DEFAULTABLE GAME OPTIONS IN A HAZARD PROCESS MODEL

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1 Introduction

In Bielecki et al. [4], we formally introduced the notion of a generic *defaultable game option*, that is, a financial contract that can be seen as an intermediate case between a general mathematical concept of a *game option* and much more specific *convertible bond*. In [4], we concentrated on developing a fairly general framework for valuing such contracts. In particular, building on results of Kifer [28] and Kallsen and Kühn [26], we showed in Section 3 of [4] that the study of an arbitrage price of a game option can be reduced to the study of the value process of the related Dynkin game under some risk-neutral measure $\mathbb{Q}$ for the primary market model. In this stochastic game, the issuer of a game option plays the role of the minimizer and the holder of the maximizer. In [4], we worked in a fairly general market model, which was assumed to be arbitrage-free, but not necessarily complete, so that the uniqueness of a risk-neutral measure was not postulated. In addition, although the default time was introduced, the way in which it is defined was not explicitly specified.

As is well known, there are two main approaches to model default risk: the structural approach and the reduced-form approach. In the latter approach (also known as the hazard process approach), the default time is modeled as an exogenous random variable, without any particular economic background. So, given the filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$ used to model the primary market, one simply assumes that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, where the filtration $\mathbb{H}$ carries the information about the default and $\mathbb{F}$ is some reference filtration.

One may object to reduced-form models for their lack of clear reference to economic fundamentals, such as the firm's value. However the possibility of choosing various parameterizations for the coefficients and calibrating these parameters to any set of CDS spreads and/or implied volatilities makes them very versatile modeling tools, well-suited to price and hedge derivatives consistently with plain-vanilla instruments. It should be acknowledged that structural models, with their solid economic background, are better suited to infer reliable debt information such as risk-neutral default probabilities or the debt's present value from equities, which are the most liquid among all financial instruments. But the structure of these models, as rich as it may be (and which can include a list of factors such as stock, spreads, default status, credit events, etc.) is never rich enough to yield consistent prices for a full set of CDS spreads and/or implied volatilities. As we ultimately aim to specify models for pricing and hedging contracts with optional features (in particular, convertible bonds), we shall favor the reduced-form approach in the sequel.

From the mathematical perspective, the goal of the present paper is twofold. First, we wish to specialize our previous results to the hazard process set-up, that is, a reduced-form approach, which is slightly more general than the intensity-based set-up. Hence we postulate that the primary market filtration $\mathbb{G}$ admits the representation $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, where the filtration $\mathbb{H}$ is generated by a default indicator process $H_t = \mathbb{1}_{\{\tau_d \le t\}}$ and where $\mathbb{F}$ is some reference filtration. The main tool employed in this section is the effective reduction of the information flow from the full filtration $\mathbb{G}$ to the reference filtration $\mathbb{F}$. The main results are Theorems 2.2 and 2.3, which give convenient pricing formulae with respect to the reference filtration $\mathbb{F}$.

The second goal of the present paper is to study the issue of hedging of a game option in a hazard process set-up. Some attempts to analyze hedging of defaultable convertible bonds within the Markovian intensity-based set-up were done by Andersen and Buffum [1] and Ayache et al. [2], who worked directly with suitable variational inequalities. In Section 3 we prove that main result of this paper (Theorem 3.1), which can be informally stated as follows: under the assumption that a related doubly reflected BSDE admits a solution under some risk-neutral measure $\mathbb{Q}$ (for which various sets of sufficient conditions are known in the literature), then the state-process of the solution multiplied by the default indicator process is the minimal (super)hedging price with $(\mathbb{G}, \mathbb{Q})$-sigma martingale cost, the latter being equal to 0 in the case of complete markets.

We use throughout this paper the *vector* (as opposed to *componentwise*) stochastic integration, as developed in Cherny and Shiryaev [10] (see also Chatelain and Stricker [9] and Jacod [23]). Given a stochastic basis satisfying the usual conditions, an $\mathbb{R}^d$-valued semimartingale integrator X and an $\mathbb{R}^{1 \otimes d}$-valued (row vector) predictable integrand H, the notion of *vector stochastic integral* $\int H \, dX$

allows one to take into account possible "interferences" of different components of a multidimensional process. Well-defined vector stochastic integrals include, in particular, all integrals with a predictable and locally bounded integrand (e.g., any integrand of the form $H = Y_-$ where Y is an adapted càdlàg process, see [22, Theorem 7.7]). Even in the one-dimensional case, the concept of vector stochastic integral is indeed more general than a stochastic integral defined as the sum of integrals of components of H with respect to the related components of X, all supposed to be well defined in the classic sense. The usual properties of stochastic integral, such as: linearity, associativity, invariance with respect to equivalent changes of measures and with respect to inclusive changes of filtrations, are known to hold for the vector stochastic integral. Moreover, unlike other kinds of stochastic integrals, vector stochastic integrals form a closed space in a suitable topology. This feature makes them well adapted to many problems arising in the mathematical finance, such as Fundamental Theorems of Asset Pricing (see [10, 4] and Section 2.1).

By default, we denote by $\int_0^t$ the integrals over $(0,t]$. Otherwise, we explicitly specify the domain of integration as a subscript of $\int$. Note also that, depending on the context, τ will stand either for a generic stopping time or it will be given as $\tau = \tau_p \wedge \tau_c$ for some specific stopping times τ_c and τ_p.

2 Valuation of Defaultable Game Options

We shall first recall the abstract set-up introduced in Bielecki et al. [4] and subsequently we shall deal with the hazard process set-up. We assume throughout that the evolution of the primary market can be modeled in terms of stochastic processes defined on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$, where $\mathbb{P}$ denotes the statistical probability measure.

2.1 Valuation in the General Set-Up

We assume that the primary market is composed of the savings account and of d risky assets, such that, given a finite horizon date $T > 0$:
• the *discount factor* process β, that is, the inverse of the savings account, is a $\mathbb{G}$-adapted, finite variation, continuous, positive and bounded process;
• the risky assets are $\mathbb{G}$-semimartingales with càdlàg sample paths.

The primary risky assets, with $\mathbb{R}^d$-valued price process X, pay dividends, whose cumulative value process, denoted by $\mathcal{D}$, is assumed to be a $\mathbb{G}$-adapted, càdlàg and $\mathbb{R}^d$-valued process of finite variation. Given the price process X, we define the *cumulative price* $\widehat{X}$ of the asset as

$$\widehat{X}_t = X_t + \beta_t^{-1} \int_{[0,t]} \beta_u \, d\mathcal{D}_u. \tag{1}$$

In the financial interpretation, the last term in (1) represents the current value at time t of all dividend payments of the asset over the period $[0,t]$, under the assumption that all dividends are immediately reinvested in the savings account. We assume that the primary market model is free of arbitrage opportunities (though presumably incomplete) in the sense that $(X_t)_{t\in[0,T]}$ is an arbitrage price for our primary market with dividends as defined in [4]. In view of the First Fundamental Theorem of Asset Pricing (see [13, 10]), this means that there exists a *risk-neutral measure* $\mathbb{Q} \in \mathcal{M}$, where $\mathcal{M}$ denotes the set of probability measures $\mathbb{Q} \sim \mathbb{P}$ for which $\beta\widehat{X}$ is a *sigma martingale* with respect to $\mathbb{G}$ under $\mathbb{Q}$.

Recall that given a standard stochastic basis, an $\mathbb{R}^d$-valued process Y is called a *sigma martingale* if there exists an $\mathbb{R}^d$-valued local martingale M and an $\mathbb{R}^d$-valued process H such that H^i belongs to the space $L(M^i)$ of M^i-integrable processes[1] and $Y^i = Y_0^i + \int H^i \, dM^i$ for $i = 1, \ldots, d$ (see Lemma 5.1(ii) in [10]). In this paper, we shall use the following well-known properties of sigma martingales.

[1]In particular, H^i is predictable, by definition (see Section 3 in [10]).

Proposition 2.1 ([10, 32, 24]) (i) *The class of sigma martingales is a vector space containing all local martingales. It is stable with respect to stochastic integration, that is, if Y is a sigma martingale and H belongs to the class $L(Y)$ of Y-integrable processes then the integral $\int H\,dY$ is a sigma martingale.*
(ii) *Any bounded from below sigma martingale is a supermartingale and any locally bounded sigma martingale is a local martingale.*

We now introduce the concept of a dividend paying game option (see also Kifer [28] or Bielecki et al. [4]). In broad terms, a *dividend paying game option* initiated at time $t = 0$ and maturing at time T, is a contract with the following cash flows that are paid by the issuer of the contract and received by the holder of the contract:
• a *dividend stream*, whose cumulative value at time t is denoted by D_t,
• a *put payment* L_t made at time $t = \tau_p$ if $\tau_p \le \tau_c$ and $\tau_p < T$; time τ_p is called the put time and is chosen by the holder,
• a *call payment* U_t made at time $t = \tau_c$ provided that $\tau_c < \tau_p \wedge T$; time τ_c, known as the call time, is chosen by the issuer and may be subject to the constraint that $\tau_c \ge \bar{\tau}$, where $\bar{\tau}$ is the lifting time of the call protection,
• a *payment at maturity* ξ made at time T provided that $T \le \tau_p \wedge \tau_c$, and subject to rules specified in the contract.

Of course, there is also the initial cash flow, namely, the purchasing price of the contract, which is paid at the initiation time by the holder and received by the issuer.

Let us now be given an $[0, +\infty]$-valued $\mathbb{G}$-stopping time τ_d representing the *default time* of a reference entity. A *defaultable* dividend paying game option is a dividend paying game option such that the contract is terminated at τ_d, if it has not been put or called and has not matured before. In particular, there are no more cash flows related to this contract after the default time. In this setting, the dividend stream D additionally includes a possible recovery payment made at the default time.

We are interested in studying the problem of the time evolution of an arbitrage price of the game option. Therefore, we formulate the problem in a dynamic way by pricing the game option at any time $t \in [0, T]$. Let 0 (respectively T) stand for the *inception date* (respectively the *maturity date*) of a game option. We write $\mathcal{G}_T^t$ to denote the set of all $\mathbb{G}$-stopping times with values in $[t, T]$, and we let $\bar{\mathcal{G}}_T^t$ stand for $\{\tau \in \mathcal{G}_T^t\,;\, \tau \wedge \tau_d \ge \bar{\tau} \wedge \tau_d\}$, where the lifting time of a call protection $\bar{\tau}$ belongs to $\mathcal{G}_T^0$. Finally, let $H_t = \mathbb{1}_{\{\tau_d \le t\}}$ denote the *default indicator process* of the reference entity.

We are in the position to state the formal definition of a defaultable game option (see Definition 4.1 and formula (14) in Bielecki et al. [4]).

Definition 2.1 A *defaultable game option* is a game option with the *ex-dividend cumulative discounted cash flows* $\beta_t \pi(t; \tau_p, \tau_c)$, where the $\mathcal{G}_{\tau \wedge \tau_d}$-measurable random variable $\pi(t; \tau_p, \tau_c)$ is given by the formula, for any $t \in [0, T]$ and $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$,

$$\beta_t \pi(t; \tau_p, \tau_c) = \int_t^\tau \beta_u\, dD_u + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau < \tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau = T\}} \xi \Big), \tag{2}$$

where $\tau = \tau_p \wedge \tau_c$ and
• the *dividend process* $D = (D_t)_{t \in [0,T]}$ equals

$$D_t = \int_{[0,t]} (1 - H_u)\, dC_u + R_u\, dH_u$$

for some *coupon process* $C = (C_t)_{t \in [0,T]}$, which is a $\mathbb{G}$-adapted càdlàg process with bounded variation, and some real-valued, $\mathbb{G}$-predictable *recovery process* $R = (R_t)_{t \in [0,T]}$,
• the *put payment* $L = (L_t)_{t \in [0,T]}$ the *call payment* $U = (U_t)_{t \in [0,T]}$ are $\mathbb{G}$-adapted, real-valued, càdlàg processes,

• the inequality $L_t \le U_t$ holds for every $t \in [\tau_d \wedge \bar{\tau}, \tau_d \wedge T)$,
• the *payment at maturity* ξ is a $\mathcal{G}_T$-measurable real random variable.

Let us denote
$$\widehat{\mathcal{L}}_t = \beta_t^{-1} \int_{[0,t]} \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > t\}} \big(\mathbb{1}_{\{t<T\}} L_t + \mathbb{1}_{\{t=T\}} \xi \big).$$

We further assume that R, L and ξ are bounded from below, so that there exists a constant c such that
$$\beta_t \widehat{\mathcal{L}}_t \ge -c, \quad t \in [0,T]. \tag{3}$$

Inequality (3) implies that the discounted cumulative payoff are bounded from below. In order to get the upper bound for this payoff, we will sometimes assume that R, U and ξ are bounded (from below and from above), or simply that
$$\beta_t \widehat{\mathcal{U}}_t \le c, \quad t \in [0,T], \tag{4}$$
where
$$\widehat{\mathcal{U}}_t = \beta_t^{-1} \int_{[0,t]} \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > t\}} \big(\mathbb{1}_{\{t<T\}} U_t + \mathbb{1}_{\{t=T\}} \xi \big).$$

The following theorem characterizes the set of arbitrage ex-dividend prices of a game option in terms of values of related Dynkin games [16, 29, 30]. The notion of arbitrage price of a game option referred to in this theorem is the dynamic notion of arbitrage price for game options, defined in Kallsen and Kühn [26], extended to the case of dividend-paying primary assets and/or game options by resorting to the transformation of prices into cumulative prices. Note that in the sequel, the statement '$(\Pi_t)_{t\in[0,T]}$ *is an arbitrage price for the game option*' is in fact to be understood as '$(X_t, \Pi_t)_{t\in[0,T]}$ *is an arbitrage price for the extended market consisting of the primary market and the game option*'.

Theorem 2.1 (Bielecki et al. [4, Theorem 4.1]) *Assume that a process Π is a $\mathbb{G}$-semimartingale and there exists $\mathbb{Q} \in \mathcal{M}$ such that Π is the value of the Dynkin game related to a game option, specifically,*
$$\begin{aligned} \operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}} \big(\pi(t; \tau_p, \tau_c) \,\big|\, \mathcal{G}_t \big) &= \Pi_t \\ = \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}} \big(\pi(t; \tau_p, \tau_c) \,\big|\, \mathcal{G}_t \big), &\quad t \in [0,T]. \end{aligned} \tag{5}$$
Then Π is an arbitrage ex-dividend price of the game option, called the $\mathbb{Q}$-price *of the game option. The converse holds true (thus any arbitrage price is a $\mathbb{Q}$-price for some $\mathbb{Q} \in \mathcal{M}$) under the following integrability assumption*
$$\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} \Big(\sup_{t\in[0,T]} \widehat{\beta}_t \mathcal{L}_t \,\Big|\, \mathcal{G}_0 \Big) < \infty, \quad \text{a.s.} \tag{6}$$

It is worth noting that the class of defaultable game options includes as special cases defaultable American options and defaultable European options.

Definition 2.2 A *defaultable American option* is a defaultable game option with $\bar{\tau} = T$. A *defaultable European option* is a defaultable game option such that $\beta_t \widehat{\mathcal{L}}_t \le \beta_T \widehat{\mathcal{L}}_T$ for $t \in [0,T]$.

In view of Theorem 2.1 (see also Theorem 4.2 in Bielecki et al. [4]), the cash flows $\phi(t)$ of a defaultable European option can be redefined by, for $t \in [0,T]$,
$$\beta_t \phi(t) = \int_t^T \beta_u \, dD_u + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi.$$

2.2 Valuation in the Hazard Process Set-Up

In this section, our objective is to derive convenient pricing formulae for an arbitrage ex-dividend price of a game option in the hazard process set-up. Given a filtered probability space $(\Omega, \mathbb{G}, \mathbb{Q})$ with $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$, and an $[0, +\infty]$-valued $\mathbb{G}$-stopping time τ_d, we assume that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, where the filtration $\mathbb{H}$ is generated by the process $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$ and $\mathbb{F}$ is some reference filtration. As expected, our approach will consist in effectively reducing the information flow from the full filtration $\mathbb{G}$ to the reference filtration $\mathbb{F}$.

We consider the right-continuous and completed versions of all filtrations, so that they satisfy the so-called 'usual conditions.' Let G stand for the process $G_t = \mathbb{Q}(\tau_d > t \,|\, \mathcal{F}_t)$ for $t \in \mathbb{R}_+$. Note that the process G is an $(\mathbb{F}, \mathbb{Q})$-supermartingale, as the optional projection on $\mathbb{F}$ of the non-increasing process $1 - H$ (see [25]).

In the sequel, we shall work under the following standing assumption.

Assumption 2.1 We assume that the process G is (strictly) positive and continuous with finite variation, so that the $(\mathbb{F}, \mathbb{Q})$-*hazard process* $\Gamma_t = -\ln(G_t)$, $t \in \mathbb{R}_+$, is well defined and continuous with finite variation.

Let us analyse the consequences of Assumption 2.1. Recall that an $(\mathbb{F}, \mathbb{Q})$-*pseudo-stopping time* τ is an $\mathbb{F}$-random time such that $\mathbb{E}_{\mathbb{Q}} M_\tau = \mathbb{E}_{\mathbb{Q}} M_0$ for every bounded $(\mathbb{F}, \mathbb{Q})$-martingale M (see Nikeghbali and Yor [31, Remark 1]).

Lemma 2.1 (i) *G is a non-increasing process.*
(ii) *The $\mathbb{G}$-stopping time τ_d is an $(\mathbb{F}, \mathbb{Q})$-pseudo-stopping time.*
(iii) *Any $(\mathbb{F}, \mathbb{Q})$-local martingale stopped at τ_d is a $(\mathbb{G}, \mathbb{Q})$-local martingale.*
(iv) *Any $(\mathbb{F}, \mathbb{Q})$-sigma martingale stopped at τ_d is a $(\mathbb{G}, \mathbb{Q})$-sigma martingale.*
(v) *The integral process of a continuous integrand with respect to an $(\mathbb{F}, \mathbb{Q})$-martingale stopped at τ_d is a well defined $(\mathbb{G}, \mathbb{Q})$-sigma martingale.*

Proof. Since G is a continuous supermartingale, it admits a Doob–Meyer decomposition $G = M - A$ with a continuous martingale component M [24, p.44, Lemma 4.24]. Hence M is in fact constant, as a continuous martingale with finite variation, and thus (i) holds. By [31, Theorem 4.5], (i) implies (ii) (note that the continuity of the filtration $\mathbb{F}$ is only used for the converse in [31, Theorem 4.5]). By [31, Theorem 4.4], (ii) implies (iii), hence (iv), which, in conjunction with Proposition 2.1(i), implies (v). $\square$

Remarks 2.1 A well known case under which the process G is non-increasing is when Hypothesis H is satisfied (see, e.g., [8]), meaning that all square integrable $(\mathbb{F}, \mathbb{Q})$-martingales are $(\mathbb{G}, \mathbb{Q})$-martingales.

Assumption 2.2 We assume that the default time τ_d avoids $\mathbb{F}$-stopping times. Hence any $(\mathbb{F}, \mathbb{Q})$-martingale M cannot jump at τ_d, that is, $\Delta M_{\tau_d} := M_{\tau_d} - M_{\tau_d -} = 0$, $\mathbb{Q}$-a.s.

Let us recall the following well-known results (see, e.g., [8]).

Lemma 2.2 (i) *Let χ be a $\mathcal{G}_\infty$-measurable random variable. For any $t \in \mathbb{R}_+$ such that one of the members of the following equality is well defined in $\overline{\mathbb{R}}$ (e.g., χ bounded from one side), the other one is well defined too, and we have*

$$\mathbb{1}_{\{t < \tau_d\}} \mathbb{E}_{\mathbb{Q}}(\chi \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t < \tau_d\}} \, e^{\Gamma_t} \, \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t < \tau_d\}} \chi \,|\, \mathcal{F}_t).$$

In particular, if χ is $\mathcal{G}_t$-measurable then $\mathbb{1}_{\{t < \tau_d\}} \chi = \mathbb{1}_{\{t < \tau_d\}} \, \widetilde{\chi}$, where the $\mathcal{F}_t$-measurable random variable $\widetilde{\chi} = e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t < \tau_d\}} \chi \,|\, \mathcal{F}_t)$.

(ii) *For any* $\mathbb{G}$*-adapted process* Y *over* $[0,T]$*, there exists a unique* $\mathbb{F}$*-adapted process* $\widetilde{Y}$ *over* $[0,T]$ *such that*

$$\mathbb{1}_{\{t<\tau_d\}} Y_t = \mathbb{1}_{\{t<\tau_d\}} \widetilde{Y}_t, \quad t \in [0,T].$$

For any $t \in [0,T]$, we denote by $\mathcal{F}_T^t$ the set of all $\mathbb{F}$-stopping times with values in $[t,T]$. Also, given a stopping time $\bar{\tau} \in \mathcal{F}_T^0$ let $\bar{\mathcal{F}}_T^t$ stand for the class $\{\tau \in \mathcal{F}_T^t\,;\, \tau \geq \bar{\tau}\}$. The following result examines the relevant properties of these classes of stopping times. The proof of Lemma 2.3 is deferred to the appendix.

Lemma 2.3 (i) *If* $\tau \in \mathcal{G}_T^t$ *for some* $t \in [0,T]$ *then there exists* $\widetilde{\tau} \in \mathcal{F}_T^t$ *such that* $\tau \wedge \tau_d = \widetilde{\tau} \wedge \tau_d$*. Moreover, if* $\bar{\tau} \in \mathcal{G}_T^0$ *and if* $\tau \in \bar{\mathcal{G}}_T^t$ *for some* $t \in [0,T]$*, then we have* $\widetilde{\tau} \wedge \tau_d \geq \bar{\tau} \wedge \tau_d$*.*
(ii) *If* $\bar{\tau} \in \mathcal{F}_T^0$ *and* $\tau \in \bar{\mathcal{G}}_T^t$ *for some* $t \in [0,T]$*, then there exists* $\widetilde{\tau} \in \bar{\mathcal{F}}_T^t$ *such that* $\tau \wedge \tau_d = \widetilde{\tau} \wedge \tau_d$*.*

Definition 2.3 The quantities $\widetilde{\chi}$, $\widetilde{Y}$ and $\widetilde{\tau}$ introduced in Lemmas 2.3 and 2.2 are called the $\mathbb{F}$-*representatives* of χ, Y and τ, respectively. In the context of credit risk, where τ_d represents the default time of a reference entity, they are also known as *pre-default values* of τ, χ and Y.

Remarks 2.2 (i) Pre-default values are unique, under Assumption 2.1, by [15, p.186]. Moreover, in the $\mathbb{G}$-predictable case, the original process Y and its pre-default value $\widetilde{Y}$ coincide up to τ_d *included*, specifically ([15], p.138),

$$\mathbb{1}_{\{t\leq\tau_d\}} Y_t = \mathbb{1}_{\{t\leq\tau_d\}} \widetilde{Y}_t, \quad t \in [0,T].$$

(ii) In the $\mathbb{G}$-adapted case, it is standard to check that $Y \geq 0$ implies $\widetilde{Y} \geq 0$.

We find it convenient to make additionally the following standing assumptions.

Assumption 2.3 (i) The discount factor process β is $\mathbb{F}$-adapted.
(ii) The coupon process C is $\mathbb{F}$-predictable.
(iii) The recovery process R is $\mathbb{F}$-predictable.
(iv) The payoff processes L, U are $\mathbb{F}$-adapted and the random variable ξ is $\mathcal{F}_T$-measurable.
(v) The call protection $\bar{\tau}$ is an $\mathbb{F}$-stopping time.

Strictly speaking, Assumption 2.3 is superfluous, in the sense that all the results below are true in general; it suffices to make use of Lemma 2.2 to reduce the problem to the case described in Assumption 2.3. Since this would make the notation heavier, without adding much value, we prefer to work under this assumption.

The next lemma shows that the computation of the lower and upper value of the Dynkin games (5) with respect to $\mathbb{G}$-stopping times can be reduced to the computation of lower and upper value with respect to $\mathbb{F}$-stopping times.

Lemma 2.4 *We have*

$$\mathrm{esssup}_{\tau_p\in\mathcal{G}_T^t}\mathrm{essinf}_{\tau_c\in\bar{\mathcal{G}}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big) = \mathrm{esssup}_{\tau_p\in\mathcal{F}_T^t}\mathrm{essinf}_{\tau_c\in\bar{\mathcal{F}}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\big|\,\mathcal{G}_t\big)$$

and

$$\mathrm{essinf}_{\tau_c\in\bar{\mathcal{G}}_T^t}\mathrm{esssup}_{\tau_p\in\mathcal{G}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big) = \mathrm{essinf}_{\tau_c\in\bar{\mathcal{F}}_T^t}\mathrm{esssup}_{\tau_p\in\mathcal{F}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\big|\,\mathcal{G}_t\big).$$

Proof. For $(\tau_p,\tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$, we have

$$\pi(t;\tau_p,\tau_c) = \pi(t;\tau_p\wedge\tau_d,\tau_c\wedge\tau_d) = \pi(t;\widetilde{\tau}_p\wedge\tau_d,\widetilde{\tau}_c\wedge\tau_d) = \pi(t;\widetilde{\tau}_p,\widetilde{\tau}_c)$$

for some stopping times $(\widetilde{\tau}_p, \widetilde{\tau}_c) \in \mathcal{F}^t_T \times \bar{\mathcal{F}}^t_T$, where the middle equality follows from Lemma 2.3, and the other two from the definition of π. Since, clearly, $\mathcal{F}^t_T \subseteq \mathcal{G}^t_T$ and $\bar{\mathcal{F}}^t_T \subseteq \bar{\mathcal{G}}^t_T$, we conclude that the lemma is valid. □

The following lemma is of independent interest. Formula (7) can be found in Dellacherie [14, T47] and part (i) can be established using (7). The proofs of statements (ii) and (iii) are deferred to the appendix.

Lemma 2.5 *For any $\mathbb{F}$-stopping time τ, we have that*

$$\mathbb{Q}(\tau_d > \tau \,|\, \mathcal{F}_\tau) = e^{-\Gamma_\tau}. \tag{7}$$

Moreover, if $\tau \in \mathcal{F}^t_T$ for some $t \in [0,T]$, then:
(i) *For any $\mathcal{F}_\tau$-measurable random variable χ such that at least one side of the following equality is well defined in $\overline{\mathbb{R}}$ (e.g., χ bounded from one side), the other one is also well defined and we have:*

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau<\tau_d\}}\chi \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau_d>t\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big(e^{-\Gamma_\tau}\chi \,\big|\, \mathcal{F}_t\big).$$

(ii) *For any $\mathbb{F}$-predictable process Z such that at least one side of the following equality is well defined in $\overline{\mathbb{R}}$ (e.g., Z is bounded from one side), the other one is also well defined and we have*

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau_d\le\tau\}} Z_{\tau_d} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau_d\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\Big(\int_t^\tau Z_u e^{-\Gamma_u}\, d\Gamma_u \,\Big|\, \mathcal{F}_t\Big).$$

(iii) *For any finite variation $\mathbb{F}$-predictable process A such that at least one side of the following equality is well defined in $\overline{\mathbb{R}}$ (e.g., the variation of A over $[0,T]$ is bounded from one side), the other one is also well defined and we have*

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{t\wedge\tau_d}^{\tau\wedge\tau_d} dA_u \,\Big|\, \mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau_d\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\Big(\int_t^\tau e^{-\Gamma_u}\, dA_u \,\Big|\, \mathcal{F}_t\Big).$$

Under our assumptions, the computation of conditional expectations of cash flows $\pi(t;\tau_p,\tau_c)$ with respect to $\mathcal{G}_t$ can be reduced to the computation of conditional expectations of $\mathbb{F}$-equivalent cash flows $\widetilde{\pi}(t;\tau_p,\tau_c)$ with respect to $\mathcal{F}_t$. Since (8) is an immediate consequence of formula (2) and Lemma 2.5, the proof of Lemma 2.6 is omitted. Note that in this lemma τ stands for $\tau_p \wedge \tau_c$.

Lemma 2.6 *For any stopping times $\tau_p \in \mathcal{F}^t_T$ and $\tau_c \in \bar{\mathcal{F}}^t_T$ we have that*

$$\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big) = \mathbb{1}_{\{t<\tau_d\}} \mathbb{E}_{\mathbb{Q}}\big(\widetilde{\pi}(t;\tau_p,\tau_c) \,\big|\, \mathcal{F}_t\big), \tag{8}$$

where the $\mathcal{F}_\tau$-measurable random variable $\widetilde{\pi}(t;\tau_p,\tau_c)$ is given by

$$\alpha_t \widetilde{\pi}(t;\tau_p,\tau_c) = \int_t^\tau \alpha_u (dC_u + R_u\, d\Gamma_u) + \alpha_\tau\big(\mathbb{1}_{\{\tau=\tau_p<T\}} L_{\tau_p} + \mathbb{1}_{\{\tau<\tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau=T\}}\xi\big). \tag{9}$$

A comparison of formulae (2) and (9) shows that we have effectively moved our considerations from the original market subject to the default risk, in which cash flows are discounted according to the discount factor β, to the fictitious default-free market, in which cash flows are discounted according to the credit risk adjusted discount factor α. Note that the original cash flows $\pi(t;\tau_p,\tau_c)$ are given as $\mathcal{G}_{\tau\wedge\tau_d}$-measurable random variables, whereas the $\mathbb{F}$-equivalent cash flows $\widetilde{\pi}(t;\tau_p,\tau_c)$ are manifestly $\mathcal{F}_\tau$-measurable and they depend of the default time τ_d only via the hazard process Γ. For the purpose of computation of ex-dividend pre-default prices of a defaultable game option these two market models are equivalent, as we now show.

Theorem 2.2 *Assuming condition (6), let Π be an arbitrage ex-dividend $\mathbb{Q}$-price for a game option. Then we have, for any $t \in [0,T]$,*

$$\Pi_t = \mathbb{1}_{\{t<\tau_d\}} \widetilde{\Pi}_t, \tag{10}$$

where $\widetilde{\Pi}_t$ *satisfies*

$$\begin{aligned}\operatorname{esssup}_{\tau_p\in\mathcal{F}^t_T}\operatorname{essinf}_{\tau_c\in\bar{\mathcal{F}}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\widetilde{\pi}(t;\tau_p,\tau_c)\,\big|\,\mathcal{F}_t\big)&=\widetilde{\Pi}_t\\&=\operatorname{essinf}_{\tau_c\in\bar{\mathcal{F}}^t_T}\operatorname{esssup}_{\tau_p\in\mathcal{F}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\widetilde{\pi}(t;\tau_p,\tau_c)\big|\,\mathcal{F}_t\big).\end{aligned}\tag{11}$$

Hence the Dynkin game with cost criterion $\mathbb{E}_{\mathbb{Q}}\big(\widetilde{\pi}(t;\tau_p,\tau_c)\,\big|\,\mathcal{F}_t\big)$ *on* $\mathcal{F}^t_T\times\bar{\mathcal{F}}^t_T$ *admits the value* $\widetilde{\Pi}_t$, *which coincides with the pre-default ex-dividend price at time* t *of the game option under the risk-neutral measure* $\mathbb{Q}$.

Proof. It suffices to combine Theorem 2.1 with Lemmas 2.4 and 2.6. □

The following result is the converse of Theorem 2.2. It follows immediately by Lemmas 2.4, 2.6 and by the 'if' part of Theorem 2.1.

Theorem 2.3 *Given a risk-neutral measure* $\mathbb{Q}\in\mathcal{M}$, *assume that* $\widetilde{\Pi}_t$ *is the value of the Dynkin game with the cost criterion* $\mathbb{E}_{\mathbb{Q}}\big(\widetilde{\pi}(t;\tau_p,\tau_c)\,\big|\,\mathcal{F}_t\big)$ *on* $\mathcal{F}^t_T\times\bar{\mathcal{F}}^t_T$, *for any* $t\in[0,T]$. *Then* Π_t *defined via (10) is the value of the Dynkin game with the cost criterion* $\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big)$ *on* $\mathcal{G}^t_T\times\bar{\mathcal{G}}^t_T$, *for any* $t\in[0,T]$. *If, in addition,* $\widetilde{\Pi}$ *is a* $\mathbb{G}$*-semimartingale then* Π *is an arbitrage ex-dividend* $\mathbb{Q}$*-price for the game option.*

The previous results allow us to reduce the study of a game option to the study of Dynkin games (11) with respect to the reference filtration $\mathbb{F}$. This essential simplification will appear crucial in the sequel.

To conclude this section, let us introduce the notions of protection and post-protection prices. The motivation for introducing these concepts is that in the Markovian set-up the related variational inequalities are different. We refer the reader to [11, 6, 5] for more about this.

Definition 2.4 Given a $\mathbb{Q}$-price Π for a game option (see Theorem 2.1):
• by *pre-default* $\mathbb{Q}$*-price*, we mean the pre-default value process $\widetilde{\Pi}$ of Π; in case $\bar{\tau}=0$, we also call $\widetilde{\Pi}$ a *no protection* $\mathbb{Q}$*-price*;
• by *protection* $\mathbb{Q}$*-price* (resp. *post-protection* $\mathbb{Q}$*-price*), we mean the process $\widetilde{\Pi}$ stopped at $\bar{\tau}$ (resp. $\widetilde{\Pi}$ restricted to the random time interval $[\bar{\tau},T]$).

So protection price refers to the pre-default price until the lifting of the call protection, whereas the post-protection price refers to the pre-default price afterwards. We shall simply note at this stage that it is intuitively clear that post-protection prices should reduce to no-protection prices. This can indeed be shown by the following simple considerations. By formula (5) applied to the $\mathbb{Q}$-price Π, and recalling that $\bar{\mathcal{G}}^t_T=\{\tau\in\mathcal{G}^t_T\,;\,\tau\wedge\tau_d\ge\bar{\tau}\wedge\tau_d\}$, we have, for $t\in[\bar{\tau},T]$:

$$\operatorname{esssup}_{\tau_p\in\mathcal{G}^t_T}\operatorname{essinf}_{\tau_c\in\mathcal{G}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big)=\Pi_t=\operatorname{essinf}_{\tau_c\in\mathcal{G}^t_T}\operatorname{esssup}_{\tau_p\in\mathcal{G}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big).$$

Therefore, we see that Π_t coincides on $[\bar{\tau},T]$ with the $\mathbb{Q}$-price of the same game option, but with $\bar{\tau}$ replaced by 0 (provided the game option modified in this way also admits a well defined $\mathbb{Q}$-price process on $[0,T]$). In this case, the *pre-default* $\mathbb{Q}$-prices of the original game option and of the game option with no call protection also coincide on $[\bar{\tau},T]\cap[0,\tau_d)$, by Lemmas 2.4 and 2.6. So, if a game option and its modification with $\bar{\tau}$ changed to 0 both admit $\mathbb{Q}$-prices, then the post-protection $\mathbb{Q}$-price of the former and the no protection $\mathbb{Q}$-price of the latter coincide on $[\bar{\tau},T]\cap[0,\tau_d)$, as was conjectured.

2.3 Valuation via BSDEs

Given a risk-neutral measure $\mathbb{Q}\in\mathcal{M}$, we shall characterize an arbitrage ex-dividend $\mathbb{Q}$-price of a game option as a solution to a suitably chosen doubly reflected BSDE. To this end, we recall some auxiliary results concerning the relationship between Dynkin games and doubly reflected BSDEs.

Let $\alpha_t := \beta_t \exp(-\Gamma_t)$ stand for the *credit-risk adjusted discount factor*. Note that α is bounded, like β. We consider the following *doubly reflected BSDE with data* $F, \xi, L, U, \bar{\tau}$ (see Cvitanić and Karatzas [12], Hamadène et al. [21], Hamadène and Hassani [20], Crépey et al. [11], Bielecki et al. [6]):

$$\begin{aligned} \alpha_t \Theta_t = \alpha_T \xi + \alpha_T F_T - \alpha_t F_t + K_T - K_t - (M_T - M_t), \quad t \in [0,T], \\ L_t \le \Theta_t \le \bar{U}_t, \quad t \in [0,T], \\ \int_0^T (\Theta_{u-} - L_{u-})\, dK_u^+ = \int_0^T (\bar{U}_{u-} - \Theta_{u-})\, dK_u^- = 0, \end{aligned} \tag{12}$$

where F is a given $\mathbb{F}$-adapted process with finite variation, and the process $\bar{U} = (\bar{U}_t)_{t\in[0,T]}$ equals

$$\bar{U}_t = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\ge\bar{\tau}\}} U_t.$$

Definition 2.5 By a $\mathbb{Q}$-*solution* to the doubly reflected BSDE (12), we mean a triplet $(\Theta, M, \mathbf{K})$ such that:
- the *state process* Θ is a real valued, $\mathbb{F}$-adapted, càdlàg process,
- M is a real-valued $(\mathbb{F}, \mathbb{Q})$-martingale vanishing at time 0,
- $\mathbf{K} = (K^+, K^-)$ is a pair of $\mathbb{F}$-adapted, non-decreasing processes (null at time 0),
- all conditions in (12) are satisfied, with $K = K^+ - K^-$ in the first line, and with the convention that $0 \times \pm\infty = 0$ in the third line.

Remarks 2.3 (i) The state process Θ in a solution to (12) is required to be an $\mathbb{F}$-semimartingale. So there are obvious cases in which (12) does *not* admit a solution: simply take $\bar{\tau} = 0$ and $L = U$, which is not an $\mathbb{F}$-semimartingale. It is also clear that a solution is not necessarily unique, unless we impose an extra condition on $\mathbf{K}$, like mutual singularity of the non-negative measures defined by K^+ and K^- (see, e.g., [20, Remark 4.1]).
(ii) In financial applications (see [5, 11, 6]), the data F is typically given as a Lebesgue integral $\alpha F = \int \alpha f\, du$ and the component M of a solution to (12) is typically searched for in the form $M = \int \alpha(z\, dN + dn)$ for some real-valued and $\mathbb{R}^q$-valued $\mathbb{F}$-martingales $\int \alpha\, dN$ and $\int \alpha\, dn$. The reader is referred to [12, 21, 20, 5, 11, 6] for various specifications of the present set-up and sets of technical assumptions ensuring existence and uniqueness either of a $\mathbb{Q}$-solution to (12) or a suitable variant of this equation (see Definition 3.5 and [5]).
(iii) Since F is a given process, the BSDE (12) can be rewritten as

$$\begin{aligned} \alpha_t \widehat{\Theta}_t = \alpha_T \widehat{\xi} + K_T - K_t - (M_T - M_t), \quad t \in [0,T], \\ \widehat{L}_t \le \widehat{\Theta}_t \le \widehat{U}_t, \quad t \in [0,T], \\ \int_0^T (\widehat{\Theta}_{u-} - \widehat{L}_{u-})\, dK_u^+ = \int_0^T (\widehat{U}_{u-} - \widehat{\Theta}_{u-})\, dK_u^- = 0, \end{aligned} \tag{13}$$

where $\widehat{\Theta}_t = \Theta_t + F_t$ and

$$\widehat{\xi} = \xi + F_T, \quad \widehat{L}_t = L_t + F_t, \quad \widehat{U}_t = \bar{U}_t + F_t.$$

This shows that the problem of solving (12) can be formally reduced to the case of $F = 0$ with suitably modified reflecting barriers $\widehat{L}$, $\widehat{U}$ and terminal condition $\widehat{\xi}$.
(iv) In the special case where all $(\mathbb{F}, \mathbb{Q})$-martingales are continuous and where the $\mathbb{F}$-semimartingale F and the barriers L and U are continuous (see [12, 21, 6]), it is natural to look for a *continuous* solution of (12), that is, a solution of (12) given by a triplet of continuous processes $(\Theta, M, \mathbf{K})$.

In order to establish the link between a $\mathbb{Q}$-solution to the related doubly reflected BSDE and an arbitrage ex-dividend $\mathbb{Q}$-price of the defaultable game option, we first recall the general relationship between doubly reflected BSDEs and Dynkin games with purely terminal cost, before applying this result to dividend-paying game options in the fictitious default-free market in Proposition 2.2.

Note that if $(\Theta, M, \mathbf{K})$ solves (12) then we have, for any stopping time $\tau \in \mathcal{F}_T^t$,

$$\alpha_t \Theta_t = \alpha_\tau \Theta_\tau + \alpha_\tau F_\tau - \alpha_t F_t + K_\tau - K_t - (M_\tau - M_t). \tag{14}$$

Proposition 2.2 (Verification Principle for a Dynkin Game) *Let $(\Theta, M, \mathbf{K})$ be a $\mathbb{Q}$-solution to (12) with $F = 0$. Then Θ_t is the value of the Dynkin game with cost criterion $\mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c) \,\big|\, \mathcal{F}_t\big)$ on $\mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$, where $\theta(t;\tau_p,\tau_c)$ is the $\mathcal{F}_\tau$-measurable random variable defined by*

$$\alpha_t \theta(t;\tau_p,\tau_c) = \alpha_\tau \left(\mathbb{1}_{\{\tau=\tau_p<T\}} L_{\tau_p} + \mathbb{1}_{\{\tau=\tau_c<\tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau=T\}} \xi\right),$$

where $\tau = \tau_p \wedge \tau_c$. Moreover, for any $t \in [0,T]$ and for any $\varepsilon > 0$, the pair of stopping times $(\tau_p^\varepsilon, \tau_c^\varepsilon) \in \mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$ given by

$$\tau_p^\varepsilon = \inf\Big\{ u \in [t,T]\,;\, \Theta_u \le L_u + \varepsilon \Big\} \wedge T, \qquad \tau_c^\varepsilon = \inf\Big\{ u \in [\bar\tau \vee t, T]\,;\, \Theta_u \ge U_u - \varepsilon \Big\} \wedge T,$$

is ε-optimal for this Dynkin game, in the sense that we have, for any $(\tau_p,\tau_c) \in \mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$,

$$\mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c^\varepsilon) \,\big|\, \mathcal{F}_t\big) - \varepsilon \le \Theta_t \le \mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p^\varepsilon,\tau_c) \,\big|\, \mathcal{F}_t\big) + \varepsilon. \tag{15}$$

If $\mathbf{K}$ is continuous then the pair of stopping times $(\tau_p^0, \tau_c^0) \in \mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$, obtained by setting $\varepsilon = 0$, is a saddle-point of the game. This means that for any $(\tau_p,\tau_c) \in \mathcal{F}_T^t \times \bar{\mathcal{F}}_T^t$, we have

$$\mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c^0) \,\big|\, \mathcal{F}_t\big) \le \Theta_t \le \mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p^0,\tau_c) \,\big|\, \mathcal{F}_t\big).$$

Proof. Except for the presence of $\bar\tau$, the result is standard (see, e.g., Lepeltier and Maingueneau [30]). The proof hinges on showing that the pair $(\tau_p^\varepsilon, \tau_c^\varepsilon)$ is ε-optimal, for any $\varepsilon > 0$. Thus, taking the supremum and infimum over stopping times in (15), we obtain

$$\begin{aligned} &\operatorname{essinf}_{\tau_c\in\bar{\mathcal{F}}_T^t}\operatorname{esssup}_{\tau_p\in\mathcal{F}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c)\,\big|\,\mathcal{F}_t\big) - \varepsilon \le \operatorname{esssup}_{\tau_p\in\mathcal{F}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c^\varepsilon)\,\big|\,\mathcal{F}_t\big) - \varepsilon \le \Theta_t \\ &\quad \le \operatorname{essinf}_{\tau_c\in\bar{\mathcal{F}}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p^\varepsilon,\tau_c)\,\big|\,\mathcal{F}_t\big) + \varepsilon \le \operatorname{esssup}_{\tau_p\in\mathcal{F}_T^t}\operatorname{essinf}_{\tau_c\in\bar{\mathcal{F}}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p,\tau_c)\,\big|\,\mathcal{F}_t\big) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, we conclude that Θ_t is the value of the Dynkin game at time t.

It remains to establish (15). Let us first check that the right-hand side inequality in (15) is valid for any $\tau_c \in \bar{\mathcal{F}}_T^t$. Let τ^ε denote $\tau_p^\varepsilon \wedge \tau_c$. By the definition of τ_p^ε, we see that K^+ equals 0 on $[t, \tau^\varepsilon]$. Since K^- is non-decreasing, (14) applied to τ^ε yields

$$\alpha_t \Theta_t \le \alpha_{\tau^\varepsilon}\Theta_{\tau^\varepsilon} - (M_{\tau^\varepsilon} - M_t).$$

Taking conditional expectations, and using also the facts that $\Theta_{\tau_p^\varepsilon} \le L_{\tau_p^\varepsilon} + \varepsilon$ if $\tau_p^\varepsilon < T$, $\Theta_{\tau_p^\varepsilon} = \xi$ if $\tau_p^\varepsilon = T$ and $\Theta_{\tau_c} \le U_{\tau_c}$ (recall that $\tau_c \in \bar{\mathcal{F}}_T^t$, so that $\tau_c \ge \bar\tau$ and $\bar{U}_{\tau_c} = U_{\tau_c}$), we obtain

$$\begin{aligned} \alpha_t\Theta_t &\le \mathbb{E}_{\mathbb{Q}}(\alpha_{\tau^\varepsilon}\Theta_{\tau^\varepsilon}\,|\,\mathcal{F}_t) \\ &\le \mathbb{E}_{\mathbb{Q}}\Big(\alpha_{\tau^\varepsilon}\Big(\mathbb{1}_{\{\tau^\varepsilon=\tau_p^\varepsilon<T\}}(L_{\tau_p^\varepsilon}+\varepsilon) + \mathbb{1}_{\{\tau^\varepsilon=\tau_c<\tau_p^\varepsilon\}}U_{\tau_c} + \mathbb{1}_{\{\tau^\varepsilon=T\}}\xi\Big)\,\Big|\,\mathcal{F}_t\Big). \end{aligned}$$

We conclude that $\Theta_t \le \mathbb{E}_{\mathbb{Q}}\big(\theta(t;\tau_p^\varepsilon,\tau_c)\,\big|\,\mathcal{F}_t\big) + \varepsilon$ for any $\tau_c \in \bar{\mathcal{F}}_T^t$. This completes the proof of the right-hand side inequality in (15). The left-hand side inequality can be shown similarly. It is in fact standard, since it does not involve $\bar\tau$, and thus we leave the details to the reader.

Finally, in the special case where $\mathbf{K}$ is continuous, ε may be taken equal to 0, since in that case the process K^+ is continuous and thus it equals 0 on $[t, \tau_p^0 \wedge \tau_c]$ for any $\tau_c \in \bar{\mathcal{F}}_T^t$ (similarly, the process K^- equals 0 on $[t, \tau_c^0 \wedge \tau_p]$ for any $\tau_p \in \mathcal{F}_T^t$). □

Let us now apply Proposition 2.2 to defaultable game options with dividends. To this end, note first that formula (9) can be rewritten as follows

$$\alpha_t \widetilde{\pi}(t;\tau_p,\tau_c) = \alpha_\tau F_\tau^0 - \alpha_t F_t^0 + \alpha_\tau\big(\mathbb{1}_{\{\tau=\tau_p<T\}} L_{\tau_p} + \mathbb{1}_{\{\tau<\tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau=T\}}\xi\big),$$

with

$$\bar{D}_t := \int_{[0,t]} dC_u + R_u d\Gamma_u, \quad F_t^0 := \alpha^{-1}\int_{[0,t]} \alpha_u \, d\bar{D}_u. \tag{16}$$

Let us then denote by (E_0) equation (13) with $F_t = F_t^0$, that is

$$\left.\begin{array}{c} \alpha_t \Theta_t^0 = \alpha_T \widehat{\xi} + K_T - K_t - (M_T - M_t), \quad t \in [0,T], \\ \widehat{L}_t \le \Theta_t^0 \le \widehat{U}_t, \quad t \in [0,T], \\ \int_0^T (\Theta_{u-}^0 - \widehat{L}_{u-})\, dK_u^+ = \int_0^T (\widehat{U}_{u-} - \Theta_{u-}^0)\, dK_u^- = 0, \end{array}\right\} (E_0)$$

with

$$\widehat{\xi} = \xi + F_T^0, \quad \widehat{L}_t = L_t + F_t^0, \quad \widehat{U}_t = \bar{U}_t + F_t^0.$$

Assumption 2.4 The doubly reflected BSDE (E_0) admits a $\mathbb{Q}$-solution $(\Theta^0, M, \mathbf{K})$.

We define, for $t \in [0,T]$,

$$\widetilde{\Pi}_t = \Theta_t^0 - F_t^0, \quad \Pi_t = \mathbb{1}_{\{t<\tau_d\}} \widetilde{\Pi}_t, \quad \bar{\Pi}_t = \widetilde{\Pi}_t + \alpha_t^{-1} \int_{[0,t]} \alpha_u \, dk_u \tag{17}$$

with $dK = \alpha \, dk$.

Let $N^d = H - \Gamma_{\cdot \wedge \tau_d}$ stand for the $\mathbb{Q}$-*compensated jump-to-default process.* Under our standing assumption that the $(\mathbb{F}, \mathbb{Q})$-hazard process Γ of τ_d is continuous the process N^d is known to be $(\mathbb{G}, \mathbb{Q})$-martingale.

Lemma 2.7 (i) *The process Y given by the formula, for $t \in [0,T]$,*

$$Y_t = \mathbb{1}_{\{t<\tau_d\}} \beta_t \bar{\Pi}_t + \int_{[0,t]} \beta_u dD_u = \mathbb{1}_{\{t<\tau_d\}} \beta_t \bar{\Pi}_t + \int_{[0,t]} (1 - H_u) \beta_u dC_u + \beta_u R_u dH_u \tag{18}$$

is a $(\mathbb{G}, \mathbb{Q})$-martingale stopped at τ_d. Moreover

$$dY_t = \mathbb{1}_{\{t \le \tau_d\}} e^{\Gamma_t} dM_t + \mathbb{1}_{\{t \le \tau_d\}} \beta_t \big(R_t - \bar{\Pi}_{t-} \big) \, dN_t^d. \tag{19}$$

(ii) *The process $\int_0^{\cdot} \mathbb{1}_{\{t \le \tau_d\}} \beta_t \big(R_t - \bar{\Pi}_{t-} \big) \, dN_t^d$ is a $(\mathbb{G}, \mathbb{Q})$-sigma martingale.*
(iii) *The process Π is a $\mathbb{G}$-semimartingale.*

Proof. (i) The triplet $(\widetilde{\Pi}, M, \mathbf{K})$ satisfies (12) with F given by (16), so that

$$\alpha_t \widetilde{\Pi}_t = \alpha_T \xi + \int_t^T \alpha_u \, d\bar{D}_u + K_T - K_t - (M_T - M_t).$$

Note that

$$\alpha_t \bar{\Pi}_t = \alpha_t \widetilde{\Pi}_t + K_t = \alpha_0 \widetilde{\Pi}_0 - \int_0^t \alpha_u \, d\bar{D}_u + M_t.$$

Therefore, the process $\bar{Y}$ given by

$$\bar{Y}_t = \alpha_t \bar{\Pi}_t + \int_0^t \alpha_u \, d\bar{D}_u = \alpha_0 \widetilde{\Pi}_0 + M_t$$

is an $(\mathbb{F}, \mathbb{Q})$-martingale. Using Lemma 2.5, it is easy to check that we have, for any $0 \le t \le u \le T$,

$$\mathbb{E}_{\mathbb{Q}}(Y_u - Y_t \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau_d\}} e^{\Gamma_t} \, \mathbb{E}_{\mathbb{Q}}(\bar{Y}_u - \bar{Y}_t \,|\, \mathcal{F}_t).$$

Since $\bar{Y}$ is an $(\mathbb{F}, \mathbb{Q})$-martingale, the process Y is a $(\mathbb{G}, \mathbb{Q})$-martingale. Expression (19) then follows from (18) by direct computations, having noted that M and N^d cannot jump together, by Assumption 2.2.

(ii) We have checked already that the process Y in (19) is a $(\mathbb{G},\mathbb{Q})$-martingale. Moreover, the process

$$\int_0^{\cdot} \mathbb{1}_{\{t\le\tau_d\}} e^{\Gamma_t}\, dM_t = \int_0^{\cdot} e^{\Gamma_t}\, dM_t^{\tau_d}$$

is a $(\mathbb{G},\mathbb{Q})$-sigma martingale, by Lemma 2.1(v). We conclude that the process

$$\int_0^{\cdot} \mathbb{1}_{\{t\le\tau_d\}} (R_t - \bar{\Pi}_{t-})\, dN_t^d$$

is a $(\mathbb{G},\mathbb{Q})$-sigma martingale, by Proposition 2.1(i).
(iii) By (18), we have that

$$Y_t = (1-H_t)\beta_t\bar{\Pi}_t + \int_{[0,t]} \beta_u\, dD_u$$

so

$$\beta_t\Pi_t = Y_t - (1-H_t)\beta_t\alpha_t^{-1}\int_{[0,t]} \alpha_u\, dk_u - \int_{[0,t]} \beta_u\, dD_u,$$

where Y is a $(\mathbb{G},\mathbb{Q})$-martingale, by (i), and where the remaining terms in the r.h.s. of the last formula are $\mathbb{G}$-adapted processes of finite variation. Therefore, Π is a $\mathbb{G}$-semimartingale. □

The following result establishes the link between $(\Theta^0, M, \mathbf{K})$ and an arbitrage ex-dividend $\mathbb{Q}$-price of the defaultable game option.

Theorem 2.4 (Verification Principle for a Defaultable Game Option) *The process Π is an arbitrage ex-dividend $\mathbb{Q}$-price for the game option. Moreover, for any $t\in[0,T]$ and $\varepsilon>0$, the pair of ε-optimal stopping times $(\tau_p^\varepsilon,\tau_c^\varepsilon)\in\mathcal{F}_T^t\times\bar{\mathcal{F}}_T^t$ for the related Dynkin game (5) on $\mathcal{G}_T^t\times\bar{\mathcal{G}}_T^t$ is given by*

$$\tau_p^\varepsilon = \inf\Big\{ u\in[t,T]\,;\,\widetilde{\Pi}_u \le L_u+\varepsilon\Big\}\wedge T, \qquad \tau_c^\varepsilon = \inf\Big\{ u\in[\bar{\tau}\vee t,T]\,;\,\widetilde{\Pi}_u \ge U_u-\varepsilon\Big\}\wedge T.$$

If $\mathbf{K}$ is continuous then the pair of stopping times $(\tau_p^0,\tau_c^0)\in\mathcal{F}_T^t\times\bar{\mathcal{F}}_T^t$, obtained by setting $\varepsilon=0$, is a saddle-point of the defaultable game option.

Proof. In view of (9), the present assumptions imply that $\widetilde{\Pi}_t$ is the value of the Dynkin game (11), by Proposition 2.2, with $(\tau_p^\varepsilon,\tau_c^\varepsilon)$ as a pair of ε-optimal stopping times. Therefore, by Lemmas 2.4 and 2.6, Π_t is the value of the Dynkin game associated with the game option on $\mathcal{G}_T^t\times\bar{\mathcal{G}}_T^t$, with $(\tau_p^\varepsilon,\tau_c^\varepsilon)$ as a pair of ε-optimal stopping times.

Moreover, Π is a $\mathbb{G}$-semimartingale, by Lemma 2.7(iii). We conclude by making use of the last part in Theorem 2.3. □

3 Hedging of Defaultable Game Options

We shall work with a definition of hedging game options adapted from successive developments, starting with the definition of hedging for American options in Karatzas [27], followed by El Karoui and Quenez [18], Kifer [28] and Hamadène [19] (see also Schweizer [33]). This definition will be shown to be consistent with the concept of arbitrage pricing of a game option.

We recall that X (resp. $\widehat{X}$) is the price process (resp. cumulative price process) of primary traded assets, as given by (1).

Definition 3.1 By a *primary strategy* we mean a triplet (V_0,ζ,Q) such that:
- V_0 is an $\mathcal{G}_0$-measurable real-valued random variable representing the *initial wealth*,
- ζ is an $\mathbb{R}^{1\otimes d}$-valued, $\beta\widehat{X}$-integrable process representing holdings in primary risky assets,

• Q is a real-valued, $\mathbb{G}$-semimartingale, with $Q_0 = 0$, representing the (generalized) *cost process.*
The *wealth process* V of a primary strategy (V_0, ζ, Q) is given by

$$d(\beta_t V_t) = \zeta_t \, d(\beta_t \widehat{X}_t) + \beta_t \, dQ_t, \quad t \in [0, T], \tag{20}$$

with the initial condition V_0.

Note that a primary strategy introduced in Definition 3.1 is not self-financing in the standard sense, unless $Q = 0$. Given the wealth process V of a primary strategy (V_0, ζ, Q), we uniquely specify a $\mathbb{G}$-predictable process ζ^0 by setting

$$V_t = \zeta_t^0 \beta_t^{-1} + \zeta_t X_t, \quad t \in [0, T].$$

The process ζ^0 represents the number of units held in the savings account at time t, starting from the initial wealth V_0 and using the strategy ζ in the primary risky assets and the cost process Q. The process V is then the wealth of the strategy defined by the pair (ζ^0, ζ) and Q is the cost process of this strategy.

An increasing process Q represents cash injected into the portfolio, whereas a decreasing process Q represents cash withdrawn from the portfolio. A finite variation part of a general $\mathbb{G}$-semimartingale Q may be used to model both withdrawals and endowments and its martingale part proves useful in the representation of unhedgeable risks in an incomplete model. In order to make Definition 3.1 more practical, we shall later impose suitable additional assumptions on the cost process of a hedging strategy for the game option.

Definition 3.2 An *issuer ε-hedge with residual cost ρ* for the game option is represented by a quadruplet $(V_0, \zeta, \rho, \tau_c)$ such that:
(i) τ_c belongs to $\bar{\mathcal{G}}_T^0$,
(ii) $(V_0, \zeta, \rho - D)$ is a primary strategy with related wealth process V such that, for $t \in [0, T]$,

$$V_{t \wedge \tau_c} - \mathbb{1}_{\{t \wedge \tau_c < \tau_d\}} \Big(\mathbb{1}_{\{t \wedge \tau_c = t < T\}} L_t + \mathbb{1}_{\{\tau_c < t\}} U_{\tau_c} + \mathbb{1}_{\{t = \tau_c = T\}} \xi \Big) \geq -\varepsilon. \tag{21}$$

A *holder ε-hedge with residual cost* ρ for the game option is a quadruplet $(V_0, \zeta, \rho, \tau_p)$ such that:
(i) τ_p belongs to $\mathcal{G}_T^0$,
(ii) $(V_0, \zeta, \rho + D)$ is a primary strategy with related wealth process V such that, for $t \in [\bar{\tau}, T]$,

$$V_{t \wedge \tau_p} + \mathbb{1}_{\{t \wedge \tau_p < \tau_d\}} \Big(\mathbb{1}_{\{\tau_p \wedge t = \tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{t < \tau_p\}} U_t + \mathbb{1}_{\{\tau_p = t = T\}} \xi \Big) \geq -\varepsilon. \tag{22}$$

For $\varepsilon = 0$, we say that we deal with an *issuer hedge* and a *holder hedge* with residual cost ρ for the game option. Issuer or holder (ε-)hedges *with no residual cost* (that is, with $\rho = 0$) are also called issuer or holder *(ε-)superhedges.*

It is clear that hedging of a game option is associated with paying (by the issuer) or receiving (by the holder) of dividends D and thus the issuer and the holder are subject to the intrinsic cost processes $-D$ and D, respectively. The residual cost process ρ is aimed to represent additional trading costs in excess of the dividend process. For instance, if $\rho = 0$ then the strategy of the holder of the option is self-financing after making account for the dividends due, in the sense that

$$d(\beta_t V_t) = \zeta_t \, d(\beta_t \widehat{X}_t) - \beta_t \, dD_t, \quad t \in [0, T].$$

A similar remark applies to the holder of the option.

3.1 Bounds on Initial Values of Hedging Strategies

Let us fix $\mathbb{Q} \in \mathcal{M}$ and let us restrict our attention to primary strategies with a $(\mathbb{G}, \mathbb{Q})$-sigma martingale residual cost Q^r. We define $\mathcal{V}_0^c$ (resp. $\mathcal{V}_0^p$) as the set of initial values V_0 such that for any

$\varepsilon > 0$ there exists an issuer (resp. holder) ε-hedge of the game option at time 0 with the initial value V_0 and a $(\mathbb{G}, \mathbb{Q})$-sigma martingale residual cost Q^r. It is worth stressing that the bounds established in Lemma 3.1 are also obviously valid when we restrict our attention to primary strategies with no residual cost, that is, with $\rho = 0$.

The following lemma gives some preliminary conclusions regarding the initial cost of a hedging strategy for the game option under very weak assumptions (note, in particular, that Assumption 2.4 is not needed here). In Corollary 3.1, we shall see that under stronger assumptions the infima are attained and we obtain equalities rather than merely inequalities in Lemma 3.1.

Lemma 3.1 (i) *We have*

$$\operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^0_T} \operatorname{esssup}_{\tau_p \in \mathcal{G}^0_T} \mathbb{E}_{\mathbb{Q}}\big(\pi(0;\tau_p,\tau_c) \,\big|\, \mathcal{G}_0\big) \leq \inf \mathcal{V}^c_0, \quad \text{a.s.} \tag{23}$$

(ii) *If inequality (4) is valid, then*

$$\operatorname{esssup}_{\tau_p \in \mathcal{G}^0_T} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^0_T} \mathbb{E}_{\mathbb{Q}}\big(\pi(0;\tau_p,\tau_c) \,\big|\, \mathcal{G}_0\big) \geq -\inf \mathcal{V}^p_0, \quad \text{a.s.} \tag{24}$$

Proof. (i) Assume that for some stopping time $\bar\tau_c \in \bar{\mathcal{G}}^0_T$ the quadruplet $(V_0, \zeta, \rho, \bar\tau_c)$ is an issuer ε-hedge with a $(\mathbb{G}, \mathbb{Q})$-sigma martingale residual cost ρ for the game option. Then (20)-(21) imply that, for any $t \in [0, T]$,

$$\begin{aligned}\beta_0 V_0 &= \int_0^{t\wedge\bar\tau_c} \beta_u (dD_u - d\rho_u) + \beta_{t\wedge\bar\tau_c} V_{t\wedge\bar\tau_c} - \int_0^{t\wedge\bar\tau_c} \zeta_u \, d(\beta_u \widehat{X}_u) \\ &\geq \beta_0 \pi(0; t, \bar\tau_c) - \beta_{t\wedge\bar\tau_c}\varepsilon - \int_0^{t\wedge\bar\tau_c} (\zeta_u \, d(\beta_u \widehat{X}_u) + \beta_u \, d\rho_u),\end{aligned} \tag{25}$$

where we also used the definition (2) of cumulative discounted cash flows $\pi(0; t, \bar\tau_c)$ and the inequality $L \leq U$.

The stochastic integral $\int_0^t \zeta_u \, d(\beta_u \widehat{X}_u)$ with respect to a $(\mathbb{G}, \mathbb{Q})$-sigma martingale $\beta \widehat{X}$ is a $(\mathbb{G}, \mathbb{Q})$-sigma martingale. Hence the stopped process $\int_0^{t\wedge\bar\tau_c} \zeta_u \, d(\beta_u \widehat{X}_u)$ and the process

$$\int_0^{t\wedge\bar\tau_c} (\zeta_u \, d(\beta_u \widehat{X}_u) + \beta_u \, d\rho_u)$$

are $(\mathbb{G}, \mathbb{Q})$-sigma martingales as well. The latter process is bounded from below (this follows from (3) and (25)), so that it is a bounded from below local martingale ([24, p.216]) and thus a supermartingale.

Moreover, for any stopping time $\tau_p \in \mathcal{G}^0_T$, the inequality in formula (25) still holds with t replaced by τ_p. By taking expectations, we obtain (recall that $\bar\tau_c$ is fixed)

$$\beta_0 V_0 \geq \mathbb{E}_{\mathbb{Q}}\big(\beta_0 \pi(0; \tau_p, \bar\tau_c) - \beta_{\tau_p\wedge\bar\tau_c}\varepsilon \,\big|\, \mathcal{G}_0\big), \quad \forall\, \tau_p \in \mathcal{G}^0_T,$$

and thus (since β is bounded and $\beta_0 > 0$)

$$V_0 \geq \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^0_T} \operatorname{esssup}_{\tau_p \in \mathcal{G}^0_T} \mathbb{E}_{\mathbb{Q}}\big(\pi(0;\tau_p,\tau_c) \,\big|\, \mathcal{G}_0\big) - K\varepsilon$$

for some constant K. Therefore, if V_0 is such that for any $\varepsilon > 0$ there exists an issuer ε-hedging strategy with the initial wealth V_0 and a $(\mathbb{G}, \mathbb{Q})$-sigma martingale residual cost Q^r then

$$V_0 \geq \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}^0_T} \operatorname{esssup}_{\tau_p \in \mathcal{G}^0_T} \mathbb{E}_{\mathbb{Q}}\big(\pi(0;\tau_p,\tau_c) \,\big|\, \mathcal{G}_0\big).$$

(ii) Let $(V_0, \zeta, \rho, \bar\tau_p)$ be a holder ε-hedge with a $(\mathbb{G}, \mathbb{Q})$-sigma martingale residual cost ρ for the game option for some stopping time $\bar\tau_p \in \mathcal{G}^0_T$. Then (20)-(22) imply that, for any $t \in [\bar\tau, T]$,

$$\begin{aligned}\beta_0 V_0 &= -\int_0^{t\wedge\bar\tau_p} \beta_u (dD_u + d\rho_u) + \beta_{t\wedge\bar\tau_p} V_{t\wedge\bar\tau_p} - \int_0^{t\wedge\bar\tau_p} \zeta_u \, d(\beta_u \widehat{X}_u) \\ &\geq -\beta_0 \pi(0; \bar\tau_p, t) - \beta_{t\wedge\bar\tau_p}\varepsilon - \int_0^{t\wedge\bar\tau_p} \big(\zeta_u \, d(\beta_u \widehat{X}_u) + \beta_u \, d\rho_u\big).\end{aligned}$$

Under condition (4), the stochastic integral in the last formula is bounded from below and thus, by the same arguments as in part (i), we conclude that it is a supermartingale. Consequently, for a fixed stopping time $\bar{\tau}_p \in \mathcal{G}_T^0$,

$$\beta_0 V_0 \geq \mathbb{E}_{\mathbb{Q}}\big(- \beta_0 \pi(0; \bar{\tau}_p, \tau_c) - \beta_{\bar{\tau}_p \wedge \tau_c} \varepsilon \,\big|\, \mathcal{G}_0 \big), \quad \text{a.s.}, \quad \forall\, \tau_c \in \bar{\mathcal{G}}_T^0,$$

so that

$$V_0 \geq -\mathrm{esssup}_{\tau_p \in \mathcal{G}_T^0} \mathrm{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^0} \mathbb{E}_{\mathbb{Q}}\big(\pi(0; \tau_p, \tau_c) \,\big|\, \mathcal{G}_0\big) - K\varepsilon, \quad \text{a.s.},$$

for some constant K. Therefore, if V_0 is such that for any $\varepsilon > 0$ there exists a holder ε-hedging strategy with the initial wealth V_0 and a $(\mathbb{G}, \mathbb{Q})$-sigma martingale residual cost Q^r then

$$V_0 \geq -\mathrm{esssup}_{\tau_p \in \mathcal{G}_T^0} \mathrm{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^0} \mathbb{E}_{\mathbb{Q}}\big(\pi(0; \tau_p, \tau_c) \,\big|\, \mathcal{G}_0\big).$$

This completes the proof. $\square$

3.2 Existence of Hedging Strategies

We now place ourselves within the hazard process set-up of Section 2.2 for some fixed $\mathbb{Q} \in \mathcal{M}$, under Assumption 2.4. Let $(\Theta^0, M, \mathbf{K})$ denote a $\mathbb{Q}$-solution to (E_0) and let $\widetilde{\Pi}, \Pi, \bar{\Pi}$ be defined as in (17). In particular, Π is an arbitrage $\mathbb{Q}$-price for the game option (by Theorem 2.4) and the left-hand side in (23) and (24) is equal to Π_0.

We denote by $\mathcal{I}$ the set of stochastic integrals with respect to $(\beta \widehat{X})^{\tau_d}$, including constants. Similarly, let $\mathcal{I}^q$ stand for the set of $\mathbb{R}^q$-valued $\mathbb{G}$-semimartingales with components in $\mathcal{I}$. In order to examine the existence of hedging strategies for a game option with a $(\mathbb{G}, \mathbb{Q})$-sigma martingale residual cost Q^r, we find it convenient to introduce the following concept.

Definition 3.3 A *decomposition* of a $\mathbb{Q}$-solution $(\Theta^0, M, \mathbf{K})$ to (E_0) with respect to the primary market model $(\beta^{-1}, \widehat{X})$ consists of:
(i) a decomposition $dM = \alpha(z\, dN + dn)$, where:
• N is an $\mathbb{R}^q$-valued $\mathbb{F}$-semimartingale with N^{τ_d} in $\mathcal{I}^q$, so that

$$\beta_t\, dN_t = \Lambda_t\, d(\beta_t \widehat{X}_t), \quad t \in [0, T \wedge \tau_d], \tag{26}$$

for some $\mathbb{R}^{q \otimes d}$-valued, (row by row) $\beta \widehat{X}$-integrable process Λ,
• z is an $\mathbb{R}^{1 \otimes q}$-valued process such that αz is N-integrable,
• n is a real-valued, $\mathbb{F}$-adapted process,
(ii) a decomposition

$$\mathbb{1}_{\{t \leq \tau_d\}} \beta_t (R_t - \bar{\Pi}_{t-})\, dN_t^d = \mathbb{1}_{\{t \leq \tau_d\}} (R_t - \bar{\Pi}_{t-}) \Lambda_t^d\, d(\beta_t \widehat{X}_t) + \mathbb{1}_{\{t \leq \tau_d\}} \beta_t\, dn_t^d, \quad t \in [0, T \wedge \tau_d], \tag{27}$$

where Λ^d is an $\mathbb{R}^{1 \otimes d}$-valued process Λ^d such that the process $(R - \bar{\Pi}_-)\Lambda^d$ is $\beta \widehat{X}$-integrable and n^d is an $\mathbb{G}$-adapted process.

The special case $n = n^d = 0$ in Definition 3.3 corresponds to a particular form of a model completeness (attainability of defaultable European options, cf. Theorem 3.2; see also [6]) in which the issuer (or the holder) of the option is able (and wishes) to hedge all risks embedded in the option. The case where either $n \neq 0$ or $n^d \neq 0$ corresponds to either model incompleteness or the situation of a complete model in which the issuer (or the holder) is able to hedge, but he *prefers* not to hedge all the risks embedded in the option, for instance, he may be *willing* to take some bets in specific risk directions. This explains why we do not postulate that we want to minimize n and/or n^d in some sense as, for instance, in Schweizer [33].

Given a decomposition of a solution $(\Theta^0, M, \mathbf{K})$ to (E_0), we define

$$N^* = \begin{bmatrix} N \\ N^d \end{bmatrix}, \quad \Lambda^* = \begin{bmatrix} \Lambda \\ \Lambda^d \end{bmatrix}, \quad \rho^* = (n + n^d)^{\tau_d}. \tag{28}$$

It is worth noting that the process ρ^* will play the role of a residual cost of ε-hedging strategies for a game option (see Theorem 3.1). Therefore, it is important to show that it is a $(\mathbb{G},\mathbb{Q})$-sigma martingale.

Lemma 3.2 *For any decomposition of a $\mathbb{Q}$-solution $(\Theta^0, M, \mathbf{K})$ to (E_0), we have that:*
(i) n^{τ_d} *is a $(\mathbb{G},\mathbb{Q})$-sigma martingale,*
(ii) $(n^d)^{\tau_d}$ *is a $(\mathbb{G},\mathbb{Q})$-sigma martingale,*
(iii) ρ^* *is a $(\mathbb{G},\mathbb{Q})$-sigma martingale.*

Proof. The process $\int \alpha^{-1}\, dM^{\tau_d}$ is a $(\mathbb{G},\mathbb{Q})$-sigma martingale, by Lemma 2.1(v). Similarly, the process βN, and thus also N and $\int z\, dN^{\tau_d}$, are $(\mathbb{G},\mathbb{Q})$-sigma martingales, by Proposition 2.1. We conclude that the process

$$n_t^{\tau_d} = \int_0^t \left(\alpha_u^{-1}\, dM_u^{\tau_d} - z_u\, dN_u^{\tau_d}\right)$$

is a $\mathbb{G}$-sigma martingale, which proves (i). Furthermore, the process

$$\int_0^t \mathbb{1}_{\{u\le\tau_d\}}(R_u - \bar{\Pi}_{u-})\, dN_u^d$$

is a $(\mathbb{G},\mathbb{Q})$-sigma martingale (by Lemma 2.7(ii) and the fact that $\beta\widehat{X}$ is a $(\mathbb{G},\mathbb{Q})$-sigma martingale). Consequently (by Proposition 2.1) so is the process

$$\int_0^t (R_u - \bar{\Pi}_{u-})\Lambda_u^d\, d(\beta_u\widehat{X}_u),$$

provided that it is well defined. Thus the $\beta\widehat{X}$-integrability of $(R-\bar{\Pi}_-)\Lambda^d$ in fact implies that the process $(n^d)^{\tau_d}$ is a $(\mathbb{G},\mathbb{Q})$-sigma martingale. Part (iii) follows from (i) and (ii). □

Remarks 3.1 (i) Note that the process

$$\int_0^t \mathbb{1}_{\{u\le\tau_d\}}(R_u - \bar{\Pi}_{u-})\, dN_u^d$$

is a $(\mathbb{G},\mathbb{Q})$-sigma martingale, by Lemma 2.7(ii). Thus it is always possible to introduce the trivial decomposition

$$dn_t := \alpha_t^{-1}\, dM_t, \quad dn_t^d := (R_t - \bar{\Pi}_{t-})\, dN_t^d, \quad t\in[0,T].$$

Needless to say that decomposition (3.1) is of a minor *practical* interest, since it implicitly assumes that either hedging is impossible or that we are not interested in hedging. Formally, this decomposition results in the trivial strategy $\zeta^* = 0$ in Theorem 3.1. This decomposition (or: the existence of a decomposition, more precisely) is used however in the proof of Corollary 3.1.
(ii) Explicit non-trivial decompositions depend on a particular model for the primary market, as well as on a game option under consideration (since for the valuation and hedging of a particular contract it usually suffices to select some traded primary assets). In particular, suitable decompositions of M will follow from a martingale representation theorem with an orthogonal component. Let $\mathcal{H}^2$ be the class of real-valued $(\mathbb{F},\mathbb{Q})$-martingales with integrable quadratic variation over $[0,T]$. If $M\in\mathcal{H}^2$ and N is as in Definition 3.3 with mutually strongly orthogonal components in $\mathcal{H}^2$ then a suitable decomposition $dM = \alpha(z\, dN + dn)$ is given by the Galtchouk-Kunita-Watanabe decomposition theorem of M with respect to N (see, e.g., Protter [32, IV.3, Corollary 1]). In this case, the integral $\int_0^\cdot \alpha\, dn$ belongs to $\mathcal{H}^2$ and is strongly orthogonal to N.
(iii) In case $n^d = 0$, decomposition (27) reduces to

$$\beta_t\, dN_t^d = \Lambda_t^d\, d(\beta_t\widehat{X}_t), \quad t\in[0,T\wedge\tau_d],$$

which has the financial meaning that some defaultable asset is traded (at least synthetically) in the primary market.

(iv) Assume that an $\mathbb{R}^q$-valued semimartingale N is given a priori. Then conditions (26) and (27) (with $n^d = 0$) are satisfied if there exists an $\mathbb{R}^{d\otimes(q+1)}$-valued, (row by row) N^*-integrable process $\beta\Xi$ such that Ξ_t is left-invertible for any $t \in [0, T \wedge \tau_d]$ and

$$d(\beta_t \widehat{X}_t) = \beta_t \Xi_t \, dN_t^*, \quad t \in [0, T \wedge \tau_d].$$

Indeed, in this case it suffices to take Λ_t^* equal to the left-inverse of Ξ_t.

We are now in the position to prove the main result of this section. Recall that the process ρ^* given by (28) is a $(\mathbb{G}, \mathbb{Q})$-sigma martingale, by Lemma 3.2. In particular, if we deal with a situation in which we have $n = n^d = 0$ in a decomposition of a solution $(\Theta^0, M, \mathbf{K})$ to (E_0), the next result yields the existence of (ε)-hedging strategies with null residual cost.

Theorem 3.1 *Given a decomposition of a $\mathbb{Q}$-solution $(\Theta^0, M, \mathbf{K})$ to (E_0), then:*
(i) *For any $\varepsilon > 0$, an issuer ε-hedge with initial value $V_0 = \Pi_0$ and residual cost ρ^* given by (28) is furnished by τ_c^ε as in Theorem 2.4 (for $t = 0$ so that $\tau_c^\varepsilon \in \bar{\mathcal{F}}_T^0$) and $\zeta = \zeta^*$ given by the formula*

$$\zeta_t^* = \mathbb{1}_{\{t \le \tau_d\}} \left[z_t, R_t - \bar{\Pi}_{t-} \right] \Lambda_t^*, \quad t \in [0, T], \tag{29}$$

where $\bar{\Pi}$ is defined by (17) and Λ^* by (28).
(ii) *For any $\varepsilon > 0$, a holder ε-hedge with initial value $V_0 = -\Pi_0$ and residual cost ρ^* given by (28) is furnished by τ_p^ε as in Theorem 2.4 (for $t = 0$ so that $\tau_p^\varepsilon \in \mathcal{F}_T^0$) and $\zeta = -\zeta^*$.*
(iii) *If the process $\mathbf{K}$ is continuous, we may set ε equal to 0 in (i)-(ii). Hence there exists an issuer hedge (respectively, a holder hedge) of the game option with initial wealth Π_0 (respectively, $-\Pi_0$) and cost ρ^*.*

Proof. (i) Let us set $V_t = \mathbb{1}_{\{t < \tau_d\}} \bar{\Pi}_t$ for $t \in [0, T]$, in particular $V_0 = \bar{\Pi}_0 = \widetilde{\Pi}_0 = \Pi_0$. We will first show that V is the wealth of some primary strategy with cost $\rho^* - D$. Lemma 2.7(i) combined with the decomposition $dM = \alpha(z \, dN + dn)$ yield

$$d(\beta_t V_t) + \beta_t \, dD_t = dY_t = \mathbb{1}_{\{t \le \tau_d\}} z_t \beta_t \, dN_t + \mathbb{1}_{\{t \le \tau_d\}} \beta_t (R_t - \bar{\Pi}_{t-}) \, dN_t^d + \mathbb{1}_{\{t \le \tau_d\}} \beta_t \, dn_t, \tag{30}$$

by (26). Thus, using also (27),

$$d(\beta_t V_t) + \beta_t \, dD_t = \zeta_t^* \, d(\beta_t \widehat{X}_t) + \beta_t \, dQ_t^*, \quad t \in [0, T].$$

This means that V is the wealth of the primary strategy with cost $\rho^* - D$ and with $\zeta = \zeta^*$ given by (29).

Let us fix $\varepsilon > 0$. In order to prove that the quadruplet $(V_0, \zeta^*, \rho^*, \tau_c^\varepsilon)$ is an issuer ε-hedge with residual cost ρ^* for the game option, where the stopping time $\tau_c^\varepsilon \in \bar{\mathcal{F}}_T^0$ is given by (see Theorem 2.4)

$$\tau_c^\varepsilon = \inf \left\{ t \in [\bar{\tau}, T] \, ; \, \widetilde{\Pi}_t \ge U_t - \varepsilon \right\} \wedge T,$$

it is enough to prove that for any $t \in [0, T]$ (cf. (21))

$$V_{t \wedge \tau_c^\varepsilon} \ge \mathbb{1}_{\{t \wedge \tau_c^\varepsilon < \tau_d\}} \Big(\mathbb{1}_{\{t \wedge \tau_c^\varepsilon = t < T\}} L_t + \mathbb{1}_{\{\tau_c^\varepsilon < t\}} U_{\tau_c^\varepsilon} + \mathbb{1}_{\{t = \tau_c^\varepsilon = T\}} \xi \Big) - \varepsilon. \tag{31}$$

Observe first that, on $[0, \tau_c^\varepsilon]$,

$$\alpha_t \widetilde{\Pi}_t = \widetilde{\Pi}_0 - \int_0^t \alpha_u \, dC_u - \int_0^t \alpha_u R_u \, d\Gamma_u + M_t - K_t = \alpha_t \bar{\Pi}_t - K_t \le \alpha_t \bar{\Pi}_t$$

since $K_t^- = 0$ on $[0, \tau_c^\varepsilon]$ (this follows from the definition of τ_c^ε and the minimality conditions in (E_0)). Hence we obtain, on $[0, \tau_c^\varepsilon]$,

$$V_t = \mathbb{1}_{\{t < \tau_d\}} \bar{\Pi}_t \ge \mathbb{1}_{\{t < \tau_d\}} \widetilde{\Pi}_t,$$

where, by (E_0), we have
$$\widetilde{\Pi}_t \geq \mathbb{1}_{\{t<T\}} L_t + \mathbb{1}_{\{t=T\}} \xi.$$
In addition, by the definition of τ_c^ε, we have that $\widetilde{\Pi}_{\tau_c^\varepsilon} \geq U_{\tau_c^\varepsilon} - \varepsilon$ on the event $\{\tau_c^\varepsilon < T\}$. It is now easy to see that (31) is satisfied, and thus $(V_0, \zeta^*, \rho^*, \tau_c^\varepsilon)$ is indeed an issuer ε-hedge.
(ii) The arguments for the existence a holder ε-hedge with $V_0 = -\Pi_0$ and residual cost ρ^* are essentially symmetrical to those for an issuer hedge examined in part (i). The details are left to the reader.
(iii) If the process $\mathbf{K}$ is continuous, one can take ε equal to 0 above and thus part (iii) holds as well. Indeed, the continuity of $\mathbf{K}$ implies that $K_t^- = 0$ on $[0, \tau_c^0]$ and $K_t^+ = 0$ on $[0, \tau_p^0]$. □

Corollary 3.1 *Under the assumptions of Theorem 3.1, we have that:*
(i) $\Pi_0 = \min \mathcal{V}_0^c$.
(ii) $-\Pi_0 \in \mathcal{V}_0^p$. *If, in addition, (4) holds then* $\Pi_0 = -\min \mathcal{V}_0^p$.

Proof. (i) By applying Theorem 3.1 to the trivial decomposition (cf. Remark 3.1(i), we get in particular that $\Pi_0 \in \mathcal{V}_0^c$, where Π_0 is also equal to the $\mathbb{Q}$-value of the related Dynkin game, by Theorem 2.4. Thus the infimum is attained and we have equality, rather than inequality, in Lemma 3.1(i).
(ii) The second claim can be proven as part (i), under assumption (4). □

Remarks 3.2 (i) Note that $\beta\widehat{X}$-integrability of ζ^* is implied by the N^*-integrability of the process $\beta[z, (R - \bar{\Pi}_-)]$. The latter property is in turn embedded in the assumption that (E_0) has a solution (cf. (30)).
(ii) In fact, it is easy to see that one could state analogous definitions and results regarding hedging a defaultable game option, starting from any $t \in [0, T]$. Otherwise said, the fact that 0 is the inception date of the option is immaterial in Lemma 3.1, Theorem 3.1 and Corollary 3.1.
(iii) Let us now discuss the dependence of these results on the choice of a particular risk-neutral measure $\mathbb{Q} \in \mathcal{M}$. Given $\mathbb{Q}, \mathbb{Q}' \in \mathcal{M}$, the existence of a $\mathbb{Q}$-solution to the $\mathbb{Q}$-related BSDE is not necessarily equivalent to the existence of a $\mathbb{Q}'$-solution to the $\mathbb{Q}'$-related BSDE. However, in the special case where both BSDEs admit solutions with decompositions such that $n = n^d = 0$, then by the minimality statement in Theorem 3.1(i) (extended to any time $t \in [0, T]$, see the remark (i) above) the related arbitrage price processes must coincide. It is not clear, however, whether the related hedging strategies must coincide as well.

3.3 Special Case of Defaultable European Options

Let us now consider the special case of a defaultable European option.

Definition 3.4 (i) An *issuer hedge* with residual cost ρ for a European option is a primary strategy $(V_0, \zeta, \rho - D)$ with wealth process V such that $V_T - \mathbb{1}_{\{\tau_d > T\}}\, \xi \geq 0$, a.s. If the inequality is replaced by equality then we deal with an *issuer replicating strategy* with residual cost ρ.
(ii) A *holder hedge* with residual cost ρ for a European option is a primary strategy $(V_0, \zeta, \rho + D)$ with wealth process V such that $V_T + \mathbb{1}_{\{\tau_d > T\}}\, \xi \geq 0$, a.s. If the inequality is replaced by equality then we deal with a *holder replicating strategy* with residual cost ρ.

In the special case of an European option, we shall consider the BSDE (E_0) with $\widehat{L}$ replaced by $\bar{L}$ such that $\alpha\bar{L} = -(c+1)$, where $-c$ is a lower bound on $\alpha_T\widehat{\xi}$. Note that under mild technical assumptions this equation has a solution $(\Theta^0, M, \mathbf{K})$ (see [5, 11])). By Proposition 2.2 we obtain
$$\alpha_t\Theta_t^0 = \operatorname{esssup}_{\tau_p \in \mathcal{F}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau_p < T\}} \alpha_{\tau_p} \bar{L}_{\tau_p} + \mathbb{1}_{\{\tau_p = T\}} \alpha_T \widehat{\xi} \,\big|\, \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(\alpha_T \widehat{\xi} \,\big|\, \mathcal{F}_t\big),$$
where we also used the definition of $\bar{L}$. So, first, the Θ^0-component of (E) is an arbitrage $\mathbb{Q}$-price of the option, and, second, we have that $\alpha\Theta^0 \geq -c$, hence $\Theta^0 > \bar{L}$ on $[0, T]$, so that necessarily $\mathbf{K} = \mathbf{0}$,

and (E) effectively reduces to an elementary BSDE with no process $\mathbf{K}$ involved in the solution. Thus in the foregoing result we may and do assume that $\mathbf{K} = \mathbf{0}$, without loss of generality.

The next theorem can be established along the similar lines as Lemma 3.1, Theorem 3.1 and Corollary 3.1. The proofs are, of course, simpler since there are in effect no barriers involved, and thus they are omitted. Note also that the obvious analogues to Remarks 3.2 can be formulated.

Theorem 3.2 *In the case of a European option, assume that the BSDE (E_0) with $\widehat{L}$ replaced by $\bar{L}$ and with $\bar{\tau} = T$ admits a $\mathbb{Q}$-solution $(\Theta^0, M, \mathbf{K} = \mathbf{0})$. Let us set $\widetilde{\Phi}_t = \Theta^0_t - F^0_t$ for $t \in [0,T]$. Then the process $\Phi_t = \mathbb{1}_{\{t<\tau_d\}}\widetilde{\Phi}_t$ is an arbitrage price process for the option, as well as a minimal issuer hedging price process with $(\mathbb{G},\mathbb{Q})$-sigma martingale residual cost. In the case where R and ξ are bounded, $-\Phi$ is also a minimal holder hedging price process with $(\mathbb{G},\mathbb{Q})$-sigma martingale residual cost. Moreover, given a decomposition of $(\Theta^0, M, \mathbf{K} = \mathbf{0})$, we have that:*
(i) *an issuer replicating strategy with initial wealth Φ_0 and residual cost ρ^* is furnished by $(\Phi_0, \zeta^*, \rho^*)$ with*

$$\zeta^*_t = \mathbb{1}_{\{t\le\tau_d\}}\left[z_t, R_t - \widetilde{\Phi}_{t-}\right]\Lambda^*_t, \quad t \in [0,T]\,,$$

(ii) *$(-\Phi_0, -\zeta^*, \rho^*)$ is a holder replicating strategy with initial wealth $-\Phi_0$ and residual cost ρ^*.*

3.4 Variants of Main Results

To deal with some practical examples, it is essential to enjoy some freedom in the choice of a doubly reflected BSDE associated with a game option. To this end, we introduce the following definition.

Definition 3.5 Given a game option with data $C, R, \xi, L, U, \bar{\tau}$, and an $\mathbb{F}$-adapted finite variation driver F with $F^0 - F$ bounded from below, we define (E) as the doubly reflected BSDE (12) with data

$$F,\ \chi = \widehat{\xi} - F_T,\ \mathcal{L} = \widehat{L} - F,\ \mathcal{U} = \widehat{U} - F,\ \bar{\tau},$$

where, as in (E_0),

$$\widehat{\xi} = \xi + F^0_T, \quad \widehat{L}_t = L_t + F^0_t, \quad \widehat{U}_t = U_t + F^0_t.$$

According to the definition above, the BSDE (E) has the following form

$$\begin{gathered}\alpha_t\Theta_t = \alpha_T\chi + \alpha_T F_T - \alpha_t F_t + K_T - K_t - (M_T - M_t), \quad t \in [0,T],\\ \mathcal{L}_t \le \Theta_t \le \bar{\mathcal{U}}_t, \quad t \in [0,T],\\ \int_0^T (\Theta_{u-} - \mathcal{L}_{u-})\, dk^+_u = \int_0^T (\bar{\mathcal{U}}_{u-} - \Theta_{u-})\, dk^-_u = 0\end{gathered}$$

with $\bar{\mathcal{U}} = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\ge\bar{\tau}\}}(\widehat{U} - F)$.

In the special case of an European option, we consider the BSDE (E) with $\mathcal{L}$ replaced by $\bar{L}$ such that $\alpha\bar{L} = -(c+1)$, where $-c$ is a lower bound on $\alpha_T\chi$.

Observe that for $F = 0$ equation (E) reduces to (E_0). As we already noted in Remark 2.3(iii), equations corresponding to various choices of a driver F are equivalent, in the sense that $(\Theta, M, \mathbf{K})$ solves (E) for some driver F if and only if $(\Theta^0, M, \mathbf{K})$ solves (E_0), where $\Theta^0 = \Theta + F$. However, as we shall see in further work (see [5, 6]), the freedom to use the most convenient driver is very useful in financial applications. This motivates us to state the following corollary to Theorem 2.4.

Corollary 3.2 *If $(\Theta, M, \mathbf{K})$ is a $\mathbb{Q}$-solution to (E) then the conclusions of Theorems 2.4 and 3.1 are still valid, provided that we set $\widetilde{\Pi} = \Theta + F - F^0$ instead of $\widetilde{\Pi} = \Theta^0 - F^0$ in (17). If $(\Theta, M, \mathbf{K} = \mathbf{0})$ is a $\mathbb{Q}$-solution to (E) with $\bar{L}$ instead of $\widehat{L}$ in $\mathcal{L}$ and with $\bar{\tau} = T$ then the conclusions of Theorem 3.2 are still valid, provided that we set $\widetilde{\Phi} = \Theta + F - F^0$ instead of $\widetilde{\Phi} = \Theta^0 - F^0$ therein.*

4 Conclusions

Assuming that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, where the filtration $\mathbb{H}$ carries the information about the default and where $\mathbb{F}$ is some reference filtration, we were actually able to we effectively reduce the information flow from the full filtration $\mathbb{G}$ to the reference filtration $\mathbb{F}$. We derived convenient pricing formulae with respect to the reference filtration $\mathbb{F}$ for a defaultable game option (including defaultable American and European options as special cases). In addition, working under suitable integrability and regularity conditions embedded in the standing assumption that a related doubly reflected BSDE (E_0) or (E) has a solution, we obtained explicit expressions for hedging strategies.

Note that while (E) is the best form of the related BSDE from the point of view of getting the most general results under minimal assumptions, however from the *practical* point of view, the most useful form of the BSDE is the *discounted form* of (E) (see [5]). Indeed the latter form is the most convenient for the direct study of the BSDEs (showing existence and uniqueness of a $\mathbb{Q}$-solution with mutually singular k^+ and k^-, see comments below Definition 2.5) and is most naturally connected to the variational approach in a Markovian set-up. For more about this, we refer the reader to [5, 11, 6].

A Appendix: Proofs of Auxiliary Lemmas

A.1 Proof of Lemma 2.3

Since τ is a $\mathbb{G}$-stopping time, by [15, p. 186, §75] there exists an $\mathbb{F}$-stopping time $\widehat{\tau}$ such that $\tau \wedge \tau_d = \widehat{\tau} \wedge \tau_d$. Moreover, since $\tau \in \mathcal{G}_T^t$, we have

$$\tau \wedge \tau_d = (\tau \vee t) \wedge \tau_d = (\tau \wedge \tau_d) \vee (t \wedge \tau_d) = (\widehat{\tau} \wedge \tau_d) \vee (t \wedge \tau_d) = (\widehat{\tau} \vee t) \wedge \tau_d,$$

so that we may take $\widetilde{\tau} = \widehat{\tau} \vee t \in \mathcal{F}_T^t$. Moreover, if $\tau \wedge \tau_d \geq \bar{\tau} \wedge \tau_d$ for some stopping time $\bar{\tau} \in \mathcal{G}_T^0$, then we also have that $\widetilde{\tau} \wedge \tau_d = \tau \wedge \tau_d \geq \bar{\tau} \wedge \tau_d$, which proves (i).

For (ii), let $\breve{\tau} \in \mathcal{F}_T^t$ be such that $\tau \wedge \tau_d = \breve{\tau} \wedge \tau_d$, by (i). Assuming that $\bar{\tau} \in \mathcal{F}_T^0$, we have that $\widetilde{\tau} = \breve{\tau} \vee \bar{\tau} \in \bar{\mathcal{F}}_T^t$. So

$$\widetilde{\tau} \wedge \tau_d = (\breve{\tau} \vee \bar{\tau}) \wedge \tau_d = (\breve{\tau} \wedge \tau_d) \vee (\bar{\tau} \wedge \tau_d) = \breve{\tau} \wedge \tau_d = \tau \wedge \tau_d,$$

where the third equality holds, since $\tau \in \bar{\mathcal{G}}_T^t$ implies that $\breve{\tau} \wedge \tau_d \geq \bar{\tau} \wedge \tau_d$, by (i). □

A.2 Proof of Lemma 2.5

(i) Since $\tau \in \mathcal{F}_t^T$, one has $\mathcal{F}_t \subset \mathcal{F}_\tau \subset \mathcal{F}_T$, hence by Lemma 2.2:

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau<\tau_d\}}\chi \,|\, \mathcal{G}_t) &= \mathbb{1}_{\{t<\tau_d\}}\, e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau<\tau_d\}}\chi \,\big|\, \mathcal{F}_t\big) \\ &= \mathbb{1}_{\{t<\tau_d\}}\, e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big(\chi\, \mathbb{Q}(\tau<\tau_d \,|\, \mathcal{F}_\tau) \,\big|\, \mathcal{F}_t\big) \\ &= \mathbb{1}_{\{\tau_d>t\}}\, e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big(\chi\, e^{-\Gamma_\tau} \,\big|\mathcal{F}_t\big), \end{aligned}$$

where the second equality follows by (7).

(ii) If suffices to prove the asserted formula for an elementary predictable process of the form $Z_s = \mathbb{1}_{]u,v]}(s)B_u$ for an arbitrary event $B_u \in \mathcal{F}_u$. For such a process, the formula follows easily from part (i).

(iii) We have that

$$\int_{t\wedge\tau_d}^{\tau\wedge\tau_d} dA_u = \mathbb{1}_{\{t<\tau_d\}} \int_{t\wedge\tau_d}^{\tau\wedge\tau_d} dA_u = \mathbb{1}_{\{\tau<\tau_d\}} \int_t^\tau dA_u + \mathbb{1}_{\{t<\tau_d\leq\tau\}} \int_t^{\tau_d} dA_u$$

where A is $\mathbb{F}$-predictable. Using parts (i) and (ii), we obtain

$$\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\tau<\tau_d\}}\int_t^\tau dA_u\,\Big|\,\mathcal{G}_t\Big)=\mathbb{1}_{\{t<\tau_d\}}\mathbb{E}_{\mathbb{Q}}\Big(e^{\Gamma_t-\Gamma_\tau}\int_t^\tau dA_u\,\Big|\,\mathcal{F}_t\Big)$$

and

$$\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_d\le\tau\}}\int_t^{\tau_d} dA_u\,\Big|\,\mathcal{G}_t\Big)=\mathbb{1}_{\{t<\tau_d\}}\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\tau\Big(\int_t^s dA_u\Big)e^{\Gamma_t-\Gamma_s}\,d\Gamma_s\,\Big|\,\mathcal{F}_t\Big),$$

where, by Fubini theorem,

$$\int_t^\tau\Big(\int_t^s dA_u\Big)e^{\Gamma_t-\Gamma_s}\,d\Gamma_s=\int_t^\tau\int_t^s dA_u\,e^{\Gamma_t-\Gamma_s}\,d\Gamma_s=\int_t^\tau e^{\Gamma_t-\Gamma_u}\,dA_u-e^{\Gamma_t-\Gamma_\tau}\int_t^\tau dA_u.$$

Hence

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{t\wedge\tau_d}^{\tau\wedge\tau_d} dA_u\,\Big|\,\mathcal{G}_t\Big)=\mathbb{1}_{\{t<\tau_d\}}\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\tau e^{\Gamma_t-\Gamma_u}\,dA_u\,\Big|\,\mathcal{F}_t\Big)=\mathbb{1}_{\{t<\tau_d\}}e^{\Gamma_t}\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\tau e^{-\Gamma_u}\,dA_u\,\Big|\,\mathcal{F}_t\Big),$$

as was expected. □

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