VALUATION AND HEDGING OF DEFAULTABLE GAME OPTIONS IN A HAZARD PROCESS MODEL

Tomasz R. Bielecki*
Department of Applied Mathematics
Illinois Institute of Technology
Chicago, IL 60616, USA

Stéphane Crépey†
Département de Mathématiques
Université d'Évry Val d'Essonne
91025 Évry Cedex, France

Monique Jeanblanc‡
Département de Mathématiques
Université d'Évry Val d'Essonne
91025 Évry Cedex, France
and
Europlace Institute of Finance

Marek Rutkowski§
School of Mathematics and Statistics
University of New South Wales
Sydney, NSW 2052, Australia
and
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-661 Warszawa, Poland

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1 Introduction

The goal of this work is to analyze valuation and hedging of defaultable contracts with game option features within a hazard process model of credit risk. Our motivation for considering American or game clauses together with defaultable features of an option is not than much a quest for generality, but rather the fact that the combination of early exercise features and defaultability is an intrinsic feature of some actively traded assets. It suffices to mention here the important class of convertible bonds, which were studied by, among others, Andersen and Buffum [2], Ayache et al. [3], Bielecki et al. [4], Davis and Lischka [14], Kallsen and Kühn [30], or Kwok and Lau [34].

In Bielecki et al. [4], we formally defined a *defaultable game option*, that is, a financial contract that can be seen as an intermediate case between a general mathematical concept of a *game option* and much more specific *convertible bond* with credit risk. We concentrated there on developing a fairly general framework for valuing such contracts. In particular, building on results of Kifer [32] and Kallsen and Kühn [30], we showed that the study of an arbitrage price of a defaultable game option can be reduced to the study of the value process of the related Dynkin game under some risk-neutral measure $\mathbb{Q}$ for the primary market model. In this stochastic game, the issuer of a game option plays the role of the minimizer and the holder of the maximizer. In [4], we dealt with a general market model, which was assumed to be arbitrage-free, but not necessarily complete, so that the uniqueness of a *risk-neutral* (or *martingale*) *measure* was not postulated. In addition, although the default time was introduced, it was left largely unspecified. An explicit specification of the default time will be an important component of the model considered in this work.

As is well known, there are two main approaches to modeling of default risk: the structural approach and the reduced-form approach. In the latter approach, also known as the hazard process approach, the default time is modeled as an exogenous random variable with no reference to any particular economic background. One may object to reduced-form models for their lack of clear reference to economic fundamentals, such as the firm's asset-to-debt ratio. However, the possibility of choosing various parameterizations for the coefficients and calibrating these parameters to any set of CDS spreads and/or implied volatilities makes them very versatile modeling tools, well-suited to price and hedge derivatives consistently with plain-vanilla instruments. It should be acknowledged that structural models, with their sound economic background, are better suited for inference of reliable debt information, such as: risk-neutral default probabilities or the present value of the firm's debt, from the equities, which are the most liquid among all financial instruments. But the structure of these models, as rich as it may be (and which can include a list of factors such as stock, spreads, default status, credit events, etc.) is never rich enough to yield consistent prices for a full set of CDS spreads and/or implied volatilities of related options. As we ultimately aim to specify models for pricing and hedging contracts with optional features (in particular, convertible bonds), we favor the reduced-form approach in the sequel.

1.1 Outline of the Paper

From the mathematical perspective, the goal of the present paper is twofold. First, we wish to specialize our previous valuation results to the hazard process set-up, that is, to a version of the reduced-form approach, which is slightly more general than the intensity-based set-up. Hence we postulate that filtration $\mathbb{G}$ modeling the information flow for the primary market admits the representation $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, where the filtration $\mathbb{H}$ is generated by a *default indicator process* $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$ and $\mathbb{F}$ is some reference filtration. The main tool employed in this section is the effective reduction of the information flow from the full filtration $\mathbb{G}$ to the reference filtration $\mathbb{F}$. The main results in this part are Theorems 3.3 and 3.4, which give convenient pricing formulas with respect to the reference filtration $\mathbb{F}$.

The second goal is to study the issue of hedging of a defaultable game option in the hazard process set-up. Some previous attempts to analyze hedging strategies for defaultable convertible bonds were done by Andersen and Buffum [2] and Ayache et al. [3], who worked directly with

suitable variational inequalities within the Markovian intensity-based set-up.

Preliminary results for hedging strategies in a hazard process set-up, Propositions 4.1 and 4.2, can be informally stated as follows: under the assumption that a related doubly reflected BSDE admits a solution (Θ, M, K) under some risk-neutral measure $\mathbb{Q}$, for which various sets of sufficient conditions are given in the literature, the state-process Θ of the solution (multiplied by the default indicator process) is the minimal *super-hedging price* up to a $(\mathbb{G}, \mathbb{Q})$-sigma (or local) martingale *cost process*, the latter being equal to 0 in the case of a complete market. This notion of a hedge with sigma (or local) martingale cost (or *hedging error*, see Corollary 4.4) thus establishes a connection between arbitrage prices and hedging in a general, incomplete market.

More explicit hedging strategies are subsequently analyzed in Propositions 5.2 and 5.3, in which we resort in the general set-up of this paper to suitable (Galtchouk-Kunita-Watanabe) *decompositions* of a solution to the related doubly reflected BSDE. It is noteworthy that these decompositions, though seemingly rather abstract in the context of a general model considered here, are by no means artificial. On the contrary, they arises naturally in the context of a Markovian set-up, which is studied in some detail in the follow-up paper by Bielecki et al. [5]. The interested reader is referred to [5] for more explicit results regarding hedging of defaultable game options. We conclude the paper by an analysis of alternative approaches to hedging and their relationships to the above-mentioned decompositions of a solution to the doubly reflected BSDE.

1.2 Conventions and Standing Notation

We use throughout this paper the *vector* (as opposed to *componentwise*) stochastic integration, as developed in Cherny and Shiryaev [9] (see also Chatelain and Stricker [8] and Jacod [27]). Given a stochastic basis satisfying the usual conditions, an $\mathbb{R}^d$-valued semimartingale integrator X and an $\mathbb{R}^{1\otimes d}$-valued (row vector) predictable integrand H, the notion of *vector stochastic integral* $\int H\,dX$ allows one to take into account possible "interferences" of different components of a multidimensional process. Well-defined vector stochastic integrals include, in particular, all integrals with a predictable and locally bounded integrand (e.g., any integrand of the form $H = Y_-$ where Y is an adapted càdlàg process, see [26, Theorem 7.7]). Even in the one-dimensional case, the concept of vector stochastic integral is indeed more general than a stochastic integral defined as the sum of integrals of components of H with respect to the related components of X, all supposed to be well defined in the classic sense. The usual properties of stochastic integral, such as: linearity, associativity, invariance with respect to equivalent changes of measures and with respect to inclusive changes of filtrations, are known to hold for the vector stochastic integral. Moreover, unlike other kinds of stochastic integrals, vector stochastic integrals form a closed space in a suitable topology. This feature makes them well adapted to many problems arising in the mathematical finance, such as Fundamental Theorems of Asset Pricing (see [9, 4] and Section 2).

By default, we denote by $\int_0^t$ the integrals over $(0, t]$. Otherwise, we explicitly specify the domain of integration as a subscript of $\int$. Note also that, depending on the context, τ will stand either for a generic stopping time or it will be given as $\tau = \tau_p \wedge \tau_c$ for some specific stopping times τ_c and τ_p. Finally, we consider the right-continuous and completed versions of all filtrations, so that they satisfy the so-called 'usual conditions.'

2 Semimartingale Set-Up

After recalling some fundamental valuation results from Bielecki et al. [4], we will examine basic features of hedging strategies for defaultable game options that are valid in a general semimartingale set-up. The important special case of a hazard process framework is studied in the next section.

We assume throughout that the evolution of the underlying primary market is modeled in terms of stochastic processes defined on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$, where $\mathbb{P}$ denotes the statistical

probability measure.

Specifically, we consider a primary market composed of the savings account and of d risky assets, such that, given a finite horizon date $T > 0$:
• the *discount factor* process β, that is, the inverse of the savings account, is a $\mathbb{G}$-adapted, finite variation, continuous and positively bounded process,
• the risky assets are $\mathbb{G}$-semimartingales with càdlàg sample paths.

The primary risky assets, with $\mathbb{R}^d$-valued price process X, pay dividends, whose cumulative value process, denoted by $\mathcal{D}$, is assumed to be a $\mathbb{G}$-adapted, càdlàg and $\mathbb{R}^d$-valued process of finite variation. Given the price process X, we define the *cumulative price* $\widehat{X}$ of primary risky assets as

$$\widehat{X}_t = X_t + \beta_t^{-1} \int_{[0,t]} \beta_u \, d\mathcal{D}_u. \tag{1}$$

In the financial interpretation, the last term in (1) represents the current value at time t of all dividend payments from the assets over the period $[0, t]$, under the assumption that we immediately reinvest all dividends in the savings account. We assume that the primary market model is free of arbitrage opportunities, though presumably incomplete. In view of the First Fundamental Theorem of Asset Pricing (see [15, 9]), and accounting in particular for the dividends on the primary market, this means that there exists a *risk-neutral measure* $\mathbb{Q} \in \mathcal{M}$, where $\mathcal{M}$ denotes the set of probability measures $\mathbb{Q} \sim \mathbb{P}$ for which $\beta\widehat{X}$ is a *sigma martingale* with respect to $\mathbb{G}$ under $\mathbb{Q}$.

Given a standard stochastic basis, an $\mathbb{R}^d$-valued process Y is called a *sigma martingale* if there exists an $\mathbb{R}^d$-valued local martingale M and an $\mathbb{R}^d$-valued process H such that H^i is a (predictable [9, section 3]) M^i-integrable process and $Y^i = Y_0^i + \int H^i \, dM^i$ for $i = 1, \ldots, d$ (see Lemma 5.1(ii) in [9]). In this paper, we shall use the following well-known properties of sigma martingales.

Proposition 2.1 ([9, 39, 28]) (i) *The class of sigma martingales is a vector space containing all local martingales. It is stable with respect to stochastic integration, that is, if Y is a sigma martingale and H is a (predictable) Y-integrable process then the integral $\int H \, dY$ is a sigma martingale.*
(ii) *Any bounded from below sigma martingale is a supermartingale and any locally bounded sigma martingale is a local martingale.*

Remark 2.1 In the same vein, we recall that stochastic integration of predictable locally bounded integrands preserves local martingales (see, e.g., [39]).

We now introduce the concept of a dividend paying game option (see also Kifer [32]). In broad terms, a *dividend paying game option* initiated at time $t = 0$ and maturing at time T, is a contract with the following cash flows that are paid by the issuer of the contract and received by the holder of the contract:
• a dividend stream with the *cumulative dividend* at time t denoted by D_t,
• a *put payment* L_t made at time $t = \tau_p$ if $\tau_p \leq \tau_c$ and $\tau_p < T$; time τ_p is called the put time and is chosen by the holder,
• a *call payment* U_t made at time $t = \tau_c$ provided that $\tau_c < \tau_p \wedge T$; time τ_c, known as the call time, is chosen by the issuer and may be subject to the constraint that $\tau_c \geq \bar{\tau}$, where $\bar{\tau}$ is the lifting time of the call protection,
• a *payment at maturity* ξ made at time T provided that $T \leq \tau_p \wedge \tau_c$ and subject to rules specified in the contract.

Of course, there is also the initial cash flow, namely, the purchasing price of the contract, which is paid at the initiation time by the holder and received by the issuer.

Let us now be given an $[0, +\infty]$-valued $\mathbb{G}$-stopping time τ_d representing the *default time* of a reference entity, with *default indicator process* $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$. A *defaultable* dividend paying game option is a dividend paying game option such that the contract is terminated at τ_d, if it has not been put or called and has not matured before. In particular, there are no more cash flows related

to this contract after the default time. In this setting, the dividend stream D additionally includes a possible recovery payment made at the default time.

We are interested in studying the problem of the time evolution of an arbitrage price of the game option. Therefore, we formulate the problem in a dynamic way by pricing the game option at any time $t \in [0,T]$. Let 0 (respectively T) stand for the *inception date* (respectively the *maturity date*) of a game option. We write $\mathcal{G}_T^t$ to denote the set of all $\mathbb{G}$-stopping times with values in $[t,T]$ and we let $\bar{\mathcal{G}}_T^t$ stand for $\{\tau \in \mathcal{G}_T^t\,;\, \tau \wedge \tau_d \geq \bar{\tau} \wedge \tau_d\}$, where the lifting time of a call protection $\bar{\tau}$ belongs to $\mathcal{G}_T^0$. The stopping time $\bar{\tau} \in \mathcal{G}_T^0$ is used to model the restriction that the issuer of a game option may be prevented from making a call on some random time interval $[0,\bar{\tau})$.

We are now in the position to state the formal definition of a defaultable game option.

Definition 2.2 A *defaultable game option with lifting time of the call protection* $\bar{\tau} \in \mathcal{G}_T^0$ is a game option with the *ex-dividend cumulative discounted cash flows* $\beta_t \pi(t;\tau_p,\tau_c)$ given by the formula, for any $t \in [0,T]$ and $(\tau_p,\tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$,

$$\beta_t \pi(t;\tau_p,\tau_c) = \int_t^\tau \beta_u \, dD_u + \mathbb{1}_{\{\tau<\tau_d\}} \beta_\tau \Big(\mathbb{1}_{\{\tau=\tau_p<T\}} L_{\tau_p} + \mathbb{1}_{\{\tau<\tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau=T\}} \xi \Big), \tag{2}$$

where $\tau = \tau_p \wedge \tau_c$ and

• the *dividend process* $D = (D_t)_{t\in[0,T]}$ equals

$$D_t = \int_{[0,t]} (1-H_u)\, dC_u + \int_{[0,t]} R_u \, dH_u \tag{3}$$

for some *coupon process* $C = (C_t)_{t\in[0,T]}$, which is a $\mathbb{G}$-predictable, càdlàg process with bounded variation, and some real-valued, $\mathbb{G}$-predictable *recovery process* $R = (R_t)_{t\in[0,T]}$,

• the *put payment* $L = (L_t)_{t\in[0,T]}$ and the *call payment* $U = (U_t)_{t\in[0,T]}$ are $\mathbb{G}$-adapted, real-valued, càdlàg processes,

• the inequality $L_t \leq U_t$ holds for every $t \in [\tau_d \wedge \bar{\tau}, \tau_d \wedge T)$,

• the *payment at maturity* ξ is a $\mathcal{G}_T$-measurable real random variable.

It is clear that, for any fixed t, $\pi(t;\tau_p,\tau_c)$ is a $\mathcal{G}_{\tau\wedge\tau_d}$-measurable random variable.

We further assume that R, L and ξ are bounded from below, so that there exists a constant c such that, for $t \in [0,T]$:

$$\int_{[0,t]} \beta_u \, dD_u + \mathbb{1}_{\{t<\tau_d\}} \beta_t \big(\mathbb{1}_{\{t<T\}} L_t + \mathbb{1}_{\{t=T\}} \xi \big) \geq -c \tag{4}$$

Symmetrically, we shall sometimes additionally assume that R, U and ξ are bounded (from below and from above), or that (4) is supplemented by, for $t \in [0,T]$:

$$\int_{[0,t]} \beta_u \, dD_u + \mathbb{1}_{\{t<\tau_d\}} \beta_t \big(\mathbb{1}_{\{t<T\}} U_t + \mathbb{1}_{\{t=T\}} \xi \big) \leq c \tag{5}$$

2.1 Valuation of a Defaultable Game Option

We will state the following fundamental pricing result without proof, referring the interested reader to [4] for more details. The goal is to characterize the set of arbitrage ex-dividend prices of a game option in terms of values of related Dynkin games [19, 33, 35]. The notion of an arbitrage price of a game option referred to in Theorem 2.2 is the dynamic notion of arbitrage price for game options, defined in Kallsen and Kühn [30], extended to the case of dividend-paying primary assets and/or game options by resorting to the transformation of prices into cumulative prices. Note that in the sequel, the statement '$(\Pi_t)_{t\in[0,T]}$ *is an arbitrage price for the game option*' is in fact to be understood as '$(X_t,\Pi_t)_{t\in[0,T]}$ *is an arbitrage price for the extended market consisting of the primary market and the game option*'.

Theorem 2.2 *Assume that a process* Π *is a* $\mathbb{G}$*-semimartingale and there exists* $\mathbb{Q} \in \mathcal{M}$ *such that* Π *is the value of the Dynkin game related to a game option, specifically,*

$$\begin{aligned} &\operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big) = \Pi_t \\ &\quad = \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big), \quad t \in [0,T]. \end{aligned} \tag{6}$$

Then Π *is an arbitrage ex-dividend price of the game option, called the* $\mathbb{Q}$-price *of the game option. The converse holds true (thus any arbitrage price is a* $\mathbb{Q}$*-price for some* $\mathbb{Q} \in \mathcal{M}$*) under the following integrability assumption*

$$\operatorname{esssup}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,T]} \int_{[0,t]} \beta_u \, dD_u + \mathbb{1}_{\{t<\tau_d\}} \beta_t \big(\mathbb{1}_{\{t<T\}} L_t + \mathbb{1}_{\{t=T\}} \xi\big) \,\Big|\, \mathcal{G}_0 \Big) < \infty, \quad \text{a.s.} \tag{7}$$

It is worth noting that the class of defaultable game options includes as special cases defaultable American options and defaultable European options.

Definition 2.3 A *defaultable American option* is a defaultable game option with $\bar{\tau} = T$. A *defaultable European option* is a defaultable American option such that $\beta_t \widehat{\mathcal{L}}_t \le \beta_T \widehat{\mathcal{L}}_T$ for every $t \in [0,T]$.

In view of Theorem 2.2, the cash flows $\phi(t)$ of a defaultable European option can be redefined by

$$\beta_t \phi(t) = \int_t^T \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi, \quad t \in [0,T]. \tag{8}$$

In the sequel we work under a fixed risk-neutral measure $\mathbb{Q} \in \mathcal{M}$***.*** All the measure-dependent notions like *(local) martingale*, *compensator*, etc., implicitly refer to the probability measure $\mathbb{Q}$.

2.2 Hedging of a Defaultable Game Option

We adopt the definition of hedging game options stemming from successive developments, starting from the hedging of American options examined by Karatzas [31], and subsequently followed by El Karoui and Quenez [22], Kifer [32], Ma and Cvitanić [36] and Hamadène [23] (see also Schweizer [40]). This definition will be later shown to be consistent with the concept of arbitrage pricing of a defaultable game option.

Recall that X (resp. $\widehat{X}$) is the price process (resp. cumulative price process) of primary traded assets, as given by (1). The following definition is standard, accounting for the dividends on the primary market.

Definition 2.4 By a (self-financing) *primary trading strategy* we mean a pair (V_0, ζ) such that:
• V_0 is a $\mathcal{G}_0$-measurable real-valued random variable representing the *initial wealth*,
• ζ is an $\mathbb{R}^{1\otimes d}$-valued, $\beta\widehat{X}$-integrable process representing holdings in primary risky assets.
The *wealth process* V of (V_0, ζ) satisfies, for $t \in [0,T]$,

$$d(\beta_t V_t) = \zeta_t \, d(\beta_t \widehat{X}_t) \tag{9}$$

with an initial condition V_0.

Given the wealth process V of a primary strategy (V_0, ζ), we uniquely specify a ($\mathbb{G}$-optional) process ζ^0 by setting

$$V_t = \zeta_t^0 \beta_t^{-1} + \zeta_t X_t, \quad t \in [0,T].$$

The process ζ^0 represents the number of units held in the savings account at time t, when we start from the initial wealth V_0 and we use the strategy ζ in the primary risky assets.

Definition 2.5 (i) An *issuer ε-hedge with cost process ρ* for the game option with ex-dividend cumulative discounted cash flows $\beta\pi$ (cf. (2)) is represented by a quadruplet $(V_0, \zeta, \rho, \tau_c)$ such that:
• (V_0, ζ) is a primary strategy with the wealth process V given by (9),
• a cost process ρ is a real-valued $\mathbb{G}$-semimartingale with $\rho_0 = 0$,
• a (fixed) call time τ_c belongs to $\bar{\mathcal{G}}^0_T$,
• the following inequality is valid, for every put time $\tau_p \in \mathcal{G}^0_T$,

$$\beta_\tau V_\tau + \int_0^\tau \beta_u \, d\rho_u \geq \beta_0 \pi(0; \tau_p, \tau_c) - \beta_\tau \varepsilon, \quad \text{a.s.} \tag{10}$$

(ii) A *holder ε-hedge with cost process ρ* for the game option is a quadruplet $(V_0, \zeta, \rho, \tau_p)$ such that:
• (V_0, ζ) is a primary strategy with the wealth process V given by (9),
• a cost process ρ is a real-valued $\mathbb{G}$-semimartingale with $\rho_0 = 0$,
• a (fixed) put time τ_p belongs to $\mathcal{G}^0_T$,
• the following inequality is valid, for every call time $\tau_c \in \bar{\mathcal{G}}^0_T$,

$$\beta_\tau V_\tau + \int_0^\tau \beta_u \, d\rho_u \geq -\beta_0 \pi(0; \tau_p, \tau_c) - \beta_\tau \varepsilon, \quad \text{a.s.} \tag{11}$$

A more explicit form of condition (10) reads: for a fixed $\tau_c \in \bar{\mathcal{G}}^0_T$ and every $\tau_p \in \mathcal{G}^0_T$

$$\begin{aligned} &V_\tau + \beta_\tau^{-1} \int_0^\tau \beta_u \, d\rho_u \\ &\quad \geq \beta_\tau^{-1} \int_0^\tau \beta_u \, dD_u + \mathbb{1}_{\{\tau < \tau_d\}} \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau < \tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau_p = \tau_c = T\}} \xi \Big) - \varepsilon, \quad \text{a.s.} \end{aligned} \tag{12}$$

The left-hand side in the last formula is the value process of a strategy with cost ρ, when the players adopt the respective exercise policies τ_p and τ_c, whereas the right-hand side represents the payoff to be done by the issuer, including past dividends and recovery at default.

By the right-continuity of the involved processes, condition (12) is in turn equivalent to the following statement: for a fixed call time $\tau_c \in \bar{\mathcal{G}}^0_T$ chosen by the issuer, the inequality

$$\begin{aligned} &V_{t\wedge\tau_c} + \beta_{t\wedge\tau_c}^{-1} \int_0^{t\wedge\tau_c} \beta_u \, d\rho_u \geq \lim_{t+} \pi(0; \cdot, \tau_c) \\ &\quad = \beta_{t\wedge\tau_c}^{-1} \int_0^{t\wedge\tau_c} \beta_u \, dD_u + \mathbb{1}_{\{t\wedge\tau_c < \tau_d\}} \Big(\mathbb{1}_{\{t < \tau_c\}} L_t + \mathbb{1}_{\{\tau_c \leq t < T\}} U_{\tau_c} + \mathbb{1}_{\{t = \tau_c = T\}} \xi \Big) - \varepsilon, \end{aligned} \tag{13}$$

is satisfied almost surely, for any $t \in [0, T]$ (or, interchangeably, for any $t \in [0, T]$, a.s.).

Likewise, condition (11) is equivalent to: for a fixed put time $\tau_p \in \mathcal{G}^0_T$ chosen by the holder, the inequality

$$\begin{aligned} &V_{\tau_p\wedge t} + \beta_{\tau_p\wedge t}^{-1} \int_0^{\tau_p\wedge t} \beta_u \, d\rho_u \geq -\lim_{t+} \pi(0; \tau_p, \cdot) = -\pi(0; \tau_p, t) \\ &\quad = -\beta_{\tau_p\wedge t}^{-1} \int_0^{\tau_p\wedge t} \beta_u \, dD_u - \mathbb{1}_{\{\tau_p\wedge t < \tau_d\}} \Big(\mathbb{1}_{\{\tau_p \leq t < T\}} L_{\tau_p} + \mathbb{1}_{\{t < \tau_p\}} U_t + \mathbb{1}_{\{t = \tau_c = T\}} \xi \Big) - \varepsilon, \end{aligned} \tag{14}$$

holds almost surely, for any $t \in [\bar{\tau}, T]$ (or, interchangeably, for any $t \in [0, T]$, a.s.).

For $\varepsilon = 0$, we say that we deal with an *issuer hedge* and a *holder hedge* with cost ρ for the game option. Issuer or holder (ε-)hedges *with no cost* (that is, with $\rho = 0$) are also called issuer or holder *(ε-)superhedges*.

Remark 2.6 (i) The process ρ is to be interpreted as the (running) *financing cost*, that is, the amount of cash added to (if $d\rho_t \geq 0$) or withdrawn from (if $d\rho_t \leq 0$) the hedging portfolio in order

to get a perfect, but no longer self-financing, hedge. In the special case where ρ is a $\mathbb{G}$-martingale we thus recover the notion of *mean self-financing* hedge in the sense of Schweizer [40].
(ii) Regarding the *admissibility issues* (see, e.g., Delbaen and Schachermayer [15]), note that the l.h.s. of (10) (discounted wealth process with financing costs included) is bounded from below for any issuer ε-hedge with a cost $(V_0, \zeta, \rho, \tau_c)$. Likewise, in the case of a bounded payoff π (that is, assuming (5)), the l.h.s. of (11) (discounted wealth process with financing costs included) is bounded from below for any holder ε-hedge with a cost $(V_0, \zeta, \rho, \tau_p)$.

We now restrict our attention to (ε-)hedges with a $\mathbb{G}$-*sigma martingale* cost ρ. We define $\mathcal{V}_0^c$ (resp. $\mathcal{V}_0^p$) as the set of initial values V_0 such that for any $\varepsilon > 0$ there exists an issuer (resp. holder) ε-hedge of the game option with the initial value V_0 (resp. $-V_0$) and with a $\mathbb{G}$-*sigma martingale* cost.

Remark 2.7 It is easy to see that $\mathcal{V}_0^c$ (resp. $\mathcal{V}_0^p$) can equivalently be defined as the set of values V_0 such that for any $\varepsilon > 0$ there exists an issuer (resp. holder) hedge of the game option at time 0 with the initial value $V_0 + \varepsilon$ (resp. $-V_0 + \varepsilon$) and with a $\mathbb{G}$-*sigma martingale* cost.

The following lemma gives some preliminary conclusions regarding the initial cost of a hedging strategy for the game option under very weak assumptions. In Proposition 4.2, we shall see that under stronger assumptions the infima are attained and we obtain equalities rather than merely inequalities in Lemma 2.3.

Lemma 2.3 (i) *We have (by convention,* $\inf \emptyset = \infty$*)*

$$\operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^0} \operatorname{esssup}_{\tau_p \in \mathcal{G}_T^0} \mathbb{E}_{\mathbb{Q}}\big(\pi(0; \tau_p, \tau_c) \,\big|\, \mathcal{G}_0\big) \leq \operatorname{essinf}_{V_0 \in \mathcal{V}_0^c} V_0, \quad \text{a.s.} \tag{15}$$

(ii) *If inequality (5) is valid then*

$$\operatorname{esssup}_{\tau_p \in \mathcal{G}_T^0} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^0} \mathbb{E}_{\mathbb{Q}}\big(\pi(0; \tau_p, \tau_c) \,\big|\, \mathcal{G}_0\big) \geq -\operatorname{essinf}_{V_0 \in \mathcal{V}_0^p} V_0, \quad \text{a.s.} \tag{16}$$

Proof. (i) Assume that for some stopping time $\bar{\tau}_c \in \bar{\mathcal{G}}_T^0$ the quadruplet $(V_0, \zeta, \rho, \bar{\tau}_c)$ is an issuer ε-hedge with a $\mathbb{G}$-sigma martingale cost ρ for the game option. Then (9)–(10) imply that, for any $t \in [0, T]$,

$$\begin{aligned} \beta_0 V_0 &= \beta_{t\wedge\bar{\tau}_c} V_{t\wedge\bar{\tau}_c} - \int_0^{t\wedge\bar{\tau}_c} \zeta_u \, d(\beta_u \widehat{X}_u) \\ &\geq \beta_0 \pi(0; t, \bar{\tau}_c) - \beta_{t\wedge\bar{\tau}_c} \varepsilon - \int_0^{t\wedge\bar{\tau}_c} \big(\zeta_u \, d(\beta_u \widehat{X}_u) + \beta_u \, d\rho_u\big). \end{aligned} \tag{17}$$

The stochastic integral $\int_0^t \zeta_u \, d(\beta_u \widehat{X}_u)$ with respect to a $\mathbb{G}$-sigma martingale $\beta \widehat{X}$ is a $\mathbb{G}$-sigma martingale. Hence the stopped process $\int_0^{t\wedge\bar{\tau}_c} \zeta_u \, d(\beta_u \widehat{X}_u)$ and the process

$$\int_0^{t\wedge\bar{\tau}_c} \big(\zeta_u \, d(\beta_u \widehat{X}_u) + \beta_u \, d\rho_u\big)$$

are $\mathbb{G}$-sigma martingales as well. The latter process is bounded from below (this follows from (2)–(4) and (17)), so that it is a bounded from below local martingale ([28, p.216]) and thus a supermartingale.

Moreover, for any stopping time $\tau_p \in \mathcal{G}_T^0$, the inequality in formula (17) still holds with t replaced by τ_p. By taking expectations, we obtain (recall that $\bar{\tau}_c$ is fixed)

$$\beta_0 V_0 \geq \mathbb{E}_{\mathbb{Q}}\big(\beta_0 \pi(0; \tau_p, \bar{\tau}_c) - \beta_{\tau_p\wedge\bar{\tau}_c} \varepsilon \,\big|\, \mathcal{G}_0\big), \quad \forall\, \tau_p \in \mathcal{G}_T^0,$$

and thus, since β is a positively bounded process,

$$V_0 \geq \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^0_T} \operatorname{esssup}_{\tau_p \in \mathcal{G}^0_T} \mathbb{E}_{\mathbb{Q}}\big(\pi(0;\tau_p,\tau_c) \,\big|\, \mathcal{G}_0\big) - k\varepsilon, \quad \text{a.s.},$$

for some constant k. Therefore, if V_0 is such that for any $\varepsilon > 0$ there exists an issuer ε-hedging strategy with the initial wealth V_0 and a $\mathbb{G}$-sigma martingale cost ρ then

$$V_0 \geq \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^0_T} \operatorname{esssup}_{\tau_p \in \mathcal{G}^0_T} \mathbb{E}_{\mathbb{Q}}\big(\pi(0;\tau_p,\tau_c) \,\big|\, \mathcal{G}_0\big), \quad \text{a.s.}$$

(ii) Let $(V_0, \zeta, \rho, \bar{\tau}_p)$ be a holder ε-hedge with a $\mathbb{G}$-sigma martingale cost ρ for the game option for some stopping time $\bar{\tau}_p \in \mathcal{G}^0_T$. Then (9)–(11) imply that, for any $t \in [\bar{\tau}, T]$,

$$\begin{aligned}
\beta_0 V_0 &= \beta_{t\wedge\bar{\tau}_p} V_{t\wedge\bar{\tau}_p} - \int_0^{t\wedge\bar{\tau}_p} \zeta_u \, d(\beta_u \widehat{X}_u) \\
&\geq -\beta_0 \pi(0;\bar{\tau}_p,t) - \beta_{t\wedge\bar{\tau}_p}\varepsilon - \int_0^{t\wedge\bar{\tau}_p} \big(\zeta_u \, d(\beta_u \widehat{X}_u) + \beta_u \, d\rho_u\big).
\end{aligned}$$

Under condition (5), the stochastic integral in the last formula is bounded from below and thus, by the same arguments as in part (i), we conclude that it is a supermartingale. Consequently, for a fixed stopping time $\bar{\tau}_p \in \mathcal{G}^0_T$,

$$\beta_0 V_0 \geq \mathbb{E}_{\mathbb{Q}}\big(- \beta_0 \pi(0;\bar{\tau}_p,\tau_c) - \beta_{\bar{\tau}_p\wedge\tau_c}\varepsilon \,\big|\, \mathcal{G}_0\big), \quad \text{a.s.}, \quad \forall\, \tau_c \in \bar{\mathcal{G}}^0_T,$$

so that

$$V_0 \geq -\operatorname{esssup}_{\tau_p \in \mathcal{G}^0_T} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^0_T} \mathbb{E}_{\mathbb{Q}}\big(\pi(0;\tau_p,\tau_c) \,\big|\, \mathcal{G}_0\big) - k\varepsilon, \quad \text{a.s.},$$

for some constant k. Therefore, if V_0 is such that for any $\varepsilon > 0$ there exists a holder ε-hedging strategy with the initial wealth V_0 and a $\mathbb{G}$-sigma martingale cost ρ then

$$V_0 \geq -\operatorname{esssup}_{\tau_p \in \mathcal{G}^0_T} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^0_T} \mathbb{E}_{\mathbb{Q}}\big(\pi(0;\tau_p,\tau_c) \,\big|\, \mathcal{G}_0\big), \quad \text{a.s.}$$

This completes the proof. □

3 Valuation in a Hazard Process Set-Up

In order to get more explicit pricing hedging results, we will now study the so-called *hazard process set-up*.

3.1 Standing Assumptions

Given an $[0,+\infty]$-valued $\mathbb{G}$-stopping time τ_d, we assume that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, where the filtration $\mathbb{H}$ is generated by the process $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$ and $\mathbb{F}$ is some reference filtration. As expected, our approach will consist in effectively reducing the information flow from the full filtration $\mathbb{G}$ to the reference filtration $\mathbb{F}$.

Let G stand for the process $G_t = \mathbb{Q}(\tau_d > t \,|\, \mathcal{F}_t)$ for $t \in \mathbb{R}_+$. The process G is an $\mathbb{F}$-supermartingale, as the optional projection on $\mathbb{F}$ of the non-increasing process $1 - H$ (see [29]).

In the sequel, we shall work under the following standing assumption.

Assumption 3.1 We assume that the process G is (strictly) positive and continuous with finite variation, so that the $\mathbb{F}$-*hazard process* $\Gamma_t = -\ln(G_t)$, $t \in \mathbb{R}_+$, is well defined and continuous with finite variation.

Remark 3.2 (i) Assuming G continuous, τ_d is a *totally inaccessible* $\mathbb{G}$-stopping time (see, e.g., [18]), which in particular avoids $\mathbb{F}$- (even $\mathbb{G}$-) predictable stopping times. More precisely, the assumption that the process G is continuous lies somewhere between assuming that τ_d avoids $\mathbb{F}$-stopping times and assuming that τ_d avoids $\mathbb{F}$-predictable stopping times.
(ii) Assuming G continuous, the further assumption that G has a finite variation in fact implies that G is non-increasing (see Lemma A.1(i)). This lies somewhere between assuming further the (stronger) $(\mathcal{H})$ Hypothesis and assuming further that τ_d is an $\mathbb{F}$-*pseudo-stopping time* (see Nikeghbali and Yor [38]). Recall that the $(\mathcal{H})$ Hypothesis means that all square-integrable $\mathbb{F}$-martingales are $\mathbb{G}$-martingales (see, e.g., [7]), or, by a standard localization argument, that all $\mathbb{F}$-local martingales are $\mathbb{G}$-local martingales, whereas τ_d is an $\mathbb{F}$-*pseudo-stopping time* iff all $\mathbb{F}$-local martingales *stopped at* τ_d are $\mathbb{G}$-local martingales (see Nikeghbali and Yor [38] and Appendix A).

More detailed consequences of Assumption 3.1 useful for this work are summarized in Lemma A.1.

The next definition refers to some auxiliary results proved in Appendix A.

Definition 3.3 The $\mathbb{F}$-stopping time, resp. r.v., resp. $\mathbb{F}$-adapted or $\mathbb{F}$-predictable process $\widetilde{\tau}, \widetilde{\chi}$ and $\widetilde{Y}$ introduced in Lemmas A.2 and A.3 are called the $\mathbb{F}$-*representatives* of τ, Y and χ, respectively. In the context of credit risk, where τ_d represents the default time of a reference entity, they are also known as the *pre-default values* of τ, χ and Y.

To simplify the presentation, we find it convenient to make an additional assumption. Strictly speaking, Assumption 3.4 is superfluous, in the sense that all the results below are true in general; it suffices to make use of Lemmas A.2 and A.3 to reduce the problem to the case described in Assumption 3.4. Since this would make the notation heavier, without adding much value, we prefer to work under this standing assumption.

Assumption 3.4 (i) The discount factor process β is $\mathbb{F}$-adapted.
(ii) The coupon process C is $\mathbb{F}$-predictable.
(iii) The recovery process R is $\mathbb{F}$-predictable.
(iv) The payoff processes L, U are $\mathbb{F}$-adapted and the random variable ξ is $\mathcal{F}_T$-measurable.
(v) The call protection $\bar{\tau}$ is an $\mathbb{F}$-stopping time.

3.2 Reduction of a Filtration

The next lemma shows that the computation of the lower and upper value of the Dynkin games (6) with respect to $\mathbb{G}$-stopping times can be reduced to the computation of the lower and upper value with respect to $\mathbb{F}$-stopping times.

Lemma 3.1 *We have*

$$\operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big) = \operatorname{esssup}_{\tau_p \in \mathcal{F}_T^t} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{F}}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \big|\, \mathcal{G}_t\big)$$

and

$$\operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big) = \operatorname{essinf}_{\tau_c \in \bar{\mathcal{F}}_T^t} \operatorname{esssup}_{\tau_p \in \mathcal{F}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \big|\, \mathcal{G}_t\big).$$

Proof. For $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$, we have

$$\pi(t;\tau_p,\tau_c) = \pi(t;\tau_p \wedge \tau_d, \tau_c \wedge \tau_d) = \pi(t;\widetilde{\tau}_p \wedge \tau_d, \widetilde{\tau}_c \wedge \tau_d) = \pi(t;\widetilde{\tau}_p, \widetilde{\tau}_c)$$

for some stopping times $(\widetilde{\tau}_p, \widetilde{\tau}_c) \in \mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$, where the middle equality follows from Lemma A.3, and the other two from the definition of π. Since, clearly, $\mathcal{F}_T^t \subseteq \mathcal{G}_T^t$ and $\bar{\mathcal{F}}_T^t \subseteq \bar{\mathcal{G}}_T^t$, we conclude that the lemma is valid. □

Under our assumptions, the computation of conditional expectations of cash flows $\pi(t;\tau_p,\tau_c)$ with respect to $\mathcal{G}_t$ can be reduced to the computation of conditional expectations of $\mathbb{F}$-equivalent cash flows $\widetilde{\pi}(t;\tau_p,\tau_c)$ with respect to $\mathcal{F}_t$. Let $\alpha_t := \beta_t \exp(-\Gamma_t)$ stand for the *credit-risk adjusted discount factor*. Note that α is bounded, like β.

Lemma 3.2 *For any stopping times $\tau_p \in \mathcal{F}_T^t$ and $\tau_c \in \bar{\mathcal{F}}_T^t$ we have that*

$$\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big) = \mathbb{1}_{\{t<\tau_d\}}\mathbb{E}_{\mathbb{Q}}\big(\widetilde{\pi}(t;\tau_p,\tau_c)\,\big|\,\mathcal{F}_t\big), \tag{18}$$

where $\widetilde{\pi}(t;\tau_p,\tau_c)$ is given by, with $\tau = \tau_p \wedge \tau_c$,

$$\alpha_t\widetilde{\pi}(t;\tau_p,\tau_c) = \int_t^\tau \alpha_u(dC_u + R_u\,d\Gamma_u) + \alpha_\tau\big(\mathbb{1}_{\{\tau=\tau_p<T\}}L_{\tau_p} + \mathbb{1}_{\{\tau<\tau_p\}}U_{\tau_c} + \mathbb{1}_{\{\tau=T\}}\xi\big). \tag{19}$$

Proof. Formula (18) is an immediate consequence of formula (2) and Lemma A.4. □

Note that $\widetilde{\pi}(t;\tau_p,\tau_c)$ is an $\mathcal{F}_\tau$-measurable random variable. A comparison of formulas (2) and (19) shows that we have effectively moved our considerations from the original market subject to the default risk, in which cash flows are discounted according to the discount factor β, to the fictitious default-free market, in which cash flows are discounted according to the credit risk adjusted discount factor α. Recall that the original cash flows $\pi(t;\tau_p,\tau_c)$ are given as $\mathcal{G}_{\tau\wedge\tau_d}$-measurable random variables, whereas the $\mathbb{F}$-equivalent cash flows $\widetilde{\pi}(t;\tau_p,\tau_c)$ are manifestly $\mathcal{F}_\tau$-measurable and they depend of the default time τ_d only via the hazard process Γ. For the purpose of computation of ex-dividend pre-default prices of a defaultable game option these two market models are equivalent, as the following result shows.

Theorem 3.3 *Assuming condition (7), let Π be the arbitrage ex-dividend $\mathbb{Q}$-price for a game option. Then we have, for any $t \in [0,T]$,*

$$\Pi_t = \mathbb{1}_{\{t<\tau_d\}}\widetilde{\Pi}_t, \tag{20}$$

where $\widetilde{\Pi}_t$ satisfies

$$\begin{aligned}\operatorname{esssup}_{\tau_p\in\mathcal{F}_T^t}\operatorname{essinf}_{\tau_c\in\bar{\mathcal{F}}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\widetilde{\pi}(t;\tau_p,\tau_c)\,\big|\,\mathcal{F}_t\big) &= \widetilde{\Pi}_t \\ &= \operatorname{essinf}_{\tau_c\in\bar{\mathcal{F}}_T^t}\operatorname{esssup}_{\tau_p\in\mathcal{F}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\widetilde{\pi}(t;\tau_p,\tau_c)\big|\,\mathcal{F}_t\big).\end{aligned} \tag{21}$$

Hence the Dynkin game with cost criterion $\mathbb{E}_{\mathbb{Q}}\big(\widetilde{\pi}(t;\tau_p,\tau_c)\,\big|\,\mathcal{F}_t\big)$ on $\mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$ admits the value $\widetilde{\Pi}_t$, which coincides with the pre-default ex-dividend price at time t of the game option under the risk-neutral measure $\mathbb{Q}$.

Proof. It suffices to combine Theorem 2.2 with Lemmas 3.1 and 3.2. □

The following result is the converse of Theorem 3.3. It follows immediately by Lemmas 3.1, 3.2 and by the 'if' part of Theorem 2.2 (noting also that Π defined by (20) is obviously a $\mathbb{G}$-semimartingale if $\widetilde{\Pi}$ is a $\mathbb{G}$-semimartingale).

Theorem 3.4 *Let $\widetilde{\Pi}_t$ be the value of the Dynkin game with the cost criterion $\mathbb{E}_{\mathbb{Q}}\big(\widetilde{\pi}(t;\tau_p,\tau_c)\,\big|\,\mathcal{F}_t\big)$ on $\mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$, for any $t \in [0,T]$. Then Π_t defined by (20) is the value of the Dynkin game with the cost criterion $\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big)$ on $\mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$, for any $t \in [0,T]$. If, in addition, $\widetilde{\Pi}$ is a $\mathbb{G}$-semimartingale then Π is the arbitrage ex-dividend $\mathbb{Q}$-price for the game option.*

Theorems 3.3 and 3.4 thus allow us to reduce the study of a game option to the study of Dynkin games (21) with respect to the reference filtration $\mathbb{F}$.

3.3 Valuation via Doubly Reflected BSDEs

In this section, we shall characterize the arbitrage ex-dividend $\mathbb{Q}$-price of a game option as a solution to a judiciously chosen doubly reflected BSDE. To this end, we first recall some auxiliary results concerning the relationship between Dynkin games and doubly reflected BSDEs.

Given an additional $\mathbb{F}$-adapted process F of finite variation, we consider the following *doubly reflected BSDE with the data* $F, \xi, L, U, \bar{\tau}$ (see Cvitanić and Karatzas [13], Hamadène and Hassani [24], Crépey et al. [12, 11], Bielecki et al. [5, 6]):

$$\begin{aligned} \alpha_t\Theta_t &= \alpha_T\xi + \alpha_T F_T - \alpha_t F_t + \textstyle\int_t^T \alpha_u dK_u - \int_t^T \alpha_u dM_u, \quad t \in [0,T], \\ L_t &\le \Theta_t \le \bar{U}_t, \quad t \in [0,T], \\ \textstyle\int_0^T (\Theta_{u-} - L_{u-})\, dK_u^+ &= \textstyle\int_0^T (\bar{U}_{u-} - \Theta_{u-})\, dK_u^- = 0, \end{aligned} \tag{22}$$

where the process $\bar{U} = (\bar{U}_t)_{t\in[0,T]}$ equals, for $t \in [0,T]$,

$$\bar{U}_t = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\ge\bar{\tau}\}}U_t.$$

Definition 3.5 By a ($\mathbb{Q}$-)*solution* to the doubly reflected BSDE (22), we mean a triplet (Θ, M, K) such that:
• the *state process* Θ is a real-valued, $\mathbb{F}$-adapted, càdlàg process,
• $\int_0^\cdot \alpha dM$ is a real-valued $\mathbb{F}$-martingale vanishing at time 0,
• K is an $\mathbb{F}$-adapted finite variation process vanishing at time 0,
• all conditions in (22) are satisfied, where in the third line K^+ and K^- denote the Jordan components of K, and where the convention that $0 \times \pm\infty = 0$ is made in the third line.
Here by *Jordan decomposition* we mean the decomposition $K = K^+ - K^-$ where the non-decreasing processes $K^\pm$ vanish at time 0 and define mutually singular measures.

The state process Θ in a solution to (22) is clearly an $\mathbb{F}$-semimartingale. So there are obvious (though rather artificial) cases in which (22) does *not* admit a solution: it suffices to take $\bar{\tau} = 0$ and $L = U$, assumed *not to be* an $\mathbb{F}$-semimartingale. It is also clear that a solution would not necessarily be unique if we did not impose the condition of a mutual singularity of the non-negative measures defined by K^+ and K^- (see, e.g., [24, Remark 4.1]).

In applications (see [5, 12, 11, 6]), the input process F is typically given as a Lebesgue integral $\alpha F = \int \alpha f\, du$ and the component M of a solution to (22) is usually searched for in the form $M = \int Z\, dN + dn$ (cf. (40)) for some real-valued and $\mathbb{R}^q$-valued square integrable $\mathbb{F}$-martingales N and n. For various concrete (including Markovian) specifications of the present set-up and definite sets of technical assumptions ensuring the existence and uniqueness of a solution to (22), we refer the reader to [12, 24, 11, 6, 13], among others.

Basically, in any model endowed with the martingale representation property, the existence (and uniqueness) of a solution to (22) (supplemented by suitable integrability conditions on the data and the solution) is equivalent to the so-called *Mokobodski condition*, namely, the existence of a quasimartingale Z such that $L \le Z \le U$ on $[0,T]$ (see, in particular, Crépey and Matoussi [12], Hamadène and Hassani [24, Theorem 4.1], and previous works in this direction, starting with [13]). It is thus satisfied when one of the barriers is a quasimartingale, and, in particular, when one of the barriers is given as $S \vee \ell$ where S is an Itô-Lévy process S with square-integrable special semimartingale decomposition components (see [12]) and ℓ is a constant in $\mathbb{R} \cup \{-\infty\}$. This covers, for instance, the call payoff of a convertible bond, see Bielecki et al. [4, 6].

Remark 3.6 (i) Since F is a given process, the BSDE (22) can be rewritten as

$$\begin{aligned} \alpha_t\widehat{\Theta}_t &= \alpha_T\widehat{\xi} + \textstyle\int_t^T \alpha_u dK_u - \int_t^T \alpha_u dM_u, \quad t \in [0,T], \\ \widehat{L}_t &\le \widehat{\Theta}_t \le \widehat{U}_t, \quad t \in [0,T], \\ \textstyle\int_0^T (\widehat{\Theta}_{u-} - \widehat{L}_{u-})\, dK_u^+ &= \textstyle\int_0^T (\widehat{U}_{u-} - \widehat{\Theta}_{u-})\, dK_u^- = 0, \end{aligned} \tag{23}$$

where $\widehat{\Theta}_t = \Theta_t + F_t$ and

$$\widehat{\xi} = \xi + F_T, \quad \widehat{L}_t = L_t + F_t, \quad \widehat{U}_t = \bar{U}_t + F_t.$$

This shows that the problem of solving (22) can be formally reduced to the case of $F = 0$ with suitably modified reflecting barriers $\widehat{L}$, $\widehat{U}$ and terminal condition $\widehat{\xi}$. Note that, in spite of this formal reduction, the freedom to choose the driver of a related BSDE associated with a game option is important from the point of view of applications (this is apparent in the follow-up papers [5, 6]). Since the related material is not directly advantageous for a reader of the present paper, it is deferred to Appendix B.

(ii) In the special case where all $\mathbb{F}$-martingales are continuous and where the $\mathbb{F}$-semimartingale F and the barriers L and U are continuous (see [13, 25, 6]), it is natural to look for a *continuous* solution of (22), that is, a solution of (22) given by a triplet of continuous processes (Θ, M, K) (the continuity of K implying in turn that of $K^{\pm}$).

(iii) More generally, in the existing literature on reflected BSDEs (including the case of models with jumps), the component K of a solution is typically sought for in the form of a *continuous* finite variation process.

(iv) In the context of Markovian set-ups, the probabilistic BSDE approach may be complemented by a related analytic *variational inequality* approach. This issue is dealt with in Bielecki et al. [5] (see also [6]). Note, however, that the variational inequality approach strongly relies on the BSDE approach. Moreover, the BSDE-based simulation method is the only efficient way of numerically solving the pricing problem if the dimension of the problem (number of model factors) is greater than three or four. Indeed in such case the computational cost of deterministic numerical schemes based on the variational inequality approach becomes prohibitive.

In order to establish a link between a solution to the related doubly reflected BSDE and the arbitrage ex-dividend $\mathbb{Q}$-price of the defaultable game option, we first recall the general relationship between doubly reflected BSDEs and Dynkin games with purely terminal cost, before applying this result to dividend-paying game options in the fictitious default-free market in Proposition 3.5.

Observe that if (Θ, M, K) solves (22) then we have, for any stopping time $\tau \in \mathcal{F}_T^t$,

$$\alpha_t \Theta_t = \alpha_\tau \Theta_\tau + \alpha_\tau F_\tau - \alpha_t F_t + \int_t^\tau \alpha_u dK_u - \int_t^\tau \alpha_u dM_u. \tag{24}$$

Proposition 3.5 (Verification Principle for a Dynkin Game) *Let (Θ, M, K) be a solution to (22) with $F = 0$. Then Θ_t is the value of the Dynkin game with cost criterion $\mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c) \,\big|\, \mathcal{F}_t\big)$ on $\mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$, where $\theta(t;\tau_p,\tau_c)$ is the $\mathcal{F}_\tau$-measurable random variable defined by*

$$\alpha_t \theta(t;\tau_p,\tau_c) = \alpha_\tau \big(\mathbb{1}_{\{\tau=\tau_p<T\}} L_{\tau_p} + \mathbb{1}_{\{\tau=\tau_c<\tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau=T\}} \xi \big),$$

where $\tau = \tau_p \wedge \tau_c$. Moreover, for any $t \in [0,T]$ and for any $\varepsilon > 0$, the pair of stopping times $(\tau_p^\varepsilon, \tau_c^\varepsilon) \in \mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$ given by

$$\tau_p^\varepsilon = \inf\Big\{ u \in [t,T]\,;\, \Theta_u \le L_u + \varepsilon \Big\} \wedge T, \quad \tau_c^\varepsilon = \inf\Big\{ u \in [\bar{\tau} \vee t, T]\,;\, \Theta_u \ge U_u - \varepsilon \Big\} \wedge T,$$

is ε-optimal for this Dynkin game, in the sense that we have, for any $(\tau_p,\tau_c) \in \mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$,

$$\mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c^\varepsilon) \,\big|\, \mathcal{F}_t\big) - \varepsilon \le \Theta_t \le \mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p^\varepsilon,\tau_c) \,\big|\, \mathcal{F}_t\big) + \varepsilon. \tag{25}$$

If K is continuous then the pair of stopping times $(\tau_p^0, \tau_c^0) \in \mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$, obtained by setting $\varepsilon = 0$, is a saddle-point of the game. This means that we have, for any $(\tau_p,\tau_c) \in \mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$,

$$\mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c^0) \,\big|\, \mathcal{F}_t\big) \le \Theta_t \le \mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p^0,\tau_c) \,\big|\, \mathcal{F}_t\big).$$

Proof. Except for the presence of $\bar{\tau}$, the result is standard (see, e.g., Lepeltier and Maingueneau [35]). The proof hinges on showing that the pair $(\tau_p^\varepsilon, \tau_c^\varepsilon)$ is ε-optimal, for any $\varepsilon > 0$. Thus, taking the supremum and infimum over stopping times in (25), we obtain

$$\begin{aligned}\operatorname{essinf}_{\tau_c \in \bar{\mathcal{F}}_T^t} \operatorname{esssup}_{\tau_p \in \mathcal{F}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c) \,\big|\, \mathcal{F}_t\big) - \varepsilon &\le \operatorname{esssup}_{\tau_p \in \mathcal{F}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c^\varepsilon) \,\big|\, \mathcal{F}_t\big) - \varepsilon \le \Theta_t \\ &\le \operatorname{essinf}_{\tau_c \in \bar{\mathcal{F}}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p^\varepsilon,\tau_c) \,\big|\, \mathcal{F}_t\big) + \varepsilon \le \operatorname{esssup}_{\tau_p \in \mathcal{F}_T^t} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{F}}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c) \,\big|\, \mathcal{F}_t\big) + \varepsilon.\end{aligned}$$

Letting ε tend to 0, we conclude that Θ_t is the value of the Dynkin game at time t.

It remains to establish (25). Let us first check that the right-hand side inequality in (25) is valid for any $\tau_c \in \bar{\mathcal{F}}_T^t$. Let τ^ε denote $\tau_p^\varepsilon \wedge \tau_c$. By the definition of τ_p^ε, we see that K^+ equals 0 on $[t, \tau^\varepsilon]$. Since K^- is non-decreasing, (24) applied to τ^ε yields

$$\alpha_t \Theta_t \le \alpha_{\tau^\varepsilon} \Theta_{\tau^\varepsilon} - \int_t^{\tau^\varepsilon} \alpha_u \, dM_u.$$

Taking conditional expectations (recall that $\int_t^\cdot \alpha_u dM_u$ is an $\mathbb{F}$-martingale), and using also the facts that $\Theta_{\tau_p^\varepsilon} \le L_{\tau_p^\varepsilon} + \varepsilon$ if $\tau_p^\varepsilon < T$, $\Theta_{\tau_p^\varepsilon} = \xi$ if $\tau_p^\varepsilon = T$ and $\Theta_{\tau_c} \le U_{\tau_c}$ (recall that $\tau_c \in \bar{\mathcal{F}}_T^t$, so that $\tau_c \ge \bar{\tau}$ and $\bar{U}_{\tau_c} = U_{\tau_c}$), we obtain

$$\begin{aligned}\alpha_t \Theta_t &\le \mathbb{E}_{\mathbb{Q}}(\alpha_{\tau^\varepsilon} \Theta_{\tau^\varepsilon} \,|\, \mathcal{F}_t) \\ &\le \mathbb{E}_{\mathbb{Q}}\Big(\alpha_{\tau^\varepsilon}\big(\mathbb{1}_{\{\tau^\varepsilon = \tau_p^\varepsilon < T\}}(L_{\tau_p^\varepsilon} + \varepsilon) + \mathbb{1}_{\{\tau^\varepsilon = \tau_c < \tau_p^\varepsilon\}} U_{\tau_c} + \mathbb{1}_{\{\tau^\varepsilon = T\}}\xi\big) \,\Big|\, \mathcal{F}_t\Big).\end{aligned}$$

We conclude that $\Theta_t \le \mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p^\varepsilon,\tau_c) \,\big|\, \mathcal{F}_t\big) + \varepsilon$ for any $\tau_c \in \bar{\mathcal{F}}_T^t$. This completes the proof of the right-hand side inequality in (25). The left-hand side inequality can be shown similarly. It is in fact standard, since it does not involve $\bar{\tau}$, and thus we leave the details to the reader.

Finally, in the special case where K is continuous, ε may be taken equal to 0, since in that case the process K^+ is continuous and thus it equals 0 on $[t, \tau_p^0 \wedge \tau_c]$ for any $\tau_c \in \bar{\mathcal{F}}_T^t$ (similarly, the process K^- equals 0 on $[t, \tau_c^0 \wedge \tau_p]$ for any $\tau_p \in \mathcal{F}_T^t$). □

Let us now apply Proposition 3.5 to a defaultable game option. To this end, we first note that formula (19) can be rewritten as follows

$$\alpha_t \widetilde{\pi}(t;\tau_p,\tau_c) = \alpha_\tau \bar{F}_\tau - \alpha_t \bar{F}_t + \alpha_\tau\big(\mathbb{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau < \tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau = T\}}\xi\big),$$

where

$$\bar{F}_t := \alpha_t^{-1} \int_{[0,t]} \alpha_u \, d\bar{D}_u \text{ with } \bar{D}_t := \int_{[0,t]} dC_u + R_u \, d\Gamma_u. \tag{26}$$

Let us denote by $(\bar{\mathcal{E}})$ equation (23) with $F_t = \bar{F}_t$, that is,

$$\left.\begin{array}{c}\alpha_t \bar{\Theta}_t = \alpha_T \widehat{\xi} + \int_t^T \alpha_u dK_u - \int_t^T \alpha_u dM_u, \quad t \in [0,T], \\ \widehat{L}_t \le \bar{\Theta}_t \le \widehat{U}_t, \quad t \in [0,T], \\ \int_0^T (\bar{\Theta}_{u-} - \widehat{L}_{u-}) \, dK_u^+ = \int_0^T (\widehat{U}_{u-} - \bar{\Theta}_{u-}) \, dK_u^- = 0,\end{array}\right\} (\bar{\mathcal{E}})$$

with

$$\widehat{\xi} = \xi + \bar{F}_T, \quad \widehat{L}_t = L_t + \bar{F}_t, \quad \widehat{U}_t = \bar{U}_t + \bar{F}_t.$$

Assumption 3.7 The doubly reflected BSDE $(\bar{\mathcal{E}})$ admits a solution $(\bar{\Theta}, M, K)$.

Remark 3.8 Let us stress that Assumption 3.7, heroic as it may seem in the general hazard process set-up of the present paper, is in fact a very mild and harmless assumption in any reasonable application one may think of (cf. comments following Definition 3.5).

We denote, for $t \in [0,T]$,

$$\widetilde{\Pi}_t = \bar{\Theta}_t - \bar{F}_t, \quad \Pi_t = \mathbb{1}_{\{t<\tau_d\}}\widetilde{\Pi}_t, \quad \widehat{\Pi}_t = \Pi_t + \beta_t^{-1}\int_{[0,t]} \beta_u \, dD_u. \tag{27}$$

The following lemma is crucial in what follows (Lemma 3.6(i) is actually the key of the proof of Proposition 4.1 below).

Lemma 3.6 (i) *The process m given by the formula, for $t \in [0,T]$,*

$$m_t = \beta_t \widehat{\Pi}_t + \int_{[0,t\wedge\tau_d]} \beta_u \, dK_u, \tag{28}$$

is a $\mathbb{G}$-martingale (stopped at τ_d).
(ii) *The process Π is a $\mathbb{G}$-semimartingale.*
(iii) *If K is a continuous process then the process $\beta\widehat{\Pi}$ is a special $\mathbb{G}$-semimartingale.*

Proof. (i) The triplet $(\widetilde{\Pi}, M, K)$ satisfies (22) with F given by $\bar{F}$ in (26), so that, for $t \in [0,T]$,

$$\alpha_t \widetilde{\Pi}_t = \alpha_T \xi + \int_t^T \alpha_u \, d\bar{D}_u + \int_t^T \alpha_u \, dK_u - \int_t^T \alpha_u \, dM_u$$

and

$$\int_0^t \alpha_u \, dM_u = \alpha_t \widetilde{\Pi}_t + \int_0^t \alpha_u \, dK_u - \alpha_0 \widetilde{\Pi}_0 + \int_0^t \alpha_u \, d\bar{D}_u. \tag{29}$$

Using Lemma A.4, it is thus easy to check that we have, for any $0 \le t \le u \le T$,

$$\mathbb{1}_{\{t<\tau_d\}} e^{\Gamma_t} \, \mathbb{E}_{\mathbb{Q}}\Big(\int_t^u \alpha_v \, dM_v \,\Big|\, \mathcal{F}_t \Big) = \mathbb{E}_{\mathbb{Q}}(m_u - m_t \,|\, \mathcal{G}_t).$$

Since the integral $\int_t^{\cdot} \alpha_v \, dM_v$ is an $\mathbb{F}$-martingale, the process m is a $\mathbb{G}$-martingale and it is manifestly stopped at τ_d.
(ii) Given (27) the process Π is a $\mathbb{G}$-semimartingale by part (i).
(iii) By (28), we have that

$$\beta_t \widehat{\Pi}_t = m_t - \int_{[0,t\wedge\tau_d]} \beta_u \, dK_u, \tag{30}$$

where m is a $\mathbb{G}$-martingale, by (i), and where the second term in the right-hand side is a $\mathbb{G}$-adapted and continuous (hence $\mathbb{G}$-predictable) processes of finite variation. □

In view of (30), in the case where K is a continuous process, the process m introduced in the first part of this lemma can equivalently be redefined as the $\mathbb{G}$-local martingale component of the *discounted cumulative $\mathbb{Q}$-value process* $\beta\widehat{\Pi}$. In that case, the processes m and $\beta\widehat{\Pi}$ are easily seen to coincide on the random interval $[0, \tau_c^0 \wedge \tau_p^0 \wedge \tau_d \wedge T]$ and thus both m and $\beta\widehat{\Pi}$ can be interpreted on this interval as the discounted cumulative $\mathbb{Q}$-value of a defaultable game option. It is thus worth stressing again that in most applications the K-component of a solution to $(\bar{\mathcal{E}})$ is sought for as a *continuous* process of finite variation (see Remark 3.6(iii)).

The following result establishes a link between $(\bar{\Theta}, M, K)$ and the arbitrage ex-dividend $\mathbb{Q}$-price of the defaultable game option.

Proposition 3.7 (Verification Principle for a Defaultable Game Option) *The process Π is the arbitrage ex-dividend $\mathbb{Q}$-price for the game option. Moreover, for any $t \in [0,T]$ and $\varepsilon > 0$, the pair of ε-optimal stopping times $(\tau_p^\varepsilon, \tau_c^\varepsilon) \in \mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$ for the related Dynkin game (6) on $\mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$ is given by*

$$\tau_p^\varepsilon = \inf\Big\{ u \in [t,T] \,;\, \widetilde{\Pi}_u \le L_u + \varepsilon \Big\} \wedge T, \quad \tau_c^\varepsilon = \inf\Big\{ u \in [\bar{\tau} \vee t, T] \,;\, \widetilde{\Pi}_u \ge U_u - \varepsilon \Big\} \wedge T.$$

If K is a continuous process then the pair of stopping times $(\tau_p^0, \tau_c^0) \in \mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$, obtained by setting $\varepsilon = 0$, is a saddle-point of the defaultable game option.

Proof. In view of (19), the present assumptions imply that $\widetilde{\Pi}_t$ is the value of the Dynkin game (21), by Proposition 3.5, with $(\tau_p^\varepsilon, \tau_c^\varepsilon)$ as a pair of ε-optimal stopping times. Therefore, by Lemmas 3.1 and 3.2, Π_t is the value of the Dynkin game associated with the game option on $\mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$, with $(\tau_p^\varepsilon, \tau_c^\varepsilon)$ as a pair of ε-optimal stopping times. Moreover, Π is a $\mathbb{G}$-semimartingale, by Lemma 3.6(ii). We conclude by making use of the last part in Theorem 3.4. □

4 Hedging in a Hazard Process Set-up

The goal of this and the next section, which constitute the central part of this work, it to examine in some detail the existence and properties of various concepts of hedging strategies for defaultable game options in a hazard process set-up.

4.1 Existence of Hedging Strategies

From now on, we shall work under Assumption 3.7. Let thus $(\bar{\Theta}, M, K)$ denote a solution to $(\bar{\mathcal{E}})$ and let $\widetilde{\Pi}$ and Π be defined as in (27). In particular, Π is the arbitrage $\mathbb{Q}$-price for the game option (by Proposition 3.7) and the left-hand sides in (15) and (16) are equal to Π_0.

For an issuer, we define the corresponding process $\rho(\zeta)$ by $\rho_0(\zeta) = 0$ and, for $t \in [0, T]$,

$$\beta_t \, d\rho_t(\zeta) = dm_t - \zeta_t \, d(\beta_t \widehat{X}_t), \tag{31}$$

where m is the $\mathbb{G}$-martingale of Lemma 3.6(i). The process $\rho(\zeta)$ is thus a $\mathbb{G}$-sigma martingale, by Lemma 3.6(i), and a $\mathbb{G}$-local martingale if $\beta\widehat{X}$ and ζ are locally bounded processes. For a holder, the corresponding process $\bar{\rho}(\zeta)$ is defined by

$$\beta_t \, d\bar{\rho}_t(\zeta) = -dm_t - \zeta_t \, d(\beta_t \widehat{X}_t). \tag{32}$$

Since trivially

$$-\beta_t \, d\rho_t(\zeta) = -dm_t + \zeta_t \, d(\beta_t \widehat{X}_t) = -dm_t - (-\zeta_t) \, d(\beta_t \widehat{X}_t),$$

we see that $\bar{\rho}(-\zeta) = -\rho(\zeta)$, so that it suffices to use the notation $\rho(\zeta)$ in what follows. For any $\varepsilon \geq 0$, let τ_c^ε and τ_p^ε be defined as in Proposition 3.7 (for $t = 0$, so that $\tau_c^\varepsilon \in \bar{\mathcal{F}}_T^0$).

Proposition 4.1 *Let ζ stand for an arbitrary $\mathbb{R}^{1\otimes d}$-valued, $\beta\widehat{X}$-integrable process. Then:*
(i) *for any $\varepsilon > 0$, $(\Pi_0, \zeta, \rho(\zeta), \tau_c^\varepsilon)$ is an issuer ε-hedge and $(-\Pi_0, -\zeta, -\rho(\zeta), \tau_p^\varepsilon)$ is a holder ε-hedge,*
(ii) *if the process K is continuous, we may set ε equal to 0 in part (i), and thus $(\Pi_0, \zeta, \rho(\zeta), \tau_c^0)$ and $(-\Pi_0, -\zeta, -\rho(\zeta), \tau_p^0)$ are an issuer hedge and a holder hedge, respectively.*

Proof. (i) For the ease of notation, we write $\rho = \rho(\zeta)$. Let us set, for $t \in [0, T]$,

$$V_t := \beta_t^{-1}\Big(m_t - \int_{[0,t]} \beta_u \, d\rho_u\Big) = \widehat{\Pi}_t + \beta_t^{-1} \int_{[0,t\wedge\tau_d]} \beta_u \, dK_u - \beta_t^{-1} \int_{[0,t]} \beta_u \, d\rho_u, \tag{33}$$

where the second equality follows from (28) and (27). Thus, in particular, $V_0 = \Pi_0$. By definition (31) of ρ, we obtain, for $t \in [0, T]$,

$$d(\beta_t V_t) = dm_t - \beta_t \, d\rho_t = \zeta_t \, d(\beta_t \widehat{X}_t). \tag{34}$$

Hence V is the wealth process of the primary strategy (Π_0, ζ). Let us fix $\varepsilon > 0$. In order to prove that the quadruplet $(\Pi_0, \zeta, \rho, \tau_c^\varepsilon)$ is an issuer ε-hedge with the cost ρ for the game option, where the stopping time $\tau_c^\varepsilon \in \bar{\mathcal{F}}_T^0$ is given by (see Proposition 3.7)

$$\tau_c^\varepsilon = \inf\Big\{ t \in [\bar{\tau}, T]\,;\, \widetilde{\Pi}_t \geq U_t - \varepsilon \Big\} \wedge T,$$

it is enough to show that we have for any $\tau_p \in \mathcal{G}_T^0$, with $\tau = \tau_p \wedge \tau_c^\varepsilon$ (cf. (12)):

$$\begin{aligned} V_\tau + \beta_\tau^{-1} \int_0^\tau \beta_u (d\rho_u - dD_u) \\ \geq \mathbb{1}_{\{\tau<\tau_d\}} \Big(\mathbb{1}_{\{\tau=\tau_p<T\}} L_{\tau_p} + \mathbb{1}_{\{\tau<\tau_p\}} U_{\tau_c^\varepsilon} + \mathbb{1}_{\{\tau_p=\tau_c^\varepsilon=T\}} \xi \Big) - \varepsilon. \end{aligned} \tag{35}$$

From the definition of τ_c^ε, the minimality conditions in $(\bar{\mathcal{E}})$ and the continuity of K^- it follows that $K^- = 0$ and thus $K \geq 0$ on $[0, \tau_c^\varepsilon]$. Since $\tau \leq \tau_c^\varepsilon$, (33) thus yields

$$V_\tau + \beta_\tau^{-1} \int_0^\tau \beta_u (d\rho_u - dD_u) = \Pi_\tau + \beta_\tau^{-1} \int_{[0,\tau\wedge\tau_d]} \beta_u \, dK_u \geq \Pi_\tau = \mathbb{1}_{\{\tau<\tau_d\}} \widetilde{\Pi}_\tau,$$

where, by $(\bar{\mathcal{E}})$, we have

$$\widetilde{\Pi}_\tau \geq \mathbb{1}_{\{\tau<T\}} L_\tau + \mathbb{1}_{\{\tau=T\}} \xi.$$

In addition, by the definition of τ_c^ε, we have that $\widetilde{\Pi}_{\tau_c^\varepsilon} \geq U_{\tau_c^\varepsilon} - \varepsilon$ on the event $\{\tau_c^\varepsilon \leq T\}$. It is now easy to see that (35) is satisfied and thus $(V_0, \zeta, \rho, \tau_c^\varepsilon)$ is indeed an issuer ε-hedge. The arguments for a holder are essentially symmetrical to those for an issuer; the details are left to the reader.

(ii) If K is a continuous process, one can take ε equal to 0 above and thus the second assertion holds as well. Indeed, the continuity of K implies that $K_t^- = 0$ on $[0, \tau_c^0]$ and $K_t^+ = 0$ on $[0, \tau_p^0]$. □

Remark 4.1 (i) The situation where ρ can be made equal to zero by the choice of a suitable strategy ζ in Proposition 4.1 corresponds to a particular form of hedgeability of a game option (cf. Proposition 4.3 below; see also [6]) in which an issuer and a holder are able to hedge all risks embedded in a defaultable game option. The case where $\rho \neq 0$ corresponds either to non-hedgeability of a game option or to the situation in which an issuer (or a holder) is able to hedge, but she *prefers* not to hedge all the risks embedded in the option, for instance, she may be *willing* to take some bets regarding specific risk directions. That is why we do not postulate a priori that ρ should be minimized in some sense as, for instance, in Schweizer [40] (see, however, Section 5.3 for a tentative unified approach).
(ii) For any $\varepsilon > 0$, it is possible to introduce the *trivial hedge* $(\Pi_0, \zeta^0, \rho^0, \tau_c^\varepsilon)$ (resp. $(-\Pi_0, -\zeta^0, -\rho^0, \tau_c^\varepsilon)$) with $\zeta^0 = 0$ and the $\mathbb{G}$-local martingale cost

$$\rho_t^0 = \int_0^t \beta_u^{-1} \, dm_u, \quad t \in [0, T].$$

Obviously, the trivial hedge is of a minor practical interest, since it implicitly assumes that either hedging is impossible or one is not interested in hedging. This hedge (or, more precisely, the existence of any hedge) is used in the proof of Proposition 4.2, however.
(iii) The situation of Proposition 4.1(ii) (a continuous K-component of a solution to $(\bar{\mathcal{E}})$) is a rule rather than an exception in applications (including the case of $\mathbb{F}$-models with jumps; see Remark 3.6(iii)).

Let us now draw some conclusions from Lemma 2.3 and Proposition 4.1.

Proposition 4.2 *Under the assumptions of Proposition 4.1, we have that:*
(i) $\Pi_0 = \operatorname{essmin} \mathcal{V}_0^c$, *so* Π_0 *is the infimum of initial wealths of an issuer hedge with a* $\mathbb{G}$*-sigma martingale cost.*

(ii) *We have that* $-\Pi_0 \in \mathcal{V}_0^p$. *If, in addition, (5) holds then* $\Pi_0 = -\mathrm{essmin}\mathcal{V}_0^p$ *and* $-\Pi_0$ *is the infimum of initial wealths of a holder hedge with a* $\mathbb{G}$*-sigma martingale cost.*
(iii) *The infima above are attained and thus they are in fact minima if* K *is continuous.*
(iv) *The above statements are also valid with* local martingale *instead of* sigma martingale *therein.*

Proof. (i) By applying Proposition 4.1 to the trivial hedge of Remark 4.1(ii), we get, in particular, that $\Pi_0 \in \mathcal{V}_0^c$, where Π_0 is also equal to the $\mathbb{Q}$-value of the related Dynkin game, by Proposition 3.7. Thus, the infimum is attained and we have equality, rather than inequality, in Lemma 2.3(i) (see also Remark 2.7).
(ii) The second claim can be proven as part (i), assuming (5).
(iii) The proof is the same as for parts (i) and (ii) above, using Proposition 4.1(ii).
(iv) This follows immediately of (i) and (ii), since the cost ρ^0 of the trivial hedge is a $\mathbb{G}$-local martingale. □

Remark 4.2 (i) Given our definition of hedging with a cost, the fact that there exists a hedge in this sense with initial wealth Π_0 (as actually with any initial wealth) was of course expected. Due now to the definition Π_0, existence of a hedge *with* $\mathbb{G}$*-sigma (or local in suitable cases) martingale cost* is also a rather natural conclusion. Finally the minimality statement establishes a connection between arbitrage prices and hedging in a general, incomplete market.
(ii) In fact, it is easy to see that one could state analogous definitions and results regarding hedging a defaultable game option, starting at any date $t \in [0, T]$. Otherwise said, the fact that 0 is the inception date of the option is immaterial in Lemma 2.3 and Propositions 4.1 and 4.2.
(iii) In case where K is continuous, an inspection of the above proofs shows that the assumption of a positively bounded discount factor process β may be relaxed into that of a positive and bounded discount factor process β.

4.2 Defaultable European Options

Let us now consider the special case of a defaultable European option.

Definition 4.3 (i) An *issuer hedge* with the cost ρ (a real-valued $\mathbb{G}$-semimartingale with $\rho_0 = 0$) for a defaultable European option is a primary strategy (V_0, ζ) with wealth process V such that (cf. (8))

$$\beta_T V_T + \int_0^T \beta_u \, d\rho_u \geq \beta_0 \phi(0), \quad \text{a.s.}$$

If the inequality may be replaced by equality then we deal with an *issuer replicating strategy* with the cost ρ.
(ii) A *holder hedge* with the cost ρ (a real-valued $\mathbb{G}$-semimartingale with $\rho_0 = 0$) for a defaultable European option is a primary strategy (V_0, ζ) with wealth process V such that (cf. (8))

$$\beta_T V_T + \int_0^T \beta_u \, d\rho_u \geq -\beta_0 \phi(0), \quad \text{a.s.}$$

If the inequality may be replaced by equality then we deal with a *holder replicating strategy* with cost ρ.

In the special case of a defaultable European option, we shall consider the BSDE $(\bar{\mathcal{E}})$ with $\widehat{L}$ replaced by $\bar{L}$ such that $\alpha \bar{L} = -(c+1)$, where $-c$ is a lower bound on $\alpha_T \widehat{\xi}$. Note that under mild technical assumptions this equation has a solution $(\bar{\Theta}, M, K)$ (see [5, 12])). By Proposition 3.5, we obtain

$$\alpha_t \bar{\Theta}_t = \mathrm{esssup}_{\tau_p \in \mathcal{F}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau_p < T\}} \alpha_{\tau_p} \bar{L}_{\tau_p} + \mathbb{1}_{\{\tau_p = T\}} \alpha_T \widehat{\xi} \,\big|\, \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(\alpha_T \widehat{\xi} \,\big|\, \mathcal{F}_t\big),$$

where we also used the definition of $\bar{L}$. So, first, the $\bar{\Theta}$-component of $(\bar{\mathcal{E}})$ is the arbitrage $\mathbb{Q}$-price of the option and, second, we have that $\alpha\bar{\Theta} \geq -c$. Hence $\bar{\Theta} > \bar{L}$ on $[0,T]$, so that necessarily $K = 0$ and $(\bar{\mathcal{E}})$ effectively reduces to an elementary BSDE with no process K involved in the solution.

The next result can be established along similar lines as Lemma 2.3, Propositions 4.1 and 4.2. The proofs are, of course, simpler since there are in effect no barriers involved, and thus they are omitted. Note also that the obvious analogues to Remark 4.2 can be formulated.

Proposition 4.3 *In the case of a defaultable European option, assume that the BSDE $(\bar{\mathcal{E}})$ with $\widehat{L}$ replaced by $\bar{L}$ and with $\bar{\tau} = T$ admits a solution $(\bar{\Theta}, M, K = 0)$. Let us set $\widetilde{\Phi}_t = \bar{\Theta}_t - \bar{F}_t$ for $t \in [0,T]$.*
(i) *The process $\Phi_t = \mathbb{1}_{\{t<\tau_d\}}\widetilde{\Phi}_t$ is the arbitrage price process for the option, as well as the minimal issuer hedging price process with $\mathbb{G}$-sigma martingale (or local martingale) cost.*
(ii) *In the case where R and ξ are bounded, $-\Phi$ is also the minimal holder hedging price process with $\mathbb{G}$-sigma martingale (or local martingale) cost.*
(iii) *Given any $\mathbb{R}^{1\otimes d}$-valued, $\beta\widehat{X}$-integrable process ζ, let the $\mathbb{G}$-sigma martingale (or $\mathbb{G}$-local martingale in case where $\beta\widehat{X}$ and ζ are locally bounded) ρ be defined as (31). Then (Π_0, ζ, ρ) and $(-\Pi_0, -\zeta, -\rho)$ are the replicating strategies with $\mathbb{G}$-sigma martingale (or local martingale) cost for an issuer and a holder, respectively.*

4.3 Hedging Error Process

In the situation of Propositions 4.1 or 4.3, the following result establishes the link between the notion of the cost process $\rho = \rho(\zeta)$ of a strategy ζ, as defined by (31), and a more practical concept of the *hedging error process* (also known as the *tracking error* (cf. [20]) or the *profit and loss process*) $e = e(\zeta)$ relative to the ex-dividend $\mathbb{Q}$-price process Π. From the perspective of an option's issuer, the discounted hedging error is defined by, for $t \in [0,T]$ (cf. (27)),

$$\beta_t e_t = \beta_0\widehat{\Pi}_0 + \int_0^t \zeta_u \, d(\beta_u \widehat{X}_u) - \beta_t\widehat{\Pi}_t. \tag{36}$$

Corollary 4.4 *We have, for $t \in [0,T]$,*

$$\beta_t e_t = \int_0^{t\wedge\tau_d} \beta_u \, dK_u - \int_0^t \beta_u \, d\rho_u. \tag{37}$$

In particular:
(i) *In the case when ρ is a $\mathbb{G}$-local martingale and K is a continuous process, then the discounted hedging error βe is a $\mathbb{G}$-special semimartingale with the canonical Doob-Meyer decomposition given by (37), where the $\mathbb{G}$-local martingale component is given by $\int_0^\cdot \beta_t \, d\rho_t$.*
(ii) *In the case of a European derivative with $K = 0$, the discounted hedging error βe is given as a $\mathbb{G}$-sigma martingale (or a $\mathbb{G}$-local martingale, in case ρ is a $\mathbb{G}$-local martingale)*

$$\beta_t e_t = -\int_0^t \beta_u \, d\rho_u.$$

Proof. Using (28) and (36), we obtain, for $t \in [0,T]$,

$$\beta_t e_t = m_0 - m_t + \int_0^{t\wedge\tau_d} \beta_u \, dK_u + \int_0^t \zeta_u \, d(\beta_u \widehat{X}_u) = \int_0^{t\wedge\tau_d} \beta_u \, dK_u - \int_0^t \beta_u \, d\rho_u$$

by definition (31) of ρ. All assertions now easily follow. □

Note that typically in applications:
• $\beta\widehat{X}$ and ζ are locally bounded processes, so that ρ is a $\mathbb{G}$-local martingale (cf. (31)),

• K is a continuous process (cf. Remark 3.6(iii)).
We are thus in the situation of Corollary 4.4(i) and thus the discounted hedging error process is a special $\mathbb{G}$-semimartingale with the local martingale component given by $\int_0^\cdot \beta_t \, d\rho_t$.

5 Analysis of Hedging Strategies

5.1 Discounted Cumulative Value Dynamics

For the next result, we will need the following technical assumption.

Assumption 5.1 The processes K and M do not jump at τ_d, that is, $\Delta K_{\tau_d} := K_{\tau_d} - K_{\tau_d-} = 0$ and $\Delta M_{\tau_d} := M_{\tau_d} - M_{\tau_d-} = 0$.

Assumption 5.1 is notably satisfied when the default time τ_d *avoids* $\mathbb{F}$-stopping times, in the sense that $\mathbb{Q}(\tau_d = \tau) = 0$ for any $\mathbb{F}$-stopping time τ. This indeed implies that an $\mathbb{F}$-adapted càdlàg process (e.g., K or M) does not jump at τ_d. This avoidance assumption is standard in the literature on the progressive enlargement of filtration (see, e.g., [1, 17]). It holds, for instance, in the case where τ_d is constructed by the *canonical construction* (cf. [7]).

Another common situation in which Assumption 5.1 is satisfied is of course when K is a continuous process and when all $\mathbb{F}$-martingales are continuous (cf. Remark 3.6(ii) and (iii)).

Let $N^d = H - \Gamma_{\cdot\wedge\tau_d}$ stand for the *compensated jump-to-default process*. Under our standing assumption that the $\mathbb{F}$-hazard process Γ of τ_d is a continuous and non-decreasing process (cf. Remark 3.2(ii)), the process N^d is known to be a $\mathbb{G}$-martingale.

An analysis of hedging strategies in the next section will rely on the following lemma, which yields the dynamics of the discounted cumulative value process of a game option or, more precisely, of its martingale component m (see the comments following the proof of Lemma 3.6).

Lemma 5.1 *Under Assumption 5.1, the $\mathbb{G}$-martingale m defined by (28) satisfies*

$$dm_t = \mathbb{1}_{\{t\le\tau_d\}}\beta_t\big(dM_t + \big(R_t - \Pi_{t-}\big)\, dN^d_t\big). \tag{38}$$

Proof. Let us introduce the *Doléans-Dade martingale* (see, e.g., [7])

$$\mathcal{E}_t = \mathbb{1}_{\{t<\tau_d\}}e^{\Gamma_t} = 1 - \int_0^t \mathcal{E}_{u-}\, dN^d_u,$$

so that $\alpha_t\mathcal{E}_t = \beta_t\mathbb{1}_{\{t<\tau_d\}}$ and $\alpha_t\mathcal{E}_{t-} = \beta_t\mathbb{1}_{\{t\le\tau_d\}}$. Then (cf. (27) and (28))

$$dm_t = d(\beta_t\widehat{\Pi}_t) + \mathbb{1}_{\{t\le\tau_d\}}\beta_t\, dK_t = d(\mathcal{E}_t\alpha_t\widetilde{\Pi}_t) + \mathbb{1}_{\{t\le\tau_d\}}\beta_t\, dK_t + \beta_t\, dD_t. \tag{39}$$

It may happen that (the $\mathbb{F}$-semimartingale) $\alpha\widetilde{\Pi}$ is not a $\mathbb{G}$-semimartingale, so a direct application of the ($\mathbb{G}$-)integration by parts formula to $\mathcal{E}\alpha\widetilde{\Pi}$ is precluded. However, by Lemma A.1(iv), the process $\alpha\widetilde{\Pi}$ *stopped at* τ_d is a $\mathbb{G}$-semimartingale. It is also clear that $\mathcal{E}\alpha\widetilde{\Pi} = \mathcal{E}\alpha_{\cdot\wedge\tau_d}\widetilde{\Pi}_{\cdot\wedge\tau_d}$. Hence, by applying the integration by parts formula to $\mathcal{E}\alpha_{\cdot\wedge\tau_d}\widetilde{\Pi}_{\cdot\wedge\tau_d}$, we obtain

$$d(\mathcal{E}_t\alpha_{t\wedge\tau_d}\widetilde{\Pi}_{t\wedge\tau_d}) = \mathcal{E}_{t-}\Big(d\big(\alpha_{t\wedge\tau_d}\widetilde{\Pi}_{t\wedge\tau_d}\big) - \alpha_t\widetilde{\Pi}_{t-}\, dN^d_t\Big) + d[\mathcal{E}, \alpha_{\cdot\wedge\tau_d}\widetilde{\Pi}_{\cdot\wedge\tau_d}]_t,$$

where, in addition, we have that $[\mathcal{E}, \alpha_{\cdot\wedge\tau_d}\widetilde{\Pi}_{\cdot\wedge\tau_d}]_t = -e^{\Gamma_{\tau_d}}\alpha_{\tau_d}\Delta\widetilde{\Pi}_{\tau_d}H_t$. Using Assumption 5.1, formula (29) and the facts that the coupon process C is $\mathbb{F}$-predictable and the hazard process Γ is

continuous, so that $\Delta C_{\tau_d} = \Delta\Gamma_{\tau_d} = 0$, we check that $\Delta\widetilde{\Pi}_{\tau_d} = 0$. Using (29), we then deduce from (39) that

$$\begin{aligned} dm_t &= \mathcal{E}_{t-}\Big(d\big(\alpha_{t\wedge\tau_d}\widetilde{\Pi}_{t\wedge\tau_d}\big) - \alpha_t\widetilde{\Pi}_{t-}\,dN^d_t\Big) + \mathbb{1}_{\{t\le\tau_d\}}\beta_t\,dK_t + \beta_t\,dD_t \\ &= \mathbb{1}_{\{t\le\tau_d\}}\beta_t\big(-dK_t - dC_t - R_t\,d\Gamma_t + dM_t - \widetilde{\Pi}_{t-}\,dN^d_t\big) + \mathbb{1}_{\{t\le\tau_d\}}\beta_t\,dK_t + \beta_t\,dD_t \\ &= \mathbb{1}_{\{t\le\tau_d\}}\beta_t\big(-dC_t - R_t\,d\Gamma_t + dM_t - \widetilde{\Pi}_{t-}\,dN^d_t\big) + \beta_t\,dD_t. \end{aligned}$$

Using (3) and the equality $\Delta C_{\tau_d} = 0$, we finally arrive at the equality

$$dm_t = \mathbb{1}_{\{t\le\tau_d\}}\beta_t\big(dM_t + (R_t - \widetilde{\Pi}_{t-})\,dN^d_t\big),$$

which is the required result. □

5.2 Hedging via Orthogonal Decompositions

In order to study non-trivial hedging strategies for a defaultable game option in the general set-up of this paper, we resort to suitable (Galtchouk-Kunita-Watanabe) *decompositions* of a solution to the related doubly reflected BSDE. Note that in a more specific Markovian set-up, a short-cut to get such decomposition will consist in using suitable versions of the Itô formula (see Bielecki et al. [5] and [6]).

We assume here that a reference $\mathbb{R}^q$-valued $\mathbb{F}$-semimartingale, denoted by N, is given a priori. In any particular application, the choice of this process will depend on the problem at hand (see [5]).

By a *decomposition of M* we mean the equality

$$dM_t = Z_t\,dN_t + dn_t, \quad t\in[0,T], \tag{40}$$

where Z is an $\mathbb{F}$-adapted, $\mathbb{R}^{1\otimes q}$-valued, N-integrable process and n is a real-valued $\mathbb{F}$-semimartingale. As it will become apparent in the sequel, n is expected to be orthogonal to N in some sense, which explains, for instance, why we cannot simply take $Z = 0$ in (40). Let us denote $\mathcal{N} = \begin{bmatrix} N \\ N^d \end{bmatrix}$. A decomposition of M combined with (38), yields, for every $t\in[0,T\wedge\tau_d]$,

$$dm_t = \beta_t Z_t\,dN_t + \beta_t(R_t - \Pi_{t-})\,dN^d_t + \beta_t\,dn_t \;=\; \beta_t[Z_t, Y_t]\,d\mathcal{N}_t + \beta_t\,dn_t \;=\; \beta_t\Xi_t\,d\mathcal{N}_t + \beta_t\,dn_t\ , \tag{41}$$

where we set $Y_t = R_t - \Pi_{t-}$ and $\Xi_t = [Z_t, Y_t]$, the concatenation of Z_t and Y_t.

Let us now focus on the discounted price process $\beta\widehat{X}$. By a *decomposition of $\beta\widehat{X}$* we mean the equality, for every $t\in[0,T\wedge\tau_d]$,

$$d(\beta_t\widehat{X}_t) = \beta_t\widehat{Z}_t\,dN_t + \beta_t\widehat{Y}_t\,dN^d_t + \beta_t\,d\widehat{n}_t \;=\; \beta_t\Sigma_t\,d\mathcal{N}_t + \beta_t\,d\widehat{n}_t \tag{42}$$

for some $\mathbb{G}$-adapted, $\mathbb{R}^{d\otimes(q+1)}$-valued, $\mathcal{N}_{\cdot\wedge\tau_d}$-integrable process $\beta\Sigma = \beta[\widehat{Z},\widehat{Y}]$ and some $\mathbb{G}$-adapted, $\mathbb{R}^d$-valued process $\widehat{n}$.

Proposition 5.2 *Assume that we are given decompositions (40) and (42). Under Assumption 5.1, for any $\mathbb{R}^{1\otimes d}$-valued, $\beta\widehat{X}$-integrable process ζ, the related cost $\rho = \rho(\zeta)$ in Proposition 4.1 satisfies, for every $t\in[0,T\wedge\tau_d]$,*

$$d\rho_t = \big(\Xi_t - \zeta_t\Sigma_t\big)\,d\mathcal{N}_t + \big(dn_t - \zeta_t\,d\widehat{n}_t\big). \tag{43}$$

(i) *Assume, in addition, that Σ is left-invertible on $[0,T\wedge\tau_d]$ with the left inverse $\widehat{\Lambda}$ and define the strategy $\widehat{\zeta}$ by the formula, for $t\in[0,T\wedge\tau_d]$,*

$$\widehat{\zeta}_t = \Xi_t\widehat{\Lambda}_t. \tag{44}$$

Then the cost $\widehat{\rho} = \rho(\widehat{\zeta})$ *satisfies, for* $t \in [0, T \wedge \tau_d]$,

$$d\widehat{\rho}_t = dn_t - \widehat{\zeta}_t \, d\widehat{n}_t. \tag{45}$$

(ii) *Alternatively to (i), let us assume additionally that* $\widehat{Y} = 0$ *and that* $\widehat{Z}$ *is left-invertible on* $[0, T \wedge \tau_d]$ *with the left inverse* $\widetilde{\Lambda}$ *and let us define the strategy* $\widetilde{\zeta}$ *by the formula, for* $t \in [0, T \wedge \tau_d]$,

$$\widetilde{\zeta}_t = Z_t \widetilde{\Lambda}_t. \tag{46}$$

Then the cost $\widetilde{\rho} = \rho(\widetilde{\zeta})$ *satisfies, for* $t \in [0, T \wedge \tau_d]$,

$$d\widetilde{\rho}_t = dn_t - \widetilde{\zeta}_t \, d\widehat{n}_t + Y_t \, dN_t^d. \tag{47}$$

Proof. All formulas follow from definition (31) of the cost process and formulas (41), (44) and (46). □

In relation with Remark 4.1(ii), note that the situation of Proposition 5.2(i) corresponds to the hedgeable case, where the cost $\widehat{\rho}$ vanishes for a strategy $\widehat{\zeta}$. The situation of Proposition 5.2(ii) corresponds to the case of unhedgeable default risk.

Practically useful decompositions of M and $\beta\widehat{X}$ will depend on a particular model for the primary market, as well as on the game option under consideration. In an abstract set-up they follow from martingale representation theorems with orthogonal components (for complementary results in the Markovian set-up we refer the interested reader to Bielecki et al. [5]). Let thus $\mathcal{H}^2(\mathbb{F})$ (resp. $\mathcal{H}^2(\mathbb{G})$) denote the class of real-valued $\mathbb{F}$ (resp. $\mathbb{G}$)-martingales with integrable quadratic variation over $[0, T]$, or, by a slight abuse of notation, the class of vector-valued processes with mutually strongly orthogonal components in $\mathcal{H}^2(\mathbb{F})$ (resp. $\mathcal{H}^2(\mathbb{G})$). It is worth noting that an $\mathbb{F}$-martingale stopped at τ_d is a $\mathbb{G}$-local martingale, by Lemma A.1(iii).

Assumption 5.2 The processes M and N belong to $\mathcal{H}^2(\mathbb{F})$, the process $\mathcal{N}$ stopped at τ_d belongs to $\mathcal{H}^2(\mathbb{G})$ and the process $\beta\widehat{X}$ belongs to $\mathcal{H}^2(\mathbb{G})$.

The Galtchouk-Kunita-Watanabe (GKW) decomposition of M with respect to N in $\mathbb{F}$ (see, e.g., Protter [39, IV.3, Corollary 1]) thus yields a decomposition (40) of M with n strongly orthogonal to N in $\mathcal{H}^2(\mathbb{F})$.

The GKW decomposition of $\beta\widehat{X}$ with respect to $\mathcal{N}_{\cdot\wedge\tau_d}$ in $\mathbb{G}$ yields likewise a decomposition (42) for some $\mathbb{R}^d$-valued process $\int_0^{\cdot} \beta_t \, d\widehat{n}_t$ strongly orthogonal to $\mathcal{N}_{\cdot\wedge\tau_d}$ in $\mathcal{H}^2(\mathbb{G})$. Alternatively, one may consider the GKW decomposition theorem of $\beta\widehat{X}$ with respect to $N_{\cdot\wedge\tau_d}$ in $\mathbb{G}$, which yields a decomposition of the form (42) with $\widehat{Y} = 0$, and for some $\mathbb{R}^d$-valued process $\int_0^{\cdot} \beta_t \, d\widehat{n}_t$ strongly orthogonal to $N_{\cdot\wedge\tau_d}$ in $\mathcal{H}^2(\mathbb{G})$.

The following proposition justifies the informal statement that the strategy $\widehat{\zeta}$ (resp. $\widetilde{\zeta}$) hedges the risk source $\mathcal{N}$ (resp. N). In this result and its proof, the symbols $[\cdot, \cdot]$ and $[\cdot, \cdot]^{\mathbb{F}}$ denote the square brackets with respect to filtrations $\mathbb{G}$ and $\mathbb{F}$, respectively.

Proposition 5.3 *Let (40) be given as the GKW decomposition of* M *with respect to* N.
(i) *In the situation of Proposition 5.2(i) with (42) given as the GKW decomposition of* $\beta\widehat{X}$ *with respect to* $\mathcal{N}_{\cdot\wedge\tau_d}$, *then the processes* $\widehat{\rho}$ *and* $\mathcal{N}_{\cdot\wedge\tau_d}$ *are* orthogonal in $\mathbb{G}$, *in the sense that* $[\widehat{\rho}, \mathcal{N}_{\cdot\wedge\tau_d}]$ *is a* $\mathbb{G}$*-sigma martingale (and a* $\mathbb{G}$*-local martingale if* $\widehat{\zeta}$ *is locally bounded).*
(ii) *In the situation of Proposition 5.2(ii) with (42) given as the GKW decomposition of* $\beta\widehat{X}$ *with respect to* $N_{\cdot\wedge\tau_d}$, *then the processes* $\widetilde{\rho}$ *and* $N_{\cdot\wedge\tau_d}$ *are* orthogonal in $\mathbb{G}$, *in the sense that* $[\widetilde{\rho}, N_{\cdot\wedge\tau_d}]$ *is a* $\mathbb{G}$*-sigma martingale (and a* $\mathbb{G}$*-local martingale if* $\widetilde{\zeta}$ *and* R *are locally bounded processes).*

Proof. Observe first that $n_{\cdot\wedge\tau_d}$ and $N_{\cdot\wedge\tau_d}$ are $\mathbb{G}$-local martingales, by Lemma A.1(iii). Since n is strongly orthogonal to N in $\mathcal{H}^2(\mathbb{F})$, the process $[n_{\cdot\wedge\tau_d}, N_{\cdot\wedge\tau_d}] = [n, N]^{\mathbb{F}}_{\cdot\wedge\tau_d}$ is a $\mathbb{G}$-local martingale, as an $\mathbb{F}$-local martingale stopped at τ_d (cf. Lemma A.1(iii)). Furthermore, by Lemma A.5, $[n_{\cdot\wedge\tau_d}, N^d]$ is a $\mathbb{G}$-local martingale. We conclude that $[n_{\cdot\wedge\tau_d}, \mathcal{N}_{\cdot\wedge\tau_d}]$ is a $\mathbb{G}$-local martingale.
In case (i), so is also $[\widehat{n}, \mathcal{N}_{\cdot\wedge\tau_d}]$, since the integral $\int_0^{\cdot} \beta_t \, d\widehat{n}_t$ is strongly orthogonal to $\mathcal{N}_{\cdot\wedge\tau_d}$ in $\mathcal{H}^2(\mathbb{G})$. Using (45), we conclude that $[\widehat{\rho}, \mathcal{N}_{\cdot\wedge\tau_d}]$ is a $\mathbb{G}$-sigma martingale and thus it follows a $\mathbb{G}$-local martingale if $\widehat{\zeta}$ is a locally bounded process.
In case (ii), the integral $\int_0^{\cdot} \beta_t \, d\widehat{n}_t$ is strongly orthogonal to $N_{\cdot\wedge\tau_d}$ in $\mathcal{H}^2(\mathbb{G})$, so the process $[\widehat{n}, N_{\cdot\wedge\tau_d}]$ is a $\mathbb{G}$-local martingale. Furthermore, by Lemma A.5, $[N_{\cdot\wedge\tau_d}, N^d]$ is a $\mathbb{G}$-local martingale. In view of (47), we conclude that $[\widetilde{\rho}, N_{\cdot\wedge\tau_d}]$ is a $\mathbb{G}$-sigma martingale and thus it follows a $\mathbb{G}$-local martingale if $\widetilde{\zeta}$ and R are locally bounded processes. □

5.3 Minimal Variance Hedging

It is rather obvious, that we provided here only a preliminary analysis of hedging strategies for defaultable game options. In order to give a unified perspective on hedging strategies, let us now consider hedging a defaultable game option as the problem of finding a strategy ζ in (31) that makes ρ in (31) $\mathbb{G}$-orthogonal to a given vector-valued $\mathbb{G}$-local martingale $\widetilde{N}$, which is, without loss of generality, stopped at τ_d. We thus consider in here a notion of hedging, which is alternative to that of Definition 2.5.

In reference to Proposition 5.3, by the $\mathbb{G}$-orthogonality, we mean here that $[\rho, \widetilde{N}]$ is a $\mathbb{G}$-local martingale. Also, in the context of this section, the process m in (31) may be defined either by (28), in reference to a solution of a related doubly reflected BSDE with respect to a reference filtration $\mathbb{F}$, or, more generally, as the $\mathbb{G}$-local martingale component of the discounted cumulative $\mathbb{Q}$-value process $\beta\widehat{\Pi}$ of a game option if $\beta\widehat{\Pi}$ is a $\mathbb{G}$-special semimartingale.

Let us thus take for granted a decomposition of the form

$$d(\beta_t \widehat{X}_t) = \beta_t \widetilde{Z}_t \, d\widetilde{N}_t + \beta_t \, d\widetilde{n}_t, \quad t \in [0, T \wedge \tau_d], \tag{48}$$

with $\widetilde{n}$ and $\widetilde{N}$ orthogonal in $\mathbb{G}$ and $\widetilde{Z}$ left-invertible on $[0, T \wedge \tau_d]$. Then, in order to have the cost ρ orthogonal to $\widetilde{N}$ in $\mathbb{G}$, it suffices to choose ζ so that $m - \int_0^{\cdot} \beta_t \zeta_t \widetilde{Z}_t \, d\widetilde{N}_t$ be $\mathbb{G}$-orthogonal to $\widetilde{N}$. Now, relying on the multi-linear regression formula, this can be achieved by setting, for $t \in [0, T \wedge \tau_d]$,

$$\zeta_t = \mathbb{C}\mathrm{ov}_t(dm_t, \beta_t \, d\widetilde{N}_t) \mathbb{V}\mathrm{ar}_t(\beta_t \, d\widetilde{N}_t)^{-1} \Lambda_t, \tag{49}$$

where Λ is the left inverse of $\widetilde{Z}$ on $[0, T \wedge \tau_d]$ and where we denote

$$\mathbb{C}\mathrm{ov}_t(dm_t, \beta_t \, d\widetilde{N}_t) = \beta_t \lim_{h \to 0} \frac{1}{h} \, \mathbb{C}\mathrm{ov}\big(m_{t+h} - m_t, \, \widetilde{N}_{t+h} - \widetilde{N}_t \,\big|\, \mathcal{G}_t\big)$$

and

$$\mathbb{V}\mathrm{ar}_t(\beta_t \, d\widetilde{N}_t) = \beta_t^2 \lim_{h \to 0} \frac{1}{h} \, \mathbb{V}\mathrm{ar}\big(\widetilde{N}_{t+h} - \widetilde{N}_t \,\big|\, \mathcal{G}_t\big).$$

So the problem of hedging the option with respect to the risk factor $\widetilde{N}$ could be solved, at least theoretically, provided one could find a decomposition (48) with the required properties. Let us conclude this short analysis by noting that there are at least two situations in which such a decomposition (48) can be obtained explicitly.

First, it may be obtained as the Galtchouk decomposition of $\beta\widehat{X}$ with respect to $\widetilde{N}$, inasmuch as the related matrix $\widetilde{Z}$ is left-invertible on $[0, T \wedge \tau_d]$. For $\widetilde{N} = \mathcal{N}_{\cdot\wedge\tau_d}$ or $N_{\cdot\wedge\tau_d}$, we leave as an exercise the issue whether the strategies (49) for $\widetilde{N} = \mathcal{N}_{\cdot\wedge\tau_d}$ or $N_{\cdot\wedge\tau_d}$ on the one hand, and (44) or (46) on the other hand, can be shown to coincide, at least in the context of specific Markovian set-ups where all the computations can be pushed further.

The second situation corresponds to the case of a process $\widetilde{N}$ such that

$$\beta_t \, d\widetilde{N}_t = \Lambda_t \, d(\beta_t \widehat{X}_t), \quad t \in [0, T \wedge \tau_d],$$

for a $\mathbb{G}$-predictable, locally bounded, $\mathbb{R}^{d\otimes d}$-valued, invertible process Λ (so in that case $\widetilde{n} = 0$ in (48)). Then formula (49) for the related strategy ζ reduces to

$$\begin{aligned} \zeta_t &= \mathbb{C}\mathrm{ov}_t(dm_t, \Lambda_t \, d(\beta_t \widehat{X}_t)) \mathbb{V}\mathrm{ar}_t(\Lambda_t \, d(\beta_t \widehat{X}_t))^{-1} \Lambda_t \\ &= \mathbb{C}\mathrm{ov}_t(dm_t, d(\beta_t \widehat{X}_t)) \mathbb{V}\mathrm{ar}_t(d(\beta_t \widehat{X}_t))^{-1} =: \zeta_t^{va}. \end{aligned}$$

We recognize here a strategy ζ^{va}, which is commonly known as the *locally minimum variance hedging strategy*. It effectively corresponds to the cost process orthogonal to prices of primary assets under the pre-selected *risk-neutral* probability $\mathbb{Q}$, in our set-up.

A Auxiliary Lemmas

Recall that an $\mathbb{F}$-*pseudo-stopping time* τ is an $\mathcal{F}_\infty$-measurable and non-negative random variable such that $\mathbb{E}_\mathbb{Q} M_\tau = \mathbb{E}_\mathbb{Q} M_0$ for every bounded $\mathbb{F}$-martingale M (see Nikeghbali and Yor [38, Remark 1]).

We work under our standing Assumption 3.1.

Lemma A.1 (i) *G is a non-increasing process.*
(ii) *The $\mathbb{G}$-stopping time τ_d is an $\mathbb{F}$-pseudo-stopping time.*
(iii) *Any $\mathbb{F}$-local martingale stopped at τ_d is a $\mathbb{G}$-local martingale.*
(iv) *Any $\mathbb{F}$-semimartingale stopped at τ_d is a $\mathbb{G}$-semimartingale.*
(v) *The integral process of a continuous integrand with respect to an $\mathbb{F}$-martingale stopped at τ_d is a $\mathbb{G}$-local martingale.*

Proof. Since G is a continuous supermartingale, it admits the Doob-Meyer decomposition $G = M - A$ with a continuous martingale component M [28, p.44, Lemma 4.24]. Hence M is in fact constant, as a continuous martingale with finite variation, and thus (i) holds. By [38, Theorem 4.5], (i) implies (ii) (note that the continuity of the filtration $\mathbb{F}$ is only used for the converse in [38, Theorem 4.5]). By [38, Theorem 4.4], (ii) implies (iii), which immediately yields (iv). As for (v), we have that an $\mathbb{F}$-martingale stopped at τ_d is a $\mathbb{G}$-local martingale, by (iii). The integral process of a continuous (hence predictable and locally bounded) integrand, with respect to an $\mathbb{F}$-martingale stopped at τ_d, is thus a $\mathbb{G}$-local martingale (cf. Remark 2.1). □

We recall the following well-known results. We refer the interested reader to Bielecki and Rutkowski [7, Lemma 5.1.2(ii) and Corollary 5.1.2] for (i) and Dellacherie et al. [17, p. 186, §75] for (ii) (see also Proposition 9.12 of Nikeghbali [37]).

Lemma A.2 (i) *Let χ be a $\mathcal{G}_\infty$-measurable random variable. For any $t \in \mathbb{R}_+$ such that one of the members of the following equality is well defined in $\overline{\mathbb{R}}$ (e.g., χ bounded from one side), the other one is well defined too, and we have*

$$\mathbb{1}_{\{t<\tau_d\}} \mathbb{E}_\mathbb{Q}(\chi \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau_d\}} \, e^{\Gamma_t} \, \mathbb{E}_\mathbb{Q}(\mathbb{1}_{\{t<\tau_d\}} \chi \,|\, \mathcal{F}_t).$$

In particular, if χ is $\mathcal{G}_t$-measurable then $\mathbb{1}_{\{t<\tau_d\}}\chi = \mathbb{1}_{\{t<\tau_d\}} \widetilde{\chi}$ where $\widetilde{\chi} = e^{\Gamma_t} \mathbb{E}_\mathbb{Q}(\mathbb{1}_{\{t<\tau_d\}}\chi \,|\, \mathcal{F}_t)$ is an $\mathcal{F}_t$-measurable random variable. So for any $\mathbb{G}$-adapted process Y over $[0, T]$, there exists an $\mathbb{F}$-adapted process $\widetilde{Y}$ over $[0, T]$ such that

$$\mathbb{1}_{\{t<\tau_d\}} Y_t = \mathbb{1}_{\{t<\tau_d\}} \widetilde{Y}_t, \quad t \in [0, T].$$

(ii) *For any* $\mathbb{G}$*-predictable process* Y *over* $[0,T]$*, there exists an* $\mathbb{F}$*-predictable process* $\widetilde{Y}$ *over* $[0,T]$ *such that*

$$\mathbb{1}_{\{t\le\tau_d\}}Y_t = \mathbb{1}_{\{t\le\tau_d\}}\widetilde{Y}_t, \quad t\in[0,T]. \tag{50}$$

Remark A.1 In the $\mathbb{G}$-predictable case, the process $\widetilde{Y}$ satisfying (50) is *uniquely defined* under Assumption 3.1, by [17, p.186].

For any $t\in[0,T]$, we denote by $\mathcal{F}_T^t$ the set of all $\mathbb{F}$-stopping times with values in $[t,T]$. Also, given a stopping time $\bar\tau\in\mathcal{F}_T^0$ let $\bar{\mathcal{F}}_T^t$ stand for the class $\{\tau\in\mathcal{F}_T^t\,;\,\tau\ge\bar\tau\}$. The following result examines the relevant properties of these classes of stopping times.

Lemma A.3 (i) *If* $\tau\in\mathcal{G}_T^t$ *for some* $t\in[0,T]$ *then there exists* $\widetilde\tau\in\mathcal{F}_T^t$ *such that* $\tau\wedge\tau_d=\widetilde\tau\wedge\tau_d$*. Moreover, if* $\bar\tau\in\mathcal{G}_T^0$ *and if* $\tau\in\bar{\mathcal{G}}_T^t$ *for some* $t\in[0,T]$ *then we have* $\widetilde\tau\wedge\tau_d\ge\bar\tau\wedge\tau_d$*.*
(ii) *If* $\bar\tau\in\mathcal{F}_T^0$ *and* $\tau\in\bar{\mathcal{G}}_T^t$ *for some* $t\in[0,T]$ *then there exists* $\widetilde\tau\in\bar{\mathcal{F}}_T^t$ *such that* $\tau\wedge\tau_d=\widetilde\tau\wedge\tau_d$*.*

Proof. Since τ is a $\mathbb{G}$-stopping time, by [17, p. 186, §75] there exists an $\mathbb{F}$-stopping time $\widehat\tau$ such that $\tau\wedge\tau_d=\widehat\tau\wedge\tau_d$. Moreover, since $\tau\in\mathcal{G}_T^t$, we have

$$\tau\wedge\tau_d=(\tau\vee t)\wedge\tau_d=(\tau\wedge\tau_d)\vee(t\wedge\tau_d)=(\widehat\tau\wedge\tau_d)\vee(t\wedge\tau_d)=(\widehat\tau\vee t)\wedge\tau_d,$$

so that we may take $\widetilde\tau=\widehat\tau\vee t\in\mathcal{F}_T^t$. Moreover, if $\tau\wedge\tau_d\ge\bar\tau\wedge\tau_d$ for some stopping time $\bar\tau\in\mathcal{G}_T^0$, then we also have that $\widetilde\tau\wedge\tau_d=\tau\wedge\tau_d\ge\bar\tau\wedge\tau_d$, which proves (i).

For (ii), let $\breve\tau\in\mathcal{F}_T^t$ be such that $\tau\wedge\tau_d=\breve\tau\wedge\tau_d$, by (i). Assuming that $\bar\tau\in\mathcal{F}_T^0$, we have that $\widetilde\tau=\breve\tau\vee\bar\tau\in\bar{\mathcal{F}}_T^t$. So

$$\widetilde\tau\wedge\tau_d=(\breve\tau\vee\bar\tau)\wedge\tau_d=(\breve\tau\wedge\tau_d)\vee(\bar\tau\wedge\tau_d)=\breve\tau\wedge\tau_d=\tau\wedge\tau_d,$$

where the third equality holds, since $\tau\in\bar{\mathcal{G}}_T^t$ implies that $\breve\tau\wedge\tau_d\ge\bar\tau\wedge\tau_d$, by (i). □

The following lemma is of independent interest. Formula (51) can be found in Dellacherie [16, T47] and part (i) can be established using (51).

Lemma A.4 *For any* $\mathbb{F}$*-stopping time* τ*, we have that*

$$\mathbb{Q}(\tau_d>\tau\,|\,\mathcal{F}_\tau)=e^{-\Gamma_\tau}. \tag{51}$$

Moreover, if $\tau\in\mathcal{F}_T^t$ *for some* $t\in[0,T]$*, then:*
(i) *For any* $\mathcal{F}_\tau$*-measurable random variable* χ *such that at least one side of the following equality is well defined in* $\overline{\mathbb{R}}$ *(e.g.,* χ *bounded from one side), the other one is also well defined and we have:*

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau<\tau_d\}}\chi\,|\,\mathcal{G}_t)=\mathbb{1}_{\{t<\tau_d\}}e^{\Gamma_t}\,\mathbb{E}_{\mathbb{Q}}\big(e^{-\Gamma_\tau}\chi\,\big|\,\mathcal{F}_t\big).$$

(ii) *For any* $\mathbb{F}$*-predictable process* Z *such that at least one side of the following equality is well defined in* $\overline{\mathbb{R}}$ *(e.g.,* Z *is bounded from one side), the other one is also well defined and we have*

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau_d\le\tau\}}Z_{\tau_d}\,|\,\mathcal{G}_t)=\mathbb{1}_{\{t<\tau_d\}}e^{\Gamma_t}\,\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\tau Z_u e^{-\Gamma_u}\,d\Gamma_u\,\Big|\,\mathcal{F}_t\Big).$$

(iii) *For any finite variation* $\mathbb{F}$*-predictable process* A *such that at least one side of the following equality is well defined in* $\overline{\mathbb{R}}$ *(e.g., the variation of* A *over* $[0,T]$ *is bounded from one side), the other one is also well defined and we have*

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{t\wedge\tau_d}^{\tau\wedge\tau_d}dA_u\,\Big|\,\mathcal{G}_t\Big)=\mathbb{1}_{\{t<\tau_d\}}e^{\Gamma_t}\,\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\tau e^{-\Gamma_u}\,dA_u\,\Big|\,\mathcal{F}_t\Big).$$

Proof. (i) Since $\tau \in \mathcal{F}_t^T$, one has $\mathcal{F}_t \subset \mathcal{F}_\tau \subset \mathcal{F}_T$, hence by Lemma A.2:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau<\tau_d\}}\chi \,|\, \mathcal{G}_t) &= \mathbb{1}_{\{t<\tau_d\}}\, e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau<\tau_d\}}\chi \,\big|\, \mathcal{F}_t\big) \\ &= \mathbb{1}_{\{t<\tau_d\}}\, e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big(\chi\, \mathbb{Q}(\tau<\tau_d \,\big|\, \mathcal{F}_\tau) \,\big|\, \mathcal{F}_t\big) \\ &= \mathbb{1}_{\{t<\tau_d\}}\, e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big(\chi\, e^{-\Gamma_\tau} \,\big|\mathcal{F}_t\big),\end{aligned}$$

where the second equality follows by (51).

(ii) If suffices to prove the formula for an elementary predictable process of the form $Z_s = \mathbb{1}_{]u,v]}(s)B_u$ for an arbitrary event $B_u \in \mathcal{F}_u$. For such a process, the formula follows easily from part (i).

(iii) We have that

$$\int_{t\wedge\tau_d}^{\tau\wedge\tau_d} dA_u = \mathbb{1}_{\{t<\tau_d\}} \int_{t\wedge\tau_d}^{\tau\wedge\tau_d} dA_u = \mathbb{1}_{\{\tau<\tau_d\}} \int_t^{\tau} dA_u + \mathbb{1}_{\{t<\tau_d\le\tau\}} \int_t^{\tau_d} dA_u,$$

where A is $\mathbb{F}$-predictable. Using parts (i) and (ii), we obtain

$$\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\tau<\tau_d\}} \int_t^{\tau} dA_u \,\Big|\, \mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau_d\}}\mathbb{E}_{\mathbb{Q}}\Big(e^{\Gamma_t-\Gamma_\tau} \int_t^{\tau} dA_u \,\Big|\, \mathcal{F}_t\Big)$$

and

$$\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_d\le\tau\}} \int_t^{\tau_d} dA_u \,\Big|\, \mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau_d\}}\mathbb{E}_{\mathbb{Q}}\Big(\int_t^{\tau} \Big(\int_t^{s} dA_u\Big) e^{\Gamma_t-\Gamma_s}\, d\Gamma_s \,\Big|\, \mathcal{F}_t\Big),$$

where, by Fubini's theorem,

$$\int_t^{\tau} \Big(\int_t^{s} dA_u\Big) e^{\Gamma_t-\Gamma_s}\, d\Gamma_s = \int_t^{\tau}\int_t^{s} dA_u\, e^{\Gamma_t-\Gamma_s}\, d\Gamma_s = \int_t^{\tau} e^{\Gamma_t-\Gamma_u}\, dA_u - e^{\Gamma_t-\Gamma_\tau}\int_t^{\tau} dA_u.$$

Hence

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{t\wedge\tau_d}^{\tau\wedge\tau_d} dA_u \,\Big|\, \mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau_d\}}\mathbb{E}_{\mathbb{Q}}\Big(\int_t^{\tau} e^{\Gamma_t-\Gamma_u}\, dA_u \,\Big|\, \mathcal{F}_t\Big) = \mathbb{1}_{\{t<\tau_d\}}e^{\Gamma_t}\mathbb{E}_{\mathbb{Q}}\Big(\int_t^{\tau} e^{-\Gamma_u}\, dA_u \,\Big|\, \mathcal{F}_t\Big),$$

as was expected. □

In the next result, $[M_{\cdot\wedge\tau_d}, N^d]$ refers to the square bracket of $M_{\cdot\wedge\tau_d}$ and N^d with respect to the filtration $\mathbb{G}$, where N^d denotes, as usual, the compensated jump-to-default process. This bracket is well defined, since N^d is a $\mathbb{G}$-martingale and $M_{\cdot\wedge\tau_d}$ is a $\mathbb{G}$-local martingale, by Lemma A.1(iii).

Lemma A.5 *For any $\mathbb{F}$-martingale M, we have that $[M_{\cdot\wedge\tau_d}, N^d]$ is a $\mathbb{G}$-local martingale.*

Proof. Let us write $H^d = (1-H)e^{\Gamma}$. Since Γ is continuous and non-decreasing, we have that $dH^d = -H^d_-\, dN^d$ (see [7]). By an application of Lemma A.4(i) with $\tau = T$ and $\chi = e^{\Gamma_T}M_T$, we obtain, for every $t \in [0,T]$,

$$H^d_t M_{t\wedge\tau_d} = \mathbb{1}_{\{\tau_d>t\}}e^{\Gamma_t}M_t = \mathbb{1}_{\{\tau_d>t\}}e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big(M_T \,\big|\, \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau_d>T\}}e^{\Gamma_T}M_T \,\big|\, \mathcal{G}_t\big),$$

so $M_{t\cdot\wedge\tau_d}H^d_t$, $t \in [0,T]$, is a $\mathbb{G}$-uniformly integrable martingale, hence $[M_{\cdot\wedge\tau_d}, H^d]$ is a $\mathbb{G}$-local martingale (since $M_{\cdot\wedge\tau_d}$ and H^d are $\mathbb{G}$-local martingales). Now we have that

$$[M_{\cdot\wedge\tau_d}, N^d]_t = \Delta M_{\tau_d}H_t = -e^{-\Gamma_t}[M_{t\wedge\tau_d}, H^d]_t,$$

hence the conclusion follows. □

B Variants of Main Results

To deal with some practical examples, it is important to enjoy some freedom in the choice of a doubly reflected BSDE associated with a game option (cf. Remark 3.6(i)). To this end, we introduce the following definition.

Definition B.1 Given a game option with data $C, R, \xi, L, U, \bar{\tau}$, and an $\mathbb{F}$-adapted finite variation driver F with $\bar{F} - F$ bounded from below, we define $(\mathcal{E})$ as the doubly reflected BSDE (22) with data

$$F,\ \chi = \widehat{\xi} - F_T,\ \mathcal{L} = \widehat{L} - F,\ \mathcal{U} = \widehat{U} - F,\ \bar{\tau},$$

where, as in $(\bar{\mathcal{E}})$,

$$\widehat{\xi} = \xi + \bar{F}_T, \quad \widehat{L}_t = L_t + \bar{F}_t, \quad \widehat{U}_t = U_t + \bar{F}_t.$$

According to the definition above, the BSDE $(\mathcal{E})$ has the following form

$$\begin{gathered}\alpha_t\Theta_t = \alpha_T\chi + \alpha_T F_T - \alpha_t F_t + \int_t^T \alpha_u dK_u - \int_t^T \alpha_u dM_u, \quad t \in [0,T], \\ \mathcal{L}_t \le \Theta_t \le \bar{\mathcal{U}}_t, \quad t \in [0,T], \\ \int_0^T (\Theta_{u-} - \mathcal{L}_{u-})\, dK_u^+ = \int_0^T (\bar{\mathcal{U}}_{u-} - \Theta_{u-})\, dK_u^- = 0\end{gathered}$$

with $\bar{\mathcal{U}} = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\ge\bar{\tau}\}}(\widehat{U} - F)$.

In the special case of an European option, we consider the BSDE $(\mathcal{E})$ with $\mathcal{L}$ replaced by $\bar{L}$ such that $\alpha\bar{L} = -(c+1)$, where $-c$ is a lower bound on $\alpha_T\chi$.

Observe that for $F = 0$ equation $(\mathcal{E})$ reduces to $(\bar{\mathcal{E}})$. As we already noted in Remark 3.6(i), equations corresponding to various choices of a driver F are equivalent, in the sense that (Θ, M, K) solves $(\mathcal{E})$ for some driver F if and only if $(\bar{\Theta}, M, K)$ solves $(\bar{\mathcal{E}})$, where $\bar{\Theta} = \Theta + F$. However, as we shall see in further work (see [5, 6]), the freedom to use the most convenient driver is useful in financial applications. This motivates us to state the following corollary to Proposition 3.7.

Corollary B.1 *If (Θ, M, K) is a solution to $(\mathcal{E})$ then the conclusions of Propositions 3.7, 4.1, 4.2 and 5.2 are still valid, provided that we set $\widetilde{\Pi} = \Theta + F - \bar{F}$ instead of $\widetilde{\Pi} = \bar{\Theta} - \bar{F}$ in (27). If $(\Theta, M, K = 0)$ is a solution to $(\mathcal{E})$ with $\bar{L}$ instead of $\widehat{L}$ in $\mathcal{L}$ and with $\bar{\tau} = T$ then the conclusions of Proposition 4.3 are still valid, provided that we set $\widetilde{\Phi} = \Theta + F - \bar{F}$ instead of $\widetilde{\Phi} = \bar{\Theta} - \bar{F}$ therein.*

C Protection and Post-protection Prices

In this section we briefly introduce the concepts of protection and post-protection prices, which really become important in the Markovian set-up in relation with the connected variational inequality approach (cf. [6, 5, 10]).

Definition C.1 Given the $\mathbb{Q}$-price Π for a game option (see Theorem 2.2):
• by the *pre-default $\mathbb{Q}$-price*, we mean the pre-default value process $\widetilde{\Pi}$ of Π; in case $\bar{\tau} = 0$, we also call $\widetilde{\Pi}$ the *no protection $\mathbb{Q}$-price*,
• by the *protection $\mathbb{Q}$-price* (resp. *post-protection $\mathbb{Q}$-price*), we mean the process $\widetilde{\Pi}$ stopped at $\bar{\tau}$ (resp. the restriction of $\widetilde{\Pi}$ to the random time interval $[\bar{\tau}, T]$).

Thus the protection price refers to the pre-default price until the lifting of the call protection, whereas the post-protection price refers to the pre-default price afterwards. It is intuitively clear that post-protection prices should reduce to no-protection prices. This can indeed be shown by the

following simple argument. By formula (6) applied to the $\mathbb{Q}$-price Π and recalling that $\bar{\mathcal{G}}_T^t = \{\tau \in \mathcal{G}_T^t \,;\, \tau \wedge \tau_d \geq \bar{\tau} \wedge \tau_d\}$, we have, for $t \in [\bar{\tau}, T]$,

$$\text{esssup}_{\tau_p \in \mathcal{G}_T^t} \text{essinf}_{\tau_c \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big) = \Pi_t = \text{essinf}_{\tau_c \in \mathcal{G}_T^t} \text{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big).$$

We thus see that Π_t coincides on $[\bar{\tau}, T]$ with the $\mathbb{Q}$-price of the same game option, but with $\bar{\tau}$ replaced by 0 (provided that the game option modified in this way also admits a well defined $\mathbb{Q}$-price process on $[0, T]$). In this case, the *pre-default* $\mathbb{Q}$-prices of the original game option and of the game option with no call protection also coincide on $[\bar{\tau}, T] \cap [0, \tau_d)$, by Lemmas 3.1 and 3.2. So, if a game option and its modification with $\bar{\tau}$ changed to 0 both admit $\mathbb{Q}$-prices then the post-protection $\mathbb{Q}$-price of the former and the no protection $\mathbb{Q}$-price of the latter coincide on $[\bar{\tau}, T] \cap [0, \tau_d)$, as was claimed.

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