Credit Risk, I.

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I. Hazard Process Approach of Credit Risk: A Toy Model

1. The Model
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The Model

The Market

We begin with the case where a riskless asset, with **deterministic** interest rate $(r(s); s \geq 0)$ is the only asset available in the default-free market.

$$R(t) = \exp\left(-\int_0^t r(s)ds\right)$$

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$$R(t) = \exp\left(-\int_0^t r(s)ds\right)$$

The time-t price $B(t,T)$ of a risk-free zero-coupon bond with maturity T is

$$B(t,T) \stackrel{def}{=} \exp\left(-\int_t^T r(s)ds\right) .$$

Default occurs at time τ, where τ is assumed to be a positive random variable with density f, constructed on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$.

$$F(t) = \mathbb{P}(\tau \leq t) = \int_0^t f(s)ds\,.$$

We assume that $F(t) < 1$, $\forall t$

Defaultable Zero-coupon with Payment at Maturity

A **defaultable zero-coupon** bond (DZC in short)- or a corporate bond- with maturity T and rebate δ paid at maturity, consists of

- The payment of one monetary unit at time T if default has not occurred before time T,
- A payment of δ monetary units, made at maturity, if $\tau < T$, where $0 \leq \delta < 1$.

Value of the defaultable zero-coupon bond

The “value” of the defaultable zero-coupon bond is defined as

$$\begin{aligned} D^{(\delta,T)}(0,T) &= \mathbb{E}\left(B(0,T)\,(\mathbb{1}_{\{T<\tau\}} + \delta\mathbb{1}_{\{\tau\le T\}})\right) \\ &= B(0,T)\,\left(1-(1-\delta)F(T)\right). \end{aligned}$$

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The value $D^{(\delta,T)}(t,T)$ of the DZC is the conditional expectation of the discounted payoff $B(t,T)\,[\mathbb{1}_{\{T<\tau\}} + \delta\mathbb{1}_{\{\tau\le T\}}]$ given the information:

$$D^{(\delta,T)}(t,T) = \mathbb{1}_{\{\tau\le t\}}B(t,T)\delta + \mathbb{1}_{\{t<\tau\}}\widetilde{D}^{(\delta,T)}(t,T)$$

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$$D^{(\delta,T)}(t,T) = 1\!\!1_{\{\tau\le t\}} B(t,T)\delta + 1\!\!1_{\{t<\tau\}} \widetilde{D}^{(\delta,T)}(t,T)$$

where the **predefault value** $\widetilde{D}^{(\delta,T)}(t,T)$ is defined as

$$\widetilde{D}^{(\delta,T)}(t,T) = \mathbb{E}\Big(B(t,T)\,(1\!\!1_{\{T<\tau\}} + \delta 1\!\!1_{\{\tau\le T\}})\,\Big|\,t<\tau\Big)$$

$$\widetilde{D}^{(\delta)}(t,T) \quad = \quad \mathbb{E}\Big(B(t,T)\left(\mathbb{1}_{\{T<\tau\}} + \delta\mathbb{1}_{\{\tau\leq T\}}\right)\Big|t<\tau\Big)$$

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&= B(t,T)\left(1-(1-\delta)\frac{\mathbb{P}(t<\tau\le T)}{\mathbb{P}(t<\tau)}\right) \\
&= B(t,T)\left(1-(1-\delta)\frac{F(T)-F(t)}{1-F(t)}\right)
\end{aligned}
$$

The formula

$$\widetilde{D}^{(\delta,T)}(t,T) = B(t,T) - B(t,T)(1-\delta)\frac{\mathbb{P}(t < \tau \leq T)}{\mathbb{P}(t < \tau)}$$

can be read as

$$\widetilde{D}^{(\delta,T)}(t,T) = B(t,T) - \text{EDLGD} \times \text{DP}$$

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$$\widetilde{D}^{(\delta,T)}(t,T) = B(t,T) - \text{EDLGD} \times \text{DP}$$

where the **Expected Discounted Loss Given Default** (EDLGD) is defined as $B(t,T)(1-\delta)$ and the **Default Probability** (DP) is

$$DP = \frac{\mathbb{P}(t < \tau \leq T)}{\mathbb{P}(t < \tau)} = \mathbb{P}(\tau \leq T | t < \tau) \; .$$

In case the payment is a function of the default time, say $\delta(\tau)$, the value of this defaultable zero-coupon is

$$\begin{aligned} D^{(\delta,T)}(0,T) &= \mathbb{E}\left(B(0,T)\,\mathbb{1}_{\{T<\tau\}} + B(0,T)\delta(\tau)\mathbb{1}_{\{\tau\leq T\}}\right) \\ &= B(0,T)\left[\mathbb{P}(T<\tau) + \int_0^T \delta(s)f(s)ds\right]. \end{aligned}$$

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The predefault price $\widetilde{D}^{(\delta,T)}(t,T)$ is

$$\begin{aligned} \widetilde{D}^{(\delta,T)}(t,T) &= B(t,T)\mathbb{E}(\,\mathbb{1}_{\{T<\tau\}} + \delta(\tau)\mathbb{1}_{\{\tau\le T\}}\big|t<\tau) \\ &= B(t,T)\left[\frac{\mathbb{P}(T<\tau)}{\mathbb{P}(t<\tau)} + \frac{1}{\mathbb{P}(t<\tau)}\int_t^T \delta(s)f(s)ds\right]. \end{aligned}$$

We introduce the increasing **hazard function** Γ defined by

$$\Gamma(t) = -\ln(1 - F(t))$$

and its derivative $\gamma(t) = \dfrac{f(t)}{1 - F(t)}$ where $f(t) = F'(t)$, i.e.,

$$1 - F(t) = e^{-\Gamma(t)} = \exp\left(-\int_0^t \gamma(s)ds\right) = \mathbb{P}(\tau > t)\,.$$

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$$1 - F(t) = e^{-\Gamma(t)} = \exp\left(-\int_0^t \gamma(s)ds\right) = \mathbb{P}(\tau > t)\,.$$

The quantity $\gamma(t)$ called the **hazard rate** is the probability that the default occurs in a small interval dt given that the default has not occured before time t

$$\gamma(t) = \lim_{h \to 0} \frac{1}{h} P(\tau \le t + h | \tau > t)\,.$$

For $\delta = 0$,

$$\widetilde{D}(t,T) = \exp\left(-\int_t^T (r+\gamma)(s)ds\right)$$

in other terms, the spot rate has to be adjusted by means of a **spread** (γ) in order to evaluate DZCs.

Defaultable Zero-coupon with Payment at Hit

Here, a defaultable zero-coupon bond with maturity T consists of

- The payment of one monetary unit at time T if default has not yet occurred,
- A payment of $\delta(\tau)$ monetary units, where δ is a deterministic function, made at time τ if $\tau < T$.

Here, we do not assume that F is differentiable.

Value of the defaultable zero-coupon

The value of this defaultable zero-coupon bond is

$$\begin{aligned} D^{(\delta)}(0,T) &= \mathbb{E}(B(0,T)\,\mathbb{1}_{\{T<\tau\}} + B(0,\tau)\delta(\tau)\mathbb{1}_{\{\tau\leq T\}}) \\ &= G(T)B(0,T) - \int_0^T B(0,s)\delta(s)dG(s)\,, \end{aligned}$$

where $G(t) = 1 - F(t) = \mathbb{P}(t < \tau)$ is the **survival probability**.

For $t < T$,

$$D^{(\delta)}(t,T) = 1\!\!1_{t<\tau} \widetilde{D}^{(\delta)}(t,T)$$

where $\widetilde{D}^{(\delta)}(t,T)$ is called the **predefault price** defined by

$$\begin{aligned} B(0,t)\widetilde{D}^{(\delta)}(t,T) &= \mathbb{E}(B(0,T)\, 1\!\!1_{\{T<\tau\}} + B(0,\tau)\delta(\tau) 1\!\!1_{\{\tau \le T\}} | t < \tau) \\ &= \frac{\mathbb{P}(T<\tau)}{\mathbb{P}(t<\tau)} B(0,T) + \frac{1}{\mathbb{P}(t<\tau)} \int_t^T B(0,s)\delta(s) dF(s)\,. \end{aligned}$$

Hence,

$$B(0,t)G(t)\widetilde{D}^{(\delta)}(t,T) = G(T)B(0,T) - \int_t^T B(0,s)\delta(s)dG(s)\,.$$

In terms of the hazard function, the time-t value $\widetilde{D}^{(\delta)}(t,T)$ satisfies:

$$B(0,t)e^{-\Gamma(t)}\widetilde{D}^{(\delta)}(t,T) = e^{-\Gamma(T)}B(0,T) + \int_t^T B(0,s)e^{-\Gamma(s)}\delta(s)d\Gamma(s)\,.$$

A particular case If F is differentiable, the function $\gamma = \Gamma'$ satisfies $f(t) = \gamma(t)e^{-\Gamma(t)}$. Then,

$$R_d(t)\widetilde{D}^{(\delta)}(t,T) = R_d(T) + \int_t^T R_d(s)\gamma(s)\delta(s)ds$$

with

$$R_d(t) = \exp\left(-\int_0^t (r(s) + \gamma(s))\, ds\right)$$

The defaultable interest rate is $r + \gamma$ and is, as expected, greater than r (the value of a DZC with $\delta = 0$ is smaller than the value of a default-free zero-coupon).

The dynamics of $\widetilde{D}^{(\delta)}(t,T)$ are

$$d\widetilde{D}^{(\delta)}(t,T) = (r(t) + \gamma(t))\widetilde{D}^{(\delta)}(t,T)dt - \delta(t)\gamma(t)dt\,.$$

The dynamics of $D^{(\delta)}$ includes a jump at time τ.

Spreads

A term structure of credit spreads associated with the zero-coupon bonds $S(t,T)$ is defined as

$$S(t,T) = -\frac{1}{T-t}\ln\frac{D(t,T)}{B(t,T)}\,.$$

In our setting, on the set $\{\tau > t\}$

$$S(t,T) = -\frac{1}{T-t}\ln\mathbb{Q}^*(\tau > T|\tau > t)\,,$$

whereas $S(t,T) = \infty$ on the set $\{\tau \le t\}$.

Toy Model and Martingales

We denote by $(H_t, t \geq 0)$ the right-continuous increasing process $H_t = 1\!\!1_{\{t \geq \tau\}}$ and by $(\mathcal{H}_t)$ its natural filtration. Any integrable $\mathcal{H}_t$-measurable r.v. H is of the form $H = h(\tau \wedge t) = h(\tau) 1\!\!1_{\{\tau \leq t\}} + h(t) 1\!\!1_{\{t < \tau\}}$ where h is a Borel function.

Key Lemma

If X is any integrable, $\mathcal{G}$-measurable r.v.

$$\mathbb{E}(X|\mathcal{H}_t)\mathbb{1}_{\{t<\tau\}} = \mathbb{1}_{\{t<\tau\}}\frac{\mathbb{E}(X\mathbb{1}_{\{t<\tau\}})}{\mathbb{P}(t<\tau)}\,.$$

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Let $Y = h(\tau)$ be a $\mathcal{H}$-measurable random variable. Then

$$\mathbb{E}(Y|\mathcal{H}_t) = \mathbb{1}_{\{\tau\leq t\}}h(\tau) + \mathbb{1}_{\{t<\tau\}}\int_t^\infty h(u)e^{\Gamma(t)-\Gamma(u)}d\Gamma(u)$$

An important Martingale

The process $(M_t, t \geq 0)$ *defined as*

$$M_t = H_t - \int_0^{\tau \wedge t} \frac{dF(s)}{1 - F(s)} = H_t - \int_0^t (1 - H_{s-}) \frac{dF(s)}{1 - F(s)}$$

is a **H**-*martingale.*

Hazard Function

The **hazard function** is

$$\Gamma(t) = -\ln(1 - F(t)) = \int_0^t \frac{dF(s)}{1 - F(s)}$$

In particular, if F is differentiable, the process

$$M_t = H_t - \int_0^{\tau \wedge t} \gamma(s) ds = H_t - \int_0^t \gamma(s)(1 - H_s) ds$$

is a martingale, where $\gamma(s) = \dfrac{f(s)}{1 - F(s)}$ is a deterministic non-negative function, called **the intensity of** τ.

The **Doob-Meyer decomposition** of the **submartingale** H_t is

$$H_t = M_t + \Gamma(t \wedge \tau)$$

The predictable process $A_t = \Gamma_{t \wedge \tau}$ is called the **compensator** of H.

The process

$$L_t \stackrel{def}{=} \mathbb{1}_{\{\tau > t\}} \exp\left(\int_0^t \gamma(s)ds\right)$$

is a $\mathbf{H}$*-martingale.*

Proof: We shall give 3 different arguments, each of which constitutes a proof.

a) Since the function γ is deterministic, for $t > s$

$$\mathbb{E}(L_t|\mathcal{H}_s) = \exp\left(\int_0^t \gamma(u)du\right) \mathbb{E}(\mathbb{1}_{\{t<\tau\}}|\mathcal{H}_s)\,.$$

From the Key Lemma

$$\mathbb{E}(\mathbb{1}_{\{t<\tau\}}|\mathcal{H}_s) = \mathbb{1}_{\{\tau>s\}}\frac{1-F(t)}{1-F(s)} = \mathbb{1}_{\{\tau>s\}}\exp\left(-\Gamma(t)+\Gamma(s)\right).$$

Hence,

$$\mathbb{E}(L_t|\mathcal{H}_s) = \mathbb{1}_{\{\tau>s\}}\exp\left(\int_0^s \gamma(u)du\right) = L_s.$$

b) Another method is to apply integration by parts formula to the process $L_t = (1 - H_t) \exp\left(\int_0^t \gamma(s)ds\right)$ If U and V are two finite variation processes, Stieltjes' integration by parts formula can be written as follows

$$\begin{aligned} U(t)V(t) &= U(0)V(0) + \int_{]0,t]} V(s-)dU(s) + \int_{]0,t]} U(s-)dV(s) \\ &\quad + \sum_{s \le t} \Delta U(s)\, \Delta V(s)\,. \end{aligned}$$

$$\begin{aligned} dL_t &= -dH_t \exp\left(\int_0^t \gamma(s)ds\right) + \gamma(t) \exp\left(\int_0^t \gamma(s)ds\right)(1 - H_t)dt \\ &= -\exp\left(\int_0^t \gamma(s)ds\right) dM_t\,. \end{aligned}$$

c) A third (sophisticated) method is to note that L is the exponential martingale of M, i.e., the solution of the SDE

$$dL_t = -L_{t-} dM_t \, , \, L_0 = 1.$$

$\triangle$

In the case where N is an inhomogeneous Poisson process with deterministic intensity λ and τ is the first time when N jumps, let $H_t = N_{t\wedge\tau}$. It is well known that $N_t - \int_0^t \lambda(s)ds$ is a martingale (see Appendix). Therefore, the process stopped at time τ is also a martingale, i.e., $H_t - \int_0^{t\wedge\tau} \lambda(s)ds$ is a martingale.

Change of probability

Let $\mathbb{P}^*$ be a probability equivalent to $\mathbb{P}$ on the space $(\Omega, \mathcal{H})$ where $\mathcal{H} = \mathcal{H}_\infty$ is the σ-algebra generated by τ. Then,

$$d\mathbb{P}^* = h(\tau)\, d\mathbb{P}$$

where h is a strictly positive fonction, such that $\mathbb{E}_{\mathbb{P}}(h(\tau)) = 1$. Let

$$g(t) = e^{\Gamma(t)} \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{t<\tau} h(\tau)))\,.$$

Let $\Gamma^*(t) = -\ln P^*(\tau > t$. *If* Γ *is continuous,* Γ^* *is continuous and*

$$d\Gamma^*(t) = \frac{h(t)}{g(t)} d\Gamma(t)$$

Proof:

$$\mathbb{P}^*(\tau > t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{t>\tau} h(\tau)) = \int_t^\infty h(u)dF(u) = e^{-\Gamma^*(t)}$$

Hence

$$e^{-\Gamma^*(t)} d\Gamma^*(t) = h(t)dF(t) = h(t)e^{-\Gamma(t)} d\Gamma(t)$$

Therefore

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{t<\tau} h(\tau)) d\Gamma^*(t) = h(t)e^{-\Gamma(t)} d\Gamma(t)$$

It follows that

$$d\Gamma^*(t) = \frac{h(t)}{e^{\Gamma(t)} \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{t<\tau} h(\tau))} d\Gamma(t) = \frac{h(t)}{g(t)} d\Gamma(t)$$

Exercices: Let $\eta_t = \mathbb{E}_P(h(\tau)|\mathcal{H}_t)$. Prove that

$$\eta_t \quad = \quad \int_0^t h(s)dH_s + (1 - H_t)g(t)$$

Prove that the martingale η admits a representation in terms of M as

$$\eta_t \quad = \quad 1 + \int_0^t (h(u) - \eta_{u-})dM_u$$

and that an explicit representation is

$$\eta_t = (1 + 1\!\!1_{\tau \le t} k(\tau)) \exp\left(-\int_0^{t \wedge \tau} k(u) d\Gamma(u)\right)$$

where $k(t) = \frac{h(t)}{g(t)} - 1$

Incompleteness of the Toy model

If the market consists only of the risk-free zero-coupon bond, there exists infinitely many e.m.m's. The discounted asset prices are constant, hence the set $\mathcal{Q}$ of equivalent martingale measures is the set of probabilities equivalent to the historical one. For any $\mathbb{Q} \in \mathcal{Q}$, we denote by F_Q the cumulative function of τ under $\mathbb{Q}$, i.e.,

$$F_Q(t) = \mathbb{Q}(\tau \leq t)\,.$$

The **range of prices** is defined as the set of prices which do not induce arbitrage opportunities. For a DZC with a constant rebate δ paid at maturity, the range of prices is equal to the set

$$\{\mathbb{E}_{\mathbb{Q}}(B(0,T)(\mathbb{1}_{\{T<\tau\}} + \delta \mathbb{1}_{\{\tau<T\}})), \mathbb{Q} \in \mathcal{Q}\}\,.$$

This set is exactly the interval $]\delta R_T, R_T[$.

Risk Neutral Probability Measures

It is usual to interpret the absence of arbitrage opportunities as the existence of an e.m.m. . If DZCs are traded, their prices are *given by the market*, and the equivalent martingale measure Q, *chosen by the market*, is such that, on the set $\{t < \tau\}$,

$$D(t,T) = B(t,T)\mathbb{E}_{\mathbb{Q}}\Big([\mathbb{1}_{T<\tau} + \delta \mathbb{1}_{t<\tau\leq T}] \Big| t < \tau \Big) .$$

Therefore, we can characterize the cumulative function of τ under Q from the market prices of the DZC as follows.

Zero Recovery If a DZC with zero recovery of maturity T is traded at a price $D(t,T)$ which belongs to the interval $]0, R_T^t[$, then, under any risk-neutral probability $\mathbb{Q}$, the process $R(t)D(t,T)$ is a martingale, the following equality holds

$$D(t,T)B(0,t) = \mathbb{E}_{\mathbb{Q}}(B(0,T)\mathbb{1}_{\{T<\tau\}}|\mathcal{H}_t) = B(0,T)\mathbb{1}_{\{t<\tau\}} \exp\left(-\int_t^T \gamma^Q(s)ds\right)$$

where $\gamma^Q(s) = \dfrac{dF_Q(s)/ds}{1 - F_Q(s)}$. The process γ^Q is the $\mathbb{Q}$-intensity of τ. Therefore the unique risk-neutral intensity can be obtained from the prices of DZCs as

$$r(t) + \gamma^Q(t) = -\partial_T \ln D(t,T)|_{T=t}$$

Fixed Payment at maturity If the prices of DZCs with different maturities are known, then)

$$\frac{B(0,T) - D(0,T)}{B(0,T)(1-\delta)} = F_Q(T)$$

where $F_Q(t) = Q(\tau \leq t)$, so that the law of τ is known under the e.m.m..

Payment at hit In this case, denoting by $\partial_T D$ the derivative of the value of the DZC at time 0 with respect to the maturity, we obtain

$$\partial_T D(0,T) = g(T)B(0,T) - G(T)B(0,T)r(T) - \delta(T)g(T)B(0,T)\,,$$

where $g(t) = G'(t)$. Therefore, solving this equation leads to

$$Q(\tau > t) = G(t) = \Delta(t)\left[1 + \int_0^t \partial_T D(0,s)\frac{1}{B(0,s)(1-\delta(s))}(\Delta(s))^{-1}ds\right]\,,$$

where $\Delta(t) = \exp\left(\int_0^t \frac{r(u)}{1-\delta(u)}du\right)$.

Representation Theorem

Let h be a (bounded) Borel function. Then, the martingale $M_t^h = \mathbb{E}(h(\tau)|\mathcal{H}_t)$ admits the representation

$$\mathbb{E}(h(\tau)|\mathcal{H}_t) = \mathbb{E}(h(\tau)) - \int_0^{t\wedge\tau} (\widetilde{h}(s) - h(s))\, dM_s\,,$$

where $M_t = H_t - \Gamma(t \wedge \tau)$ and $\widetilde{h}(s) = -\frac{\int_t^\infty h(u)dG(u)}{G(t)}$.

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where $M_t = H_t - \Gamma(t \wedge \tau)$ and $\widetilde{h}(s) = -\frac{\int_t^\infty h(u)dG(u)}{G(t)}$.

Note that $\widetilde{h}(s) = M_s^h$ on $s < \tau$.

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where $M_t = H_t - \Gamma(t \wedge \tau)$ and $\widetilde{h}(s) = -\frac{\int_t^\infty h(u)dG(u)}{G(t)}$.
Note that $\widetilde{h}(s) = M_s^h$ on $s < \tau$.
In particular, any square integrable $\mathcal{H}$-martingale $(X_t, t \geq 0)$ can be written as $X_t = X_0 + \int_0^t x_s dM_s$ where $(x_t, t \geq 0)$ is a predictable process.

PROOF: A proof consists in computing the conditional expectation

$$\mathbb{E}(h(\tau)|\mathcal{H}_t) = h(\tau)H_t + (1 - H_t)e^{-\Gamma(t)} \int_t^\infty h(s)dF(s)$$

and to use integration by parts formula.

Partial information: Duffie and Lando's model

Duffie and Lando study the case where $\tau = \inf\{t : V_t \leq m\}$ where V satisfies

$$dV_t = \mu(t, V_t)dt + \sigma(t, V_t)dW_t \,.$$

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Here the process W is a Brownian motion. If the information is the Brownian filtration, the time τ is a stopping time w.r.t. a Brownian filtration, therefore is predictable and admits no intensity. If the agent does not know the behavior of V, but only the minimal information $\mathcal{H}_t$, i.e. he knows when the default appears, the price of a zero-coupon is, in the case where the default is not yet occurred, $\exp\left(-\int_t^T \gamma(s)ds\right)$ where $\gamma(s) = \dfrac{f(s)}{G(s)}$ and $G(s) = \mathbb{P}(\tau > s), f = -G'$, as soon as the cumulative function of τ is differentiable.

Valuation and Trading Defaultable Claims

We assume that the market has chosen a riskneutral probability $\mathbb{Q}$ and that M and γ are computed w.r.t. $\mathbb{Q}$. We assume here that the interest rate r is constant.

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Price dynamics of a survival claim $(X, 0, \tau)$.

Let $(X, 0, \tau)$ be a *survival claim*. The price of the payoff $1\!\!1_{\{T<\tau\}} X$ that settles at time T is

$$Y_t = e^{rt}\, \mathbb{E}_{\mathbb{Q}}(1\!\!1_{\{T<\tau\}} e^{-rT} X \,|\, \mathcal{H}_t).$$

We assume that the market has chosen a riskneutral probability $\mathbb{Q}$ and that M and γ are computed w.r.t. $\mathbb{Q}$. We assume here that the interest rate r is constant.

Price dynamics of a survival claim $(X, 0, \tau)$**.**

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The dynamics of the price process is

$$dY_t = rY_t\,dt - (X\,e^{-r(T-t)} - Y_{t-})dM_t$$

Price dynamics of a recovery claim $(0, Z, \tau)$.

The recovery Z is paid at the time of default. The **cum-dividend** price process Y of $(0, Z, \tau)$ is

$$Y_t = e^{rt}\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{T \geq \tau\}} e^{-r\tau} Z(\tau) \,|\, \mathcal{H}_t\big)\,,$$

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The **ex-dividend** price is
$S_t = e^{rt}\,\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{T \geq \tau > t\}} e^{-r\tau} Z(\tau) \,|\, \mathcal{H}_t) = Y_t - e^{rt} Z(\tau) e^{-r\tau} \mathbb{1}_{\{\tau \leq t\}}.$

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$$dS_t = (rS_t - Z(t)\gamma(t))dt - (S_{t-} - Z(t))dM_t\,.$$

Valuation of a Credit Default Swap

A credit default swap (CDS) is a contract between two counterparties. B agrees to pay a default payment Z to A if a default of the obligor C occurs. If there is no default until the maturity of the default swap, B pays nothing. A pays a fee for the default protection. The fee can be either a fee paid till the maturity or till the default event.

A can not cancelled the contract. He can at any time before the default transfer the contract to D: D will pay the fee and receive the default payment if any. As we shall see, it can happen that D will require an amount of cash to accept to receive the contract. Usually, the fee consists of C_i paid at time T_i (this is the fixed leg). However, here we shall consider a continuous payment. The default payment is called the default leg.

A stylized credit default swap is formally introduced through the following definition.

A ***credit default swap*** *with a constant spread* κ *and* ***recovery at default*** *is a defaultable claim* $(0, C, Z, \tau)$*, where* $Z_t \equiv \delta(t)$ *and* $C_t = -\kappa t$ *for every* $t \in [0, T]$*. An RCLL function* $\delta : [0, T] \to \mathbb{R}$ *represents the* ***protection payment*** *and a constant* $\kappa \in \mathbb{R}$ *is termed the* ***spread*** *(or the premium) of a CDS.*

For simplicity, we assume that the interest rate $r = 0$, so that the price of a savings account $B_t = 1$ for every t. Our results can be easily extended to the case of a constant r.

Ex-dividend Price of a CDS

The ex-dividend price of a CDS maturing at T with spread κ is given by the formula

$$S_t(\kappa) = \mathbb{E}_{\mathbb{Q}^*}\Big(\delta(\tau)\mathbb{1}_{\{t<\tau\leq T\}} - \mathbb{1}_{\{t<\tau\}}\kappa\big((\tau \wedge T) - t\big) \,\Big|\, \mathcal{H}_t\Big).$$

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The ex-dividend price at time $t \in [s, T]$ of a credit default swap with spread κ and recovery at default equals

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\,\frac{1}{G(t)}\left(-\int_t^T \delta(u)\,dG(u) - \kappa\int_t^T G(u)\,du\right).$$

PROOF: We have, on the set $\{t < \tau\}$,

$$
\begin{aligned}
S_t(\kappa) &= -\frac{\int_t^T \delta(u)\, dG(u)}{G(t)} - \kappa \left(\frac{-\int_t^T u\, dG(u) + TG(T)}{G(t)} - t \right) \\
&= \frac{1}{G(t))} \left(-\int_t^T \delta(u)\, dG(u) - \kappa \Big(TG(T) - tG(t) - \int_t^T u\, dG(u) \Big) \right).
\end{aligned}
$$

It remains to note that

$$
\int_t^T G(u)\, du = TG(T) - tG(t) - \int_t^T u\, dG(u),
$$

△

The ex-dividend price of a CDS can also be represented as follows

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa), \quad \forall\, t \in [0, T],$$

where $\widetilde{S}_t(\kappa)$ stands for the *ex-dividend pre-default price* of a CDS.

Market CDS Spreads

Assume now that a CDS was initiated at some date $s \leq t$ and its initial price was equal to zero. *A market CDS started at s is a CDS initiated at time s whose initial value is equal to zero. A T-maturity CDS market spread at time s is the level of the spread $\kappa = \kappa(s,T)$ that makes a T-maturity CDS started at s worthless at its inception. A CDS market spread at time s is thus determined by the equation $S_s(\kappa(s,T)) = 0$.*

The T-maturity market spread $\kappa(s,T)$ is a solution to the equation

$$\int_s^T \delta(u)\,dG(u) + \kappa(s,T)\int_s^T G(u)\,du = 0,$$

and thus for every $s \in [0,T]$,

$$\kappa(s,T) = -\frac{\int_s^T \delta(u)\,dG(u)}{\int_s^T G(u)\,du}.$$

Standing assumptions. We fix the maturity date T, and we write briefly $\kappa(s)$ instead of $\kappa(s, T)$. In addition, we assume that all credit default swaps have a common recovery function δ.

Note that the ex-dividend pre-default value at time $t \in [0, T]$ We have the following result, in which the quantity $\nu(t, s) = \kappa(t) - \kappa(s)$ represents the *calendar CDS market spread* (for a given maturity T).

The ex-dividend price of a market CDS started at s with recovery δ at default and maturity T equals, for every $t \in [s, T]$,

$$S_t(\kappa(s)) = \mathbb{1}_{\{t<\tau\}} \left(\kappa(t) - \kappa(s)\right) \frac{\int_t^T G(u)\, du}{G(t)} = \mathbb{1}_{\{t<\tau\}} \, \nu(t, s) \, \frac{\int_t^T G(u)\, du}{G(t)},$$

or more explicitly,

$$S_t(\kappa(s)) = \mathbb{1}_{\{t<\tau\}} \frac{\int_t^T G(u)\, du}{G(t)} \left(\frac{\int_s^T \delta(u)\, dG(u)}{\int_s^T G(u)\, du} - \frac{\int_t^T \delta(u)\, dG(u)}{\int_t^T G(u)\, du} \right).$$

Price Dynamics of a CDS

In what follows, we assume that

$$G(t) = \mathbb{Q}(\tau > t) = \exp\left(-\int_0^t \gamma(u)\,du\right)$$

where the default intensity $\gamma(t)$ under $\mathbb{Q}$ is deterministic. We first focus on the dynamics of the ex-dividend price of a CDS with spread κ started at some date $s < T$.

The dynamics of the ex-dividend price $S_t(\kappa)$ *on* $[s, T]$ *are*

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt,$$

where the **H**-*martingale* M *under* $\mathbb{Q}$ *is given by the formula*

$$M_t = H_t - \int_0^t (1 - H_u)\gamma(u)\, du, \quad \forall\, t \in \mathbb{R}_+.$$

PROOF: It suffices to recall that

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\widetilde{S}_t(\kappa) = (1 - H_t)\widetilde{S}_t(\kappa)$$

so that

$$dS_t(\kappa) = (1 - H_t)\, d\widetilde{S}_t(\kappa) - \widetilde{S}_{t-}(\kappa)\, dH_t.$$

Using the explicit expression of $\widetilde{S}_t$, we find easily that we have

$$d\widetilde{S}_t(\kappa) = \gamma(t)\widetilde{S}_t(\kappa)\, dt + (\kappa(s) - \delta(t)\gamma(t))\, dt.$$

The SDE for S follows.

Trading Strategies with a CDS

A strategy $\phi_t = (\phi_t^0, \phi_t^1)$, $t \in [0, T]$, is self-financing if the wealth process $U(\phi)$, defined as

$$U_t(\phi) = \phi_t^0 + \phi_t^1 S_t(\kappa),$$

satisfies

$$dU_t(\phi) = \phi_t^1 \, dS_t(\kappa) + \phi_t^1 \, dD_t,$$

where $S(\kappa)$ is the ex-dividend price of a CDS with the dividend stream D. A strategy ϕ replicates a contingent claim Y if $U_T(\phi) = Y$.

Hedging of a Contingent Claim in the CDS Market

Our aim is to find **a replicating strategy for the defaultable claim** $(X, 0, Z, \tau)$, where X is a constant and $Z_t = z(t)$.

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$$\widetilde{y}(t) = \frac{1}{G(t)} \left(\int_0^t z(s)\,dG(s) + XG(T) \right)$$

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Let $\phi_t^0 = V_t(\phi) - \phi^1(t) S_t(\kappa)$, where $V_t(\phi) = \mathbb{E}_Q(Y|\mathcal{H}_t)$ and

$$Y = \mathbb{1}_{\{T \geq \tau\}} z(\tau) + \mathbb{1}_{\{T < \tau\}} X$$

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Then the self-financing strategy $\phi = (\phi^0, \phi^1)$ based on the savings account and the CDS is a replicating strategy.

Proof: The terminal value of the wealth is

$$Y = z(\tau)\mathbb{1}_{\{\tau<T\}} + X\mathbb{1}_{\{T<\tau\}}$$

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On the one hand

$$\begin{aligned} E(Y|\mathcal{H}_t) = Y_t &= z(\tau)\mathbb{1}_{\{\tau\le t\}} + \mathbb{1}_{\{\tau<t\}}\frac{1}{G(t)}\left(XG(T) + \int_0^t z(s)dG(s)\right) \\ &= \int_0^t z(s)dH_s + (1-H_t)\frac{1}{G(t)}\left(XG(T) + \int_0^t z(s)dG(s)\right) \end{aligned}$$

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hence $dY_t = (z(t) - \widetilde{y}(t))\, dM_t$ with $\widetilde{y}(t) = \frac{1}{G(t)}(\int_0^t z(s)dG(s) + XG(T))$.

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hence $dY_t = (z(t) - \widetilde{y}(t)) \, dM_t$ with $\widetilde{y}(t) = \frac{1}{G(t)} (\int_0^t z(s) dG(s) + XG(T))$.

On the other hand,

$dY_t = \phi_t^1 \left(dS_t(\kappa) - \kappa(1 - H_t) dt + \delta(t) dH_t \right) = \phi_t^1 (\delta(t) - S_{t-}(\kappa)) \, dM_t.$