Credit Risk IV

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IV. Hazard process Approach

1. General case
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General case

The model

Two kinds of information :

the information from the asset's prices, denoted as $(\mathcal{F}_t, t \geq 0)$

the information from the default time τ modeled by the filtration $\mathcal{H}_t$ generated by the default process $H_t = 1\!\!1_{\tau \leq t}$.

We denote by $\mathcal{G}_t \stackrel{def}{=} \mathcal{F}_t \vee \mathcal{H}_t$.

Key lemma

Any $\mathcal{G}_t$-random variable is equal, on the set $\{\tau > t\}$, to an $\mathcal{F}_t$-measurable random variable.

We denote by $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$ the conditional law of τ given the information $\mathcal{F}_t$, and $G_t = 1 - F_t$.

Let X be an $\mathcal{F}_T$-measurable integrable r.v. Then,

$$\mathbb{E}(X \mathbb{1}_{T<\tau}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}(X \mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}(X e^{-\Gamma_T}|\mathcal{F}_t).$$

where $\Gamma_t \stackrel{def}{=} -\ln(1 - F_t) = -\ln G_t$

Let h be an $\mathbf{F}$*-predictable process. Then,*

$$\mathbb{E}(h_\tau \mathbb{1}_{\tau<T}|\mathcal{G}_t) = h_\tau \mathbb{1}_{\{\tau<t\}} + \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}\left(\int_t^T h_u dF_u | \mathcal{F}_t\right).$$

Martingales

We assume for simplicity that F is continuous.

(i) The process $L_t \stackrel{def}{=} (1 - H_t)e^{\Gamma_t}$ *is a* **G**-***martingale***.

(ii) The process $M_t \stackrel{def}{=} H_t - \Gamma_{t\wedge\tau}$ *is a* **G** *-martingale as soon as* F *(or* Γ*) is increasing.*

The submartingale F admits a **decomposition** as $F = Z + A$ where Z is a **martingale** and A a **predictable increasing** process.

(iii) The process

$$M_t \stackrel{def}{=} H_t - \int_0^{t\wedge\tau} \frac{dA_u}{1 - F_u}$$

is a **G**-*martingale.*

Proofs: *The process* $L_t = (1 - H_t)e^{\Gamma_t}$ *is a* **G**-***martingale***.

From the key lemma, for $t > s$

$$\begin{aligned}
\mathbb{E}(L_t|\mathcal{G}_s) &= \mathbb{E}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}|\mathcal{G}_s) = \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}\mathbb{E}(\mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}|\mathcal{F}_s) \\
&= \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}\mathbb{E}(\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)e^{\Gamma_t}|\mathcal{F}_s) = \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s}\mathbb{E}(e^{-\Gamma_t}e^{\Gamma_t}|\mathcal{F}_s) \\
&= \mathbb{1}_{\{\tau>s\}}e^{\Gamma_s} = L_s
\end{aligned}$$

The process $M_t = H_t - \Gamma_{t\wedge\tau}$ is a $\mathbf{G}$ -martingale as soon as F (or Γ) is increasing.

From integration by parts formula :

$$dL_t = (1 - H_t)e^{\Gamma_t} d\Gamma_t - e^{\Gamma_t} dH_t$$

and the process $M_t = H_t - \Gamma(t \wedge \tau)$ can be written

$$M_t \stackrel{def}{=} \int_{]0,t]} dH_u - \int_{]0,t]} (1 - H_u) d\Gamma_u = -\int_{]0,t]} e^{-\Gamma_u} dL_u$$

and is a $\mathbf{G}$-martingale since L is $\mathbf{G}$-martingale.

The process

$$M_t = H_t - \int_0^{t\wedge\tau} \frac{dA_u}{1-F_u}$$

is a $\mathbf{G}$*-martingale.*

Let $s < t$. We give the proof in two steps, using the Doob-Meyer decomposition of F as $F_t = Z_t + A_t$.

First step: we prove

$$\mathbb{E}(H_t|\mathcal{G}_s) = H_s + \mathbb{1}_{s<\tau}\frac{1}{1-F_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

Indeed,

$$\begin{aligned}
\mathbb{E}(H_t|\mathcal{G}_s) &= 1 - \mathbb{P}(t < \tau|\mathcal{G}_s) = 1 - \mathbb{1}_{s<\tau}\frac{1}{1-F_s}\mathbb{E}(1 - F_t|\mathcal{F}_s) \\
&= 1 - \mathbb{1}_{s<\tau}\frac{1}{1-F_s}\mathbb{E}(1 - Z_t - A_t|\mathcal{F}_s) \\
&= 1 - \mathbb{1}_{s<\tau}\frac{1}{1-F_s}(1 - Z_s - A_s - \mathbb{E}(A_t - A_s|\mathcal{F}_s)) \\
&= 1 - \mathbb{1}_{s<\tau}\frac{1}{1-F_s}(1 - F_s - \mathbb{E}(A_t - A_s|\mathcal{F}_s)) \\
&= \mathbb{1}_{\tau\le s} + \mathbb{1}_{s<\tau}\frac{1}{1-F_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)
\end{aligned}$$

In a second step, we prove that, setting $\Lambda_t = \int_0^t (1 - H_s) \frac{dA_s}{1-F_s}$,

$$\mathbb{E}(\Lambda_{t\wedge\tau}|\mathcal{G}_s) = \Lambda_{s\wedge\tau} + \mathbb{1}_{s<\tau} \frac{1}{1-F_s} \mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

From the key formula,

$$\begin{aligned}
\mathbb{E}(\Lambda_{t\wedge\tau}|\mathcal{G}_s) &= \Lambda_{s\wedge\tau} \mathbb{1}_{\tau\leq s} + \mathbb{1}_{s<\tau} \frac{1}{1-F_s} \mathbb{E}\left(\int_s^\infty \Lambda_{t\wedge u} dF_u |\mathcal{F}_s \right) \\
&= \Lambda_{s\wedge\tau} \mathbb{1}_{\tau\leq s} + \mathbb{1}_{s<\tau} \frac{1}{1-F_s} \mathbb{E}\left(\int_s^t \Lambda_u dF_u + \int_t^\infty \Lambda_t dF_u |\mathcal{F}_s \right) \\
&= \Lambda_{s\wedge\tau} \mathbb{1}_{\tau\leq s} + \mathbb{1}_{s<\tau} \frac{1}{1-F_s} \mathbb{E}\left(\int_s^t \Lambda_u dF_u + \Lambda_t (1 - F_t) |\mathcal{F}_s \right)
\end{aligned}$$

We now use IP formula, using that Λ is bounded variation and continuous

$$d(\Lambda_t(1-F_t)) = -\Lambda_t dF_t + (1-F_t)d\Lambda_t = -\Lambda_t dF_t + dA_t$$

hence

$$\begin{aligned}\int_s^t \Lambda_u dF_u + \Lambda_t(1-F_t) &= -\Lambda_t(1-F_t) + \Lambda_s(1-F_s) + A_t - A_s + \Lambda_t(1-F_t)\\ &= \Lambda_s(1-F_s) + A_t - A_s\end{aligned}$$

From

$$\mathbb{E}(\Lambda_{t\wedge\tau}|\mathcal{G}_s) = \Lambda_{s\wedge\tau}1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}\left(\int_s^t \Lambda_u dF_u + \Lambda_t(1-F_t)|\mathcal{F}_s\right)$$

and

$$\int_s^t \Lambda_u dF_u + \Lambda_t(1-F_t) = \Lambda_s(1-F_s) + A_t - A_s$$

it follows that

$$\begin{aligned}\mathbb{E}(\Lambda_{t\wedge\tau}|\mathcal{G}_s) &= \Lambda_{s\wedge\tau}1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}\left(\Lambda_s(1-F_s) + A_t - A_s|\mathcal{F}_s\right)\\ &= \Lambda_{s\wedge\tau} + 1\!\!1_{s<\tau}\frac{1}{1-F_s}\mathbb{E}\left(A_t - A_s|\mathcal{F}_s\right)\ .\end{aligned}$$

If A is absolutely continuous wrt the Lebesgue measure, there exists an $\mathbf{F}$-adapted process γ, called the intensity such that the process

$$H_t - \int_0^{t\wedge\tau} \gamma_u du = H_t - \int_0^t (1 - H_u)\gamma_u du$$

is a $\mathbf{G}$-martingale. The process γ satisfies

$$\gamma_t = \lim_{h\to 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t + h|\mathcal{F}_t)}{\mathbb{P}(t < \tau|\mathcal{F}_t)}$$

Let $\widetilde{V}$ and R be **F**-predictable processes. The process

$$V_t = \widetilde{V}_t \mathbb{1}_{\{t<\tau\}} + R_\tau \mathbb{1}_{\{\tau \le t\}}$$

is a **G**-martingale if and only if the process

$$\widetilde{V}_t e^{-\Gamma_t} + \int_0^t R_u e^{-\Gamma_u} d\Gamma_u$$

is an **F**-martingale

PROOF: The direct part comes from the fact that if V is a $\mathbf{G}$-martingale, then $\mathbb{E}_{\mathbb{Q}}(V_t|\mathcal{F}_t)$ is an $\mathbf{F}$-martingale. The converse is an immediate application of the Key Lemma $\triangle$

Let P be the ex-dividend price process of a claim which delivers R_τ at default time and pays a cumulative coupon C till the default time, i.e. the discounted cum-dividend process

$$\beta_t P_t + 1\!\!1_{\{\tau \le t\}} \beta_\tau R_\tau + \int_0^{t\wedge\tau} \beta_u dC_u$$

is a **G**-martingale. Let $\widetilde{P}_t$ be the predefault price of the process P, i.e., $\tilde{P}$ is **F**-predictable and $P_t = 1\!\!1_{\{t<\tau\}} \widetilde{P}_t$. Then the process

$$P_t^* = \alpha_t \widetilde{P}_t + \int_0^t \alpha_s dC_s + \int_0^t R_u \alpha_u \, d\Gamma_u$$

is an **F**-martingale.

Conversely, if $\widetilde{V}$ is an **F**-predictable process such that the process $\alpha_t \widetilde{V}_t + \int_0^t \alpha_s dC_s + \int_0^t R_u \alpha_u \, d\Gamma_u$ is an **F**-martingale, then (the discounted cum-dividend) process

$$\beta_t \widetilde{V}_t \mathbb{1}_{\{t<\tau\}} + \mathbb{1}_{\{\tau \le t\}} \beta_\tau R_\tau + \int_0^{t\wedge\tau} \beta_u dC_u$$

is a **G**-martingale.

PROOF: This is an application of the previous Lemma . △

Computation in a restricted filtration

Let $\widetilde{\mathbf{F}} \subset \mathbf{F}$ and $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$.

From

$$F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$$

we deduce

$$\widetilde{F}_t = \mathbb{P}(\tau \leq t | \widetilde{\mathcal{F}}_t) = \mathbb{E}(F_t | \widetilde{\mathcal{F}}_t)$$

The computation of the intensity is more difficult, the $\widetilde{\mathbf{F}}$- intensity in the restricted filtration is not the conditional expectation of the $\mathbf{F}$-intensity

$(\mathcal{H})$ Hypothesis

Complete model case

Let S be a semi-martingale on $(\Omega, \mathcal{G}, \mathbb{P})$ such that there exists a **unique** probability $\mathbb{Q}$, equivalent to $\mathbb{P}$ on $\mathcal{F}_T$, where $\mathcal{F}_t = \mathcal{F}_t^S = \sigma(S_s, s \leq t)$ such that $(\widetilde{S}_t = S_t R_t, 0 \leq t \leq T)$ is an $\mathbf{F}^S$-martingale under the probability Q.
We assume that there exists a probability $\tilde{\mathbb{Q}}$, equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ such that $(\widetilde{S}_t, 0 \leq t \leq T)$ is a $\mathbf{G}$-martingale under the probability $\tilde{\mathbb{Q}}$.
Then, **any $(\mathbf{F}, \mathbb{Q})$-martingale is a $(\mathbf{G}, \mathbb{Q})$-martingale** and the restriction of $\tilde{\mathbb{Q}}$ to $\mathcal{F}_T$ is equal Q.

Definition and Properties of immersion

We shall now examine the immersion property (or $(\mathcal{H})$-hypothesis) which reads:

$(\mathcal{H})$ **Every F square-integrable martingale is a G square integrable martingale.**

This hypothesis implies that the **F**-Brownian motion remains a Brownian motion in the enlarged filtration and that every **F**-local martingale is a **G**-local martingale .

Assume that $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$, where $\mathbf{F}$ is an arbitrary filtration and $\mathbf{H}$ is generated by the process $H_t = \mathbb{1}_{\{\tau \leq t\}}$. Then the following conditions are equivalent to the hypothesis $(\mathcal{H})$.

(i) For any $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_\infty).$$

(ii) For any $t \in \mathbb{R}_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$ under $\mathbb{P}$, that is,

$$\mathbb{E}_{\mathbb{P}}(\xi\,\eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)\, \mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t)$$

for any bounded, $\mathcal{F}_\infty$-measurable random variable ξ and bounded, $\mathcal{G}_t$-measurable random variable η.

(iii) For any $t \in \mathbb{R}_+$ and any bounded, $\mathcal{F}_\infty$-measurable random variable ξ: $\mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)$.

Change of a probability measure

Kusuoka shows, by means of a counter-example, that the hypothesis $(\mathcal{H})$ is not invariant with respect to an equivalent change of the underlying probability measure, in general.

Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ for every $t \in I\!R_+$, with the associated Radon-Nikodým density process η. If the **density process η is F-adapted** then we have

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_t)$$

for every $t \in I\!R_+$. Hence, the hypothesis $(\mathcal{H})$ is also valid under $\mathbb{Q}$ and the **F**-intensities of τ under $\mathbb{Q}$ and under $\mathbb{P}$ coincide.

PROOF:

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \le t\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_t)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \le t\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_\infty)}$$

$$= \frac{\mathbb{E}_{\mathbb{P}}(\eta_\infty \mathbb{1}_{\{\tau \le t\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{F}_\infty)} = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_\infty).$$

Stochastic Barrier

Suppose that

$$P(\tau \leq t | \mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where Γ is an arbitrary continuous strictly increasing $\mathbf{F}$-adapted process. There exists a random variable Θ, independent of $\mathcal{F}_\infty$, with exponential law of parameter 1, such that $\tau \stackrel{law}{=} \inf\{t \geq 0 : \Gamma_t > \Theta\}$. In fact $\Theta \stackrel{def}{=} \Gamma_\tau$.

PROOF: : Suppose that

$$P(\tau \leq t|\mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where Γ is an arbitrary continuous strictly increasing **F**-adapted process. Let us set $\Theta \stackrel{def}{=} \Gamma_\tau$. Then

$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where C is the right inverse of Γ, so that $\Gamma_{C_t} = t$. Therefore

$$P(\Theta > u|\mathcal{F}_\infty) = e^{-\Gamma_{C_u}} = e^{-u}.$$

We have thus established the required properties, namely, the probability law of Θ and its independence of the σ-field $\mathcal{F}_\infty$. Furthermore, $\tau = \inf\{t : \Gamma_t > \Gamma_\tau\} = \inf\{t : \Gamma_t > \Theta\}$.

Representation theorem

Kusuoka establishes the following representation theorem. Under $(\mathcal{H})$, any $\mathbf{G}$-square integrable martingale admits a representation as the sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale M.

Suppose that hypothesis $(\mathcal{H})$ holds under $\mathbb{P}$ and that any **F**-martingale is continuous. Then, the martingale $M_t^h = \mathbb{E}_{\mathbb{P}}(h_\tau | \mathcal{G}_t)$, where h is an **F**-predictable process such that $\mathbb{E}(h_\tau) < \infty$, admits the following decomposition

$$M_t^h = m_0^h + \int_0^{t\wedge\tau} e^{\Gamma_u} dm_u^h + \int_{]0,t\wedge\tau]} (h_u - J_u)\, dM_u,$$

where m^h is the continuous **F**-martingale

$$m_t^h = \mathbb{E}_{\mathbb{P}}\Big(\int_0^\infty h_u dF_u \,|\, \mathcal{F}_t \Big),$$

$J_t = e^{\Gamma_t}(m_t^h - \int_0^t h_u dF_u)$ and M is the discontinuous **G**-martingale $M_t = H_t - \Gamma_{t\wedge\tau}$.

PROOF: : We know that

$$\begin{aligned} M_t^h &= \mathbb{E}(h_\tau \,|\, \mathcal{G}_t) \\ &= \mathbb{1}_{\{\tau \le t\}} h_\tau + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}\Big(\int_t^\infty h_u dF_u \,\Big|\, \mathcal{F}_t \Big) \\ &= \mathbb{1}_{\{\tau \le t\}} h_\tau + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \Big(m_t^h - \int_0^t h_u dF_u \Big) \\ &= \int_0^t h_u dH_u + \mathbb{1}_{\{\tau > t\}} J_t \,. \end{aligned}$$

From the facts that Γ is an increasing process

m^h a continuous martingale

and using the integration by parts formula, we deduce that

$$dJ_t = e^{\Gamma_t} dm_t^h + (J_t - h_t) \frac{dF_t}{1 - F_t}$$

Partial information

As pointed out by Jamshidian, *"one may wish to apply the general theory perhaps as an intermediate step, to a subfiltration that is not equal to the default-free filtration. In that case,* $\mathbf{F}$ *rarely satisfies hypothesis* $(\mathcal{H})$*"*.

Information at discrete times

Assume that

$$dV_t = V_t(\mu dt + \sigma dW_t),\ V_0 = v$$

i.e., $V_t = ve^{\sigma(W_t+\nu t)} = ve^{\sigma X_t}$. The default time is assumed to be the first hitting time of α with $\alpha < v$, i.e.,

$$\tau = \inf\{t : V_t \le \alpha\} = \inf\{t : X_t \le a\}$$

where $a = \sigma^{-1}\ln(\alpha/v)$.

Here, $\mathbf{F}$ is the filtration of the observations of V at discrete times $t_1, \cdots t_n$ where $t_n \leq t < t_{n+1}$, i.e.,

$$\mathcal{F}_t = \sigma(V_{t_1}, \cdots, V_{t_n}, t_i \leq t)$$

The process $F_t = P(\tau \leq t | \mathcal{F}_t)$ F is **continuous and increasing in** $[t_i, t_{i+1}[$ but is **not increasing**.

Lemma 0.1 *The process ζ defined by*

$$\zeta_t = \sum_{i, t_i \leq t} \Delta F_{t_i}.$$

is an $\mathbf{F}$-martingale.

The Doob-Meyer decomposition of F is

$$F_t = \zeta_t + (F_t - \zeta_t),$$

where ζ is an $\mathbf{F}$-martingale and $F_t - \zeta_t$ is a predictable increasing process.

From

$$P(\inf_{s\le t} X_s > z) = \Phi(\nu, t, z)\,,$$

where

$$\begin{aligned}
\Phi(\nu,t,z) &= \mathcal{N}\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z}\mathcal{N}\left(\frac{z+\nu t}{\sqrt{t}}\right), \qquad \text{for } z<0,\ t>0,\\
&= 0, \qquad \text{for } z\ge 0,\ t\ge 0,\\
\Phi(\nu,0,z) &= 1, \qquad \text{for } z<0
\end{aligned}$$

we obtain (we skip the parameter ν in the definition of Φ) for $t_1 < t < t_2$ and $X_{t_1} > a$

$$F_t = 1 - \Phi(t-t_1, a - X_{t_1})\left[1 - \exp\left(-\frac{2\,a}{t_1}\,(a - X_{t_1})\right)\right].$$

The case $X_{t_1} \le a$ corresponds to default: for $X_{t_1} \le a$, $F_t = 1$.

Another example, related with Parisian stopping times is presented in Çetin et al.

Delayed information

Guo et al. suggested to start form a structural model with delayed information, i.e. the reference filtration is $\mathcal{F}_t = \sigma(S_u, u \leq t - \delta)$. In that case, (H) hypothesis is not satisfied.

Intensity approach

In the so-called intensity approach, the default time τ is a $\mathbf{G}$-stopping time. The intensity is defined as any non-negative process λ, such that

$$M_t \stackrel{def}{=} H_t - \int_0^{t\wedge\tau} \lambda_s ds$$

is a $\mathbf{G}$-martingale.

The existence of the intensity relies on the fact that H is a sub-martingale and can be written as $M + A$ where M is a martingale M and A a predictable increasing process. The increasing process A is such that $A_t 1\!\!1_{t\geq\tau} = A_\tau 1\!\!1_{t\geq\tau}$.

The intensity exists only if τ is a totally inaccessible stopping time.

We emphasize that, in that setting the intensity is not well defined after time τ, i.e., if λ is an intensity, for any non-negative predictable process g the process $\tilde{\lambda}_t = \lambda_t 1\!\!1_{t\leq\tau} + g_t 1\!\!1_{\{t>\tau\}}$ is also an intensity.

If the process $Y_t = \mathbb{E}\left(X \exp\left(-\int_0^T \lambda_u du\right) |\mathcal{G}_t\right)$ ***is continuous at time*** τ**,** *then, setting* $L_t = 1\!\!1_{\{t<\tau\}} e^{\Gamma_t}$

$$\mathbb{E}(X 1\!\!1_{\{T<\tau\}} |\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} \mathbb{E}\left(X \exp\left(-\int_t^T \lambda_u du\right) |\mathcal{G}_t\right) = L_t Y_t$$

If Y *is not continuous*

$$\mathbb{E}(X 1\!\!1_{\{T<\tau\}} |\mathcal{G}_t) = L_t Y_t - \mathbb{E}(\Delta Y_\tau 1\!\!1_{\tau<T} |\mathcal{G}_t) \,.$$

It can be mentioned that the continuity of the process depends on the choice of λ after time τ.

If the process Y is not continuous, then setting

$$U_t = L_t Y_t = \mathbb{1}_{t<\tau} \exp\left(\int_0^t \lambda_s ds\right) \mathbb{E}\left(X \exp\left(-\int_0^T \lambda_u du\right) |\mathcal{G}_t\right)$$

we have $U_T = X\mathbb{1}_{\{T<\tau\}}$ and

$$dU_t = L_{t-}dY_t + Y_{t-}dL_t + d[L,Y]_t = L_{t-}dY_t + Y_{t-}dL_t + \Delta L_t \Delta Y_t$$

and

$$\mathbb{E}(U_T|\mathcal{G}_t) = \mathbb{E}(X\mathbb{1}_{\{T<\tau\}}|\mathcal{G}_t) = U_t - \mathbb{E}(\Delta Y_\tau \mathbb{1}_{\tau<T}|\mathcal{G}_t)\,.$$

Then, for any $X \in \mathcal{G}_T$:

$$\mathbb{E}(X \mathbb{1}_{T<\tau} | \mathcal{G}_t) = \mathbb{1}_{\tau>t} \left(e^{\Lambda_t} \mathbb{E}(e^{-\Lambda_T} X | \mathcal{G}_t) - \mathbb{E}(e^{\Lambda_\tau} \Delta Y_\tau \mathbb{1}_{\tau<T} | \mathcal{G}_t) \right)$$

where $Y_t = \mathbb{E}\left(X \exp\left(-\Lambda_T\right) | \mathcal{G}_t\right)$ and $\Lambda_t = \int_0^t \lambda_u du$

CDS Price, General case

The ex-dividend price of a credit default swap, with a rate process κ and a protection payment δ_τ at default, equals, for every $t \in [s, T]$,

$$
\begin{aligned}
S_t(\kappa) &= \mathbb{E}_Q\Big(\mathbb{1}_{\{t<\tau\le T\}}\delta_\tau \,\Big|\, \mathcal{G}_t\Big) - \mathbb{E}_Q\Big(\mathbb{1}_{\{t<\tau\}}\int_t^{\tau\wedge T}\kappa_s ds \,\Big|\, \mathcal{G}_t\Big), \\
&= \mathbb{1}_{\{t<\tau\}}\frac{1}{G_t}\,\mathbb{E}_Q\left(-\int_t^T \delta_u\, dG_u + \int_t^\infty dG_u \int_t^{u\wedge T}\kappa_v\, dv \,\Big|\, \mathcal{F}_t\right).
\end{aligned}
$$

We now assume that **(H) hypothesis holds** between $\mathbf{F}$ and $\mathbf{G}$, that is $\mathbf{F}$-martingales are $\mathbf{G}$-martingales. Then, F is increasing and the process

$$M_t = H_t - \int_0^{t\wedge\tau} \gamma_u \, du,$$

with $\gamma_t dt = \frac{dF_t}{G_t}$ is a $\mathbf{G}$-martingale.

The dynamics of the ex-dividend price $S_t(\kappa)$ are

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1-H_t)B_t G_t^{-1}\, dm_t + (1-H_t)(r_t S_t(\kappa) + \kappa - \delta_t \gamma_t)\, dt,$$

where m is the $(Q, \mathbf{F})$-martingale given by

$$m_t = \mathbb{E}_Q \left(\int_0^T B_u^{-1} \delta_u G_u \gamma_u \, du - \kappa \int_0^T B_u^{-1} G_u \, du \,\bigg|\, \mathcal{F}_t \right).$$

Hedging defaultable claims

Our aim is to hedge

$$Y = \mathbb{1}_{\{T \geq \tau\}} Z_\tau + \mathbb{1}_{\{T < \tau\}} X.$$

using two CDS with maturities T_i, rates κ_i and protection payment δ^i. We assume $r = 0$. Let ζ_t^i defined as

$$m_t^i = \mathbb{E}_Q \left(\int_0^T \delta_u^i G_u \gamma_u \, du - \kappa_i \int_0^T G_u \, du \,\bigg|\, \mathcal{F}_t \right) \,,\; dm_t^i = \zeta_t^i dW_t$$

and

$$m_t^Z = \mathbb{E}_Q(-\int_0^\infty Z_u dG_u + G_T X | \mathcal{F}_t),\; dm_t^Z = \zeta_t^Z dW_t$$

Assume that there exist **F**-predictable processes ϕ^1, ϕ^2 such that

$$\sum_{i=1}^{2} \phi_t^i \big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big) = Z_t - \widetilde{y}_t, \quad \sum_{i=1}^{2} \phi_t^i \zeta_t^i = \zeta_t,$$

where $\widetilde{y}$ is given by

$$\widetilde{y}_t = \frac{1}{G_t} \mathbb{E}_Q \left(- \int_t^T Z_u \, dG_u + G_T X \,\Big|\, \mathcal{F}_t \right).$$

Let $\phi_t^0 = V_t(\phi) - \sum_{i=1}^{2} \phi_t^i S_t^i(\kappa_i)$, where the process $V(\phi)$ is given by

$$dV_t(\phi) = \sum_{i=1}^{2} \phi_t^i \big(dS_t^i(\kappa_i) + dD_t^i\big)$$

with the initial condition $V_0(\phi) = \mathbb{E}_Q(Y)$. Then the self-financing trading strategy $\phi = (\phi^0, \phi^1, \phi^2)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$.