Credit Risk VI.

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Several default times

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Basket Credit Derivatives

Basket credit derivatives are credit derivatives deriving their cash flows values (and thus their values) from credit risks of several reference entities (or prespecified credit events).

Standing assumptions. We assume that:

- We are given a collection of default times $\tau_1, \ldots, \tau_n$ defined on a common probability space $(\Omega, \mathcal{G}, Q)$.
- $Q\{\tau_i = 0\} = 0$ and $Q\{\tau_i > t\} > 0$ for every i and t.
- $Q\{\tau_i = \tau_j\} = 0$ for arbitrary $i \neq j$ (in a continuous time setup).

We associate with the collection $\tau_1, \ldots, \tau_n$ of default times the ordered sequence $\tau_{(1)} < \tau_{(2)} < \cdots < \tau_{(n)}$, where $\tau_{(i)}$ stands for the random time of the i^{th} default. Formally,

$$\tau_{(1)} = \min\{\tau_1, \tau_2, \ldots, \tau_n\}$$

and for $i = 2, \ldots, n$

$$\tau_{(i)} = \min\{\tau_k : k = 1, \ldots, n,\ \tau_k > \tau_{(i-1)}\}.$$

In particular,

$$\tau_{(n)} = \max\{\tau_1, \tau_2, \ldots, \tau_n\}.$$

The i^{th}-to-Default Contingent Claims

A general i^{th}-to-default contingent claim which matures at time T is specified by the following covenants:

- If $\tau_{(i)} = \tau_k \leq T$ for some $k = 1, \ldots, n$ it pays at time $\tau_{(i)}$ the amount $Z^k_{\tau_{(i)}}$ where Z^k is an $\mathbf{F}$-predictable recovery process.
- If $\tau_{(i)} > T$ it pays at time T an $\mathcal{F}_T$-measurable promised amount X.

Case of Two Entities

For the sake of notational simplicity, we shall frequently consider the case of two reference credit risks.

Cash flows of the first-to-default contract (FDC):

- If $\tau_{(1)} = \min\{\tau_1, \tau_2\} = \tau_i \le T$ for $i = 1, 2$, the claim pays at time τ_i the amount $Z^i_{\tau_i}$.
- If $\min\{\tau_1, \tau_2\} > T$, it pays at time T the amount X.

Cash flows of the last-to-default contract (LDC):

- If $\tau_{(2)} = \max\{\tau_1, \tau_2\} = \tau_i \le T$ for $i = 1, 2$, the claim pays at time τ_i the amount $Z^i_{\tau_i}$.
- If $\max\{\tau_1, \tau_2\} > T$, it pays at time T the amount X.

Values of FDC and LDC

The value at time t of the **FDC** equals:

$$\begin{aligned} S_t^{(1)} &= \beta_t \, E_Q\Big(\beta_{\tau_1}^{-1} Z_{\tau_1}^1 \, \mathbb{1}_{\{\tau_1<\tau_2,\, t<\tau_1\leq T\}} \,\Big|\, \mathcal{G}_t\Big) \\ &\quad + \beta_t \, E_Q\Big(\beta_{\tau_2}^{-1} Z_{\tau_2}^2 \, \mathbb{1}_{\{\tau_2<\tau_1,\, t<\tau_2\leq T\}} \,\Big|\, \mathcal{G}_t\Big) \\ &\quad + \beta_t \, E_Q\Big(\beta_T^{-1} X \, \mathbb{1}_{\{T<\tau_{(1)}\}} \,\Big|\, \mathcal{G}_t\Big). \end{aligned}$$

The value at time t of the **LDC** equals:

$$\begin{aligned} S_t^{(2)} &= \beta_t \, E_Q\Big(\beta_{\tau_1}^{-1} Z_{\tau_1}^1 \, \mathbb{1}_{\{\tau_2<\tau_1 ,\, t<\tau_1\le T\}} \,\Big|\, \mathcal{G}_t\Big) \\ &\quad + \beta_t \, E_Q\bigg(\beta_{\tau_2}^{-1} Z_{\tau_2}^2 \, \mathbb{1}_{\{\tau_1<\tau_2 ,\, t<\tau_2\le T\}} \,\Big|\, \mathcal{G}_t\bigg) \\ &\quad + \beta_t \, E_Q\Big(\beta_T^{-1} X \, \mathbb{1}_{\{T<\tau_{(2)}\}} \,\Big|\, \mathcal{G}_t\Big). \end{aligned}$$

Independent Default Times

Suppose that $\tau_1, \ldots, \tau_n$ are independent random times under P. Let $F_k(t) = P\{\tau_k \le t\}$ and $\tau_{(1)} < \cdots < \tau_{(1)}$ the ranked sequence of the τ_i's. The cumulative distribution functions of $\tau_{(1)}$ and $\tau_{(n)}$ are:

$$F_{(1)}(t) = P\{\tau_{(1)} \le t\} = 1 - \prod_{k=1}^{n} (1 - F_k(t))$$

and

$$F_{(n)}(t) = P\{\tau_{(n)} \le t\} = \prod_{k=1}^{n} F_k(t).$$

Suppose, in addition, that the default times $\tau_1, \ldots, \tau_n$ admit deterministic intensity functions $\gamma_1(t), \ldots, \gamma_n(t)$, such that

$$H_t^i - \int_0^{t \wedge \tau_i} \gamma_i(s) ds$$

are $\mathbf{H}^i$-martingales. Then,

$$P\{\tau_{(1)} > t\} = \prod P\{\tau_i > t\} = e^{-\int_0^t \gamma_{(1)}(v)\, dv}.$$

where

$$\gamma_{(1)}(t) = \gamma_1(t) + \ldots + \gamma_n(t)$$

hence

$$H_t^{(1)} - \int_0^{t \wedge \tau_{(1)}} \gamma_{(1)}(t) dt$$

is a $\mathbf{H}^{(1)}$-martingale, where $\mathcal{H}_t^{(1)} = \sigma(\tau_{(1)} \wedge t)$.

Copula-Based Approaches

The concept of a *copula function* allows to produce various multidimensional probability distributions with prespecified univariate marginal laws.

A function $C : [0,1]^n \to [0,1]$ is called a **copula** if the following conditions are satisfied:

(i) $C(1, \ldots, 1, v_i, 1, \ldots, 1) = v_i$ for any i and any $v_i \in [0,1]$,

(ii) $C(u_1, \ldots, u_n)$ is increasing with respect to each component u_i

(iii) For any $a, b \in [0,1]^n$ with $a \le b$ (i.e., $a_i \le b_i$, $\forall i$)

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\ldots+i_n} C(u_{1,i_1}, \ldots, u_{n,i_n}) \ge 0\,,$$

where $u_{j,1} = a_j$, $u_{j,2} = b_j$.

Let us give few examples of copulas:

- Product copula: $\Pi(u_1, \ldots, u_n) = \Pi_{i=1}^n u_i$,
- Gumbel copula: for $\theta \in [1, \infty)$ we set

$$C(u_1, \ldots, u_n) = \exp\left(- \left[\sum_{i=1}^n (-\ln u_i)^\theta \right]^{1/\theta} \right),$$

- Gaussian copula:

$$C(u_1, \ldots, u_n) = N_\Sigma^n \left(N^{-1}(u_1), \ldots, N^{-1}(u_n) \right),$$

  where N_Σ^n is the c.d.f for the n-variate central normal distribution with the linear correlation matrix Σ, and N^{-1} is the inverse of the c.d.f. for the univariate standard normal distribution.

Sklar Theorem:

For any cumulative distribution function F on $\mathbb{R}^n$ there exists a copula function C such that

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

where F_i is the i^{th} marginal cumulative distribution function.

If, in addition, F is continuous then C is unique.

Direct Application

Let F_i be the probability distribution for τ_i. A copula function C is chosen in order to introduce a dependence structure of the random vector $(\tau_1, \tau_2, \ldots, \tau_n)$. The joint distribution of the random vector $(\tau_1, \tau_2, \ldots, \tau_n)$ is derived by

$$P\{\tau_i \le t_i,\, i = 1, 2, \ldots, n\} = C\big(F_1(t_1), \ldots, F_n(t_n)\big).$$

Indirect Application

Assume that the cumulative distribution function of $(\xi_1, \dots, \xi_n)$ is given by an n-dimensional copula C, and that the univariate marginal laws are uniform on $[0, 1]$. We postulate that $(\xi_1, \dots, \xi_n)$ are independent of $\mathbf{F}$, and we set

$$\tau_i = \inf \{ t : \Gamma_t^i \geq - \ln \xi_i \}.$$

Then, $\{\tau_i > t_i\} = \{e^{-\Gamma^i_{t_i}} > \xi_i\}$.

Then:

- The case of default times conditionally independent with respect to $\mathbf{F}$ corresponds to the choice of the product copula Π. In this case, for $t_1, \ldots, t_n \leq T$ we have

$$P\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = \Pi(Z^1_{t_1}, \ldots, Z^n_{t_n}),$$

  where we set $Z^i_t = e^{-\Gamma^i_t}$.

- In general, for $t_1, \ldots, t_n \leq T$ we obtain

$$P\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = C(Z^1_{t_1}, \ldots, Z^n_{t_n}),$$

  where C is the copula used in the construction of $\xi_1, \ldots, \xi_n$.

An example

This example describes the use of one-factor Gaussian copula (Bank of International Settlements (BIS) standard).

Let q_i be a decreasing function taking values in $[0,1]$ with $q_i(0) = 1$.

$$\tau_i = \inf\{t : q_i(t) < U_i\}$$

Then, $q_i(t) = P(\tau_i > t) = 1 - p_i(t)$.

Correlation specification of the thresholds U_i: Let $Y_1, \cdots, Y_n$ and Y be independent random variables and $X_i = \rho_i Y + \sqrt{1-\rho_i^2} Y_i$.

The default thresholds are defined by $U_i = 1 - F_i(X_i)$ where F_i is the cumulative distribution function of X_i. Then

$$\tau_i = \inf\left\{t : \rho_i Y + \sqrt{1-\rho_i^2} Y_i \leq F_i^{-1}(1 - q_i(t))\right\}.$$

Conditioned on the common factor Y,

$$p^i(t|Y) = F_i^Y\left(\frac{F_i^{-1}(p_i(t)) - \rho_i Y}{\sqrt{1-\rho_i^2}}\right)$$

where F_i^Y is the cumulative distribution function of Y_i.

Let us consider the particular case where

$$X_i = \rho_i Y + \sqrt{1-\rho_i^2} Y_i,$$

where Y, Y_i, $i = 1, 2, \ldots, n$, are independent standard Gaussian variables. In that case, X_i is also a standard Gaussian law and

$$p^i(t|Y) = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(p_i(t)) - \rho_i Y}{\sqrt{1-\rho_i^2}}\right)$$

and

$$P(\tau_i \le t_i, \forall i \le n) = \int \prod_i \mathcal{N}\left(\frac{\mathcal{N}^{-1}(F_i(t_i)) - \rho_i y}{\sqrt{1-\rho_i^2}}\right) f(y) dy \,.$$

where f is the density of Y

The cumulative loss on the underlying portfolio is
$L_t = \sum_{i=1}^{n} N_i(1 - R_i)1\!\!1_{\tau_i \leq t}$ where N_i is the nominal value of each firm and R_i is the (constant) recovery rate. The first defaults only affect the equity tranche until the cumulative loss has arrived the total nominal amount of the equity tranche and the loss on the tranche is given by

$$L_t^E = L_t 1\!\!1_{0,n^E}(L_t) + n^E 1\!\!1_{n^E,\infty}(L_t) = L_t - (L_t - n^E)^+$$

Conditional on the common factor Y , we can rewrite

$$L_T = \sum N_i(1 - R_i)1\!\!1_{Y_i \leq \,(\,F_i^{-1}(p_i(T)) - \rho_i Y)\,/\,(1-\rho_i^2)}$$

Hence, the conditional total loss L_T on the factor Y can be written as the sum of independent Bernoulli random variables, each with probability $p_i(T|Y)$

Survival Intensities

For arbitrary $s \leq t$ on the set $\{\tau_1 > s, \ldots, \tau_n > s\} = \{\tau_{(1)} > s\}$ we have

$$P\{\tau_i > t \,|\, \mathcal{G}_s\} = E_P \left(\frac{C(Z_s^1, \ldots, Z_t^i, \ldots, Z_s^n)}{C(Z_s^1, \ldots, Z_s^n)} \,\Big|\, \mathcal{F}_s \right).$$

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Proof: The proof is straightforward, and follows from the key lemma

$$P\{\tau_i > t \,|\, \mathcal{G}_s\} 1\!\!1_{\{\tau_{(1)} > s\}} = 1\!\!1_{\{\tau_{(1)} > s\}} \frac{P(\tau_1 > s, \ldots, \tau_i > t, \ldots, \tau_n > s \,|\, \mathcal{F}_s)}{P(\tau_1 > s, \ldots, \tau_i > s, \ldots, \tau_n > s \,|\, \mathcal{F}_s)}$$

△

Consequently, assuming that the derivatives $\gamma_t^i = \frac{d\Gamma_t^i}{dt}$ exist, the i^{th} intensity of survival equals, on the set $\{\tau_1 > t, \ldots, \tau_n > t\}$,

$$\lambda_t^i = \gamma_t^i \, Z_t^i \, \frac{\frac{\partial}{\partial v_i} \, C(Z_t^1, \ldots, Z_t^n)}{C(Z_t^1, \ldots, Z_t^n)} = \gamma_t^i \, Z_t^i \, \frac{\partial}{\partial v_i} \ln C(Z_t^1, \ldots, Z_t^n),$$

where λ_t^i is understood as the limit:

$$\lambda_t^i = \lim_{h \downarrow 0} h^{-1} \, Q\{t < \tau_i \le t + h \,|\, \mathcal{F}_t, \tau_1 > t, \ldots, \tau_n > t\}.$$

It appears that, in general, the i^{th} intensity of survival jumps at time t, if the j^{th} entity defaults at time t for some $j \neq i$. In fact, it holds that

$$\lambda_t^{i,j} = \gamma_t^i \, Z_t^i \, \frac{\frac{\partial^2}{\partial v_i \partial v_j} C(Z_t^1, \ldots, Z_t^n)}{\frac{\partial}{\partial v_j} C(Z_t^1, \ldots, Z_t^n)} \,,$$

where

$$\lambda_t^{i,j} = \lim_{h \downarrow 0} h^{-1} \, Q\{t < \tau_i \leq t + h \,|\, \mathcal{F}_t, \tau_k > t, k \neq j, \tau_j = t\}.$$

Schönbucher and Schubert (2001) also examine the intensities of survival after the default times of some entities. Let us fix s, and let $t_i \leq s$ for $i = 1, 2, \dots, k < n$, and $T_i \geq s$ for $i = k+1, k+2, \dots, n$. Then,

$$Q\Big\{\tau_i > T_i,\, i = k+1, k+2, \dots, n \,|\, \mathcal{F}_s,\, \tau_j = t_j,\, j = 1, 2, \dots, k,$$

$$\tau_i > s,\, i = k+1, k+2, \dots, n\Big\}$$

$$= \frac{E_Q\left(\frac{\partial^k}{\partial v_1 \dots \partial v_k} C(Z^1_{t_1}, \dots, Z^k_{t_k}, Z^{k+1}_{T_{k+1}}, \dots, Z^n_{T_n}) \,\Big|\, \mathcal{F}_s \right)}{\frac{\partial^k}{\partial v_1 \dots \partial v_k} C(Z^1_{t_1}, \dots, Z^k_{t_k}, Z^{k+1}_s, \dots, Z^n_s)}.$$

Toy model

We study the case with two random times such that $P(\tau_1 = \tau_2) = 0$

- $(H^i_t, t \geq 0)$ is the default processes associated with τ_i,
- $\mathbf{H}^i$ is the completed filtration generated by the process H^i
- $\mathbf{H}$ is the completed filtration generated by the processes H^1 and H^2,

$$\mathcal{H}_t = \sigma(\tau_1 \wedge t) \vee \sigma(\tau_2 \wedge t)$$

.

Let $G(t, s) = P(\tau_1 > t, \tau_2 > s)$. Then,

$$P(t < \tau_1 | \tau_2) = \frac{\partial_2 G(t, \tau_2)}{\partial_2 G(0, \tau_2)}$$

Let $\tau_{(1)} = \inf(\tau_1, \tau_2)$ and $\tau_{(2)} = \sup(\tau_1, \tau_2)$

A $\mathcal{H}_t$-measurable random variable is equal to

- a constant on the set $t < \tau_{(1)}$,
- a $\sigma(\tau_{(1)})$-measurable random variable on the set $\tau_{(1)} \leq t < \tau_{(2)}$,
- a $\sigma(\tau_{(1)}, \tau_{(2)})$-measurable random variable on the set $\tau_{(2)} \leq t$.

Note that **H** is strictly greater than the filtration generated by $H_t^1 + H_t^2$.

Exemple: assume that τ_1 and τ_2 are independent and identically distributed. Then, obviously, for $u < t$

$$P(\tau_1 < \tau_2 | \tau_{(1)} = u, \tau_{(2)} = t) = 1/2\,,$$

hence $\sigma(\tau_1, \tau_2) \neq \sigma(\tau_{(1)}, \tau_{(2)})$.

Pricing in a toy model

Noting

$$\begin{aligned} DZC_t^i &= E(\mathbb{1}_{\{T<\tau_i\}}|\mathcal{H}_t) \\ CD_t &= E(\delta_1 \mathbb{1}_{\{0<\tau_{(1)}\le T\}} + \delta_2 \mathbb{1}_{\{0<\tau_{(2)}\le T\}}|\mathcal{H}_t) \end{aligned}$$

we obtain

$$\begin{aligned} DZC_t^1 &= \mathbb{1}_{\{\tau_1>t\}} \left(\mathbb{1}_{\{\tau_2\le t\}} \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} + \mathbb{1}_{\{\tau_2>t\}} \frac{G(T,t)}{G(t,t)} \right) \\ DZC_t^2 &= \mathbb{1}_{\{\tau_2>t\}} \left(\mathbb{1}_{\{\tau_1\le t\}} \frac{\partial_1 G(\tau_1,T)}{\partial_2 G(\tau_1,t)} + \mathbb{1}_{\{\tau_1>t\}} \frac{G(t,T)}{G(t,t)} \right) \end{aligned}$$

$$
\begin{aligned}
CD_t \quad = \quad & \delta_1 1\!\!1_{\{\tau_{(1)}>t\}} \left(\frac{G(t,t) - G(T,T)}{G(t,t)} \right) + \delta_2 1\!\!1_{\{\tau_{(2)}\le t\}} + \delta_1 1\!\!1_{\{\tau_{(1)}\le t\}} \\
& + \delta_2 1\!\!1_{\{\tau_{(2)}>t\}} \left\{ I_t(0,1) \left(1 - \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} \right) + I_t(1,0) \left(1 - \frac{\partial_1 G(\tau_1,T)}{\partial_1 G(\tau_1,t)} \right) \right. \\
& \qquad \left. + I_t(0,0) \left(1 - \frac{G(t,T) + G(T,t) - G(T,T)}{G(t,t)} \right) \right\}
\end{aligned}
$$

where

$$
I_t(1,1) = 1\!\!1_{\{\tau_1\le t, \tau_2 \le t\}} \ , \qquad I_t(0,0) = 1\!\!1_{\{\tau_1 > t, \tau_2 > t\}}
$$

$$
1\!\!1_t(1,0) = 1\!\!1_{\{\tau_1\le t, \tau_2 > t\}} \ , \qquad I_t(0,1) = 1\!\!1_{\{\tau_1 > t, \tau_2 \le t\}}
$$

First to default claims

A **first-to-default claim** (an *FtD claim*, for short) on a basket of 2 credit names is a defaultable claim $(X, 0, Z, \tau_{(1)})$, where X is a constant amount payable at maturity if no default occurs, and $Z = (Z_1, Z_2)$, where a **function** $Z_i : [0, T] \to \mathbb{R}$ specifies the recovery payment made at the time τ_i if the ith firm was the first defaulted firm, that is, on the event $\{\tau_i = \tau_{(1)} \leq T\}$.

The pre-default price of a FtD claim $(X, 0, Z, \tau_{(1)})$, where $Z = (Z_1, Z_2)$, equals

$$\frac{1}{G(t,t)}\left(-\int_t^T Z_1(u)\,G(du,u) - \int_t^T Z_2(v)G(v,dv) + XG(T,T)\right).$$

The pre-default price of a LtD claim $(X, 0, Z, \tau_{(2)})$ is

$$\frac{1}{G(t,t)}\left(\int_t^T Z_1(u)\,F(du,u) + \int_t^T Z_2(v)F(v,dv) + XF(T,T)\right).$$

Martingales

- **Filtration $\mathbf{H}^i$** The processes

$$M_t^i = H_t^i - \int_0^{t \wedge \tau_i} \frac{f_i(s)}{1 - F_i(s)} ds$$

where $F_i(s) = P(\tau_i \le s) = \int_0^s f_i(u)du$ are $\mathcal{H}_t^i$-martingales.

- **Filtration H** From our previous computation applied to

$$\mathcal{G}_t = \mathcal{H}_t = \mathcal{F}_t \vee \mathcal{H}^1_t = \mathcal{H}^2_t \vee \mathcal{H}^1_t \,,$$

the process

$$H^1_t - \int_0^{t\wedge\tau_1} \frac{a_s}{1 - F^{1;2}_{s-}} ds$$

is a **H**-martingale

Here $F^{1;2}$ is the submartingale $F^{1;2}_t = P(\tau_1 \le t | \mathcal{H}^2_t)$ with decomposition $F^{1;2}_t = Z^{1;2}_t + \int_0^t a_s ds$ where $Z^{1;2}$ is a $\mathbf{H}^2$ martingale.

In general, $F^{1;2}$ is not increasing.

The previous computation enables us to write

$$\begin{aligned} F_t^{1;2} &= H_t^2 P(\tau_1 \le t | \tau_2) + (1 - H_t^2) \frac{P(\tau_1 \le t < \tau_2)}{P(\tau_2 > t)} \\ &= H_t^2 h(t, \tau_2) + (1 - H_t^2) \frac{G(0,t) - G(t,t)}{G(0,t)} \end{aligned}$$

where

$$h(t, v) = 1 - \frac{\partial_2 G(t, v)}{\partial_2 G(0, v)} .$$

It follows that

$$dF_t^{1;2} = \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} \right) dM_t^2 + \left(H_t^2 \partial_1 h(t, \tau_2) - (1 - H_t^2) \frac{\partial_1 G(t,t)}{G(0,t)} \right) dt$$

This process has a non null martingale part. This shows again that the intensity is not the good tool to work with.

The process

$$M_t^{1;2} = H_t^1 - \int_0^{t\wedge\tau_1} \frac{a(s)}{1 - F^{1;2}(s)} ds$$

where $a(t) = H_t^2 \partial_1 h(t, \tau_2) - (1 - H_t^2) \frac{\partial_1 G(t,t)}{G(0,t)}$ is a **H**-martingale.

Note that

$$\begin{aligned} M_t^{1;2} &= H_t^1 - \int_0^{t\wedge\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)} ds - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \frac{\partial_1 h(s,\tau_2)}{1 - h(s,\tau_2)} ds \\ &= H_t^1 - \int_0^{t\wedge\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)} ds - \ln \frac{1 - h(t\wedge\tau_1, \tau_2)}{1 - h(t\wedge\tau_1\wedge\tau_2, \tau_2)} \end{aligned}$$

Pricing CDS

Let $S_t^i(\kappa_i)$ the price of a CDS associated with τ_i, and $\widetilde{S}_t^i(\kappa_i)$ its predefault price. It is rather easy to find the dynamics of S^1. One starts from the fact that, on the set $\{\tau_1 > t, \tau_2 > t\}$

$$\begin{aligned} S_t^1 &= \frac{1}{G(t,t)}\left(-\int_t^T \delta(u)G(du,t) - \kappa\int_t^T du\, G(u,t)\right) \\ &= V^1(t) \end{aligned}$$

and, on the set $\{\tau_1 > t > \tau_2\}$

$$\begin{aligned} S_t^1 &= \frac{1}{\partial_2 G(t,\tau_2)}\left(-\int_t^T du\,\delta(u)f(u,\tau_2) - \kappa\int_t^T du\,\partial_2 G(u,\tau_2)\right) \\ &= V^2(t,\tau_2) \end{aligned}$$

Hence

$$S_t^1 = (1 - H_t^1)(1 - H_t^2)\, V^1(t) + (1 - H_t^1)\, H_t^2\, V^2(t, \tau_2)$$

and

$$\begin{aligned} dS_t^1 &= (1 - H_t^1)(1 - H_t^2)\, dV^1(t) + (1 - H_t^1)\, H_t^2\, dV^2(t, \tau_2) \\ &\quad - S_{t-}^1\, dH_t^1 - (1 - H_t^1)\{V^1(t) - V^2(t, \tau_2)\}\, dH_t^2 \end{aligned}$$

where

$$\begin{aligned} dV^1(t) &= \Big((\gamma_1(t) + \gamma_2(t))\, V^1(t) + \kappa_1 - \delta_1(t)\gamma_1(t) - S_{t|2}^1(\kappa_1)\gamma_2(t) \Big)\, dt \\ dV^2(t, \tau_2) &= \Big(\gamma^{1|2}(t, \tau_2) V^2(t, \tau_2) - \gamma^{1|2}(t, \tau_2)\delta_1(t) + \kappa_1 \Big)\, dt \end{aligned}$$

and the function $S^1_{t|2}(\kappa_1)$ equals

$$S^1_{t|2}(\kappa_1) = \frac{\int_t^T \delta_1(u) f(u,t) du}{\int_t^\infty f(u,t)\, du} - \kappa_1 \frac{\int_t^T du \int_u^\infty dz f(z,t)}{\int_t^\infty f(u,t)\, du}.$$

Note that $V^2(\tau_2, \tau_2) = S^1_{\tau_2|2}(\kappa_1)$

Let $S_t^1(\kappa_1)1\!\!1_{\{t<\tau_{(1)}\}} = \widetilde{S}_t^1(\kappa_1)1\!\!1_{\{t<\tau_{(1)}\}}$ where $\tau_{(1)} = \tau_1 \wedge \tau_2$.

The dynamics of the pre-default price $\widetilde{S}_t^1(\kappa_1)$ are

$$d\widetilde{S}_t^1(\kappa_1) = (\gamma_1(t) + \gamma_2(t))\widetilde{S}_t^1(\kappa_1)\,dt + \big(\kappa_1 - \delta_1(t)\gamma_1(t) - S_{t|2}^1(\kappa_1)\gamma_2(t)\big)\,dt,$$

Example: Jarrow and Yu's Model

Let $\tau_i = \inf\{t : \Lambda_i(t) \geq \Theta_i\}, i = 1, 2$ where $\Lambda_i(t) = \int_0^t \lambda_i(s)ds$ and Θ_i are independent random variables with exponential law of parameter 1. Jarrow and Yu study the case where λ_1 is a constant and

$$\lambda_2(t) = \lambda_2 + (\alpha_2 - \lambda_2)\mathbb{1}_{\{\tau_1 \leq t\}} = \lambda_2 \mathbb{1}_{\{t < \tau_1\}} + \alpha_2 \mathbb{1}_{\{\tau_1 \leq t\}}.$$

Assume for simplicity that $r = 0$ and compute the value of a defaultable zero-coupon with default time τ_i, with a rebate δ_i:

$$D_{i,d}(t, T) = E(\mathbb{1}_{\{\tau_i > T\}} + \delta_i \mathbb{1}_{\{\tau_i < T\}} | \mathcal{G}_t), \text{ for } \mathcal{G}_t = \mathcal{D}_t^1 \vee \mathcal{D}_t^2 .$$

Let $G(s, t) = P(\tau_1 > s, \tau_2 > t)$

Case $t \leq s$ For $t < s < \tau_1$, one has $\lambda_2(t) = \lambda_2 t$. Hence, the following equality

$$\begin{aligned} \{\tau_1 > s\} \cap \{\tau_2 > t\} &= \{\tau_1 > s\} \cap \{\Lambda_2(t) < \Theta_2\} = \{\tau_1 > s\} \cap \{\lambda_2 t < \Theta_2\} \\ &= \{\lambda_1 s < \Theta_1\} \cap \{\lambda_2 t < \Theta_2\} \end{aligned}$$

leads to

$$\text{for } \ t < s, \ P(\tau_1 > s, \tau_2 > t) = e^{-\lambda_1 s} e^{-\lambda_2 t}$$

Case $t > s$

$$\{\tau_1 > s\} \cap \{\tau_2 > t\} = \{\{t > \tau_1 > s\} \cap \{\tau_2 > t\}\} \cup \{\cap\{\tau_1 > t\} \cap \{\tau_2 > t\}\}$$

$$\{t > \tau_1 > s\} \cap \{\tau_2 > t\} = \{t > \tau_1 > s\} \cap \{\Lambda_2(t) < \Theta_2\}$$

$$= \{t > \tau_1 > s\} \cap \{\lambda_2 \tau_1 + \alpha_2(t - \tau_1) < \Theta_2\}$$

The independence between Θ_1 and Θ_2 implies that the r.v. τ_1 is independent from Θ_2, hence

$$P(t > \tau_1 > s, \tau_2 > t) = E\left(1\!\!1_{\{t>\tau_1>s\}} e^{-(\lambda_2\tau_1 + \alpha_2(t-\tau_1))}\right)$$

$$= \int du\, 1\!\!1_{\{t>u>s\}} e^{-(\lambda_2 u + \alpha_2(t-u))} \lambda_1 e^{-\lambda_1 u}$$

$$= \frac{1}{\lambda_1 + \lambda_2 - \alpha_2} \lambda_1 e^{-\alpha_2 t} \left(e^{-s(\lambda_1+\lambda_2-\alpha_2)} - e^{-t(\lambda_1+\lambda_2-\alpha_2)}\right)$$

Setting $\Delta = \lambda_1 + \lambda_2 - \alpha_2$, it follows that

$$P(\tau_1 > s, \tau_2 > t) = \frac{1}{\Delta}\lambda_1 e^{-\alpha_2 t}\left(e^{-s\Delta} - e^{-t\Delta}\right) + e^{-\lambda_1 t}e^{-\lambda_2 t}\,.$$

In particular, for $s = 0$,

$$P(\tau_2 > t) = \frac{1}{\Delta}\left(\lambda_1\left(e^{-\alpha_2 t} - e^{-(\lambda_1+\lambda_2)t}\right) + \Delta e^{-\lambda_1 t}\right)$$

- The computation of $D_{1,d}$ reduces to that of

$$P(\tau_1 > T|\mathcal{G}_t) = P(\tau_1 > T|\mathcal{F}_t \vee \mathcal{D}_t^1)$$

where $\mathcal{F}_t = \mathcal{D}_t^2$. From the key lemma,

$$P(\tau_1 > T|\mathcal{F}_t \vee \mathcal{D}_t^1) = 1\!\!1_{\{t<\tau_1\}} \frac{P(\tau_1 > T|\mathcal{F}_t)}{P(\tau_1 > t|\mathcal{F}_t)}\,.$$

Therefore,

$$P_{1,d}(t,T) = \delta_1 + 1\!\!1_{\{\tau_1>t\}}(1-\delta_1)e^{-\lambda_1(T-t)}\,.$$

One can also use that

$$P(\tau_1 > T|\mathcal{G}_t) = 1 - DZC_t^1 \quad = \quad 1\!\!1_{\{\tau_1>t\}}\left(1\!\!1_{\{\tau_2\le t\}}\frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} + 1\!\!1_{\{\tau_2>t\}}\frac{G(T,t)}{G(t,t)}\right)$$

- The computation of $D_{2,d}$ follows

$$
\begin{aligned}
D_{2,d}(t,T) \quad &= \quad \delta_2 + (1-\delta_2)\mathbb{1}_{\{\tau_2>t\}}\Big(\mathbb{1}_{\{\tau_1\le t\}}e^{-\alpha_2(T-t)} \\
&\quad +\mathbb{1}_{\{\tau_1>t\}}\frac{1}{\Delta}(\lambda_1 e^{-\alpha_2(T-t)} + (\lambda_2-\alpha_2)e^{-(\lambda_1+\lambda_2)(T-t)})\Big)
\end{aligned}
$$