DEFAULTABLE OPTIONS IN A MARKOVIAN INTENSITY MODEL OF CREDIT RISK

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1 Introduction

In Bielecki et al. [4], we studied the valuation and hedging of defaultable game options in a very general reduced-form model of credit risk. Given a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$ used to model the primary market, it was assumed in [4] that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ where the filtration $\mathbb{H}$ carries the information about the default and where $\mathbb{F}$ is some reference filtration. The main technique employed in [4] was the effective reduction of the information flow from the full filtration $\mathbb{G}$ to the reference filtration $\mathbb{F}$. Under suitable conditions on the $(\mathbb{F}, \mathbb{Q})$-optional projection of the *default indicator process* $H_t = \mathbb{1}_{\{\tau_d \le t\}}$, we derived convenient pricing formulae with respect to the reference filtration $\mathbb{F}$. Also, we proved that, under suitable integrability and regularity conditions embedded in the standing assumption that a related doubly reflected BSDE denoted by (E) admits a solution under some risk-neutral measure $\mathbb{Q}$, the default indicator process multiplied by the state-process of the solution is the minimal (super)hedging price with $(\mathbb{G}, \mathbb{Q})$-sigma martingale[1] residual cost. In the case of complete markets, the residual cost of hedging strategies vanishes, so that they are self-financing in the usual sense.

In order to apply these results to practical models for pricing and hedging financial derivatives (in particular, convertible bonds, see e.g. [5]), we propose here a generic *Markovian pre-default intensity model* of credit risk. It encompasses, in particular, the industry standard jump-diffusion model that is studied in detail in [5].

As a prerequisite, we first derive in Theorem 2.1 a variant of the main result in [4] under the slightly stronger assumption that the so-called *discounted form* of the doubly reflected backward stochastic differential equation associated with a defaultable game option has a solution. Let us note that various sets of sufficient conditions for this are given in the literature.

In our previous work [4], we took a primary market arbitrage price process X as given, satisfying all our assumptions. In Section 3.1 (Proposition 3.1) we shall show a generic way to *construct* such a primary market arbitrage price process X. In particular, we give in Lemma 3.2 a general arbitrage drift consistency condition that determines the pre-default model drift coefficients. Now, in order to have a practical use, a (dynamic) pricing model needs to be constructive, or *Markovian* in some sense, relative to a given option. This will be achieved by assuming that the related BSDE $(\mathcal{E})$ is in fact a *Markovian Forward Backward SDE* (Section 3.2). Under a rather generic specification for the infinitesimal generator of the underlying Markov process, we shall derive in Section 3.3 a related *variational inequality approach* to pricing and hedging the option.

Finally, in Section 4, we illustrate our study on the case of convertible bonds.

1.1 General Set-Up

For a finite horizon date $T > 0$, we assume that the primary market is composed of the savings account and d risky assets with price processes defined on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$ where $\mathbb{P}$ denotes the statistical probability measure. We postulate that (cf. [3]):

- the *discount factor* process β, that is, the inverse of the savings account, is a $\mathbb{G}$-adapted, finite variation, continuous, positive and bounded process;
- the prices of risky assets are $\mathbb{G}$-semimartingales with càdlàg sample paths.

The primary risky assets, with $\mathbb{R}^d$-valued price process X, are assumed to pay dividends, whose cumulative value process, denoted by $\mathcal{D}$, is modeled as a $\mathbb{G}$-adapted, càdlàg and $\mathbb{R}^d$-valued process of finite variation. Given the price process X, we define the *cumulative price* $\widehat{X}$ of the asset as

$$\widehat{X}_t = X_t + \beta_t^{-1} \int_{[0,t]} \beta_u \, d\mathcal{D}_u. \tag{1}$$

We assume that the primary market model is free of arbitrage opportunities (though presumably incomplete), in the sense that there exists a *risk-neutral measure* $\mathbb{Q} \in \mathcal{M}$, where $\mathcal{M}$ denotes the set

[1] For the definition and properties of a sigma martingale, see, for instance, [8, 19, 25].

of probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ for which $\beta\widehat{X}$ is a $(\mathbb{G},\mathbb{Q})$-sigma martingale. It is worth stressing that in this paper, similarly as in [3, 4], we work with the notion of vector (as opposed to componentwise) stochastic integral (see [4] or Cherny and Shiryaev [8]). By convention, we denote by $\int_0^t$ the integral over $(0,t]$; otherwise, we explicitly specify the domain of integration as a subscript of $\int$.

Note that in what follows we in fact deal with right-continuous and completed versions of all relevant filtrations, so that all the filtrations under consideration satisfy the so-called 'usual conditions.'

2 Valuation and Hedging of Defaultable Options in the Hazard Process Set-Up: A User's Guide

In [4], we derived general hedging results for a game option under fairly general assumptions in the so-called hazard process set-up. In the same framework, and under slightly stronger assumptions (see Remark 2.3), we shall now derive variants of these results that are required in practical applications of the general theory.

2.1 Hazard Process Set-Up

Given a $[0,+\infty]$-valued $\mathbb{G}$-stopping time τ_d representing the *default time* of a reference entity, we assume as in [4] that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, where the filtration $\mathbb{H}$ is generated by the *default indicator process* $H_t = \mathbb{1}_{\{\tau_d \le t\}}$ and $\mathbb{F}$ is some reference filtration. Moreover, we assume that the process G given by $G_t = \mathbb{Q}(\tau_d > t \,|\, \mathcal{F}_t)$ for $t \in \mathbb{R}_+$ is (strictly) positive, continuous and non-increasing. Hence the $(\mathbb{F},\mathbb{Q})$-*hazard process* $\Gamma_t = -\ln(G_t)$, $t \in \mathbb{R}_+$, is well defined, continuous and non-decreasing. Finally, we assume that the default time τ_d avoids $\mathbb{F}$-stopping times. Hence any $(\mathbb{F},\mathbb{Q})$-martingale M cannot jump at τ_d, that is, $\Delta M_{\tau_d} := M_{\tau_d} - M_{\tau_d -} = 0$, $\mathbb{Q}$-a.s.

We shall sometimes assume, in addition, that the discount factor β and the hazard process Γ are absolutely continuous with respect to the Lebesgue measure, namely,
- $\beta_t = \exp(-\int_0^t r_u \, du)$ for an $\mathbb{F}$-adapted bounded from below *short-term interest rate* process r,
- $\Gamma_t = \int_0^t \gamma_u \, du$ with non-negative $(\mathbb{F},\mathbb{Q})$-*intensity process* γ.

A set-up satisfying these assumptions is referred to as a *default intensity set-up*.

Let $\alpha_t = \beta_t \exp(-\Gamma_t)$ stand for the *credit-risk adjusted discount factor* (note that the process α is bounded, like β). Also, let us denote $B_t = -\ln \alpha_t$. In the default intensity set-up we obtain $B_t = \int_0^t \mu_u \, du$ with $\mu = r + \gamma$.

For any $t \in [0,T]$, let $\mathcal{F}_T^t$ (resp. $\mathcal{G}_T^t$) denote the set of $[t,T]$-valued $\mathbb{F}$ (resp. $\mathbb{G}$)-stopping times.

Lemma 2.1 (Bielecki et al. [4]) (i) *Any $(\mathbb{F},\mathbb{Q})$-local martingale stopped at τ_d is a $(\mathbb{G},\mathbb{Q})$-local martingale.*
(ii) *For any $\mathbb{G}$-adapted process Y over $[0,T]$ there exists a unique $\mathbb{F}$-adapted process $\widetilde{Y}$ over $[0,T]$ such that $\mathbb{1}_{\{t<\tau_d\}} Y_t = \mathbb{1}_{\{t<\tau_d\}} \widetilde{Y}_t$ for $t \in [0,T]$.*
(iii) *If $\tau \in \mathcal{G}_T^0$ then there exists a unique $\widetilde{\tau} \in \mathcal{F}_T^0$ such that $\tau \wedge \tau_d = \widetilde{\tau} \wedge \tau_d$.*

The quantities $\widetilde{\tau}$ and $\widetilde{Y}$ are called the *pre-default values* of τ and Y, respectively. For any $\bar{\tau} \in \mathcal{F}_T^0$, let $\bar{\mathcal{G}}_T^t$ stand for $\{\tau \in \mathcal{G}_T^t \,;\, \tau \wedge \tau_d \ge \bar{\tau} \wedge \tau_d\}$.

The next result is an immediate consequence of Lemma 2.5 in Bielecki et al. [4]. This is the cornerstone for the reduction of the information flow from the full filtration $\mathbb{G}$ to the reference filtration $\mathbb{F}$.

Lemma 2.2 *Let $\tau \in \mathcal{F}_T^t$ for some $t \in [0,T]$. For any $\mathcal{F}_\tau$-measurable random variable χ, $\mathbb{F}$-predictable process Z and finite variation $\mathbb{F}$-predictable process A such that the conditional expectation of each of the three terms of the sums on the l.h.s. or on the r.h.s. of (2) is well defined in $\overline{\mathbb{R}}$, we have:*

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\Big(\int_{t\wedge\tau_d}^{\tau\wedge\tau_d} dA_u + \mathbb{1}_{\{t<\tau_d\le\tau\}} Z_{\tau_d} + \mathbb{1}_{\{\tau<\tau_d\}}\chi \,\Big|\, \mathcal{G}_t\Big) \qquad (2)\\ &\quad = \mathbb{1}_{\{t<\tau_d\}} e^{\Gamma_t}\,\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\tau e^{-\Gamma_u} dA_u + \int_t^\tau e^{-\Gamma_u} Z_u\, d\Gamma_u + e^{-\Gamma_\tau}\chi \,\Big|\, \mathcal{F}_t\Big).\end{aligned}$$

We now recall the concept of a (dividend paying) *defaultable game option* (see [21, 20, 3, 4]) with inception date 0 and maturity date T.

Definition 2.1 (Bielecki et al. [3, 4]) A *defaultable game option* is a game option with the *ex-dividend cumulative discounted cash flows* $\beta_t\pi(t;\tau_p,\tau_c)$, where the $\mathcal{G}_{\tau\wedge\tau_d}$-measurable random variable $\pi(t;\tau_p,\tau_c)$ is given by the formula, for any $t \in [0,T]$ and $(\tau_p,\tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$,

$$\beta_t\pi(t;\tau_p,\tau_c) = \int_t^\tau \beta_u\, dD_u + \mathbb{1}_{\{\tau_d>\tau\}}\beta_\tau\Big(\mathbb{1}_{\{\tau=\tau_p<T\}} L_{\tau_p} + \mathbb{1}_{\{\tau<\tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau=T\}}\xi\Big),$$

where $\tau = \tau_p \wedge \tau_c$ and
• the *dividend process* $D = (D_t)_{t\in[0,T]}$ equals

$$D_t = \int_{[0,t]} (1-H_u)\, dC_u + R_u\, dH_u$$

for some *coupon process* $C = (C_t)_{t\in[0,T]}$, which is a $\mathbb{G}$-adapted càdlàg process with bounded variation, and some real-valued, $\mathbb{G}$-predictable *recovery process* $R = (R_t)_{t\in[0,T]}$,
• the *put payment* $L = (L_t)_{t\in[0,T]}$ the *call payment* $U = (U_t)_{t\in[0,T]}$ are $\mathbb{G}$-adapted, real-valued, càdlàg processes,
• the inequality $L_t \le U_t$ holds for every $t \in [\tau_d\wedge\bar\tau, \tau_d\wedge T)$, for some *lifting time of a call protection* $\bar\tau \in \mathcal{F}_T^0$, and
• the *payment at maturity* ξ is a $\mathcal{G}_T$-measurable real random variable.

We further assume that R, L and ξ are bounded from below, so that the cumulative discounted payoff is bounded from below. Specifically, there exists a constant c such that

$$\beta_t\widehat{\mathcal{L}}_t := \int_{[0,t]} \beta_u\, dD_u + \mathbb{1}_{\{\tau_d>t\}}\beta_t\Big(\mathbb{1}_{\{t<T\}} L_t + \mathbb{1}_{\{t=T\}}\xi\Big) \ge -c, \quad t\in[0,T].$$

In order to get the upper bound for this payoff, we will sometimes assume that R, U and ξ are bounded (from below and from above), or simply that there exists a constant c such that

$$\beta_t\widehat{\mathcal{U}}_t := \int_{[0,t]} \beta_u\, dD_u + \mathbb{1}_{\{\tau_d>t\}}\beta_t\Big(\mathbb{1}_{\{t<T\}} U_t + \mathbb{1}_{\{t=T\}}\xi\Big) \le c, \quad t\in[0,T]. \qquad (3)$$

The class of defaultable game options covers as special cases *defaultable American options* (case $\bar\tau = T$). It can be shown that the latter class includes *defaultable European options* as a special case (sub-case $\beta\widehat{\mathcal{L}}$ maximum at T, see [3]). Defaultable European options can equivalently be redefined as contracts with cash flows $\phi(t)$ given by, for $t\in[0,T]$,

$$\beta_t\phi(t) = \int_t^T \beta_u\, dD_u + \mathbb{1}_{\{\tau_d>T\}}\beta_T\xi.$$

We are in the position to introduce the concept of hedging of a game option (cf. [4]). Recall that X (resp. $\widehat{X}$) is the price process (resp. cumulative price process) of primary traded assets defined in (1).

Definition 2.2 By a *primary strategy* we mean a triplet (V_0, ζ, Q) such that:
• V_0 is an $\mathcal{G}_0$-measurable real-valued random variable representing the *initial wealth*,
• ζ is a $\mathbb{R}^{1\otimes d}$-valued, $\beta\widehat{X}$-integrable process representing holdings in primary risky assets,
• Q is a real-valued, $\mathbb{G}$-semimartingale, with $Q_0 = 0$, representing the (generalized) *cost process.*
The *wealth process* V of a primary strategy (V_0, ζ, Q) is given by

$$d(\beta_t V_t) = \zeta_t \, d(\beta_t \widehat{X}_t) + \beta_t \, dQ_t, \quad t \in [0, T],$$

with the initial condition V_0.

Note that a primary strategy introduced in Definition 2.2 is not self-financing in the standard sense, unless $Q = 0$. Given the wealth process V of a primary strategy (V_0, ζ, Q), we uniquely specify a $\mathbb{G}$-predictable process ζ^0 by setting

$$V_t = \zeta^0_t \beta^{-1}_t + \zeta_t X_t, \quad t \in [0, T].$$

The process ζ^0 represents the number of units held in the savings account at time t, starting from the initial wealth V_0 and using the strategy ζ in the primary risky assets and the cost process Q (see [4] for more comments on Definitions 2.2 and 2.3).

Definition 2.3 An *issuer hedge with residual cost* ρ for the game option is represented by a quadruplet $(V_0, \zeta, \rho, \tau_c)$ such that:
(i) τ_c belongs to $\bar{\mathcal{G}}^0_T$,
(ii) $(V_0, \zeta, \rho - D)$ is a primary strategy with related wealth process V such that, for $t \in [0, T]$,

$$V_{t\wedge\tau_c} - \mathbb{1}_{\{t\wedge\tau_c<\tau_d\}}\Big(\mathbb{1}_{\{t\wedge\tau_c=t<T\}}L_t + \mathbb{1}_{\{\tau_c<t\}}U_{\tau_c} + \mathbb{1}_{\{t=\tau_c=T\}}\xi\Big) \geq 0.$$

A *holder hedge with residual cost* ρ for the game option is a quadruplet $(V_0, \zeta, \rho, \tau_p)$ such that:
(i) τ_p belongs to $\mathcal{G}^0_T$,
(ii) $(V_0, \zeta, \rho + D)$ is a primary strategy with related wealth process V such that, for $t \in [\bar{\tau}, T]$,

$$V_{t\wedge\tau_p} + \mathbb{1}_{\{t\wedge\tau_p<\tau_d\}}\Big(\mathbb{1}_{\{\tau_p\wedge t=\tau_p<T\}}L_{\tau_p} + \mathbb{1}_{\{t<\tau_p\}}U_t + \mathbb{1}_{\{\tau_p=t=T\}}\xi\Big) \geq 0.$$

Issuer or holder hedges *with no residual cost* (that is, with $\rho = 0$) are also called issuer or holder *superhedges.*

Let us now consider the special case of a defaultable European option.

Definition 2.4 (i) An *issuer hedge* with residual cost ρ for an European option is a primary strategy $(V_0, \zeta, \rho - D)$ with wealth process V such that $V_T - \mathbb{1}_{\{\tau_d>T\}}\,\xi \geq 0$. If the inequality is replaced by equality then we deal with an *issuer replicating strategy* with residual cost ρ.
(ii) A *holder hedge* with residual cost ρ for a European option is a primary strategy $(V_0, \zeta, \rho + D)$ with wealth process V such that $V_T + \mathbb{1}_{\{\tau_d>T\}}\,\xi \geq 0$. If the inequality is replaced by equality then we deal with a *holder replicating strategy* with residual cost ρ.

Remarks 2.1 Note that in [4] we defined more general notions of ε-hedges, that were pertaining in the case where there may be jumps in the process $\mathbf{k}$ to be defined below. Since in practice $\mathbf{k}$ is always found to be a continuous process (see Remark 2.2), we only consider hedges in this paper, not ε-hedges. Note however that the following developments can be extended to possible jumps in $\mathbf{k}$, using the generalized notion of ε-hedge defined in [4].

2.2 Valuation and Hedging Results

Given a risk-neutral measure $\mathbb{Q} \in \mathcal{M}$, we shall now study valuation and hedging of a game option under suitable integrability and regularity conditions. These assumptions are implicitly embedded in the standing assumption that a related doubly reflected BSDE admits a solution. We introduce below a doubly reflected BSDE $(\mathcal{E})$ with respect to $(\Omega, \mathbb{F}, \mathbb{Q})$, with data defined from the pre-default value processes of the data of a game option. Assuming that $(\mathcal{E})$ has a solution (for which various sets of sufficient regularity and integrability conditions are known in the literature), we will deduce explicit hedging strategies with minimal initial wealth and $(\mathbb{G}, \mathbb{Q})$-sigma martingale residual cost for the game option.

Let us define the $\mathbb{F}$-adapted processes $\bar{D}$ and F^0 of finite variation by setting, for $t \in [0, T]$,

$$\bar{D}_t = \int_{[0,t]} dC_u + R_u \, d\Gamma_u, \quad F_t^0 := \alpha_t^{-1} \int_{[0,t]} \alpha_u \, d\bar{D}_u. \tag{4}$$

Given the *driver* F, an $\mathbb{F}$-adapted finite variation process with $F^0 - F$ bounded from below, we consider the following *doubly reflected BSDE* $(\mathcal{E})$ with data $F, \chi, \mathcal{L}, \mathcal{U}, \bar{\tau}$:

$$\left.\begin{array}{c} \Theta_t = \chi + (F_T - F_t) - \int_t^T (F_u + \Theta_u)\, dB_u + k_T - k_t - (m_T - m_t), \quad t \in [0, T], \\ \mathcal{L}_t \le \Theta_t \le \bar{\mathcal{U}}_t, \quad t \in [0, T], \\ \int_0^T (\Theta_{u-} - \mathcal{L}_{u-})\, dk_u^+ = \int_0^T (\bar{\mathcal{U}}_{u-} - \Theta_{u-})\, dk_u^- = 0, \end{array}\right\} (\mathcal{E})$$

where we denote

$$\chi = \xi + F_T^0 - F_T, \quad \mathcal{L} = L + F^0 - F, \quad \bar{\mathcal{U}} = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t \ge \bar{\tau}\}}\mathcal{U} \text{ with } \mathcal{U} = U + F^0 - F\ .$$

We shall see below that $(\mathcal{E})$ is the so-called *discounted form* of equation (E) in [4].

Definition 2.5 By a $\mathbb{Q}$-*solution* to $(\mathcal{E})$, we mean a triplet $(\Theta, m, \mathbf{k})$ such that:
• the *state process* Θ is a real valued, $\mathbb{F}$-adapted, càdlàg process,
• m is a real-valued $(\mathbb{F}, \mathbb{Q})$-martingale vanishing at time 0,
• $\mathbf{k} = (k^+, k^-)$ is a pair of $\mathbb{F}$-adapted, non-decreasing, continuous processes (null at time 0),
• all conditions in $(\mathcal{E})$ are satisfied with $k = k^+ - k^-$ in the first line and with the convention that $0 \times \pm\infty = 0$ in the third line.

For various specifications of the present set-up and sets of technical assumptions ensuring existence and uniqueness of a $\mathbb{Q}$-solution to $(\mathcal{E})$ (at least in the default intensity set-up, so that $dB_t = \mu_t \, dt$ with $\mu = r + \gamma$ in BSDE $(\mathcal{E})$), we refer the reader to [10, 18, 17, 9, 5].

Remarks 2.2 Since in all existing works on doubly reflected BSDEs, the process $\mathbf{k}$ is actually found to be continuous (see the above references), for simplicity of presentation we have imposed the continuity of $\mathbf{k}$ in Definition 2.5. Note, however, that the results presented in this paper can be extended to possible jumps in $\mathbf{k}$, using a generalized notion of ε-hedge (see Remark 2.1 and [4]).

Note that equation $(\mathcal{E})$ and the developments that follow are implicitly parameterized by the choice of a driver F in $(\mathcal{E})$. Equations $(\mathcal{E})$ corresponding to various choices of a driver F are essentially equivalent, in the sense that $(\Theta, m, \mathbf{k})$ solves $(\mathcal{E})$ for some driver F if and only if $(\widehat{\Theta}, m, \mathbf{k})$ solves $(\mathcal{E})$ for $F = 0$, where $\widehat{\Theta} = \Theta + F$. However, as we shall see below, this freedom to use the most convenient driver is essential in financial applications. So a particular form of F may be selected in order to deal with the most tractable BSDE, namely, the BSDE with the simplest form of reflecting barriers, which are the most difficult point to tackle with, from the point of view of solving the BSDE (see Section 4 and [9, 5]). In the case of Markov models (see later sections and [5]), this freedom will allow us to deal with the related variational inequalities of the most tractable structure.

In [4], we introduced the following equation (E)

$$\alpha_t\Theta_t = \alpha_T\chi + \alpha_T F_T - \alpha_t F_t + K_T - K_t - (M_T - M_t), \quad t \in [0,T],$$
$$\mathcal{L}_t \le \Theta_t \le \bar{\mathcal{U}}_t, \quad t \in [0,T],$$
$$\int_0^T (\Theta_{u-} - \mathcal{L}_{u-})\, dk_u^+ = \int_0^T (\bar{\mathcal{U}}_{u-} - \Theta_{u-})\, dk_u^- = 0 \,.$$

The definition of a $\mathbb{Q}$-solution of (E) is the obvious analogue of Definition 2.5 (see [4]). We shall now see that the doubly reflected BSDE in the *intrinsic form* (E) and in the *discounted form* $(\mathcal{E})$ are essentially equivalent. To this end, we define the one-to-one correspondence between processes m and M (resp. $\mathbf{k}$ and $\mathbf{K}$) by setting $dM_t = \alpha_t\, dm_t$ and $d\mathbf{K}_t = \alpha_t\, d\mathbf{k_t}$ with $M_0 = 0$ and $\mathbf{K}_0 = \mathbf{0}$. Let $\mathcal{H}^2$ denote the set of $(\mathbb{F},\mathbb{Q})$-martingales with integrable quadratic variation over $[0,T]$.

Lemma 2.3 *If $(\Theta, m, \mathbf{k})$ is a $\mathbb{Q}$-solution to $(\mathcal{E})$ with $m \in \mathcal{H}^2$ then $(\Theta, M, \mathbf{K})$ is a $\mathbb{Q}$-solution to (E) with $M \in \mathcal{H}^2$. The converse is true in the case of positively bounded α.*

Proof. The proof of the lemma relies on standard computations. Let us only mention that α is bounded, and thus if m is an $(\mathbb{F},\mathbb{Q})$-martingale in $\mathcal{H}^2$ then so is M. The converse implication is valid in the case of positively bounded α. □

Remarks 2.3 Equation (E) is the best form of the related BSDE from the point of view of getting the most general results under minimal assumptions (see [4]). However, as already mentioned, the *discounted form* $(\mathcal{E})$ is more convenient for practical purposes. For this reason, in the present paper, the postulate that equation (E) has a solution in [4], will be replaced by the slightly stronger assumption that equation $(\mathcal{E})$ has a solution $(\Theta, m, \mathbf{k})$ with $m \in \mathcal{H}^2$. Except for this mild strengthening, assumptions in this section are identical as in [4].

For any $\mathbb{G}$-adapted process Y, let Y^{τ_d} stand for the process Y stopped at τ_d. Let $\mathcal{I}$ denote the set of vector stochastic integrals, including constants, with respect to $(\beta\widehat{X})^{\tau_d}$. Given $q \in \mathbb{N}$, let also $\mathcal{H}^{2,q}$ (resp. $\mathcal{I}^q$) stand for the set of $\mathbb{R}^q$-valued $\mathbb{G}$-semimartingales with components in $\mathcal{H}^2$ (resp. $\mathcal{I}$). Let $N^d = H - \Gamma_{\cdot\wedge\tau_d}$ stand for the $\mathbb{Q}$-*compensated jump-to-default process*. Under our standing assumption that the $(\mathbb{F},\mathbb{Q})$-hazard process Γ of τ_d is continuous the process N^d is known to be $(\mathbb{G},\mathbb{Q})$-martingale.

Given a solution $(\Theta, m, \mathbf{k})$ to $(\mathcal{E})$, we define, for $t \in [0,T]$,

$$\widetilde{\Pi}_t = \Theta_t + F_t - F_t^0, \quad \Pi_t = \mathbb{1}_{\{t<\tau_d\}}\widetilde{\Pi}_t, \quad \bar{\Pi}_t = \widetilde{\Pi}_t + \alpha_t^{-1}\int_{[0,t]} \alpha_u\, dk_u \tag{5}$$

$$\tau_p^* = \inf\Big\{\, u \in [t,T]\,;\, \widetilde{\Pi}_u \le L_u \Big\} \wedge T, \quad \tau_c^* = \inf\Big\{\, u \in [\bar{\tau} \vee t, T]\,;\, \widetilde{\Pi}_u \ge U_u \Big\} \wedge T\,. \tag{6}$$

The notion of an arbitrage price of a game option referred to in the next result is a suitable extension to game options of the *No Free Lunch with Vanishing Risk* (NFVLR) condition of Delbaen and Schachermayer [11] (see also [8, 20, 3]).

Theorem 2.1 *Assume that the BSDE $(\mathcal{E})$ has a solution $(\Theta, m, \mathbf{k})$ with $m \in \mathcal{H}^2$. Then Π defined in (5) is an arbitrage price process for the game option. Moreover:*
(i) Π_0 *is the minimal initial wealth of an issuer hedge with $(\mathbb{G},\mathbb{Q})$-sigma martingale residual cost and, under condition (3), $-\Pi_0$ is the minimal initial wealth of a holder hedge with a $(\mathbb{G},\mathbb{Q})$-sigma martingale residual cost.*
(ii) *Let Λ be some $\mathbb{R}^{q\otimes d}$-valued, (row by row) $\beta\widehat{X}$-integrable process Λ such that the process N given by*

$$N_t = \int_0^t \beta_u^{-1}\Lambda_u\, d(\beta_u\widehat{X}_u), \quad t \in [0,T],$$

belongs to $\mathcal{H}^{2,q}$, let z be some $\mathbb{R}^{1\otimes q}$-valued, $\mathbb{F}$-predictable process such that

$$\sum_{j=1}^q \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T (z_t^j)^2\, d[N^j, N^j]_t \,\Big|\, \mathcal{F}_0\Big) < \infty \quad \text{a.s.},$$

and let Λ^d be some $\mathbb{R}^{1\otimes d}$-valued process such that $(R-\bar{\Pi}_-)\Lambda^d$ is $\beta\widehat{X}$-integrable. We define the following real-valued processes:

$$
\begin{aligned}
n_t &= m_t - \int_0^t z_u \, dN_u, \\
n_t^d &= \int_0^{t\wedge\tau_d} (R_u - \bar{\Pi}_{u-})\Big(dN_u^d - \beta_u^{-1}\Lambda_u^d \, d(\beta_u \widehat{X}_u)\Big), \\
\rho_t^* &= (n+n^d)_t^{\tau_d}.
\end{aligned}
\tag{7}
$$

Then:

- $(\Pi_0, \zeta^*, \rho^*, \tau_c^*)$ *with*

$$\zeta_t^* = \mathbb{1}_{\{t\le\tau_d\}} \left[z_t \ , \ R_t - \bar{\Pi}_{t-} \right] \Lambda_t^*, \quad t\in[0,T],$$

where $\bar{\Pi}$ and τ_c^* are defined by (5)–(6) and $\Lambda^* = \begin{bmatrix} \Lambda \\ \Lambda^d \end{bmatrix}$, is an issuer hedge with initial wealth Π_0 and $(\mathbb{G},\mathbb{Q})$-sigma martingale residual cost ρ^*.

- $(-\Pi_0, -\zeta^*, \rho^*, \tau_p^*)$*, where τ_p^* is defined by (6), is a holder hedge with initial wealth $-\Pi_0$ and $(\mathbb{G},\mathbb{Q})$-sigma martingale residual cost ρ^*.*

Proof. Note that n^d and ρ^* are well defined $\mathbb{G}$-adapted processes (see [4]). In view of Lemma 2.3, the theorem follows by application of Corollary 3.2 in [4]. □

In the case of an European option, we consider the BSDE $(\mathcal{E})$ with $\mathcal{L}$ replaced by $\bar{\mathcal{L}}$ such that $\alpha\bar{\mathcal{L}} = -(c+1)$, where $-c$ is a lower bound on $\alpha_T\chi$. Note that under mild technical assumptions this equation has a solution $(\Theta, m, \mathbf{k}=\mathbf{0})$ (see [4, 9])), so that $(\mathcal{E})$ effectively reduces to a standard BSDE with no process $\mathbf{k}$ involved.

Theorem 2.2 *In the case of an European option, assume that the BSDE $(\mathcal{E})$ with $\mathcal{L}$ replaced by $\bar{\mathcal{L}}$ and with $\bar{\tau}=T$ has a $\mathbb{Q}$-solution $(\Theta, m, \mathbf{k}=\mathbf{0})$ with $m\in\mathcal{H}^2$. Then $\Phi=(1-H)\widetilde{\Phi}$ with $\widetilde{\Phi}=\Theta+F-F^0$ is an arbitrage price process for the option. Moreover:*
(i) *Φ_0 is the minimal initial wealth of an issuer hedge with $(\mathbb{G},\mathbb{Q})$-sigma martingale residual cost and, for bounded R and ξ, $-\Phi_0$ is the minimal initial wealth of an holder hedge with $(\mathbb{G},\mathbb{Q})$-sigma martingale residual cost.*
(ii) *For any $z, N, \Lambda, \Lambda^d, \Lambda^*, n, n^d$ and ρ^* as in Theorem 2.1(ii), define*

$$\zeta_t^* = \mathbb{1}_{\{t\le\tau_d\}} \left[z_t \ , \ R_t - \widetilde{\Phi}_{t-} \right] \Lambda_t^*, \quad t\in[0,T].$$

Then $(\Phi_0, \zeta^*, \rho^*)$ is an issuer hedge with $(\mathbb{G},\mathbb{Q})$-sigma martingale residual cost ρ^* and $(-\Phi_0, -\zeta^*, \rho^*)$ is a holder hedge with $(\mathbb{G},\mathbb{Q})$-sigma martingale residual cost ρ^*.

Remarks 2.4 (i) The special case $n=n^d=0$ corresponds to a particular form of a model completeness (attainability of defaultable European options, cf. Theorem 2.2; see also [5]) in which the issuer (or the holder) of the option is able (and wishes) to hedge all risks embedded in the option. The case where either $n\neq 0$ or $n^d\neq 0$ corresponds to either model incompleteness or the situation of a complete model in which the issuer (or the holder) is able to hedge, but he *prefers* not to hedge all the risks embedded in the option, for instance, he may be *willing* to take some bets in specific risk directions.
(ii) In cases where n and n^d may be taken equal to 0 in Theorem 2.1 or 2.2, the minimality statements in parts (i) of these theorems may be used to prove uniqueness of the related arbitrage prices (see [4]).
(iii) When $n^d=0$ the second equation in (7) reduces to

$$dN_t^d = \beta_t^{-1}\Lambda_t^d \, d(\beta_t\widehat{X}_t), \quad t\in[0, T\wedge\tau_d],$$

which effectively means that some defaultable asset is traded (at least synthetically) in the primary market.

3 Market Model Factory

3.1 Construction of a Primary Market Model

In the previous sections we took a primary market model satisfying all assumptions as exogenously given. The goal of this section is to present a generic *construction* of an arbitrage-free primary market model in a default intensity set-up.

We assume that we are given a stochastic basis $(\Omega, \mathbb{F}, \mathbb{Q})$ endowed with the following processes:
• an $\mathbb{F}$-adapted, bounded from below and locally time-integrable process r, intended to represent *short-term interest rate*,
• an $\mathbb{F}$-adapted, non-negative and locally time-integrable process γ, intended to represent the *default intensity*,
• an $\mathbb{R}^d$-valued càdlàg $(\mathbb{F}, \mathbb{Q})$-semimartingale Y, which is aimed to model the *pre-default prices* of primary assets, as well as the associated *coupon process* $\mathcal{C}$ and *recovery process* $\mathcal{R}$, such that:
— $\mathcal{C}$ is an $\mathbb{R}^d$-valued, $\mathbb{F}$-adapted process of integrable variation, with the density c with respect to the Lebesgue measure,[2]
— $\mathcal{R}$ is an $\mathbb{R}^d$-valued, $\mathbb{F}$-predictable and bounded from below process.

Relevant ways to *construct* such primary data $(\Omega, \mathbb{F}, \mathbb{Q})$, r, γ, Y, $\mathcal{C}$, $\mathcal{R}$ will be given later in the paper (see Remark 3.1(i)).

Given these primary data, the construction of the primary market model goes as follows. First, we define the discount factor $\beta_t = e^{-\int_0^t r_u du}$. Next, the so-called *canonical construction* (see, e.g., [7]) yields a convenient method of defining a random time τ_d on an enlarged probability spaces $(\Omega, \mathcal{G}, \mathbb{Q})$, so that:
• τ_d is a $\mathbb{G}$-stopping time with respect to $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, where $\mathbb{H}$ is the filtration generated by the *default indicator process* $H_t = \mathbb{1}_{\{\tau_d \le t\}}$,
• the process γ is the $(\mathbb{F}, \mathbb{Q})$-intensity process of τ_d,
• τ_d avoids all $\mathbb{F}$-stopping times.
Finally, since Y is intended to model the pre-default prices of primary assets, we set $X = \mathbb{1}_{\{t<\tau_d\}} Y$, so that clearly $\widetilde{X} = Y$. Let us observe that Y^{τ_d} is a $(\mathbb{G}, \mathbb{Q})$-semimartingale (by Lemma 2.1(i)) and thus X is an $\mathbb{R}^d$-valued, $(\mathbb{G}, \mathbb{Q})$-semimartingale on $[0, T]$, which is null on $[\tau_d \wedge T, T]$. The last feature reflects the fact that any residual value at τ_d is embedded in the recovery part of the dividend process $\mathcal{D}$ for X, given as

$$\mathcal{D}_t = \mathcal{C}_{t\wedge\tau_d} + \mathbb{1}_{\{\tau_d \le t\}} \mathcal{R}_{\tau_d}. \tag{8}$$

We further define, for $t \in [0, T]$,

$$\bar{\mathcal{D}}_t = \int_0^t d\mathcal{C}_u + \mathcal{R}_u \, d\Gamma_u = \int_0^t (c_u + \gamma_u \mathcal{R}_u) \, du \tag{9}$$

and the *pre-default cumulative price* $\bar{X}$

$$\bar{X}_t = \widetilde{X}_t + \alpha_t^{-1} \int_0^t \alpha_u \, d\bar{\mathcal{D}}_u. \tag{10}$$

Finally, we define the *cumulative price* $\widehat{X}$ by setting, for $t \in [0, T]$,

$$\beta_t \widehat{X}_t = \mathbb{1}_{\{t<\tau_d\}} \beta_t \widetilde{X}_t + \int_{[0,t\wedge\tau_d]} \beta_u \, d\mathcal{D}_u = \mathbb{1}_{\{t<\tau_d\}} \beta_t \Big(\bar{X}_t - \alpha_t^{-1} \int_0^t \alpha_u \, d\bar{\mathcal{D}}_u \Big) + \int_{[0,t\wedge\tau_d]} \beta_u \, d\mathcal{D}_u. \tag{11}$$

The proof of the following auxiliary result is deferred to the appendix.

[2] As opposed to the case of a game option, we do not assume the variation of $\mathcal{C}$ to be bounded, in order to cover typical examples, see e.g. [5].

Lemma 3.1 *The process $\alpha\bar{X}$ is an $(\mathbb{F},\mathbb{Q})$-local martingale if and only if the process $\beta\widehat{X}$ is a $(\mathbb{G},\mathbb{Q})$-local martingale.*

We assume in the sequel that Y (recall that $Y=\widetilde{X}$) is a *special* $\mathbb{F}$-semimartingale (see, e.g., Protter [25]) with Lebesgue absolutely continuous predictable finite variation component and related density b, so that the canonical decomposition of Y reads $Y_t = Y_0 + \int_0^t b_u\,du + M_t^Y$ for some local martingale M^Y.

Lemma 3.2 *Assume that Y satisfies the following* arbitrage $\mathbb{Q}$-consistency condition:

$$b_t = \mu_t Y_t - c_t - \gamma_t \mathcal{R}_t, \quad t\in[0,T]. \tag{12}$$

Then $\beta\widehat{X}$ is a $(\mathbb{G},\mathbb{Q})$-local martingale.

Proof. By (10), we have

$$d(\alpha_t\bar{X}_t) = d(\alpha_t\widetilde{X}_t) + \alpha_t\,d\bar{\mathcal{D}}_t = \alpha_t(d\widetilde{X}_t - \mu_t\widetilde{X}_t\,dt) + \alpha_t\,d\bar{\mathcal{D}}_t,$$

where by (12) the drift coefficient of $\widetilde{X}=Y$ is $b=\mu\widetilde{X}-c-\gamma\mathcal{R}$. Given (9), this implies that $\alpha\bar{X}$ is an $(\mathbb{F},\mathbb{Q})$-local martingale, hence $\beta\,\widehat{X}$ is a $(\mathbb{G},\mathbb{Q})$-local martingale, by Lemma 3.1. □

The following proposition is to be read as a constructive procedure for building primary market model arbitrage price processes.

Proposition 3.1 *Let us be given a stochastic basis $(\Omega,\mathbb{F},\mathbb{Q})$, an $\mathbb{F}$-adapted bounded from below and locally time-integrable process r, and a $\mathbb{F}$-adapted non-negative locally time-integrable process γ. Let τ_d, $\mathbb{H}$ and $\mathbb{G}$ be accordingly defined by canonical construction. In addition, let us be given an $\mathbb{R}^d$-valued càdlàg $(\mathbb{F},\mathbb{Q})$-special semimartingale Y and an $\mathbb{R}^d$-valued primary dividend process $\mathcal{D}$ as in (8), such that the arbitrage $\mathbb{Q}$-consistency condition (12) is satisfied. Then the discount factor$\beta_t = e^{-\int_0^t r_u du}$ and the primary market risky price process $X_t = \mathbb{1}_{\{t<\tau_d\}}Y_t$ define a primary market with* arbitrage *price process X, with pre-default process $\widetilde{X}=Y$, for any statistical probability $\mathbb{P}\sim\mathbb{Q}$.*

Proof. Most of the proposition follows by construction of the model. The only point that remains to be justified is that X is an *arbitrage* price process for the underlying market. But this results from Lemma 3.2, which tells us that X is a $(\mathbb{G},\mathbb{Q})$-local martingale, hence a $(\mathbb{G},\mathbb{Q})$-sigma martingale (recall that any local martingale is a sigma martingale [8]). Therefore $\mathbb{Q}$ belongs to the class $\mathcal{M}$ of probability measures $\widetilde{\mathbb{Q}}\sim\mathbb{P}$ for which $\beta\widehat{X}$ is a $(\mathbb{G},\widetilde{\mathbb{Q}})$-sigma martingale, thus $\mathcal{M}$ is non-void, hence (see Section 1) X is an arbitrage price process for the underlying market. □

A primary market arbitrage price process X constructed in this way shall be called a *canonical* $(\Omega,\mathbb{F},\mathbb{Q})$*-intensity market model.*

3.2 Markovian FBSDE

The market model introduced above is fairly generic and thus it is not yet suitable for practical purposes. In particular, for computational purposes, it is important to impose some *Markovian* structure on a market model. This will be achieved, relative to a given option, by assuming that the related BSDE $(\mathcal{E})$ is in fact a *Markovian Forward Backward SDE* (see, e.g., Section 4 of [13]).

Let us thus be given a (game) option with data $C,R,L,U,\xi,\bar{\tau}$, in a canonical $(\Omega,\mathbb{F},\mathbb{Q})$-intensity market model. Let us also be given a driver F of the form

$$F_t = \alpha_t^{-1}\int_0^t \alpha_u f_u\,du, \tag{13}$$

for some $\mathbb{F}$-adapted time-integrable process f over $[0,T]$ (which will sometimes also be called *driver* in the sequel), such that process $F^0 - F$ (cf. (4) is bounded from below. Let χ, $\mathcal{L}$, $\mathcal{U}$ and $\bar{\tau}$ denote the data of the related BSDE ($\mathcal{E}$). Note that since $d(\alpha_t F_t) = \alpha_t(dF_t - F_t dB_t)$, we have

$$F_T - F_t - \int_t^T F_u\, dB_u = \int_t^T \alpha_u^{-1} d(\alpha_u F_u) = \int_t^T f_u du,$$

and the first line of ($\mathcal{E}$) can be rewritten as

$$\Theta_t = \chi + \int_t^T (f_u - \mu_u \Theta_u) du + k_T - k_t - (m_T - m_t), \quad t \in [0,T].$$

Definition 3.1 We say that the BSDE ($\mathcal{E}$) is a *decoupled Markovian Forward-BSDE* (Markovian FBSDE, for short), if:
• the input data $\mu = r+\gamma$, f, χ, $\mathcal{L}$ and $\mathcal{U}$ of ($\mathcal{E}$) are given by Borel functions of some $(\Omega, \mathbb{F}, \mathbb{Q})$-Markov process $\mathcal{Z}$ with values in a suitable (finite-dimensional) state space, so

$$r_t = r(\mathcal{Z}_t)\,,\ \gamma_t = \gamma(\mathcal{Z}_t)$$
$$f_t = f(\mathcal{Z}_t)\,,\ \chi = \chi(\mathcal{Z}_T)\,,\ \mathcal{L}_t = \mathcal{L}(\mathcal{Z}_t)\,,\ \mathcal{U}_t = \mathcal{U}(\mathcal{Z}_t)$$

for some Borel functions denoted as the related processes;
• $\bar{\tau}$ is the entry time of a given domain by $\mathcal{Z}$.

In particular, the system made of the specification of a forward dynamics for $\mathcal{Z}$, together with the BSDE ($\mathcal{E}$), constitutes a decoupled *Markovian forward-backward system of equations* in $(\mathcal{Z}, \Theta, m, \mathbf{k})$. The system is decoupled in the sense that the forward component of the system serves as an input for the backward component($\mathcal{Z}$ is an input to ($\mathcal{E}$), cf. (14)), but not the other way round.

Of course, the possibility to find such a process $\mathcal{Z}$ and the nature of $\mathcal{Z}$ obviously depend on the driver f in ($\mathcal{E}$), so the following developments are, again, parameterized by the choice of the process f in (13).

From the point of view of interpretation, the components of $\mathcal{Z}$ are observable *factors*. The first component of $\mathcal{Z}$ will typically be given as time t. As for the other components of $\mathcal{Z}$, they are intimately, though non-trivially, connected with the canonical market primary pre-default price process $\widetilde{X} = Y$ as follows:
• Most components of Y will typically be given by some components of $\mathcal{Z}$. Note that, typically, there will be some extra primary risky assets in X that are not represented in $\mathcal{X}$. The reason for this is that if d is the number of assets used for hedging the game option in the real world (filtration $\mathbb{G}$), including, if this is wished, hedging default risk, then the dimension of the pre-default problem in the fictitious default-free $\mathbb{F}$-market will typically be $d-1$. Thus, $d-1$ primary pre-default price processes will be enough in the pre-default model (see Proposition 4.2 or [5] for a concrete example);
• The components of $\mathcal{Z}$ that are not included in Y (if any) are to be understood as simple factors that may be required to 'Markovianize' the payoffs of a game option (e.g., factors accounting for path dependence in the option's payoff and/or non-traded factors such as stochastic volatility in the dynamics of the assets underlying the option);
• There exists a well defined and constructive mapping from a collection of meaningful and 'directly observable' economic variables to $\mathcal{Z}$.

Regarding the last point, note that due to the nature of the model, observability of the factor process $\mathcal{Z}$ in the mathematical sense of $\mathbb{F}$-adaptedness is not sufficient in practice. In order for the model to be usable in practice, a constructive mapping from a collection of meaningful and directly observable economic variables to $\mathcal{Z}$ is really needed. Otherwise, the model is useless.

Since the model is defined under a risk-neutral probability $\mathbb{Q} \in \mathcal{M}$, this mapping will typically be achieved in practice by calibration of $\mathcal{Z}$ to a set of observed prices of traded derivatives. Of course this 'calibration' is obvious for the time component (if any) of $\mathcal{Z}$ or the components of $\mathcal{Z}$ which are represented in $\widetilde{X}$: simply fix these components equal to the current time or to the current market values of the related assets.

3.3 Variational Inequality Approach in a Jump–Diffusion Setting with Regimes

Under a rather generic specification for the Markov factor process $\mathcal{Z}$, we shall now derive the associated decoupled forward-backward system of stochastic differential equations (FBSDE), as well as the related partial integro-differential variational inequality (PIDVI).

To this end, given an integer p and a finite set E of size l, we define the following operator $\mathcal{A}$ acting on regular functions $\Theta = \Theta(t,x,y)$, $(t,x,y) \in [0,T] \times \mathbb{R}^p \times E$:

$$\begin{aligned}\mathcal{A}_t\Theta(t,x,y) &= \tfrac{1}{2}\sum_{i,j=1}^{p} a_{ij}(t,x,y)\partial^2_{x_ix_j}\Theta(t,x,y) \\ &+ \sum_{i=1}^{p}\Big(b_i(t,x,y) - g(t,x,y)\int_{\mathbb{R}^p} u_i(t,x,y,x')h(t,x,y,dx')\Big)\partial_{x_i}\Theta(t,x,y) \\ &+ g(t,x,y)\int_{\mathbb{R}^p}\big(\Theta(t,x+u(t,x,y,x') - \Theta(t,x,y)\big)h(t,x,y,dx') \\ &+ \sum_{y'\in E}\lambda(t,x,y,y')(\Theta(t,x,y') - \Theta(t,x,y)),\end{aligned} \tag{14}$$

where:
- the $a(t,x,y)$ are symmetric non-negative matrices,
- the $b(t,x,y)$ are drift vector coefficients,
- the jump intensity function $g(t,x,y)$ is non-negative, the $h(t,x,y,\cdot)$ are probability measures on $\mathbb{R}^p$, and $u(t,x,y,x')$ is the jump size function,
- the intensity matrix function $[\lambda(t,x,y,y')]_{y,y'\in E}$ is such that $\lambda(t,x,y,y') \geq 0$ whenever $y \neq y'$, and $\lambda(t,x,y,y) = -\sum_{y'\in E\setminus\{y\}}\lambda(t,x,y,y')$, by convention.

Under appropriate technical conditions on the coefficients of $\mathcal{A}$, the existence and uniqueness (in law) of a Markov process $\mathcal{Z}$ with the generator $\mathcal{A}$ follows from the respective results regarding martingale problems, see e.g. Theorems 4.1 and 5.4 in Chapter 4 of Ethier and Kurtz [15]. Equivalently, under these conditions, there exists a stochastic basis $(\Omega, \mathbb{F}, \mathbb{Q})$ and an $(\Omega, \mathbb{F}, \mathbb{Q})$-Markov process $\mathcal{Z} = (t, \mathcal{X}, \mathcal{Y})$, such that:
- The $\mathbb{R}^p$-valued process $\mathcal{X}$ satisfies the SDE, for $t \in \mathbb{R}_+$,

$$d\mathcal{X}_t = b(t,\mathcal{X}_t,\mathcal{Y}_t)\,dt + \sigma(t,\mathcal{X}_t,\mathcal{Y}_t)\,dW_t + \int_{\mathbb{R}^p} u(t,\mathcal{X}_{t-},\mathcal{Y}_{t-},x)\,P(dt,\mathcal{X}_{t-},\mathcal{Y}_{t-},dx), \tag{15}$$

where the *dispersion matrix* function $\sigma(t,x,y)$ satisfies the equality $\sigma(t,x,y)\sigma(t,x,y)^{\mathsf{T}} = a(t,x,y)$; and

$$P(dt,\mathcal{X}_{t-},\mathcal{Y}_{t-},dx) = J(dt,\mathcal{X}_{t-},\mathcal{Y}_{t-},dx) - g(t,\mathcal{X}_t,\mathcal{Y}_t)h(t,\mathcal{X}_t,\mathcal{Y}_t,dx)\,dt,$$

is the $\mathbb{Q}$-compensated martingale measure associated with a Poisson random measure J with $\mathbb{Q}$-*intensity measure* (or $\mathbb{Q}$-*compensator measure*) $g(t,\mathcal{X}_t,\mathcal{Y}_t)h(t,\mathcal{X}_t,\mathcal{Y}_t,dx)\,dt$.
- the E-valued Markov chain $\mathcal{Y}$ satisfies

$$dY_t = \sum_{y\in E}(y - \mathcal{Y}_{t-})\,dI_t(y)$$

where I is the jump-compensated counting measure corresponding to the Markov chain $\mathcal{Y}$, so

$$dI_t(y) = d\nu_t(y) - \mathbb{1}_{\{\mathcal{Y}_t\neq y\}}\lambda(t,\mathcal{X}_t,\mathcal{Y}_t,y)\,dt, \quad y \in E, \tag{16}$$

with $\nu_t(y) =$ number of transitions to state y since time 0.

We then have the following Itô formula, in which ∂_x denotes the row-gradient of $\Theta(t,x,y)$ with respect to x,

$$\begin{aligned}
d\Theta(\mathcal{Z}_t) &= (\partial_t + \mathcal{A}_t)\Theta(t,\mathcal{X}_t,\mathcal{Y}_t)\,dt + \partial_x\Theta(t,\mathcal{X}_t,\mathcal{Y}_t)\sigma(t,\mathcal{X}_t,\mathcal{Y}_t)\,dW_t \\
&+ \int_{\mathbb{R}^p} \big(\Theta(t,\mathcal{X}_{t-} + u(t,\mathcal{X}_{t-},\mathcal{Y}_{t-},x),\mathcal{Y}_{t-}) - \Theta(t,\mathcal{X}_{t-},\mathcal{Y}_{t-})\big)P(dt,\mathcal{X}_{t-},\mathcal{Y}_{t-},dx) \\
&+ \sum_{y\in E} (\Theta(t,\mathcal{X}_{t-}y) - \Theta(t,\mathcal{X}_{t-},\mathcal{Y}_{t-}))dI_t(y),
\end{aligned}$$

for regular enough functions Θ. So $\mathcal{A}$ is indeed the infinitesimal generator of $\mathcal{Z}$.

Note that $\mathcal{X}$ in (15) is a special semimartingale with predictable finite variation component $\int_0^\cdot b(t,\mathcal{X}_t,\mathcal{Y}_t)\,dt$. So for those $\mathcal{X}^i$ that will be represented in $\widetilde{X}$, assuming that the related dividend coefficients are *Markovian*, that is,

$$c_t^i = c_i(\mathcal{Z}_t)\ ,\ \mathcal{R}_t^i = \mathcal{R}_i(\mathcal{Z}_{t-})\ , \tag{17}$$

the arbitrage $\mathbb{Q}$-consistency condition (12) is satisfied if the *drift coefficient* b in (14)–(15) equals

$$b_i(t,x,y) = \mu(t,x,y)x_i - c_i(t,x,y) - \gamma(t,x,y)\mathcal{R}_i(t,x,y), \quad (t,x,y)\in[0,T]\times\mathbb{R}^p\times E, \tag{18}$$

for such i.

Remarks 3.1 (i) Given such a factor process $\mathcal{Z}$ and suitable Borel functions r and γ, the related stochastic basis $(\Omega,\mathbb{F},\mathbb{Q})$ and processes $r_t = r(\mathcal{Z}_t), \gamma_t = \gamma(\mathcal{Z}_t)$ can be used as starting points in the construction of a canonical intensity model with respect to $(\Omega,\mathbb{F},\mathbb{Q})$, cf. Proposition 3.1. Process Y and related primary dividends in Proposition 3.1 may then be defined in terms of $\mathcal{Z}$ (with Markovian dividend coefficients as in (17)), without forgetting to take care about the availability of a well-defined and constructive mapping between $\mathcal{Z}$ and Y (cf. section 3.2).
(ii) Note that if we suppose that the intensity matrix of $\mathcal{Y}$ does not depend on t,x, then $\mathcal{Y}$ is an homogenous Markov chain with finite state space E. Alternatively, if we take $g(t,x,y,x') = x'$, and we suppose that the coefficients σ,b,u,g and h do not depend on t,x,y, then $\mathcal{X}$ is a Poisson-Lévy process. For simplicity we do not consider the "infinite activity" case, that is, the case when the Lévy jump measure gh is not a finite measure. We thus defined a rather generic class (except for the exclusion of 'small jumps' in $\mathcal{X}$) of Markov processes $\mathcal{Z} = (t,\mathcal{X},\mathcal{Y})$, to be used as factor processes in intensity pre-default $(\Omega,\mathbb{F},\mathbb{Q})$-models for an option, in the form of a $\mathcal{Y}$-modulated Lévy-like component $\mathcal{X}$ and an $\mathcal{X}$-modulated Markov chain component $\mathcal{Y}$.
(iii) From the point of view of interpretation, process $\mathcal{Y}$ represents *regimes* that modulate the dynamics of the risk-neutral pricing process. For the sake of calibrability of the model, this means in particular that the various model regimes $y\in E$ will have to be associated with non-overlapping intervals for the other model parameters.

Given such an intensity pre-default $(\Omega,\mathbb{F},\mathbb{Q})$-model with related Markovian FBSDE $(\mathcal{E})$ for a given option (so the inputs of $(\mathcal{E})$ are functions of $\mathcal{Z}$), let us further define $N\in\mathcal{H}^{2,q}$ by $N^{\mathsf{T}} = (W^{\mathsf{T}}, I^{\mathsf{T}})$, with $q = p + l$; in case when $l = 1$, that is in case when the regime indicator process is constant, the one-dimensional process I in (16) is trivially null and plays no role whatsoever (see Section 4.3 for a concrete example), and so, we may take $q = p$. Denote by $\mathcal{P}$ the $\mathbb{F}$-predictable σ-algebra on $\Omega\times[0,T]$ and by $\mathcal{P}(E)$ the σ-algebra of all subsets of E. A solution $(\Theta, m, \mathbf{k})$ to $(\mathcal{E})$ is then typically sought for with m in the form

$$m_t = \int_0^t z_u\,dN_u + n_t = \int_0^t z_u\,dN_u + \int_0^t \int_{\mathbb{R}^p} v_u(x)\,P(du,\mathcal{X}_{u-},\mathcal{Y}_{u-},dx), \tag{19}$$

for an $\mathbb{F}$-predictable process z and a $\mathcal{P}\otimes\mathcal{P}(E)$ measurable map $v:\Omega\times[0,T]\times E\mapsto\mathbb{R}$ such that, denoting $E = \{y_1,\ldots,y_l\}$,

$$\sum_{j=1}^q \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T (z_t^j)^2 dt\,\Big|\,\mathcal{F}_0\Big) + \sum_{j=1}^l \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T (z_t^{m+j})^2 \lambda(t,\mathcal{X}_t,\mathcal{Y}_t,y^j)dt\,\Big|\,\mathcal{F}_0\Big) < \infty, \quad \text{a.s.}$$

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_0^T\int_E v_t^2(x) f(t,\mathcal{X}_t,\mathcal{Y}_t) g(t,\mathcal{X}_t,\mathcal{Y}_t,dx)dt\,\Big|\,\mathcal{F}_0\Big) < \infty, \quad \text{a.s.}$$

Assuming further that $N^{\tau_d} \in \mathcal{I}^q$, and given an additional $\beta\widehat{X}$-integrable process $(R - \bar{\Pi}_-)\Lambda^d$, then all assumptions are then satisfied in Theorem 2.1. We are thus led to look for solutions $(\Theta, m, \mathbf{k})$ to $(\mathcal{E})$ with m in the form (19), where z and v are part of the solution. We thus get a decoupled forward-backward system of stochastic differential equations in the unknowns $(\mathcal{Z}, \Theta, z, v, \mathbf{k})$ (see, e.g., Ma and Yong [24]).

The issues of solving $(\mathcal{E})$ under rather general assumptions covering in particular the cases corresponding to typical financial applications (e.g. convertible bonds, see [3, 4]), and the related variational inequality approach in the Markovian FBSDE case, are addressed in [9] (see also [5]). In particular, we prove in [9] that, under mild regularity conditions, $(\mathcal{E})$ has a unique solution $(\Theta, m, \mathbf{k})$ with m in the form (19) in some L^2-spaces (and with mutually singular $dk^{\pm}$, see [17, Remark 4.1]). In the Markovian FBSDE case, we establish the relation between this solution and the unique solution in some sense (viscosity solution with growth conditions or weak solution in weighted Sobolev spaces), $\Theta(t, x, y)$, to an associated PIDVI problem.

In the simplest case where $\bar{\tau} = 0$ (no call protection), the PIDVI problem is as follows (it is actually a *system* of l coupled PIDVIs, in space-dimension m):

$$\begin{aligned} \max\big(\min\big(&-\partial_t\Theta(t,x,y) - \mathcal{A}_t\Theta(t,x,y) - f(t,x,y) + \mu(t,x,y)\Theta(t,x,y), \\ &\Theta(t,x,y) - \mathcal{L}(t,x,y)\big), \Theta(t,x,y) - \mathcal{U}(t,x,y)\big) = 0, \quad t < T, (x,y) \in \mathbb{R}^p \times E, \end{aligned} \tag{20}$$

with terminal condition $\Theta(T, x, y) = \chi(x, y)$. Then the above-mentioned relationship writes: $\Theta_t = \Theta(\mathcal{Z}_t)$ for $t \in [0, T]$, plus extra relations between the z-component of the solution of $(\mathcal{E})$ and $\partial_x\Theta(\mathcal{Z}_t)$, in regular cases. The related hedging strategies with residual cost ρ^* in Theorem 2.1, can then be expressed in terms of the solution to the PIDE problem (20).

4 Illustration on Defaultable Convertible Bonds

We conclude this paper by applying results of the previous sections to the case of a defaultable convertible bond with underlying S, one of the primary risky assets.

4.1 Specification of the Payoffs

To describe the covenants of a typical *convertible bond* (CB), we need to introduce some additional notation:

$\bar{N}$: the par (nominal) value,

S: the price process of the asset underlying the CB,

$\widetilde{S}$: the pre-default value process of S,

c^{cb}: the continuous coupon rate process; a bounded, $\mathbb{F}$-progressively measurable process,

T_i, c^i, $i = 0, 1, \ldots, K$ ($T_0 = c^0 = 0$): coupon dates and amounts; the coupon dates $T_0, \ldots, T_K$ are deterministic fixed times with $T_{K-1} < T \le T_K$; the coupon amounts c^i are bounded, $\mathcal{F}_{T_{i-1}}$-measurable random variables, for $i = 1, 2, \ldots, K$,

A_t: the accrued interest at time t, specifically,

$$A_t = \frac{t - T_{i_t - 1}}{T_{i_t} - T_{i_t - 1}} c^{i_t},$$

where i_t is the integer satisfying $T_{i_t - 1} \le t < T_{i_t}$; in view of our assumptions, the process A is an $\mathbb{F}$-adapted, càdlàg process with finite variation.

$\bar{R}$: the recovery process on the CB upon default of the issuer; an $\mathbb{F}$-predictable, bounded process,

κ : the bond's conversion factor,

$\bar{P} \leq \bar{C}$: the put and call nominal payments, respectively; by assumption $\bar{P} \leq \bar{N} \leq \bar{C}$.

For a more detailed description of covenants of convertible bonds, see [3]. Real-life convertible bonds typically include a positive *call notice period* so that they may continue to live some time beyond the call time τ_c. This feature makes such bonds difficult to price directly, since they are not covered by the definition of a game option. To overcome this obstacle, we developed in [3] a recursive approach to valuation of a CB with positive call notice period. In the first step of this procedure, a CB is valued upon call. Subsequently, we use this price as the payoff at call time of a CB with no call notice period. In this way, a CB with positive call notice period can be priced as a *reduced convertible bond* (see [3]).

Definition 4.1 A *reduced convertible bond* is a game option with coupon process C, recovery process R^{cb} and payoffs L^{cb}, U^{cb}, ξ^{cb} such that

$$C_t = \int_0^t c_u^{cb}\,du + \sum_{0\leq T_i\leq t} c^i, \quad R_t^{cb} = (1-\eta)\kappa\widetilde{S}_{t-} \vee \bar{R}_t,$$
$$L_t^{cb} = \bar{P} \vee \kappa\widetilde{S}_t + A_t, \quad \xi^{cb} = \bar{N} \vee \kappa\widetilde{S}_T + A_T,$$

and where $(U_t^{cb})_{t\in[0,T]}$ is some càdlàg process satisfying the following inequality

$$U_t^{cb} \geq \bar{C} \vee \kappa\widetilde{S}_t + A_t, \quad t \in [0,T]. \tag{21}$$

Reduced convertible bonds are special cases of more general *convertible securities* considered in [3]. In the financial interpretation, U_t^{cb} represents the pre-default value of the reduced convertible bond upon a call at time t. In particular, a convertible bond without call notice period is a reduced convertible bond with $U_t^{cb} = \bar{C} \vee \kappa\widetilde{S}_t + A_t$ for $t \in [0,T]$.

Under our assumption that $\bar{P} \leq \bar{N} \leq \bar{C}$, we obtain, by (21),

$$L_T^{cb} = \bar{P} \vee \kappa\widetilde{S}_T + A_T \leq \bar{N} \vee \kappa\widetilde{S}_T + A_T = \xi^{cb} \leq \bar{C} \vee \kappa\widetilde{S}_T + A_T \leq U_T^{cb}.$$

4.2 Clean Price

Definition 4.2 For a pre-default $\mathbb{Q}$-price $\widetilde{\Pi}$ of a convertible bond, by the *clean price* of this bond we mean the difference $\widetilde{\Pi} - A$.

The notion of the clean price is consistent with the market convention for bonds, which hinges on subtracting the accrued interest from the trading (*dirty*) price. Market quotations for bonds are usually given in terms of clean prices (or bond yields), in order to avoid coupon-related discontinuities in quotations.

Let us set $a_t = \frac{c^{i_t}}{T_{i_t} - T_{i_t - 1}}$ for $t \in [0,T]$. Then

$$A_t = \int_0^t a_u\,du - \sum_{0\leq T_i\leq t} c^i$$

and the integration by part formula yields

$$\alpha_t A_t = \int_0^t \alpha_u \big(a_u\,du - A_u\,dB_u\big) - \sum_{0\leq T_i\leq t} \alpha_{T_i} c^i. \tag{22}$$

Let us fix some risk-neutral measure $\mathbb{Q}$.

Proposition 4.1 (i) *Considering a reduced convertible bond, let us choose the driver*

$$F = F^{cb} := F^0 + A, \tag{23}$$

where F^0 was defined in (4). Then the F-price of a convertible bond for $F = F^{cb}$ is equal to the clean price of this bond, and the data of the doubly reflected BSDE ($\mathcal{E}$) take the following form:

$$F^{cb},\ \chi = \xi^{cb} - A_T = \bar{N} \vee \kappa \widetilde{S}_T,$$
$$\mathcal{L} = L^{cb} - A = \bar{P} \vee \kappa \widetilde{S},\ \mathcal{U} = (U^{cb} - A),\ \bar{\tau}.$$

(ii) *In the default intensity set-up, we have*

$$F_t^{cb} - \int_0^t F_u^{cb}\, dB_u = \int_0^t f_u^{cb}\, du \tag{24}$$

with

$$f^{cb} = \gamma R^{cb} + c^{cb} + a - \mu A. \tag{25}$$

Hence in ($\mathcal{E}$) we obtain

$$F_T - F_t - \int_t^T F_u\, dB_u = \int_t^T f_u^{cb}\, du.$$

(iii) *Assume that the pre-default value process $\widetilde{S}$ is continuous. Then the lower barrier process $\mathcal{L}$ is continuous. Moreover, the upper barrier process $\bar{\mathcal{U}}$ is continuous after $\bar{\tau}$, in the case of a convertible bond with no call notice period.*

Proof. (i) We have, by (5),

$$\Theta = \widetilde{\Pi} + F^0 - F^{cb} = \widetilde{\Pi} - A,$$

in view of the definition of F^{cb}. Also,

$$\mathcal{L} = \widehat{L}^{cb} - F^{cb} = L^{cb} + F - (F + A) = L^{cb} - A.$$

The other identities can be shown similarly.
(ii) Using the definition f^{cb} and (22) with $dB_u = \mu_u\, du$, we obtain in an intensity default model

$$\begin{aligned}\int_0^t \alpha_u f_u^{cb}\, du &= \int_0^t \alpha_u \gamma_u R_u^{cb}\, du + \int_0^t \alpha_u c_u^{cb}\, du + \int_0^t \alpha_u \big(a_u - \mu_u A_u\big) du \\ &= \alpha_t A_t + \int_0^t \alpha_u\, dC_u + \int_0^t \alpha_u \gamma_u R_u^{cb}\, du = \alpha_t (A_t + F_t) = \alpha_t F_t.\end{aligned}$$

Thus

$$F_t - \int_0^t F_u\, dB_u = \int_0^t \alpha_u^{-1} d(\alpha F)_u = \int_0^t f_u^{cb}\, du.$$

(iii) It suffices to note that for a convertible bond with no call notice period we have, after $\bar{\tau}$, $\bar{\mathcal{U}}_t = \mathcal{U}_t = U_t^{cb} - A_t = \bar{C} \vee \kappa \widetilde{S}_t$. □

Let us summarize our findings at this point of this section. First, we have shown that by solving the doubly reflected BSDE ($\mathcal{E}$) with the driver $F = F^{cb}$ given by (25), we obtain the clean price of a reduced convertible bond as its F-price, that is, as the state process Θ of a solution to ($\mathcal{E}$).

Second, the related driver terms in ($\mathcal{E}$) are then given as integrals with respect to the Lebesgue measure, which is the standard form in the BSDE literature.

Third, under mild assumptions, the lower and upper barriers for this choice of the driver F are given as continuous processes, and thus the state process process Θ of a solution to the doubly reflected BSDE ($\mathcal{E}$) is continuous, provided the martingale M is continuous (e.g., in the special case where N is a multi-dimensional Wiener process and $n = 0$).

4.3 A Simple Model

The previous observations prove useful in the practical implementation of a jump-to-default intensity model with a Markovian structure in Bielecki et al. [5] (see also Ayache et al. [2] or Andersen and Buffum [1]). In [5], the filtration $\mathbb{F}$ is generated by a standard Brownian motion W under $\mathbb{Q}$, and we consider a primary market model composed of the savings account and $d = 2$ risky assets:
• the first primary risky asset is the stock of a reference firm with price process S and default time represented by τ_d,
• the second primary risky asset is the credit default swap (CDS) written on the reference entity.

The pre-default stock price $\widetilde{S}$ is the unique strong solution to the following SDE

$$d\widetilde{S}_t = \widetilde{S}_t\Big(\big(r(t) - q(t) + \eta\gamma(t, \widetilde{S}_t)\big)\,dt + \sigma(t, \widetilde{S}_t)\,dW_t\Big) \tag{26}$$

with $\widetilde{S}_0$ given as a real-valued, $\mathcal{F}_0$-measurable random variable, where:
• the riskless short interest rate $r(t)$, the equity dividend yield $q(t)$, and the local default intensity $\gamma(t, S) \geq 0$ are bounded Borel functions,
• the fractional recovery on S upon default, η, is a non-negative constant,
• the local volatility $\sigma(t, S)$ is a positively bounded Borel function,
• the functions $\gamma(t, S)S$ and $\sigma(t, S)S$ are Lipschitz in S.

It is further postulated that:
• the *coupon process*

$$C_t = C(t) := \int_{[0,t]} c^{cb}_u\,du + \sum_{0\leq T_i\leq t} c^i$$

for a bounded Borel *continuous coupon rate* c^{cb} and deterministic *discrete coupon* dates and amounts, with $T_0 = 0$ and $T_{K-1} < T \leq T_K$;
• the recovery process $\bar{R}_t$ is of the form $\bar{R}(t, S_{t-})$ for a Borel function $\bar{R}$.

We say that we deal with the *hard call protection* if the lifting time of call protection $\bar{\tau} = \bar{T}$ for some $\bar{T} \leq T$. The *standard soft call protection* corresponds to the lifting time of call protection given as $\bar{\tau} = \inf\{t > 0\,;\, \widetilde{S}_t \geq \bar{S}\} \wedge T$ for some $\bar{S} \in \mathbb{R}^\star_+$.

Proposition 4.2 *Let us assume either a hard call protection or a standard soft call protection. Then, choosing the driver $f = f^{cb}$ as in (25), the related BSDE ($\mathcal{E}$) is a Markovian FBSDE with respect to the state vector $\mathcal{Z} = (t, \widetilde{S})$ (so $\mathcal{X} = \widetilde{S}$ and no regimes; $l = 1$ and $p = q = 1 < 2 = d$). Moreover, $\widetilde{S}$ satisfies the related arbitrage $\mathbb{Q}$-consistency condition (18).*

Proof. Except for the last point, the statements follows by construction of the model. Moreover, condition (18) obviously holds for $\widetilde{S}$ since (we omit indices, since $\mathcal{X}$ is reduced to $\mathcal{X}^1$):

$$b(t, x) = (r(t) - q(t) + \eta\gamma(t, x))x,\ \mu(t, x) = r(t) + \gamma(t, x),\ c(t, x) = q(t)x,\ \mathcal{R}(t, x) = (1 - \eta)x.$$

□

A Appendix: Proof of Lemma 3.1

Let us first observe that by combining (11) with Lemma 2.2, we obtain, for any $0 \leq t \leq u \leq T$ and any $\mathbb{F}$-stopping time $\tau \in \mathcal{F}^t_T$,

$$\mathbb{E}_{\mathbb{Q}}\big(\beta_{u\wedge\tau}\widehat{X}_{u\wedge\tau} - \beta_t\widehat{X}_t \,\big|\, \mathcal{G}_t\big) = \mathbb{1}_{\{t<\tau_d\}}e^{\Gamma_t}\,\mathbb{E}_{\mathbb{Q}}\big(\alpha_{u\wedge\tau}\bar{X}_{u\wedge\tau} - \alpha_t\bar{X}_t \,\big|\, \mathcal{F}_t\big). \tag{27}$$

Note also that for a *càdlàg adapted* process to be a local martingale, it suffices that there exists a localizing sequence of stopping time for which the stopped processes are martingales (the usual uniform integrability condition in the definition of a local martingale is not necessary for càdlàg adapted processes, see [25, I,6,Theorem 50]).

($\Rightarrow$) Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence of $\mathbb{F}$-stopping times for the $(\mathbb{F},\mathbb{Q})$-local martingale $\alpha\bar{X}$. Let us take arbitrary $0 \le t \le u \le T$. We claim that

$$\mathbb{E}_{\mathbb{Q}}\big((\beta_u \widehat{X}_u)^{\tau_n} - (\beta_t \widehat{X}_t)^{\tau_n} \,\big|\, \mathcal{G}_t\big) = 0. \tag{28}$$

To establish (28), it suffices to apply (27) to $\tau = \tau_n \vee t \in \mathcal{F}^t_T$ and use the fact that the stopped process $(\alpha\bar{X})^{\tau_n}$ is an $(\mathbb{F},\mathbb{Q})$-martingale. We thus see that $(\beta\widehat{X})^{\tau_n}$ is a $(\mathbb{G},\mathbb{Q})$-martingale and τ_n is a localizing sequence of $\mathbb{G}$-stopping times (recall that $\mathbb{F}\subset\mathbb{G}$) for $\beta\widehat{X}$. We conclude that $\beta\widehat{X}$ is a $(\mathbb{G},\mathbb{Q})$-local martingale.

($\Leftarrow$) To prove the converse implication, let us assume that $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence of $\mathbb{G}$-stopping times for the $(\mathbb{G},\mathbb{Q})$-local martingale $\beta\widehat{X}$ and let us denote by $(\widetilde{\tau}_n)_{n\in\mathbb{N}}$ the related sequence of pre-default values, as defined in Lemma 2.1(iii), so that $\tau_n \wedge \tau_d = \widetilde{\tau}_n \wedge \tau_d$. We claim that $(\widetilde{\tau}_n)_{n\in\mathbb{N}}$ is a localizing sequence of $\mathbb{G}$-stopping times for the $(\mathbb{F},\mathbb{Q})$-local martingale $\beta\bar{X}$.

To check this claim, let us fix $n\in\mathbb{N}$ and let us consider arbitrary $0\le t\le u\le T$. We need to show that for any $n\in\mathbb{N}$

$$\mathbb{E}_{\mathbb{Q}}\big((\alpha_u \bar{X}_u)^{\widetilde{\tau}_n} - (\alpha_t \bar{X}_t)^{\widetilde{\tau}_n} \,\big|\, \mathcal{F}_t\big) = 0. \tag{29}$$

By applying (27) to $\tau = \widetilde{\tau}_n \vee t \in \mathcal{F}^t_T$, we obtain

$$\begin{aligned}
&\mathbb{1}_{\{t<\tau_d\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big((\alpha_u \bar{X}_u)^{\widetilde{\tau}_n} - (\alpha_t \bar{X}_t)^{\widetilde{\tau}_n} \,\big|\, \mathcal{F}_t\big)\\
&\quad = \mathbb{1}_{\{t<\tau_d\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big((\alpha\bar{X})_{u\wedge\tau} - (\alpha\bar{X})_t \,\big|\, \mathcal{F}_t\big) = \mathbb{1}_{\{t<\tau_d\}}\, \mathbb{E}_{\mathbb{Q}}\big((\beta\widehat{X})_{u\wedge\tau} - (\beta\widehat{X})_t \,\big|\, \mathcal{G}_t\big).
\end{aligned}$$

Furthermore, since the process $\beta\widehat{X}$ is stopped at τ_d and $\tau_n\wedge\tau_d = \widetilde{\tau}_n\wedge\tau_d$, we obtain

$$\begin{aligned}
\mathbb{1}_{\{t<\tau_d\}}\, \mathbb{E}_{\mathbb{Q}}\big((\beta\widehat{X})_{u\wedge\widetilde{\tau}_n\vee t} - (\beta\widehat{X})_t \,\big|\, \mathcal{G}_t\big) &= \mathbb{1}_{\{t<\tau_d\}}\, \mathbb{E}_{\mathbb{Q}}\big((\beta\widehat{X})_{u\wedge\widetilde{\tau}_n\wedge\tau_d\vee t} - (\beta\widehat{X})_t \,\big|\, \mathcal{G}_t\big)\\
&= \mathbb{1}_{\{t<\tau_d\}}\, \mathbb{E}_{\mathbb{Q}}\big((\beta\widehat{X})_{u\wedge\tau_n\wedge\tau_d\vee t} - (\beta\widehat{X})_t \,\big|\, \mathcal{G}_t\big)\\
&= \mathbb{1}_{\{t<\tau_d\}}\, \mathbb{E}_{\mathbb{Q}}\big((\beta_u\widehat{X}_u)^{\tau_n} - (\beta_t\widehat{X}_t)^{\tau_n} \,\big|\, \mathcal{G}_t\big) = 0.
\end{aligned}$$

By the uniqueness of pre-default values (see Lemma 2.1), this implies that (29) holds. □

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