i, $$

where $D^i = D(\kappa_i, \delta^i, T_i, \tau)$ for some fixed spread κ_i.

The ex-dividend price ${}^*S^i$ **of the** T_i**-maturity market CDS equals zero for any** $t \in [0, T_i]$.

Since market CDSs are traded on the ex-dividend basis, to describe the self-financing trading strategies in the savings account B and the market CDSs with ex-dividend prices ${}^*S^i$.

A strategy $\phi = (\phi^0, \ldots, \phi^k)$ in the savings account B and the market CDSs with dividends ${}^*D^i$ is said to be *self-financing* if its wealth $V_t(\phi) = \phi_t^0 B_t$ satisfies $V_t(\phi) = V_0(\phi) + G_t(\phi)$ for every $t \in [0, T]$, where the gains process $G(\phi)$ is defined as follows

$$G_t(\phi) = \int_{]0,t]} \phi_u^0 \, dB_u + \sum_{i=1}^{k} \int_{]0,t]} \phi_u^i \, d\, {}^*D_u^i.$$

Let ϕ be a self-financing strategy in the savings account B and ex-dividend prices $S^i(\kappa_i)$, $i = 1, \ldots, k$.

Then the strategy $\psi = (\psi^0, \ldots, \psi^k)$ where $\psi^i = \phi^i$ for $i = 1, \ldots, k$ and $\psi_t^0 = B_t^{-1} V_t(\phi)$ is a self-financing strategy in the savings account B and the market CDSs with dividends ${}^*D^i$ and its wealth process satisfies $V(\psi) = V(\phi)$.

The cumulative price of the T_i-maturity market CDS satisfies

$$
\begin{aligned}
{}^*S_t^{c,i} &= {}^*S_t^i + B_t \int_{]0,t]} B_u^{-1}\, d\,{}^*D_u^i \\
&= \mathbb{1}_{\{t<\tau\}}(\kappa_t^i - \kappa_i)\widetilde{A}(t,T) - B_t S_0^i(\kappa_i) + B_t \int_{]0,t]} B_u^{-1}\, dD_u^i
\end{aligned}
$$

where

$$
\widetilde{A}(t,T) = \frac{B_t}{G_t}\,\mathbb{E}_{Q^*}\left(\int_t^{T\wedge\tau} B_u^{-1} du \,\Big|\, \mathcal{F}_t\right).
$$

If we choose $\kappa_i = \kappa_0^i$ then

$$ {}^*S_t^{c,i} = 1\!\!1_{\{t<\tau\}}(\kappa_t^i - \kappa_0^i)\widetilde{A}(t,T) + B_t \int_{]0,t]} B_u^{-1}\, dD_u^i = S_t^{c,i}(\kappa_0^i). $$

Assume that there exist $\mathbf{F}$-predictable processes $\phi^1, \ldots, \phi^k$ satisfying the following conditions, for any $t \in [0, T]$,

$$\sum_{i=1}^{k} \phi_t^i \big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big) = Z_t - \widetilde{S}_t, \quad \sum_{i=1}^{k} \phi_t^i \zeta_t^i = \xi_t.$$

Let the process $V(\phi)$ be given by

$$dV_t(\phi) = \sum_{i=1}^{k} \phi_t^i \Big(\big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big)\, dM_t + (1 - H_t) B_t G_t^{-1} dn_t^i \Big)$$

with the initial condition $V_0(\phi) = Y_0$ and let ϕ^0 be given by, for $t \in [0, T]$,

$$\phi_t^0 = B_t^{-1} V_t(\phi).$$

Then the self-financing trading strategy $\phi = (\phi^0, \ldots, \phi^k)$ in the savings account B and market CDSs with dividends ${}^*D^i$, $i = 1, \ldots, k$ replicates the defaultable claim (X, A, Z, τ).

Intuitively, one can think of the market CDS as a stream of CDSs that are continuously entered into and immediately unwound. Consequently, one can assume an accounting convention according to which one never holds a non-market CDS: suppose at time 0 one goes long the market CDS with spread κ_0^i. If one still owns it at time $t > 0$, the convention dictates that one owns at time t the market CDS with spread κ_t^i, but that it has already paid the cumulative dividends. In this way, we avoid any problem with considering the short-sale positions: what would be a short-sale position in an on-the-run (i.e., non-market) CDS becomes a short position in the corresponding market CDS. This mathematical convention is actually consistent with the market practice where default protection is bought or sold and then nullified, that is, CDSs are longed or shorted and then unwound, as needed.