CREDIT RISK MODELING

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Introduction

The goal of this book is to provide a survey of mathematical techniques that are used in the area of credit risk modeling and to review some recent developments in this field. Special emphasis is put on the important issue of hedging defaultable claims. In this respect, the present text is largely based on some of our previous works, in particular:

- Modelling and valuation of credit risk. In: *Stochastic Methods in Finance,* M. Frittelli and W. Runggaldier, eds., Springer, 2004, 27–126,
- Hedging of defaultable claims. In: *Paris-Princeton Lectures on Mathematical Finance 2003,* R. Carmona et al., eds. Springer, 2004, 1–132,
- PDE approach to valuation and hedging of credit derivatives. *Quantitative Finance* 5 (2005), 257–270,
- Hedging of credit derivatives in models with totally unexpected default. In: *Stochastic Processes and Applications to Mathematical Finance,* J. Akahori et al., eds., World Scientific, 2006, 35–100,
- Hedging of basket credit derivatives in credit default swap market. *Journal of Credit Risk* 3 (2007), 91–132.
- Pricing and trading credit default swaps in a hazard process model. *Annals of Applied Probability* 18 (2008), 2495–2529.

Credit risk embedded in a financial transaction is the risk that at least one of the parties involved in the transaction will suffer a financial loss due to default or decline in the creditworthiness either of the counterparty or of some third party (reference name). To give just a few examples:

- A holder of a corporate bond bears a risk that the market value of the bond will decline due to decline in credit rating of the issuer.
- A bank may suffer a loss if a bank's debtor defaults on payment of the interest due and/or the principal amount of the loan.
- A party involved in a trade of a credit derivative, such as a credit default swap (CDS), may suffer a loss if a reference credit event occurs.
- The market value of individual tranches constituting a collateralized debt obligation (CDO) may decline as a result of changes in the correlation between the default times of the underlying defaultable securities (that is, the collateral assets or the reference credit default swaps).

The most extensively studied form of credit risk is the *default risk* – that is, the risk that a counterparty in a financial contract will not fulfil a contractual commitment to meet her/his obligations stated in the contract. For this reason, the main tool in the area of credit risk modeling is a judicious specification of the random time of default. A large part of the present text is devoted to this issue. Our main goal is to present a comprehensive introduction to the most important mathematical tools that are used in arbitrage valuation of defaultable claims, which are also known under the name of credit derivatives. We also examine in some detail the important issue of hedging these claims. The book is organized as follows:

- In Chapter 1, we provide a concise summary of the main developments within the so-called *structural approach* to modeling and valuation of credit risk. In particular, we present the classic structural models, put forward by Merton [143] and Black and Cox [28], and we mention some variants and extensions of these models. We also study very succinctly the case of a structural model with a random default triggering barrier.
- Chapter 2 is devoted to the study of an elementary model of credit risk within the *hazard function* framework. We focus here on the derivation of pricing formulae for defaultable claims and the dynamics of their prices. We also deal here with the issue of replication of single- and multi-name credit derivatives in the stylized credit default swap market. Results of this chapter should be seen as a first step toward more practical approaches that are presented in the foregoing chapters.
- Chapter 3 deals with the alternative *reduced-form approach* in which the main modeling tool is the *hazard process*. We examine the pricing formulae for defaultable claims in the reduced-form setup with stochastic hazard rate and we examine the behavior of the stochastic intensity when the reference filtration is reduced. Special emphasis is put on the so-called *hypothesis (H)* and its invariance with respect to an equivalent change of a probability measure. As an application of mathematical results, we present here an extension of hedging results established in Chapter 2 in the case of deterministic pre-default intensities to the case of stochastic default intensities.
- Chapter 4 is devoted to a study of hedging strategies for defaultable claims. We first present some theoretical results on replication of defaultable claims in an abstract semimartingale market model. Next, we develop the PDE approach to the valuation and hedging of defaultable claims in a Markovian framework. For clarity of presentation, we focus on the case of a market model with three traded primary assets and we deal with a single default time. An extension of the PDE method to the case of any finite number of traded assets and several default times is readily available, however.

- Chapter 5 provides an introduction to the area of modeling dependent defaults and, more generally, to modeling of dependent credit rating migrations for a portfolio of reference credit names. We present here some applications of these models to the valuation of real-life examples of actively traded credit derivatives, such as: credit default swaps and swaptions, first-to-default swaps, credit default index swaps and tranches of collateralized debt obligations.
- For the reader's convenience, we present in the appendix some well-known results regarding the Poisson process and its generalizations, such as: the inhomogeneous Poisson process and the conditional Poisson process. We also recall there the definition and basic properties of the Doléans exponential of a semimartingale.

The detailed proofs of most results can be found in Bielecki, Jeanblanc and Rutkowski [13, 14, 17], Bielecki and Rutkowski [21, 23], and Jeanblanc and Rutkowski [111, 112, 113]. We also quote some of the seminal papers but, unfortunately, we were not able to provide here a survey of an extensive research in the area of credit risk modeling. For more information on credit risk modeling and related issues, the interested reader is thus referred to original papers by other authors, as well as to monographs by Ammann [2], Bluhm, Overbeck and Wagner [31], Bielecki and Rutkowski [22], Cossin and Pirotte [62], Duffie and Singleton [77], McNeil, Frey and Embrechts [142], Lando [128], and Schönbucher [162].

It is assumed that the reader has some familiarity with fundamental concepts and results of arbitrage pricing theory for financial derivatives and term structure modeling; see, e.g., Björk [27], Brigo and Mercurio [41], Dana and Jeanblanc [63], Elliott and Kopp [80], Hunt and Kennedy [103], Karatzas and Shreve [118], Musiela and Rutkowski [146], Shiryaev [164], and Shreve [165, 166].

Last but not least, some knowledge of the Itô stochastic calculus is also expected; we refer, e.g., to the monographs by Elliott [78], Jeanblanc, Yor and Chesney [114], Karatzas and Shreve [117], Klebaner [123], Kuo [124], Øksendal [150], Protter [153], and Revuz and Yor [154].

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Chapter 1

Structural Approach

We start by presenting a rather succinct overview of the *structural approach* to credit risk modeling. Since it is based on the modeling of the behavior of the total value of the firm's assets, it is also known as the *value-of-the-firm approach.* In order to model credit events (the default event, in particular), this methodology refers directly to economic fundamentals, such as the capital structure of a company. As we shall see in what follows, the two major driving concepts in the structural modeling are: the total value of the firm's assets and the default triggering barrier. Historically, this was the first approach used in this area – it can be traced back to the fundamental papers by Black and Scholes [29] and Merton [143]. The presentation in this chapter is based on Chapters 2 and 3 in Bielecki and Rutkowski [22]; the interested reader is encouraged to consult [22] for a more detailed exposition.

1.1 Notation and Definitions

We fix a finite horizon date $T^* > 0$. The underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is endowed with some reference filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$, and is sufficiently rich to support the following random quantities:

- the *short-term interest rate* process r and thus also a default-free term structure model,
- the *value of the firm process* V, which is interpreted as a stochastic model for the total value of the firm's assets,
- the *barrier process* v, which is used to specify the default time τ,
- the *promised contingent claim* X representing the liabilities to be redeemed to the holder of a defaultable claim at maturity date $T \leq T^*$,
- the process A, which models the *promised dividends,* that is, the liabilities that are redeemed continuously or discretely over time to the holder of a defaultable claim,
- the *recovery claim* $\widetilde{X}$ representing the recovery payoff received at time T if default occurs prior to or at the claim's maturity date T,

- the *recovery process* Z, which specifies the recovery payoff at time of default if it occurs prior to or at the maturity date T.

The probability measure $\mathbb{P}$ is aimed to represent the *real-world* (or *statistical*) probability, as opposed to a *martingale measure* (also known as a *risk-neutral probability*). Any *martingale measure* will be denoted by $\mathbb{Q}$ in what follows.

1.1.1 Defaultable Claims

We postulate that the processes V, Z, A and v are progressively measurable with respect to the filtration $\mathbb{F}$, and that the random variables X and $\widetilde{X}$ are $\mathcal{F}_T$-measurable. In addition, A is assumed to be a process of finite variation with $A_0 = 0$. We assume without mentioning that all random objects introduced above satisfy suitable integrability conditions. Within the structural approach, the default time τ is typically defined in terms of the firm's value process V and the barrier process v. We set

$$\tau = \inf \{ t > 0 : t \in \mathcal{T} \text{ and } V_t \leq v_t \}$$

with the usual convention that the infimum over the empty set equals $+\infty$. Typically, the set $\mathcal{T}$ is the interval $[0, T]$ (or $[0, \infty)$ in the case of perpetual claims). In classic *first-passage-time* structural models, the default time τ is given by the formula

$$\tau = \inf \{ t > 0 : t \in [0, T] \text{ and } V_t \leq \bar{v}(t) \},$$

where $\bar{v} : [0, T] \to \mathbb{R}_+$ is some deterministic function, termed the *barrier*.

Remark 1.1.1. In most structural models, the underlying filtration $\mathbb{F}$ is generated by a standard Brownian motion. In that case, the default time τ will be an $\mathbb{F}$-predictable stopping time (as any stopping time with respect to a Brownian filtration), meaning that there exists a strictly increasing sequence of $\mathbb{F}$-stopping times announcing the default time.

Provided that default has not occurred before or at time T, the promised claim X is received in full at the claim's maturity date T. Otherwise, depending on the market convention regarding a particular contract, either the amount $\widetilde{X}$ is received at maturity T, or the amount Z_τ is received at time τ. If default occurs at maturity of the claim, that is, on the event $\{\tau = T\}$, we adopt the convention that only the recovery payment $\widetilde{X}$ is received.

It is sometimes convenient to consider simultaneously both kinds of recovery payoff. Therefore, in this chapter, a generic *defaultable claim* is formally defined as a quintuplet $(X, A, \widetilde{X}, Z, \tau)$. In other chapters, we set $\widetilde{X} = 0$ and we consider a quadruplet (X, A, Z, τ), formally identified with a claim $(X, A, 0, Z, \tau)$. In some cases, we will also set $A = 0$ so that a defaultable claim will reduce to a triplet (X, Z, τ), to be identified with $(X, 0, Z, \tau)$.

1.1.2 Risk-Neutral Valuation Formula

Suppose that our financial market model is arbitrage-free, in the sense that there exists a *martingale measure* (*risk-neutral probability*) $\mathbb{Q}$, meaning that price process of any tradeable security, which pays no coupons or dividends, becomes an $\mathbb{F}$-martingale under $\mathbb{Q}$, when discounted by the *savings account* B, given as

$$B_t = \exp\Big(\int_0^t r_u\,du\Big).$$

We introduce the *default indicator process* $H_t = \mathbb{1}_{\{t\geq\tau\}}$ and we denote by D the process modeling all cash flows received by the owner of a defaultable claim. Let us also denote

$$X_T^d = X\mathbb{1}_{\{\tau>T\}} + \widetilde{X}\mathbb{1}_{\{\tau\leq T\}}.$$

Definition 1.1.1. The *dividend process* D of a defaultable contingent claim $(X, A, \widetilde{X}, Z, \tau)$ with maturity date T equals, for every $t \in \mathbb{R}_+$,

$$D_t = X_T^d\mathbb{1}_{[T,\infty[}(t) + \int_{]0,t]}(1-H_u)\,dA_u + \int_{]0,t]} Z_u\,dH_u.$$

It is easy to check that the dividend process D is of finite variation, and

$$\int_{]0,t]}(1-H_u)\,dA_u = \int_{]0,t]}\mathbb{1}_{\{u<\tau\}}\,dA_u = A_{\tau-}\mathbb{1}_{\{t\geq\tau\}} + A_t\mathbb{1}_{\{t<\tau\}}.$$

Note that if default occurs at some date t, the promised dividend payment $A_t - A_{t-}$, which was expected to be occur on this date, is not paid to the holder of the claim. It is also easily seen that

$$\int_{]0,t]} Z_u\,dH_u = Z_\tau\mathbb{1}_{\{t\geq\tau\}}.$$

Remark 1.1.2. In principle, the promised payoff X could be easily incorporated into the promised dividends process A. This alternative convention would be more difficult to handle, however, since in practice the recovery rules concerning the promised dividends A and the promised claim X are likely to be different.

For instance, in the case of a defaultable coupon bond, it is frequently postulated that if default occurs then the future coupons are foregone, whereas a strictly positive fraction of the face value is received by the bondholder.

We are in a position to define the risk-neutral ex-dividend price S_t of a defaultable claim. In the financial interpretation, any time t, the random variable S_t is aimed to represent the current market value of all future cash flows associated with a given defaultable claim.

Definition 1.1.2. For any date $t \in [0,T]$, the *ex-dividend price* of a defaultable claim $(X, A, \widetilde{X}, Z, \tau)$ is given as

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{F}_t \Big). \tag{1.1}$$

Note that the *discounted ex-dividend price* $S_t^* = S_t B_t^{-1}$ satisfies, for every $t \in [0,T]$,

$$S_t^* = \mathbb{E}_{\mathbb{Q}}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{F}_t \Big) - \int_{]0,t]} B_u^{-1} \, dD_u.$$

It is easy to see that S^* is a supermartingale (submartingale, respectively) under $\mathbb{Q}$ if and only if the dividend process D is increasing (decreasing, respectively). The *cumulative price* S^c of $(X, A, \widetilde{X}, Z, \tau)$ equals, for $t \in [0,T]$,

$$S_t^c = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{F}_t \Big) = S_t + B_t \int_{]0,t]} B_u^{-1} \, dD_u.$$

Hence the *discounted cumulative price* $S_t^{c*} = S_t^c B_t^{-1}$ is a martingale under $\mathbb{Q}$.

1.1.3 Defaultable Zero-Coupon Bond

Assume that $A = 0$, $Z = 0$ and $X = L$ for some positive constant $L > 0$. Then the value process S represents the price of a *defaultable zero-coupon bond* (also referred to as the *corporate bond*) with the face value L and the recovery payoff $\widetilde{X}$ at maturity. The ex-dividend price $S_t = D(t,T)$ of this bond equals, for $t < T$,

$$D(t,T) = B_t \, \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} (L \mathbb{1}_{\{\tau > T\}} + \widetilde{X} \mathbb{1}_{\{\tau \le T\}}) \, \big| \, \mathcal{F}_t \big).$$

The last formula can be rewritten as follows

$$D(t,T) = L B_t \, \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} (\mathbb{1}_{\{\tau > T\}} + \delta(T) \mathbb{1}_{\{\tau \le T\}}) \, \big| \, \mathcal{F}_t \big),$$

where the random variable $\delta(T) = \widetilde{X}/L$ represents the *recovery rate upon default*. For the corporate bond, it is natural to assume that $0 \le \widetilde{X} \le L$, so that the random variable $\delta(T)$ satisfies $0 \le \delta(T) \le 1$.

Alternatively, we may re-express the bond price as follows

$$D(t,T) = L\Big(B(t,T) - B_t \, \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} w(T) \mathbb{1}_{\{\tau \le T\}} \, \big| \, \mathcal{F}_t \big) \Big),$$

where

$$B(t,T) = B_t \, \mathbb{E}_{\mathbb{Q}}(B_T^{-1} \, | \, \mathcal{F}_t)$$

is the price of a unit default-free zero-coupon bond and $w(T) = 1 - \delta(T)$ is the *writedown rate upon default*. Generally speaking, the value of a corporate bond depends on the joint probability distribution under $\mathbb{Q}$ of the three-dimensional random variable $(B_T, \delta(T), \tau)$ or, equivalently, $(B_T, w(T), \tau)$.

Example 1.1.1. According to the Merton [143] model, the recovery payoff upon default (that is, on the event $\{V_T < L\}$) equals $\tilde{X} = V_T$, where the random variable V_T is the firm's value at maturity date T of a corporate bond. Consequently, the random recovery rate upon default is equal here to $\delta(T) = V_T/L$ and the writedown rate upon default equals $w(T) = 1 - V_T/L$.

For simplicity, we assume that the savings account B is non-stochastic, that is, that the short-term interest rate r is deterministic. Then the price of a default-free zero-coupon bond equals $B(t,T) = B_t B_T^{-1}$ and the price of a zero-coupon corporate bond satisfies

$$D(t,T) = L_t(1 - w^*(t,T)),$$

where $L_t = LB(t,T)$ is the present value of future liabilities and $w^*(t,T)$ is the *conditional expected writedown rate* under $\mathbb{Q}$. The quantity $w^*(t,T)$ is given by the following expression

$$w^*(t,T) = \mathbb{E}_{\mathbb{Q}}\big(w(T)\mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{F}_t\big).$$

The *conditional expected writedown rate upon default* equals, under $\mathbb{Q}$,

$$w_t^* = \frac{w^*(t,T)}{p_t^*} = \frac{\mathbb{E}_{\mathbb{Q}}\big(w(T)\mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{F}_t\big)}{\mathbb{Q}(\tau \le T \,|\, \mathcal{F}_t)},$$

where $p_t^* = \mathbb{Q}(\tau \le T \,|\, \mathcal{F}_t)$ is the *conditional risk-neutral probability of default.* Finally, $\delta_t^* = 1 - w_t^*$ is the *conditional expected recovery rate upon default* under $\mathbb{Q}$. In terms of the quantities p_t^*, δ_t^* and w_t^*, we obtain

$$D(t,T) = L_t(1 - p_t^*) + L_t p_t^* \delta_t^* = L_t(1 - p_t^* w_t^*).$$

If the random variables $w(T)$ and τ are conditionally independent with respect to the σ-field $\mathcal{F}_t$ under $\mathbb{Q}$ then we have that $w_t^* = \mathbb{E}_{\mathbb{Q}}(w(T) \,|\, \mathcal{F}_t)$.

Example 1.1.2. It is common to assume that the recovery rate is deterministic. Let the recovery rate $\delta(T)$ be constant, specifically, $\delta(T) = \delta$ for some real number δ. In this case, the writedown rate $w(T) = w = 1 - \delta$ is deterministic as well. Then $w^*(t,T) = wp_t^*$ and $w_t^* = w$ for every $t \in [0,T]$. Furthermore, the price of a defaultable bond has the following representation

$$D(t,T) = L_t(1 - p_t^*) + \delta L_t p_t^* = L_t(1 - wp_t^*).$$

For more details on various conventions regarding the recovery schemes for corporate bonds, we refer to Section 2.1.

1.2 The Merton Model

The standing assumption in most classic structural models is that the risk-neutral dynamics for the value process V of the firm's assets are given by the following stochastic differential equation (SDE)

$$dV_t = V_t \big((r - \kappa)\, dt + \sigma_V \, dW_t \big)$$

with $V_0 > 0$, where κ is the constant *payout ratio* (*dividend yield*) and the process W is a standard Brownian motion under the martingale measure $\mathbb{Q}$. The positive constant σ_V represents the *volatility* of the firm's value.

According to the classic model put forward by Merton [143], the valuation of the corporate bond is based on the following postulates:

- a firm has a single liability with the notional value L, interpreted as a zero-coupon bond with maturity T and face value $L > 0$,
- the ability of the firm to redeem its debt is determined exclusively by the level V_T of the total value firm's assets at time T,
- the default event may occur at the debt's maturity date T only, and it corresponds to the event $\{V_T < L\}$.

Formally, the default time τ in the Merton model is given by the expression

$$\tau = T \mathbb{1}_{\{V_T < L\}} + \infty \mathbb{1}_{\{V_T \geq L\}}.$$

Using the present notation, the corporate bond can be described by setting $A = 0$, $Z = 0$, and

$$X_T^d = V_T \mathbb{1}_{\{V_T < L\}} + L \mathbb{1}_{\{V_T \geq L\}}$$

so that $\tilde{X} = V_T$. In other words, the bond's payoff at maturity date T equals

$$D(T,T) = \min (V_T, L) = L - \max (L - V_T, 0) = L - (L - V_T)^+.$$

The last equality makes it clear that the valuation of the corporate bond in the Merton model is equivalent to the valuation of a European put option written on the firm's value with the strike equal to the bond's face value.

Let $D(t,T)$ denote the price at time $t < T$ of the corporate bond. Using the option-like features of a corporate bond, Merton [143] derived a closed-form expression for $D(t,T)$. Let N denote the standard Gaussian cumulative distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \quad \forall\, x \in \mathbb{R}.$$

Proposition 1.2.1. *For every $t \in [0, T[$, the value $D(t,T)$ of the corporate bond equals*

$$D(t,T) = V_t e^{-\kappa(T-t)} N\big(-d_+(V_t, T-t)\big) + L\,B(t,T) N\big(d_-(V_t, T-t)\big)$$

where

$$d_\pm(V_t, T-t) = \frac{\ln(V_t/L) + \big(r - \kappa \pm \frac{1}{2}\sigma_V^2\big)(T-t)}{\sigma_V \sqrt{T-t}}.$$

The unique replicating strategy for the corporate bond involves holding, at any time $t \in [0, T[$, $\phi_t^1 V_t$ units of cash invested in the firm's value and $\phi_t^2 B(t,T)$ units of cash invested in default-free bonds, where

$$\phi_t^1 = e^{-\kappa(T-t)} N\big(-d_+(V_t, T-t)\big)$$

and

$$\phi_t^2 = \frac{D(t,T) - \phi_t^1 V_t}{B(t,T)} = LN\big(d_-(V_t, T-t)\big).$$

It is clear that the value $D(V_t) = D(t,T)$ of the firm's debt admits the following representation $D(V_t) = LB(t,T) - P_t$, where P_t is the price of a put option with strike L and expiration date T. Hence the value $E(V_t)$ of the firm's equity at time t equals

$$E(V_t) = V_t e^{-\kappa(T-t)} - D(V_t) = V_t e^{-\kappa(T-t)} - LB(t,T) + P_t = C_t,$$

where C_t stands for the price at time t of a call option written on the firm's assets, with strike price L and exercise date T. To justify the last equality above, we may also observe that at time T we have

$$E(V_T) = V_T - D(V_T) = V_T - \min(V_T, L) = (V_T - L)^+.$$

We conclude that the firm's shareholders can be seen as holders of the call option with strike L and expiry T on the total value of the firm's assets.

1.2.1 Credit Spreads

Let us now examine *credit spreads* in the Merton model. For notational simplicity, we set $\kappa = 0$. Then Merton's formula becomes

$$D(t,T) = LB(t,T)\big(\Gamma_t N(-d) + N(d - \sigma_V \sqrt{T-t})\big),$$

where we denote $\Gamma_t = V_t / LB(t,T)$ and

$$d = d_+(V_t, T-t) = \frac{\ln \Gamma_t + \frac{1}{2}\sigma_V^2 (T-t)}{\sigma_V \sqrt{T-t}}.$$

Since $LB(t,T)$ represents the current value of the face value of the firm's debt, the quantity Γ_t can be seen as a proxy of the *asset-to-debt ratio* $V_t/D(t,T)$. It can be easily verified that the inequality $D(t,T) < LB(t,T)$ is valid. This condition is in turn equivalent to the strict positivity of the corresponding credit spread, as defined by formula (1.2) below.

In the present setup, the *continuously compounded yield* $r(t,T)$ at time t on the T-maturity Treasury zero-coupon bond is constant and equal to the short-term interest rate r since

$$B(t,T) = e^{-r(t,T)(T-t)} = e^{-r(T-t)}.$$

Let us denote by $r^d(t,T)$ the *continuously compounded yield* at time $t < T$ on the corporate bond, so that

$$D(t,T) = Le^{-r^d(t,T)(T-t)}.$$

From the last equality, it follows that

$$r^d(t,T) = -\frac{\ln D(t,T) - \ln L}{T-t}.$$

The *credit spread* $S(t,T)$ is defined as the excess return on the corporate bond, that is, for any $t < T$,

$$S(t,T) = r^d(t,T) - r(t,T) = \frac{1}{T-t} \ln \frac{LB(t,T)}{D(t,T)}. \tag{1.2}$$

In the Merton model, the credit spread $S(t,T)$ is given by the following expression

$$S(t,T) = -\frac{\ln \big(N(d - \sigma_V \sqrt{T-t}) + \Gamma_t N(-d) \big)}{T-t} > 0.$$

The property that $S(t,T) > 0$ is consistent with the real-life feature that corporate bonds have an expected return in excess of the risk-free interest rate. Indeed, the observed yields on bonds issued by companies are systematically higher than yields on Treasury bonds with matching notional amounts and maturities. It was frequently argued in the financial literature that, for realistic values of model's parameters, the credit spreads produced by the Merton model for bonds with short maturities are far below the corporate bonds spreads observed in the market (see, however, the recent paper by Hull et al. [99] for an alternative implementation of Merton's model).

Note that when time t converges to maturity date T then the *forward short credit spread* at time T in the Merton model tends either to infinity or to 0, depending on whether $V_T < L$ or $V_T > L$. To be more specific

$$S(T,T) := \lim_{t \uparrow T} S(t,T) = \begin{cases} 0, & \text{on the event } \{V_T > L\}, \\ \infty, & \text{on the event } \{V_T < L\}. \end{cases}$$

1.3 First Passage Times

Before we proceed to an extension of the Merton model, put forward by Black and Cox [28], we present some auxiliary mathematical results regarding the first passage times, which will prove useful in what follows.

Let W be a standard one-dimensional Brownian motion under $\mathbb{Q}$ with respect to its natural filtration $\mathbb{F}$. We define an auxiliary process Y by setting, for every $t \in \mathbb{R}_+$,

$$Y_t = y_0 + \nu t + \sigma W_t, \tag{1.3}$$

for some constants $\nu \in \mathbb{R}$ and $\sigma > 0$. It is clear that Y inherits from W the strong Markov property with respect to the filtration $\mathbb{F}$.

1.3.1 Distributions of First Passage Times

Let τ stand for the *first passage time to zero* by the process Y, that is,

$$\tau = \inf\{\, t \in \mathbb{R}_+ : Y_t = 0\}. \tag{1.4}$$

Recall that well-known fact that in an arbitrarily small interval $[0, t]$ the sample path of the Brownian motion started at 0 passes through origin infinitely many times. Using the Girsanov theorem and the strong Markov property of the Brownian motion (see, e.g., Karatzas and Shreve [117]), it is thus easy to deduce that the first passage time by Y to zero coincides with the first crossing time by Y of the level 0, that is, with probability 1,

$$\tau = \inf\{\, t \in \mathbb{R}_+ : Y_t < 0\} = \inf\{\, t \in \mathbb{R}_+ : Y_t \le 0\}.$$

Let us denote $X_t = \nu t + \sigma W_t$ for every $t \in \mathbb{R}_+$.

Lemma 1.3.1. *Let $\sigma > 0$ and $\nu \in \mathbb{R}$. Then for every $x > 0$ we have*

$$\mathbb{Q}\big(\sup_{0 \le u \le s} X_u \le x\big) = N\left(\frac{x - \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu\sigma^{-2}x} N\left(\frac{-x - \nu s}{\sigma\sqrt{s}}\right) \tag{1.5}$$

and for every $x < 0$

$$\mathbb{Q}\big(\inf_{0 \le u \le s} X_u \ge x\big) = N\left(\frac{-x + \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu\sigma^{-2}x} N\left(\frac{x + \nu s}{\sigma\sqrt{s}}\right). \tag{1.6}$$

Proof. Assume first that $\sigma = 1$. Let $\widetilde{\mathbb{Q}}$ be the probability measure on $(\Omega, \mathcal{F}_s)$ given by

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = e^{-\nu W_s - \frac{\nu^2}{2}s}, \quad \mathbb{Q}\text{-a.s.},$$

so that the process $W_t^* := X_t = W_t + \nu t$, $t \in [0, s]$, is a standard Brownian motion under $\widetilde{\mathbb{Q}}$. Also

$$\frac{d\mathbb{Q}}{d\widetilde{\mathbb{Q}}} = e^{\nu W_s^* - \frac{\nu^2}{2} s}, \quad \widetilde{\mathbb{Q}}\text{-a.s.}$$

Moreover, for any $x > 0$,

$$\mathbb{Q}\big(\sup_{0 \le u \le s} X_u > x,\, X_s \le x \big) = \mathbb{E}_{\widetilde{\mathbb{Q}}} \Big(e^{\nu W_s^* - \frac{\nu^2}{2} s}\, \mathbb{1}_{\{ \sup_{0 \le u \le s} W_u^* > x,\, W_s^* \le x \}} \Big).$$

We set $\tau_x = \inf\{ t \ge 0 : W_t^* = x \}$ and we define an auxiliary process $(\widetilde{W}_t,\, t \in [0, s])$ by setting

$$\widetilde{W}_t = W_t^* \mathbb{1}_{\{\tau_x \ge t\}} + (2x - W_t^*) \mathbb{1}_{\{\tau_x < t\}}. \tag{1.7}$$

By virtue of the reflection principle, the process $\widetilde{W}$ is a standard Brownian motion under $\widetilde{\mathbb{Q}}$. Moreover, it is easy to check that

$$\{ \sup_{0 \le u \le s} \widetilde{W}_u > x,\, \widetilde{W}_s \le x \} = \{ W_s^* \ge x \} \subset \{ \tau_x \le s \}. \tag{1.8}$$

Let us denote

$$J = \mathbb{Q}\big(\sup_{0 \le u \le s} (W_u + \nu u) \le x \big).$$

We first observe that

$$\begin{aligned}
J &= \mathbb{Q}(X_s \le x) - \mathbb{Q}\big(\sup_{0 \le u \le s} X_u > x,\, X_s \le x \big) \\
&= \mathbb{Q}(X_s \le x) - \mathbb{E}_{\widetilde{\mathbb{Q}}} \Big(e^{\nu W_s^* - \frac{\nu^2}{2} s}\, \mathbb{1}_{\{ \sup_{0 \le u \le s} W_u^* > x,\, W_s^* \le x \}} \Big) \\
&= \mathbb{Q}(X_s \le x) - \mathbb{E}_{\widetilde{\mathbb{Q}}} \Big(e^{\nu \widetilde{W}_s - \frac{\nu^2}{2} s}\, \mathbb{1}_{\{ \sup_{0 \le u \le s} \widetilde{W}_u > x,\, \widetilde{W}_s \le x \}} \Big)
\end{aligned}$$

since $\widetilde{W}$ is also a Brownian motion under $\widetilde{\mathbb{Q}}$. In view of (1.7) and (1.8), we thus obtain

$$\begin{aligned}
J &= \mathbb{Q}(X_s \le x) - \mathbb{E}_{\widetilde{\mathbb{Q}}} \Big(e^{\nu (2x - W_s^*) - \frac{\nu^2}{2} s}\, \mathbb{1}_{\{ W_s^* \ge x \}} \Big) \\
&= \mathbb{Q}(X_s \le x) - e^{2\nu x}\, \mathbb{E}_{\widetilde{\mathbb{Q}}} \Big(e^{\nu W_s^* - \frac{\nu^2}{2} s}\, \mathbb{1}_{\{ W_s^* \le -x \}} \Big) \\
&= \mathbb{Q}(W_s + \nu s \le x) - e^{2\nu x}\, \mathbb{Q}(W_s + \nu s \le -x) \\
&= N\left(\frac{x - \nu s}{\sqrt{s}} \right) - e^{2\nu x} N\left(\frac{-x - \nu s}{\sqrt{s}} \right).
\end{aligned}$$

This ends the proof of the first equality for $\sigma = 1$. For any $\sigma > 0$, we have

$$\mathbb{Q}\big(\sup_{0 \le u \le s} (\sigma W_u + \nu u) \le x \big) = \mathbb{Q}\big(\sup_{0 \le u \le s} (W_u + \nu \sigma^{-1} u) \le x \sigma^{-1} \big),$$

and this implies (1.5). Since the process $-W$ is also a standard Brownian motion under $\mathbb{Q}$, we also have that, for any $x < 0$,

$$\mathbb{Q}\big(\inf_{0\le u\le s}(\sigma W_u + \nu u) \ge x\big) = \mathbb{Q}\big(\sup_{0\le u\le s}(\sigma W_u - \nu u) \le -x\big),$$

and thus (1.6) easily follows from (1.5). □

Proposition 1.3.1. *The first passage time τ given by (1.4) has the inverse Gaussian probability distribution under $\mathbb{Q}$. Specifically, for any $0 < s < \infty$,*

$$\mathbb{Q}(\tau \le s) = \mathbb{Q}(\tau < s) = N(h_1(s)) + e^{-2\nu\sigma^{-2}y_0}N(h_2(s)), \tag{1.9}$$

where N is the standard Gaussian cumulative distribution function and

$$h_1(s) = \frac{-y_0 - \nu s}{\sigma\sqrt{s}}, \quad h_2(s) = \frac{-y_0 + \nu s}{\sigma\sqrt{s}}.$$

Proof. Notice first that

$$\mathbb{Q}(\tau \ge s) = \mathbb{Q}\big(\inf_{0\le u\le s} Y_u \ge 0\big) = \mathbb{Q}\big(\inf_{0\le u\le s} X_u \ge -y_0\big), \tag{1.10}$$

where $X_u = \nu u + \sigma W_u$. From Lemma 1.3.1, we have that, for every $x < 0$,

$$\mathbb{Q}\big(\inf_{0\le u\le s} X_u \ge x\big) = N\left(\frac{-x+\nu s}{\sigma\sqrt{s}}\right) - e^{2\nu\sigma^{-2}x}N\left(\frac{x+\nu s}{\sigma\sqrt{s}}\right),$$

and this yields (1.9), when combined with (1.10). □

The following corollary is a consequence of Proposition 1.3.1 and the strong Markov property of the process Y with respect to the filtration $\mathbb{F}$.

Corollary 1.3.1. *For any $t < s$ we have, on the event $\{t < \tau\}$,*

$$\mathbb{Q}(\tau \le s \,|\, \mathcal{F}_t) = N\left(\frac{-Y_t - \nu(s-t)}{\sigma\sqrt{s-t}}\right) + e^{-2\nu\sigma^{-2}Y_t}N\left(\frac{-Y_t + \nu(s-t)}{\sigma\sqrt{s-t}}\right).$$

We are in a position to apply the foregoing results to specific examples of default times. We first examine the case of a constant lower threshold.

Example 1.3.1. Suppose that the short-term interest rate is constant, that is, $r_t = r$ for every $t \in \mathbb{R}_+$. Let the value of the firm process V obey the SDE

$$dV_t = V_t\big((r-\kappa)\,dt + \sigma_V\,dW_t\big)$$

with constant coefficients $\kappa \in \mathbb{R}$ and $\sigma_V > 0$. Let us also assume that the barrier process v is constant and equal to $\bar{v}$, where the constant $\bar{v}$ satisfies $\bar{v} < V_0$, so that the default time is given as

$$\tau = \inf\{\, t \in \mathbb{R}_+ : V_t \le \bar{v}\} = \inf\{\, t \in \mathbb{R}_+ : V_t < \bar{v}\}.$$

We now set $Y_t = \ln(V_t/\bar{v})$ so that $\nu = r - \kappa - \frac{1}{2}\sigma_V^2$ and $\sigma = \sigma_V$ in formula (1.3). By applying Corollary 1.3.1, we obtain, for every $s > t$ on the event $\{t < \tau\}$,

$$\mathbb{Q}(\tau \le s \,|\, \mathcal{F}_t) = N\left(\frac{\ln\frac{\bar{v}}{V_t} - \nu(s-t)}{\sigma_V\sqrt{s-t}}\right) + \left(\frac{\bar{v}}{V_t}\right)^{2a} N\left(\frac{\ln\frac{\bar{v}}{V_t} + \nu(s-t)}{\sigma_V\sqrt{s-t}}\right),$$

where we denote

$$a = \frac{\nu}{\sigma_V^2} = \frac{r - \kappa - \frac{1}{2}\sigma_V^2}{\sigma_V^2}.$$

This result was used, in particular, in the paper by Leland and Toft [136].

Example 1.3.2. Let the value process V and the short-term interest rate r be as in Example 1.3.1. For a strictly positive constant K and an arbitrary $\gamma \in \mathbb{R}_+$, let the barrier function be defined as $\bar{v}(t) = Ke^{-\gamma(T-t)}$ for $t \in \mathbb{R}_+$. In other words, the function $\bar{v}(t)$ satisfies

$$d\bar{v}(t) = \gamma\bar{v}(t)\,dt, \quad \bar{v}(0) = Ke^{-\gamma T}.$$

We now set $Y_t = \ln(V_t/\bar{v}(t))$, so that the coefficients in formula (1.3) are $\widetilde{\nu} = r - \kappa - \gamma - \frac{1}{2}\sigma_V^2$ and $\sigma = \sigma_V$. We define the default time τ by setting $\tau = \inf\{\, t \ge 0 : V_t \le \bar{v}(t)\}$. From Corollary 1.3.1, we obtain, for every $t < s$ on the event $\{t < \tau\}$,

$$\mathbb{Q}(\tau \le s \,|\, \mathcal{F}_t) = N\left(\frac{\ln\frac{\bar{v}(t)}{V_t} - \widetilde{\nu}(s-t)}{\sigma_V\sqrt{s-t}}\right) + \left(\frac{\bar{v}(t)}{V_t}\right)^{2\widetilde{a}} N\left(\frac{\ln\frac{\bar{v}(t)}{V_t} + \widetilde{\nu}(s-t)}{\sigma_V\sqrt{s-t}}\right),$$

where

$$\widetilde{a} = \frac{\widetilde{\nu}}{\sigma_V^2} = \frac{r - \kappa - \gamma - \frac{1}{2}\sigma_V^2}{\sigma_V^2}.$$

This formula was employed in the classic paper by Black and Cox [28].

1.3.2 Extensions to Joint Distributions

The next step is to find the joint probability distribution, for every $y \ge 0$ and $s > t$,

$$I := \mathbb{Q}(Y_s \ge y,\, \tau \ge s \,|\, \mathcal{F}_t) = \mathbb{Q}(Y_s \ge y,\, \tau > s \,|\, \mathcal{F}_t),$$

where τ is given by (1.4). Let us denote by M^W and m^W the running maximum and minimum of a one-dimensional standard Brownian motion W, respectively. More explicitly, $M_s^W = \sup_{0\le u\le s} W_u$ and $m_s^W = \inf_{0\le u\le s} W_u$.

It is well known that for every $s > 0$ we have

$$\mathbb{Q}(M_s^W > 0) = 1, \quad \mathbb{Q}(m_s^W < 0) = 1.$$

The following classic result, commonly referred to as the *reflection principle,* is a straightforward consequence of the strong Markov property of the Brownian motion.

Lemma 1.3.2. *We have that, for every $s > 0$, $y \geq 0$ and $x \leq y$,*

$$\mathbb{Q}(W_s \leq x,\, M_s^W \geq y) = \mathbb{Q}(W_s \geq 2y - x) = \mathbb{Q}(W_s \leq x - 2y). \tag{1.11}$$

We need to examine the Brownian motion with non-zero drift. Consider the process X that equals $X_t = \nu t + \sigma W_t$. We write $M_s^X = \sup_{0 \leq u \leq s} X_u$ and $m_s^X = \inf_{0 \leq u \leq s} X_u$. By virtue of the Girsanov theorem, the process X is a Brownian motion, up to an appropriate re-scaling, under an equivalent probability measure and thus we have, for any $s > 0$,

$$\mathbb{Q}(M_s^X > 0) = 1, \quad \mathbb{Q}(m_s^X < 0) = 1.$$

Lemma 1.3.3. *For every $s > 0$, the joint distribution of (X_s, M_s^X) is given by the expression*

$$\mathbb{Q}(X_s \leq x,\, M_s^X \geq y) = e^{2\nu y \sigma^{-2}}\, \mathbb{Q}(X_s \geq 2y - x + 2\nu s)$$

for every $x, y \in \mathbb{R}$ such that $y \geq 0$ and $x \leq y$.

Proof. Let us define $X_t^\sigma = \sigma^{-1} X_t = W_t + \sigma^{-1}\nu t$. It is easy to check that the following equality is valid

$$I := \mathbb{Q}\big(X_s \leq x,\, M_s^X \geq y\big) = \mathbb{Q}\big(X_s^\sigma \leq x\sigma^{-1},\, M_s^{X^\sigma} \geq y\sigma^{-1}\big).$$

We thus see that we may and do assume, without loss of generality, that $\sigma = 1$. From the Girsanov theorem, it follows that X is a standard Brownian motion under the probability measure $\widetilde{\mathbb{Q}}$, which is given on $(\Omega, \mathcal{F}_s)$ by the Radon-Nikodým density (recall that $\sigma = 1$)

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = e^{-\nu W_s - \frac{\nu^2}{2}s}, \quad \mathbb{Q}\text{-a.s.}$$

Note also that

$$\frac{d\mathbb{Q}}{d\widetilde{\mathbb{Q}}} = e^{\nu W_s^* - \frac{\nu^2}{2}s}, \quad \widetilde{\mathbb{Q}}\text{-a.s.},$$

where the process $(W_t^* = X_t = W_t + \nu t,\, t \in [0, s])$ is a standard Brownian motion under $\widetilde{\mathbb{Q}}$. It is easily seen that

$$I = \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(e^{\nu W_s^* - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{X_s \leq x,\, M_s^X \geq y\}}\Big) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(e^{\nu W_s^* - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{W_s^* \leq x,\, M_s^{W^*} \geq y\}}\Big).$$

Since W^* is a standard Brownian motion under $\widetilde{\mathbb{Q}}$, an application of the reflection principle gives

$$\begin{aligned} I &= \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(e^{\nu(2y-W^*_s)-\frac{\nu^2}{2}s}\,\mathbb{1}_{\{2y-W^*_s\le x,\,M^{W^*}_s\ge y\}}\Big)\\ &= \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(e^{\nu(2y-W^*_s)-\frac{\nu^2}{2}s}\,\mathbb{1}_{\{W^*_s\ge 2y-x\}}\Big)\\ &= e^{2\nu y}\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(e^{-\nu W^*_s-\frac{\nu^2}{2}s}\,\mathbb{1}_{\{W^*_s\ge 2y-x\}}\Big),\end{aligned}$$

since clearly $2y-x\ge y$.

Let us define one more equivalent probability measure, $\widetilde{\mathbb{P}}$ say, by setting

$$\frac{d\widetilde{\mathbb{P}}}{d\widetilde{\mathbb{Q}}} = e^{-\nu W^*_s-\frac{\nu^2}{2}s}, \quad \mathbb{P}\text{-a.s.}$$

Is is clear that

$$I = e^{2\nu y}\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(e^{-\nu W^*_s-\frac{\nu^2}{2}s}\,\mathbb{1}_{\{W^*_s\ge 2y-x\}}\Big) = e^{2\nu y}\,\widetilde{\mathbb{P}}(W^*_s\ge 2y-x).$$

Furthermore, the process $(\widetilde{W}_t = W^*_t+\nu t,\ t\in[0,s])$ is a standard Brownian motion under $\widetilde{\mathbb{P}}$ and we have that

$$I = e^{2\nu y}\,\widetilde{\mathbb{P}}(\widetilde{W}_s+\nu s\ge 2y-x+2\nu s).$$

The last equality easily yields the asserted formula. □

It is worthwhile to observe that (a similar remark applies to all formulae below)

$$\mathbb{Q}(X_s\le x,\,M^X_s\ge y) = \mathbb{Q}(X_s< x,\,M^X_s> y).$$

The following result is a straightforward consequence of Lemma 1.3.3.

Proposition 1.3.2. *For any $x,y\in\mathbb{R}$ satisfying $y\ge 0$ and $x\le y$, we have that*

$$\mathbb{Q}\big(X_s\le x,\,M^X_s\ge y\big) = e^{2\nu y\sigma^{-2}}\,N\left(\frac{x-2y-\nu s}{\sigma\sqrt{s}}\right).$$

Hence

$$\mathbb{Q}\big(X_s\le x,\,M^X_s\le y\big) = N\left(\frac{x-\nu s}{\sigma\sqrt{s}}\right) - e^{2\nu y\sigma^{-2}}\,N\left(\frac{x-2y-\nu s}{\sigma\sqrt{s}}\right)$$

for every $x,y\in\mathbb{R}$ such that $x\le y$ and $y\ge 0$.

Proof. For the first equality, note that

$$\mathbb{Q}(X_s\ge 2y-x+2\nu s) = \mathbb{Q}(-\sigma W_s\le x-2y-\nu s) = N\left(\frac{x-2y-\nu s}{\sigma\sqrt{s}}\right),$$

since $-\sigma W_t$ has Gaussian law with zero mean and variance $\sigma^2 t$. For the second formula, it is enough to observe that

$$\mathbb{Q}(X_s \le x,\, M_s^X \le y) + \mathbb{Q}(X_s \le x,\, M_s^X \ge y) = \mathbb{Q}(X_s \le x)$$

and to apply the first equality. □

It is clear that

$$\mathbb{Q}(M_s^X \ge y) = \mathbb{Q}(X_s \ge y) + \mathbb{Q}(X_s \le y,\, M_s^X \ge y)$$

for every $y \ge 0$, and thus

$$\mathbb{Q}(M_s^X \ge y) = \mathbb{Q}(X_s \ge y) + e^{2\nu y \sigma^{-2}}\, \mathbb{Q}(X_s \ge y + 2\nu s).$$

Consequently,

$$\mathbb{Q}(M_s^X \le y) = 1 - \mathbb{Q}(M_s^X \ge y) = \mathbb{Q}(X_s \le y) - e^{2\nu y \sigma^{-2}}\, \mathbb{Q}(X_s \ge y + 2\nu s).$$

This leads to the following corollary.

Corollary 1.3.2. *The following equality is valid, for every $s > 0$ and $y \ge 0$,*

$$\mathbb{Q}(M_s^X \le y) = N\left(\frac{y - \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu y \sigma^{-2}}\, N\left(\frac{-y - \nu s}{\sigma\sqrt{s}}\right).$$

We will now focus on the distribution of the minimal value of X. Observe that we have, for any $y \le 0$,

$$\mathbb{Q}\big(\sup_{0 \le u \le s} (\sigma W_u - \nu u) \ge -y\big) = \mathbb{Q}\big(\inf_{0 \le u \le s} X_u \le y\big),$$

where we have used the symmetry of the Brownian motion. Consequently, for every $y \le 0$ we have $\mathbb{Q}(m_s^X \le y) = \mathbb{Q}(M_s^{\widetilde{X}} \ge -y)$, where the process $\widetilde{X}$ equals $\widetilde{X}_t = \sigma W_t - \nu t$. It is thus not difficult to establish the following result.

Proposition 1.3.3. *The joint probability distribution of (X_s, m_s^X) satisfies, for every $s > 0$,*

$$\mathbb{Q}(\, X_s \ge x,\, m_s^X \ge y) = N\left(\frac{-x + \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu y \sigma^{-2}}\, N\left(\frac{2y - x + \nu s}{\sigma\sqrt{s}}\right)$$

for every $x, y \in \mathbb{R}$ such that $y \le 0$ and $y \le x$.

Corollary 1.3.3. *The following equality is valid, for every $s > 0$ and $y \le 0$,*

$$\mathbb{Q}(m_s^X \ge y) = N\left(\frac{-y + \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu y \sigma^{-2}}\, N\left(\frac{y + \nu s}{\sigma\sqrt{s}}\right).$$

Recall that we denote $Y_t = y_0 + X_t$, where $X_t = \nu t + \sigma W_t$. We write

$$m_s^X = \inf_{0 \le u \le s} X_u, \quad m_s^Y = \inf_{0 \le u \le s} Y_u.$$

Corollary 1.3.4. *We have that, for any $s > 0$ and $y \ge 0$,*

$$\mathbb{Q}(Y_s \ge y,\, \tau \ge s) = N\left(\frac{-y + y_0 + \nu s}{\sigma\sqrt{s}}\right) - e^{-2\nu\sigma^{-2}y_0}\, N\left(\frac{-y - y_0 + \nu s}{\sigma\sqrt{s}}\right).$$

Proof. Since

$$\mathbb{Q}(Y_s \ge y,\, \tau \ge s) = \mathbb{Q}(\, Y_s \ge y,\, m_s^Y \ge 0) = \mathbb{Q}(\, X_s \ge y - y_0,\, m_s^X \ge -y_0),$$

the asserted formula is rather obvious. □

More generally, the Markov property of Y justifies the following result.

Lemma 1.3.4. *We have that, for any $t < s$ and $y \ge 0$, on the event $\{t < \tau\}$,*

$$\begin{aligned}\mathbb{Q}(Y_s \ge y,\, \tau \ge s \,|\, \mathcal{F}_t) &= N\left(\frac{-y + Y_t + \nu(s-t)}{\sigma\sqrt{s-t}}\right) \\ &- e^{-2\nu\sigma^{-2}Y_t} N\left(\frac{-y - Y_t + \nu(s-t)}{\sigma\sqrt{s-t}}\right).\end{aligned}$$

Example 1.3.3. Assume that the dynamics of the value of the firm process V are

$$dV_t = V_t\big((r-\kappa)\,dt + \sigma_V\,dW_t\big) \tag{1.12}$$

and set $\tau = \inf\{\, t \ge 0 : V_t \le \bar{v}\}$, where the constant $\bar{v}$ satisfies $\bar{v} < V_0$. By applying Lemma 1.3.4 to $Y_t = \ln(V_t/\bar{v})$ and $y = \ln(x/\bar{v})$, we obtain the following equality, which holds for $x \ge \bar{v}$ on the event $\{t < \tau\}$,

$$\begin{aligned}\mathbb{Q}(V_s \ge x,\, \tau \ge s \,|\, \mathcal{F}_t) &= N\left(\frac{\ln(V_t/x) + \nu(s-t)}{\sigma\sqrt{s-t}}\right) \\ &- \left(\frac{\bar{v}}{V_t}\right)^{2a} N\left(\frac{\ln \bar{v}^2 - \ln(xV_t) + \nu(s-t)}{\sigma\sqrt{s-t}}\right),\end{aligned}$$

where $\nu = r - \kappa - \frac{1}{2}\sigma_V^2$ and $a = \nu\sigma_V^{-2}$.

Example 1.3.4. We consider the setup of Example 1.3.2, so that the value process V satisfies (1.12) and the barrier function equals $\bar{v}(t) = Ke^{-\gamma(T-t)}$ for some constants $K > 0$ and $\gamma \in \mathbb{R}$.

Making use again of Lemma 1.3.4, but this time with $Y_t = \ln(V_t/\bar{v}(t))$ and $y = \ln(x/\bar{v}(s))$, we find that, for every $t < s \le T$ and an arbitrary

$x \geq \bar{v}(s)$, the following equality holds on the event $\{t < \tau\}$

$$\begin{aligned}\mathbb{Q}(V_s \geq x,\, \tau \geq s \,|\, \mathcal{F}_t) &= N\left(\frac{\ln(V_t/\bar{v}(t)) - \ln(x/\bar{v}(s)) + \widetilde{\nu}(s-t)}{\sigma_V\sqrt{s-t}}\right) \\ &- \left(\frac{\bar{v}(t)}{V_t}\right)^{2\widetilde{a}} N\left(\frac{-\ln(V_t/\bar{v}(t)) - \ln(x/\bar{v}(s)) + \widetilde{\nu}(s-t)}{\sigma_V\sqrt{s-t}}\right),\end{aligned}$$

where $\widetilde{\nu} = r - \kappa - \gamma - \frac{1}{2}\sigma_V^2$ and $\widetilde{a} = \widetilde{\nu}\sigma_V^{-2}$. Upon simplification, this yields

$$\begin{aligned}\mathbb{Q}(V_s \geq x,\, \tau \geq s \,|\, \mathcal{F}_t) &= N\left(\frac{\ln(V_t/x) + \nu(s-t)}{\sigma_V\sqrt{s-t}}\right) \\ &- \left(\frac{\bar{v}(t)}{V_t}\right)^{2\widetilde{a}} N\left(\frac{\ln \bar{v}^2(t) - \ln(xV_t) + \nu(s-t)}{\sigma_V\sqrt{s-t}}\right),\end{aligned}$$

where $\nu = r - \kappa - \frac{1}{2}\sigma_V^2$.

Remark 1.3.1. Note that if we take $x = \bar{v}(s) = Ke^{-\gamma(T-s)}$ then clearly

$$1 - \mathbb{Q}(V_s \geq \bar{v}(s),\, \tau \geq s \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau < s \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \leq s \,|\, \mathcal{F}_t).$$

But we also have that

$$1 - N\left(\frac{\ln(V_t/\bar{v}(s)) + \nu(s-t)}{\sigma_V\sqrt{s-t}}\right) = N\left(\frac{\ln(\bar{v}(t)/V_t) - \widetilde{\nu}(s-t)}{\sigma_V\sqrt{s-t}}\right)$$

and

$$N\left(\frac{\ln \bar{v}^2(t) - \ln(\bar{v}(s)V_t) + \nu(s-t)}{\sigma_V\sqrt{s-t}}\right) = N\left(\frac{\ln(\bar{v}(t)/V_t) + \widetilde{\nu}(s-t)}{\sigma_V\sqrt{s-t}}\right).$$

By setting $x = \bar{v}(s)$, we rediscover the formula established in Example 1.3.2.

1.4 The Black and Cox Model

By construction, the original Merton model does not allow for a premature default, in the sense that the default may only occur at the maturity of the claim. Several authors have put forward various structural models for valuation of a corporate debt in which this restrictive and unrealistic feature was relaxed.

In most of these models, the time of default was defined as the *first passage time* of the value process V to either deterministic or random barrier. In principle, the bond's default may thus occur at any time before or on the maturity date T. The challenge is to appropriately specify the lower threshold v, the recovery process Z, and to explicitly evaluate the conditional expectation that appears on the right-hand side of the risk-neutral valuation formula

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{F}_t \Big),$$

which is valid for $t \in [0, T[$. As one might easily guess, this is a non-trivial mathematical problem, in general. In addition, the practical problem of the lack of direct observations of the value process V largely limits the applicability of the first-passage-time models based on the firm value process V.

1.4.1 Safety Covenants

Black and Cox [28] generalize the Merton [143] approach in many aspects by taking into account such real-life features of debt contracts as: safety covenants, debt subordination, and restrictions on the sale of assets.

As in the Merton model, they assume that the firm's stockholders receive continuous dividend payments proportional to the current value of firm's assets, so that the risk-neutral dynamics of the value process are

$$dV_t = V_t\big((r - \kappa)\, dt + \sigma_V \, dW_t\big), \quad V_0 > 0,$$

where W is a Brownian motion under the risk-neutral probability $\mathbb{Q}$, the constant $\kappa \geq 0$ represents the payout ratio and $\sigma_V > 0$ is the constant volatility. The short-term interest rate r is assumed to be constant.

The so-called *safety covenants* provide the bondholders with the right to force the firm to bankruptcy or reorganization if the firm is doing poorly according to some gauge. The standard for a poor performance is set by Black and Cox in terms of a time-dependent deterministic barrier $\bar{v}(t) = Ke^{-\gamma(T-t)}$, $t \in [0, T[$, for some constant $K > 0$. As soon as the total value of firm's assets hits this lower threshold, the bondholders take over the firm. Otherwise, default either occurs at maturity date T or not, depending on whether the inequality $V_T < L$ holds or not.

Let us set

$$v_t = \begin{cases} \bar{v}(t), & \text{for } t < T, \\ L, & \text{for } t = T. \end{cases}$$

Formally, the default event occurs at the first time $t \in [0, T]$ at which the firm's value V_t falls below the level v_t, or the default event does not occur at all. Hence the default time τ is now given by the expression (by convention $\inf \emptyset = +\infty$)

$$\tau = \inf \{\, t \in [0, T] : V_t \leq v_t \}.$$

The recovery process Z and the recovery payoff $\widetilde{X}$ are proportional to the value process, specifically, $Z = \beta_2 V$ and $\widetilde{X} = \beta_1 V_T$ for some constants

$\beta_1, \beta_2 \in [0,1]$. Though the case examined by Black and Cox [28] corresponded to $\beta_1 = \beta_2 = 1$, the extension of their approach to the case of arbitrary β_1 and β_2 is trivial.

To sum up, the corporate bond is now given as the following defaultable claim

$$X = L,\ A = 0,\ \widetilde{X} = \beta_1 V_T,\ Z = \beta_2 V,\ \tau = \bar{\tau} \wedge \widehat{\tau},$$

where the *early default time* $\bar{\tau}$ equals

$$\bar{\tau} = \inf\{\, t \in [0,T[\, : V_t \leq \bar{v}(t)\}$$

and $\widehat{\tau}$ stands for Merton's default time, that is, $\widehat{\tau} = T\mathbb{1}_{\{V_T < L\}} + \infty \mathbb{1}_{\{V_T \geq L\}}$.

1.4.2 Corporate Bond

Similarly as in the Merton model, it is assumed that the short-term interest rate is deterministic and equal to a positive constant r. It is postulated, in addition, that $\bar{v}(t) \leq LB(t,T)$ for every $t \in [0,T]$ or, more explicitly,

$$Ke^{-\gamma(T-t)} \leq Le^{-r(T-t)},$$

so that, in particular, $K \leq L$. This additional condition is imposed in order to guarantee that the payoff to the bondholder at the default time τ will never exceed the face value of the debt, discounted at a risk-free rate.

Since the dynamics for the value process V are given in terms of a diffusion process, a suitable partial differential equation can be used to characterize the value process of the corporate bond. Let us write $D(t,T) = u(V_t,t)$. Then the pricing function $u = u(v,t)$ of a corporate bond satisfies the following PDE

$$u_t(v,t) + (r-\kappa)vu_v(v,t) + \frac{1}{2}\sigma_V^2 v^2 u_{vv}(v,t) - ru(v,t) = 0$$

on the domain

$$\{(v,t) \in \mathbb{R}_+ \times \mathbb{R}_+ :\ 0 < t < T,\ v > Ke^{-\gamma(T-t)}\}$$

with the boundary condition

$$u(Ke^{-\gamma(T-t)}, t) = \beta_2 Ke^{-\gamma(T-t)}$$

and the terminal condition $u(v,T) = \min(\beta_1 v, L)$.

Alternatively, the price $D(t,T) = u(V_t,t)$ of a defaultable bond has the following probabilistic representation, on the event $\{t < \tau\} = \{t < \bar{\tau}\}$,

$$\begin{aligned} D(t,T) = \mathbb{E}_{\mathbb{Q}}\Big(&Le^{-r(T-t)}\mathbb{1}_{\{\bar{\tau} \geq T,\, V_T \geq L\}} \,\Big|\, \mathcal{F}_t\Big) \\ &+ \beta_1 \mathbb{E}_{\mathbb{Q}}\Big(V_T e^{-r(T-t)}\mathbb{1}_{\{\bar{\tau} \geq T,\, V_T < L\}} \,\Big|\, \mathcal{F}_t\Big) \\ &+ \beta_2 K\, \mathbb{E}_{\mathbb{Q}}\Big(e^{-\gamma(T-\bar{\tau})} e^{-r(\bar{\tau}-t)}\mathbb{1}_{\{t < \bar{\tau} < T\}} \,\Big|\, \mathcal{F}_t\Big) \end{aligned}$$

where $\mathcal{F}_t = \mathcal{F}_t^V = \mathcal{F}_t^W$ for every $t \in [0, T]$. After default, that is, on the event $\{t \geq \tau\} = \{t \geq \bar{\tau}\}$, the bond price is obviously given by

$$D(t,T) = \beta_2 \bar{v}(\tau) B^{-1}(\tau, T) B(t,T) = \beta_2 K e^{-\gamma(T-\tau)} e^{r(t-\tau)}.$$

We wish find explicit expressions for the conditional expectations arising in the above probabilistic representation of the price $D(t,T)$. To this end, we observe that:

- the first two conditional expectations can be computed by using the formula for the joint conditional probability $\mathbb{Q}(V_s \geq x,\, \tau \geq s \,|\, \mathcal{F}_t)$,
- to evaluate the third conditional expectation, it suffices to employ the conditional probability law of the first passage time of the process V to the barrier $\bar{v}(t)$.

1.4.3 The Black and Cox Formula

Before stating the bond valuation formula established by Black and Cox [28], let us recall the standing notation introduced in Section 1.3 (see, in particular, Examples 1.3.2 and 1.3.4)

$$\begin{aligned} \nu &= r - \kappa - \frac{1}{2}\sigma_V^2, \\ \widetilde{\nu} &= \nu - \gamma = r - \kappa - \gamma - \frac{1}{2}\sigma_V^2, \\ \widetilde{a} &= \widetilde{\nu}\sigma_V^{-2}. \end{aligned}$$

For the sake of brevity, in the statement of Proposition 1.4.1 we shall write σ instead of σ_V. As already mentioned, the probabilistic proof of this result relies on the knowledge of the probability law of the first passage time of the geometric (that is, exponential) Brownian motion to an exponential barrier. All relevant results regarding this issue were already established in Section 1.3 (once again the reader is referred to Examples 1.3.2 and 1.3.4).

Proposition 1.4.1. *Assume that $\widetilde{\nu}^2 + 2\sigma^2(r-\gamma) > 0$. Prior to default, that is, on the event $\{t < \tau\}$, the price process $D(t,T) = u(V_t, t)$ of a defaultable bond equals*

$$\begin{aligned} D(t,T) &= LB(t,T)\big(N\big(h_1(V_t, T-t)\big) - R_t^{2\widetilde{a}} N\big(h_2(V_t, T-t)\big)\big) \\ &\quad + \beta_1 V_t e^{-\kappa(T-t)}\big(N\big(h_3(V_t, T-t)\big) - N\big(h_4(V_t, T-t)\big)\big) \\ &\quad + \beta_1 V_t e^{-\kappa(T-t)} R_t^{2\widetilde{a}+2}\big(N\big(h_5(V_t, T-t)\big) - N\big(h_6(V_t, T-t)\big)\big) \\ &\quad + \beta_2 V_t\big(R_t^{\theta+\zeta} N\big(h_7(V_t, T-t)\big) + R_t^{\theta-\zeta} N\big(h_8(V_t, T-t)\big)\big), \end{aligned}$$

where $R_t = \bar{v}(t)/V_t$, $\theta = \widetilde{a} + 1$, $\zeta = \sigma^{-2}\sqrt{\widetilde{\nu}^2 + 2\sigma^2(r-\gamma)}$ *and*

$$\begin{aligned}
h_1(V_t, T-t) &= \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\
h_2(V_t, T-t) &= \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}}, \\
h_3(V_t, T-t) &= \frac{\ln(L/V_t) - (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_4(V_t, T-t) &= \frac{\ln(K/V_t) - (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_5(V_t, T-t) &= \frac{\ln \bar{v}^2(t) - \ln(LV_t) + (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_6(V_t, T-t) &= \frac{\ln \bar{v}^2(t) - \ln(KV_t) + (\nu+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
h_7(V_t, T-t) &= \frac{\ln(\bar{v}(t)/V_t) + \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \\
h_8(V_t, T-t) &= \frac{\ln(\bar{v}(t)/V_t) - \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}$$

Before proceeding to the proof of Proposition 1.4.1, we need to establish an elementary lemma.

Lemma 1.4.1. *For any* $a \in \mathbb{R}$ *and* $b > 0$ *we have, for every* $y > 0$,

$$\int_0^y x\, dN\left(\frac{\ln x + a}{b}\right) = e^{\frac{1}{2}b^2 - a}\, N\left(\frac{\ln y + a - b^2}{b}\right) \tag{1.13}$$

and

$$\int_0^y x\, dN\left(\frac{-\ln x + a}{b}\right) = e^{\frac{1}{2}b^2 + a}\, N\left(\frac{-\ln y + a + b^2}{b}\right). \tag{1.14}$$

Let $a, b, c \in \mathbb{R}$ *satisfy* $b < 0$ *and* $c^2 > 2a$. *Then we have, for every* $y > 0$,

$$\int_0^y e^{ax}\, dN\left(\frac{b - cx}{\sqrt{x}}\right) = \frac{d+c}{2d}\, g(y) + \frac{d-c}{2d}\, h(y), \tag{1.15}$$

where $d = \sqrt{c^2 - 2a}$ *and where we denote*

$$g(y) = e^{b(c-d)}\, N\left(\frac{b - dy}{\sqrt{y}}\right), \quad h(y) = e^{b(c+d)}\, N\left(\frac{b + dy}{\sqrt{y}}\right).$$

Proof. The derivation of equalities (1.13)–(1.14) is standard. For (1.15), we first observe that

$$f(y) := \int_0^y e^{ax}\, dN\left(\frac{b-cx}{\sqrt{x}}\right) = \int_0^y e^{ax}\, n\left(\frac{b-cx}{\sqrt{x}}\right)\left(-\frac{b}{2x^{3/2}} - \frac{c}{2\sqrt{x}}\right) dx,$$

where n is the probability density function of the standard Gaussian law. Note also that

$$\begin{aligned} g'(x) &= e^{b(c-\sqrt{c^2-2a})}\, n\left(\frac{b-\sqrt{c^2-2a}x}{\sqrt{x}}\right)\left(-\frac{b}{2x^{3/2}} - \frac{\sqrt{c^2-2a}}{2\sqrt{x}}\right) \\ &= e^{ax}\, n\left(\frac{b-cx}{\sqrt{x}}\right)\left(-\frac{b}{2x^{3/2}} - \frac{d}{2\sqrt{x}}\right) \end{aligned}$$

and

$$\begin{aligned} h'(x) &= e^{b(c+\sqrt{c^2-2a})}\, n\left(\frac{b+\sqrt{c^2-2a}x}{\sqrt{x}}\right)\left(-\frac{b}{2x^{3/2}} + \frac{\sqrt{c^2-2a}}{2\sqrt{x}}\right) \\ &= e^{ax}\, n\left(\frac{b-cx}{\sqrt{x}}\right)\left(-\frac{b}{2x^{3/2}} + \frac{d}{2\sqrt{x}}\right). \end{aligned}$$

Consequently,

$$g'(x) + h'(x) = -e^{ax}\frac{b}{x^{3/2}}\, n\left(\frac{b-cx}{\sqrt{x}}\right)$$

and

$$g'(x) - h'(x) = -e^{ax}\frac{d}{x^{1/2}}\, n\left(\frac{b-cx}{\sqrt{x}}\right).$$

Hence f can be represented as follows

$$f(y) = \frac{1}{2}\int_0^y \left(g'(x) + h'(x) + \frac{c}{d}\,(g'(x) - h'(x))\right) dx.$$

Since $\lim_{y\to 0+} g(y) = \lim_{y\to 0+} h(y) = 0$, we conclude that we have, for every $y > 0$,

$$f(y) = \frac{1}{2}(g(y) + h(y)) + \frac{c}{2d}\,(g(y) - h(y)).$$

This ends the proof of the lemma. □

Proof of Proposition 1.4.1. To establish the asserted bond valuation formula, it suffices to evaluate the following conditional expectations:

$$\begin{aligned} D_1(t,T) &= LB(t,T)\,\mathbb{Q}(V_T \ge L,\, \bar\tau \ge T \,|\, \mathcal{F}_t), \\ D_2(t,T) &= \beta_1 B(t,T)\,\mathbb{E}_{\mathbb{Q}}\big(V_T \mathbb{1}_{\{V_T < L,\, \bar\tau \ge T\}} \,\big|\, \mathcal{F}_t\big), \\ D_3(t,T) &= K\beta_2 B_t e^{-\gamma T}\,\mathbb{E}_{\mathbb{Q}}\big(e^{(\gamma - r)\bar\tau}\mathbb{1}_{\{t<\bar\tau<T\}} \,\big|\, \mathcal{F}_t\big). \end{aligned}$$

For the sake of notational convenience, we will focus on the case $t = 0$. The general result will follow easily from the Markov property of V.

Let us first evaluate $D_1(0,T)$, that is, the part of the bond value corresponding to no-default event. From Example 1.3.4, we know that if $L \geq \bar{v}(T) = K$ then

$$\mathbb{Q}(V_T \geq L,\, \bar{\tau} \geq T) = N\left(\frac{\ln\frac{V_0}{L} + \nu T}{\sigma\sqrt{T}}\right) - R_0^{2\tilde{a}} N\left(\frac{\ln\frac{\bar{v}^2(0)}{LV_0} + \nu T}{\sigma\sqrt{T}}\right)$$

with $R_0 = \bar{v}(0)/V_0$. It is thus clear that

$$D_1(0,T) = LB(0,T)\big(N\big(h_1(V_0,T)\big) - R_0^{2\tilde{a}} N\big(h_2(V_0,T)\big)\big).$$

Let us now examine $D_2(0,T)$ – that is, the part of the bond's value associated with default at time T. We note that

$$\frac{D_2(0,T)}{\beta_1 B(0,T)} = \mathbb{E}_{\mathbb{Q}}\big(V_T \mathbb{1}_{\{V_T < L,\, \bar{\tau} \geq T\}}\big) = \int_K^L x\, d\mathbb{Q}(V_T < x,\, \bar{\tau} \geq T).$$

Using again Example 1.3.4 and the fact that the probability $\mathbb{Q}(\bar{\tau} \geq T)$ does not depend on x, we obtain, for every $x \geq K$,

$$d\mathbb{Q}(V_T < x,\, \bar{\tau} \geq T) = dN\left(\frac{\ln\frac{x}{V_0} - \nu T}{\sigma\sqrt{T}}\right) + R_0^{2\tilde{a}}\, dN\left(\frac{\ln\frac{\bar{v}^2(0)}{xV_0} + \nu T}{\sigma\sqrt{T}}\right).$$

Let us denote

$$K_1(0) = \int_K^L x\, dN\left(\frac{\ln x - \ln V_0 - \nu T}{\sigma\sqrt{T}}\right)$$

and

$$K_2(0) = \int_K^L x\, dN\left(\frac{2\ln \bar{v}(0) - \ln x - \ln V_0 + \nu T}{\sigma\sqrt{T}}\right).$$

Using (1.13)–(1.14), we obtain

$$K_1(0) = V_0 e^{(r-\kappa)T}\left(N\left(\frac{\ln\frac{L}{V_0} - \widehat{\nu} T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\frac{K}{V_0} - \widehat{\nu} T}{\sigma\sqrt{T}}\right)\right),$$

where $\widehat{\nu} = \nu + \sigma^2 = r - \kappa + \frac{1}{2}\sigma^2$. Similarly,

$$K_2(0) = V_0 R_0^2 e^{(r-\kappa)T}\left(N\left(\frac{\ln\frac{\bar{v}^2(0)}{LV_0} + \widehat{\nu} T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\frac{\bar{v}^2(0)}{KV_0} + \widehat{\nu} T}{\sigma\sqrt{T}}\right)\right).$$

Since

$$D_2(0,T) = \beta_1 B(0,T)\big(K_1(0) + R_0^{\tilde{a}} K_2(0)\big),$$

we conclude that $D_2(0,T)$ is equal to

$$\begin{aligned}D_2(0,T) &= \beta_1 V_0 e^{-\kappa T}\big(N\big(h_3(V_0,T)\big) - N\big(h_4(V_0,T)\big)\big)\\ &\quad + \beta_1 V_0 e^{-\kappa T} R_0^{2\widetilde{a}+2}\big(N\big(h_5(V_0,T)\big) - N\big(h_6(V_0,T)\big)\big).\end{aligned}$$

It remains to evaluate $D_3(0,T)$, that is, the part of the bond value associated with the possibility of the forced bankruptcy before the maturity date T. To this end, it suffices to calculate the following expected value

$$\bar{v}(0)\,\mathbb{E}_{\mathbb{Q}}\big(e^{(\gamma-r)\bar{\tau}}\mathbb{1}_{\{\bar{\tau}<T\}}\big) = \bar{v}(0)\int_0^T e^{(\gamma-r)s}\,d\mathbb{Q}(\bar{\tau}\le s),$$

where (see Example 1.3.2)

$$\mathbb{Q}(\bar{\tau}\le s) = N\left(\frac{\ln(\bar{v}(0)/V_0) - \widetilde{\nu}s}{\sigma\sqrt{s}}\right) + \left(\frac{\bar{v}(0)}{V_0}\right)^{2\widetilde{a}} N\left(\frac{\ln(\bar{v}(0)/V_0) + \widetilde{\nu}s}{\sigma\sqrt{s}}\right).$$

Note that $\bar{v}(0) < V_0$ and thus $\ln(\bar{v}(0)/V_0) < 0$. Using (1.15), we obtain

$$\begin{aligned}&\bar{v}(0)\int_0^T e^{(\gamma-r)s}\,dN\left(\frac{\ln(\bar{v}(0)/V_0) - \widetilde{\nu}s}{\sigma\sqrt{s}}\right)\\ &\quad = \frac{V_0(\widetilde{a}+\zeta)}{2\zeta}\,R_0^{\theta-\zeta}\,N\big(h_8(V_0,T)\big) - \frac{V_0(\widetilde{a}-\zeta)}{2\zeta}\,R_0^{\theta+\zeta}\,N\big(h_7(V_0,T)\big)\end{aligned}$$

and

$$\begin{aligned}&\frac{\bar{v}(0)^{2\widetilde{a}+1}}{V_0^{2\widetilde{a}}}\int_0^T e^{(\gamma-r)s}\,dN\left(\frac{\ln(\bar{v}(0)/V_0) + \widetilde{\nu}s}{\sigma\sqrt{s}}\right)\\ &\quad = \frac{V_0(\widetilde{a}+\zeta)}{2\zeta}\,R_0^{\theta+\zeta}\,N\big(h_7(V_0,T)\big) - \frac{V_0(\widetilde{a}-\zeta)}{2\zeta}\,R_0^{\theta-\zeta}\,N\big(h_8(V_0,T)\big).\end{aligned}$$

Consequently,

$$D_3(0,T) = \beta_2 V_0\big(R_0^{\theta+\zeta}N\big(h_7(V_0,T)\big) + R_0^{\theta-\zeta}N\big(h_8(V_0,T)\big)\big).$$

Upon summation, this completes the proof for $t = 0$. □

Let us consider some special cases of the Black and Cox pricing formula. Assume that $\beta_1 = \beta_2 = 1$ and the barrier function $\bar{v}$ is such that $K = L$. Then necessarily $\gamma \ge r$. It can be checked that for $K = L$ the pricing formula reduces to $D(t,T) = D_1(t,T) + D_3(t,T)$, where

$$D_1(t,T) = LB(t,T)\big(N\big(h_1(V_t,T-t)\big) - R_t^{2\hat{a}}N\big(h_2(V_t,T-t)\big)\big),$$

$$D_3(t,T) = V_t\big(R_t^{\theta+\zeta}N\big(h_7(V_t,T-t)\big) + R_t^{\theta-\zeta}N\big(h_8(V_t,T-t)\big)\big).$$

• **Case $\gamma = r$.** If we assume, in addition, that $\gamma = r$ then $\zeta = -\sigma^{-2}\hat{\nu}$ and thus

$$V_t R_t^{\theta+\zeta} = LB(t,T), \quad V_t R_t^{\theta-\zeta} = V_t R_t^{2\hat{a}+1} = LB(t,T) R_t^{2\hat{a}}.$$

It is also easy to see that in this case

$$h_1(V_t, T-t) = \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}} = -h_7(V_t, T-t)$$

and

$$h_2(V_t, T-t) = \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}} = h_8(V_t, T-t).$$

We conclude that if $\bar{v}(t) = Le^{-r(T-t)} = LB(t,T)$ then $D(t,T) = LB(t,T)$. This valuation result is intuitively obvious. Indeed, a corporate bond with the safety covenant represented by the barrier function $\bar{v}$, which is given by the bond's face value discounted at the risk-free rate, is manifestly equivalent to a default-free bond with the same face value and maturity.
• **Case $\gamma > r$.** For $K = L$ and $\gamma > r$, it can be checked that $D(t,T)$ is strictly smaller than $LB(t,T)$. In addition, one can show that when the value of the parameter γ tends to infinity (all other parameters being fixed), then the Black and Cox price of a corporate bond converges to Merton's value of the bond.

1.4.4 Corporate Coupon Bond

We now postulate that the short-term rate $r > 0$ and that a corporate bond, with a fixed maturity date T and the face value L, pays continuously coupons at a constant rate c, so that $A_t = ct$ for every $t \in \mathbb{R}_+$. It is natural to postulate that the coupon payments are discontinued as soon as the default event occurs. Formally, we thus consider here a defaultable claim specified as follows

$$X = L, \ A_t = ct, \ \widetilde{X} = \beta_1 V_T, \ Z = \beta_2 V, \ \tau = \inf\{ t \in [0,T] : V_t < v_t \}$$

with the Black and Cox barrier v. Let us denote by $D_c(t,T)$ the value of such a claim at time $t < T$. It is clear that $D_c(t,T) = D(t,T) + A(t,T)$, where $A(t,T)$ stands for the discounted value of future coupon payments. The value of $A(t,T)$ can be computed as follows

$$A(t,T) = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T ce^{-r(s-t)}\, \mathbb{1}_{\{\bar{\tau}>s\}}\, ds \,\Big|\, \mathcal{F}_t \Big) = ce^{rt} \int_t^T e^{-rs}\, \mathbb{Q}(\bar{\tau} > s \,|\, \mathcal{F}_t)\, ds.$$

Setting $t = 0$, we thus obtain

$$D_c(0,T) = D(0,T) + c\int_0^T e^{-rs}\, \mathbb{Q}(\bar{\tau} > s)\, ds = D(0,T) + A(0,T),$$

where (recall that we write σ instead of σ_V)

$$\mathbb{Q}(\bar{\tau} > s) = N\left(\frac{\ln(V_0/\bar{v}(0)) + \widetilde{\nu}s}{\sigma\sqrt{s}}\right) - \left(\frac{\bar{v}(0)}{V_0}\right)^{2\widetilde{a}} N\left(\frac{\ln(\bar{v}(0)/V_0) + \widetilde{\nu}s}{\sigma\sqrt{s}}\right).$$

An integration by parts formula yields

$$\int_0^T e^{-rs}\,\mathbb{Q}(\bar{\tau} > s)\,ds = \frac{1}{r}\Big(1 - e^{-rT}\mathbb{Q}(\bar{\tau} > T)\; + \int_0^T e^{-rs}\,d\mathbb{Q}(\bar{\tau} > s)\Big).$$

We assume, as usual, that $V_0 > \bar{v}(0)$, so that $\ln(\bar{v}(0)/V_0) < 0$. Arguing in a similar way as in the last part of the proof of Proposition 1.4.1 (specifically, using formula (1.15)), we obtain

$$\begin{aligned}\int_0^T e^{-rs}\,d\mathbb{Q}(\bar{\tau} > s) = &-\left(\frac{\bar{v}(0)}{V_0}\right)^{\widetilde{a}+\widetilde{\zeta}} N\left(\frac{\ln(\bar{v}(0)/V_0) + \widetilde{\zeta}\sigma^2 T}{\sigma\sqrt{T}}\right)\\ &-\left(\frac{\bar{v}(0)}{V_0}\right)^{\widetilde{a}-\widetilde{\zeta}} N\left(\frac{\ln(\bar{v}(0)/V_0) - \widetilde{\zeta}\sigma^2 T}{\sigma\sqrt{T}}\right),\end{aligned}$$

where

$$\widetilde{\nu} = r - \kappa - \gamma - \frac{1}{2}\sigma^2, \quad \widetilde{a} = \widetilde{\nu}\sigma^{-2},$$

and

$$\widetilde{\zeta} = \sigma^{-2}\sqrt{\widetilde{\nu}^2 + 2\sigma^2 r}.$$

Although we have focused on the case when $t = 0$, it is clear that the derivation of the general valuation formula for any $t < T$ hinges on essentially the same arguments. We are thus in the position to state the following result.

Proposition 1.4.2. *Consider a defaultable bond with face value L, which pays continuously coupons at a constant rate c. The price of such a bond equals $D_c(t,T) = D(t,T) + A(t,T)$, where $D(t,T)$ is the value of a defaultable zero-coupon bond given by Proposition 1.4.1 and $A(t,T)$ equals, on the event $\{t < \tau\} = \{t < \bar{\tau}\}$,*

$$\begin{aligned}A(t,T) = \frac{c}{r}\Big\{&1 - B(t,T)\Big(N\big(k_1(V_t, T-t)\big) - R_t^{2\widetilde{a}} N\big(k_2(V_t, T-t)\big)\Big)\\ &- R_t^{\widetilde{a}+\widetilde{\zeta}} N\big(g_1(V_t, T-t)\big) - R_t^{\widetilde{a}-\widetilde{\zeta}} N\big(g_2(V_t, T-t)\big)\Big\},\end{aligned}$$

where $R_t = \bar{v}(t)/V_t$ *and*

$$k_1(V_t, T-t) = \frac{\ln(V_t/\bar{v}(t)) + \widetilde{\nu}(T-t)}{\sigma\sqrt{T-t}},$$
$$k_2(V_t, T-t) = \frac{\ln(\bar{v}(t)/V_t) + \widetilde{\nu}(T-t)}{\sigma\sqrt{T-t}},$$
$$g_1(V_t, T-t) = \frac{\ln(\bar{v}(t)/V_t) + \widetilde{\zeta}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$
$$g_2(V_t, T-t) = \frac{\ln(\bar{v}(t)/V_t) - \widetilde{\zeta}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Some authors apply the general result to the special case when the default triggering barrier is assumed to be a constant level $\bar{v} \geq L$. Note that the firm's insolvency at maturity T is now excluded. In this special case, the coefficient γ equals zero. Consequently, $\widetilde{\nu} = \nu, \widetilde{a} = a = \nu\sigma^{-2}$ and

$$\widetilde{\zeta} = \sigma^{-2}\sqrt{\nu^2 + 2\sigma^2 r} = \zeta.$$

For the sake of the reader's convenience, we state the following immediate corollary to Propositions 1.4.1 and 1.4.2.

Corollary 1.4.1. *Assume that* $\gamma = 0$ *so that the default barrier is constant. Specifically, let* $v_t = \bar{v} \geq L$ *and* $R_t = \bar{v}/V_t$*. Then the price of a defaultable coupon bond equals, on the event* $\{t < \tau\} = \{t < \bar{\tau}\}$,

$$\begin{aligned} D_c(t,T) = \frac{c}{r} + B(t,T)\Big(L - \frac{c}{r}\Big)\Big(N\big(k_1(V_t, T-t)\big) - R_t^{2a} N\big(k_2(V_t, T-t)\big)\Big) \\ + \Big(\beta_2\bar{v} - \frac{c}{r}\Big)\Big(R_t^{a+\zeta} N\big(g_1(V_t, T-t)\big) + R_t^{a-\zeta} N\big(g_2(V_t, T-t)\big)\Big). \end{aligned}$$

It is worth mentioning that the valuation formula of Corollary 1.4.1 coincides with expression (3) in Leland and Toft [136]. Letting the bond maturity T tend to infinity, we obtain the following representation of the price of a *consol bond* (that is, a *perpetual* coupon bond with infinite maturity)

$$D_c(t) = D_c(t, \infty) = \frac{c}{r}\left(1 - \left(\frac{\bar{v}}{V_t}\right)^{a+\zeta}\right) + \beta_2\bar{v}\left(\frac{\bar{v}}{V_t}\right)^{a+\zeta}. \tag{1.16}$$

1.4.5 Optimal Capital Structure

Following Black and Cox [28], we will now present an example of an analysis of the optimal capital structure of a firm. Let us consider a firm that has an interest paying bonds outstanding. We assume that it is a consol bond, which pays continuously coupon rate c, and we postulate, in addition, that $r > 0$ and the payout rate κ is equal to zero.

The condition $\kappa = 0$ can be given the financial interpretation as the restriction on the sale of assets, as opposed to issuing of new equity. Equivalently, we may think about a situation in which the stockholders will make payments to the firm to cover the interest payments. However, they have the right to stop making payments at any time and either turn the firm over to the bondholders or pay them a lump payment of c/r per unit of the bond's notional amount.

Recall that we denote by $E(V_t)$ ($D(V_t)$, respectively) the value at time t of the firm equity (debt, respectively), hence the total value of the firm's assets satisfies $V_t = E(V_t) + D(V_t)$. Black and Cox [28] argue that there is a critical level of the value of the firm, denoted as v^*, below which no more equity can be sold. The critical value v^* will be chosen by stockholders, whose aim is to minimize the value of the bonds (equivalently, to maximize the value of the equity). Let us observe that v^* is nothing else than a constant default barrier in the problem under consideration. The optimal default time τ^* is thus assumed to be given by the formula (see, however, Décamps and Villeneuve [65] for a critique of this postulate in the context of Leland's [135] model)

$$\tau^* = \inf \{ t \in \mathbb{R}_+ : V_t \leq v^* \}.$$

To find the critical value v^*, let us first fix the bankruptcy level $\bar{v}$. Then the ODE for the pricing function $u^\infty = u^\infty(V)$ of a consol bond takes the following form (recall that $\sigma = \sigma_V$)

$$\frac{1}{2} V^2 \sigma^2 u^\infty_{VV} + rV u^\infty_V + c - r u^\infty = 0,$$

subject to the lower boundary condition $u^\infty(\bar{v}) = \min(\bar{v}, c/r)$ and the upper boundary condition

$$\lim_{V \to \infty} u^\infty_V(V) = 0.$$

For the last condition, observe that when the firm's value grows to infinity, the possibility of default becomes meaningless, so that the value of the defaultable consol bond tends to the value c/r of the default-free consol bond. The general solution to our problem has the following form

$$u^\infty(V) = \frac{c}{r} + K_1 V + K_2 V^{-\alpha},$$

where we denote $\alpha = 2r/\sigma^2$ and K_1, K_2 are some constants, to be determined from boundary conditions. We find easily that $K_1 = 0$ and

$$K_2 = \begin{cases} \bar{v}^{\alpha+1} - (c/r)\bar{v}^\alpha, & \text{if } \bar{v} < c/r, \\ 0, & \text{if } \bar{v} \geq c/r. \end{cases}$$

This means that the bond price equals (cf. (1.16))

$$u^{\infty}(V_t) = \frac{c}{r}\left(1 - \left(\frac{\bar{v}}{V_t}\right)^{\alpha}\right) + \bar{v}\left(\frac{\bar{v}}{V_t}\right)^{\alpha}.$$

It is in the interest of the stockholders to select the bankruptcy level in such a way that the value of the debt, represented here by $D(V_t) = u^{\infty}(V_t)$, is minimized, so that the value of the firm's equity

$$E(V_t) = V_t - D(V_t) = V_t - \frac{c}{r}(1 - \bar{q}_t) - \bar{v}\bar{q}_t$$

is maximized. It is not difficult to check that the optimal level of the barrier does not depend on the current value of the firm, and it equals

$$v^* = \frac{c}{r}\frac{\alpha}{\alpha + 1} = \frac{c}{r + \sigma^2/2}.$$

Given the optimal strategy of the stockholders, the price process of the firm's debt (i.e., of a consol bond) takes the form, on the event $\{\tau^* > t\}$,

$$D^*(V_t) = \frac{c}{r} - \frac{1}{\alpha V_t^{\alpha}}\left(\frac{c}{r + \sigma^2/2}\right)^{\alpha+1} = \frac{c}{r}(1 - q_t^*) + v^* q_t^*,$$

where

$$q_t^* = \left(\frac{v^*}{V_t}\right)^{\alpha} = \frac{1}{V_t^{\alpha}}\left(\frac{c}{r + \sigma^2/2}\right)^{\alpha}.$$

For other important developments in the area of the optimal capital structure, we refer to Leland [135], Leland and Toft [136], Christensen et al. [57], and Décamps and Villeneuve [65]. Chen and Kou [53] and Hilberink and Rogers [97] study analogous problems, but they model the firm's value as a diffusion process with jumps. This extension is aimed to eliminate an undesirable feature of a typical structural model that the credit spread for a corporate bond converges to zero for short maturities.

1.5 Extensions of the Black and Cox Model

The Black and Cox first-passage-time approach was later developed by, among others: Brennan and Schwartz [35, 36] – an analysis of convertible bonds, Nielsen et al. [148] – a random barrier and random interest rates, Leland [135], Leland and Toft [136] – a study of an optimal capital structure, bankruptcy costs and tax benefits, Longstaff and Schwartz [138] – a constant barrier combined with random interest rates, Fouque et al. [84, 85] – a stochastic volatility model and its extension to a multi-name case.

In general, the default time can be given as

$$\tau = \inf \{ t \in \mathbb{R}_+ : V_t \leq v(t) \},$$

where $v : \mathbb{R}_+ \to \mathbb{R}$ is an arbitrary function and the value of the firm V is modeled as a geometric Brownian motion.

Moraux [144] (see also Çetin et al. [51] and Yıldırım [172] for extensions) proposes to model the default time as a *Parisian stopping time*. For a continuous process V and any $t > 0$, we define the random variable $g_t^b(V)$ representing the last moment before t when the process V was at a level b by setting

$$g_t^b(V) = \sup \{ 0 \leq s \leq t : V_s = b \}.$$

The *Parisian stopping time* is the first time at which the process V is below the level b for a time period of length greater or equal to a constant α. Formally, the default time τ is given by the formula

$$\tau = \inf \{ t \in \mathbb{R}_+ : (t - g_t^b(V)) \mathbb{1}_{\{V_t < b\}} \geq \alpha \}.$$

In the case of the process V governed by the Black and Scholes dynamics, it is possible to find the joint probability distribution of (τ, V_τ) by means of the Laplace transform. Another plausible choice for the default time is the first moment when the process V has spent more than α units of time below a predetermined level, that is,

$$\tau = \inf \{ t \in \mathbb{R}_+ : A_t^V > \alpha \},$$

where we denote $A_t^V = \int_0^t \mathbb{1}_{\{V_u < b\}} \, du$. The probability distribution of this random time is related to the so-called *cumulative options*.

Campi and Sbuelz [47] assume that the default time is given by the first hitting time of 0 by the CEV process and they study the difficult problem of pricing an equity default swap. More precisely, they assume that the dynamics under $\mathbb{Q}$ of the firm's value are

$$dV_t = V_{t-} \Big((r - \kappa) \, dt + \sigma V_t^\beta \, dW_t - dM_t \Big),$$

where W is a Brownian motion and M the compensated martingale of a Poisson process (i.e., $M_t = N_t - \lambda t$), and they set $\tau = \inf \{ t \in \mathbb{R}_+ : V_t \leq 0 \}$. Put another way, Campi and Sbuelz [47] define the default time by setting $\tau = \tau^\beta \wedge \tau^N$, where τ^N is the first jump of the Poisson process and τ^β is defined as $\tau^\beta = \inf \{ t \in \mathbb{R}_+ : X_t \leq 0 \}$, where in turn the process X obeys the following SDE

$$dX_t = X_{t-} \Big((r - \kappa + \lambda) \, dt + \sigma X_t^\beta \, dW_t \Big).$$

Using the well-known fact that the CEV process can be expressed in terms of a time-changed Bessel process and results on the hitting time of zero for a Bessel process of dimension smaller than 2, they obtain closed-form solutions (see also Campi et al. [48] and Carr and Linetsky [50]).

Zhou [176] examines the case where the dynamics under $\mathbb{Q}$ of the firm are

$$dV_t = V_{t-}\Big(\big(r - \lambda\nu\big)\,dt + \sigma\,dW_t + dX_t\Big),$$

where W is a standard Brownian motion and X is a compound Poisson process. Specifically, we set $X_t = \sum_{i=1}^{N_t}\big(e^{Y_i} - 1\big)$, where N is a Poisson process with a constant intensity λ, random variables Y_i are independent and have the Gaussian distribution $N(a, b^2)$. We also set $\nu = \exp(a + b^2/2) - 1$, since for this choice of ν the process $V_t e^{-rt}$ is a martingale. Zhou [176] studies the Merton problem in this setup and gives an approximation for the first passage time problem.

1.5.1 Stochastic Interest Rates

In this section, we present a generalization of the Black and Cox valuation formula for a corporate bond to the case of random interest rates. We assume that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$, supports the short-term interest rate process r and the value process V. The dynamics under the martingale measure $\mathbb{Q}$ of the firm's value and of the price of a default-free zero-coupon bond $B(t,T)$ are

$$dV_t = V_t\big((r_t - \kappa(t))\,dt + \sigma(t)\,dW_t\big)$$

and

$$dB(t,T) = B(t,T)\big(r_t\,dt + b(t,T)\,dW_t\big)$$

respectively, where W is a d-dimensional standard $\mathbb{Q}$-Brownian motion. Furthermore, $\kappa : [0,T] \to \mathbb{R}$, $\sigma : [0,T] \to \mathbb{R}^d$ and $b(\cdot,T) : [0,T] \to \mathbb{R}^d$ are assumed to be bounded functions. The *forward value* $F_V(t,T) = V_t/B(t,T)$ of the firm satisfies under the *forward martingale measure* $\mathbb{Q}_T$

$$dF_V(t,T) = -\kappa(t)F_V(t,T)\,dt + F_V(t,T)\big(\sigma(t) - b(t,T)\big)\,dW_t^T,$$

where the process $W_t^T = W_t - \int_0^t b(u,T)\,du$, $t \in [0,T]$, is a d-dimensional Brownian motion under $\mathbb{Q}_T$. We set, for any $t \in [0,T]$,

$$F_V^\kappa(t,T) = F_V(t,T)e^{-\int_t^T \kappa(u)\,du}.$$

Then

$$dF_V^\kappa(t,T) = F_V^\kappa(t,T)\big(\sigma(t) - b(t,T)\big)\,dW_t^T.$$

Furthermore, it is apparent that $F_V^\kappa(T,T) = F_V(T,T) = V_T$. We consider the following modification of the Black and Cox approach

$$X = L,\ Z_t = \beta_2 V_t,\ \tilde{X} = \beta_1 V_T,\ \tau = \inf\{\, t \in [0,T] : V_t < v_t\},$$

where $\beta_1,\ \beta_2 \in [0,1]$ are constants and the barrier v is given by the formula

$$v_t = \begin{cases} KB(t,T)e^{\int_t^T \kappa(u)\,du}, & \text{for } \ t < T, \\ L, & \text{for } \ t = T, \end{cases}$$

with the constant K satisfying $0 < K \le L$. Let us denote, for any $t \le T$,

$$\kappa(t,T) = \int_t^T \kappa(u)\,du, \quad \sigma^2(t,T) = \int_t^T |\sigma(u) - b(u,T)|^2\,du,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$. For brevity, we write $F_t = F_V^\kappa(t,T)$ and we denote

$$\eta_\pm(t,T) = \kappa(t,T) \pm \frac{1}{2}\sigma^2(t,T).$$

The following result was established in Rutkowski [155].

Proposition 1.5.1. *The forward price $F_D(t,T) = D(t,T)/B(t,T)$ of the defaultable bond equals, for every $t \in [0,T[$ on the event $\{\tau > t\}$,*

$$\begin{aligned} & L\big(N\big(\widehat{h}_1(F_t,t,T)\big) - (F_t/K)e^{-\kappa(t,T)}N\big(\widehat{h}_2(F_t,t,T)\big)\big) \\ & \quad + \beta_1 F_t e^{-\kappa(t,T)}\big(N\big(\widehat{h}_3(F_t,t,T)\big) - N\big(\widehat{h}_4(F_t,t,T)\big)\big) \\ & \quad + \beta_1 K\big(N\big(\widehat{h}_5(F_t,t,T)\big) - N\big(\widehat{h}_6(F_t,t,T)\big)\big) \\ & \quad + \beta_2 K J_+(F_t,t,T) + \beta_2 F_t e^{-\kappa(t,T)} J_-(F_t,t,T), \end{aligned}$$

where

$$\begin{aligned} \widehat{h}_1(F_t,t,T) &= \frac{\ln(F_t/L) - \eta_+(t,T)}{\sigma(t,T)}, \\ \widehat{h}_2(F_t,T,t) &= \frac{2\ln K - \ln(LF_t) + \eta_-(t,T)}{\sigma(t,T)}, \\ \widehat{h}_3(F_t,t,T) &= \frac{\ln(L/F_t) + \eta_-(t,T)}{\sigma(t,T)}, \\ \widehat{h}_4(F_t,t,T) &= \frac{\ln(K/F_t) + \eta_-(t,T)}{\sigma(t,T)}, \\ \widehat{h}_5(F_t,t,T) &= \frac{2\ln K - \ln(LF_t) + \eta_+(t,T)}{\sigma(t,T)}, \\ \widehat{h}_6(F_t,t,T) &= \frac{\ln(K/F_t) + \eta_+(t,T)}{\sigma(t,T)}, \end{aligned}$$

and where we write, for $F_t > 0$ *and* $t \in [0, T[$,

$$J_\pm(F_t, t, T) = \int_t^T e^{\kappa(u,T)}\, dN\left(\frac{\ln(K/F_t) + \kappa(t,T) \pm \frac{1}{2}\sigma^2(t,u)}{\sigma(t,u)}\right).$$

In the special case when $\kappa = 0$, the formula of Proposition 1.5.1 covers as a special case the valuation result established by Briys and de Varenne [44]. In some other recent studies of first passage time models, in which the triggering barrier is assumed to be either a constant or an unspecified stochastic process, typically no closed-form solution for the value of a corporate debt is available and thus a numerical approach is required (see, for instance, Longstaff and Schwartz [138], Nielsen et al. [148], or Saá-Requejo and Santa-Clara [158]).

1.6 Random Barrier

In the case of the full information and the Brownian filtration, the first hitting time of a deterministic barrier is a predictable stopping time. This is no longer the case when we deal with an incomplete information (as, e.g., in Duffie and Lando [73]), or when an additional source of randomness is present. We present here a formula for credit spreads arising in a special case of a totally inaccessible time of default. For a more detailed study we refer to Babbs and Bielecki [9] and Giesecke [92]. As we shall see, the method used here is in fact fairly close to the general method presented in Chapter 3. We now postulate that the barrier which triggers default is represented by a random variable η defined on the underlying probability space. The default time τ is given as $\tau = \inf\{\, t \in \mathbb{R}_+ : V_t \le \eta\}$, where V is the value of the firm and, for simplicity, $V_0 = 1$. Note that $\{t < \tau\} = \{\inf_{u \le t} V_u > \eta\}$. We shall denote by m^V the *running minimum* of the continuous process V, that is, $m^V_t = \inf_{u \le t} V_u$. With this notation, we have that $\{\tau > t\} = \{m^V_t > \eta\}$. Note that m^V is manifestly a decreasing, continuous process.

1.6.1 Independent Barrier

We assume that, under the risk-neutral probability $\mathbb{Q}$, a random variable η modeling the barrier is independent of the value of the firm. We denote by F_η the cumulative distribution function of η, that is, $F_\eta(z) = \mathbb{Q}(\eta \le z)$. We assume that F_η is a differentiable function and we denote by f_η its derivative (with $f_\eta(z) = 0$ for $z > V_0$).

Lemma 1.6.1. *Let us set* $F_t = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t)$ *and* $\Gamma_t = -\ln(1 - F_t)$. *Then*

$$\Gamma_t = -\int_0^t \frac{f_\eta(m^V_u)}{F_\eta(m^V_u)}\, dm^V_u.$$

Proof. If a random variable η is independent of $\mathcal{F}_\infty$ then

$$F_t = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{Q}(m_t^V \le \eta \,|\, \mathcal{F}_t) = 1 - F_\eta(m_t^V).$$

The process m^V is decreasing and thus $\Gamma_t = -\ln F_\eta(m_t^V)$. We conclude that

$$\Gamma_t = -\int_0^t \frac{f_\eta(m_u^V)}{F_\eta(m_u^V)}\, dm_u^V,$$

as required. □

Let us postulate, in addition, that the value process V is modeled by a geometric Brownian motion with a drift. Specifically, we set $V_t = e^{X_t}$, where $X_t = \mu t + \sigma W_t$. It is clear that $\tau = \inf\{\, t \in \mathbb{R}_+ : m_t^X \le \psi\}$, where $\psi = \ln \eta$ and m^X is the running minimum of the process X, that is, $m_t^X = \inf\{\, X_s : 0 \le s \le t\}$. We choose the Brownian filtration as the reference filtration, that is, we set $\mathbb{F} = \mathbb{F}^W$. This means that we assume that the value of the firm process V (hence also the process X) is perfectly observed. The barrier ψ is not observed, however. We only postulate that an investor can observe the occurrence of the default time. In other words, he can observe the process $H_t = \mathbb{1}_{\{t \ge \tau\}} = \mathbb{1}_{\{m_t^X \le \psi\}}$. We denote by $\mathbb{H}$ the natural filtration of the process H. The information available to the investor is thus represented by the joint filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.

If the default time τ and interest rates are independent under $\mathbb{Q}$ then it is possible to establish the following result (for the proof, the interested reader is referred to Giesecke [92] or Babbs and Bielecki [9]).

Proposition 1.6.1. *Under the assumptions stated above, we deal with a unit corporate bond with zero recovery. Then the credit spread $S(t,T)$ is given as, for every $t \in [0,T[$,*

$$S(t,T) = -\mathbb{1}_{\{t<\tau\}} \frac{1}{T-t} \ln \mathbb{E}_{\mathbb{Q}} \Big\{ \exp\Big(\int_t^T \frac{f_\psi(m_u^X)}{F_\psi(m_u^X)}\, dm_u^X \Big) \,\Big|\, \mathcal{F}_t \Big\}.$$

Note that the process m^X is decreasing, so that the stochastic integral with respect to this process can be interpreted as a pathwise Stieltjes integral. In Chapter 3, we will examine the notion of a hazard process of a random time with respect to a reference filtration $\mathbb{F}$. It is thus worth mentioning that for the default time τ defined above, the $\mathbb{F}$-hazard process Γ exists and it is given by the formula

$$\Gamma_t = -\int_0^t \frac{f_\psi(m_u^X)}{F_\psi(m_u^X)}\, dm_u^X.$$

Since this process is manifestly continuous, the default time τ is in fact a totally inaccessible stopping time with respect to the filtration $\mathbb{G}$.

Chapter 2

Hazard Function Approach

The goal of this chapter is to provide a detailed analysis of a relatively simple case of the *reduced-form* methodology, when the information flow available to an agent is reduced to the observations of the random time representing the default event of some credit name. The emphasis is put on the evaluation of conditional expectations with respect to the filtration generated by a default time with the use of the hazard function. We also study hedging strategies based on credit default swaps in the single-name setup and in the case of several credit names. We conclude this chapter by dealing with examples of copula-based credit risk models with several default times.

2.1 Elementary Market Model

We begin with the simple case where risk-free zero-coupon bonds, driven by a deterministic short-term interest rate $(r(t),\, t \in \mathbb{R}_+)$, are the only traded assets in the default-free market model. Recall that in that case the price at time t of the *risk-free zero-coupon bond* with maturity T equals

$$B(t,T) = \exp\Big(- \int_t^T r(u)\, du \Big) = \frac{B(t)}{B(T)},$$

where $B(t) = \exp\Big(\int_0^t r(u)\, du \Big)$ is the value at time t of the *savings account*.

Definition 2.1.1. By the *default time* τ we mean an arbitrary positive random variable defined on some underlying probability space $(\Omega, \mathcal{G}, \mathbb{Q})$.

Let F be the cumulative distribution function of a random variable τ so that

$$F(t) = \mathbb{Q}(\tau \le t) = \int_0^t f(u)\, du,$$

where the second equality holds provided that the distribution of the random time τ admits the probability density function f.

It is assumed throughout that the inequality $F(t) < 1$ holds for every $t \in \mathbb{R}_+$. Otherwise, there would exist a finite date t_0 for which $F(t_0) = 1$, so that the default event would occur either before or at t_0 with probability 1.

We emphasize that the random payoff of the form $\mathbb{1}_{\{T<\tau\}}$ cannot be perfectly hedged with deterministic zero-coupon bonds, which are the only traded primary assets in our elementary market model. To hedge the default risk, we shall later postulate that some defaultable assets are traded, e.g., a defaultable zero-coupon bond or a credit default swap. In the first step, we will postulate that the "fair value" of a defaultable asset is given by the risk-neutral valuation formula with respect to $\mathbb{Q}$. Let us note in this regard that in practice the risk-neutral distribution of default time is inferred from market quotes of traded defaultable assets, rather than postulated a priori.

2.1.1 Hazard Function and Hazard Rate

Recall the standing assumption that $F(t) < 1$ for every $t \in \mathbb{R}_+$.

Definition 2.1.2. The *hazard function* $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ of τ is given by the formula, for every $t \in \mathbb{R}_+$,

$$\Gamma(t) = -\ln(1 - F(t)).$$

Note that Γ is a non-decreasing function with the initial value $\Gamma(0) = 0$ and with the limit $\lim_{t\to+\infty} \Gamma(t) = +\infty$. The following elementary result is easy to prove.

Lemma 2.1.1. *If the cumulative distribution function F is absolutely continuous with respect to the Lebesgue measure, so that $F(t) = \int_0^t f(u)\, du$ where f is the probability density function of τ, then the hazard function Γ is absolutely continuous as well. Specifically, $\Gamma(t) = \int_0^t \gamma(u)\, du$ where $\gamma(t) = f(t)(1 - F(t))^{-1}$ for every $t \in \mathbb{R}_+$.*

The function γ is called the *hazard rate* or the *intensity function* of default time τ. When τ admits the hazard rate γ, we have that, for every $t \in \mathbb{R}_+$,

$$\mathbb{Q}(\tau > t) = 1 - F(t) = e^{-\Gamma(t)} = \exp\Big(-\int_0^t \gamma(u)\, du\Big).$$

The interpretation of the hazard rate is that it represents the conditional probability of the occurrence of default in a small time interval $[t, t + dt]$, given that default has not occurred by time t. More formally, for almost every $t \in \mathbb{R}_+$,

$$\gamma(t) = \lim_{h\to 0} \frac{1}{h}\, \mathbb{Q}(t < \tau \le t + h \,|\, \tau > t).$$

Remark 2.1.1. Let τ be the moment of the first jump of an inhomogeneous Poisson process with a deterministic intensity $(\lambda(t),\, t \in \mathbb{R}_+)$. It is then well known that the probability density function of τ equals

$$f(t) = \frac{\mathbb{Q}(\tau \in dt)}{dt} = \lambda(t) \exp\left(-\int_0^t \lambda(u)\, du\right) = \lambda(t) e^{-\Lambda(t)},$$

where $\Lambda(t) = \int_0^t \lambda(u)\, du$ and thus $F(t) = \mathbb{Q}(\tau \le t) = 1 - e^{-\Lambda(t)}$. The hazard function Γ is thus equal to the *compensator* of the Poisson process, that is, $\Gamma(t) = \Lambda(t)$ for every $t \in \mathbb{R}_+$. In other words, the compensated Poisson process $N_t - \Gamma(t) = N_t - \Lambda(t)$ is a martingale with respect to the filtration generated by the Poisson process N.

Conversely, if τ is a random time with the probability density function f, setting $\Lambda(t) = -\ln(1 - F(t))$ allows us to interpret τ as the moment of the first jump of an inhomogeneous Poisson process with the intensity function equal to the derivative of Λ.

Remark 2.1.2. It is not difficult to generalize the study presented in what follows to the case where τ does not admit a density, by dealing with the right-continuous version of the cumulative function. The case where τ is bounded can also be studied along the same method.

2.1.2 Defaultable Bond with Recovery at Maturity

We denote by $H = (H_t,\, t \in \mathbb{R}_+)$ the right-continuous increasing process $H_t = \mathbb{1}_{\{t \ge \tau\}}$, referred to as the *default indicator process*. Let $\mathbb{H}$ stand for the natural filtration of the process H. It is clear that the filtration $\mathbb{H}$ is the smallest filtration which makes τ a stopping time. More explicitly, for any $t \in \mathbb{R}_+$, the σ-field $\mathcal{H}_t$ is generated by the events $\{s \ge \tau\}$ for $s \le t$. The key observation is that any $\mathcal{H}_t$-measurable random variable X has the form

$$X = h(\tau)\mathbb{1}_{\{t \ge \tau\}} + c\mathbb{1}_{\{t < \tau\}},$$

where $h : \mathbb{R}_+ \to \mathbb{R}$ is a Borel measurable function and c is a constant.

Remark 2.1.3. It is worth mentioning that if the cumulative distribution function F is continuous then the random time τ is known to be a totally inaccessible stopping time with respect to $\mathbb{H}$ (see, e.g., Dellacherie [66] or Dellacherie and Meyer [69], Page 107). We are not going to use explicitly this important property in what follows, however.

Our next goal is to derive some useful valuation formulae for corporate bonds with various conventions regarding recovery schemes in the case of default prior to maturity.

For the sake of simplicity, we will first assume that a bond is represented by a single payoff at its maturity T. Therefore, it is possible to value a bond as a European contingent claim X maturing at T, by applying the standard *risk-neutral valuation formula*

$$\pi_t(X) = B(t)\,\mathbb{E}_{\mathbb{Q}}\left(\frac{X}{B(T)}\,\Big|\,\mathcal{H}_t\right) = B(t,T)\,\mathbb{E}_{\mathbb{Q}}(X\,|\,\mathcal{H}_t).$$

For the ease of notation, we will consider, without loss of generality, a defaultable bond with the face value $L = 1$. It is worth recalling that the corporate bond price computed within the structural approach was not proportional to its face value L. This feature is manifestly true for the bond valuation formulae obtained within the reduced-form approach.

Constant Recovery at Maturity

A unit *defaultable zero-coupon bond* (DZC) with maturity T and recovery value δ paid at maturity, is represented by the following cash flows:

- the payment of one monetary unit at time T if default has not occurred before T, i.e., if $\tau > T$,
- the payment of δ monetary units, made at maturity, if $\tau \le T$, where $\delta \in [0,1]$ is a constant.

The price at time 0 of the defaultable zero-coupon bond is formally defined as the expectation under $\mathbb{Q}$ of the discounted payoff, so that

$$D^{\delta}(0,T) = B(0,T)\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{T<\tau\}} + \delta\mathbb{1}_{\{\tau\le T\}}\big).$$

Consequently,

$$D^{\delta}(0,T) = B(0,T) - (1-\delta)B(0,T)F(T).$$

The value of the defaultable zero-coupon bond is thus equal to the value of the default-free zero-coupon bond minus the discounted value of the expected loss computed under the risk-neutral probability. Of course, for $\delta = 1$ we recover, as expected, the price of a default-free zero-coupon bond. Obviously, the price defined above is not a *hedging price*, since the payoff at maturity of the defaultable bond cannot be replicated by trading in primary assets; recall that only default-free zero-coupon bonds are traded in the present setup. Therefore, we deal with an incomplete market model and the risk-neutral pricing formula for the defaultable zero-coupon bond is thus postulated, rather than derived from replication.

The value of the bond at any date $t \in [0,T]$ depends whether or not default has happened before this time.

On the one hand, if default has occurred before or at time t, the constant payment of δ will surely be made at maturity date T and thus the price of the DZC is obviously equal to $\delta B(t,T)$.

On the other hand, if default has not yet occurred before or at time t, the date of its occurrence is uncertain. It is thus natural in this situation to define the *ex-dividend price* $D^\delta(t,T)$ at time $t \in [0,T[$ of the DZC maturing at T as the conditional expectation under $\mathbb{Q}$ of the discounted payoff

$$B(t,T)\left(\mathbb{1}_{\{T<\tau\}} + \delta \mathbb{1}_{\{\tau \le T\}}\right), \tag{2.1}$$

given the information, which is available at time t, that is, given the no-default event $\{\tau > t\}$.

In view of specification (2.1) of the bond's payoff, we thus obtain

$$D^\delta(t,T) = \mathbb{1}_{\{t\ge\tau\}}\delta B(t,T) + \mathbb{1}_{\{t<\tau\}}\widetilde{D}^\delta(t,T),$$

where the *pre-default value* $\widetilde{D}^\delta(t,T)$, $t \in [0,T]$, is defined as

$$\widetilde{D}^\delta(t,T) = \mathbb{E}_{\mathbb{Q}}\big(B(t,T)\,(\mathbb{1}_{\{T<\tau\}} + \delta\mathbb{1}_{\{\tau\le T\}})\,\big|\, t<\tau\big).$$

To compute $\widetilde{D}^\delta(t,T)$, we observe that

$$\begin{aligned}
\widetilde{D}^\delta(t,T) &= B(t,T)\Big(1-(1-\delta)\mathbb{Q}(\tau \le T \,|\, t<\tau)\Big) \\
&= B(t,T)\left(1-(1-\delta)\frac{\mathbb{Q}(t<\tau\le T)}{\mathbb{Q}(t<\tau)}\right) \\
&= B(t,T)\left(1-(1-\delta)\frac{G(t)-G(T)}{G(t)}\right),
\end{aligned} \tag{2.2}$$

where $G(t) = 1 - F(t)$ is the *survival function.* Let us define, for every $t \in [0,T]$,

$$B^\gamma(t,T) = B(t,T)\frac{G(T)}{G(t)} = \exp\Big(-\int_t^T (r(u)+\gamma(u))\,du\Big).$$

Then pre-default value of the bond can be represented as follows

$$\widetilde{D}^\delta(t,T) = B^\gamma(t,T) + \delta\big(B(t,T) - B^\gamma(t,T)\big).$$

In particular, for $\delta = 0$, that is, for the bond with *zero recovery*, we obtain the equality $\widetilde{D}^0(t,T) = B^\gamma(t,T)$, and thus the price $D^0(t,T)$ satisfies

$$D^0(t,T) = \mathbb{1}_{\{t<\tau\}}\widetilde{D}^0(t,T) = \mathbb{1}_{\{t<\tau\}}B^\gamma(t,T).$$

It is worth noting that the value of the DZC is discontinuous at default time τ since we have, on the event $\{\tau \le T\}$,

$$D^{\delta}(\tau,T) - D^{\delta}(\tau-,T) = \delta B(\tau,T) - \widetilde{D}^{\delta}(\tau,T) = (\delta-1)B^{\gamma}(t,T) < 0,$$

where the last inequality holds for any $\delta < 1$. Recall that for $\delta = 1$ the DZC is simply a default-free zero-coupon bond.

For practical purposes, equality (2.2) can be rewritten as follows

$$\widetilde{D}^{\delta}(t,T) = B(t,T)(1 - \text{LGD } \times \text{DP}),$$

where the *loss given default* (LGD) is defined as $1-\delta$ and the conditional *default probability* (DP) is given by the formula

$$\text{DP} = \frac{\mathbb{Q}(t<\tau\le T)}{\mathbb{Q}(t<\tau)} = \mathbb{Q}(\tau \le T \,|\, t<\tau).$$

If the hazard rate $\gamma \ge 0$ is constant then the *pre-default credit spread* equals

$$\widetilde{S}(t,T) = \frac{1}{T-t}\ln\frac{B(t,T)}{\widetilde{D}^{\delta}(t,T)} = \gamma - \frac{1}{T-t}\ln\Big(1+\delta(e^{\gamma(T-t)}-1)\Big).$$

It is thus easily seen that the pre-default credit spread converges to the constant $\gamma(1-\delta)$ when time to maturity $T-t$ tends to zero. It is thus strictly positive when $\gamma > 0$ and $0 \le \delta < 1$.

Recall that for $\delta = 0$, the equality $\widetilde{D}^{0}(t,T) = B^{\gamma}(t,T)$ is valid. Hence the short-term interest rate has simply to be adjusted by adding the credit spread (equal here to γ) in order to price the DZC with zero recovery using the formula for default-free bonds. The *default-risk-adjusted interest rate* equals $\widehat{r} = r + \gamma$ and thus it is higher than the risk-free interest rate r if γ is positive. This corresponds to the real-life feature that the value of a DZC with zero recovery is strictly smaller than the value of a default-free zero-coupon with the same par value and maturity provided, of course, that the real-life probability of default event during the bond's lifetime is positive.

General Recovery at Maturity

Let us now assume that the payment is a deterministic function of the default time, denoted as $\delta : \mathbb{R}_+ \to \mathbb{R}_+$. Then the value at time 0 of this defaultable zero-coupon is

$$D^{\delta}(0,T) = B(0,T)\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{T<\tau\}} + \delta(\tau)\mathbb{1}_{\{\tau\le T\}}\big)$$

or, more explicitly,

$$D^{\delta}(0,T) = B(0,T)\Big(G(T) + \int_0^T \delta(s)f(s)\,ds\Big),$$

where, as before, $G(t) = 1 - F(t)$ stands for the survival probability. More generally, the ex-dividend price is given by the formula, for every $t \in [0,T[$,

$$D^{\delta}(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{T<\tau\}} + \delta(\tau)\mathbb{1}_{\{\tau\le T\}}\,\big|\,\mathcal{H}_t\big).$$

The following result furnishes an explicit representation for the bond's price in the present setup.

Lemma 2.1.2. *The price of the bond satisfies, for every* $t \in [0,T[$,

$$D^{\delta}(t,T) = \mathbb{1}_{\{t<\tau\}}\,\widetilde{D}^{\delta}(t,T) + \mathbb{1}_{\{t\ge\tau\}}\,\delta(\tau)B(t,T), \tag{2.3}$$

where the pre-default value $\widetilde{D}^{\delta}(t,T)$ *equals*

$$\begin{aligned}
\widetilde{D}^{\delta}(t,T) &= B(t,T)\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{T<\tau\}} + \delta(\tau)\mathbb{1}_{\{\tau\le T\}}\,\big|\,t<\tau\big)\\
&= B(t,T)\frac{G(T)}{G(t)} + \frac{B(t,T)}{G(t)}\int_t^T \delta(u)f(u)\,du\\
&= B^{\gamma}(t,T) + \frac{B^{\gamma}(t,T)}{G(T)}\int_t^T \delta(u)f(u)\,du\\
&= B^{\gamma}(t,T) + B^{\gamma}(t,T)\int_t^T \delta(u)\gamma(u)e^{\int_u^T \gamma(v)\,dv}\,du.
\end{aligned}$$

The dynamics of the process $(\widetilde{D}^{\delta}(t,T),\, t\in[0,T])$ *are*

$$d\widetilde{D}^{\delta}(t,T) = (r(t)+\gamma(t))\widetilde{D}^{\delta}(t,T)\,dt - B(t,T)\gamma(t)\delta(t)\,dt. \tag{2.4}$$

The proof of the lemma is based on straightforward computations. To derive the dynamics of $\widetilde{D}^{\delta}(t,T)$, it is useful to observe, in particular, that

$$dB^{\gamma}(t,T) = (r(t)+\gamma(t))B^{\gamma}(t,T)\,dt.$$

The risk-neutral dynamics of the discontinuous process $D^{\delta}(t,T)$ involve also the $\mathbb{H}$-martingale M introduced in Section 2.2 below (see Example 2.2.2).

2.1.3 Defaultable Bond with Recovery at Default

Let us now consider a corporate bond with recovery at default. A holder of a defaultable zero-coupon bond with maturity T is now entitled to:

- the payment of one monetary unit at time T if default has not yet occurred,
- the payment of $\delta(\tau)$ monetary units, where δ is a deterministic function; note that this payment is made at time τ if $\tau \le T$.

The price at time 0 of this defaultable zero-coupon bond is

$$\begin{aligned} D^\delta(0,T) &= \mathbb{E}_\mathbb{Q}\big(B(0,T)\,\mathbb{1}_{\{T<\tau\}} + B(0,\tau)\delta(\tau)\mathbb{1}_{\{\tau\le T\}}\big) \\ &= \mathbb{Q}(T<\tau)B(0,T) + \int_0^T B(0,u)\delta(u)\,dF(u) \\ &= G(T)B(0,T) + \int_0^T B(0,u)\delta(u)f(u)\,du. \end{aligned}$$

Obviously, if the default has occurred before time t, the value of the DZC is null (this was not the case for the recovery payment made at the bond's maturity) since, unless explicitly stated otherwise, we adopt throughout the ex-dividend price convention for all assets.

Lemma 2.1.3. *The price of the bond satisfies, for every $t \in [0,T[$,*

$$D^\delta(t,T) = \mathbb{1}_{\{t<\tau\}}\widetilde{D}^\delta(t,T), \tag{2.5}$$

where the pre-default value $\widetilde{D}^\delta(t,T)$ equals

$$\begin{aligned} \widetilde{D}^\delta(t,T) &= \mathbb{E}_\mathbb{Q}\big(B(t,T)\,\mathbb{1}_{\{T<\tau\}} + B(t,\tau)\delta(\tau)\mathbb{1}_{\{\tau\le T\}} \,\big|\, t<\tau\big) \\ &= B(t,T)\frac{G(T)}{G(t)} + \frac{1}{G(t)}\int_t^T B(t,u)\delta(u)\,dF(u) \\ &= B^\gamma(t,T) + \frac{1}{G(t)}\int_t^T B(t,u)\delta(u)f(u)\,du \\ &= B^\gamma(t,T) + \int_t^T B^\gamma(t,u)\delta(u)\gamma(u)\,du. \end{aligned}$$

The dynamics of the process $(\widetilde{D}^\delta(t,T),\, t\in[0,T])$ are

$$d\widetilde{D}^\delta(t,T) = (r(t)+\gamma(t))\widetilde{D}^\delta(t,T)\,dt - \delta(t)\gamma(t)\,dt. \tag{2.6}$$

As expected, the dynamics of the price process $D^\delta(t,T)$ will also include a jump with a negative value occurring at time τ (see Proposition 2.2.2).

Fractional Recovery of Par Value

Assume that a DZC pays a constant recovery δ at default. The pre-default value of the bond is here the same as for the recovery at maturity scheme with the function $\delta B^{-1}(t,T)$. This observation follows from a simple reasoning, but it can also be deduced from the formulae established in Lemmas 2.1.2 and 2.1.3.

Fractional Recovery of Treasury Value

We now consider the recovery $\delta(t) = \delta B(t,T)$ at the moment of default. The pre-default value is in this case the same as for a defaultable bond with a constant recovery δ at maturity date T. Once again, this property is also a consequence of Lemmas 2.1.2 and 2.1.3. Under this convention, we obtain the following expressions for the pre-default value of the bond

$$\begin{aligned}\widetilde{D}^{\delta}(t,T) &= e^{-\int_t^T (r(u)+\gamma(u))\,du} + \frac{\delta B(t,T)}{G(t)} \int_t^T \gamma(u) G(u)\,du \\ &= \widetilde{D}^{0}(t,T) + \delta B(t,T) \int_t^T \gamma(u) e^{-\int_t^u \gamma(v)\,dv}\,du.\end{aligned}$$

Fractional Recovery of Market Value

Let us finally assume that the recovery is paid at the moment of default and it equals $\delta(t)\widetilde{D}^{\delta}(t,T)$, where δ is a deterministic function. Equivalently, the recovery payoff is given as $\delta(\tau)D^{\delta}(\tau-,T)$. The dynamics of the pre-default value $\widetilde{D}^{\delta}(t,T)$ are now given by (see Duffie and Singleton [75])

$$d\widetilde{D}^{\delta}(t,T) = \big(r(t) + \gamma(t)(1-\delta(t))\big)\widetilde{D}^{\delta}(t,T)\,dt$$

with the terminal condition $\widetilde{D}^{\delta}(t,T) = 1$. This yields, for every $t \in [0,T]$,

$$\widetilde{D}^{\delta}(t,T) = \exp\Big(-\int_t^T r(u)\,du - \int_t^T \gamma(u)(1-\delta(u))\,du\Big).$$

2.2 Martingale Approach

We shall work under the standing assumption that $F(t) = \mathbb{Q}(\tau \le t) < 1$ for every $t \in \mathbb{R}_+$, but we do not impose any further restrictions on the cumulative distribution function F of a default time τ under $\mathbb{Q}$ at this stage. In particular, we do not postulate, in general, that F is a continuous function.

2.2.1 Conditional Expectations

We first give an elementary formula for the computation of the conditional expectation with respect to the σ-field $\mathcal{H}_t$, as presented, for instance, in Brémaud [33], Dellacherie [66, 67], or Elliott [78].

Lemma 2.2.1. *For any $\mathbb{Q}$-integrable and $\mathcal{G}$-measurable random variable X we have that*

$$\mathbb{1}_{\{t<\tau\}}\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{H}_t) = \mathbb{1}_{\{t<\tau\}} \frac{\mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{t<\tau\}})}{\mathbb{Q}(t<\tau)}. \tag{2.7}$$

Proof. The conditional expectation $\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{H}_t)$ is, obviously, $\mathcal{H}_t$-measurable. Therefore, it can be represented as follows

$$\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{H}_t) = h(\tau)\mathbb{1}_{\{t \geq \tau\}} + c\mathbb{1}_{\{t<\tau\}} \tag{2.8}$$

for some Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$ and some constant c. By multiplying both members by $\mathbb{1}_{\{t<\tau\}}$ and taking the expectation, we obtain

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\}}\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{H}_t)) = \mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{t<\tau\}}) = c\mathbb{Q}(t < \tau),$$

so that $c = (\mathbb{Q}(t < \tau))^{-1}\mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{t<\tau\}})$. By combining this equality with (2.8), we get the desired result. □

Let us recall the notion of the hazard function (cf. Definition 2.1.2).

Definition 2.2.1. The *hazard function* Γ of a default time τ is defined by the formula $\Gamma(t) = -\ln(1 - F(t))$ for every $t \in \mathbb{R}_+$.

Corollary 2.2.1. *Assume that X is an $\mathcal{H}_\infty$-measurable and $\mathbb{Q}$-integrable random variable, so that $X = h(\tau)$ for some Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$ such that $\mathbb{E}_{\mathbb{Q}}|h(\tau)| < +\infty$. If the hazard function Γ of τ is continuous then*

$$\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{H}_t) = \mathbb{1}_{\{t \geq \tau\}}h(\tau) + \mathbb{1}_{\{t<\tau\}} \int_t^\infty h(u)e^{\Gamma(t)-\Gamma(u)}\, d\Gamma(u). \tag{2.9}$$

If, in addition, τ admits the intensity function γ then

$$\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{H}_t) = \mathbb{1}_{\{t \geq \tau\}}h(\tau) + \mathbb{1}_{\{t<\tau\}} \int_t^\infty h(u)\gamma(u)e^{-\int_t^u \gamma(v)\, dv}\, du.$$

In particular, we have, for any $t \leq s$,

$$\mathbb{Q}(s < \tau \mid \mathcal{H}_t) = \mathbb{1}_{\{t<\tau\}}e^{-\int_t^s \gamma(v)\, dv}$$

and

$$\mathbb{Q}(t < \tau < s \mid \mathcal{H}_t) = \mathbb{1}_{\{t<\tau\}}\Big(1 - e^{-\int_t^s \gamma(v)\, dv}\Big).$$

2.2.2 Compensator of Default Indicator Process

We first consider the general case of a possibly discontinuous cumulative distribution function F of the default time τ.

Proposition 2.2.1. *The process $(M_t,\, t \in \mathbb{R}_+)$ defined as*

$$M_t = H_t - \int_{]0,t\wedge\tau]} \frac{dF(u)}{1 - F(u-)} \tag{2.10}$$

is an $\mathbb{H}$-martingale.

Proof. Let $t < s$. Then, on the one hand, we obtain

$$\mathbb{E}_{\mathbb{Q}}(H_s - H_t \,|\, \mathcal{H}_t) = \mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\leq s\}} \,|\, \mathcal{H}_t) = \mathbb{1}_{\{t<\tau\}} \frac{F(s) - F(t)}{1 - F(t)}, \quad (2.11)$$

where the second equality follows from equality (2.7) with $X = \mathbb{1}_{\{s \geq \tau\}}$.

On the other hand, by applying once again formula (2.7), we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}} \left(\int_{]t\wedge\tau, s\wedge\tau]} \frac{dF(u)}{1 - F(u-)} \,\Big|\, \mathcal{H}_t \right) &= \mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}} \left(\int_{]t\wedge\tau, s\wedge\tau]} \frac{dF(u)}{1 - F(u-)} \,\Big|\, \mathcal{H}_t \right) \\
&= \mathbb{1}_{\{t<\tau\}} \frac{1}{\mathbb{Q}(t<\tau)} \, \mathbb{E}_{\mathbb{Q}} \left(\int_{]t,s]} \mathbb{1}_{\{u\leq\tau\}} \frac{dF(u)}{1 - F(u-)} \right) \\
&= \mathbb{1}_{\{t<\tau\}} \frac{1}{\mathbb{Q}(t<\tau)} \int_{]t,s]} \mathbb{Q}(u \leq \tau) \, \frac{dF(u)}{1 - F(u-)} \\
&= \mathbb{1}_{\{t<\tau\}} \frac{1}{\mathbb{Q}(t<\tau)} \int_{]t,s]} (1 - F(u-)) \, \frac{dF(u)}{1 - F(u-)} \\
&= \mathbb{1}_{\{t<\tau\}} \frac{1}{1 - F(t)} \int_{]t,s]} dF(u) \\
&= \mathbb{1}_{\{t<\tau\}} \frac{F(s) - F(t)}{1 - F(t)}.
\end{aligned}$$

In view of (2.11), this proves the result. □

Assume now that the cumulative distribution function F is continuous. Then the process $(M_t,\, t \in \mathbb{R}_+)$, defined as

$$M_t = H_t - \int_0^{\tau\wedge t} \frac{dF(u)}{1 - F(u)},$$

is an $\mathbb{H}$-martingale.

Moreover, we have that

$$\int_0^t \frac{dF(u)}{1 - F(u)} = -\ln(1 - F(t)) = \Gamma(t).$$

These observations yield the following corollary to Proposition 2.2.1.

Corollary 2.2.2. *Assume that F (and thus also Γ) is a continuous function. Then the process $M_t = H_t - \Gamma(t \wedge \tau)$, $t \in \mathbb{R}_+$, is an $\mathbb{H}$-martingale.*

In particular, if F is an absolutely continuous function then the process

$$M_t = H_t - \int_0^{\tau\wedge t} \gamma(u)\, du = H_t - \int_0^t \gamma(u)(1 - H_u)\, du \quad (2.12)$$

is an $\mathbb{H}$-martingale.

Remark 2.2.1. From Corollary 2.2.2, we obtain the Doob-Meyer decomposition of the submartingale H as $H_t = M_t + \Gamma(t \wedge \tau)$. The predictable increasing process $A_t = \Gamma(t \wedge \tau)$ is called the *compensator* (or the *dual predictable projection*) of increasing and $\mathbb{H}$-adapted process H.

Example 2.2.1. In the case where N is an inhomogeneous Poisson process with deterministic intensity λ and τ is the moment of the first jump of N, let $H_t = N_{t\wedge\tau}$. It is well known that $N_t - \int_0^t \lambda(u)\,du$ is a martingale with respect to the natural filtration of N. Therefore, the process stopped at time τ is also a martingale, i.e., $H_t - \int_0^{t\wedge\tau} \lambda(u)\,du$ is a martingale. Furthermore, we have seen in Remark 2.1.1 that we can reduce our attention to this case, since any random time in the present setup can be viewed as the moment of the first jump of an inhomogeneous Poisson process.

We are in a position to derive the dynamics of a defaultable zero-coupon bond with recovery $\delta(\tau)$ paid at default. We will use the property that the process M is an $\mathbb{H}$-martingale under the risk-neutral probability $\mathbb{Q}$. For convenience, we shall work under the assumption that τ admits the hazard rate γ. We emphasize that we are working here under a risk-neutral probability. In the sequel, we shall see how to compute the risk-neutral default intensity from the historical one, using a suitable Radon-Nikodým density process.

Proposition 2.2.2. *Assume that τ admits the hazard rate γ. Then the risk-neutral dynamics of a DZC with recovery $\delta(\tau)$ paid at default, where $\delta : \mathbb{R}_+ \to \mathbb{R}$ is a Borel measurable function, are*

$$dD^\delta(t,T) = \big(r(t)D^\delta(t,T) - \delta(t)\gamma(t)(1-H_t)\big)\,dt - \widetilde{D}^\delta(t,T)\,dM_t$$

where the $\mathbb{H}$-martingale M is given by (2.12).

Proof. Combining the equality (cf. (2.5))

$$D^\delta(t,T) = \mathbb{1}_{\{t<\tau\}}\widetilde{D}^\delta(t,T) = (1-H_t)\widetilde{D}^\delta(t,T)$$

with dynamics (2.6) of the pre-default value $\widetilde{D}^\delta(t,T)$, we obtain

$$\begin{aligned} dD^\delta(t,T) &= (1-H_t)\,d\widetilde{D}^\delta(t,T) - \widetilde{D}^\delta(t,T)\,dH_t \\ &= (1-H_t)\Big(\big(r(t)+\gamma(t)\big)\widetilde{D}^\delta(t,T) - \delta(t)\gamma(t)\Big)\,dt - \widetilde{D}^\delta(t,T)\,dH_t \\ &= \big(r(t)D^\delta(t,T) - \delta(t)\gamma(t)(1-H_t)\big)\,dt - \widetilde{D}^\delta(t,T)\,dM_t, \end{aligned}$$

as required. □

Example 2.2.2. Assume that τ admits the hazard rate γ. By combining the pricing formula (2.3) with the pre-default dynamics (2.4), it is possible to show that the risk-neutral dynamics of the price $D^{\delta}(t,T)$ of a DZC with recovery $\delta(\tau)$ paid at maturity are

$$dD^{\delta}(t,T) = r(t)D^{\delta}(t,T)\,dt + \big(\delta(t)B(t,T) - \widetilde{D}^{\delta}(t,T)\big)\,dM_t. \tag{2.13}$$

From the last formula, one may also derive the integral representation of the $\mathbb{H}$-martingale $B(0,t)D^{\delta}(t,T)$ for $t \in [0,T[$, which gives the discounted price of the bond, in terms of the $\mathbb{H}$-martingale M associated with τ (see also Proposition 2.2.6 in this regard).

2.2.3 Martingales Associated with Default Time

We are going to furnish a few more examples of $\mathbb{H}$-martingales.

Proposition 2.2.3. *The process* $(L_t,\, t \in \mathbb{R}_+)$*, given by the formula*

$$L_t = \mathbb{1}_{\{t<\tau\}} e^{\Gamma(t)} = (1 - H_t)e^{\Gamma(t)}, \tag{2.14}$$

is an $\mathbb{H}$*-martingale. If the hazard function* Γ *is continuous then the process* L *satisfies*

$$L_t = 1 - \int_{]0,t]} L_{u-}\, dM_u, \tag{2.15}$$

where the $\mathbb{H}$*-martingale* M *is given by the formula* $M_t = H_t - \Gamma(t \wedge \tau)$*.*

Proof. We will first show that L is an $\mathbb{H}$-martingale. We have that, for any $t < s$,

$$\mathbb{E}_{\mathbb{Q}}(L_s \,|\, \mathcal{H}_t) = e^{\Gamma(s)}\, \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}} \,|\, \mathcal{H}_t).$$

By applying (2.7) to $X = 1$, we obtain

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}} \,|\, \mathcal{H}_t) = \mathbb{1}_{\{t<\tau\}} \frac{1 - F(s)}{1 - F(t)} = \mathbb{1}_{\{t<\tau\}} e^{\Gamma(t)-\Gamma(s)},$$

which means that, for every $t \le s$,

$$\mathbb{E}_{\mathbb{Q}}(L_s \,|\, \mathcal{H}_t) = \mathbb{1}_{\{t<\tau\}} e^{\Gamma(t)} = L_t,$$

and thus L is an $\mathbb{H}$-martingale. To establish (2.15), it suffices to note that $L_0 = 1$ and to apply the integration by parts formula for the product of two functions of finite variation. Since Γ is assumed to be continuous, we obtain

$$\begin{aligned} dL_t &= -e^{\Gamma(t)}\, dH_t + (1 - H_t)e^{\Gamma(t)}\, d\Gamma(t) = -e^{\Gamma(t)}\, dM_t \\ &= -\mathbb{1}_{\{t\le\tau\}} e^{\Gamma(t)}\, dM_t = -L_{t-}\, dM_t. \end{aligned}$$

Alternatively, it is possible to show directly that the process L given by (2.14) is the Doléans exponential of M, that is, that L is the unique solution of the SDE

$$dL_t = -L_{t-}\, dM_t, \quad L_0 = 1.$$

Note that this SDE can be solved pathwise, since M is manifestly a process of finite variation (for more details, see Appendix A). □

Proposition 2.2.4. *Assume that the hazard function* Γ *is continuous. Let* $h : \mathbb{R}_+ \to \mathbb{R}$ *be a Borel measurable function such that the random variable* $h(\tau)$ *is* $\mathbb{Q}$*-integrable. Then the process* $(\bar{M}^h_t,\, t \in \mathbb{R}_+)$*, given by the formula*

$$\bar{M}^h_t = \mathbb{1}_{\{t \geq \tau\}} h(\tau) - \int_0^{t\wedge\tau} h(u)\, d\Gamma(u), \tag{2.16}$$

is an $\mathbb{H}$*-martingale. Moreover, for every* $t \in \mathbb{R}_+$*,*

$$\bar{M}^h_t = \int_{]0,t]} h(u)\, dM_u = -\int_{]0,t]} e^{-\Gamma(u)} h(u)\, dL_u. \tag{2.17}$$

Proof. The proof given below provides an alternative proof of Corollary 2.2.2. We wish to establish, through direct calculations, the martingale property of the process $\bar{M}^h$ given by formula (2.16). On the one hand, formula (2.9) in Corollary 2.2.1 yields

$$\mathbb{E}_{\mathbb{Q}}\big(h(\tau)\mathbb{1}_{\{t<\tau\leq s\}} \,|\, \mathcal{H}_t\big) = \mathbb{1}_{\{t<\tau\}} e^{\Gamma(t)} \int_t^s h(u) e^{-\Gamma(u)}\, d\Gamma(u).$$

On the other hand, we note that

$$J := \mathbb{E}_{\mathbb{Q}}\Big(\int_{t\wedge\tau}^{s\wedge\tau} h(u)\, d\Gamma(u)\Big) = \mathbb{E}_{\mathbb{Q}}\big(\tilde{h}(\tau)\mathbb{1}_{\{t<\tau\leq s\}} + \tilde{h}(s)\mathbb{1}_{\{s<\tau\}} \,|\, \mathcal{H}_t\big),$$

where we write $\tilde{h}(s) = \int_t^s h(u)\, d\Gamma(u)$. Consequently, using again (2.9), we obtain

$$J = \mathbb{1}_{\{t<\tau\}} e^{\Gamma(t)} \Big(\int_t^s \tilde{h}(u) e^{-\Gamma(u)}\, d\Gamma(u) + e^{-\Gamma(s)} \tilde{h}(s)\Big).$$

To conclude the proof, it is enough to observe that the Fubini theorem yields

$$\begin{aligned}
&\int_t^s e^{-\Gamma(u)} \int_t^u h(v)\, d\Gamma(v)\, d\Gamma(u) + e^{-\Gamma(s)} \tilde{h}(s) \\
&= \int_t^s h(u) \int_u^s e^{-\Gamma(v)}\, d\Gamma(v)\, d\Gamma(u) + e^{-\Gamma(s)} \int_t^s h(u)\, d\Gamma(u) \\
&= \int_t^s h(u) e^{-\Gamma(u)}\, d\Gamma(u),
\end{aligned}$$

as required. The proof of formula (2.17) is left to the reader. □

Corollary 2.2.3. *Assume that the hazard function* Γ *of* τ *is continuous. Let* $h : \mathbb{R}_+ \to \mathbb{R}$ *be a Borel measurable function such that the random variable* $e^{h(\tau)}$ *is* $\mathbb{Q}$*-integrable. Then the process* $(\widetilde{M}^h_t,\, t \in \mathbb{R}_+)$*, given by the formula*

$$\widetilde{M}^h_t = \exp\big(\mathbb{1}_{\{t\geq\tau\}} h(\tau)\big) - \int_0^{t\wedge\tau} (e^{h(u)} - 1)\, d\Gamma(u),$$

is an $\mathbb{H}$*-martingale.*

Proof. It suffices to observe that

$$\exp\big(\mathbb{1}_{\{t\geq\tau\}} h(\tau)\big) = \mathbb{1}_{\{t\geq\tau\}} e^{h(\tau)} + \mathbb{1}_{\{t<\tau\}} = \mathbb{1}_{\{t\geq\tau\}}(e^{h(\tau)} - 1) + 1,$$

and to apply Proposition 2.2.4 to the function $e^h - 1$. □

Proposition 2.2.5. *Assume that the hazard function* Γ *of* τ *is continuous. Let* $h : \mathbb{R}_+ \to \mathbb{R}$ *be a Borel measurable function such that* $h \geq -1$ *and, for every* $t \in \mathbb{R}_+$*,*

$$\int_0^t h(u)\, d\Gamma(u) < +\infty.$$

Then the process $(\widehat{M}^h_t,\, t \in \mathbb{R}_+)$*, given by the formula*

$$\widehat{M}^h_t = \big(1 + \mathbb{1}_{\{t\geq\tau\}} h(\tau)\big) \exp\Big(-\int_0^{t\wedge\tau} h(u)\, d\Gamma(u)\Big),$$

is a non-negative $\mathbb{H}$*-martingale.*

Proof. We start by noting that

$$\begin{aligned}
\widehat{M}^h_t &= \exp\Big(-\int_0^t (1 - H_u) h(u)\, d\Gamma(u)\Big) \\
&\quad + \mathbb{1}_{\{t\geq\tau\}} h(\tau) \exp\Big(-\int_0^\tau (1 - H_u) h(u)\, d\Gamma(u)\Big) \\
&= \exp\Big(-\int_0^t (1 - H_u) h(u)\, d\Gamma(u)\Big) \\
&\quad + \int_{]0,t]} h(u) \exp\Big(-\int_0^u (1 - H_s) h(s)\, d\Gamma(s)\Big)\, dH_u.
\end{aligned}$$

Using Itô's formula, we thus obtain

$$\begin{aligned}
d\widehat{M}^h_t &= \exp\Big(-\int_0^t (1 - H_u) h(u)\, d\Gamma(u)\Big)\big(h(t)\, dH_t - (1 - H_t) h(t)\, d\Gamma(t)\big) \\
&= h(t) \exp\Big(-\int_0^t (1 - H_u) h(u)\, d\Gamma(u)\Big)\, dM_t.
\end{aligned}$$

This shows that $\widehat{M}^h$ is a non-negative local $\mathbb{H}$-martingale and thus a supermartingale. It can be checked directly that $\mathbb{E}_{\mathbb{Q}}(\widehat{M}^h_t) = 1$ for every $t \in \mathbb{R}_+$. Hence the process $\widehat{M}^h$ is indeed an $\mathbb{H}$-martingale. □

2.2.4 Predictable Representation Theorem

In this subsection, we assume that the hazard function Γ is continuous, so that the process $M_t = H_t - \Gamma(t \wedge \tau)$ is an $\mathbb{H}$-martingale. The next result shows that the martingale M has the predictable representation property for the filtration $\mathbb{H}$ generated by the default process. Let us observe that this filtration is also generated by the martingale M.

Proposition 2.2.6. *Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a Borel measurable function such that the random variable $h(\tau)$ is integrable under $\mathbb{Q}$. Then the martingale $M_t^h = \mathbb{E}_{\mathbb{Q}}(h(\tau) \,|\, \mathcal{H}_t)$ admits the representation*

$$M_t^h = M_0^h + \int_{]0,t]} (h(u) - g(u))\, dM_u, \tag{2.18}$$

where

$$g(t) = \frac{1}{G(t)} \int_t^\infty h(u)\, dF(u) = e^{\Gamma(t)}\, \mathbb{E}_{\mathbb{Q}}(h(\tau)\mathbb{1}_{\{t<\tau\}}) = \mathbb{E}_{\mathbb{Q}}(h(\tau) \,|\, t < \tau).$$

Moreover, g is a continuous function and $g(t) = M_t^h$ on $\{t < \tau\}$, so that

$$M_t^h = M_0^h + \int_{]0,t]} (h(u) - M_{u-}^h)\, dM_u.$$

Proof. From Lemma 2.2.1, we obtain

$$\begin{aligned} M_t^h &= h(\tau)\mathbb{1}_{\{t\geq\tau\}} + \mathbb{1}_{\{t<\tau\}} \frac{\mathbb{E}_{\mathbb{Q}}(h(\tau)\mathbb{1}_{\{t<\tau\}})}{\mathbb{Q}(t<\tau)} \\ &= h(\tau)\mathbb{1}_{\{t\geq\tau\}} + \mathbb{1}_{\{t<\tau\}} e^{\Gamma(t)}\, \mathbb{E}_{\mathbb{Q}}(h(\tau)\mathbb{1}_{\{t<\tau\}}). \end{aligned}$$

We first consider the event $\{t < \tau\}$. On this event, we clearly have that $M_t^h = g(t)$. The integration by parts formula yields

$$\begin{aligned} M_t^h = g(t) &= e^{\Gamma(t)}\, \mathbb{E}_{\mathbb{Q}}\big(h(\tau)\mathbb{1}_{\{t<\tau\}}\big) = e^{\Gamma(t)} \int_t^\infty h(u)\, dF(u) \\ &= \int_0^\infty h(u)\, dF(u) - \int_0^t e^{\Gamma(u)} h(u)\, dF(u) + \int_0^t e^{-\Gamma(u)} g(u)\, de^{\Gamma(u)} \\ &= \int_0^\infty h(u)\, dF(u) - \int_0^t e^{\Gamma(u)} h(u)\, dF(u) + \int_0^t g(u)\, d\Gamma(u). \end{aligned}$$

On the other hand, the right-hand side of (2.18) yields, on the event $\{t < \tau\}$,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}(h(\tau)) - \int_0^t (h(u) - g(u))\, d\Gamma(u) \\ &\quad = \int_0^\infty h(u)\, dF(u) - \int_0^t e^{\Gamma(u)} h(u)\, dF(u) + \int_0^t g(u)\, d\Gamma(u), \end{aligned}$$

where we used the equality $d\Gamma(u) = e^{\Gamma(u)}\,dF(u)$. Hence equality (2.18) is established on the event $\{t < \tau\}$.

To prove that (2.18) holds on the event $\{t \geq \tau\}$ as well, it suffices to note that the process M^h and the process given by the right-hand side of (2.18) are constant on this event (that is, they are stopped at τ) and the jump at time τ of both processes are identical.

Specifically, on the event $\{t \geq \tau\}$ we have that

$$\Delta M^h_\tau = M^h_\tau - M^h_{\tau-} = h(\tau) - g(\tau).$$

This completes the proof. □

Assume that the default time τ admits the intensity function γ. Then an alternative derivation of (2.18) consists in computing the conditional expectation

$$\begin{aligned} M^h_t &= \mathbb{E}_{\mathbb{Q}}(h(\tau)\,|\,\mathcal{H}_t) = h(\tau)\mathbb{1}_{\{t\geq\tau\}} + \mathbb{1}_{\{t<\tau\}}e^{\Gamma(t)}\int_t^\infty h(u)\,dF(u) \\ &= \int_{]0,t]} h(u)\,dH_u + (1-H_t)e^{\Gamma(t)}\int_t^\infty h(u)\,dF(u) \\ &= \int_{]0,t]} h(u)\,dH_u + (1-H_t)g(t). \end{aligned}$$

Noting that

$$dF(t) = e^{-\Gamma(t)}\,d\Gamma(t) = e^{-\Gamma(t)}\gamma(t)\,dt,$$

we obtain

$$dg(t) = \mathbb{E}_{\mathbb{Q}}(h(\tau)\mathbb{1}_{\{t<\tau\}})\,de^{\Gamma(t)} - e^{\Gamma(t)}h(t)e^{-\Gamma(t)}\gamma(t)\,dt = (g(t)-h(t))\gamma(t)\,dt.$$

Consequently, the Itô formula yields

$$dM^h_t = (h(t)-g(t))\,dH_t + (1-H_t)(g(t)-h(t))\gamma(t)\,dt = (h(t)-g(t))\,dM_t,$$

since, obviously,

$$dM_t = dH_t - \gamma(t)(1-H_t)\,dt.$$

The following corollary to Proposition 2.2.6 emphasizes the important role played by the basic martingale M.

Corollary 2.2.4. *Any* $\mathbb{H}$*-martingale* $(X_t,\, t\in\mathbb{R}_+)$ *can be represented as* $X_t = X_0 + \int_{]0,t]}\zeta_s\,dM_s$*, where* $(\zeta_t,\, t\in\mathbb{R}_+)$ *is an* $\mathbb{H}$*-predictable process.*

Remark 2.2.2. Assume that the hazard function Γ is only right-continuous. It is then possible to establish the following formula

$$\mathbb{E}_{\mathbb{Q}}(h(\tau)\,|\,\mathcal{H}_t) = \mathbb{E}_{\mathbb{Q}}(h(\tau)) - \int_{]0,t\wedge\tau]} e^{\Delta\Gamma(u)}(g(u)-h(u))\,dM_u,$$

where $\Delta\Gamma(u) = \Gamma(u) - \Gamma(u-)$ and g is defined in Proposition 2.2.6.

2.2.5 The Girsanov Theorem

Let τ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. We denote by F the cumulative distribution function of τ under $\mathbb{Q}$. It is assumed throughout that $F(t) < 1$ for every $t \in \mathbb{R}_+$, so that the hazard function Γ of τ under $\mathbb{Q}$ is well defined.

Let $\mathbb{P}$ be an arbitrary probability measure on $(\Omega, \mathcal{H}_\infty)$, which is *absolutely continuous* with respect to $\mathbb{Q}$. Let η stand for the $\mathcal{H}_\infty$-measurable Radon-Nikodým density of $\mathbb{P}$ with respect to $\mathbb{Q}$

$$\eta := \frac{d\mathbb{P}}{d\mathbb{Q}} = h(\tau) \geq 0, \quad \mathbb{Q}\text{-a.s.,} \tag{2.19}$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function satisfying

$$\mathbb{E}_{\mathbb{Q}}(h(\tau)) = \int_{[0,\infty[} h(u)\, dF(u) = 1. \tag{2.20}$$

The probability measure $\mathbb{P}$ is *equivalent* to $\mathbb{Q}$ if and only if the inequality in formula (2.19) is strict, $\mathbb{Q}$-a.s.

Let $\widehat{F}$ be the cumulative distribution function of τ under $\mathbb{P}$, that is,

$$\widehat{F}(t) := \mathbb{P}(\tau \leq t) = \int_{[0,t]} h(u)\, dF(u).$$

We assume $\widehat{F}(t) < 1$ for any $t \in \mathbb{R}_+$ or, equivalently, that

$$\mathbb{P}(\tau > t) = 1 - \widehat{F}(t) = \int_{]t,\infty[} h(u)\, dF(u) > 0. \tag{2.21}$$

Therefore, the hazard function $\widehat{\Gamma}$ of τ under $\mathbb{P}$ is well defined (of course, this property always holds if $\mathbb{P}$ is equivalent to $\mathbb{Q}$).

Put another way, we assume that

$$g(t) = e^{\Gamma(t)} \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{t<\tau\}} h(\tau)\big) = e^{\Gamma(t)} \int_{]t,\infty[} h(u)\, dF(u) = e^{\Gamma(t)}\, \mathbb{P}(\tau > t) > 0.$$

Our first goal is to examine the relationship between the hazard functions $\widehat{\Gamma}(t) = -\ln(1 - \widehat{F}(t))$ and $\Gamma(t) = -\ln(1 - F(t))$. The first result is an immediate consequence of the definition of the hazard function.

Lemma 2.2.2. *We have, for every $t \in \mathbb{R}_+$,*

$$\frac{\widehat{\Gamma}(t)}{\Gamma(t)} = \frac{\ln\left(\int_{]t,\infty[} h(u)\, dF(u)\right)}{\ln(1 - F(t))}.$$

From now on, we assume, in addition, that F is a continuous function. The following result can be seen as an elementary counterpart of the celebrated Girsanov theorem for a Brownian motion.

Lemma 2.2.3. *Assume that the cumulative distribution function F of τ under $\mathbb{Q}$ is continuous. Then the cumulative distribution function $\widehat{F}$ of τ under $\mathbb{P}$ is continuous and we have that, for every $t \in \mathbb{R}_+$,*

$$\widehat{\Gamma}(t) = \int_0^t \widehat{h}(u)\, d\Gamma(u),$$

where the function $\widehat{h} : \mathbb{R}_+ \to \mathbb{R}_+$ is given by the formula $\widehat{h}(t) = h(t)/g(t)$. Hence the process $(\widehat{M}_t,\, t \in \mathbb{R}_+)$, which is given by the formula

$$\widehat{M}_t := H_t - \int_0^{t\wedge\tau} \widehat{h}(u)\, d\Gamma(u) = M_t - \int_0^{t\wedge\tau} (\widehat{h}(u) - 1)\, d\Gamma(u),$$

is an $\mathbb{H}$-martingale under $\mathbb{P}$.

Proof. Indeed, if F (and thus $\widehat{F}$) is continuous, we obtain

$$d\widehat{\Gamma}(t) = \frac{d\widehat{F}(t)}{1 - \widehat{F}(t)} = \frac{d(1 - e^{-\Gamma(t)} g(t))}{e^{-\Gamma(t)} g(t)} = \frac{g(t)\, d\Gamma(t) - dg(t)}{g(t)} = \widehat{h}(t)\, d\Gamma(t),$$

where we used the following, easy to check, equalities $1 - \widehat{F}(t) = e^{-\Gamma(t)} g(t)$ and $dg(t) = (g(t) - h(t))\, d\Gamma(t)$. □

Remark 2.2.3. Since $\widehat{\Gamma}$ is the hazard function of τ under $\mathbb{P}$, we necessarily have

$$\lim_{t\to+\infty} \widehat{\Gamma}(t) = \int_0^\infty \widehat{h}(t)\, d\Gamma(t) = +\infty. \tag{2.22}$$

Conversely, if a continuous function Γ is the hazard function of τ under $\mathbb{Q}$ and $\widehat{h} : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that, for every $t \in \mathbb{R}_+$,

$$\widehat{\Gamma}(t) := \int_0^t \widehat{h}(u)\, d\Gamma(u) < +\infty \tag{2.23}$$

and (2.22) holds, then it is possible to find a probability measure $\mathbb{P}$ absolutely continuous with respect to $\mathbb{Q}$ such that $\widehat{\Gamma}$ is the hazard function of τ under $\mathbb{P}$ (see Remark 2.2.4).

In the special case when F is an absolutely continuous function, so that the intensity function γ of τ under $\mathbb{Q}$ is well defined, the cumulative distribution function $\widehat{F}$ of τ under $\mathbb{P}$ equals

$$\widehat{F}(t) = \int_0^t h(u) f(u)\, du,$$

so that $\widehat{F}$ is an absolutely continuous function as well. Therefore, the intensity function $\widehat{\gamma}$ of τ under $\mathbb{P}$ exists and it is given by the formula

$$\widehat{\gamma}(t) = \frac{h(t)f(t)}{1 - \widehat{F}(t)} = \frac{h(t)f(t)}{1 - \int_0^t h(u)f(u)\,du}.$$

From Lemma 2.2.3, it follows that $\widehat{\gamma}(t) = \widehat{h}(t)\gamma(t)$. To re-derive this result, observe that

$$\begin{aligned}\widehat{\gamma}(t) &= \frac{h(t)f(t)}{1 - \widehat{F}(t)} = \frac{h(t)f(t)}{1 - \int_0^t h(u)f(u)\,du} = \frac{h(t)f(t)}{\int_t^\infty h(u)f(u)\,du} = \frac{h(t)f(t)}{e^{-\Gamma(t)}g(t)} \\ &= \widehat{h}(t)\,\frac{f(t)}{1 - F(t)} = \widehat{h}(t)\gamma(t).\end{aligned}$$

Let us now examine the Radon-Nikodým density process $(\eta_t,\, t \in \mathbb{R}_+)$, which is given by the formula

$$\eta_t := \frac{d\mathbb{P}}{d\mathbb{Q}}\,|\mathcal{H}_t = \mathbb{E}_{\mathbb{Q}}(\eta \,|\, \mathcal{H}_t).$$

Proposition 2.2.7. *Assume that F is a continuous function and let $\mathcal{E}$ stand for the Doléans exponential (see Section A.4). Then*

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-}(\widehat{h}(u) - 1)\,dM_u \tag{2.24}$$

or, equivalently,

$$\eta_t = \mathcal{E}_t\Big(\int_{]0,\,\cdot\,]} (\widehat{h}(u) - 1)\,dM_u\Big). \tag{2.25}$$

Proof. Note that $\eta_t = M_t^h$ where $M_t^h = \mathbb{E}_{\mathbb{Q}}(h(\tau)\,|\,\mathcal{H}_t)$. Using Proposition 2.2.6 and noting that $\eta_0 = M_0^h = 1$, we thus obtain (cf. (2.18))

$$\begin{aligned}\eta_t &= \eta_0 + \int_{]0,t]} (h(u) - g(u))\,dM_u = 1 + \int_{]0,t]} (h(u) - \eta_{u-})\,dM_u \\ &= 1 + \int_{]0,t]} \eta_{u-}(\widehat{h}(u) - 1)\,dM_u.\end{aligned}$$

Formula (2.25) follows from the definition of the Doléans exponential. □

It is worth noting that

$$\eta_t = \mathbb{1}_{\{t \geq \tau\}} h(\tau) + \mathbb{1}_{\{t < \tau\}} \int_t^\infty h(u) e^{\Gamma(t) - \Gamma(u)}\,d\Gamma(u),$$

but also (this can be deduced from (2.24))

$$\eta_t = \big(1 + \mathbb{1}_{\{t \geq \tau\}} \kappa(\tau)\big) \exp\Big(- \int_0^{t\wedge\tau} \kappa(u)\, d\Gamma(u)\Big),$$

where we write $\kappa = \widehat{h} - 1$. Since $\widehat{h}$ is a non-negative function, it is clear that the inequality $\kappa \geq -1$ holds.

Remark 2.2.4. Let κ be any Borel measurable function $\kappa \geq -1$ ($\kappa > -1$, respectively) such that the inequality $\int_0^t \kappa(u)\, d\Gamma(u) < +\infty$ holds for every $t \in \mathbb{R}_+$. Then, by virtue of Proposition 2.2.5, the process

$$\eta_t^\kappa := \mathcal{E}_t\Big(\int_{]0,\,\cdot\,]} \kappa(u)\, dM_u\Big)$$

follows a non-negative (positive, respectively) $\mathbb{H}$-martingale under $\mathbb{Q}$. If, in addition, we have that

$$\int_0^\infty (1 + \kappa(u))\, d\Gamma(u) = +\infty$$

then $\eta_t^\kappa = \mathbb{E}_{\mathbb{Q}}(\eta_\infty^\kappa \,|\, \mathcal{H}_t)$, where $\eta_\infty^\kappa = \lim_{t\to\infty} \eta_t^\kappa$. In that case, we may define a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{H}_\infty)$ by setting $d\mathbb{P} = \eta_\infty^\kappa\, d\mathbb{Q}$. The hazard function $\widehat{\Gamma}$ of τ under $\mathbb{P}$ satisfies $d\widehat{\Gamma}(t) = (1 + \kappa(t))\, d\Gamma(t)$. Note also that in terms of κ we obtain (cf. Theorem 3.4.1)

$$\widehat{M}_t := M_t - \int_0^{t\wedge\tau} \kappa(u)\, d\Gamma(u) = H_t - \int_0^{t\wedge\tau} (1 + \kappa(u))\, d\Gamma(u).$$

2.2.6 Range of Arbitrage Prices

In order to study the important feature of model completeness, we first need to specify the class of primary traded assets. In our elementary model, the primary traded assets are risk-free zero-coupon bonds with deterministic prices and thus there exists infinitely many *equivalent martingale measures* (EMMs). Indeed, the discounted asset prices are constant and thus the class $\mathcal{Q}$ of all EMMs coincide with the set of all probability measures equivalent to the historical probability.

Let us assume that under the historical probability measure $\mathbb{P}$ the default time is an unbounded random variable with a strictly positive probability density function. For any $\mathbb{Q} \in \mathcal{Q}$, we denote by F_Q the cumulative distribution function of τ under $\mathbb{Q}$, that is,

$$F_Q(t) = \mathbb{Q}(\tau \leq t) = \int_0^t f_Q(u)\, du.$$

The range of arbitrage prices is defined as the set of all potential viable prices, that is, prices that do not induce arbitrage opportunities. For instance, in the case of a DZC with a constant recovery $\delta \in [0,1[$ paid at maturity, the range of arbitrage prices is equal to the set

$$\{B(0,T)\,\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{T<\tau\}} + \delta \mathbb{1}_{\{\tau \le T\}}),\ \mathbb{Q} \in \mathcal{Q}\}.$$

It is easy to check that this set is exactly the open interval $]\delta B(0,T), B(0,T)[$. Let us note that this range of arbitrage prices is manifestly too wide to be useful for practical purposes.

2.2.7 Implied Risk-Neutral Default Intensity

The absence of arbitrage opportunities in a financial model is commonly interpreted in terms of the existence of an EMM. If defaultable zero-coupon bonds issued by a given firm are traded, their prices are observed in the bond market. Therefore, the equivalent martingale measure $\mathbb{Q}$, to be used for pricing purposes for other credit derivatives with the same reference credit name, is in some sense selected by the market, rather than arbitrarily postulated. To support this claim, we will show that it is possible to derive the cumulative distribution function of τ under an *implied martingale measure* $\mathbb{Q}$ from market quotes for default-free and defaultable zero-coupon bonds, that is, from observed Treasury and corporate yield curves.

It is important to stress that, in the present setup, no specific relationship between the risk-neutral default intensity and the historical one is expected to hold, in general. In particular, the risk-neutral default intensity can be either higher or lower than the historical one. The historical default intensity can be deduced from observation of default times for a cohort of credit names, whereas the risk-neutral one is obtained from prices of traded defaultable claims for a given credit name.

Zero Recovery

If a defaultable zero-coupon bond with zero recovery and maturity T is traded at some price $D^0(t,T)$ belonging to the interval $]0, B(t,T)[$ then the process $B(0,t)D^0(t,T)$ is a martingale under a risk-neutral probability $\mathbb{Q}$. We do not postulate that the market model is complete, so we do not assume that an equivalent martingale measure is unique. The following equalities are thus valid under some martingale measure $\mathbb{Q} \in \mathcal{Q}$

$$\begin{aligned} B(0,t)D^0(t,T) &= \mathbb{E}_{\mathbb{Q}}(B(0,T)\mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{H}_t) \\ &= B(0,T)\mathbb{1}_{\{t<\tau\}} \exp\Big(-\int_t^T \gamma^Q(u)\,du \Big), \end{aligned}$$

where $\gamma^Q(u) = f_Q(u)(1 - F_Q(u))^{-1}$. Note that the knowledge of the implied intensity γ^Q is manifestly sufficient for the computation of the implied cumulative distribution function F_Q.

Let us now consider $t = 0$. It is easily seen that if for any maturity date T the price $D^0(0,T)$ belongs to the range of arbitrage prices $]0, B(0,T)[$ then the function γ^Q is strictly positive and the converse implication holds as well. Assuming that prices $D^0(0,T)$, $T > 0$, are observed, the function γ^Q that satisfies, for every $T > 0$,

$$D^0(0,T) = B(0,T) \exp\Big(- \int_0^T \gamma^Q(u)\, du \Big)$$

is the *implied risk-neutral default intensity*, that is, the unique $\mathbb{Q}$-intensity of τ that is consistent with the market data for DZCs. More precisely, the value of the integral $\int_0^T \gamma^Q(u)\, du$ is known for any $T > 0$ as soon as defaultable zero-coupon bonds with all maturities are traded at time 0.

The unique risk-neutral intensity can be formally obtained from the market quotes for DZCs by differentiation with respect to maturity date T, specifically,

$$r(t) + \gamma^Q(t) = -\partial_T \ln D^0(0,T)\,|_{\,T=t}.$$

Of course, the last formula is valid provided that the partial derivative in the right-hand side of this formula is well defined.

Recovery at Maturity

Assume that the prices of DZCs with different maturities and fixed recovery δ at maturity, are known. Then we deduce from (2.1.2)

$$F_Q(T) = \frac{B(0,T) - D^\delta(0,T)}{B(0,T)(1-\delta)}.$$

Hence the probability distribution of τ under the EMM implied by the market quotes of DZCs is uniquely determined. However, as observed by Hull and White [100], extracting risk-neutral default probabilities from bond prices is in practice more complicated, since most corporate bonds are coupon-bearing bonds, rather than zero-coupons.

Recovery at Default

In this case, the cumulative distribution function can also be obtained by differentiation of the defaultable zero-coupon curve with respect to the maturity. Indeed, denoting by $\partial_T D^\delta(0,T)$ the derivative of the value of the

DZC at time 0 with respect to the maturity and assuming that $G = 1 - F$ is differentiable, we obtain from (2.5)

$$\partial_T D^\delta(0,T) = g(T)B(0,T) - G(T)B(0,T)r(T) - \delta(T)g(T)B(0,T),$$

where we write $g(t) = G'(t)$. By solving this equation, we obtain

$$\mathbb{Q}(\tau > t) = G(t) = K(t)\left(1 + \int_0^t \partial_T D^\delta(0,u)\frac{(K(u))^{-1}}{B(0,u)(1-\delta(u))}\,du\right),$$

where we denote $K(t) = \exp\Big(\int_0^t \frac{r(u)}{1-\delta(u)}\,du\Big)$.

2.2.8 Price Dynamics of Simple Defaultable Claims

This section examines the dynamics of prices of some simple defaultable claims. For the sake of simplicity, we postulate here that the interest rate r is constant and we assume that the default intensity γ is well defined.

Recovery at Maturity

Let S be the price of an asset that only delivers a recovery $Z(\tau)$ at time T for some function Z. Formally, this corresponds to the defaultable claim $(0, 0, Z(\tau), 0, \tau)$, that is, $\widetilde{X} = Z(\tau)$. We know already that the process

$$M_t = H_t - \int_0^t (1 - H_u)\gamma(u)\,du$$

is an $\mathbb{H}$-martingale. Recall that $\gamma(t) = f(t)/G(t)$, where f is the probability density function of τ. Observe that

$$\begin{aligned} e^{-rt}S_t &= \mathbb{E}_{\mathbb{Q}}(Z(\tau)e^{-rT} \,|\, \mathcal{H}_t) \\ &= \mathbb{1}_{\{t\geq\tau\}}e^{-rT}Z(\tau) + \mathbb{1}_{\{t<\tau\}}e^{-rT}\frac{\mathbb{E}_{\mathbb{Q}}(Z(\tau)\mathbb{1}_{\{t<\tau\leq T\}})}{G(t)} \\ &= e^{-rT}\int_{]0,t]} Z(u)\,dH_u + \mathbb{1}_{\{t<\tau\}}e^{-rT}\widetilde{Z}(t), \end{aligned}$$

where the function $\widetilde{Z} : [0,T] \to \mathbb{R}$ is given by the formula

$$\widetilde{Z}(t) = \frac{\mathbb{E}_{\mathbb{Q}}(Z(\tau)\mathbb{1}_{\{t<\tau\leq T\}})}{G(t)} = \frac{\int_t^T Z(u)f(u)\,du}{G(t)}.$$

It is easily seen that

$$d\widetilde{Z}(t) = f(t)\,\frac{\int_t^T Z(u)f(u)\,du}{G^2(t)}\,dt - \frac{Z(t)f(t)}{G(t)}\,dt = \widetilde{Z}(t)\frac{f(t)}{G(t)}\,dt - \frac{Z(t)f(t)}{G(t)}\,dt$$

and thus

$$\begin{aligned} d(e^{-rt}S_t) &= e^{-rT}\left(Z(t)\,dH_t + (1-H_t)\frac{f(t)}{G(t)}\big(\widetilde{Z}(t)-Z(t)\big)\,dt - \widetilde{Z}(t-)\,dH_t\right) \\ &= \big(e^{-rT}Z(t) - e^{-rt}S_{t-}\big)\big(dH_t - (1-H_t)\gamma(t)\,dt\big) \\ &= e^{-rt}\big(e^{-r(T-t)}Z(t) - S_{t-}\big)\,dM_t. \end{aligned}$$

The discounted price is here an $\mathbb{H}$-martingale under the risk-neutral probability $\mathbb{Q}$ and the price S does not vanish (unless Z equals zero).

Recovery at Default

Assume now that the recovery payoff is received at default time. Hence we deal here with the defaultable claim $(0,0,0,Z,\tau)$ and thus the price of this claim is obviously equal to zero after τ. In general, we have

$$e^{-rt}S_t = \mathbb{E}_{\mathbb{Q}}(e^{-r\tau}Z(\tau)\mathbb{1}_{\{t<\tau\le T\}}\,|\,\mathcal{H}_t) = \mathbb{1}_{\{t<\tau\}}\frac{\mathbb{E}_{\mathbb{Q}}(e^{-r\tau}Z(\tau)\mathbb{1}_{\{t<\tau\le T\}})}{G(t)},$$

so that $e^{-rt}S_t = \mathbb{1}_{\{t<\tau\}}\widehat{Z}(t)$, where the function $\widehat{Z} : [0,T] \to \mathbb{R}$ equals

$$\widehat{Z}(t) = \frac{1}{G(t)}\int_t^T Z(u)e^{-ru}f(u)\,du.$$

Note that

$$\begin{aligned} d\widehat{Z}(t) &= -Z(t)e^{-rt}\frac{f(t)}{G(t)}\,dt + f(t)\frac{\int_t^T Z(u)e^{-ru}f(u)du}{G^2(t)}\,dt \\ &= -Z(t)e^{-rt}\frac{f(t)}{G(t)}\,dt + \widehat{Z}(t)\frac{f(t)}{G(t)}\,dt \\ &= \gamma(t)\big(\widehat{Z}(t) - Z(t)e^{-rt}\big)\,dt. \end{aligned}$$

Consequently,

$$\begin{aligned} d(e^{-rt}S_t) &= (1-H_t)\gamma(t)\big(\widehat{Z}(t) - Z(t)e^{-rt}\big)\,dt - \widehat{Z}(t)\,dH_t \\ &= \big(Z(t)e^{-rt} - \widehat{Z}(t)\big)\,dM_t - Z(t)e^{-rt}(1-H_t)\gamma(t)\,dt \\ &= e^{-rt}(Z(t) - S_{t-})\,dM_t - Z(t)e^{-rt}(1-H_t)\gamma(t)\,dt. \end{aligned}$$

In that case, the discounted price is not an $\mathbb{H}$-martingale under the risk-neutral probability. By contrast, the process

$$S_t e^{-rt} + \int_0^{t\wedge\tau} e^{-ru}Z(u)\gamma(u)\,du$$

is an $\mathbb{H}$-martingale. It is also worth noting that the recovery can be formally interpreted as a dividend stream paid at the rate $Z\gamma$ up to time $\tau\wedge T$.

2.3 Pricing of General Defaultable Claims

We will now examine the behavior of the arbitrage price of a general defaultable claim. Let us first recall the standing notation. A strictly positive random variable τ, defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, is called the *random time*. In view of its interpretation, it will be later referred to as the *default time*. We introduce the default indicator process $H_t = \mathbb{1}_{\{t \geq \tau\}}$ associated with τ and we denote by $\mathbb{H}$ the filtration generated by this process. We assume from now on that we are given, in addition, some auxiliary filtration $\mathbb{F}$ and we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, meaning that we have $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$. Note that $\mathbb{P}$ is aimed to represent the real-life probability measure.

We simplify slightly the definition of a defaultable claim of Section 1.1.1 by setting $\widetilde{X} = 0$, so that a generic defaultable claim is now formally reduced to a quadruplet (X, A, Z, τ).

Definition 2.3.1. By a *defaultable claim* maturing at time T we mean a quadruplet (X, A, Z, τ), where X is an $\mathcal{F}_T$-measurable random variable, A is an $\mathbb{F}$-adapted process of finite variation, Z is an $\mathbb{F}$-predictable process, and τ is a random time.

As in Section 1.1.1, the role of each component of a defaultable claim will become clear from the definition of the dividend process D (cf. Definition 1.1.1), which describes all cash flows associated with a defaultable claim over the lifespan $]0, T]$, that is, after the contract was initiated at time 0. Of course, the choice of 0 as the date of inception is merely a convention.

Definition 2.3.2. The *dividend process* D of a defaultable claim maturing at T equals, for every $t \in [0, T]$,

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T,\infty[}(t) + \int_{]0,t]} (1 - H_u) \, dA_u + \int_{]0,t]} Z_u \, dH_u.$$

The financial interpretation of the process D justifies the following terminology (cf. Section 1.1):

- X is the *promised payoff* at maturity T,
- A represents the process of *promised dividends*,
- the *recovery process* Z specifies the recovery payoff at default.

It is worth stressing that we maintain here the convention that the cash payment (premium) at time 0 is not included in the dividend process D associated with a defaultable claim.

Example 2.3.1. When dealing with a credit default swap (CDS), it is natural to assume that the premium paid at time 0 is equal to zero and the process A represents the fee (annuity), which is paid in instalments up to maturity date or default, whichever comes first. For instance, if $A_t = -\kappa t$ for some constant $\kappa > 0$, then the market quote of a stylized credit default swap is formally represented by this constant, referred to as the continuously paid *CDS spread* or *premium* (see Section 2.4.1 for more details).

If the other covenants of the contract are known (i.e., the payoff X and recovery Z are given), the valuation of a credit default swap is equivalent to finding the level of the rate κ that makes the swap valueless at inception.

Typically, in a credit default swap we have $X = 0$, whereas the *default protection* process Z is specified in reference to recovery rate of an underlying credit name. In a more realistic approach, the process A is discontinuous, with jumps occurring at the premium payment dates. In this text, we will only deal with a stylized CDS with a continuously paid premium. For a discussion of market conventions for CDSs, see, for instance, Brigo [38].

Let us return to the general setup. It is clear that the dividend process D follows a process of finite variation on $[0,T]$. Since

$$\int_{]0,t]} (1 - H_u)\, dA_u = \int_{]0,t]} \mathbb{1}_{\{u<\tau\}}\, dA_u = A_{\tau-}\mathbb{1}_{\{t\geq\tau\}} + A_t\mathbb{1}_{\{t<\tau\}},$$

it is also apparent that if default occurs at some date t, the promised dividend $A_t - A_{t-}$ that is due to be received or paid at this date is canceled. We have also that

$$\int_{]0,t]} Z_u\, dH_u = Z_\tau \mathbb{1}_{\{t\geq\tau\}}.$$

Let us stress that the process $D_u - D_t$, $u \in [t,T]$, represents all cash flows from a defaultable claim to be received by an investor who has purchased it at time t. Of course, the process $D_u - D_t$ may depend on the past behavior of the claim (e.g., through some intrinsic parameters, such as credit spreads) as well as on the history of the market prior to t. The past cash flows from a claim are not valued by the market, however, so that the current *market value* at time t of a claim (that is, the price at which it is traded at time t) depends only on future cash flows to be either paid or received over the time interval $]t,T]$.

We will work under the standing assumption that our underlying financial market model is *arbitrage-free*, in the sense that there exists a *spot martingale measure* $\mathbb{Q}$ (also referred to as a *risk-neutral probability*), meaning that $\mathbb{Q}$ is equivalent to the real-life probability $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ and the price process of any traded security, paying no coupons or dividends, follows a $\mathbb{G}$-martingale under $\mathbb{Q}$, when discounted by the *savings account* B, which is, as usual, given by the formula

$$B_t = \exp\left(\int_0^t r_u\, du\right).$$

2.3.1 Buy-and-Hold Strategy

We write S^i, $i = 1, 2, \ldots, k$ to denote the price processes of k primary securities in an arbitrage-free financial model. We make the standard assumption that the processes S^i, $i = 1, 2, \ldots, k-1$ follow semimartingales. In addition, we set $S^k = B$ so that S^k represents the value process of the savings account.

The last assumption is not necessary, however. One may assume, for instance, that S^k is the price of a T-maturity risk-free zero-coupon bond, or choose any other strictly positive price process as a *numéraire*.

For the sake of convenience, we assume that S^i, $i = 1, 2, \ldots, k-1$ are non-dividend-paying assets and we introduce the discounted price processes S^{i*} by setting $S^{i*} = B^{-1} S^i$. All processes are assumed to be given on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$, where $\mathbb{P}$ is the real-life (i.e., statistical) probability measure.

Let us now assume that we have an additional traded security that pays dividends during its lifespan, assumed to be the time interval $[0, T]$, according to a process of finite variation D, with $D_0 = 0$. Let S denote a (yet unspecified) price process of this security. In particular, we do not postulate a priori that S follows a semimartingale. It is not necessary to interpret S as a price process of a defaultable claim, though we have here this particular interpretation in mind.

Let a $\mathbb{G}$-predictable, $\mathbb{R}^{k+1}$-valued process $\phi = (\phi^0, \phi^1, \ldots, \phi^k)$ represent a generic trading strategy, where ϕ^j_t represents the number of shares of the jth asset held at time t. We identify here S^0 with S, so that S is the 0th asset. In order to derive a pricing formula for this asset, it suffices to examine a simple trading strategy involving S, namely, the buy-and-hold strategy.

Suppose that one unit of the 0th asset was purchased at time 0, at the initial price S_0, and it was held until time T. We assume all dividends are immediately reinvested in the savings account B. Formally, we consider a *buy-and-hold* strategy $\psi = (1, 0, \ldots, 0, \psi^k)$, where ψ^k is a $\mathbb{G}$-predictable process. The *wealth process* $V(\psi)$ of ψ equals, for every $t \in [0, T]$,

$$V_t(\psi) = S_t + \psi^k_t B_t. \tag{2.26}$$

Definition 2.3.3. We say that a strategy $\psi = (1, 0, \ldots, 0, \psi^k)$ is *self-financing* if

$$dV_t(\psi) = dS_t + dD_t + \psi^k_t \, dB_t,$$

or more explicitly, for every $t \in [0, T]$,

$$V_t(\psi) - V_0(\psi) = S_t - S_0 + D_t + \int_0^t \psi^k_u \, dB_u. \tag{2.27}$$

We assume from now on that the process ψ^k is chosen in such a way (with respect to S, D and B) that a buy-and-hold strategy ψ is self-financing.

In view of (2.26)–(2.27) this means that, for every $t \in [0,T]$,

$$\psi^k_t B_t = V_0(\psi) - S_0 + D_t + \int_0^t \psi^k_u \, dB_u.$$

In addition, we make the standing assumption that the random variable Y defined by the equality $Y = \int_{]0,T]} B_u^{-1} dD_u$ is $\mathbb{Q}$-integrable, where $\mathbb{Q}$ is a martingale measure.

Lemma 2.3.1. *The discounted wealth process $V^*(\psi) = B^{-1}V(\psi)$ of any self-financing buy-and-hold trading strategy ψ satisfies, for every $t \in [0,T]$,*

$$V^*_t(\psi) = V^*_0(\psi) + S^*_t - S^*_0 + \int_{]0,t]} B_u^{-1} dD_u. \tag{2.28}$$

Therefore, we have, for every $t \in [0,T]$,

$$V^*_T(\psi) - V^*_t(\psi) = S^*_T - S^*_t + \int_{]t,T]} B_u^{-1} \, dD_u. \tag{2.29}$$

Proof. We define an auxiliary process $\widehat{V}(\psi)$ by setting $\widehat{V}_t(\psi) = V_t(\psi) - S_t = \psi^k_t B_t$ for $t \in [0,T]$. In view of (2.27), we have

$$\widehat{V}_t(\psi) = \widehat{V}_0(\psi) + D_t + \int_0^t \psi^k_u \, dB_u,$$

and thus the process $\widehat{V}(\psi)$ follows a semimartingale. An application of Itô's product rule yields

$$\begin{aligned} d\big(B_t^{-1}\widehat{V}_t(\psi)\big) &= B_t^{-1} d\widehat{V}_t(\psi) + \widehat{V}_t(\psi) \, dB_t^{-1} \\ &= B_t^{-1} dD_t + \psi^k_t B_t^{-1} dB_t + \psi^k_t B_t \, dB_t^{-1} \\ &= B_t^{-1} dD_t, \end{aligned}$$

where we have used the obvious identity $B_t^{-1} dB_t + B_t \, dB_t^{-1} = 0$. By integrating the last equality, we obtain

$$B_t^{-1}\big(V_t(\psi) - S_t\big) = B_0^{-1}\big(V_0(\psi) - S_0\big) + \int_{]0,t]} B_u^{-1} dD_u,$$

and this immediately yields (2.28). □

Let us note that Lemma 2.3.1 remains valid if the assumption that S^k represents the savings account B is relaxed. It suffices to assume that S^k is a *numéraire*, that is, a strictly positive continuous semimartingale.

For the sake of brevity, we write $S^k = \beta$. We say that $\psi = (1, 0, \dots, 0, \psi^k)$ is self-financing if the wealth process $V(\psi)$, defined as

$$V_t(\psi) = S_t + \psi_t^k \beta_t,$$

satisfies, for every $t \in [0, T]$,

$$V_t(\psi) - V_0(\psi) = S_t - S_0 + D_t + \int_0^t \psi_u^k \, d\beta_u.$$

Lemma 2.3.2. *The relative wealth* $V_t^*(\psi) = \beta_t^{-1} V_t(\psi)$ *of a self-financing trading strategy* ψ *satisfies, for every* $t \in [0, T]$,

$$V_t^*(\psi) = V_0^*(\psi) + S_t^* - S_0^* + \int_{]0,t]} \beta_u^{-1} dD_u,$$

where $S^* = \beta_t^{-1} S_t$.

Proof. The proof proceeds along the same lines as the proof of Lemma 2.3.1. It suffices to note that the equality $\beta_t^{-1} d\beta_t + \beta_t \, d\beta_t^{-1} + d\langle \beta, \beta^{-1} \rangle_t = 0$ holds for every $t \in [0, T]$. □

2.3.2 Spot Martingale Measure

Our next goal is to derive the risk-neutral valuation formula for the ex-dividend price process S from the no-arbitrage principle. Recall that we have assumed that our market model is *arbitrage-free*, meaning that it admits a (not necessarily unique) martingale measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, which is associated with the choice of the savings account B as a numéraire. Let us recall the definition of a *spot martingale measure*.

Definition 2.3.4. We say that $\mathbb{Q}$ is a *spot martingale measure* if the discounted price $S^{i*} = S^i B^{-1}$ of any non-dividend paying traded security S^i follows a $\mathbb{Q}$-martingale with respect to the filtration $\mathbb{G}$.

It is well known that the discounted wealth process $V^*(\phi) = V(\phi) B^{-1}$ of any self-financing trading strategy $\phi = (0, \phi^1, \phi^2, \dots, \phi^k)$ is a local martingale under any martingale measure $\mathbb{Q}$. In what follows, we only consider *admissible* trading strategies, that is, strategies for which the discounted wealth process $V^*(\phi)$ is a martingale under some martingale measure $\mathbb{Q}$.

A market model in which only admissible trading strategies are allowed is *arbitrage-free*, that is, there are no arbitrage opportunities in this model.

Following this line of arguments, we now postulate, in addition, that the trading strategy ψ introduced in Section 2.3.1 is also *admissible*, so that its discounted wealth process $V^*(\psi)$ is a martingale under $\mathbb{Q}$ with respect to $\mathbb{G}$.

This assumption is fairly natural, since we wish to preclude arbitrage opportunities from the extended model of the financial market. Indeed, since we postulate that S is traded, the corresponding wealth process $V(\psi)$ can formally be seen as an additional non-dividend paying traded security.

To derive the pricing formula for a defaultable claim, we make a natural assumption that the market value at time t of the 0th security is based exclusively on its future dividends, that is, on the cash flows that occur in the interval $]t,T]$. Since the overall lifespan of S is $[0,T]$, this amounts to postulate that $S_T = S^*_T = 0$. To emphasize this property, we shall refer to S as the *ex-dividend price* of the 0th asset.

Definition 2.3.5. A process S with $S_T = 0$ is the *ex-dividend price* of the 0th asset if the discounted wealth $V^*(\psi)$ of any self-financing buy-and-hold strategy ψ follows a $\mathbb{G}$-martingale under a martingale measure $\mathbb{Q}$.

As a special case, we obtain the ex-dividend price a defaultable claim with maturity T.

Proposition 2.3.1. *The ex-dividend price process S associated with the dividend process D satisfies, for every $t \in [0,T]$,*

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t \Big). \tag{2.30}$$

Proof. The postulated martingale property of the discounted wealth process $V^*(\psi)$ yields, for every $t \in [0,T]$,

$$\mathbb{E}_{\mathbb{Q}}\big(V^*_T(\psi) - V^*_t(\psi) \,\big|\, \mathcal{G}_t \big) = 0.$$

Taking into account (2.29), we thus obtain

$$S^*_t = \mathbb{E}_{\mathbb{Q}}\Big(S^*_T + \int_{]t,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t \Big).$$

Since, by virtue of the definition of the ex-dividend price, the equalities $S_T = S^*_T = 0$ are valid, the last formula yields (2.30). □

It is not difficult to show that the ex-dividend price S satisfies the equality $S_t = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t$ for $t \in [0,T]$, where the process $\widetilde{S}$ represents the *ex-dividend pre-default price* of a defaultable claim. The *cumulative price* process S^c associated with the dividend process D is given by the formula, for every $t \in [0,T]$,

$$S^c_t = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t \Big). \tag{2.31}$$

The corresponding *discounted cumulative price* process, $S^{c*} := B^{-1} S^c$, is a $\mathbb{G}$-martingale under $\mathbb{Q}$.

Remark 2.3.1. The savings account B can be substituted with an arbitrary numéraire β. The corresponding valuation formula becomes, for every $t \in [0,T]$,

$$S_t = \beta_t \, \mathbb{E}_{\mathbb{Q}^\beta}\Big(\int_{]t,T]} \beta_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big),$$

where $\mathbb{Q}^\beta$ is a martingale measure on $(\Omega, \mathcal{G}_T)$ associated with a numéraire β, that is, a probability measure on $(\Omega, \mathcal{G}_T)$ given by the formula

$$\frac{d\mathbb{Q}^\beta}{d\mathbb{Q}} = \frac{B_0 \beta_T}{\beta_0 B_T}, \quad \mathbb{Q}\text{-a.s.}$$

2.3.3 Self-Financing Trading Strategies

Let us now consider a general trading strategy $\phi = (\phi^0, \phi^1, \dots, \phi^k)$ with $\mathbb{G}$-predictable components. The associated *wealth process* $V(\phi)$ is given by the equality $V_t(\phi) = \sum_{i=0}^k \phi_t^i S_t^i$, where, as before $S^0 = S$. A strategy ϕ is said to be *self-financing* if $V_t(\phi) = V_0(\phi) + G_t(\phi)$ for every $t \in [0,T]$, where the *gains process* $G(\phi)$ is defined as follows, for every $t \in [0,T]$,

$$G_t(\phi) = \int_{]0,t]} \phi_u^0 \, dD_u + \sum_{i=0}^k \int_{]0,t]} \phi_u^i \, dS_u^i.$$

Corollary 2.3.1. *Let $S^k = B$. Then for any self-financing trading strategy ϕ, the discounted wealth process $V^*(\phi) = B^{-1} V(\phi)$ is a martingale under $\mathbb{Q}$.*

Proof. Since B is a continuous process of finite variation, the Itô product rule yields $dS_t^{i*} = S_t^i \, dB_t^{-1} + B_t^{-1} dS_t^i$ for $i = 0, 1, \dots, k$. Consequently,

$$\begin{aligned} dV_t^*(\phi) &= V_t(\phi) \, dB_t^{-1} + B_t^{-1} dV_t(\phi) \\ &= V_t(\phi) \, dB_t^{-1} + B_t^{-1} \Big(\sum_{i=0}^k \phi_t^i \, dS_t^i + \phi_t^0 \, dD_t \Big) \\ &= \sum_{i=0}^k \phi_t^i \big(S_t^i \, dB_t^{-1} + B_t^{-1} dS_t^i \big) + \phi_t^0 B_t^{-1} dD_t \\ &= \sum_{i=1}^{k-1} \phi_t^i \, dS_t^{i*} + \phi_t^0 \big(dS_t^* + B_t^{-1} dD_t \big) = \sum_{i=1}^{k-1} \phi_t^i \, dS_t^{i*} + \phi_t^0 \, dS_t^{c*}, \end{aligned}$$

where the auxiliary process S^{c*} is given by the following expression

$$S_t^{c*} = S_t^* + \int_{]0,t]} B_u^{-1} \, dD_u.$$

To conclude, it suffices to observe that in view of (2.30) the process S^{c*} satisfies

$$S_t^{c*} = \mathbb{E}_{\mathbb{Q}}\Big(\int_{]0,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t \Big), \tag{2.32}$$

and thus it is a martingale under $\mathbb{Q}$. □

It is worth noting that S_t^{c*}, as given by formula (2.32), represents the *discounted cumulative price* at time t of the 0th asset, that is, the arbitrage price at time t of all past and future dividends associated with the 0th asset over its lifespan. To check this, let us consider a buy-and-hold strategy such that $\psi_0^k = 0$. Then, in view of (2.29), the terminal wealth at time T of this strategy equals

$$V_T(\psi) = B_T \int_{]0,T]} B_u^{-1}\, dD_u.$$

It is clear that $V_T(\psi)$ represents all dividends from S in the form of a single payoff at time T. The *arbitrage price* $\pi_t(\widehat{Y})$ at time $t \in [0, T[$ of the claim $\widehat{Y} = V_T(\psi)$ equals (under the assumption that this claim is attainable)

$$\pi_t(\widehat{Y}) = B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]0,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t \Big)$$

and thus $S_t^{c*} = B_t^{-1}\pi_t(\widehat{Y})$. It is clear that discounted cumulative price follows a martingale under $\mathbb{Q}$ (under the standard integrability assumption).

Remarks 2.3.1. (i) Under the assumption of uniqueness of a spot martingale measure $\mathbb{Q}$, any $\mathbb{Q}$-integrable contingent claim is attainable, and the valuation formula established above can be justified by means of replication.
(ii) Otherwise – that is, when a martingale probability measure $\mathbb{Q}$ is not uniquely determined by the model $(S^1, S^2, \ldots, S^k)$ – the right-hand side of (2.30) may depend on the choice of a particular martingale probability, in general. In this case, a process defined by (2.30) for an arbitrarily chosen spot martingale measure $\mathbb{Q}$ can be taken as the no-arbitrage price process of a defaultable claim. In some cases, a market model can be completed by postulating that S is also a traded asset.

2.3.4 Martingale Properties of Arbitrage Prices

In the next result, we summarize the martingale properties of arbitrage prices of a generic defaultable claim.

Corollary 2.3.2. *The discounted cumulative price process* $(S_t^{c*},\, t \in [0,T])$ *of a defaultable claim is a* $\mathbb{Q}$*-martingale with respect to* $\mathbb{G}$*. The discounted* *ex-dividend price* $(S^*_t,\, t \in [0,T])$ *satisfies, for every* $t \in [0,T]$,

$$S^*_t = S^{c*}_t - \int_{]0,t]} B_u^{-1}\, dD_u$$

and thus it follows a supermartingale under $\mathbb{Q}$ *if and only if the dividend process* D *is increasing.*

In an application considered in Section 2.4, the finite variation process A is interpreted as the positive premium paid in instalments by the claim-holder to the counterparty in exchange for a positive recovery. It is thus natural to assume that A is a decreasing process, whereas other components of the dividend process are increasing processes (that is, $X \geq 0$ and $Z \geq 0$). It is rather clear that, under these assumptions, the discounted ex-dividend price S^* is neither a super- nor submartingale under $\mathbb{Q}$, in general.

Assume now that $A = 0$, so that the premium for a defaultable claim is paid upfront at time 0 and it is not accounted for in the dividend process D. We postulate, as before, that $X \geq 0$ and $Z \geq 0$. In this case, the dividend process D is manifestly increasing and thus the discounted ex-dividend price S^* is a supermartingale under $\mathbb{Q}$. This feature is quite natural since the discounted expected value of future dividends decreases when time elapses.

2.4 Single-Name Credit Derivatives

Following Bielecki et al. [19] (see also Schmidt [159]), we will now apply the general theory to a widely particular class of credit derivatives, namely, to credit default swaps. We do not need to specify explicitly the underlying market model at this stage, but we make the following standing assumption.

Assumption 2.4.1. We assume throughout that:

- the underlying probability measure $\mathbb{Q}$ represents a spot martingale measure on $(\Omega, \mathcal{H}_T)$,
- the short-term interest rate $r = 0$, so that $B_t = 1$ for every $t \in \mathbb{R}_+$.

2.4.1 Stylized Credit Default Swap

A stylized T-maturity credit default swap is formally introduced through the following definition.

Definition 2.4.1. A *credit default swap* (CDS) with a constant rate κ and protection at default is a defaultable claim $(0, A, Z, \tau)$ where $Z(t) = \delta(t)$ and $A(t) = -\kappa t$ for every $t \in [0,T]$. A function $\delta : [0,T] \to \mathbb{R}$ represents the *default protection* whereas κ is the *CDS spread* (also termed the *rate, premium* or *annuity* of a CDS).

As usual, we denote by F the cumulative distribution function of default time τ under $\mathbb{Q}$ and we assume that F is a continuous function, with $F(0) = 0$ and $F(T) < 1$. Also, we write $G = 1 - F$ to denote the *survival probability function* of τ, so that the inequality $G(t) > 0$ is valid for every $t \in [0, T]$.

Remark 2.4.1. Note that the choice of $\mathbb{Q}$ is reflected in the cumulative distribution function F; in particular, in the default intensity if F admits a probability density function. In practical applications of reduced-form models, the choice of F is done by calibration.

Since the ex-dividend price of a CDS is the price at which the contract is actually traded, we shall refer to the ex-dividend price as the *price* in what follows. Recall that we have also introduced the *cumulative price*, which encompasses also all past payoffs from a CDS, assumed to be reinvested in the savings account.

Let $s \in [0, T]$ be a fixed date. We consider a stylized T-maturity credit default swap with a constant spread κ and default protection function δ, initiated at time s and maturing at T.

The dividend process of a CDS equals

$$D_t = \int_{]s,t]} \delta(u)\, dH_u - \kappa \int_{]s,t]} (1 - H_u)\, du \tag{2.33}$$

and thus, in view of (2.30), the ex-dividend price of this contract equals, for every $t \in [s, T]$,

$$S_t(\kappa, \delta, T) = \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau\leq T\}}\delta(\tau)\,\Big|\,\mathcal{H}_t\Big) - \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau\}}\kappa\big((\tau \wedge T) - t\big)\,\Big|\,\mathcal{H}_t\Big),$$

where the first conditional expectation represents the current value of the *default protection stream* (or simply the *protection leg*) and the second expectation is the value of the *survival annuity stream* (or the *fee leg*). To alleviate notation, we shall write $S_t(\kappa)$ instead of $S_t(\kappa, \delta, T)$ in what follows.

Lemma 2.4.1. *The ex-dividend price at time $t \in [s, T]$ of a credit default swap started at s, with spread κ and default protection δ, equals*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\, \frac{1}{G(t)}\Big(- \int_t^T \delta(u)\, dG(u) - \kappa \int_t^T G(u)\, du\Big). \tag{2.34}$$

Proof. We have, on the event $\{t < \tau\}$,

$$\begin{aligned} S_t(\kappa) &= -\frac{\int_t^T \delta(u)\, dG(u)}{G(t)} - \kappa \left(\frac{-\int_t^T u\, dG(u) + TG(T)}{G(t)} - t \right) \\ &= \frac{1}{G(t)}\Big(- \int_t^T \delta(u)\, dG(u) - \kappa\Big(TG(T) - tG(t) - \int_t^T u\, dG(u)\Big)\Big). \end{aligned}$$

Since

$$\int_t^T G(u)\,du = TG(T) - tG(t) - \int_t^T u\,dG(u),$$

we conclude that (2.34) holds. □

The *pre-default price* is defined as the unique function $\widetilde{S}(\kappa)$ such that we have, for every $t \in [0,T]$ (see Lemma 2.5.1 with $n = 1$)

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\widetilde{S}_t(\kappa). \tag{2.35}$$

Combining (2.34) with (2.35), we find that the pre-default price of the CDS equals, for $t \in [s,T]$,

$$\widetilde{S}_t(\kappa) = \frac{1}{G(t)}\Big(- \int_t^T \delta(u)\,dG(u) - \kappa \int_t^T G(u)\,du\Big) \tag{2.36}$$

so that $\widetilde{S}_t(\kappa) = \widetilde{P}(t,T) - \kappa\widetilde{A}(t,T)$, where

$$\widetilde{P}(t,T) = -\frac{1}{G(t)}\int_t^T \delta(u)\,dG(u)$$

is the pre-default price at time t of the protection leg, and

$$\widetilde{A}(t,T) = \frac{1}{G(t)}\int_t^T G(u)\,du$$

represents the pre-default price at time t of the fee leg for the period $[t,T]$ per one unit of the CDS spread κ. We shall refer henceforth to $\widetilde{A}(t,T)$ as the *CDS annuity* (it is also known as the present value of one basis point of a CDS). Note that, under our standing assumption that the survival function G is continuous, the pre-default price $\widetilde{S}(\kappa)$ is a continuous function.

2.4.2 Market CDS Spread

A CDS that has null value at its inception plays an important role as a benchmark CDS and thus we introduce a formal definition, in which it is implicitly assumed that a protection function δ of a CDS is given and that we are on the event $\{s < \tau\}$, that is, the default of the reference name has not yet occurred prior to or at time s.

Definition 2.4.2. A *market CDS started at* s is the CDS initiated at time s whose initial value is equal to zero. The T-maturity *market CDS spread* (also known as the *fair CDS spread*) at time s is the fixed level of the spread $\kappa = \kappa(s,T)$ that makes the T-maturity CDS started at s valueless at its inception. The market CDS spread at time s is thus determined by the equation $\widetilde{S}_s(\kappa(s,T)) = 0$ where $\widetilde{S}_s(\kappa)$ is given by the formula (2.36).

Under the present assumptions, by virtue of (2.36), the T-maturity market CDS spread $\kappa(s,T)$ equals, for every $s \in [0,T]$,

$$\kappa(s,T) = \frac{\widetilde{P}(s,T)}{\widetilde{A}(s,T)} = -\frac{\int_s^T \delta(u)\, dG(u)}{\int_s^T G(u)\, du}. \tag{2.37}$$

Example 2.4.1. Assume that $\delta(t) = \delta$ is constant, and $F(t) = 1 - e^{-\gamma t}$ for some constant default intensity $\gamma > 0$ under $\mathbb{Q}$. In that case, the valuation formulae for a CDS can be further simplified. In view of Lemma 2.4.1, the ex-dividend price of a (spot) CDS with spread κ equals, for every $t \in [0,T]$,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}(\delta\gamma - \kappa)\gamma^{-1}\Big(1 - e^{-\gamma(T-t)}\Big).$$

The last formula (or the general formula (2.37)) yields $\kappa(s,T) = \delta\gamma$ for every $s < T$, so that the market spread $\kappa(s,T)$ is here independent of s. As a consequence, the ex-dividend price of a market CDS started at s equals zero not only at the inception date s, but indeed at any time $t \in [s,T]$, both prior to and after default. Hence this process is trivially a martingale under $\mathbb{Q}$. As we shall see in what follows, this martingale property of the ex-dividend price of a market CDS is an exception, in the sense so that it fails to hold if the default intensity varies over time.

In what follows, we fix a maturity date T and we assume that credit default swaps with different inception dates have a common default protection δ. We shall write briefly $\kappa(s)$ instead of $\kappa(s,T)$. Then we have the following result, in which the quantity $\nu(t,s) = \kappa(t) - \kappa(s)$ represents the *calendar CDS market spread* for a given maturity T.

Proposition 2.4.1. *The price of a market CDS started at s with protection δ at default and maturity T equals, for every $t \in [s,T]$,*

$$S_t(\kappa(s)) = \mathbb{1}_{\{t<\tau\}}\,(\kappa(t) - \kappa(s))\,\widetilde{A}(t,T) = \mathbb{1}_{\{t<\tau\}}\,\nu(t,s)\widetilde{A}(t,T). \tag{2.38}$$

Proof. It suffices to observe that $S_t(\kappa(s)) = S_t(\kappa(s)) - S_t(\kappa(t))$, since $S_t(\kappa(t)) = 0$, and to use (2.36) with $\kappa = \kappa(t)$ and $\kappa = \kappa(s)$. □

Note that formula (2.38) can be extended to any value of κ, specifically,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}(\kappa(t) - \kappa)\widetilde{A}(t,T),$$

assuming that the CDS with spread κ was initiated at some date $s \in [0,t]$. The last representation shows that the price of a CDS can take negative values. The negative value occurs whenever the current market spread is lower than the contracted spread.

2.4.3 Price Dynamics of a CDS

In the remainder of Section 2.4, we assume that the hazard function satisfies

$$G(t) = \mathbb{Q}(\tau > t) = \exp\left(- \int_0^t \gamma(u)\, du \right), \quad \forall\, t \in [0, T],$$

where the default intensity $\gamma(t)$ under $\mathbb{Q}$ is a strictly positive deterministic function. Recall that the process M, given by the formula, for $t \in [0, T]$,

$$M_t = H_t - \int_0^t (1 - H_u)\gamma(u)\, du, \tag{2.39}$$

is an $\mathbb{H}$-martingale under $\mathbb{Q}$.

We first focus on the dynamics of the price of a CDS, with spread κ, which was initiated at some date $s < T$.

Lemma 2.4.2. *(i) The dynamics of the price $S_t(\kappa)$, $t \in [s, T]$, are*

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt. \tag{2.40}$$

(ii) The cumulative price $S^c_t(\kappa)$, $t \in [s, T]$, is an $\mathbb{H}$-martingale under $\mathbb{Q}$, specifically,

$$dS^c_t(\kappa) = \big(\delta(t) - S_{t-}(\kappa)\big)\, dM_t. \tag{2.41}$$

Proof. To prove (i), it suffices to recall that

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa) = (1 - H_t)\widetilde{S}_t(\kappa)$$

so that the integration by parts formula yields

$$dS_t(\kappa) = (1 - H_t)\, d\widetilde{S}_t(\kappa) - \widetilde{S}_{t-}(\kappa)\, dH_t.$$

Using formula (2.34), we find easily that

$$d\widetilde{S}_t(\kappa) = \gamma(t)\widetilde{S}_t(\kappa)\, dt + (\kappa - \delta(t)\gamma(t))\, dt.$$

In view of (2.39) and the fact that

$$S_{\tau-}(\kappa) = \widetilde{S}_{\tau-}(\kappa)$$

and $S_t(\kappa) = 0$ for $t \geq \tau$, the derivation of dynamics (2.40) is completed. To prove part (ii), we note that in the case of $B = 1$ the formulae (2.30) and (2.31) yield the following relationship, for every $t \in [s, T]$,

$$S^c_t(\kappa) = S_t(\kappa) + D_t.$$

Therefore, the cumulative price $S^c_t(\kappa)$, $t \in [s,T]$ can be represented as follows

$$\begin{aligned} S^c_t(\kappa) &= S_t(\kappa) + \int_{]s,t]} \delta(u)\, dH_u - \kappa \int_s^t (1-H_u)\, du \\ &= S_t(\kappa) + \int_{]s,t]} \delta(u)\, dM_u - \int_s^t (1-H_u)(\kappa - \delta(u)\gamma(u))\, du \\ &= S^c_s(\kappa) + \int_{]s,t]} \big(\delta(u) - S_{u-}(\kappa)\big)\, dM_u, \end{aligned}$$

where the last equality follows from (2.40) and the equality $S_s(\kappa) = S^c_s(\kappa)$, which in turns is clear since $D_s = 0$ (cf. (2.33)). □

Equality (2.40) emphasizes the fact that a single cash flow of $\delta(\tau)$ occurring at time τ can be formally treated as a dividend stream at the rate $\delta(t)\gamma(t)$ paid continuously prior to default. It is clear that we also have

$$dS_t(\kappa) = -\widetilde{S}_{t-}(\kappa)\, dM_t + (1-H_t)(\kappa - \delta(t)\gamma(t))\, dt.$$

2.4.4 Replication of a Defaultable Claim

Our goal is to show that, in order to replicate a general defaultable claim, it suffices to trade dynamically in two assets: a CDS maturing at T and the savings account B, assumed here to be constant. Since one may always work with discounted values, the last assumption is not restrictive. Moreover, it is also possible to take a CDS with any maturity $U \geq T$.

Let ϕ^0, ϕ^1 be $\mathbb{H}$-predictable processes and let $C : [0,T] \to \mathbb{R}$ be a function of finite variation with $C(0) = 0$. We say that $(\phi, C) = (\phi^0, \phi^1, C)$ is a *self-financing trading strategy with dividend stream* C if the wealth process $V(\phi, C)$, defined as

$$V_t(\phi, C) = \phi^0_t + \phi^1_t S_t(\kappa),$$

where $S_t(\kappa)$ is the price of a CDS at time t, satisfies

$$dV_t(\phi, C) = \phi^1_t \big(dS_t(\kappa) + dD_t\big) - dC(t) = \phi^1_t\, dS^c_t(\kappa) - dC(t),$$

where the dividend process D of a CDS is in turn given by (2.33). Note that C represents both outflows and infusions of funds. It will be used to cover the running cash flows associated with a claim we wish to replicate.

Consider a defaultable claim (X, A, Z, τ) where X is a constant, A is a continuous function of finite variation, and Z is some recovery function. In order to define replication of a defaultable claim (X, A, Z, τ), it suffices to consider trading strategies on the random interval $[0, \tau \wedge T]$.

Definition 2.4.3. We say that a trading strategy (ϕ, C) *replicates a defaultable claim* (X, A, Z, τ) if:
(i) the processes $\phi = (\phi^0, \phi^1)$ and $V(\phi, C)$ are stopped at $\tau \wedge T$,
(ii) $C(\tau \wedge t) = A(\tau \wedge t)$ for every $t \in [0, T]$,
(iii) the equality $V_{\tau\wedge T}(\phi, C) = Y$ holds, where the random variable Y equals

$$Y = X\mathbb{1}_{\{\tau>T\}} + Z(\tau)\mathbb{1}_{\{\tau\le T\}}. \tag{2.42}$$

Remark 2.4.2. Alternatively, one may say that a self-financing trading strategy $\phi = (\phi, 0)$ (i.e., a trading strategy with $C = 0$) replicates a defaultable claim (X, A, Z, τ) if and only if $V_{\tau\wedge T}(\phi) = \widehat{Y}$, where we set

$$\widehat{Y} = X\mathbb{1}_{\{\tau>T\}} + A(\tau \wedge T) + Z(\tau)\mathbb{1}_{\{\tau\le T\}}. \tag{2.43}$$

However, in the case of non-zero (possibly random) interest rates, it is more convenient to define replication of a defaultable claim via Definition 2.4.3, since the running payoffs specified by A are distributed over time and thus, in principle, they need to be discounted accordingly (this does not show in (2.43), since it is assumed here that $r = 0$).

Let us denote, for every $t \in [0, T]$,

$$\widetilde{Z}(t) = \frac{1}{G(t)}\left(XG(T) - \int_t^T Z(u)\,dG(u)\right)$$

and

$$\widetilde{A}(t) = \frac{1}{G(t)}\int_t^T G(u)\,dA(u).$$

Let π and $\widetilde{\pi}$ be the risk-neutral value and the pre-default risk-neutral value of a defaultable claim under $\mathbb{Q}$, so that $\pi_t = \mathbb{1}_{\{t<\tau\}}\widetilde{\pi}(t)$ for every $t \in [0, T]$. Also, let $\widehat{\pi}$ stand for its risk-neutral cumulative price. It is clear that the equalities $\widetilde{\pi}(0) = \pi(0) = \widehat{\pi}(0) = \mathbb{E}_{\mathbb{Q}}(\widehat{Y})$ are valid.

Proposition 2.4.2. *The pre-default risk-neutral value of a defaultable claim* (X, A, Z, τ) *equals* $\widetilde{\pi}(t) = \widetilde{Z}(t) + \widetilde{A}(t)$ *for every* $t \in [0, T[$ *(clearly,* $\widetilde{\pi}(T) = 0$*). Therefore, for every* $t \in [0, T[$,

$$d\widetilde{\pi}(t) = \gamma(t)(\widetilde{\pi}(t) - Z(t))\,dt - dA(t). \tag{2.44}$$

Moreover

$$d\pi_t = -\widetilde{\pi}(t-)\,dM_t - \gamma(t)(1 - H_t)Z(t)\,dt - dA(t \wedge \tau) \tag{2.45}$$

and

$$d\widehat{\pi}_t = (Z(t) - \widetilde{\pi}(t-))\,dM_t.$$

Proof. The proof of equality $\widetilde{\pi}(t) = \widetilde{Z}(t) + \widetilde{A}(t)$ is similar to the derivation of formula (2.36). We have, for $t \in [0, T[$,

$$\begin{aligned}\pi_t &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau\}} Y + A(\tau \wedge T) - A(\tau \wedge t) \,\Big|\, \mathcal{H}_t\Big)\\ &= \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \Big(XG(T) - \int_t^T Z(u)\, dG(u)\Big)\\ &\quad + \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \int_t^T G(u)\, dA(u)\\ &= \mathbb{1}_{\{t<\tau\}} (\widetilde{Z}(t) + \widetilde{A}(t)) = \mathbb{1}_{\{t<\tau\}} \widetilde{\pi}(t).\end{aligned}$$

By elementary computations, we obtain the following equalities

$$d\widetilde{Z}(t) = \gamma(t)(\widetilde{Z}(t) - Z(t))\, dt$$

and

$$d\widetilde{A}(t) = \gamma(t)\widetilde{A}(t)\, dt - dA(t),$$

so that (2.44) holds. Formula (2.45) follows easily from (2.44) and the integration by parts formula applied to the equality $\pi_t = (1 - H_t)\widetilde{\pi}(t)$ (see the proof of Lemma 2.4.2 for similar computations). The last formula is also easy to check. □

The next proposition shows that the risk-neutral value of a defaultable claim is also its replication price, that is, a defaultable claim derives its value from the price of the traded CDS.

Theorem 2.4.1. *Assume that the inequality $\widetilde{S}_t(\kappa) \neq \delta(t)$ holds for every $t \in [0, T]$. Let $\phi^1_t = \widetilde{\phi}_1(\tau \wedge t)$, where the function $\widetilde{\phi}_1 : [0, T] \to \mathbb{R}$ is given by the formula*

$$\widetilde{\phi}_1(t) = \frac{Z(t) - \widetilde{\pi}(t-)}{\delta(t) - \widetilde{S}_t(\kappa)} \tag{2.46}$$

and let $\phi^0_t = V_t(\phi, A) - \phi^1_t S_t(\kappa)$, where the process $V(\phi, A)$ is given by the formula

$$V_t(\phi, A) = \widetilde{\pi}(0) + \int_{]0, \tau\wedge t]} \widetilde{\phi}_1(u)\, dS^c_u(\kappa) - A(t \wedge \tau). \tag{2.47}$$

Then the strategy (ϕ^0, ϕ^1, A) replicates the defaultable claim (X, A, Z, τ).

Proof. Assume first that a trading strategy $\phi = (\phi^0, \phi^1, C)$ is a replicating strategy for (X, A, Z, τ). By virtue of condition (i) in Definition 2.4.3 we have $C = A$ and thus, by combining (2.47) with (2.41), we obtain

$$dV_t(\phi, A) = \phi^1_t (\delta(t) - \widetilde{S}_t(\kappa))\, dM_t - dA(\tau \wedge t)$$

For ϕ^1 given by (2.46), we thus obtain

$$dV_t(\phi, A) = (Z(t) - \widetilde{\pi}(t-))\, dM_t - dA(\tau \wedge t).$$

It is thus clear that if we take $\phi^1_t = \widetilde{\phi}_1(\tau \wedge t)$ with $\widetilde{\phi}_1$ given by (2.46), and the initial condition $V_0(\phi, A) = \widetilde{\pi}(0) = \pi_0$, then we have that $V_t(\phi, A) = \widetilde{\pi}(t)$ for every $t \in [0, T[$ on the event $\{t < \tau\}$. By examining, in particular, the jump of the wealth process $V(\phi, A)$ at the moment of default, one may check that all conditions of Definition 2.4.3 are indeed satisfied. □

Remark 2.4.3. Of course, if we take as (X, A, Z, τ) a CDS with spread κ and protection function δ, then we have $Z(t) = \delta(t)$ and $\widetilde{\pi}(t-) = \widetilde{\pi}(t) = \widetilde{S}_t(\kappa)$, so that $\phi^1_t = 1$ for every $t \in [0, T]$.

2.5 Basket Credit Derivatives

In this section, we shall examine hedging of first-to-default basket claims with single-name credit default swaps on the underlying n credit names, denoted as $1, 2, \dots, n$ (see Bielecki et al. [19] and Schmidt and Ward [160]). The standing Assumption 2.4.1 is maintained throughout this section.

Let the random times $\tau_1, \tau_2, \dots, \tau_n$, given on a common probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, represent the default times of n reference credit names. We denote by

$$\tau_{(1)} = \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n = \min(\tau_1, \tau_2, \dots, \tau_n)$$

the moment of the first default, so that no defaults are observed on the event $\{t < \tau_{(1)}\}$. Let

$$F(t_1, t_2, \dots, t_n) = \mathbb{Q}(\tau_1 \le t_1, \tau_2 \le t_2, \dots, \tau_n \le t_n)$$

be the joint probability distribution function of default times. We assume that the probability distribution of default times is jointly continuous, and we write $f(t_1, t_2, \dots, t_n)$ to denote the joint probability density function. Also, let

$$G(t_1, t_2, \dots, t_n) = \mathbb{Q}(\tau_1 > t_1, \tau_2 > t_2, \dots, \tau_n > t_n)$$

stand for the joint probability that the names $1, 2, \dots, n$ have survived up to times $t_1, t_2, \dots, t_n$. In particular, the *joint survival function* is given by the formula, for every $t \in \mathbb{R}_+$,

$$G(t, \dots, t) = \mathbb{Q}(\tau_1 > t, \tau_2 > t, \dots, \tau_n > t) = \mathbb{Q}(\tau_{(1)} > t) = G_{(1)}(t).$$

For $i = 1, 2, \dots, n$, we define the *default indicator process* $H^i_t = \mathbb{1}_{\{t \ge \tau_i\}}$ and the corresponding filtration $\mathbb{H}^i = (\mathcal{H}^i_t)_{t \in \mathbb{R}_+}$ where $\mathcal{H}^i_t = \sigma(H^i_u : u \le t)$.

Let $\mathbb{G}$ be the joint filtration generated by default indicator processes $H^1, H^2, \ldots, H^n$, so that $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2 \vee \cdots \vee \mathbb{H}^n$. It is clear that $\tau_{(1)}$ is a $\mathbb{G}$-stopping time, as the minimum of a finite family of $\mathbb{G}$-stopping times.

Finally, we define the process $H_t^{(1)} = \mathbb{1}_{\{t \geq \tau_{(1)}\}}$ and the associated filtration $\mathbb{H}^{(1)} = (\mathcal{H}_t^{(1)})_{t \in \mathbb{R}_+}$ where $\mathcal{H}_t^{(1)} = \sigma(H_u^{(1)} : u \leq t)$.

Since we postulate that $\mathbb{Q}(\tau_i = \tau_j) = 0$ for any $i \neq j$, $i, j = 1, 2, \ldots, n$, we also have that

$$H_t^{(1)} = H_{t \wedge \tau_{(1)}}^{(1)} = \sum_{i=1}^{n} H_{t \wedge \tau_{(1)}}^i .$$

We now fix a finite horizon date $T > 0$, and we make the standing assumption that

$$G_{(1)}(T) = \mathbb{Q}(\tau_{(1)} > T) > 0.$$

For any $t \in [0, T]$, the event $\{t < \tau_{(1)}\}$ is an atom of the σ-field $\mathcal{G}_t$. Hence the following simple, but useful, result.

Lemma 2.5.1. *Let X be a $\mathbb{Q}$-integrable stochastic process on $(\Omega, \mathcal{G}, \mathbb{Q})$. Then*

$$\mathbb{1}_{\{t < \tau_{(1)}\}} \, \mathbb{E}_{\mathbb{Q}}(X_t \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t < \tau_{(1)}\}} \widetilde{X}(t),$$

where the function $\widetilde{X} : [0, T] \to \mathbb{R}$ is given by the formula

$$\widetilde{X}(t) = \frac{\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{t < \tau_{(1)}\}} X_t\big)}{G_{(1)}(t)}.$$

If X is a $\mathbb{G}$-adapted, $\mathbb{Q}$-integrable stochastic process then, for every $t \in [0, T]$,

$$X_t = \mathbb{1}_{\{t < \tau_{(1)}\}} \widetilde{X}(t) + \mathbb{1}_{\{t \geq \tau_{(1)}\}} X_t .$$

By convention, the function $\widetilde{X} : [0, T] \to \mathbb{R}$ is called the *pre-default value* of the process X.

2.5.1 First-to-Default Intensities

In this section, we introduce the notion of the *first-to-default intensity.* This natural concept will prove useful in the valuation and hedging of a first-to-default claim.

Definition 2.5.1. The function $\widetilde{\lambda}_i : \mathbb{R}_+ \to \mathbb{R}_+$, given by the formula

$$\widetilde{\lambda}_i(t) = \lim_{h \downarrow 0} \frac{1}{h} \, \mathbb{Q}(t < \tau_i \leq t + h \,|\, \tau_{(1)} > t),$$

is called the *ith first-to-default intensity*. The function $\widetilde{\lambda} : \mathbb{R}_+ \to \mathbb{R}_+$, given by

$$\widetilde{\lambda}(t) = \lim_{h \downarrow 0} \frac{1}{h} \, \mathbb{Q}(t < \tau_{(1)} \leq t + h \,|\, \tau_{(1)} > t), \tag{2.48}$$

is called the *first-to-default intensity*.

Let us denote

$$\partial_i G(t, \ldots, t) = \frac{\partial G(t_1, t_2, \ldots, t_n)}{\partial t_i}\Big|_{t_1 = t_2 = \cdots = t_n = t}.$$

Then we have the following elementary lemma summarizing the properties of first-to-default intensities $\widetilde{\lambda}_i$ and $\widetilde{\lambda}$.

Lemma 2.5.2. *The ith first-to-default intensity $\widetilde{\lambda}_i$ satisfies*

$$\widetilde{\lambda}_i(t) = \frac{\int_t^\infty \ldots \int_t^\infty f(u_1, \ldots, u_{i-1}, t, u_{i+1}, \ldots, u_n) \, du_1 \ldots du_{i-1} du_{i+1} \ldots du_n}{G(t, \ldots, t)}$$

$$= \frac{\int_t^\infty \ldots \int_t^\infty F(du_1, \ldots, du_{i-1}, t, du_{i+1}, \ldots, du_n)}{G_{(1)}(t)} = -\frac{\partial_i G(t, \ldots, t)}{G_{(1)}(t)}.$$

The first-to-default intensity $\widetilde{\lambda}$ satisfies

$$\widetilde{\lambda}(t) = -\frac{1}{G_{(1)}(t)} \frac{dG_{(1)}(t)}{dt} = \frac{f_{(1)}(t)}{G_{(1)}(t)},$$

where $f_{(1)}(t)$ is the probability density function of the random time $\tau_{(1)}$. The equality $\widetilde{\lambda}(t) = \sum_{i=1}^n \widetilde{\lambda}_i(t)$ holds for every $t \in \mathbb{R}_+$.

Proof. Clearly

$$\widetilde{\lambda}_i(t) = \lim_{h \downarrow 0} \frac{1}{h} \frac{\int_t^\infty \ldots \int_t^{t+h} \ldots \int_t^\infty f(u_1, \ldots, u_i, \ldots, u_n) \, du_1 \ldots du_i \ldots du_n}{G(t, \ldots, t)}$$

and thus the first asserted formula follows. The second equality follows directly from (2.48) and the definition of the joint survival function $G_{(1)}$. Finally, equality $\widetilde{\lambda}(t) = \sum_{i=1}^n \widetilde{\lambda}_i(t)$ is equivalent to the equality

$$\lim_{h \downarrow 0} \frac{1}{h} \sum_{i=1}^n \mathbb{Q}(t < \tau_i \leq t + h \,|\, \tau_{(1)} > t) = \lim_{h \downarrow 0} \frac{1}{h} \, \mathbb{Q}(t < \tau_{(1)} \leq t + h \,|\, \tau_{(1)} > t),$$

which in turn is easy to establish. □

Remarks 2.5.1. The ith first-to-default intensity $\widetilde{\lambda}_i$ should not be confused with the *marginal intensity function* λ_i of τ_i, which is defined as

$$\lambda_i(t) = \frac{f_i(t)}{G_i(t)}, \quad \forall\, t \in \mathbb{R}_+,$$

where f_i is the marginal probability density function of τ_i, that is,

$$f_i(t) = \int_0^\infty \dots \int_0^\infty f(u_1, \dots, u_{i-1}, t, u_{i+1}, \dots, u_n)\, du_1 \dots du_{i-1} du_{i+1} \dots du_n$$

and where $G_i(t) = 1 - F_i(t) = \int_t^\infty f_i(u)\, du$. Indeed, we have that $\widetilde{\lambda}_i \neq \lambda_i$, in general. However, if $\tau_1, \dots, \tau_n$ are mutually independent under $\mathbb{Q}$ then $\widetilde{\lambda}_i = \lambda_i$, that is, the first-to-default and marginal default intensities coincide.

It is also rather clear that the first-to-default intensity $\widetilde{\lambda}$ is not equal to the sum of marginal default intensities, that is, we have that $\widetilde{\lambda}(t) \neq \sum_{i=1}^n \lambda_i(t)$, in general.

2.5.2 First-to-Default Representation Theorem

We will now prove an integral representation theorem for any $\mathbb{G}$-martingale stopped at $\tau_{(1)}$ with respect to some finite collection of $\mathbb{G}$-martingales stopped at $\tau_{(1)}$. To this end, we define, for every $i = 1, 2, \dots n$,

$$\widehat{M}^i_t = H^i_{t \wedge \tau_{(1)}} - \int_0^{t \wedge \tau_{(1)}} \widetilde{\lambda}_i(u)\, du, \quad \forall\, t \in \mathbb{R}_+. \tag{2.49}$$

Then we have the following result, referred to as the *first-to-default predictable representation theorem.*

Proposition 2.5.1. *Consider the $\mathbb{G}$-martingale $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}}(Y \,|\, \mathcal{G}_t)$, $t \in [0, T]$, where Y is a $\mathbb{Q}$-integrable random variable given by the expression*

$$Y = \sum_{i=1}^n Z_i(\tau_i) \mathbb{1}_{\{\tau_i \le T,\, \tau_i = \tau_{(1)}\}} + X \mathbb{1}_{\{\tau_{(1)} > T\}} \tag{2.50}$$

for some functions $Z_i : [0, T] \to \mathbb{R}$, $i = 1, 2, \dots, n$ and some constant X. Then $\widehat{M}$ admits the following representation

$$\widehat{M}_t = \mathbb{E}_{\mathbb{Q}}(Y) + \sum_{i=1}^n \int_{]0,t]} h_i(u)\, d\widehat{M}^i_u \tag{2.51}$$

where the functions h_i, $i = 1, 2, \dots, n$ are given by

$$h_i(t) = Z_i(t) - \widehat{M}_{t-} = Z_i(t) - \widetilde{M}(t-), \quad \forall\, t \in [0, T], \tag{2.52}$$

where $\widetilde{M}$ *is the unique function such that* $\widehat{M}_t \mathbb{1}_{\{t<\tau_{(1)}\}} = \widetilde{M}(t)\mathbb{1}_{\{t<\tau_{(1)}\}}$ *for every* $t \in [0,T]$. *The function* $\widetilde{M}$ *satisfies* $\widetilde{M}_0 = \mathbb{E}_{\mathbb{Q}}(Y)$ *and*

$$d\widetilde{M}(t) = \sum_{i=1}^{n} \widetilde{\lambda}_i(t)\big(\widetilde{M}(t) - Z_i(t)\big)\,dt. \tag{2.53}$$

More explicitly,

$$\widetilde{M}(t) = \mathbb{E}_{\mathbb{Q}}(Y)\exp\Big(\int_0^t \widetilde{\lambda}(s)\,ds\Big) - \int_0^t \sum_{i=1}^{n} \widetilde{\lambda}_i(s)Z_i(s)\exp\Big(\int_s^t \widetilde{\lambda}(u)\,du\Big)ds.$$

Proof. To alleviate notation, we provide the proof of this result in a bivariate setting only, so that $\tau_{(1)} = \tau_1 \wedge \tau_2$ and $\mathcal{G}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2$. We start by noting that

$$\begin{aligned}\widehat{M}_t &= \mathbb{E}_{\mathbb{Q}}(Z_1(\tau_1)\mathbb{1}_{\{\tau_1\le T,\,\tau_2>\tau_1\}}\,|\,\mathcal{G}_t) + \mathbb{E}_{\mathbb{Q}}(Z_2(\tau_2)\mathbb{1}_{\{\tau_2\le T,\,\tau_1>\tau_2\}}\,|\,\mathcal{G}_t)\\ &\quad + \mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{\tau_{(1)}>T\}}\,|\,\mathcal{G}_t),\end{aligned}$$

and thus (see Lemma 2.5.1)

$$\mathbb{1}_{\{t<\tau_{(1)}\}}\widehat{M}_t = \mathbb{1}_{\{t<\tau_{(1)}\}}\widetilde{M}(t) = \mathbb{1}_{\{t<\tau_{(1)}\}}\sum_{i=1}^{3}\widetilde{Y}^i(t)$$

where the auxiliary functions $\widetilde{Y}^i : [0,T] \to \mathbb{R}$, $i = 1,2,3$, are given by

$$\begin{aligned}\widetilde{Y}^1(t) &= \frac{\int_t^T du Z_1(u)\int_u^\infty dv f(u,v)}{G_{(1)}(t)},\\ \widetilde{Y}^2(t) &= \frac{\int_t^T dv Z_2(v)\int_v^\infty du f(u,v)}{G_{(1)}(t)},\\ \widetilde{Y}^3(t) &= \frac{XG_{(1)}(T)}{G_{(1)}(t)}.\end{aligned}$$

By elementary calculations and using Lemma 2.5.2, we obtain

$$\begin{aligned}\frac{d\widetilde{Y}^1(t)}{dt} &= -\frac{Z_1(t)\int_t^\infty dv f(t,v)}{G_{(1)}(t)} - \frac{\int_t^T du Z_1(u)\int_u^\infty dv f(u,v)}{G^2_{(1)}(t)}\,\frac{dG_{(1)}(t)}{dt}\\ &= -Z_1(t)\,\frac{\int_t^\infty dv f(t,v)}{G_{(1)}(t)} - \frac{\widetilde{Y}^1(t)}{G_{(1)}(t)}\,\frac{dG_{(1)}(t)}{dt}\\ &= -Z_1(t)\widetilde{\lambda}_1(t) + \widetilde{Y}^1(t)(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)),\end{aligned} \tag{2.54}$$

and thus, by the symmetry of the problem,

$$\frac{d\widetilde{Y}^2(t)}{dt} = -Z_2(t)\widetilde{\lambda}_2(t) + \widetilde{Y}^2(t)(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)). \tag{2.55}$$

Moreover,

$$\frac{d\widetilde{Y}^3(t)}{dt} = -\frac{XG_{(1)}(T)}{G^2_{(1)}(t)}\,\frac{dG_{(1)}(t)}{dt} = \widetilde{Y}^3(t)(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)). \tag{2.56}$$

Therefore, recalling that $\widetilde{M}(t) = \sum_{i=1}^3 \widetilde{Y}^i(t)$, we obtain from (2.54)–(2.56)

$$d\widetilde{M}(t) = -\widetilde{\lambda}_1(t)\big(Z_1(t) - \widetilde{M}(t)\big)\,dt - \widetilde{\lambda}_2(t)\big(Z_2(t) - \widetilde{M}(t)\big)\,dt. \tag{2.57}$$

Consequently, since the function $\widetilde{M}$ is continuous, we deduce that, on the event $\{\tau_{(1)} > t\}$,

$$d\widehat{M}_t = -\widetilde{\lambda}_1(t)\big(Z_1(t) - \widehat{M}_{t-}\big)\,dt - \widetilde{\lambda}_2(t)\big(Z_2(t) - \widehat{M}_{t-}\big)\,dt.$$

We shall now check that both sides of equality (2.51) coincide at time $\tau_{(1)}$ on the event $\{\tau_{(1)} \le T\}$. To this end, we note that, on the event $\{\tau_{(1)} \le T\}$,

$$\widehat{M}_{\tau_{(1)}} = Z_1(\tau_1)\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} + Z_2(\tau_2)\mathbb{1}_{\{\tau_{(1)}=\tau_2\}},$$

whereas the right-hand side in (2.51) is equal to

$$\begin{aligned}
\widehat{M}_0 &+ \int_{]0,\tau_{(1)}[} h_1(u)\,d\widehat{M}^1_u + \int_{]0,\tau_{(1)}[} h_2(u)\,d\widehat{M}^2_u \\
&+ \mathbb{1}_{\{\tau_{(1)}=\tau_1\}} \int_{[\tau_{(1)}]} h_1(u)\,dH^1_u + \mathbb{1}_{\{\tau_{(1)}=\tau_2\}} \int_{[\tau_{(1)}]} h_2(u)\,dH^2_u \\
&= \widetilde{M}(\tau_{(1)}-) + \big(Z_1(\tau_1) - \widetilde{M}(\tau_{(1)}-)\big)\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} \\
&+ \big(Z_2(\tau_2) - \widetilde{M}(\tau_{(1)}-)\big)\mathbb{1}_{\{\tau_{(1)}=\tau_2\}} \\
&= Z_1(\tau_1)\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} + Z_2(\tau_2)\mathbb{1}_{\{\tau_{(1)}=\tau_2\}}
\end{aligned}$$

as $\widetilde{M}(\tau_{(1)}-) = \widehat{M}_{\tau_{(1)}-}$. Since the processes on both sides of equality (2.51) are stopped at time $\tau_{(1)}$, we conclude that equality (2.51) is valid for every $t \in [0,T]$. Let us finally observe that formula (2.53) was also established in the proof (cf. formula (2.57)). □

The next result shows that the processes $\widehat{M}^i$ are in fact $\mathbb{G}$-martingales. They will be referred to as the *basic first-to-default martingales.*

Corollary 2.5.1. *For each $i = 1, 2, \ldots, n$, the process $\widehat{M}^i$ given by the formula (2.49) is a $\mathbb{G}$-martingale stopped at time $\tau_{(1)}$.*

Proof. Let us fix $k \in \{1, 2, \ldots, n\}$. We start by noting that the process $\widehat{M}^k$ is manifestly stopped at $\tau_{(1)}$. We also observe that

$$\widetilde{M}^k(t) = -\int_0^t \widetilde{\lambda}_i(u)\,du$$

is the unique function such that, for every $t \in [0, T]$,

$$\mathbb{1}_{\{t<\tau_{(1)}\}}\widehat{M}_t^i = \mathbb{1}_{\{t<\tau_{(1)}\}}\widetilde{M}^k(t).$$

Let us take $h_k(t) = 1$ and $h_i(t) = 0$ for any $i \neq k$ in formula (2.51) or, equivalently, let us set

$$Z_k(t) = 1 + \widetilde{M}^k(t), \quad Z_i(t) = \widetilde{M}^k(t),\ i \neq k,$$

in definition (2.50) of the random variable Y. Moreover, let a constant X in (2.50) be chosen in such a way that the random variable Y satisfies $\mathbb{E}_{\mathbb{Q}}(Y) = \widehat{M}_0^k$. Then we may deduce from (2.51) that $\widehat{M}^k = \widehat{M}$ and thus we deduce from Proposition 2.5.1 that $\widehat{M}^k$ is indeed a $\mathbb{G}$-martingale. □

2.5.3 Price Dynamics of Credit Default Swaps

As primary traded assets in the market model under consideration, we take the constant savings account and a family of single-name CDSs with default protections δ_i and spreads κ_i for $i = 1, 2, \ldots, n$.

For convenience, we assume that the CDSs have the same maturity T, but this assumption can be easily relaxed. The ith traded CDS is formally defined by its dividend process $(D_t^i,\ t \in [0, T])$, which is given by the formula

$$D_t^i = \int_{]0,t]} \delta_i(u)\,dH_u^i - \kappa_i(t \wedge \tau_i).$$

Consequently, the price at time t of the ith CDS equals

$$S_t^i(\kappa_i) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau_i\leq T\}}\delta_i(\tau_i)\,|\,\mathcal{G}_t) - \kappa_i\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{t<\tau_i\}}\big((\tau_i \wedge T) - t\big)\,\big|\,\mathcal{G}_t\big).$$

To replicate a first-to-default claim, we only need to examine the dynamics of each CDS on the interval $[0, \tau_{(1)} \wedge T]$. The following lemma will prove useful in this regard.

Lemma 2.5.3. *We have, on the event* $\{t < \tau_{(1)}\}$,

$$\begin{aligned} S_t^i(\kappa_i) = {} & \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\leq T\}}\delta_i(\tau_{(1)}) + \sum_{j\neq i}\mathbb{1}_{\{t<\tau_{(1)}=\tau_j\leq T\}}S^i_{\tau_{(1)}}(\kappa_i)\,\Big|\,\mathcal{G}_t\Big) \\ & - \mathbb{E}_{\mathbb{Q}}\Big(\kappa_i\mathbb{1}_{\{t<\tau_{(1)}\}}(\tau_{(1)} \wedge T - t)\,\Big|\,\mathcal{G}_t\Big). \end{aligned}$$

Proof. We first note that the price $S_t^i(\kappa_i)$ can be represented as follows, on the event $\{t < \tau_{(1)}\}$,

$$\begin{aligned} S_t^i(\kappa_i) &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\le T\}}\delta_i(\tau_{(1)})\,\Big|\,\mathcal{G}_t\Big) \\ &\quad + \sum_{j\neq i}\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_{(1)}=\tau_j\le T\}}\mathbb{1}_{\{\tau_{(1)}<\tau_i\le T\}}\delta_i(\tau_i\wedge T)\,\Big|\,\mathcal{G}_t\Big) \\ &\quad - \kappa_i\sum_{j\neq i}\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_{(1)}=\tau_j\le T\}}\mathbb{1}_{\{\tau_{(1)}<\tau_i\}}(\tau_i-\tau_{(1)})\,\Big|\,\mathcal{G}_t\Big) \\ &\quad - \kappa_i\,\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{t<\tau_{(1)}\}}(\tau_{(1)}\wedge T - t)\,\big|\,\mathcal{G}_t\big). \end{aligned}$$

By conditioning first on the σ-field $\mathcal{G}_{\tau_{(1)}}$, we obtain the stated expression for $S_t^i(\kappa_i)$. The details are left to the reader. □

The representation established in Lemma 2.5.3 is by no means surprising; it merely shows that in order to compute the price of a CDS prior to the first default, we can either do the computations in a single step, by considering the cash flows occurring on $]t, \tau_i \wedge T]$ or, alternatively, we can first compute the price of the contract at time $\tau_{(1)} \wedge T$ and subsequently value all cash flows occurring between t and $\tau_{(1)} \wedge T$.

In view of Lemma 2.5.3, we can argue that in what follows, instead of considering the original ith CDS maturing at T, we can deal with the corresponding synthetic CDS contract with the random maturity $\tau_{(1)} \wedge T$.

Similarly as in Section 2.4.1, we will write $S_t^i(\kappa_i) = \mathbb{1}_{\{t<\tau_{(1)}\}}\widetilde{S}_t^i(\kappa_i)$, where the pre-default price $\widetilde{S}_t^i(\kappa_i)$ satisfies

$$\widetilde{S}_t^i(\kappa_i) = \widetilde{P}^i(t,T) - \kappa_i \widetilde{A}^i(t,T),$$

where $\widetilde{P}^i(t,T)$ and $\kappa_i\widetilde{A}^i(t,T)$ stand for the pre-default values of the protection leg and the fee leg, respectively.

For any $j \neq i$, we define a function $S_{t|j}^i(\kappa_i) : [0,T] \to \mathbb{R}$, which represents the price of the ith CDS at time t on the event $\{\tau_{(1)} = \tau_j = t\}$. Formally, this quantity is defined as the unique function satisfying

$$\mathbb{1}_{\{\tau_{(1)}=\tau_j\le T\}}S_{\tau_{(1)}}^i(\kappa_i) = \mathbb{1}_{\{\tau_{(1)}=\tau_j\le T\}}S_{\tau_{(1)}|j}^i(\kappa_i),$$

so that

$$\mathbb{1}_{\{\tau_{(1)}\le T\}}S_{\tau_{(1)}}^i(\kappa_i) = \sum_{j\neq i}\mathbb{1}_{\{\tau_{(1)}=\tau_j\le T\}}S_{\tau_{(1)}|j}^i(\kappa_i).$$

Let us examine, for instance, the case of two credit names. Then the function $S_{t|2}^1(\kappa_1)$, $t \in [0,T]$, represents the price of the first CDS at time t on the event $\{\tau_{(1)} = \tau_2 = t\}$.

Lemma 2.5.4. *The function* $S^1_{v|2}(\kappa_1)$, $v \in [0,T]$, *equals*

$$S^1_{v|2}(\kappa_1) = \frac{\int_v^T \delta_1(u) f(u,v) du}{\int_v^\infty f(u,v)\, du} - \kappa_1 \frac{\int_v^T du \int_u^\infty dz f(z,v)}{\int_v^\infty f(u,v)\, du}. \tag{2.58}$$

Proof. Note that the conditional cumulative distribution function of τ_1 given that $\tau_1 > \tau_2 = v$ equals, for $u \in [v,\infty]$,

$$\mathbb{Q}(\tau_1 \le u \,|\, \tau_1 > \tau_2 = v) = F_{\tau_1|\tau_1>\tau_2=v}(u) = \frac{\int_v^u f(z,v)\, dz}{\int_v^\infty f(z,v)\, dz},$$

so that the conditional tail equals, for $u \in [v,\infty]$,

$$G_{\tau_1|\tau_1>\tau_2=v}(u) = 1 - F_{\tau_1|\tau_1>\tau_2=v}(u) = \frac{\int_u^\infty f(z,v)\, dz}{\int_v^\infty f(z,v)\, dz}.$$

Let J be the right-hand side of (2.58). It is clear that

$$J = -\int_v^T \delta_1(u)\, dG_{\tau_1|\tau_1>\tau_2=v}(u) - \kappa_1 \int_v^T G_{\tau_1|\tau_1>\tau_2=v}(u)\, du.$$

Combining Lemma 2.4.1 with the fact that $S^1_{\tau_{(1)}}(\kappa_i)$ is equal to the conditional expectation with respect to σ-field $\mathcal{G}_{\tau_{(1)}}$ of the cash flows of the ith CDS on $]\tau_{(1)} \vee \tau_i, \tau_i \wedge T]$, we conclude that J coincides with $S^1_{v|2}(\kappa_1)$, the price of the first CDS on the event $\{\tau_{(1)} = \tau_2 = v\}$. □

The following result extends Lemma 2.4.2.

Lemma 2.5.5. *The dynamics of the pre-default price* $\widetilde{S}^i_t(\kappa_i)$ *are*

$$d\widetilde{S}^i_t(\kappa_i) = \widetilde{\lambda}(t)\widetilde{S}^i_t(\kappa_i)\, dt + \Big(\kappa_i - \delta_i(t)\widetilde{\lambda}_i(t) - \sum_{j\neq i}^n S^i_{t|j}(\kappa_i)\widetilde{\lambda}_i(t)\Big)dt \tag{2.59}$$

where $\widetilde{\lambda}(t) = \sum_{i=1}^n \widetilde{\lambda}_i(t)$ *or, equivalently,*

$$\begin{aligned} d\widetilde{S}^i_t(\kappa_i) &= \widetilde{\lambda}_i(t)\big(\widetilde{S}^i_t(\kappa_i) - \delta_i(t)\big)\, dt \\ &\quad + \sum_{j\neq i} \widetilde{\lambda}_j(t)\big(\widetilde{S}^i_t(\kappa_i) - S^i_{t|j}(\kappa_i)\big)\, dt + \kappa_i\, dt. \end{aligned} \tag{2.60}$$

The cumulative price of the ith CDS stopped at $\tau_{(1)}$ *satisfies*

$$\begin{aligned} S^{c,i}_t(\kappa_i) &= S^i_t(\kappa_i) + \int_{]0,t]} \delta_i(u)\, dH^i_{u\wedge\tau_{(1)}} \\ &\quad + \sum_{j\neq i} \int_{]0,t]} S^i_{u|j}(\kappa_i)\, dH^j_{u\wedge\tau_{(1)}} - \kappa_i(\tau_{(1)} \wedge t), \end{aligned} \tag{2.61}$$

and thus

$$dS_t^{c,i}(\kappa_i) = \big(\delta_i(t) - \widetilde{S}^i_{t-}(\kappa_i)\big)\, d\widehat{M}^i_t + \sum_{j\neq i} \big(S^i_{t|j}(\kappa_i) - \widetilde{S}^i_{t-}(\kappa_i)\big)\, d\widehat{M}^j_t. \quad (2.62)$$

Proof. We shall consider the case $n = 2$. Using the formula derived in Lemma 2.5.3, we obtain

$$\widetilde{P}^1(t,T) = \frac{\int_t^T du\, \delta_1(u) \int_u^\infty dv f(u,v)}{G_{(1)}(t)} + \frac{\int_t^T dv\, S^1_{v|2}(\kappa_1) \int_v^\infty du f(u,v)}{G_{(1)}(t)}.$$

By adapting equality (2.54), we get

$$d\widetilde{P}^1(t,T) = \Big((\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t))\widetilde{g}_1(t) - \widetilde{\lambda}_1(t)\delta_1(t) - \widetilde{\lambda}_2(t) S^1_{t|2}(\kappa_1)\Big) dt.$$

To establish (2.59)–(2.60), we need also to examine the fee leg. Its price equals

$$\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_{(1)}\}}\kappa_1\big((\tau_{(1)} \wedge T) - t\big) \,\Big|\, \mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau_{(1)}\}}\kappa_1 \widetilde{A}^i(t,T),$$

To evaluate the conditional expectation above, it suffices to use the cumulative distribution function $F_{(1)}$ of the random time $\tau_{(1)}$. As in Section 2.4.1 (see the proof of Lemma 2.4.1), we obtain

$$\widetilde{A}^i(t,T) = \frac{1}{G_{(1)}(t)} \int_t^T G_{(1)}(u)\, du, \quad (2.63)$$

and thus

$$d\widetilde{A}^i(t,T) = \big(1 + (\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t))\widetilde{A}^i(t,T)\big)\, dt.$$

Since $\widetilde{S}^1_t(\kappa_1) = \widetilde{P}^i(t,T) - \kappa_i \widetilde{A}^i(t,T)$, the formulae (2.59) and (2.60) follow. Formula (2.61) is rather clear. Finally, dynamics (2.62) can be deduced easily from (2.60) and (2.61). □

2.5.4 Valuation of a First-to-Default Claim

In this section, we shall analyze the risk-neutral valuation of first-to-default claims on a basket of n credit names.

Definition 2.5.2. A *first-to-default claim* (*FTDC*) with maturity T is a defaultable claim $(X, A, Z, \tau_{(1)})$ where X is a constant amount payable at maturity if no default occurs, $A : [0,T] \to \mathbb{R}$ with $A_0 = 0$ is a continuous function of bounded variation representing the dividend stream up to $\tau_{(1)}$, and $Z = (Z_1, Z_2, \ldots, Z_n)$ is the vector of functions $Z_i : [0,T] \to \mathbb{R}$ where $Z_i(\tau_{(1)})$ specifies the recovery received at time $\tau_{(1)}$ if the ith name is the first defaulted name, that is, on the event $\{\tau_i = \tau_{(1)} \leq T\}$.

We define the *risk-neutral value* π of an FTDC by setting

$$\pi_t = \sum_{i=1}^{n} \mathbb{E}_{\mathbb{Q}}\Big(Z_i(\tau_i)\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\le T\}} + \mathbb{1}_{\{t<\tau_{(1)}\}} \int_t^T (1-H_u^{(1)})\, dA(u) \,\Big|\, \mathcal{G}_t\Big) \\ + \mathbb{E}_{\mathbb{Q}}\Big(X\mathbb{1}_{\{\tau_{(1)}>T\}} \,\Big|\, \mathcal{G}_t\Big)$$

and the *risk-neutral cumulative value* $\widehat{\pi}$ of an FTDC by the formula

$$\widehat{\pi}_t = \sum_{i=1}^{n} \mathbb{E}_{\mathbb{Q}}\Big(Z_i(\tau_i)\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\le T\}} + \mathbb{1}_{\{t<\tau_{(1)}\}} \int_t^T (1-H_u^{(1)})\, dA(u) \,\Big|\, \mathcal{G}_t\Big) \\ + \mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{\tau_{(1)}>T\}}|\mathcal{G}_t) + \sum_{i=1}^{n}\int_{]0,t]} Z_i(u)\, dH^i_{u\wedge\tau_{(1)}} + \int_0^t (1-H_u^{(1)})\, dA(u)$$

where the last two terms represent the past dividends. Let us stress that the risk-neutral valuation of an FTDC will be later supported by replication arguments (see Theorem 2.5.1) and thus risk-neutral value π of an FTDC will be shown to be its replication price.

By the *pre-default risk-neutral value* associated with a $\mathbb{G}$-adapted process π, we mean the function $\widetilde{\pi}$ such that $\pi_t\mathbb{1}_{\{t<\tau_{(1)}\}} = \widetilde{\pi}(t)\mathbb{1}_{\{t<\tau_{(1)}\}}$ for every $t\in[0,T]$. Direct calculations lead to the following result, which can also be deduced from Proposition 2.5.1.

Lemma 2.5.6. *The pre-default risk-neutral value of an FTDC equals*

$$\widetilde{\pi}(t) = \sum_{i=1}^{n} \frac{\Psi_i(t)}{G_{(1)}(t)} + \frac{1}{G_{(1)}(t)}\int_t^T G_{(1)}(u)\, dA(u) + X\frac{G_{(1)}(T)}{G_{(1)}(t)} \tag{2.64}$$

where

$$\Psi_i(t) = \int_{u_i=t}^{T}\int_{u_1=u_i}^{\infty}\dots\int_{u_{i-1}=u_i}^{\infty}\int_{u_{i+1}=u_i}^{\infty}\dots\int_{u_n=u_i}^{\infty} Z_i(u_i) \\ F(du_1,\dots,du_{i-1},du_i,du_{i+1},\dots,du_n).$$

The next result extends Proposition 2.4.2 to the multi-name setup. Its proof is similar to the proof of Lemma 2.5.5 and thus it is omitted.

Proposition 2.5.2. *The pre-default risk-neutral value of an FTDC satisfies*

$$d\widetilde{\pi}(t) = \sum_{i=1}\widetilde{\lambda}_i(t)\big(\widetilde{\pi}(t) - Z_i(t)\big)\, dt - dA(t).$$

Moreover, the risk-neutral value of an FTDC satisfies

$$d\pi_t = -\sum_{i=1}^{n}\widetilde{\pi}(t-)\, d\widehat{M}^i_u - dA(\tau_{(1)}\wedge t) \tag{2.65}$$

and the risk-neutral cumulative value $\widehat{\pi}$ *of an FTDC satisfies*

$$d\widehat{\pi}_t = \sum_{i=1}^{n} \left(Z_i(t) - \widetilde{\pi}(t-)\right) d\widehat{M}_u^i.$$

2.5.5 Replication of a First-to-Default Claim

Let the savings account with the price $B = 1$ and single-name credit default swaps with prices $S^1(\kappa_1), \ldots, S^n(\kappa_n)$ be primary traded assets. We say that a $\mathbb{G}$-predictable process $\phi = (\phi^0, \phi^1, \ldots, \phi^n)$ and a function C of finite variation with $C(0) = 0$ define a *self-financing strategy with dividend stream* C if the wealth process $V(\phi, C)$, defined as

$$V_t(\phi, C) = \phi_t^0 + \sum_{i=1}^{n} \phi_t^i S_t^i(\kappa_i),$$

satisfies

$$dV_t(\phi, C) = \sum_{i=1}^{n} \phi_t^i \big(dS_t^i(\kappa_i) + dD_t^i\big) - dC(t) = \sum_{i=1}^{n} \phi_t^i \, dS_t^{c,i}(\kappa_i) - dC(t) \quad (2.66)$$

where $S^i(\kappa_i)$ ($S^{c,i}(\kappa_i)$, respectively) is the price (cumulative price, respectively) of the ith traded CDS.

Definition 2.5.3. We say that a trading strategy (ϕ, C) *replicates an FTDC* $(X, A, Z, \tau_{(1)})$ whenever the following conditions are satisfied:
(i) the processes $\phi = (\phi^0, \phi^1, \ldots, \phi^n)$ and $V(\phi, C)$ are stopped at $\tau_{(1)} \wedge T$,
(ii) $C(\tau_{(1)} \wedge t) = A(\tau_{(1)} \wedge t)$ for every $t \in [0, T]$,
(iii) the equality $V_{\tau_{(1)} \wedge T}(\phi, C) = Y$ holds, where the random variable Y equals

$$Y = X \mathbb{1}_{\{\tau_{(1)} > T\}} + \sum_{i=1}^{n} Z_i(\tau_{(1)}) \mathbb{1}_{\{\tau_i = \tau_{(1)} \le T\}}.$$

We are now in a position to extend Theorem 2.4.1 to the case of a first-to-default claim written on a basket of n reference credit names.

Theorem 2.5.1. *Assume that* $\det N(t) \neq 0$ *for every* $t \in [0, T]$*, where*

$$N(t) = \begin{bmatrix} \delta_1(t) - \widetilde{S}_t^1(\kappa_1) & S_{t|1}^2(\kappa_2) - \widetilde{S}_t^2(\kappa_2) & \ldots & S_{t|1}^n(\kappa_n) - \widetilde{S}_t^n(\kappa_n) \\ S_{t|2}^1(\kappa_1) - \widetilde{S}_t^1(\kappa_1) & \delta_2(t) - \widetilde{S}_t^2(\kappa_2) & \ldots & S_{t|2}^n(\kappa_n) - \widetilde{S}_t^n(\kappa_n) \\ \vdots & \vdots & \ddots & \vdots \\ S_{t|n}^1(\kappa_1) - \widetilde{S}_t^1(\kappa_1) & S_{t|n}^2(\kappa_1) - \widetilde{S}_t^2(\kappa_1) & \ldots & \delta_n(t) - \widetilde{S}_t^n(\kappa_n) \end{bmatrix}$$

For every $t \in [0,T]$, *let* $\widetilde{\phi}(t) = (\widetilde{\phi}_1(t), \widetilde{\phi}_2(t), \dots, \widetilde{\phi}_n(t))$ *be the unique solution to the linear equation* $N(t)\widetilde{\phi}(t) = h(t)$ *where* $h(t) = (h_1(t), h_2(t), \dots, h_n(t))$ *with* $h_i(t) = Z_i(t) - \widetilde{\pi}(t-)$ *and where* $\widetilde{\pi}$ *is given by Lemma 2.5.6. More explicitly, the functions* $\widetilde{\phi}_1, \widetilde{\phi}_2, \dots, \widetilde{\phi}_n$ *satisfy, for* $t \in [0,T]$ *and* $i = 1, 2, \dots, n$,

$$\widetilde{\phi}_i(t)\big(\delta_i(t) - \widetilde{S}^i_t(\kappa_i)\big) + \sum_{j \neq i} \widetilde{\phi}_j(t)\big(S^j_{t|i}(\kappa_j) - \widetilde{S}^j_t(\kappa_j)\big) = Z_i(t) - \widetilde{\pi}(t-). \quad (2.67)$$

Let us set $\phi^i_t = \widetilde{\phi}_i(\tau_{(1)} \wedge t)$ *for* $i = 1, 2, \dots, n$ *and let, for every* $t \in [0,T]$,

$$\phi^0_t = V_t(\phi, A) - \sum_{i=1}^n \phi^i_t S^i_t(\kappa_i), \quad (2.68)$$

where the process $V(\phi, A)$ *is given by the formula*

$$V_t(\phi, A) = \widetilde{\pi}(0) + \sum_{i=1}^n \int_{]0, \tau_{(1)} \wedge t]} \widetilde{\phi}_i(u)\, dS^{c,i}_u(\kappa_i) - A(\tau_{(1)} \wedge t). \quad (2.69)$$

Then the trading strategy (ϕ, A) *replicates the FTDC* $(X, A, Z, \tau_{(1)})$.

Proof. The proof is based on similar arguments to those used in the proof of Theorem 2.4.1. It suffices to check that under the assumption of the theorem, for a trading strategy (ϕ, A) stopped at $\tau_{(1)}$, we obtain from (2.62) and (2.66) that

$$\begin{aligned} dV_t(\phi, A) = \sum_{i=1}^n \phi^i_t \Big(\big(\delta_i(t) - \widetilde{S}^i_{t-}(\kappa_i)\big)\, d\widehat{M}^i_t + \sum_{j \neq i} \big(S^i_{t|j}(\kappa_i) - \widetilde{S}^i_{t-}(\kappa_i)\big)\, d\widehat{M}^j_t \Big) \\ - dA(\tau_{(1)} \wedge t). \end{aligned}$$

For $\phi^i_t = \widetilde{\phi}_i(\tau_{(1)} \wedge t)$, where the functions $\widetilde{\phi}_1, \widetilde{\phi}_2, \dots, \widetilde{\phi}_n$ solve (2.67), we thus obtain

$$dV_t(\phi, A) = \sum_{i=1}^n (Z_i(t) - \widetilde{\pi}(t-))\, d\widehat{M}^i_t - dA(\tau_{(1)} \wedge t).$$

By comparing the last formula with (2.65), we conclude that if, in addition, $V_0(\phi, A) = \pi_0 = \widetilde{\pi}_0$ and ϕ^0 is given by (2.68) then the strategy (ϕ, A) replicates an FTDC $(X, A, Z, \tau_{(1)})$. □

2.5.6 Conditional Default Distributions

In the case of first-to-default claims, it was enough to consider the unconditional distribution of default times. As expected, in order to deal with a general basket defaultable claim, we need to analyze conditional distributions of default times. It is possible to extend the approach presented in the preceding sections and to explicitly derive the dynamics of all processes of interest on the time interval $[0,T]$. However, since we deal here with a simple model of joint defaults, it suffices to make a non-restrictive assumption that we work on the canonical space $\Omega = \mathbb{R}^n$ and to use simple arguments based on the conditioning with respect to past defaults.

Suppose that k names out of a total of n names have already defaulted. To introduce a convenient notation, we adopt the convention that the $n-k$ non-defaulted names are in their original order $j_1 < \cdots < j_{n-k}$, whereas the k defaulted names $i_1, \dots, i_k$ are ordered in such a way that $u_1 < \cdots < u_k$. For the sake of brevity, we write $D_k = \{\tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k\}$ to denote the *information structure* of the past k defaults.

Definition 2.5.4. The *joint conditional distribution function* of default times $\tau_{j_1}, \dots, \tau_{j_{n-k}}$ equals, for every $t_1, \dots, t_{n-k} > u_k$,

$$
\begin{aligned}
&F(t_1, \dots, t_{n-k} \,|\, \tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k) \\
&= \mathbb{Q}\big(\tau_{j_1} \le t_1, \dots, \tau_{j_{n-k}} \le t_{n-k} \,|\, \tau_{i_1} u_1, \dots, \tau_{i_k} = u_k\big).
\end{aligned}
$$

The *joint conditional survival function* of default times $\tau_{j_1}, \dots, \tau_{j_{n-k}}$ is given by the expression

$$
\begin{aligned}
&G(t_1, \dots, t_{n-k} \,|\, \tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k) \\
&= \mathbb{Q}\big(\tau_{j_1} > t_1, \dots, \tau_{j_{n-k}} > t_{n-k} \,|\, \tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k\big)
\end{aligned}
$$

for every $t_1, \dots, t_{n-k} > u_k$.

As expected, the conditional first-to-default intensities are defined using the joint conditional distributions, instead of the joint (unconditional) distribution of default times. We will denote $G_{(1)}(t \,|\, D_k) = G(t, \dots, t \,|\, D_k)$.

Definition 2.5.5. Given the event D_k, for any $j_l \in \{j_1, \dots, j_{n-k}\}$ the *conditional first-to-default intensity* of a surviving name j_l is denoted by $\widetilde{\lambda}_{j_l}(t \,|\, D_k) = \widetilde{\lambda}_{j_l}(t \,|\, \tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k)$. It is given by the formula

$$
\widetilde{\lambda}_{j_l}(t \,|\, D_k) = \frac{\int_t^\infty \int_t^\infty \dots \int_t^\infty dF(t_1, \dots, t_{l-1}, t, t_{l+1}, \dots, t_{n-k} | D_k)}{G_{(1)}(t \,|\, D_k)}
$$

for every $t \in [u_k, T]$.

In Section 2.5.3, we introduced the processes $S^i_{t|j}(\kappa_j)$ representing the value of the ith CDS at time t on the event $\{\tau_{(1)} = \tau_j = t\}$. According to the notation introduced above, we thus dealt with the conditional value of the ith CDS with respect to the event $D_1 = \{\tau_j = t\}$. It is clear that to value a CDS for each surviving name, one can proceed as prior to the first default, except that one should now use the conditional distribution

$$F(t_1, \dots, t_{n-1} \,|\, D_1) = F(t_1, \dots, t_{n-1} \,|\, \tau_j = j), \quad \forall\, t_1, \dots, t_{n-1} \in [t, T],$$

rather than the unconditional distribution $F(t_1, \dots, t_n)$, which was employed in Proposition 2.5.6. The same argument can be applied to any default event D_k. The corresponding conditional version of Proposition 2.5.6 is rather easy to formulate and prove and thus we decided not to provide an explicit conditional pricing formula here.

The conditional first-to-default intensities introduced in Definition 2.5.5 will allow us to construct the conditional first-to-default martingales in a similar way as we defined the first-to-default martingales M^i associated with the first-to-default intensities $\widetilde{\lambda}_i$. However, since any name can default at any time, we need to introduce an entire family of conditional martingales, whose compensators are based on intensities conditioned on the information about the past defaults.

Definition 2.5.6. Given the default event $D_k = \{\tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k\}$, for each surviving name $j_l \in \{j_1, \dots, j_{n-k}\}$, we define the *basic conditional first-to-default martingale* $\widehat{M}^{j_l}_{t|D_k}$ by setting, for $t \in [u_k, T]$,

$$\widehat{M}^{j_l}_{t|D_k} = H^{j_l}_{t \wedge \tau_{(k+1)}} - \int_{u_k}^{t} \mathbb{1}_{\{u < \tau_{(k+1)}\}} \widetilde{\lambda}_{j_l}(u \,|\, D_k)\, du. \tag{2.70}$$

The process $\widehat{M}^{j_l}_{t|D_k}$, $t \in [u_k, T]$, is a martingale under the *conditional probability measure* $\mathbb{Q}|D_k$, that is, the probability measure $\mathbb{Q}$ conditioned on the event D_k and with respect to the filtration generated by default processes of the surviving names, that is, the filtration $\mathcal{G}^{D_k}_t := \mathcal{H}^{j_1}_t \vee \dots \vee \mathcal{H}^{j_{n-k}}_t$ for $t \in [u_k, T]$.

Conditionally on the event D_k, we have $\tau_{(k+1)} = \tau_{j_1} \wedge \tau_{j_2} \wedge \dots \wedge \tau_{j_{n-k}}$, so that $\tau_{(k+1)}$ is the first default for all surviving names. Formula (2.70) is thus a rather straightforward generalization of formula (2.49). In particular, for $k = 0$ we obtain $\widehat{M}^i_{t|D_0} = \widehat{M}^i_t$, $t \in [0, T]$, for any $i = 1, 2, \dots, n$.

The martingale property of the process $\widehat{M}^{j_l}_{t|D_k}$, as stated in Definition 2.5.6, follows from Proposition 2.5.3; this property can also be seen as a conditional version of Corollary 2.5.1.

We are in a position to state the conditional version of the first-to-default predictable representation theorem of Section 2.5.2. Formally, this result is nothing else than a restatement of the martingale representation formula of Proposition 2.5.1 in terms of conditional first-to-default intensities and conditional first-to-default martingales.

Let us fix an event D_k and let us write $\mathbb{G}^{D_k} = \mathbb{H}^{j_1} \vee \cdots \vee \mathbb{H}^{j_{n-k}}$.

Proposition 2.5.3. *Let Y be a random variable given by the formula*

$$Y = \sum_{l=1}^{n-k} Z_{j_l|D_k}(\tau_{j_l}) \mathbb{1}_{\{\tau_{j_l} \le T,\, \tau_{j_l} = \tau_{(k+1)}\}} + X \mathbb{1}_{\{\tau_{(k+1)} > T\}}$$

for some functions $Z_{j_l|D_k} : [u_k, T] \to \mathbb{R}$, $l = 1, 2, \ldots, n-k$ and some constant X (possibly dependent on D_k). Let us define, for $t \in [u_k, T]$,

$$\widehat{M}_{t|D_k} = \mathbb{E}_{\mathbb{Q}|D_k}(Y \,|\, \mathcal{G}_t^{D_k}).$$

Then the process $\widehat{M}_{t|D_k}$, $t \in [u_k, T]$, is a $\mathbb{G}^{D_k}$-martingale with respect to the conditional probability measure $\mathbb{Q}|D_k$.

Furthermore, $\widehat{M}_{t|D_k}$ admits the following representation, for $t \in [u_k, T]$,

$$\widehat{M}_{t|D_k} = \widehat{M}_{0|D_k} + \sum_{l=1}^{n-k} \int_{]u_k, t]} h_{j_l}(u|D_k)\, d\widehat{M}^{j_l}_{u|D_k},$$

where the processes h_{j_l} are given by

$$h_{j_l}(t\,|D_k) = Z_{j_l|D_k}(t) - \widehat{M}_{t-|D_k}, \quad \forall\, t \in [u_k, T].$$

Proof. The proof relies on a rather straightforward extension of arguments used in the proof of Proposition 2.5.1 to the context of conditional default distributions. Therefore, we leave the details to the reader. □

2.5.7 Recursive Valuation of a Basket Claim

We are ready to extend the results developed in the context of first-to-default claims to value and hedge general basket claims. A generic basket claim is any contingent claim that pays a specified amount on each default from a basket of n credit names and a constant amount at maturity T if no defaults have occurred prior to or at T.

Definition 2.5.7. A *basket claim* associated with a family of n credit names is given as $(X, A, \bar{Z}, \bar{\tau})$ where X is a constant amount payable at maturity only if no default occurs prior to or at T, the vector $\bar{\tau} = (\tau_1, \ldots, \tau_n)$ represents default times and the time-dependent matrix $\bar{Z}$ represents the recovery payoffs at defaults, specifically,

$$\bar{Z} = \begin{bmatrix} Z_1(t\,|D_0) & Z_2(t\,|D_0) & \ldots & Z_n(t\,|D_0) \\ Z_1(t\,|D_1) & Z_2(t\,|D_1) & \ldots & Z_n(t\,|D_1) \\ \vdots & \vdots & \ddots & \vdots \\ Z_1(t\,|D_{n-1}) & Z_2(t\,|D_{n-1}) & \ldots & Z_n(t\,|D_{n-1}) \end{bmatrix}.$$

Note that the above matrix $\bar{Z}$ is presented in the shorthand notation. In fact, in each row one needs to specify, for an arbitrary choice of the event $D_k = \{\tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k\}$ and any name $j_l \notin \{i_1, \dots, i_k\}$, the *conditional payoff function* at the moment of the $(k+1)$th default, that is,

$$Z_{j_l}(t\,|D_k) = Z_{j_l}(t\,|\,\tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k), \quad \forall\, t \in [u_k, T].$$

In the financial interpretation, the function $Z_{j_l}(t\,|D_k)$ specifies the recovery payment at the default of the name j_l, conditional on the event D_k and on the event $\{\tau_{j_l} = \tau_{(k+1)} = t\}$, that is, assuming that the name j_l is the first defaulted name among all surviving names.

In particular, $Z_i(t\,|D_0) := Z_i(t)$ represents the recovery payment at the default of the ith name at time $t \in [0,T]$, given that no defaults have occurred prior to t, that is, at the moment of the first default. We will use the symbol D_0 to denote the situation where no defaults have occurred prior to time t.

Example 2.5.1. Let us consider the kth-to-default claim for some fixed $k \in \{1, 2, \dots, n\}$. Assume that the payoff at the kth default depends only on the moment of the kth default and the identity of the kth defaulted name. Then all rows of the matrix $\bar{Z}$ are equal to zero, except for the kth row, which equals, for every $t \in [0,T]$,

$$[Z_1(t\,|\,k-1), Z_2(t\,|\,k-1), \dots, Z_n(t\,|\,k-1)].$$

We write here $k-1$, rather than D_{k-1}, in order to emphasize that the knowledge of timings and identities of the k defaulted names is not relevant under the present assumptions.

More generally, for a generic basket claim in which the payoff at the ith default depends on the time of the ith default and identity of the ith defaulted name only, the recovery matrix $\bar{Z}$ reads

$$\bar{Z} = \begin{bmatrix} Z_1(t) & Z_2(t) & \dots & Z_n(t) \\ Z_1(t\,|1) & Z_2(t\,|1) & \dots & Z_n(t\,|1) \\ \vdots & \vdots & \ddots & \vdots \\ Z_1(t\,|n-1) & Z_2(t\,|n-1) & \dots & Z_n(t\,|n-1) \end{bmatrix}$$

where $Z_j(t\,|k-1)$ represents the payoff at the moment $\tau_{(k)} = t$ of the kth default if j is the kth defaulting name, that is, on the event $\{\tau_j = \tau_{(k)} = t\}$. This shows that in several practically important examples of basket credit derivatives, the matrix $\bar{Z}$ of recoveries will have a relatively simple structure.

It is rather clear that any basket claim can be represented as a static portfolio of kth-to-default claims for $k = 1, 2, \dots, n$. However, this decomposition does not seem to be advantageous for the purposes of dynamic hedging.

In what follows, we prefer to represent a basket claim as a sequence of *conditional first-to-default claims*, with the same value between any two defaults as a basket claim under consideration. Using this approach, we will be able to directly apply previously developed results for the case of first-to-default claims and thus to produce a rather straightforward recursive algorithm for the valuation and hedging of a basket claim.

Instead of stating a formal result, which would require heavy notation, we prefer to focus first on the computational algorithm for valuation and hedging of a basket claim. An important concept in this algorithm is the *conditional pre-default price*

$$\widetilde{Z}(t\,|D_k) = \widetilde{Z}(t\,|\,\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k), \quad \forall\, t \in [u_k, T],$$

of a conditional first-to-default claim. The function $\widetilde{Z}(t\,|D_k)$, $t \in [u_k, T]$, is defined as the risk-neutral value of a conditional FTDC on $n-k$ surviving names, with the following recovery payoffs upon the first default at any date $t \in [u_k, T]$

$$\widehat{Z}_{j_l}(t\,|\,D_k) = Z_{j_l}(t\,|\,D_k) + \widetilde{Z}(t\,|\,D_k, \tau_{j_l} = t). \tag{2.71}$$

Assume for the moment that for any name $j_m \notin \{i_1, \ldots, i_k, j_l\}$ the conditional recovery payoff $\widehat{Z}_{j_m}(t\,|\,\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k, \tau_{j_l} = u_{k+1})$ upon the first default after date u_{k+1} is known. Then we can compute the function

$$\widetilde{Z}(t\,|\,\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k, \tau_{j_l} = u_{k+1}), \quad \forall\, t \in [u_{k+1}, T],$$

as in Lemma 2.5.6, but using the conditional default distribution. The assumption that the conditional payoffs are known is not restrictive, since the functions appearing in right-hand side of (2.71) are known from the previous step in the following recursive pricing algorithm.

- **First step.** We first derive the value of a basket claim assuming that all but one defaults have already occurred. Let

  $$D_{n-1} = \{\tau_{i_1} = u_1, \ldots, \tau_{i_{n-1}} = u_{n-1}\}.$$

  For any $t \in [u_{n-1}, T]$, we deal with the payoffs

  $$\widehat{Z}_{j_1}(t\,|D_{n-1}) = Z_{j_1}(t\,|D_{n-1}) = Z_{j_1}(t\,|\,\tau_{i_1} = u_1, \ldots, \tau_{i_{n-1}} = u_{n-1}),$$

  for $j_1 \notin \{i_1, \ldots, i_{n-1}\}$ where the recovery payment $Z_{j_1}(t\,|D_{n-1})$ for $t \in [u_{n-1}, T]$ is given by the specification of the basket claim. Hence we can evaluate the pre-default value $\widetilde{Z}(t\,|D_{n-1})$ at any time $t \in [u_{n-1}, T]$, as a value of a conditional first-to-default claim with the said payoff, using the conditional distribution under $\mathbb{Q}|D_{n-1}$ of the random time $\tau_{j_1} = \tau_{i_n}$ on the interval $[u_{n-1}, T]$.

- **Second step.** In this step, we assume that all but two names have already defaulted. Let

$$D_{n-2} = \{\tau_{i_1} = u_1, \dots, \tau_{i_{n-2}} = u_{n-2}\}.$$

For each surviving name $j_1, j_2 \notin \{i_1, \dots, i_{n-2}\}$, the payoff $\widehat{Z}_{j_l}(t\,|D_{n-2})$ for $t \in [u_{n-2}, T]$, of a basket claim at the moment of the next default formally comprises the recovery payoff from the defaulted name j_l, which is equal to $Z_{j_l}(t\,|D_{n-2})$, as well as the pre-default value $\widetilde{Z}(t\,|\,D_{n-2}, \tau_{j_l} = t)$, $t \in [u_{n-2}, T]$, which was computed in the first step. Therefore, we have, for every $t \in [u_{n-2}, T]$,

$$\widehat{Z}_{j_l}(t\,|\,D_{n-2}) = Z_{j_l}(t\,|\,D_{n-2}) + \widetilde{Z}(t\,|\,D_{n-2}, \tau_{j_l} = t).$$

To find the value of a basket claim between the moments of the $(n-2)$th and the $(n-1)$th default, it suffices to compute the pre-default value of the conditional FTDC associated with the two surviving names, $j_1, j_2 \notin \{i_1, \dots, i_{n-2}\}$. Since the conditional payoffs $\widehat{Z}_{j_1}(t\,|D_{n-2})$ and $\widehat{Z}_{j_2}(t\,|D_{n-2})$ are already known at this stage, it is sufficient to compute the expectation under the conditional probability measure $\mathbb{Q}|D_{n-2}$ in order to find the pre-default value of this conditional FTDC for any $t \in [u_{n-2}, T]$.

- **General induction step.** We now assume that exactly k defaults have occurred, that is, we assume that we are working on the event

$$D_k = \{\tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k\}.$$

From the preceding step, we know the function $\widetilde{Z}(t\,|D_{k+1})$ where the event D_{k+1} is given as $D_{k+1} = \{\tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k, \tau_{j_l} = u_{k+1}\}$.

In order to evaluate $\widetilde{Z}(t\,|D_k)$, we set, for $t \in [u_k, T]$,

$$\widehat{Z}_{j_l}(t\,|\,D_k) = Z_{j_l}(t\,|\,D_k) + \widetilde{Z}(t\,|\,D_k, \tau_{j_l} = t), \qquad (2.72)$$

for any $j_1, \dots, j_{n-k} \notin \{i_1, \dots, i_k\}$ and we compute $\widetilde{Z}(t\,|D_k)$ for every $t \in [u_k, T]$ as the risk-neutral value under the conditional probability $\mathbb{Q}|D_k$ of the conditional FTDC with payoffs given by (2.72).

We are in a position to state the valuation result for a basket claim, which can be formally established using the reasoning outlined above.

Proposition 2.5.4. *The risk-neutral value at time $t \in [0, T]$ of a basket claim $(X, A, \bar{Z}, \bar{\tau})$ equals, for $t \in [0, T]$,*

$$\pi_t = \sum_{k=0}^{n-1} \widetilde{Z}(t\,|D_k)\mathbb{1}_{[\tau_{(k)}\wedge T, \tau_{(k+1)}\wedge T[}(t),$$

where $D_k = D_k(\omega) = \{\tau_{i_1}(\omega) = u_1, \dots, \tau_{i_k}(\omega) = u_k\}$ *for* $k = 1, 2, \dots, n$ *and* D_0 *means that no defaults have yet occurred.*

Proof. Assume that we are at some date $t \in [0, T]$ and suppose that exactly k names (for some $k = 1, 2, \dots, n$) have already defaulted, hence the set D_k is known to us (so that $t \geq u_k$). Form the point of view of valuation, the basket claim can be seen this point of time as a conditional FTDC with the conditional payoff $\widehat{Z}(t \,|D_k) = Z(t \,|D_k) + \widetilde{Z}(t \,|D_{k+1})$. We can now use the pricing formula of Proposition 2.5.6 (using conditional distribution) for an FTDC in order to derive the value of $\widetilde{Z}(t \,|D_k)$ for every $t \in [u_k, T]$. □

2.5.8 Recursive Replication of a Basket Claim

From our discussion, it is clear that a basket claim can be conveniently interpreted as a specific sequence of conditional first-to-default claims and thus the replication of a basket claim relies on hedging of a sequence of conditional first-to-default claims. In the next result, we denote $\tau_{(0)} = 0$.

Theorem 2.5.2. *For any* $k = 0, 1, \dots, n$*, the replicating strategy* ϕ *for a basket claim* $(X, A, \bar{Z}, \bar{\tau})$ *on the time interval* $[\tau_k \wedge T, \tau_{k+1} \wedge T]$ *coincides with the replicating strategy for the conditional FTDC with payoffs* $\widehat{Z}(t \,|D_k)$ *given by (2.72). The replicating strategy* $\phi = (\phi^0, \phi^{j1}, \dots, \phi^{j_{n-k}}, A)$*, corresponding to the units of savings account and units of CDS on each surviving name at time* t*, has the wealth process*

$$V_t(\phi, A) = \phi^0_t + \sum_{l=1}^{n-k} \phi^{j_l}_t S^{j_l}_t(\kappa_{j_l}),$$

where the processes ϕ^{j_l}, $l = 1, 2, \dots, n-k$ *can be computed by the conditional version of Theorem 2.5.1.*

Proof. We know that the basket claim can be decomposed into a series of conditional first-to-default claims. So, at any given moment of time $t \in [0, T]$, assuming that k defaults have already occurred, our basket claim is equivalent to the conditional FTDC with payoffs $\widehat{Z}(t \,|\, D_k)$ and the pre-default value $\widetilde{Z}(t \,|D_k)$. This conditional FTDC is alive up to the next default $\tau_{(k+1)}$ or maturity T, whichever comes first.

It is thus clear that the replicating strategy of a basket claim over the random interval $[\tau_k \wedge T, \tau_{k+1} \wedge T]$ need to coincide with the replicating strategy for this conditional first-to-default claim and thus it can be found by proceeding along the same lines as in Theorem 2.5.1, but using the conditional distribution of defaults for surviving names under $\mathbb{Q}|D_k$. □

2.6 Applications to Copula-Based Models

We will now apply the general results to simple models, in which some *copula functions* (see Section 5.4 for the definition) are used to describe the dependence of default times. For various applications of copula functions to credit risk modeling and to valuation of credit derivatives, the interested reader is referred to, for instance, Andersen and Sidenius [4], Burtschell et al. [45, 46], Cherubini and Luciano [55], Cherubini et al. [56], Embrechts et al. [82], Frey et al. [89], Gennheimer [90], Giesecke [91], Kijima et al. [121], Laurent and Gregory [134], Li [137], McNeil et al. [142], and Schönbucher and Schubert [161].

For simplicity of exposition, we only consider the bivariate situation and we work under the following standing assumptions.

Assumption 2.6.1. We assume that:

- we are given a first-to-default claim $(X, A, Z, \tau_{(1)})$ where $Z = (Z_1, Z_2)$ for some constants Z_1, Z_2 and X,
- the default times τ_1 and τ_2 have exponential marginal distributions with parameters λ_1 and λ_2,
- the protection δ_i of the ith credit default swap is constant and $\kappa_i = \lambda_i \delta_i$ for $i = 1, 2$.

2.6.1 Independent Default Times

Let us first consider the case where the default times τ_1 and τ_2 are independent (of course, this corresponds to the product copula $C(u, v) = uv$). In view of independence, the marginal intensities and the first-to-default intensities can be easily shown to coincide. We have, for $i = 1, 2$,

$$G_i(u) = \mathbb{Q}(\tau_i > u) = e^{-\lambda_i u},$$

and thus the joint survival probability equals, for every $(u, v) \in \mathbb{R}_+^2$,

$$G(u, v) = G_1(u) G_2(v) = e^{-\lambda_1 u} e^{-\lambda_2 v}.$$

Consequently, we obtain

$$F(du, dv) = G(du, dv) = \lambda_1 \lambda_2 e^{-\lambda_1 u} e^{-\lambda_2 v} \, dudv = f(u, v) \, dudv$$

and

$$G(du, u) = -\lambda_1 e^{-(\lambda_1 + \lambda_2) u} \, du.$$

Proposition 2.6.1. *Assume that the default times τ_1 and τ_2 are independent. Then the replicating strategy for an FTDC $(X, 0, Z, \tau_{(1)})$ is given as*

$$\widetilde{\phi}^1(t) = \frac{Z_1 - \widetilde{\pi}(t)}{\delta_1}, \quad \widetilde{\phi}^2(t) = \frac{Z_2 - \widetilde{\pi}(t)}{\delta_2},$$

where

$$\widetilde{\pi}(t) = \frac{(Z_1\lambda_1 + Z_2\lambda_2)}{\lambda_1 + \lambda_2}(1 - e^{-(\lambda_1+\lambda_2)(T-t)}) + Xe^{-(\lambda_1+\lambda_2)(T-t)}.$$

Proof. From the previous remarks, we obtain

$$\begin{aligned}
\widetilde{\pi}(t) &= \frac{Z_1 \int_t^T \int_u^\infty dF(u,v)}{G(t,t)} + \frac{Z_2 \int_t^T \int_v^\infty dF(u,v)}{G(t,t)} + X\frac{G(T,T)}{G(t,t)} \\
&= \frac{Z_1\lambda_1 \int_t^T e^{-(\lambda_1+\lambda_2)u} du}{e^{-(\lambda_1+\lambda_2)t}} + \frac{Z_2\lambda_2 \int_t^T e^{-(\lambda_1+\lambda_2)v} dv}{e^{-(\lambda_1+\lambda_2)t}} + X\frac{G(T,T)}{G(t,t)} \\
&= \frac{Z_1\lambda_1}{(\lambda_1+\lambda_2)}(1 - e^{-(\lambda_1+\lambda_2)(T-t)}) + \frac{Z_2\lambda_2}{(\lambda_1+\lambda_2)}(1 - e^{-(\lambda_1+\lambda_2)(T-t)}) \\
&\quad + X\frac{G(T,T)}{G(t,t)},
\end{aligned}$$

and thus

$$\widetilde{\pi}(t) = \frac{(Z_1\lambda_1 + Z_2\lambda_2)}{\lambda_1 + \lambda_2}(1 - e^{-(\lambda_1+\lambda_2)(T-t)}) + Xe^{-(\lambda_1+\lambda_2)(T-t)}.$$

Under the assumption of independence of default times, we also have that $S^i_{t|j}(\kappa_i) = \widetilde{S}^i_t(\kappa_i)$ for $i, j = 1, 2$ and $i \neq j$. Furthermore, from Example 2.4.1, we have that $\widetilde{S}^i_t(\kappa_i) = 0$ for $t \in [0, T]$ and thus the matrix $N(t)$ in Theorem 2.5.1 reduces to

$$N(t) = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}.$$

The replicating strategy can be found easily by solving the linear equation $N(t)\widetilde{\phi}(t) = h(t)$ where $h(t) = (h_1(t), h_2(t))$ with the function h_i given by the formula

$$h_i(t) = Z_i - \widetilde{\pi}(t-) = Z_i - \widetilde{\pi}(t)$$

for $i = 1, 2$. □

As an important example of a first-to-default claim, we will now consider the case of a *first-to-default swap* (FTDS). A stylized FTDS is formally defined by setting $X = 0$, $A(t) = -\kappa_{(1)}t$ where $\kappa_{(1)}$ is the *swap spread* and $Z_i(t) = \delta_i \in [0, 1)$ for some constants δ_i, $i = 1, 2$. Hence an FTDS can be equivalently seen as the FTDC $(0, -\kappa_{(1)}t, (\delta_1, \delta_2), \tau_{(1)})$. Under the present assumptions, we obtain

$$\pi_0 = \widetilde{\pi}(0) = \frac{1 - e^{\lambda T}}{\lambda} \Big((\lambda_1 \delta_1 + \lambda_2 \delta_2) - \kappa_{(1)} \Big)$$

where we denote $\lambda = \lambda_1 + \lambda_2$. The *FTDS market spread* is the level of $\kappa_{(1)}$ that makes the FTDS valueless at initiation. Hence, in this elementary example, this spread equals $\lambda_1 \delta_1 + \lambda_2 \delta_2$. In addition, it can be shown that, under the present assumptions, we have that $\widetilde{\pi}(t) = 0$ for every $t \in [0, T]$.

Suppose that we wish to hedge the short position in the FTDS using two CDSs, say CDS^i, $i = 1, 2$, with respective default times τ_i, protection payments δ_i and spreads $\kappa_i = \lambda_i \delta_i$. Recall that in the present setup we have that, for every $t \in [0, T]$,

$$S^i_{t|j}(\kappa_i) = \widetilde{S}^i_t(\kappa_i) = 0, \quad i, j = 1, 2, \ i \neq j. \tag{2.73}$$

Consequently, we have here that $h_i(t) = -Z_i(t) = -\delta_i$ for every $t \in [0, T]$. It then follows from equation $N(t)\widetilde{\phi}(t) = h(t)$ that $\widetilde{\phi}_1(t) = \widetilde{\phi}_2(t) = 1$ for every $t \in [0, T]$ and thus $\phi^0_t = 0$ for every $t \in [0, T]$. This result is by no means surprising; we hedge a short position in the FTDS by holding a static portfolio of two single-name CDSs since, under the present assumptions, the FTDS is equivalent to such a portfolio of the corresponding single-name CDSs. Of course, one would not expect that this feature will still hold in a general case of dependent default times.

The first equality in (2.73) is due to the standing assumption of independence of default times τ_1 and τ_2 and thus it will no longer be true for other copulae. The second equality follows from our simplifying postulate that the risk-neutral marginal distributions of default times are exponential. In practice, the risk-neutral marginal distributions of default times are obtained by calibrating a model to market data (i.e., market prices of single-name CDSs) and thus, typically, they are not exponential.

2.6.2 Archimedean Copulae

We now proceed to the case of exponentially distributed, but dependent, default times. The mutual dependence will be specified by a choice of some *Archimedean copula*. Recall that a bivariate Archimedean copula is defined as $C(u, v) = \varphi^{-1}(\varphi(u), \varphi(v))$, where φ is called the *generator* of a copula.

Clayton Copula

Recall that the generator of the *Clayton copula* is given as $\varphi(s) = s^{-\theta} - 1$ for every $s \in \mathbb{R}_+$, for some strictly positive parameter θ and thus the bivariate

Clayton copula can be represented as follows

$$C(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}.$$

Under the present assumptions, the corresponding joint survival function $G(u,v)$ equals

$$G(u,v) = C(G_1(u), G_2(v)) = (e^{\lambda_1 u\theta} + e^{\lambda_2 v\theta} - 1)^{-\frac{1}{\theta}},$$

so that

$$\frac{G(u,dv)}{dv} = -\lambda_2 e^{\lambda_2 v\theta}(e^{\lambda_1 u\theta} + e^{\lambda_2 v\theta} - 1)^{-\frac{1}{\theta}-1}$$

and

$$f(u,v) = \frac{G(du,dv)}{dudv} = (\theta+1)\lambda_1\lambda_2 e^{\lambda_1 u\theta + \lambda_2 v\theta}(e^{\lambda_1 u\theta} + e^{\lambda_2 v\theta} - 1)^{-\frac{1}{\theta}-2}.$$

We only provide explicit formulae for $\widetilde{\phi}_1$ and $S^1_{v|2}(\kappa_1)$, since the quantities $\widetilde{\phi}_2$ and $S^2_{u|1}(\kappa_2)$ are given by symmetric expressions.

Proposition 2.6.2. *Let the joint distribution of (τ_1, τ_2) be given by the Clayton copula with some $\theta > 0$. Then the replicating strategy for an FTDC $(X, 0, Z, \tau_{(1)})$ is given by the expression*

$$\widetilde{\phi}_1(t) = \frac{\delta_2(Z_1 - \widetilde{\pi}(t)) + S^2_{t|1}(\kappa_2)(Z_2 - \widetilde{\pi}(t))}{\delta_1\delta_2 - S^1_{t|2}(\kappa_1)S^2_{t|1}(\kappa_2)}, \tag{2.74}$$

where

$$\begin{aligned}\widetilde{\pi}(t) = {} & Z_1 \frac{\int_{e^{\lambda_1\theta t}}^{e^{\lambda_1\theta T}} (s + s^{\frac{\lambda_2}{\lambda_1}} - 1)^{-\frac{1}{\theta}-1}\, ds}{\theta(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}} + Z_2 \frac{\int_{e^{\lambda_2\theta t}}^{e^{\lambda_2\theta T}} (s + s^{\frac{\lambda_1}{\lambda_2}} - 1)^{-\frac{1}{\theta}-1}\, ds}{\theta(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}} \\ & + X \frac{(e^{\lambda_1\theta T} + e^{\lambda_2\theta T} - 1)^{-\frac{1}{\theta}}}{(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}}\end{aligned}$$

and

$$\begin{aligned}S^1_{v|2}(\kappa_1) = {} & \delta_1 \frac{[(e^{\lambda_1\theta T} + e^{\lambda_2\theta T} - 1)^{-\frac{1}{\theta}-1} - (e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}]}{(e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}} \\ & - \kappa_1 \frac{\int_v^T (e^{\lambda_1\theta u} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1} du}{(e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}}.\end{aligned}$$

Proof. Observe that

$$\int_t^T du \int_u^\infty f(u,v)dv = \int_t^T \lambda_1 e^{\lambda_1 u\theta}(e^{\lambda_1 u\theta} + e^{\lambda_2 u\theta} - 1)^{-\frac{1}{\theta}-1}\, du$$
$$= \frac{1}{\theta}\int_{e^{\lambda_1\theta t}}^{e^{\lambda_1\theta T}} (s + s^{\frac{\lambda_2}{\lambda_1}} - 1)^{-\frac{1}{\theta}-1}\, ds$$

and thus, by symmetry,

$$\int_t^T dv \int_v^\infty f(u,v)du = \frac{1}{\theta}\int_{e^{\lambda_2\theta t}}^{e^{\lambda_2\theta T}} (s + s^{\frac{\lambda_1}{\lambda_2}} - 1)^{-\frac{1}{\theta}-1}\, ds.$$

Consequently,

$$\widetilde{\pi}(t) = \frac{Z_1\int_t^T\int_u^\infty dG(u,v)}{G(t,t)} + \frac{Z_2\int_t^T\int_v^\infty dG(u,v)}{G(t,t)} + X\frac{G(T,T)}{G(t,t)}$$
$$= Z_1\frac{\int_{e^{\lambda_1\theta t}}^{e^{\lambda_1\theta T}} (s + s^{\frac{\lambda_2}{\lambda_1}} - 1)^{-\frac{1}{\theta}-1}\, ds}{\theta(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}} + Z_2\frac{\int_{e^{\lambda_2\theta t}}^{e^{\lambda_2\theta T}} (s + s^{\frac{\lambda_1}{\lambda_2}} - 1)^{-\frac{1}{\theta}-1}\, ds}{\theta(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}}$$
$$+ X\frac{(e^{\lambda_1\theta T} + e^{\lambda_2\theta T} - 1)^{-\frac{1}{\theta}}}{(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}}.$$

We are in a position to determine the replicating strategy. Under the standing assumption that $\kappa_i = \lambda_i\delta_i$ for $i = 1,2$ we still have that $\widetilde{S}^i_t(\kappa_i) = 0$ for $i = 1,2$ and for $t \in [0,T]$. Hence the matrix $N(t)$ reduces to

$$N(t) = \begin{bmatrix} \delta_1 & -S^2_{t|1}(\kappa_2) \\ -S^1_{t|2}(\kappa_1) & \delta_2 \end{bmatrix}$$

where

$$S^1_{v|2}(\kappa_1) = \delta_1\frac{\int_v^T f(u,v)\,du}{\int_v^\infty f(u,v)\,du} - \kappa_1\frac{\int_v^T\int_u^\infty f(z,v)\,dzdu}{\int_v^\infty f(u,v)\,du}$$
$$= \delta_1\frac{[(e^{\lambda_1\theta T} + e^{\lambda_2\theta T} - 1)^{-\frac{1}{\theta}-1} - (e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}]}{(e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}}$$
$$- \kappa_1\frac{\int_v^T (e^{\lambda_1\theta u} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}\, du}{(e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}}.$$

The expression for $S^2_{u|1}(\kappa_2)$ can be found by analogous computations. By solving the equation $N(t)\widetilde{\phi}(t) = h(t)$, we obtain the required expression (2.74) for the replicating strategy. □

Gumbel Copula

As another example of an Archimedean copula, we consider the *Gumbel copula* with the generator $\varphi(s) = (-\ln s)^\theta$ for every $s \in \mathbb{R}_+$ where the parameter θ satisfies $\theta \geq 1$. The bivariate Gumbel copula can thus be written as

$$C(u,v) = e^{-[(-\ln u)^\theta + (-\ln v)^\theta]^{\frac{1}{\theta}}}.$$

Under our standing assumptions, the corresponding joint survival function $G(u,v)$ equals

$$G(u,v) = C(G_1(u), G_2(v)) = e^{-(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}.$$

Consequently, the partial derivatives of our interest satisfy

$$\frac{dG(u,v)}{dv} = -G(u,v)\lambda_2^\theta v^{\theta-1}(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1}$$

and

$$\frac{dG(u,v)}{dudv} = G(u,v)(\lambda_1\lambda_2)^\theta (uv)^{\theta-1}(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-2}\big((\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}} + \theta - 1\big).$$

As in the case of the Clayton copula, it is enough to derive the formulae for $\widetilde{\phi}_1$ and $S^1_{v|2}(\kappa_1)$, since $\widetilde{\phi}_2$ and $S^2_{u|1}(\kappa_2)$ are given by symmetric expressions.

Proposition 2.6.3. *Assume that the joint distribution of (τ_1, τ_2) is given by the Gumbel copula with $\theta \geq 1$. Then the replicating strategy for an FTDC $(X, 0, Z, \tau_{(1)})$ is given by*

$$\widetilde{\phi}_1(t) = \frac{\delta_2(Z_1 - \widetilde{\pi}(t)) + S^2_{t|1}(\kappa_2)(Z_2 - \widetilde{\pi}(t))}{\delta_1\delta_2 - S^1_{t|2}(\kappa_1)S^2_{t|1}(\kappa_2)},$$

where

$$\widetilde{\pi}(t) = (Z_1\lambda_1^\theta + Z_2\lambda_2^\theta)\lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}) + Xe^{-\lambda(T-t)}$$

with $\lambda = (\lambda_1^\theta + \lambda_2^\theta)^{\frac{1}{\theta}}$ and

$$\begin{aligned} S^1_{v|2}(\kappa_1) = \delta_1 & \frac{e^{-(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1} - e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}}{e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}} \\ & - \kappa_1 \frac{\int_v^T e^{-(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1}\,du}{e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}}. \end{aligned}$$

Proof. Let us denote $\lambda = (\lambda_1^\theta + \lambda_2^\theta)^{\frac{1}{\theta}}$. We have that

$$\begin{aligned} \int_t^T \int_u^\infty dG(u,v) &= \int_t^T \lambda_1^\theta(\lambda_1^\theta + \lambda_2^\theta)^{\frac{1}{\theta}-1} e^{-(\lambda_1^\theta + \lambda_2^\theta)^{\frac{1}{\theta}} u}\,du \\ &= (-\lambda_1^\theta \lambda^{-\theta} e^{-\lambda u})|_{u=t}^{u=T} = \lambda_1^\theta\lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}). \end{aligned}$$

Similarly, we also obtain

$$\int_t^T \int_v^\infty dG(u,v) = \lambda_2^\theta \lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}).$$

Furthermore, we have that $G(T,T) = e^{-\lambda T}$ and $G(t,t) = e^{-\lambda t}$. Hence

$$\widetilde{\pi}(t) = Z_1 \frac{\int_t^T \int_u^\infty dG(u,v)}{G(t,t)} + Z_2 \frac{\int_t^T \int_v^\infty dG(u,v)}{G(t,t)} + X \frac{G(T,T)}{G(t,t)}$$

or, more explicitly,

$$\widetilde{\pi}(t) = Z_1 \lambda_1^\theta \lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}) + Z_2 \lambda_2^\theta \lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}) + X e^{-\lambda(T-t)}.$$

We conclude that

$$\widetilde{\pi}(t) = (Z_1 \lambda_1^\theta + \delta_2 Z_2^\theta)\lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}) + X e^{-\lambda(T-t)}.$$

In order to find the replicating strategy, we proceed as in the proof of Proposition 2.6.2. Under the present assumptions, we obtain the following expression for $S^1_{v|2}(\kappa_1)$

$$\begin{aligned}
S^1_{v|2}(\kappa_1) &= \delta_1 \frac{\int_v^T f(u,v)du}{\int_v^\infty f(u,v)du} - \kappa_1 \frac{\int_v^T \int_u^\infty f(z,v)dzdu}{\int_v^\infty f(u,v)du} \\
&= \delta_1 \frac{e^{-(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}} (\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1} - e^{-\lambda v}\lambda^{1-\theta} v^{1-\theta}}{e^{-\lambda v}\lambda^{1-\theta} v^{1-\theta}} \\
&\quad - \kappa_1 \frac{\int_v^T e^{-(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}} (\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1} du}{e^{-\lambda v}\lambda^{1-\theta} v^{1-\theta}}.
\end{aligned}$$

By the symmetry of the model, a similar expression is valid for the value $S^2_{u|1}(\kappa_2)$. This completes the proof of the proposition. □

Prior to the recent global credit crisis, copula-based models were widely used by the financial industry for modeling of dependent defaults. In particular, one of such models (the one-factor Gaussian copula model proposed by Li [137] and presented in Section 5.5) was adopted by practitioners as the market convention for valuing tranches of CDOs. It is this important to observe that copula-based models suffer a major shortcoming of being inherently static models. Therefore, their practical use should at best be limited to the risk-neutral valuation of credit derivatives, as opposed to the arbitrage pricing of defaultable claims, which relies on the concept of dynamic replication of a given credit derivative with traded assets, or to the credit risk management. More realistic models and approaches to credit risk are presented in the foregoing chapters.

Chapter 3

Hazard Process Approach

In the general *reduced-form* (or *hazard process*) *approach*, we deal with two kinds of information: the information conveyed by assets prices and other economic factors, denoted as $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$, and the information about the occurrence of the default time, that is, the knowledge of the time where the default occurred in the past, if the default has indeed already happen. As we already know, the latter information is modeled by the filtration $\mathbb{H}$ generated by the *default process* H.

At the intuitive level, the *reference filtration* $\mathbb{F}$ is generated by prices of some assets, or by other economic factors (such as, e.g., interest rates). This filtration can also be a sub-filtration of the filtration generated by the asset prices. The case where $\mathbb{F}$ is the trivial filtration is exactly what we have studied in the previous chapter. Though in a typical example $\mathbb{F}$ is chosen to be the Brownian filtration, most theoretical results do not rely on a particular choice of the reference filtration $\mathbb{F}$. We denote by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ the *full filtration* (sometimes referred to as the *enlarged filtration*).

Special attention will be paid in what follows to the so-called *hypothesis (H)*. In the present context, it postulates the preservation of the martingale property with respect to the enlargement of $\mathbb{F}$ by the observations of default time. It is important to note that this hypothesis is not preserved under an equivalent change of a probability measure, in general.

In order to examine the precise meaning of market completeness in a defaultable security market model and to derive the hedging strategies for credit derivatives, we shall also establish a suitable version of the predictable representation theorem.

Most results presented in Sections 3.1–3.6 can be found, for instance, in survey papers by Jeanblanc and Rutkowski [111, 112]; see also the papers by Artzner and Delbaen [6], Bélanger et al. [11], Jarrow and Turnbull [106], Lando [126], and Wong [170].

Sections 3.7–3.8 are based on the paper by Bielecki et al. [20].

3.1 Hazard Process and its Applications

The concepts introduced in the Chapter 2 will now be extended to a more general setup, in which an additional flow of information, which will be formally represented hereafter by some filtration $\mathbb{F}$, is available.

We denote by τ a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, satisfying $\mathbb{Q}(\tau = 0) = 0$ and $\mathbb{Q}(\tau > t) > 0$ for any $t \in \mathbb{R}_+$. We introduce the right-continuous *default indicator* process H by setting $H_t = \mathbb{1}_{\{t \geq \tau\}}$ for $t \in \mathbb{R}_+$ and we write $\mathbb{H}$ to denote the filtration generated by the process H, so that $\mathcal{H}_t = \sigma(H_u : u \leq t)$ for every $t \in \mathbb{R}_+$.

We assume that we are given an auxiliary *reference filtration* $\mathbb{F}$ such that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, that is, $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for any $t \in \mathbb{R}_+$. For each $t \in \mathbb{R}_+$, the total information available at time t is captured by the σ-field $\mathcal{G}_t$.

All filtrations considered in what follows are implicitly assumed to satisfy the 'usual conditions' of right-continuity and completeness. For the sake of simplicity, we assume that the σ-field $\mathcal{F}_0$ is trivial. Since $\mathbb{Q}(\tau = 0) = 0$ this implies that $\mathcal{G}_0$ is the trivial σ-field as well.

The process H is obviously $\mathbb{G}$-adapted, but it is not necessarily $\mathbb{F}$-adapted. In other words, the random time τ is a $\mathbb{G}$-stopping time, but it may fail to be an $\mathbb{F}$-stopping time.

Lemma 3.1.1. *Assume that the filtration $\mathbb{G}$ satisfies $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. Then $\mathbb{G} \subseteq \mathbb{G}^*$ where the filtration $\mathbb{G}^* = (\mathcal{G}^*_t)_{t \in \mathbb{R}_+}$ is defined as follows*

$$\mathcal{G}^*_t := \big\{ A \in \mathcal{G} \, : \, A \cap \{\tau > t\} = B \cap \{\tau > t\} \text{ for some } B \in \mathcal{F}_t \, \big\}.$$

Proof. It is rather clear that the class $\mathcal{G}^*_t$ is a sub-σ-field of $\mathcal{G}$. Therefore, it is enough to check that $\mathcal{H}_t \subseteq \mathcal{G}^*_t$ and $\mathcal{F}_t \subseteq \mathcal{G}^*_t$ for every $t \in \mathbb{R}_+$. Put another way, we need to verify that if either $A = \{\tau \leq u\}$ for some $u \leq t$ or $A \in \mathcal{F}_t$ then there exists an event $B \in \mathcal{F}_t$ such that $A \cap \{\tau > t\} = B \cap \{\tau > t\}$. In the former case, we may take $B = \emptyset$ and in the latter $B = A$. □

For any $t \in \mathbb{R}_+$, we write $F_t = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t)$ and we denote by G the $\mathbb{F}$-*survival process* of τ with respect to the filtration $\mathbb{F}$, given as

$$G_t := 1 - F_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t).$$

For any $0 \leq t \leq s$ the inclusion $\{\tau \leq t\} \subseteq \{\tau \leq s\}$ holds, and thus

$$\mathbb{E}_{\mathbb{Q}}(F_s \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}\big(\mathbb{Q}(\tau \leq s \,|\, \mathcal{F}_s) \,\big|\, \mathcal{F}_t\big) = \mathbb{Q}(\tau \leq s \,|\, \mathcal{F}_t) \geq \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = F_t.$$

This shows that the process F (G, respectively) follows a bounded and non-negative $\mathbb{F}$-submartingale ($\mathbb{F}$-supermartingale, respectively) under $\mathbb{Q}$ and thus we may deal with the right-continuous modifications of F and G with finite left-hand limits. It is worth noting that $F_0 = 0$ and $\lim_{t \to \infty} F_t = 1$.

The next definition introduces a straightforward generalization of the concept of the hazard function (see Definition 2.2.1).

Definition 3.1.1. Assume that $F_t < 1$ for $t \in \mathbb{R}_+$. The $\mathbb{F}$-*hazard process* of τ under $\mathbb{Q}$, denoted by Γ, is defined through the formula $1 - F_t = e^{-\Gamma_t}$. Equivalently, $\Gamma_t = -\ln G_t = -\ln(1 - F_t)$ for every $t \in \mathbb{R}_+$.

Since $G_0 = 1$, it is clear that $\Gamma_0 = 0$. Moreover, $\lim_{t\to\infty} \Gamma_t = \infty$ since $\lim_{t\to\infty} G_t = 0$. For the sake of conciseness, we shall refer briefly to Γ as the $\mathbb{F}$-hazard process, rather than the $\mathbb{F}$-hazard process under $\mathbb{Q}$, unless there is a danger of misunderstanding.

Throughout this chapter, we will work under the standing assumption that the inequality $F_t < 1$ holds for every $t \in \mathbb{R}_+$, so that the $\mathbb{F}$-hazard process Γ is well defined. Therefore, the case when τ is an $\mathbb{F}$-stopping time (that is, the case when $\mathbb{F} = \mathbb{G}$) is not dealt with here.

3.1.1 Conditional Expectations

We will first focus on the conditional expectation $\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\}} X \,|\, \mathcal{G}_t)$, where X is a $\mathbb{Q}$-integrable random variable. We start by extending the formula established in Lemma 2.2.1.

Lemma 3.1.2. *For any $\mathcal{G}$-measurable and $\mathbb{Q}$-integrable random variable X we have, for any $t \in \mathbb{R}_+$,*

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\}} X \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \frac{\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\}} X \,|\, \mathcal{F}_t)}{\mathbb{Q}(t < \tau \,|\, \mathcal{F}_t)}. \tag{3.1}$$

In particular, for any $t \le s$

$$\mathbb{Q}(t < \tau \le s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \frac{\mathbb{Q}(t < \tau \le s \,|\, \mathcal{F}_t)}{\mathbb{Q}(t < \tau \,|\, \mathcal{F}_t)} = \mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}}(1 - e^{\Gamma_t - \Gamma_s} \,|\, \mathcal{F}_t).$$

Proof. Since $\mathcal{F}_t \subseteq \mathcal{G}_t$, it suffices to check that

$$\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_C X \mathbb{Q}(C \,|\, \mathcal{F}_t) \,\big|\, \mathcal{G}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_C \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C X \,|\, \mathcal{F}_t) \,\big|\, \mathcal{G}_t\big),$$

where we denote $C = \{t < \tau\}$. Put another way, we need to show that for any $A \in \mathcal{G}_t$ we have

$$\int_A \mathbb{1}_C X \mathbb{Q}(C \,|\, \mathcal{F}_t)\, d\mathbb{Q} = \int_A \mathbb{1}_C \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C X \,|\, \mathcal{F}_t)\, d\mathbb{Q}. \tag{3.2}$$

In view of Lemma 3.1.1, for any $A \in \mathcal{G}_t$ we have $A \cap C = B \cap C$ for some event $B \in \mathcal{F}_t$, and so

$$\begin{aligned}\int_A \mathbb{1}_C X \mathbb{Q}(C \,|\, \mathcal{F}_t)\, d\mathbb{Q} &= \int_{A\cap C} X \mathbb{Q}(C \,|\, \mathcal{F}_t)\, d\mathbb{Q} = \int_{B\cap C} X \mathbb{Q}(C \,|\, \mathcal{F}_t)\, d\mathbb{Q} \\ &= \int_B \mathbb{1}_C X \mathbb{Q}(C \,|\, \mathcal{F}_t)\, d\mathbb{Q} = \int_B \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C X \,|\, \mathcal{F}_t) \mathbb{Q}(C \,|\, \mathcal{F}_t)\, d\mathbb{Q} \\ &= \int_B \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C X \,|\, \mathcal{F}_t) \,|\, \mathcal{F}_t)\, d\mathbb{Q} = \int_{B\cap C} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C X \,|\, \mathcal{F}_t)\, d\mathbb{Q} \\ &= \int_{A\cap C} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C X \,|\, \mathcal{F}_t)\, d\mathbb{Q} = \int_A \mathbb{1}_C \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_C X \,|\, \mathcal{F}_t)\, d\mathbb{Q}.\end{aligned}$$

We thus conclude that (3.2) holds. □

The following corollary to Lemma 3.1.2 is rather straightforward.

Corollary 3.1.1. *Let X be a $\mathcal{F}_T$-measurable and $\mathbb{Q}$-integrable random variable. Then, for every $t \le T$,*

$$\mathbb{E}_{\mathbb{Q}}(X \mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \frac{\mathbb{E}_{\mathbb{Q}}(X \mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\}} \,|\, \mathcal{F}_t)} = \mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}}(X e^{\Gamma_t - \Gamma_T} \,|\, \mathcal{F}_t).$$

The following result will be used in valuation of a recovery payoff that occurs at default.

Lemma 3.1.3. *Assume that Z is an $\mathbb{F}$-predictable process such that the random variable $Z_\tau \mathbb{1}_{\{\tau \le T\}}$ is $\mathbb{Q}$-integrable. Then we have, for every $t \le T$,*

$$\mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} Z_u\, dF_u \,\Big|\, \mathcal{F}_t \Big). \tag{3.3}$$

Let $F = N + C$ be the Doob-Meyer decomposition of F, where N is an $\mathbb{F}$-martingale, and C is an $\mathbb{F}$-predictable increasing process. Then, for every $t \le T$,

$$\mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} Z_u\, dC_u \,\Big|\, \mathcal{F}_t \Big). \tag{3.4}$$

If F is a continuous, increasing process then $F = C = e^{-\Gamma_t}$ so that the equality $dF_t = e^{-\Gamma_t}\, d\Gamma_t$ is valid. Consequently,

$$\mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\, \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T Z_u e^{\Gamma_t - \Gamma_u}\, d\Gamma_u \,\Big|\, \mathcal{F}_t \Big).$$

Proof. We start by noting that (3.3) implies that

$$\mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\, e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{t<\tau \le T\}} \,|\, \mathcal{F}_t).$$

Let us first assume that Z is a stepwise $\mathbb{F}$-predictable process; specifically, $Z_u = \sum_{i=0}^n Z_{t_i} \mathbb{1}_{\{t_i < u \le t_{i+1}\}}$ for $t < u \le T$, where $t_0 = t < \cdots < t_{n+1} = T$, and Z_{t_i} is an $\mathcal{F}_{t_i}$-measurable random variable for $i = 0, \ldots, n$. Then we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{t<\tau \le T\}} \,|\, \mathcal{F}_t) &= \mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{t<\tau \le T\}} \,|\, \mathcal{F}_t) \\
&= \mathbb{E}_{\mathbb{Q}}\Big(\sum_{i=0}^n \mathbb{1}_{\{t_i<\tau\le t_{i+1}\}} Z_{t_i} \,\Big|\, \mathcal{F}_t \Big) \\
&= \mathbb{E}_{\mathbb{Q}}\Big(\sum_{i=0}^n Z_{t_i} (F_{t_{i+1}} - F_{t_i}) \,\Big|\, \mathcal{F}_t \Big).
\end{aligned}$$

Hence for any stepwise, bounded, $\mathbb{F}$-predictable process Z we have

$$\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{t<\tau\le T\}} Z_\tau \,|\, \mathcal{F}_t \big) = \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} Z_u \, dF_u \,\Big|\, \mathcal{F}_t \Big). \tag{3.5}$$

In the next step, Z is approximated by a suitable sequence of bounded, stepwise, $\mathbb{F}$-predictable processes. The sum under the sign of the conditional expectation converges to the Itô integral (or to the Lebesque-Stieltjes integral if the process F is of finite variation). The assumption that Z and F are bounded is a sufficient condition for the convergence of sequence of conditional expectations. □

The next auxiliary result will prove useful in valuation of defaultable securities that pay dividends prior to the default time.

Proposition 3.1.1. *Assume that A is a bounded, $\mathbb{F}$-predictable process of finite variation. Then, for every $t \le T$,*

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} (1 - H_u) \, dA_u \,\Big|\, \mathcal{G}_t \Big) = \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} (1 - F_u) \, dA_u \,\Big|\, \mathcal{F}_t \Big)$$

or, equivalently,

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} (1 - H_u) \, dA_u \,\Big|\, \mathcal{G}_t \Big) = \mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} e^{\Gamma_t - \Gamma_u} \, dA_u \,\Big|\, \mathcal{F}_t \Big).$$

Proof. For a fixed, but arbitrary, $t \le T$, we introduce an auxiliary process $\widehat{A}$ by setting $\widehat{A}_u = A_u - A_t$ for $u \in [t,T]$. It is clear that $\widehat{A}$ is a bounded and $\mathbb{F}$-predictable process of finite variation; the same remark applies to its left-continuous version $\widehat{A}_{t-}$.

Therefore,

$$\begin{aligned} J_t \;&=\; \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} (1-H_u)\, dA_u \,\Big|\, \mathcal{G}_t \Big) \\ &= \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} \mathbb{1}_{\{\tau>u\}}\, d\widehat{A}_u \,\Big|\, \mathcal{G}_t \Big) \\ &= \mathbb{E}_{\mathbb{Q}}\Big(\widehat{A}_{\tau-}\mathbb{1}_{\{t<\tau\le T\}} + \widehat{A}_T \mathbb{1}_{\{\tau>T\}} \,\Big|\, \mathcal{G}_t \Big) \\ &= \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} \widehat{A}_{u-}\, dF_u + \widehat{A}_T(1-F_T) \,\Big|\, \mathcal{F}_t \Big), \end{aligned}$$

where the last equality follows from formulae (3.1) and (3.3). Using an obvious equality $G_t = 1 - F_t$, we obtain

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} \widehat{A}_{u-}\, dF_u + \widehat{A}_T(1-F_T) \,\Big|\, \mathcal{F}_t \Big) = \mathbb{E}_{\mathbb{Q}}\Big(-\int_{]t,T]} \widehat{A}_{u-}\, dG_u + \widehat{A}_T G_T \,\Big|\, \mathcal{F}_t \Big).$$

Since $\widehat{A}$ is a process of finite variation (so that its continuous martingale part vanishes), the following version of Itô's product rule is valid

$$\widehat{A}_T G_T = \widehat{A}_t G_t + \int_{]t,T]} \widehat{A}_{u-}\, dG_u + \int_{]t,T]} G_u\, d\widehat{A}_u.$$

But $\widehat{A}_t = 0$, and thus

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} \widehat{A}_{u-}\, dF_u + \widehat{A}_T(1-F_T) \,\Big|\, \mathcal{F}_t \Big) = \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} (1-F_u)\, dA_u \,\Big|\, \mathcal{F}_t \Big).$$

This proves the first asserted formula. The second equality is merely a restatement of the first one. □

3.1.2 Hazard Rate

Let the process F be absolutely continuous, that is, $F_t = \int_0^t f_u\, du$ for some $\mathbb{F}$-progressively measurable, non-negative process f. Then necessarily F is an increasing process and thus Γ is an absolutely continuous and increasing process. Specifically, it is easy to check that Γ admits the $\mathbb{F}$-*hazard rate* γ, that is, $\Gamma_t = \int_0^t \gamma_u\, du$ where in turn the $\mathbb{F}$-progressively measurable, non-negative process γ is given by the formula $\gamma_t = (1-F_t)^{-1} f_t$. We will sometimes refer to γ as the $\mathbb{F}$-*intensity* (or simply *stochastic intensity*) of default time τ (see Section 3.1.6).

3.1.3 Valuation of Defaultable Claims

Our next goal is to establish a convenient representation for the pre-default value of a defaultable claim in terms of the hazard process Γ of the default time. We postulate that $\mathbb{Q}$ represents a martingale measure associated with the choice of the savings account B as a discount factor (or a *numéraire*). Therefore, in the present setup, the *risk-neutral valuation formula* reads (for a justification of this formula, see Section 2.3)

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}} \Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big), \tag{3.6}$$

where S is the ex-dividend price process, B is the savings account and D is the dividend process associated with a defaultable claim (see Section 1.1.2), that is,

$$D_t = X_T^d \mathbb{1}_{[T,\infty[}(t) + \int_{]0,t]} (1 - H_u) \, dA_u + \int_{]0,t]} Z_u \, dH_u. \tag{3.7}$$

For the sake of conciseness, we will write

$$I_t = B_t \, \mathbb{E}_{\mathbb{Q}} \Big(\int_{]t,T]} B_u^{-1} (1 - H_u) \, dA_u \, \Big| \, \mathcal{G}_t \Big)$$

$$J_t = B_t \, \mathbb{E}_{\mathbb{Q}} \big(\mathbb{1}_{\{t < \tau \leq T\}} B_\tau^{-1} Z_\tau \, \big| \, \mathcal{G}_t \big),$$

and

$$K_t = B_t \, \mathbb{E}_{\mathbb{Q}} \big(B_T^{-1} X \mathbb{1}_{\{T < \tau\}} \, \big| \, \mathcal{G}_t \big).$$

In view of (3.6)–(3.7), it is clear that the ex-dividend price of a generic defaultable claim (X, A, Z, τ) (cf. Definition 2.3.1) can be represented as follows $S_t = I_t + J_t + K_t$. It is noteworthy that the default time τ does not appear explicitly in the conditional expectation in the right-hand side of pricing formulae of Proposition 3.1.2.

Proposition 3.1.2. *For every $t \in [0, T]$, the ex-dividend price of a defaultable claim (X, A, Z, τ) admits the following representation*

$$S_t = \mathbb{1}_{\{t < \tau\}} G_t^{-1} B_t \, \mathbb{E}_{\mathbb{Q}} \Big(\int_{]t,T]} B_u^{-1} (G_u \, dA_u - Z_u \, dG_u) + G_T B_T^{-1} X \, \Big| \, \mathcal{F}_t \Big).$$

If F (and thus also Γ) is an increasing, continuous process then

$$S_t = \mathbb{1}_{\{t < \tau\}} B_t \, \mathbb{E}_{\mathbb{Q}} \Big(\int_{]t,T]} B_u^{-1} e^{\Gamma_t - \Gamma_u} \, (dA_u + Z_u \, d\Gamma_u) + B_T^{-1} X e^{\Gamma_t - \Gamma_T} \, \Big| \, \mathcal{F}_t \Big).$$

Proof. By applying Proposition 3.1.1 to the process of finite variation $\int_{]0,t]} B_u^{-1}\, dA_u$, we obtain

$$I_t = \mathbb{1}_{\{t<\tau\}} G_t^{-1} B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1} G_u\, dA_u \,\Big|\, \mathcal{F}_t\Big)$$

or, equivalently,

$$I_t = \mathbb{1}_{\{t<\tau\}} B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1} e^{\Gamma_t - \Gamma_u}\, dA_u \,\Big|\, \mathcal{F}_t\Big).$$

Furthermore, Lemma 3.1.3 yields

$$J_t = \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t} B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1} Z_u\, dF_u \,\Big|\, \mathcal{F}_t\Big).$$

If, in addition, the hazard process Γ is an increasing continuous process then

$$J_t = \mathbb{1}_{\{t<\tau\}} B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T B_u^{-1} e^{\Gamma_t - \Gamma_u} Z_u\, d\Gamma_u \,\Big|\, \mathcal{F}_t\Big).$$

Finally, it follows from (3.1) that

$$K_t = \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t} B_t\, \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau>T\}} B_T^{-1} X \,|\, \mathcal{F}_t).$$

Since the random variables X and B_T are $\mathcal{F}_T$-measurable, we also have

$$K_t = \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t} B_t\, \mathbb{E}_{\mathbb{Q}}(G_T B_T^{-1} X \,|\, \mathcal{F}_t) = \mathbb{1}_{\{t<\tau\}} B_t\, \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} X e^{\Gamma_t - \Gamma_T} \,|\, \mathcal{F}_t\big).$$

Both formulae of the proposition are obtained upon summation. □

Let us note that $S_t = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t$, where the $\mathbb{F}$-adapted process $\widetilde{S}$ represents the *pre-default value* of a defaultable claim (X, A, Z, τ). The next result is a straightforward consequence of Proposition 3.1.2.

Corollary 3.1.2. *Assume that F (and thus also Γ) is an increasing, continuous process. Then the pre-default value of a defaultable claim (X, A, Z, τ) coincides with the pre-default value of a defaultable claim $(X, \widehat{A}, 0, \tau)$, where the process $\widehat{A}$ is given by the formula $\widehat{A}_t = A_t + \int_0^t Z_u\, d\Gamma_u$ for $t \in [0, T]$.*

Let us consider the case of a default time τ that admits the $\mathbb{F}$-intensity process γ. The second formula in Proposition 3.1.2 now becomes

$$\begin{aligned} S_t &= \mathbb{1}_{\{t<\tau\}}\, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} e^{-\int_t^u (r_v+\gamma_v)dv}\, (dA_u + \gamma_u Z_u\, du) \,\Big|\, \mathcal{F}_t\Big) \\ &\quad + \mathbb{1}_{\{t<\tau\}}\, \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T (r_v+\gamma_v)dv} X \,\Big|\, \mathcal{F}_t\Big). \end{aligned}$$

To obtain a more intuitive representation for the last expression, we introduce the *default-risk-adjusted interest rate* $\widehat{r} = r + \gamma$ and the associated *default-risk-adjusted savings account* $\widehat{B}$, which is given by the formula

$$\widehat{B}_t = \exp\Big(\int_0^t \widehat{r}_u \, du\Big). \tag{3.8}$$

Although the process $\widehat{B}$ does not represent the price of a tradeable security, it enjoys the features of the savings account B. Specifically, $\widehat{B}$ is an $\mathbb{F}$-adapted, continuous process of finite variation (typically, though not necessarily, an increasing process). In terms of the process $\widehat{B}$, we have

$$S_t = \mathbb{1}_{\{t<\tau\}} \widehat{B}_t \, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} \widehat{B}_u^{-1} \, dA_u + \int_t^T \widehat{B}_u^{-1} Z_u \gamma_u \, du + \widehat{B}_T^{-1} X \,\Big|\, \mathcal{F}_t\Big). \tag{3.9}$$

3.1.4 Defaultable Bonds

Consider a defaultable zero-coupon bond with the par (face) value L and maturity date T. We will re-examine the following recovery schemes: the fractional recovery of par value and the fractional recovery of Treasury value; recall that these schemes were already studied in Section 2.1 in the case of deterministic intensity. The fractional recovery of market value scheme is more difficult to deal with, though it is still tractable (cf. Duffie et al. [74] and Duffie and Singleton [75]).

We assume in this subsection that τ admits the $\mathbb{F}$-hazard rate γ.

Fractional Recovery of Par Value

Under this scheme, a fixed fraction of the bond face value is paid to the bondholders at the time of default. Formally, we deal here with a defaultable claim $(X, 0, Z, \tau)$, which settles at time T, with the promised payoff $X = L$, where L stands for the bond's face value and with the constant recovery process $Z = \delta L$ for some $\delta \in [0, 1]$. The ex-dividend price at time $t \in [0, T]$ of the bond is thus given by the following expression

$$D^{\delta}(t,T) = LB_t \, \mathbb{E}_{\mathbb{Q}}\big(\delta B_\tau^{-1} \mathbb{1}_{\{t<\tau\le T\}} + B_T^{-1} \mathbb{1}_{\{T<\tau\}} \,\big|\, \mathcal{G}_t\big).$$

If τ admits the $\mathbb{F}$-intensity process γ then the pre-default value of the bond equals

$$\widetilde{D}^{\delta}(t,T) = L\widehat{B}_t \, \mathbb{E}_{\mathbb{Q}}\Big(\delta \int_t^T \widehat{B}_u^{-1} \gamma_u \, du + \widehat{B}_T^{-1} \,\Big|\, \mathcal{F}_t\Big). \tag{3.10}$$

Fractional Recovery of Treasury Value

According to this convention, the fixed fraction of the face value is paid to bondholders at maturity date T. A corporate zero-coupon bond is now given by a defaultable claim $(X, 0, Z, \tau)$ with the promised payoff $X = L$ and the recovery process $Z_t = \delta L B(t,T)$ where, as usual, $B(t,T)$ stands for the price at time t of a unit zero-coupon Treasury bond with maturity T. The defaultable bond is here equivalent to a single contingent claim Y, which settles at time T and equals

$$Y = L\big(\mathbb{1}_{\{\tau > T\}} + \delta \mathbb{1}_{\{\tau \le T\}}\big).$$

The ex-dividend price $D^\delta(t,T)$ of this claim at time $t < T$ thus equals

$$D^\delta(t,T) = LB_t\, \mathbb{E}_{\mathbb{Q}}\left(B_T^{-1}(\delta \mathbb{1}_{\{T \ge \tau\}} + \mathbb{1}_{\{T<\tau\}}) \,\middle|\, \mathcal{G}_t\right)$$

or, equivalently,

$$S_t = LB_t\, \mathbb{E}_{\mathbb{Q}}\left(\delta B_\tau^{-1} B(\tau,T) \mathbb{1}_{\{t<\tau\le T\}} + B_T^{-1}\mathbb{1}_{\{T<\tau\}} \,\middle|\, \mathcal{G}_t\right).$$

The pre-default value $\widetilde{D}^\delta(t,T)$ of a defaultable bond that is subject to the fractional recovery of Treasury value scheme is given by the expression

$$\widetilde{D}^\delta(t,T) = L\widehat{B}_t\, \mathbb{E}_{\mathbb{Q}}\Big(\delta \int_t^T \widehat{B}_u^{-1} B(u,T)\gamma_u\, du + \widehat{B}_T^{-1} \,\Big|\, \mathcal{F}_t\Big).$$

3.1.5 Compensator of Default Indicator Process

We will now examine the compensator of the default indicator process.

Proposition 3.1.3. *(i) The process $L_t = (1 - H_t)e^{\Gamma_t}$ is a $\mathbb{G}$-martingale.*
(ii) If X is an $\mathbb{F}$-martingale and the process XL is integrable then it is a $\mathbb{G}$-martingale.
(iii) If the process F (or, equivalently, Γ) is increasing and continuous then the process $M_t = H_t - \Gamma(t \wedge \tau)$ is a $\mathbb{G}$-martingale.

Proof. (i) From Lemma 3.1.2, we obtain, for any $t \le s$,

$$\mathbb{E}_{\mathbb{Q}}(L_s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}} e^{\Gamma_s} \,|\, \mathcal{F}_t) = \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t} = L_t,$$

since the tower rule yields (obviously, $\mathcal{F}_t \subset \mathcal{F}_s$)

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}} e^{\Gamma_s} | \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\mathbb{Q}(\tau > s \,|\, \mathcal{F}_s) e^{\Gamma_s} | \mathcal{F}_t) = 1.$$

(ii) Using again Lemma 3.1.2, we get, for any $t \le s$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(L_s X_s \,|\, \mathcal{G}_t) &= \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}} L_s X_s \,|\, \mathcal{G}_t) \\
&= \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{s<\tau\}} e^{\Gamma_s} X_s \,\big|\, \mathcal{F}_t\big) \\
&= \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big(\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}} \,|\, \mathcal{F}_s) e^{\Gamma_s} X_s \,\big|\, \mathcal{F}_t\big) \\
&= L_t X_t.
\end{aligned}$$

(iii) Note that H is a process of finite variation and Γ is an increasing, continuous process. Hence from the integration by parts formula, we obtain

$$dL_t = (1 - H_t) e^{\Gamma_t}\, d\Gamma_t - e^{\Gamma_t}\, dH_t.$$

Moreover, the process $M_t = H_t - \Gamma(t \wedge \tau)$ can be represented as follows

$$M_t = \int_{]0,t]} dH_u - \int_0^t (1 - H_u)\, d\Gamma_u = - \int_{]0,t]} e^{-\Gamma_u}\, dL_u,$$

and thus it is a $\mathbb{G}$-martingale, since L is $\mathbb{G}$-martingale and $e^{-\Gamma_t}$ is a bounded process. It should be noted that if the hazard process Γ is not assumed to be increasing then the Itô differential de^{Γ_t} becomes more complicated. □

It is worth stressing that the process F (or, equivalently, Γ) is not of finite variation, in general. This means that part (iii) in Proposition 3.1.3 does not yield the general form of the Doob-Meyer decomposition of the submartingale H.

For simplicity, in the next result we shall assume that F is a continuous process. It is worth noting that part (iii) in Proposition 3.1.3 is a consequence of Proposition 3.1.4, since for a continuous and increasing F we have that $F = C = 1 - e^{-\Gamma}$.

Proposition 3.1.4. *Assume that F is a continuous process with the Doob-Meyer decomposition $F = N + C$. Then the process $M = (M_t,\, t \in \mathbb{R}_+)$, which is given by the formula*

$$M_t = H_t - \int_0^{t\wedge\tau} \frac{dC_u}{1 - F_u}, \tag{3.11}$$

is a $\mathbb{G}$-martingale.

Proof. We split the proof into two steps.
First step. We shall prove that, for any $t \le s$,

$$\mathbb{E}_{\mathbb{Q}}(H_s \,|\, \mathcal{G}_t) = H_t + \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}(C_s - C_t \,|\, \mathcal{F}_t). \tag{3.12}$$

Indeed, we have that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(H_s \,|\, \mathcal{G}_t) &= 1 - \mathbb{Q}(s < \tau \,|\, \mathcal{G}_t) = 1 - \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}(1 - F_s \,|\, \mathcal{F}_t) \\
&= 1 - \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}(1 - N_s - C_s \,|\, \mathcal{F}_t) \\
&= 1 - \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t} \big(1 - N_t - C_t - \mathbb{E}_{\mathbb{Q}}(C_s - C_t \,|\, \mathcal{F}_t)\big) \\
&= 1 - \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t} \big(1 - F_t - \mathbb{E}_{\mathbb{Q}}(C_s - C_t \,|\, \mathcal{F}_t)\big) \\
&= \mathbb{1}_{\{t\geq\tau\}} + \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}(C_s - C_t \,|\, \mathcal{F}_t).
\end{aligned}$$

Second step. Let us denote

$$U_t = \int_0^t \frac{dC_u}{1 - F_u} = \int_0^t e^{\Gamma_u}\, dC_u.$$

We shall prove that, for any $t \leq s$,

$$\mathbb{E}_{\mathbb{Q}}(U_{s\wedge\tau} \,|\, \mathcal{G}_t) = U_{t\wedge\tau} + \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}(C_s - C_t \,|\, \mathcal{F}_t).$$

From Lemma 3.1.3, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(U_{s\wedge\tau} \,|\, \mathcal{G}_t) &= U_{t\wedge\tau} \mathbb{1}_{\{t\geq\tau\}} + \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}} \left(\int_t^\infty U_{s\wedge u}\, dF_u \,\Big|\, \mathcal{F}_t \right) \\
&= U_{t\wedge\tau} \mathbb{1}_{\{t\geq\tau\}} + \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}} \left(\int_t^s U_u\, dF_u + \int_s^\infty U_s\, dF_u \,\Big|\, \mathcal{F}_t \right) \\
&= U_{t\wedge\tau} \mathbb{1}_{\{t\geq\tau\}} + \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}} \left(\int_t^s U_u\, dF_u + U_s(1 - F_s) \,\Big|\, \mathcal{F}_t \right).
\end{aligned}$$

Using the integration by parts formula and the fact that U is a continuous process of finite variation, we obtain

$$d(U_t(1 - F_t)) = -U_t\, dF_t + (1 - F_t)\, dU_t = -U_t\, dF_t + dC_t.$$

Consequently,

$$\begin{aligned}
\int_t^s U_u\, dF_u + U_s(1 - F_s) &= -U_s(1 - F_s) + U_t(1 - F_t) + C_s - C_t \\
&\quad + U_s(1 - F_s) = U_t(1 - F_t) + C_s - C_t.
\end{aligned}$$

It follows that, for any $t \leq s$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(U_{s\wedge\tau} \,|\, \mathcal{G}_t) &= \mathbb{1}_{\{t\geq\tau\}} U_{t\wedge\tau} + \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\big(U_t(1 - F_t) + C_s - C_t \,|\, \mathcal{F}_t\big) \\
&= U_{t\wedge\tau} + \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}(C_s - C_t \,|\, \mathcal{F}_t).
\end{aligned}$$

By combining the formula above with (3.12), we conclude that the process M given by (3.11) is a $\mathbb{G}$-martingale. □

Proposition 3.1.5. *Assume that the bounded submartingale F admits the Doob-Meyer decomposition $F = N + C$. Then the process $M = (M_t,\, t \in \mathbb{R}_+)$, which is given by the formula*

$$M_t = H_t - \int_0^{t\wedge\tau} \frac{dC_u}{1 - F_{u-}}, \tag{3.13}$$

is a $\mathbb{G}$-martingale.

Proof. In the first part of the proof, we proceed along the same lines as in the proof of Proposition 2.2.1. In view of Lemma 3.1.2, we find that, in the present case, it is enough to show that the following equalities hold, for every $t \le s$,

$$I := \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,s\wedge\tau]} \frac{dC_u}{1-F_{u-}} \,\Big|\, \mathcal{F}_t \Big) = \mathbb{E}_{\mathbb{Q}}(F_s - F_t \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(C_s - C_t \,|\, \mathcal{F}_t),$$

where the second equality is simply a consequence of the definition of C. We have

$$\begin{aligned}
I &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{s<\tau\}} \int_{]t,s]} \frac{dC_u}{1-F_{u-}} + \mathbb{1}_{\{t<\tau\le s\}} \int_{]t,s\wedge\tau]} \frac{dC_u}{1-F_{u-}} \,\Big|\, \mathcal{F}_t \Big) \\
&= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{s<\tau\}} \int_{]t,s]} \frac{dC_u}{1-F_{u-}} \,\Big|\, \mathcal{F}_s \Big) + \mathbb{1}_{\{t<\tau\le s\}} \int_{]t,s\wedge\tau]} \frac{dC_u}{1-F_{u-}} \,\Big|\, \mathcal{F}_t \Big) \\
&= \mathbb{E}_{\mathbb{Q}}\Big((1-F_s) \int_{]t,s]} \frac{dC_u}{1-F_{u-}} + \int_{]t,s]} \int_{]t,u]} \frac{dC_v}{1-F_{v-}}\, dC_u \,\Big|\, \mathcal{F}_t \Big) \\
&= \mathbb{E}_{\mathbb{Q}}\Big((\Lambda_s - \Lambda_t)(1-F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dC_u \,\Big|\, \mathcal{F}_t \Big),
\end{aligned}$$

where the third equality follows from formula (3.5) and where we denote, for every $r \in \mathbb{R}_+$,

$$\Lambda_t = \int_{]0,t]} \frac{dC_u}{1-F_{u-}}. \tag{3.14}$$

Since Λ is an $\mathbb{F}$-predictable process and N is an $\mathbb{F}$-martingale, we obtain

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,s]} (\Lambda_u - \Lambda_t)\, dN_u \,\Big|\, \mathcal{F}_t \Big) = 0,$$

and this in turn yields

$$\begin{aligned}
I &= \mathbb{E}_{\mathbb{Q}}\Big((\Lambda_s - \Lambda_t)(1-F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dC_u \,\Big|\, \mathcal{F}_t \Big) \\
&= \mathbb{E}_{\mathbb{Q}}\Big((\Lambda_s - \Lambda_t)(1-F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u \,\Big|\, \mathcal{F}_t \Big).
\end{aligned}$$

Recall that our goal is to show that $I = \mathbb{E}_{\mathbb{Q}}(C_s - C_t \,|\, \mathcal{F}_t)$. To this end, we observe that

$$\int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u = -\Lambda_t (F_s - F_t) + \int_{]t,s]} \Lambda_u \, dF_u.$$

Since Λ is a process of finite variation, Itô's product rule yields

$$\int_{]t,s]} \Lambda_u \, dF_u = \Lambda_s F_s - \Lambda_t F_t - \int_{]t,s]} F_{u-} \, d\Lambda_u. \tag{3.15}$$

Finally, it follows from (3.14) that

$$\int_{]t,s]} F_{u-} \, d\Lambda_u = \Lambda_s - \Lambda_t - C_s + C_t.$$

Combining the above formulae, we conclude that

$$(\Lambda_s - \Lambda_t)(1 - F_s) + \int_{]t,s]} (\Lambda_u - \Lambda_t)\, dF_u = C_s - C_t. \tag{3.16}$$

This completes the proof. □

3.1.6 $\mathbb{F}$-Intensity of Default Time

Assume that F admits the Doob-Meyer decomposition $F = N + C$, where the process C is absolutely continuous with respect to the Lebesgue measure, so that $C_t = \int_0^t c_u \, du$ for some $\mathbb{F}$-progressively measurable process c.

Definition 3.1.2. The $\mathbb{F}$-*intensity* of default time τ is a non-negative and $\mathbb{F}$-progressively measurable process λ such that M is a $\mathbb{G}$-martingale, where M is given by

$$M_t = H_t - \int_0^{t \wedge \tau} \lambda_u \, du.$$

Under the present assumptions the $\mathbb{F}$-intensity is given by the formula $\lambda_t = c_t (1 - F_t)^{-1}$ for every $t \in \mathbb{R}_+$ (note that since C is absolutely continuous we have that $(1 - F_{t-})^{-1} dC_t = (1 - F_t)^{-1} dC_t$). If we assume that the process F is absolutely continuous, then we recover the definition of the hazard rate of Section 3.1.2, since manifestly the equality $\lambda = \gamma$ holds in that case. The proof of the next lemma is left to the reader.

Lemma 3.1.4. *The* $\mathbb{F}$-*intensity of default time satisfies, for almost every* $t \in \mathbb{R}_+$,

$$\lambda_t = \lim_{h \to 0} \frac{1}{h} \frac{\mathbb{Q}(t < \tau < t + h \,|\, \mathcal{F}_t)}{\mathbb{Q}(t < \tau \,|\, \mathcal{F}_t)}.$$

3.1.7 Reduction of Information

In this section, we follow Jeanblanc and LeCam [109]. Suppose that $\widetilde{\mathbb{F}}$ is a sub-filtration of $\mathbb{F}$, so that $\widetilde{\mathcal{F}}_t \subset \mathcal{F}_t$ for every $t \in \mathbb{R}_+$. We define the full filtration $\widetilde{\mathbb{G}}$ by setting $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$ for every $t \in \mathbb{R}_+$. The hazard process of τ with respect to $\widetilde{\mathbb{F}}$ is given by $\widetilde{\Gamma}_t = -\ln \widetilde{G}_t$ with

$$\widetilde{G}_t = \mathbb{Q}(t < \tau \,|\, \widetilde{\mathcal{F}}_t) = \mathbb{E}_{\mathbb{Q}}(G_t \,|\, \widetilde{\mathcal{F}}_t).$$

For any $\mathbb{Q}$-integrable random variable Y, Lemma 3.1.2 implies that

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\}} Y \,|\, \widetilde{\mathcal{G}}_t) = \mathbb{1}_{\{t<\tau\}} e^{\widetilde{\Gamma}_t}\, \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\}} Y \,|\, \widetilde{\mathcal{F}}_t).$$

In particular, if Y is a $\widetilde{\mathcal{F}}_s$-measurable random variable then, for every $t \leq s$,

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}} Y \,|\, \widetilde{\mathcal{G}}_t) = \mathbb{1}_{\{t<\tau\}} e^{\widetilde{\Gamma}_t}\, \mathbb{E}_{\mathbb{Q}}(\widetilde{G}_s Y \,|\, \widetilde{\mathcal{F}}_t).$$

From the obvious equality

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}} Y \,|\, \widetilde{\mathcal{G}}_t) = \mathbb{E}_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}} Y \,|\, \mathcal{G}_t) \,|\, \widetilde{\mathcal{G}}_t),$$

we also obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{s<\tau\}} Y \,|\, \widetilde{\mathcal{G}}_t) &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}(G_s Y \,|\, \mathcal{F}_t) \,|\, \widetilde{\mathcal{G}}_t\right) \\ &= \mathbb{1}_{\{t<\tau\}} e^{\widetilde{\Gamma}_t}\, \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}(G_s Y \,|\, \mathcal{F}_t) \,\middle|\, \widetilde{\mathcal{F}}_t\right).\end{aligned}$$

From the uniqueness of the pre-default $\mathbb{F}$-adapted process, it can now be deduced that the following result is true.

Lemma 3.1.5. *For any $\mathbb{Q}$-integrable and $\widetilde{\mathcal{F}}_s$-measurable random variable Y we have, for every $t \leq s$,*

$$\mathbb{E}_{\mathbb{Q}}(\widetilde{G}_s Y \,|\, \widetilde{\mathcal{F}}_t) = \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}(G_s Y \,|\, \mathcal{F}_t) \,\middle|\, \widetilde{\mathcal{F}}_t\right).$$

Proof. We provide a direct proof of the asserted formula. We have

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{Q}}(G_s Y \,|\, \mathcal{F}_t) e^{\Gamma_t} \,\middle|\, \widetilde{\mathcal{F}}_t\right) \\ &\quad = \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\}} \,|\, \mathcal{F}_t) e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}(G_s Y \,|\, \mathcal{F}_t) \,\middle|\, \widetilde{\mathcal{F}}_t\right) \\ &\quad = \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(G_s Y \,|\, \mathcal{F}_t) \,\middle|\, \widetilde{\mathcal{F}}_t\right) = \mathbb{E}_{\mathbb{Q}}(G_s Y \,|\, \widetilde{\mathcal{F}}_t) \\ &\quad = \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(G_s \,|\, \widetilde{\mathcal{F}}_s) Y \,\middle|\, \widetilde{\mathcal{F}}_t\right) = \mathbb{E}_{\mathbb{Q}}(\widetilde{G}_s Y \,|\, \widetilde{\mathcal{F}}_t),\end{aligned}$$

since we assumed that Y is $\widetilde{\mathcal{F}}_s$-measurable. □

Let $F = N + C$ be the Doob-Meyer decomposition of the submartingale F with respect to $\mathbb{F}$ and let us assume that C is absolutely continuous with respect to t, that is, $C_t = \int_0^t c_u\,du$. Since C is an increasing process, it is easily seen that the process $\widetilde{C}_t = \mathbb{E}_{\mathbb{Q}}(C_t \,|\, \widetilde{\mathcal{F}}_t)$ is a submartingale with respect to $\widetilde{\mathbb{F}}$. Let us denote by $\widetilde{C} = \widetilde{z} + \widetilde{\alpha}$ its Doob-Meyer decomposition with respect to $\widetilde{\mathbb{F}}$ and let us set $\widetilde{N}_t = \mathbb{E}_{\mathbb{Q}}(N_t \,|\, \widetilde{\mathcal{F}}_t)$. Since $\widetilde{N}$ is an $\widetilde{\mathbb{F}}$-martingale, we see that the submartingale

$$\widetilde{F}_t = \mathbb{Q}(t \geq \tau \,|\, \widetilde{\mathcal{F}}_t) = \mathbb{E}_{\mathbb{Q}}(F_t \,|\, \widetilde{\mathcal{F}}_t)$$

admits the Doob-Meyer decomposition $\widetilde{F} = \widetilde{m} + \widetilde{\alpha}$, where the $\widetilde{\mathbb{F}}$-martingale part equals $\widetilde{m} = \widetilde{N} + \widetilde{z}$. The next lemma furnishes an explicit relationship between the increasing processes C and $\widetilde{\alpha}$.

Lemma 3.1.6. *Let $C_t = \int_0^t c_u\,du$ be the $\mathbb{F}$-predictable increasing process in the Doob-Meyer decomposition of the $\mathbb{F}$-submartingale F. Then the $\widetilde{\mathbb{F}}$-predictable increasing process in the Doob-Meyer decomposition $\widetilde{F} = \widetilde{m} + \widetilde{\alpha}$ of the $\widetilde{\mathbb{F}}$-submartingale $\widetilde{F}$ equals, for every $t \in \mathbb{R}_+$,*

$$\widetilde{\alpha}_t = \int_0^t \mathbb{E}_{\mathbb{Q}}(c_u \,|\, \widetilde{\mathcal{F}}_u)\,du. \tag{3.17}$$

Proof. To establish (3.17), we will show that the process

$$M_t^F = \mathbb{E}_{\mathbb{Q}}(F_t \,|\, \widetilde{\mathcal{F}}_t) - \int_0^t \mathbb{E}_{\mathbb{Q}}(c_u \,|\, \widetilde{\mathcal{F}}_u)\,du$$

is an $\widetilde{\mathbb{F}}$-martingale. Clearly, the process M^F is integrable and $\widetilde{\mathbb{F}}$-adapted. Moreover, for every $t \leq s$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(M_s^F \,|\, \widetilde{\mathcal{F}}_t) &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{E}_{\mathbb{Q}}(F_s \,|\, \widetilde{\mathcal{F}}_s) - \int_0^s \mathbb{E}_{\mathbb{Q}}(c_u \,|\, \widetilde{\mathcal{F}}_u)\,du \,\Big|\, \widetilde{\mathcal{F}}_t\Big) \\
&= \mathbb{E}_{\mathbb{Q}}(F_s \,|\, \widetilde{\mathcal{F}}_t) - \mathbb{E}_{\mathbb{Q}}\Big(\int_0^t \mathbb{E}_{\mathbb{Q}}(c_u \,|\, \widetilde{\mathcal{F}}_u)\,du \,\Big|\, \widetilde{\mathcal{F}}_t\Big) \\
&\quad - \mathbb{E}_{\mathbb{Q}}\Big(\int_t^s \mathbb{E}_{\mathbb{Q}}(c_u \,|\, \widetilde{\mathcal{F}}_u)\,du \,\Big|\, \widetilde{\mathcal{F}}_t\Big) \\
&= \widetilde{N}_t + \mathbb{E}_{\mathbb{Q}}\Big(\int_0^t c_u\,du \,\Big|\, \widetilde{\mathcal{F}}_t\Big) + \mathbb{E}_{\mathbb{Q}}\Big(\int_t^s c_u\,du \,\Big|\, \widetilde{\mathcal{F}}_t\Big) \\
&\quad - \int_0^t \mathbb{E}_{\mathbb{Q}}(c_u \,|\, \widetilde{\mathcal{F}}_u)\,du - \mathbb{E}_{\mathbb{Q}}\Big(\int_t^s \mathbb{E}_{\mathbb{Q}}(c_u \,|\, \widetilde{\mathcal{F}}_u)\,du \,\Big|\, \widetilde{\mathcal{F}}_t\Big)
\end{aligned}$$

and thus

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(M^F_s \,|\, \widetilde{\mathcal{F}}_t) &= M^F_t + \mathbb{E}_{\mathbb{Q}}\Big(\int_t^s c_u\,du \,\Big|\, \widetilde{\mathcal{F}}_t\Big) - \mathbb{E}_{\mathbb{Q}}\Big(\int_t^s \mathbb{E}_{\mathbb{Q}}(c_u \,|\, \widetilde{\mathcal{F}}_u)\,du \,\Big|\, \widetilde{\mathcal{F}}_t\Big)\\ &= M^F_t + \int_t^s \mathbb{E}_{\mathbb{Q}}(c_u \,|\, \widetilde{\mathcal{F}}_t)\,du - \int_t^s \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{E}_{\mathbb{Q}}(c_u \,|\, \widetilde{\mathcal{F}}_u) \,\Big|\, \widetilde{\mathcal{F}}_t\Big)\,du\\ &= M^F_t + \int_t^s \mathbb{E}_{\mathbb{Q}}(c_u \,|\, \widetilde{\mathcal{F}}_t)\,du - \int_t^s \mathbb{E}_{\mathbb{Q}}(c_u \,|\, \widetilde{\mathcal{F}}_t)\,du = M^F_t.\end{aligned}$$

We have thus shown that the process M^F is an $\widetilde{\mathbb{F}}$-martingale. Moreover, the $\widetilde{\mathbb{F}}$-adapted process $\widetilde{\alpha}$, given by (3.17) is manifestly continuous, and thus it is $\widetilde{\mathbb{F}}$-predictable. By virtue of uniqueness of the Doob-Meyer decomposition, we conclude that $M^F = \widetilde{m}$ and formula (3.17) is valid. □

Corollary 3.1.3. *Let us denote* $\widetilde{c}_t = \mathbb{E}_{\mathbb{Q}}(c_t \,|\, \widetilde{\mathcal{F}}_t)$*. The process*

$$\widetilde{M}_t = H_t - \int_0^{t\wedge\tau} \frac{\widetilde{c}_u}{1-\widetilde{F}_u}\,du$$

is a $\widetilde{\mathbb{G}}$*-martingale and the* $\widetilde{\mathbb{F}}$*-intensity of* τ *is equal to* $\widetilde{\lambda}_t = \widetilde{c}_t \widetilde{G}_t^{-1}$*.*

Remark 3.1.1. It is worth noting that, typically, the inequality

$$\mathbb{E}_{\mathbb{Q}}(\lambda_t \,|\, \widetilde{\mathcal{F}}_t) = \mathbb{E}_{\mathbb{Q}}(c_t G_t^{-1} | \, \widetilde{\mathcal{F}}_t) \neq \mathbb{E}_{\mathbb{Q}}(c_t \,|\, \widetilde{\mathcal{F}}_t)\widetilde{G}_t^{-1} = \widetilde{\lambda}_t$$

holds. This means that the $\widetilde{\mathbb{F}}$-intensity of τ is not given by the optional projection of the $\mathbb{F}$-intensity on the reduced filtration $\widetilde{\mathbb{F}}$, in general.

3.1.8 General Enlargement of Filtration

Assume that $\mathbb{G}$ is any enlarged filtration, that is, $\mathbb{F} \vee \mathbb{H} \subset \mathbb{G}$. Then we may work directly with the filtration $\mathbb{G}$, provided that the decomposition of any $\mathbb{F}$-martingale in this filtration is known up to time τ. For example, if W is an $\mathbb{F}$-Brownian motion, then it is not necessarily a $\mathbb{G}$-martingale and its Doob-Meyer decomposition with respect to the filtration $\mathbb{G}$ up to time τ reads

$$W_{t\wedge\tau} = \beta_{t\wedge\tau} + \int_0^{t\wedge\tau} \frac{d\langle W, G\rangle_u}{G_{u-}},$$

where $(\beta_{t\wedge\tau},\, t \in \mathbb{R}_+)$ is a continuous $\mathbb{G}$-martingale with the increasing process $t\wedge\tau$. Suppose, for instance, that the dynamics of an asset S are given by

$$dS_t = S_t\big(r_t\,dt + \sigma_t\,dW_t\big)$$

in the default-free framework, that is, with respect to the filtration $\mathbb{F}$. Then its dynamics with respect to the enlarged filtration $\mathbb{G}$ are

$$dS_t = S_t\Big(r_t\,dt + \sigma_t \frac{d\langle W, G\rangle_t}{G_{t-}} + \sigma_t\,d\beta_t\Big)$$

provided that we restrict our attention to the behavior of S prior to default. We conclude that the possibility of default changes the drift term in the price dynamics. The interested reader is referred to Mansuy and Yor [140] for more information.

3.2 Hypothesis (H)

As already mentioned above, an arbitrary $\mathbb{F}$-martingale does not remain a $\mathbb{G}$-martingale, in general. We shall now study a particular case in which this *martingale invariance property* (also known as the *immersion property* between $\mathbb{F}$ and $\mathbb{G}$) actually holds.

3.2.1 Equivalent Forms of the Hypothesis (H)

Once again we consider a general situation where $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ for some reference filtration $\mathbb{F}$. We shall now examine the so-called *hypothesis (H)* which can be stated as follows.

Hypothesis (H) Every $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale.

This hypothesis implies, in particular, that any $\mathbb{F}$-Brownian motion remains a Brownian motion with respect to the filtration $\mathbb{G}$. It was studied, among others, by Brémaud and Yor [34], Jeanblanc and Le Cam [108], Mazziotto and Szpirglas [141], Kusuoka [125] and Nikeghbali and Yor [149].

Let us first examine some equivalent forms of hypothesis (H) (for conditional independence of σ-fields, see, e.g., Dellacherie and Meyer [68]).

Lemma 3.2.1. *Assume that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F}$ is an arbitrary filtration and $\mathbb{H}$ is generated by the process $H_t = \mathbb{1}_{\{t \geq \tau\}}$. Then the following conditions are equivalent to the hypothesis (H).*
(i) For any $t, h \in \mathbb{R}_+$, we have

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_{t+h}). \tag{3.18}$$

(i′) For any $t \in \mathbb{R}_+$, we have

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty). \tag{3.19}$$

(ii) For any $t \in \mathbb{R}_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$ under $\mathbb{Q}$. This means that the equality

$$\mathbb{E}_{\mathbb{Q}}(\xi\eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\xi \,|\, \mathcal{F}_t)\, \mathbb{E}_{\mathbb{Q}}(\eta \,|\, \mathcal{F}_t)$$

holds for any bounded, $\mathcal{F}_\infty$-measurable random variable ξ and any bounded, $\mathcal{G}_t$-measurable random variable η.
(iii) For any $t \in \mathbb{R}_+$ and any $u \ge t$, the σ-fields $\mathcal{F}_u$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$.
(iv) For any $t \in \mathbb{R}_+$ and any bounded, $\mathcal{F}_\infty$-measurable random variable ξ, we have that $\mathbb{E}_{\mathbb{Q}}(\xi \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(\xi \,|\, \mathcal{F}_t)$.
(v) For any $t \in \mathbb{R}_+$, and any bounded, $\mathcal{G}_t$-measurable random variable η, we have that $\mathbb{E}_{\mathbb{Q}}(\eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\eta \,|\, \mathcal{F}_\infty)$.

Proof. If the hypothesis (H) holds then (3.19) is valid as well. If (3.19) holds then the fact that $\mathcal{H}_t$ is generated by the events $\{\tau \le s\}$, $s \le t$, proves that the σ-fields $\mathcal{F}_\infty$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$. The desired property now follows. The equivalence between (3.19) and (3.18) is left to the reader.

Using the monotone class theorem, it can be shown that conditions (i) and (i′) are equivalent. The proof of equivalence of conditions (i′)–(v) can be found, for instance, in Section 6.1.1 of Bielecki and Rutkowski [22] (for related results, see Elliott et al. [79]).

Let us show, for instance, that condition (iv) and the hypothesis (H) are equivalent.

Assume first that the hypothesis (H) is valid and consider an arbitrary bounded, $\mathcal{F}_\infty$-measurable random variable ξ. Let $M_t = \mathbb{E}_{\mathbb{Q}}(\xi \,|\, \mathcal{F}_t)$ be the martingale associated with ξ. Of course, M is also a local martingale with respect to $\mathbb{F}$. Then the hypothesis (H) implies that M is a local martingale with respect to $\mathbb{G}$ and thus a $\mathbb{G}$-martingale, since M is bounded (any bounded local martingale is known to be a martingale). We conclude that $M_t = \mathbb{E}_{\mathbb{Q}}(\xi \,|\, \mathcal{G}_t)$ and thus (iv) holds.

Suppose now that (iv) holds. First, we note that the standard truncation argument shows that the boundedness of a random variable ξ in condition (iv) can be replaced by the assumption that ξ is $\mathbb{Q}$-integrable. Hence any $\mathbb{F}$-martingale M is an $\mathbb{G}$-martingale, since any $\mathbb{F}$-martingale M is clearly $\mathbb{G}$-adapted and we have that, for every $t \le s$,

$$M_t = \mathbb{E}_{\mathbb{Q}}(M_s \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(M_s \,|\, \mathcal{G}_t),$$

where the second equality is a consequence of (iv).

Suppose now that M is an $\mathbb{F}$-local martingale. Then there exists an increasing sequence of $\mathbb{F}$-stopping times τ_n such that $\lim_{n\to\infty} \tau_n = \infty$ and for any n the stopped process M^{τ_n} is a uniformly integrable $\mathbb{F}$-martingale.

Hence M^{τ_n} is also a uniformly integrable $\mathbb{G}$-martingale and this means that M is a $\mathbb{G}$-local martingale. □

Remarks 3.2.1. (i) Equality (3.19) appears in numerous papers on default risk, typically without any reference to the hypothesis (H). For example, in Madan and Unal [139], the main theorem follows from the fact that (3.19) holds (see the proof of B9 in the appendix of [139]). This is also the case for the model studied by Wong [170].
(ii) If τ is $\mathcal{F}_\infty$-measurable and (3.19) holds then τ is an $\mathbb{F}$-stopping time. If τ is an $\mathbb{F}$-stopping time then equality (3.18) holds.
(iii) Though the hypothesis (H) is not necessarily valid, in general, it is satisfied when τ is constructed through the so-called canonical approach (or for Cox processes). It also holds when τ is independent of $\mathcal{F}_\infty$ (see Greenfield [94]).
(iv) If the hypothesis (H) holds then from the condition

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_\infty), \quad \forall\, t \in \mathbb{R}_+,$$

we deduce easily that F is an increasing process. The property that F is increasing is equivalent to the fact that any $\mathbb{F}$-martingale stopped at time τ is a $\mathbb{G}$-martingale. Nikeghbali and Yor [149] proved that this is equivalent to $\mathbb{E}_{\mathbb{Q}}(M_\tau) = M_0$ for any bounded $\mathbb{F}$-martingale M.
(v) The hypothesis (H) was also studied by Florens and Fougère [83], who coined the term *noncausality*. For more comments on the hypothesis (H), we refer to Elliott et al. [79].

Proposition 3.2.1. *Assume that the hypothesis (H) holds. If a process X is an $\mathbb{F}$-martingale then the processes XL and $[L, X]$ are $\mathbb{G}$-local martingales.*

Proof. From Proposition 3.1.3(ii), the process XL is a $\mathbb{G}$-martingale. Since

$$[L, X]_t = L_t X_t - \int_{]0,t]} L_{u-}\, dX_u - \int_{]0,t]} X_{u-}\, dL_u,$$

and the process X is an $\mathbb{F}$-martingale (and thus also a $\mathbb{G}$-martingale), we conclude that the process $[L, X]$ is a $\mathbb{G}$-local martingale, as the sum of three $\mathbb{G}$-local martingales. □

3.2.2 Canonical Construction of Default Time

We now briefly describe the commonly used construction of a default time associated with a given a priori hazard process Γ. It should be stressed that the random time obtained through this particular method – which will be called the *canonical construction* in what follows – has certain specific features that are not necessarily shared by all random times with the same $\mathbb{F}$-hazard process Γ.

We assume that we are given an $\mathbb{F}$-adapted, right-continuous, increasing process Γ defined on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. As usual, we assume that $\Gamma_0 = 0$ and $\Gamma_\infty = +\infty$. In many instances, Γ is given by the equality, for every $t \in \mathbb{R}_+$,

$$\Gamma_t = \int_0^t \gamma_u \, du$$

for some non-negative, $\mathbb{F}$-progressively measurable intensity process γ.

To construct a random time τ, we postulate that the underlying probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ is sufficiently rich to support a random variable ξ, which is uniformly distributed on the interval $[0,1]$ and independent of the filtration $\mathbb{F}$ under $\mathbb{Q}$. In this version of the canonical construction, Γ represents the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}$.

We define the random time $\tau : \Omega \to \mathbb{R}_+$ by setting

$$\tau = \inf \{ \, t \in \mathbb{R}_+ : \, e^{-\Gamma_t} \leq \xi \, \} = \inf \{ \, t \in \mathbb{R}_+ : \, \Gamma_t \geq \eta \, \},$$

where the random variable $\eta = -\ln \xi$ has a unit exponential law under $\mathbb{Q}$. It is not difficult to find the process $F_t = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t)$. Indeed, since clearly $\{\tau > t\} = \{\xi < e^{-\Gamma_t}\}$ and the random variable Γ_t is $\mathcal{F}_\infty$-measurable, we obtain

$$\mathbb{Q}(\tau > t \,|\, \mathcal{F}_\infty) = \mathbb{Q}(\xi < e^{-\Gamma_t} \,|\, \mathcal{F}_\infty) = \mathbb{Q}(\xi < e^{-x})_{x=\Gamma_t} = e^{-\Gamma_t}.$$

Consequently, we have

$$1 - F_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}\big(\mathbb{Q}(\tau > t \,|\, \mathcal{F}_\infty) \,\big|\, \mathcal{F}_t\big) = e^{-\Gamma_t},$$

and so F is an $\mathbb{F}$-adapted, right-continuous, increasing process. It is also clear that the process Γ represents the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}$. As an immediate consequence of the last two formulae, we obtain the following property of the canonical construction of the default time (cf. (3.19))

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t), \quad \forall \, t \in \mathbb{R}_+. \tag{3.20}$$

To summarize, we have that

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_u) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = 1 - e^{-\Gamma_t}$$

for arbitrary dates $0 \leq t \leq u$.

3.2.3 Stochastic Barrier

Suppose that

$$\mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_\infty) = 1 - e^{-\Gamma_t},$$

where Γ is a continuous, strictly increasing and $\mathbb{F}$-adapted process. Our goal is to show that there exists a random variable Θ, independent of $\mathcal{F}_\infty$, with exponential distribution of parameter 1, such that $\tau = \inf\{t \geq 0 : \Gamma_t > \Theta\}$. Let us set $\Theta := \Gamma_\tau$. Then

$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where C is the right inverse of Γ, so that $\Gamma_{C_t} = t$. Therefore

$$\mathbb{Q}(\Theta > u \,|\, \mathcal{F}_\infty) = e^{-\Gamma_{C_u}} = e^{-u}.$$

We have thus established the required properties, namely, the probability distribution of Θ and its independence of the σ-field $\mathcal{F}_\infty$. Furthermore,

$$\tau = \inf\{\, t \in \mathbb{R}_+ : \Gamma_t > \Gamma_\tau\} = \inf\{\, t \in \mathbb{R}_+ : \Gamma_t > \Theta\}.$$

3.3 Predictable Representation Theorem

Kusuoka [125] established the following representation theorem in which the reference filtration $\mathbb{F}$ is generated by a Brownian motion.

Theorem 3.3.1. *Assume that the hypothesis (H) is satisfied under $\mathbb{Q}$. Then any square-integrable martingale with respect to $\mathbb{G}$ admits a representation as the sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale M associated with τ.*

To derive a version of the predictable representation theorem, we will assume, for simplicity, that F is continuous and $F_t < 1$ for every $t \in \mathbb{R}_+$. If the hypothesis (H) is assumed to hold, F is also an increasing process and thus

$$dF_t = e^{-\Gamma_t}\, d\Gamma_t, \quad de^{\Gamma_t} = e^{\Gamma_t}\, d\Gamma_t. \tag{3.21}$$

The following result extends Proposition 2.2.6 to the case of the reference filtration $\mathbb{F}$ that only supports continuous martingale; in particular, this result covers the case when $\mathbb{F}$ is the Brownian filtration.

Theorem 3.3.2. *Suppose that the hypothesis (H) holds under $\mathbb{Q}$ and that any $\mathbb{F}$-martingale is continuous. Then the martingale $M^h_t = \mathbb{E}_{\mathbb{Q}}(h_\tau |\, \mathcal{G}_t)$, where h is an $\mathbb{F}$-predictable process such that $\mathbb{E}_{\mathbb{Q}}|h_\tau| < \infty$, admits the following decomposition in the sum of a continuous martingale and a discontinuous martingale*

$$M^h_t = m^h_0 + \int_0^{t\wedge\tau} e^{\Gamma_u}\, dm^h_u + \int_{]0,t\wedge\tau]} (h_u - M^h_{u-})\, dM_u, \tag{3.22}$$

where m^h is the continuous $\mathbb{F}$-martingale given by

$$m_t^h = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^\infty h_u\,dF_u \,\Big|\, \mathcal{F}_t\Big)$$

and M is the discontinuous $\mathbb{G}$-martingale defined as $M_t = H_t - \Gamma_{t\wedge\tau}$.

Proof. We start by noting that

$$\begin{aligned} M_t^h &= \mathbb{E}_{\mathbb{Q}}(h_\tau \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t\geq\tau\}} h_\tau + \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{Q}}\Big(\int_t^\infty h_u\,dF_u \,\Big|\, \mathcal{F}_t\Big) \\ &= \mathbb{1}_{\{t\geq\tau\}} h_\tau + \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t}\Big(m_t^h - \int_0^t h_u\,dF_u\Big). \end{aligned} \tag{3.23}$$

We will sketch two slightly different derivations of (3.22).

First derivation. Let the process J be given by the formula, for $t \in \mathbb{R}_+$,

$$J_t = e^{\Gamma_t}\Big(m_t^h - \int_0^t h_u\,dF_u\Big).$$

Noting that Γ is a continuous increasing process and m^h is a continuous martingale, we deduce from the Itô integration by parts formula that

$$\begin{aligned} dJ_t &= e^{\Gamma_t}\,dm_t^h - e^{\Gamma_t} h_t\,dF_t + \Big(m_t^h - \int_0^t h_u\,dF_u\Big) e^{\Gamma_t}\,d\Gamma_t \\ &= e^{\Gamma_t}\,dm_t^h - e^{\Gamma_t} h_t\,dF_t + J_t\,d\Gamma_t. \end{aligned}$$

Therefore, from (3.21),

$$dJ_t = e^{\Gamma_t}\,dm_t^h + (J_t - h_t)\,d\Gamma_t$$

or, in the integrated form,

$$J_t = M_0^h + \int_0^t e^{\Gamma_u}\,dm_u^h + \int_0^t (J_u - h_u)\,d\Gamma_u.$$

Note that $J_t = M_t^h = M_{t-}^h$ on the event $\{t < \tau\}$. Therefore, on the event $\{t < \tau\}$,

$$M_t^h = M_0^h + \int_0^{t\wedge\tau} e^{\Gamma_u}\,dm_u^h + \int_0^{t\wedge\tau} (M_{u-}^h - h_u)\,d\Gamma_u.$$

From (3.23), the jump of M^h at time τ equals

$$h_\tau - J_\tau = h_\tau - M_{\tau-}^h = M_\tau^h - M_{\tau-}^h.$$

Equality (3.22) now easily follows.

Second derivation. Equality (3.23) can be re-written as follows

$$M_t^h = \int_0^t h_u \, dH_u + (1 - H_t) e^{\Gamma_t} \Big(m_t^h - \int_0^t h_u \, dF_u \Big).$$

Hence formula (3.22) can be obtained directly by the Itô integration by parts formula. □

3.4 The Girsanov Theorem

We now start by defining a random time τ on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ and we postulate that it admits the continuous $\mathbb{F}$-hazard process Γ under $\mathbb{Q}$. Hence, from Proposition 3.1.4, we know that the process $M_t = H_t - \Gamma_{t \wedge \tau}$ is a $\mathbb{G}$-martingale. We postulate that the hypothesis (H) holds under $\mathbb{Q}$. Finally, we postulate that the reference filtration $\mathbb{F}$ is generated by an $\mathbb{F}$- (hence also $\mathbb{G}$-) Brownian motion under $\mathbb{Q}$.

Let us fix $T > 0$. For a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ we introduce the $\mathbb{G}$-martingale $(\eta_t,\, t \in [0,T])$ by setting

$$\eta_t := \frac{d\mathbb{P}}{d\mathbb{Q}} \Big| \, \mathcal{G}_t = \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}_t), \quad \mathbb{Q}\text{-a.s.} \tag{3.24}$$

Note that $X = \eta_T$ is here some $\mathcal{G}_T$-measurable random variable such that $\mathbb{Q}(X > 0) = 1$ and $\mathbb{E}_{\mathbb{Q}} X = 1$.

Using Theorem 3.3.1, we deduce that the Radon-Nikodým density process η admits the following representation, for every $t \in [0,T]$,

$$\eta_t = 1 + \int_0^t \xi_u \, dW_u + \int_{]0,t]} \zeta_u \, dM_u,$$

where ξ and ζ are $\mathbb{G}$-predictable stochastic processes. Since η is a strictly positive martingale, by setting $\theta_t = \xi_t \eta_{t-}^{-1}$ and $\kappa_t = \zeta_t \eta_{t-}^{-1}$, we obtain

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-} \big(\theta_u \, dW_u + \kappa_u \, dM_u \big) \tag{3.25}$$

where θ and κ are $\mathbb{G}$-predictable processes, with $\kappa > -1$. This means the process η is the Doléans exponential or, more explicitly,

$$\eta_t = \mathcal{E}_t \left(\int_0^{\cdot} \theta_u \, dW_u \right) \mathcal{E}_t \left(\int_{]0,\,\cdot\,]} \kappa_u \, dM_u \right) = \eta_t^{(1)} \eta_t^{(2)}, \tag{3.26}$$

where we write

$$\eta_t^{(1)} = \mathcal{E}_t \left(\int_0^{\cdot} \theta_u \, dW_u \right) = \exp \left(\int_0^t \theta_u \, dW_u - \frac{1}{2} \int_0^t \theta_u^2 \, du \right)$$

and

$$\eta_t^{(2)} = \mathcal{E}_t\Big(\int_{]0,\,\cdot\,]} \kappa_u \, dM_u\Big) \tag{3.27}$$
$$= \exp\Big(\int_{]0,t]} \ln(1+\kappa_u)\, dH_u - \int_0^{t\wedge\tau} \kappa_u\gamma_u \, du\Big).$$

Then we have the following extension of the classic Girsanov theorem for a Brownian motion.

Theorem 3.4.1. *Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{G}_T)$ equivalent to $\mathbb{Q}$. If the Radon-Nikodým density of $\mathbb{P}$ with respect to $\mathbb{Q}$ is given by (3.24) with η satisfying (3.25) then the process $(\widehat{W}_t,\, t \in [0,T])$, given by*

$$\widehat{W}_t = W_t - \int_0^t \theta_u \, du,$$

is a Brownian motion with respect to the filtration $\mathbb{G}$ under $\mathbb{P}$ and the process $(\widehat{M}_t,\, t \in [0,T])$, given by

$$\widehat{M}_t := M_t - \int_0^{t\wedge\tau} \kappa_u \, d\Gamma_u = H_t - \int_0^{t\wedge\tau} (1+\kappa_u)\, d\Gamma_u,$$

is a $\mathbb{G}$-martingale orthogonal to $\widehat{W}$ under $\mathbb{P}$.

Proof. Note first that, for every $t \in [0,T]$, we have

$$\begin{aligned} d(\eta_t \widehat{W}_t) &= \widehat{W}_t \, d\eta_t + \eta_{t-}\, d\widehat{W}_t + d[\widehat{W}, \eta]_t \\ &= \widehat{W}_t \, d\eta_t + \eta_{t-}\, dW_t - \eta_{t-}\theta_t \, dt + \eta_{t-}\theta_t \, d[W,W]_t \\ &= \widehat{W}_t \, d\eta_t + \eta_{t-}\, dW_t. \end{aligned}$$

This shows that $\widehat{W}$ is a $\mathbb{G}$-local martingale under $\mathbb{P}$. Since the quadratic variation of $\widehat{W}$ under $\mathbb{P}$ equals $[\widehat{W}, \widehat{W}]_t = t$ and $\widehat{W}$ is continuous, using the Lévy characterization theorem, we conclude that $\widehat{W}$ follows a Brownian motion under $\mathbb{P}$. Similarly, for every $t \in [0,T]$,

$$\begin{aligned} d(\eta_t \widehat{M}_t) &= \widehat{M}_t \, d\eta_t + \eta_{t-}\, d\widehat{M}_t + d[\widehat{M}, \eta]_t \\ &= \widehat{M}_t \, d\eta_t + \eta_{t-}\, dM_t - \eta_{t-}\kappa_t \, d\Gamma_{t\wedge\tau} + \eta_{t-}\kappa_t \, dH_t \\ &= \widehat{M}_t \, d\eta_t + \eta_{t-}(1+\kappa_t)\, dM_t. \end{aligned}$$

This is turn shows that $\widehat{M}$ is a $\mathbb{G}$-martingale under $\mathbb{P}$. To complete the proof of the proposition, it suffices to observe that $\widehat{W}$ is a continuous process whereas $\widehat{M}$ is manifestly a process of finite variation. Hence $\widehat{W}$ and $\widehat{M}$ are orthogonal $\mathbb{G}$-martingales under $\mathbb{P}$. □

Corollary 3.4.1. *Let Y be a $\mathbb{G}$-martingale under $\mathbb{P}$, where the probability measure $\mathbb{P}$ is defined in Theorem 3.4.1. Then Y admits the following integral representation*

$$Y_t = Y_0 + \int_0^t \xi^*_u \, d\widehat{W}_u + \int_{]0,t]} \zeta^*_u \, d\widehat{M}_u, \tag{3.28}$$

where ξ^* and ζ^* are $\mathbb{G}$-predictable stochastic processes.

Proof. Consider the process $\widetilde{Y}$ given by the formula

$$\widetilde{Y}_t = \int_{]0,t]} \eta^{-1}_{u-} \, d(\eta_u Y_u) - \int_{]0,t]} \eta^{-1}_{u-} Y_{u-} \, d\eta_u.$$

It is clear that $\widetilde{Y}$ is a $\mathbb{G}$-local martingale under $\mathbb{Q}$. Notice also that Itô's formula yields

$$\eta^{-1}_{u-} \, d(\eta_u Y_u) = dY_u + \eta^{-1}_{u-} Y_{u-} \, d\eta_u + \eta^{-1}_{u-} \, d[Y, \eta]_u,$$

and thus

$$Y_t = Y_0 + \widetilde{Y}_t - \int_{]0,t]} \eta^{-1}_{u-} \, d[Y, \eta]_u. \tag{3.29}$$

From the predictable representation theorem, we know that the process $\widetilde{Y}$ admits the following integral representation

$$\widetilde{Y}_t = Y_0 + \int_0^t \widetilde{\xi}_u \, dW_u + \int_{]0,t]} \widetilde{\zeta}_u \, dM_u \tag{3.30}$$

for some $\mathbb{G}$-predictable processes $\widetilde{\xi}$ and $\widetilde{\zeta}$. Consequently,

$$dY_t = \widetilde{\xi}_t \, dW_t + \widetilde{\zeta}_t \, dM_t - \eta^{-1}_{t-} \, d[Y, \eta]_t = \widetilde{\xi}_t \, d\widehat{W}_t + \widetilde{\zeta}_t (1 + \kappa_t)^{-1} \, d\widehat{M}_t,$$

since (3.25) combined with (3.29)–(3.30) yield

$$\eta^{-1}_{t-} \, d[Y, \eta]_t = \widetilde{\xi}_t \theta_t \, dt + \widetilde{\zeta}_t \kappa_t (1 + \kappa_t)^{-1} \, dH_t.$$

To derive the last equality we observe, in particular, that in view of (3.29) we have (we take into account continuity of Γ)

$$\Delta[Y, \eta]_t = \eta_{t-} \widetilde{\zeta}_t \kappa_t \, dH_t - \kappa_t \Delta[Y, \eta]_t.$$

We conclude that Y satisfies (3.28) with $\xi^* = \widetilde{\xi}$ and $\zeta^* = \widetilde{\zeta}(1 + \kappa)^{-1}$, where in turn the processes $\widetilde{\xi}$ and $\widetilde{\zeta}$ are given by (3.30). □

3.5 Invariance of the Hypothesis (H)

Kusuoka [125] shows by means of a counter-example (see Example 3.5.1) that the hypothesis (H) is not invariant with respect to an equivalent change of the underlying probability measure, in general. It is worth noting that his counter-example is based on two filtrations, $\mathbb{H}^1$ and $\mathbb{H}^2$, generated by the two random times τ^1 and τ^2 and $\mathbb{H}^1$ is chosen to play the role of the reference filtration $\mathbb{F}$. We shall argue that in the case where $\mathbb{F}$ is generated by a Brownian motion, the above-mentioned invariance property is valid under mild technical assumptions.

Let us first examine a general setup in which $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F}$ is an arbitrary filtration and $\mathbb{H}$ is generated by the default process H. We say that $\mathbb{Q}$ is locally equivalent to $\mathbb{P}$ if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ for every $t \in \mathbb{R}_+$. Then there exists the Radon-Nikodým density process η such that, for every $t \in \mathbb{R}_+$,

$$d\mathbb{Q}\,|\,_{\mathcal{G}_t} = \eta_t \, d\mathbb{P}\,|\,_{\mathcal{G}_t}. \tag{3.31}$$

For part (i) in Lemma 3.5.1, we refer to Blanchet-Scalliet and Jeanblanc [30] or Proposition 2.2 in Jamshidian [104]. For part (ii), see Jeulin and Yor [115].

In this section, we will work under the standing assumption that the hypothesis (H) is valid under $\mathbb{P}$.

Lemma 3.5.1. *(i) Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ for every $t \in \mathbb{R}_+$, with the associated Radon-Nikodým density process η. If the density process η is $\mathbb{F}$-adapted then we have that, for every $t \in \mathbb{R}_+$,*

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_t).$$

Hence the hypothesis (H) is also valid under $\mathbb{Q}$ and the $\mathbb{F}$-intensities of τ under $\mathbb{Q}$ and under $\mathbb{P}$ coincide.
(ii) Assume that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$ and $d\mathbb{Q} = \eta_\infty \, d\mathbb{P}$, so that $\eta_t = \mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{G}_t)$. Then the hypothesis (H) is valid under $\mathbb{Q}$ whenever we have, for every $t \in \mathbb{R}_+$,

$$\frac{\mathbb{E}_{\mathbb{P}}(\eta_\infty H_t \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{F}_\infty)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t H_t \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_\infty)}. \tag{3.32}$$

Proof. To prove (i), assume that the density process η is $\mathbb{F}$-adapted. We have for each $t \le s \in \mathbb{R}_+$

$$\begin{aligned} \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{t \ge \tau\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_t)} = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_t) \\ &= \mathbb{P}(\tau \le t \,|\, \mathcal{F}_s) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_s), \end{aligned}$$

where the last equality follows by another application of the Bayes formula. The assertion now follows from part (i) in Lemma 3.2.1.

To prove part (ii), it suffices to establish the equality

$$\widehat{F}_t := \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_\infty), \quad \forall\, t \in \mathbb{R}_+.$$

Note that since the random variables $\eta_t \mathbb{1}_{\{t \ge \tau\}}$ and η_t are $\mathbb{P}$-integrable and $\mathcal{G}_t$-measurable, using the Bayes formula, part (v) in Lemma 3.2.1, and assumed equality (3.32), we obtain the following chain of equalities

$$\begin{aligned}\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{t \ge \tau\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_t)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{t \ge \tau\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_\infty)} \\ &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_\infty \mathbb{1}_{\{t \ge \tau\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{F}_\infty)} = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_\infty).\end{aligned}$$

We conclude that the hypothesis (H) holds under $\mathbb{Q}$ if and only if the equality (3.32) is valid. □

Unfortunately, a straightforward verification of condition (3.32) is rather cumbersome. For this reason, we shall provide alternative sufficient conditions for the preservation of the hypothesis (H) under a locally equivalent probability measure.

3.5.1 Case of the Brownian Filtration

Let W be a Brownian motion under $\mathbb{P}$ and let $\mathbb{F}$ be its natural filtration. Since we work under the standing assumption that the hypothesis (H) is satisfied under $\mathbb{P}$, the process W is also a $\mathbb{G}$-martingale, where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. Hence W is a Brownian motion with respect to $\mathbb{G}$ under $\mathbb{P}$. Our goal is to show that the hypothesis (H) is still valid under $\mathbb{Q} \in \mathcal{Q}$ for a large class $\mathcal{Q}$ of (locally) equivalent probability measures. We postulate that τ admits the hazard rate γ with respect to $\mathbb{F}$ under $\mathbb{P}$.

Let $\mathbb{Q}$ be an arbitrary probability measure locally equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$. The predictable representation theorem implies that there exist $\mathbb{G}$-predictable processes θ and $\kappa > -1$ such that the Radon-Nikodým density η of $\mathbb{Q}$ with respect to $\mathbb{P}$ satisfies the following SDE

$$d\eta_t = \eta_{t-}\big(\theta_t \, dW_t + \kappa_t \, dM_t\big)$$

with the initial value $\eta_0 = 1$. This means that the density process η is given by formula (3.26). By virtue of a suitable version of the Girsanov theorem, the processes $\widehat{W}$ and $\widehat{M}$ are $\mathbb{G}$-martingales under $\mathbb{Q}$, where we set

$$\widehat{W}_t = W_t - \int_0^t \theta_u \, du, \quad \widehat{M}_t = M_t - \int_0^{t \wedge \tau} \gamma_u \kappa_u \, du.$$

Proposition 3.5.1. *Assume that the hypothesis (H) holds under $\mathbb{P}$. Let $\mathbb{Q}$ be a probability measure locally equivalent to $\mathbb{P}$ with the associated Radon-Nikodým density process η given by formula (3.26). If the process θ is $\mathbb{F}$-adapted then the hypothesis (H) is valid under $\mathbb{Q}$ and the $\mathbb{F}$-intensity of τ under $\mathbb{Q}$ equals $\widehat{\gamma}_t = (1+\widetilde{\kappa}_t)\gamma_t$, where $\widetilde{\kappa}$ is the unique $\mathbb{F}$-predictable process such that the equality $\widetilde{\kappa}_t \mathbb{1}_{\{t\le\tau\}} = \kappa_t \mathbb{1}_{\{t\le\tau\}}$ holds for every $t\in\mathbb{R}_+$.*

Proof. Let $\widetilde{\mathbb{P}}$ be the probability measure locally equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{G})$, given by

$$d\widetilde{\mathbb{P}}\,|\,_{\mathcal{G}_t} = \mathcal{E}_t\bigg(\int_{]0,\,\cdot\,]} \kappa_u\,dM_u\bigg)\,d\mathbb{P}\,|\,_{\mathcal{G}_t} = \eta_t^{(2)}\,d\mathbb{P}\,|\,_{\mathcal{G}_t}. \tag{3.33}$$

We claim that the hypothesis (H) holds under $\widetilde{\mathbb{P}}$. From the Girsanov theorem, the process W follows a Brownian motion under $\widetilde{\mathbb{P}}$ with respect to both $\mathbb{F}$ and $\mathbb{G}$. Moreover, from the predictable representation property of W under $\widetilde{\mathbb{P}}$, we deduce that any $\mathbb{F}$-local martingale L under $\widetilde{\mathbb{P}}$ can be written as a stochastic integral with respect to W. Specifically, there exists an $\mathbb{F}$-predictable process ξ such that

$$L_t = L_0 + \int_0^t \xi_u\,dW_u.$$

This shows that L is also a $\mathbb{G}$-local martingale, and thus the hypothesis (H) holds under $\widetilde{\mathbb{P}}$. Since we have that

$$d\mathbb{Q}\,|\,_{\mathcal{G}_t} = \mathcal{E}_t\bigg(\int_0^{\cdot} \theta_u\,dW_u\bigg)\,d\widetilde{\mathbb{P}}\,|\,_{\mathcal{G}_t},$$

we conclude, by virtue of part (i) in Lemma 3.5.1, that the hypothesis (H) is valid under $\mathbb{Q}$ as well. The last claim in the statement of the lemma can be deduced from the fact that the hypothesis (H) holds under $\mathbb{Q}$ and, by the Girsanov theorem, the process $\widehat{M}$, given by the formula

$$\widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}}\gamma_u\kappa_u\,du = H_t - \int_0^t \mathbb{1}_{\{u<\tau\}}(1+\widetilde{\kappa}_u)\gamma_u\,du,$$

is a $\mathbb{Q}$-martingale. □

We claim that the equality $\widetilde{\mathbb{P}} = \mathbb{P}$ holds on the filtration $\mathbb{F}$. Indeed, we have that $d\widetilde{\mathbb{P}}\,|\,_{\mathcal{F}_t} = \widetilde{\eta}_t\,d\mathbb{P}\,|\,_{\mathcal{F}_t}$, where we write $\widetilde{\eta}_t = \mathbb{E}_{\mathbb{P}}(\eta_t^{(2)}\,|\,\mathcal{F}_t)$. Furthermore, for every $t\in\mathbb{R}_+$,

$$\mathbb{E}_{\mathbb{P}}(\eta_t^{(2)}\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\bigg(\mathcal{E}_t\bigg(\int_{]0,\,\cdot\,]} \kappa_u\,dM_u\bigg)\,\bigg|\,\mathcal{F}_\infty\bigg) = 1, \tag{3.34}$$

where the first equality follows from part (v) in Lemma 3.2.1. To establish the second equality in (3.34), we first note that since the process M is stopped at

τ, we may assume, without loss of generality, that $\kappa = \widetilde{\kappa}$, where the process $\widetilde{\kappa}$ is $\mathbb{F}$-predictable. Moreover, the conditional cumulative distribution function of τ given $\mathcal{F}_\infty$ has the form $1 - \exp(-\Gamma_t(\omega))$. Hence, for arbitrarily selected sample paths of processes κ and Γ, the claimed equality can be seen as a consequence of the martingale property of the Doléans exponential.

3.5.2 Extension to Orthogonal Martingales

Equality (3.34) suggests that Proposition 3.5.1 can be extended to the case of arbitrary orthogonal local martingales. Such a generalization is convenient, if we wish to cover the situation considered in Kusuoka's counter-example. Let N be a local martingale under $\mathbb{P}$ with respect to the filtration $\mathbb{F}$. It is also a $\mathbb{G}$-local martingale, since we maintain the assumption that the hypothesis (H) holds under $\mathbb{P}$. Let $\mathbb{Q}$ be an arbitrary probability measure locally equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$. We assume that the Radon-Nikodým density process η of $\mathbb{Q}$ with respect to $\mathbb{P}$ equals

$$d\eta_t = \eta_{t-}\big(\theta_t\, dN_t + \kappa_t\, dM_t\big)$$

for some $\mathbb{G}$-predictable processes θ and $\kappa > -1$ (the properties of the process θ depend, of course, on the choice of a local martingale N). The next result covers the case where N and M are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$, so that the product MN is a $\mathbb{G}$-local martingale.

Proposition 3.5.2. *Assume that the following conditions hold:*
(a) N and M are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$,
(b) N has the predictable representation property under $\mathbb{P}$ with respect to $\mathbb{F}$, in the sense that any $\mathbb{F}$-local martingale L under $\mathbb{P}$ there exists an $\mathbb{F}$-predictable process ξ such that, for every $t \in \mathbb{R}_+$,

$$L_t = L_0 + \int_{]0,t]} \xi_u\, dN_u,$$

(c) $\widetilde{\mathbb{P}}$ is a probability measure on $(\Omega, \mathcal{G})$ such that (3.33) holds.
Then we have:
(i) the hypothesis (H) is valid under $\widetilde{\mathbb{P}}$,
(ii) if the process θ is $\mathbb{F}$-adapted then the hypothesis (H) is valid under $\mathbb{Q}$.

Lemma 3.5.2. *Under the assumptions of Proposition 3.5.2, we have:*
(i) N is a $\mathbb{G}$-local martingale under $\widetilde{\mathbb{P}}$,
(ii) N has the predictable representation property for $\mathbb{F}$-local martingales under $\widetilde{\mathbb{P}}$.

Proof. In view of (c), we have $d\widetilde{\mathbb{P}}\,|\,_{\mathcal{G}_t} = \eta_t^{(2)}\, d\mathbb{P}\,|\,_{\mathcal{G}_t}$, where the density process $\eta^{(2)}$ is given by (3.27), so that $d\eta_t^{(2)} = \eta_{t-}^{(2)}\kappa_t\, dM_t$. From the assumed orthogonality of N and M, it follows that N and $\eta^{(2)}$ are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$ and thus $N\eta^{(2)}$ is a $\mathbb{G}$-local martingale under $\mathbb{P}$ as well. This means that N is a $\mathbb{G}$-local martingale under $\widetilde{\mathbb{P}}$, so that (i) holds.

To establish part (ii) in the lemma, we first define the auxiliary process $\widetilde{\eta}$ by setting $\widetilde{\eta}_t = \mathbb{E}_{\mathbb{P}}(\eta_t^{(2)}\,|\,\mathcal{F}_t)$. Then manifestly $d\widetilde{\mathbb{P}}\,|\,_{\mathcal{F}_t} = \widetilde{\eta}_t\, d\mathbb{P}\,|\,_{\mathcal{F}_t}$, and thus in order to show that any $\mathbb{F}$-local martingale under $\widetilde{\mathbb{P}}$ is an $\mathbb{F}$-local martingale under $\mathbb{P}$, it suffices to check that $\widetilde{\eta}_t = 1$ for every $t \in \mathbb{R}_+$, so that $\widetilde{\mathbb{P}} = \mathbb{P}$ on $\mathbb{F}$. To this end, we note that, for every $t \in \mathbb{R}_+$,

$$\mathbb{E}_{\mathbb{P}}(\eta_t^{(2)}\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\bigg(\mathcal{E}_t\bigg(\int_{]0,\,\cdot\,]} \kappa_u\, dM_u\bigg)\,\bigg|\,\mathcal{F}_\infty\bigg) = 1,$$

where the first equality follows from part (v) in Lemma 3.2.1 and the second one can established similarly as the second equality in (3.34).

We are in a position to prove (ii). Let L be an $\mathbb{F}$-local martingale under $\widetilde{\mathbb{P}}$. Then it follows also an $\mathbb{F}$-local martingale under $\mathbb{P}$ and thus, by virtue of (b), it admits an integral representation with respect to N under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$. This shows that N has the predictable representation property with respect to $\mathbb{F}$ under $\widetilde{\mathbb{P}}$. □

Proof of Proposition 3.5.2. We shall argue along the similar lines as in the proof of Proposition 3.5.1. To prove (i), note that by part (ii) in Lemma 3.5.2 we know that any $\mathbb{F}$-local martingale under $\widetilde{\mathbb{P}}$ admits the integral representation with respect to N. But, by part (i) in Lemma 3.5.2, N is a $\mathbb{G}$-local martingale under $\widetilde{\mathbb{P}}$. We conclude that L is a $\mathbb{G}$-local martingale under $\widetilde{\mathbb{P}}$ and thus the hypothesis (H) is valid under $\widetilde{\mathbb{P}}$. Assertion (ii) now follows from part (i) in Lemma 3.5.1. □

Example 3.5.1. Kusuoka [125] presents a counter-example based on the two independent random times τ_1 and τ_2 given on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We write $M_t^i = H_t^i - \int_0^{t\wedge\tau_i} \gamma_i(u)\, du$, where $H_t^i = \mathbb{1}_{\{t\geq\tau_i\}}$ and γ_i is the deterministic intensity function of τ_i under $\mathbb{P}$. Let us set $d\mathbb{Q}\,|\,_{\mathcal{G}_t} = \eta_t\, d\mathbb{P}\,|\,_{\mathcal{G}_t}$, where $\eta_t = \eta_t^{(1)}\eta_t^{(2)}$ and, for $i = 1, 2$ and every $t \in \mathbb{R}_+$,

$$\eta_t^{(i)} = 1 + \int_0^t \eta_{u-}^{(i)}\kappa_u^{(i)}\, dM_u^i = \mathcal{E}_t\bigg(\int_{]0,\,\cdot\,]} \kappa_u^{(i)}\, dM_u^i\bigg)$$

for some $\mathbb{G}$-predictable processes $\kappa^{(i)}$, $i = 1, 2$, where $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$. We set $\mathbb{F} = \mathbb{H}^1$ and $\mathbb{H} = \mathbb{H}^2$. Manifestly, the hypothesis (H) holds under $\mathbb{P}$.

Moreover, in view of Proposition 3.5.2, it is still valid under the equivalent probability measure $\widetilde{\mathbb{P}}$ given by $d\widetilde{\mathbb{P}}\,|\,_{\mathcal{G}_t} = \eta_t^{(2)}\, d\mathbb{P}\,|\,_{\mathcal{G}_t}$. It is clear that $\widetilde{\mathbb{P}} = \mathbb{P}$

on $\mathbb{F}$, since we have that, for every $t \in \mathbb{R}_+$,

$$\mathbb{E}_{\mathbb{P}}(\eta_t^{(2)} \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\bigg(\mathcal{E}_t\bigg(\int_{]0,\,\cdot\,]} \kappa_u^{(2)}\, dM_u^2\bigg) \,\bigg|\, \mathcal{H}_t^1\bigg) = 1.$$

However, the hypothesis (H) is not necessarily valid under $\mathbb{Q}$ if the process $\kappa^{(1)}$ fails to be $\mathbb{F}$-adapted. In Kusuoka's counter-example, the process $\kappa^{(1)}$ was chosen to be explicitly dependent on both random times and it was shown that the hypothesis (H) fails to hold under $\mathbb{Q}$.

For an alternative approach to Kusuoka's example, through an absolutely continuous change of a probability measure, the interested reader may consult Collin-Dufresne et al. [58].

3.6 $\mathbb{G}$-Intensity of Default Time

In an alternative approach to modeling of default time, we start by assuming that we are given a default time τ and some filtration $\mathbb{G}$ such that τ is a $\mathbb{G}$-stopping time. In this setup, the default intensity is defined as follows.

Definition 3.6.1. A $\mathbb{G}$-*intensity* of default time τ is any non-negative and $\mathbb{G}$-predictable process $(\lambda_t,\, t \in \mathbb{R}_+)$ such that the process $(M_t,\, t \in \mathbb{R}_+)$, which is given as

$$M_t = H_t - \int_0^{t\wedge\tau} \lambda_u\, du,$$

is a $\mathbb{G}$-martingale.

The existence of a $\mathbb{G}$-intensity of τ hinges on the fact that H is a bounded increasing process, therefore a bounded sub-martingale, and thus, by the Doob-Meyer decomposition, it can be written as a sum of a martingale M and a $\mathbb{G}$-predictable, increasing process A, which is stopped at τ. In the case where τ is a predictable stopping time, obviously $A = H$. In fact, it is known that the $\mathbb{G}$-intensity exists only if τ is a *totally inaccessible* stopping time with respect to $\mathbb{G}$. In the present setup, the default intensity is not well defined after time τ. Specifically, if λ is a $\mathbb{G}$-intensity then for any non-negative, $\mathbb{G}$-predictable process g the process $\widetilde{\lambda}$, given by the expression

$$\widetilde{\lambda}_t = \lambda_t \mathbb{1}_{\{t\le\tau\}} + g_t \mathbb{1}_{\{t>\tau\}},$$

is also a version of a $\mathbb{G}$-intensity. Let us write $\Lambda_t = \int_0^t \lambda_u\, du$. The following result is a counterpart of Lemma 3.1.3(i).

Lemma 3.6.1. *The process $L_t = \mathbb{1}_{\{t<\tau\}} e^{\Lambda_t}$ for $t \in \mathbb{R}_+$ is a $\mathbb{G}$-martingale.*

Proof. From the integration by parts formula, we get

$$dL_t = e^{\Lambda_t}\big((1-H_t)\lambda_t\,dt - dH_t\big) = -e^{\Lambda_t}\,dM_t.$$

This shows that L is a $\mathbb{G}$-martingale. □

The following result is due to Duffie et al. [74].

Proposition 3.6.1. *For any $\mathcal{G}_T$-measurable and $\mathbb{Q}$-integrable random variable X we have*

$$\mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{T<\tau\}}\,|\,\mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}e^{\Lambda_t}\,\mathbb{E}_{\mathbb{Q}}(Xe^{-\Lambda_T}\,|\,\mathcal{G}_t) - \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\leq T\}}\Delta Y_\tau e^{\Lambda_\tau}\,|\,\mathcal{G}_t),$$

where the process Y is defined by setting, for every $t\in\mathbb{R}_+$,

$$Y_t = \mathbb{E}_{\mathbb{Q}}\big(Xe^{-\Lambda_T}\,\big|\,\mathcal{G}_t\big).$$

Proof. Let us denote $U = LY$. The Itô integration by parts formula yields

$$dU_t = L_{t-}\,dY_t + Y_{t-}\,dL_t + d[L,Y]_t = L_{t-}\,dY_t + Y_{t-}\,dL_t + \Delta L_t\Delta Y_t.$$

Since L and Y are $\mathbb{G}$-martingales, we obtain

$$\mathbb{E}_{\mathbb{Q}}(U_T\,|\,\mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}\big(X\mathbb{1}_{\{T<\tau\}}\,|\,\mathcal{G}_t\big) = U_t - \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{t<\tau\leq T\}}\Delta Y_\tau e^{\Lambda_\tau}\,|\,\mathcal{G}_t\big).$$

Consequently,

$$\mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{T<\tau\}}\,|\,\mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}e^{\Lambda_t}\mathbb{E}_{\mathbb{Q}}\big(Xe^{-\Lambda_T}\,|\,\mathcal{G}_t\big) - \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{t<\tau\leq T\}}e^{\Lambda_\tau}\,\Delta Y_\tau\,|\,\mathcal{G}_t\big)$$

as required. □

It is worthwhile to compare the next result with the formula established in Corollary 3.1.1.

Corollary 3.6.1. *Assume that the process $Y_t = \mathbb{E}_{\mathbb{Q}}\big(Xe^{-\Lambda_T}\,\big|\,\mathcal{G}_t\big)$ is continuous at time τ, that is, $\Delta Y_\tau = 0$. Then for any $\mathcal{G}_T$-measurable, $\mathbb{Q}$-integrable random variable X*

$$\mathbb{E}_{\mathbb{Q}}(X\mathbb{1}_{\{T<\tau\}}\,|\,\mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\,\mathbb{E}_{\mathbb{Q}}\big(Xe^{\Lambda_t-\Lambda_T}\,\big|\,\mathcal{G}_t\big).$$

It should be stressed that the continuity of the process Y at time τ depends on the choice of λ after time τ and that this condition is rather difficult to verify, in general. Furthermore, the jump size ΔY_τ is usually quite hard to compute explicitly (see, e.g., Çetin et al. [51]). It is thus worth noting that Collin-Dufresne et al. [58] apply an absolutely continuous change of a probability measure that leads to an essential simplification of the formula of Proposition 3.6.1. In a recent paper by Jeanblanc and Le Cam [110], the authors provide a detailed comparison of the two alternative approaches to the modeling of default time.

3.7 Single-Name CDS Market

A strictly positive random variable τ defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ is termed a *random time.* In view of its financial interpretation, we will refer to it as a *default time.* We define the default indicator process $H_t = \mathbb{1}_{\{t \geq \tau\}}$ and we denote by $\mathbb{H}$ the filtration generated by this process. We assume that we are given, in addition, some auxiliary filtration $\mathbb{F}$ and we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, meaning that we have $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$. The filtration $\mathbb{G}$ is referred to as to the *full filtration.* It is clear that τ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time (but not necessarily an $\mathbb{F}$-stopping time).

All processes are defined on the space $(\Omega, \mathbb{G}, \mathbb{P})$, where $\mathbb{P}$ is to be interpreted as the real-life (i.e., statistical) probability measure. Unless otherwise stated, they are assumed to be $\mathbb{G}$-adapted and with càdlàg sample paths.

3.7.1 Standing Assumptions

We assume that the underlying market model is arbitrage-free, meaning that it admits a *spot martingale measure* $\mathbb{Q}$ (not necessarily unique) equivalent to $\mathbb{Q}$. A *spot martingale measure* is associated with the choice of the savings account B as a numéraire, in the sense that the price process of any traded security, which pays no coupons or dividends, is a $\mathbb{G}$-martingale under $\mathbb{Q}$ when discounted by the *savings account* B. As usual, B is given by

$$B_t = \exp\Big(\int_0^t r_u\, du\Big), \quad \forall\, t \in \mathbb{R}_+,$$

where the short-term r is assumed to follow an $\mathbb{F}$-progressively measurable stochastic process. The choice of a suitable term structure model is arbitrary and it is not discussed in the present work.

Recall that $G_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t)$ is the *survival process* of τ with respect to a filtration $\mathbb{F}$. We postulate that $G_0 = 1$ and $G_t > 0$ for every $t \in \mathbb{R}_+$ (hence the case where τ is an $\mathbb{F}$-stopping time is excluded) so that the *hazard process* $\Gamma = -\ln G$ of τ with respect to the filtration $\mathbb{F}$ is well defined.

Clearly, the process G is a bounded $\mathbb{F}$-supermartingale and thus it admits the unique Doob-Meyer decomposition $G = \mu - \nu$, where μ is an $\mathbb{F}$-martingale with $\mu_0 = 1$ and ν is an $\mathbb{F}$-predictable, increasing process. If $F = N + C$ is the Doob-Meyer decomposition of F then, of course, $\mu = 1 - N$ and $C = \nu$. We shall work throughout under the following standing assumption.

Assumption 3.7.1. We postulate that G is a continuous process and the increasing process C in its Doob-Meyer decomposition is absolutely continuous with respect to the Lebesgue measure, so that $dC_t = c_t\, dt$ for some $\mathbb{F}$-progressively measurable, non-negative process c.

Let λ be the $\mathbb{F}$-progressively measurable process defined as $\lambda_t = G_t^{-1} c_t$. For a further reference, let us note that, under Assumption 3.7.1, we have $dG_t = d\mu_t - \lambda_t G_t \, dt$, where the $\mathbb{F}$-martingale μ is continuous. Moreover, in view of the Lebesgue dominated convergence theorem, the continuity of G implies that the expected value $\mathbb{E}_{\mathbb{Q}}(G_t) = \mathbb{Q}(\tau > t)$ is a continuous function, and thus $\mathbb{Q}(\tau = t) = 0$ for any fixed $t \in \mathbb{R}_+$.

We already know that, under Assumption 3.7.1, the process M, which is given by the formula

$$M_t = H_t - \Lambda_{t\wedge\tau} = H_t - \int_0^{t\wedge\tau} \lambda_u \, du, \tag{3.35}$$

is a $\mathbb{G}$-martingale. The increasing, absolutely continuous, $\mathbb{F}$-adapted process Λ satisfies the following equalities

$$\Lambda_t = \int_0^t G_u^{-1} \, dC_u = \int_0^t \lambda_u \, du.$$

Let us finally recall that the $\mathbb{F}$-progressively measurable process λ is called the $\mathbb{F}$-intensity of a default time τ.

3.7.2 Valuation of a Defaultable Claim

Let us first recall the concept of a generic defaultable claim (cf. Section 1.1.1 and Definition 2.3.1). In this section, we work within a single-name framework, so that τ is the moment of default of a reference credit name. A generic defaultable claim is now specified by the following extension of Definition 2.3.1 (note that, similarly as in Definition 2.3.1, we set $\widetilde{X} = 0$).

Definition 3.7.1. By a *defaultable claim* with maturity date T we mean a quadruplet (X, A, Z, τ) where X is an $\mathcal{F}_T$-measurable random variable, $(A_t, \, t \in [0,T])$ is an $\mathbb{F}$-adapted, continuous process of finite variation with $A_0 = 0$, $(Z_t, \, t \in [0,T])$ is an $\mathbb{F}$-predictable process and τ is a random time.

As usual, the financial interpretation of components of a defaultable claim can be inferred from the specification of the *dividend process* D describing all cash flows associated with a defaultable claim over its lifespan $]0,T]$, that is, excluding the initial premium, if any. We follow here our standard convention that the date 0 is the inception date of a defaultable contract.

Definition 3.7.2. The *dividend process* $(D_t, \, t \in \mathbb{R}_+)$ of a defaultable claim (X, A, Z, τ) maturing at T equals, for every $t \in \mathbb{R}_+$,

$$D_t = X \mathbb{1}_{\{T<\tau\}} \mathbb{1}_{[T,\infty[}(t) + \int_0^{t\wedge T} (1 - H_u) \, dA_u + \int_{]0,t\wedge T]} Z_u \, dH_u.$$

It is clear that the dividend process D is an $\mathbb{F}$-adapted process of finite variation on $[0, T]$.

Let us recall the financial interpretation of D is as follows: X is the *promised payoff*, the process A represents the *promised dividends* and the process Z, termed the *recovery process*, specifies the recovery payoff at default. As already mentioned above, according to our convention, a possible cash payment (premium) at time 0 is not included in the dividend process D associated with a defaultable claim.

3.7.3 Price Dynamics of a Defaultable Claim

For any fixed $t \in [0, T]$, the process $D_u - D_t$, $u \in [t, T]$, represents all cash flows from a defaultable claim received by an investor who purchased it at time t. In general, the process $D_u - D_t$ may depend on the past prices of underlying assets and on the history of the market prior to t. The past dividends are not valued by the market, however, so that the current market value at time $t \in [0, T]$ of a defaultable claim – that is, the price at which it is traded at time t – will only reflect future cash flows over the time interval $]t, T]$. This leads to the following definition of the ex-dividend price of a defaultable claim (cf. formula (3.6))

Definition 3.7.3. The *ex-dividend price* process S of a defaultable claim (X, A, Z, τ) equals, for every $t \in [0, T]$,

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}} \Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big). \tag{3.36}$$

Obviously, $S_T = 0$ for any dividend process D. We work throughout under the natural integrability assumptions: $\mathbb{E}_{\mathbb{Q}} |B_T^{-1} X| < \infty$,

$$\mathbb{E}_{\mathbb{Q}} \Big| \int_0^T B_u^{-1} (1 - H_u) \, dA_u \Big| < \infty$$

and

$$\mathbb{E}_{\mathbb{Q}} |B_{\tau \wedge T}^{-1} Z_{\tau \wedge T}| < \infty,$$

which ensure that the ex-dividend price S_t is well defined for any $t \in [0, T]$. We will later need the following technical assumption

$$\mathbb{E}_{\mathbb{Q}} \Big(\int_0^T (B_u^{-1} Z_u)^2 \, d\langle \mu \rangle_u \Big) < \infty. \tag{3.37}$$

We first derive a convenient representation for the ex-dividend price S of a defaultable claim.

Proposition 3.7.1. *The ex-dividend price of a defaultable claim* (X, A, Z, τ) *equals, for* $t \in [0, T[$,

$$S_t = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} G_T X + \int_t^T B_u^{-1} G_u \big(Z_u \lambda_u \, du + dA_u\big) \, \Big| \, \mathcal{F}_t\Big).$$

Proof. For any $t \in [0, T[$, the ex-dividend price is given by the conditional expectation

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} X \mathbb{1}_{\{T<\tau\}} + \int_{t\wedge\tau}^{T\wedge\tau} B_u^{-1} \, dA_u + B_\tau^{-1} Z_\tau \mathbb{1}_{\{t<\tau\leq T\}} \, \Big| \, \mathcal{G}_t\Big).$$

Let us fix t and let us introduce two auxiliary processes $Y = (Y_u)_{u\in[t,T]}$ and $R = (R_u)_{u\in[t,T]}$ by setting

$$Y_u = \int_t^u B_v^{-1} \, dA_v, \quad R_u = B_u^{-1} Z_u + \int_t^u B_v^{-1} \, dA_v = B_u^{-1} Z_u + Y_u.$$

Then S_t can be represented as follows

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} X \mathbb{1}_{\{T<\tau\}} + \mathbb{1}_{\{T<\tau\}} Y_T + R_\tau \mathbb{1}_{\{t<\tau\leq T\}} \, \Big| \, \mathcal{G}_t\Big).$$

We use the formula of Corollary 3.1.1, to evaluate the conditional expectations

$$B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{T<\tau\}} B_T^{-1} X \, \Big| \, \mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} G_T X \, \Big| \, \mathcal{F}_t\Big),$$

and

$$B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{T<\tau\}} Y_T \, \Big| \, \mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}}\Big(G_T Y_T \, \Big| \, \mathcal{F}_t\Big).$$

In addition, we will use of the following formula

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{t<\tau\leq T\}} R_\tau \, | \, \mathcal{G}_t) = -\mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T R_u \, dG_u \, \Big| \, \mathcal{F}_t\Big),$$

which is known to hold for any $\mathbb{F}$-predictable process R such that $\mathbb{E}_{\mathbb{Q}}|R_\tau| < \infty$. We thus obtain, for any $t \in [0, T[$,

$$S_t = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} G_T X + G_T Y_T - \int_t^T (B_u^{-1} Z_u + Y_u) \, dG_u \, \Big| \, \mathcal{F}_t\Big),$$

Moreover, since $dG_t = d\mu_t - \lambda_t G_t \, dt$, where μ is an $\mathbb{F}$-martingale, we also obtain

$$\mathbb{E}_{\mathbb{Q}}\Big(-\int_t^T B_u^{-1} Z_u \, dG_u \, \Big| \, \mathcal{F}_t\Big) = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T B_u^{-1} G_u Z_u \lambda_u \, du \, \Big| \, \mathcal{F}_t\Big),$$

where we have used the assumed inequality (3.37).

To complete the proof, it remains to observe that G is a continuous semimartingale and Y is a continuous process of finite variation with $Y_t = 0$, so that the Itô integration by parts formula yields

$$G_T Y_T - \int_t^T Y_u \, dG_u = \int_t^T G_u \, dY_u = \int_t^T B_u^{-1} G_u \, dA_u,$$

where the second equality follows from the definition of Y. We conclude that the asserted formula holds for any $t \in [0, T[$, as required. □

Proposition 3.7.1 implies that the ex-dividend price S satisfies, for every $t \in [0, T]$,

$$S_t = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t$$

for some $\mathbb{F}$-adapted process $\widetilde{S}$, which is termed the *ex-dividend pre-default price* of a defaultable claim. Note that S may not be continuous at time T, in which case $S_{T-} \neq S_T = 0$.

Definition 3.7.4. The *cumulative price* process S^c associated with the dividend process D is defined by setting, for every $t \in [0, T]$,

$$S_t^c = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t \Big) = S_t + B_t \int_{]0,t]} B_u^{-1} \, dD_u. \tag{3.38}$$

Note that the discounted cumulative price process $S^{c*} = B^{-1} S^c$ follows a $\mathbb{G}$-martingale under $\mathbb{Q}$. We deduce immediately from Proposition 3.7.1 and Definition 3.7.4 that the following corollary is valid.

Corollary 3.7.1. *The cumulative price of a defaultable claim (X, A, Z, τ) equals, for $t \in [0, T]$,*

$$S_t^c = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \, \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} G_T X \mathbb{1}_{\{t<T\}} + \int_t^T B_u^{-1} G_u \big(Z_u \lambda_u \, du + dA_u \big) \,\Big|\, \mathcal{F}_t \Big) + B_t \int_{]0,t]} B_u^{-1} \, dD_u.$$

The *pre-default cumulative price* is the unique $\mathbb{F}$-adapted process $\widetilde{S}^c$ that satisfies, for every $t \in [0, T]$,

$$\mathbb{1}_{\{t<\tau\}} S_t^c = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t^c. \tag{3.39}$$

Our next goal is to derive the dynamics under $\mathbb{Q}$ for the (pre-default) price of a defaultable claim in terms of some $\mathbb{G}$-martingales and $\mathbb{F}$-martingales. To simplify the presentation, we shall work from now on under the following standing assumption.

Assumption 3.7.2. Any $\mathbb{F}$-martingale is a continuous process.

The following auxiliary result is well known (see, for instance, Lemma 5.1.6 in [22]). Recall that μ is the $\mathbb{F}$-martingale appearing in the Doob-Meyer decomposition of G.

Lemma 3.7.1. *Let n be any $\mathbb{F}$-martingale. Then the process $\check{n}$ given by*

$$\check{n}_t = n_{t\wedge\tau} - \int_0^{t\wedge\tau} G_u^{-1}\, d\langle n, \mu\rangle_u \tag{3.40}$$

is a continuous $\mathbb{G}$-martingale.

In particular, the process $\check{\mu}$ given by

$$\check{\mu}_t = \mu_{t\wedge\tau} - \int_0^{t\wedge\tau} G_u^{-1}\, d\langle \mu, \mu\rangle_u \tag{3.41}$$

is a continuous $\mathbb{G}$-martingale.

In the next result, we deal with the dynamics of the ex-dividend price process S. Recall that the $\mathbb{G}$-martingale M is given by formula (3.35).

Proposition 3.7.2. *The dynamics of the ex-dividend price S on $[0,T]$ are*

$$\begin{aligned} dS_t = {} & -S_{t-}\, dM_t + (1-H_t)\big((r_tS_t - \lambda_t Z_t)\, dt - dA_t\big) \\ & + (1-H_t)G_t^{-1}\big(B_t dm_t - S_t d\mu_t\big) + (1-H_t)G_t^{-2}\big(S_t d\langle\mu\rangle_t - B_t d\langle \mu, m\rangle_t\big) \end{aligned} \tag{3.42}$$

where the continuous $\mathbb{F}$-martingale m is given by the formula

$$m_t = \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1}G_T X + \int_0^T B_u^{-1}G_u\big(Z_u\lambda_u\, du + dA_u\big)\,\Big|\, \mathcal{F}_t\Big). \tag{3.43}$$

Proof. We shall first derive the dynamics of the pre-default ex-dividend price $\widetilde{S}$. In view of Proposition 3.7.1, the price S can be represented as follows, for $t \in [0,T[$,

$$S_t = \mathbb{1}_{\{t<\tau\}}\widetilde{S}_t = \mathbb{1}_{\{t<\tau\}}B_tG_t^{-1}U_t,$$

where the auxiliary process U equals

$$U_t = m_t - \int_0^t B_u^{-1}G_uZ_u\lambda_u\, du - \int_0^t B_u^{-1}G_u\, dA_u,$$

where in turn the continuous $\mathbb{F}$-martingale m is given by (3.43). It is thus obvious that $\widetilde{S} = BG^{-1}U$ for $t\in[0,T[$ (of course, $\widetilde{S}_T = 0$). Since $G = \mu - C$, an application of Itô's formula leads to

$$\begin{aligned} d(G_t^{-1}U_t) = {} & G_t^{-1}\, dm_t - B_t^{-1}Z_t\lambda_t\, dt - B_t^{-1}\, dA_t \\ & + U_t\Big(G_t^{-3}\, d\langle\mu\rangle_t - G_t^{-2}\,(d\mu_t - dC_t)\Big) - G_t^{-2}\, d\langle\mu, m\rangle_t. \end{aligned}$$

Therefore, since under the present assumptions $dC_t = \lambda_t G_t\,dt$, using again Itô's formula, we obtain

$$d\widetilde{S}_t = \big((\lambda_t + r_t)\widetilde{S}_t - \lambda_t Z_t\big)\,dt - dA_t + G_t^{-1}\big(B_t\,dm_t - \widetilde{S}_t\,d\mu_t\big) \\ + G_t^{-2}\big(\widetilde{S}_t\,d\langle\mu\rangle_t - B_t\,d\langle\mu,m\rangle_t\big). \tag{3.44}$$

Note that, under the present assumptions, the pre-default ex-dividend price $\widetilde{S}$ follows on $[0,T[$ a continuous process with dynamics given by (3.44). This means that $S_{t-} = \widetilde{S}_t$ on $\{t \le \tau\}$ for any $t \in [0,T[$. Moreover, since G is continuous, we clearly have that $\mathbb{Q}(\tau = T) = 0$. Hence for the process $S_t = (1 - H_t)\widetilde{S}_t$ we obtain, for every $t \in [0,T]$,

$$dS_t = -S_{t-}\,dM_t + (1 - H_t)\big((r_t S_t - \lambda_t Z_t)\,dt - dA_t\big) \\ + (1 - H_t)G_t^{-1}\big(B_t dm_t - S_t d\mu_t\big) + (1 - H_t)G_t^{-2}\big(S_t d\langle\mu\rangle_t - B_t d\langle\mu,m\rangle_t\big)$$

as expected. □

Let us now examine the dynamics of the cumulative price. As expected, the discounted cumulative price $S^{c*} = B^{-1}S^c$ is a $\mathbb{G}$-martingale under $\mathbb{Q}$ (see formula (3.46) below).

Corollary 3.7.2. *The dynamics of the cumulative price S^c on $[0,T]$ are*

$$dS^c_t = r_t S^c_t\,dt + (Z_t - S_{t-})\,dM_t + (1 - H_t)G_t^{-1}\big(B_t\,dm_t - S_t\,d\mu_t\big) \\ + (1 - H_t)G_t^{-2}\big(S_t\,d\langle\mu\rangle_t - B_t\,d\langle\mu,m\rangle_t\big) \tag{3.45}$$

with the $\mathbb{F}$-martingale m given by (3.43). Equivalently,

$$dS^c_t = r_t S^c_t\,dt + (Z_t - S_{t-})\,dM_t + G_t^{-1}(B_t\,d\check{m}_t - S_t\,d\check{\mu}_t), \tag{3.46}$$

where the $\mathbb{G}$-martingales $\check{m}$ and $\check{\mu}$ are given by (3.40) and (3.41) respectively. The pre-default cumulative price $\widetilde{S}^c$ satisfies, for $t \in [0,T]$,

$$d\widetilde{S}^c_t = r_t \widetilde{S}^c_t\,dt + \lambda_t(\widetilde{S}_t - Z_t)\,dt + G_t^{-1}\big(B_t\,dm_t - \widetilde{S}_t\,d\mu_t\big) \\ + G_t^{-2}\big(\widetilde{S}_t\,d\langle\mu\rangle_t - B_t\,d\langle\mu,m\rangle_t\big). \tag{3.47}$$

Proof. Formula (3.38) yields

$$dS^c_t = dS_t + d\Big(B_t \int_{]0,t]} B_u^{-1}\,dD_u\Big) = dS_t + r_t(S^c_t - S_t)\,dt + dD_t \\ = dS_t + r_t(S^c_t - S_t)\,dt + (1 - H_t)\,dA_t + Z_t\,dH_t. \tag{3.48}$$

By combining (3.48) with (3.42), we obtain (3.45). Formulae (3.46) and (3.47) are immediate consequences of (3.40), (3.41) and (3.45). □

Dynamics under the hypothesis (H). Let us now consider the special case where the hypothesis (H) is satisfied under $\mathbb{Q}$ between the filtrations $\mathbb{F}$ and $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. This means that the immersion property holds for the filtrations $\mathbb{F}$ and $\mathbb{G}$, in the sense that any $\mathbb{F}$-martingale under $\mathbb{Q}$ is also a $\mathbb{G}$-martingale under $\mathbb{Q}$. In that case, the survival process G of τ with respect to $\mathbb{F}$ is known to be non-increasing, so that $G = -C$. In other words, the continuous martingale μ in the Doob-Meyer decomposition of G vanishes. Consequently, formula (3.42) becomes

$$dS_t = -S_{t-}\, dM_t + (1 - H_t)\big((r_t S_t - \lambda_t Z_t)\, dt - dA_t\big) + (1 - H_t) B_t G_t^{-1}\, dm_t.$$

Similarly, (3.45) reduces to

$$dS_t^c = r_t S_t^c\, dt + (Z_t - \widetilde{S}_t)\, dM_t + (1 - H_t) G_t^{-1} B_t\, dm_t$$

and (3.47) becomes

$$d\widetilde{S}_t^c = r_t \widetilde{S}_t^c\, dt + \lambda_t (\widetilde{S}_t - Z_t)\, dt + G_t^{-1} B_t\, dm_t.$$

Remark 3.7.1. The hypothesis (H) is a rather natural assumption in the present context. Indeed, it can be shown that it is necessarily satisfied under the postulate that the underlying $\mathbb{F}$-market model is complete and arbitrage-free, and the extended $\mathbb{G}$-market model is arbitrage-free (see Blanchet-Scalliet and Jeanblanc [30]).

3.7.4 Price Dynamics of a CDS

In Definition 3.7.5 of a stylized T-maturity credit default swap, we follow the convention adopted in Section 2.4. Unlike in Section 2.4, the default protection stream is now represented by an $\mathbb{F}$-predictable process δ. We assume that the default protection payment is received at the time of default and it equals δ_t if default occurs at time t prior to or at maturity date T. Note that δ_t represents the protection payment, so that according to our notational convention the recovery rate equals $1 - \delta_t$ rather than δ_t. The notional amount of the CDS is equal to one monetary unit.

Definition 3.7.5. The stylized T-maturity *credit default swap* (CDS) with a constant *spread* κ and *protection at default* is a defaultable claim $(0, A, Z, \tau)$ in which we set $Z_t = \delta_t$ and $A_t = -\kappa t$ for every $t \in [0, T]$. An $\mathbb{F}$-predictable process $\delta : [0, T] \to \mathbb{R}$ represents the *default protection* and a constant κ is the fixed *CDS spread* (also termed the *rate* or *premium* of the CDS).

A credit default swap is thus a particular defaultable claim in which the promised payoff X is null and the recovery process Z is determined in reference to the estimated recovery rate of the reference credit name.

We denote by $D(\kappa, \delta, T, \tau)$ the dividend process of a CDS. It follows immediately from Definition 3.7.2 that the dividend process $D(\kappa, \delta, T, \tau)$ of a stylized CDS equals, for every $t \in \mathbb{R}_+$,

$$D_t(\kappa, \delta, T, \tau) = \delta_\tau \mathbb{1}_{\{t \geq \tau\}} - \kappa(t \wedge T \wedge \tau). \tag{3.49}$$

In a more realistic approach, the process A is discontinuous, with jumps occurring at the premium payment dates. In this section, we shall only deal with a stylized CDS with a continuously paid premium.

Let us first examine the valuation formula for a stylized T-maturity CDS. Since we now have $X = 0, Z = \delta$ and $A_t = -\kappa t$, we deduce easily from (3.36) that the ex-dividend price of such CDS contract equals, for every $t \in [0, T]$,

$$S_t(\kappa, \delta, T, \tau) = \mathbb{1}_{\{t<\tau\}} \big(\widetilde{P}(t, T) - \kappa \widetilde{A}(t, T)\big), \tag{3.50}$$

where we denote, for any $t \in [0, T]$,

$$\widetilde{P}(t, T) = \frac{B_t}{G_t} \, \mathbb{E}_{\mathbb{Q}} \Big(\mathbb{1}_{\{t<\tau \leq T\}} B_\tau^{-1} \delta_\tau \, \Big| \, \mathcal{F}_t \Big)$$

and

$$\widetilde{A}(t, T) = \frac{B_t}{G_t} \, \mathbb{E}_{\mathbb{Q}} \Big(\int_t^{T \wedge \tau} B_u^{-1} du \, \Big| \, \mathcal{F}_t \Big).$$

The quantity $\widetilde{P}(t, T)$ is the pre-default value at time t of the protection leg, whereas $\widetilde{A}(t, T)$ represents the pre-default present value at time t of one risky basis point paid up to the maturity T or the default time τ, whichever comes first. For ease of notation, we shall write $S_t(\kappa)$ in place of $S_t(\kappa, \delta, T, \tau)$ in what follows. Note that the quantities $\widetilde{P}(t, T)$ and $\widetilde{A}(t, T)$ are well defined at any date $t \in [0, T]$, and not only prior to default as the terminology 'pre-default values' might suggest.

We are in a position to state the following immediate corollary to Proposition 3.7.1.

Corollary 3.7.3. *The ex-dividend price of a CDS equals, for every* $t \in [0, T]$,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \, \mathbb{E}_{\mathbb{Q}} \Big(\int_t^T B_u^{-1} G_u (\delta_u \lambda_u - \kappa) \, du \, \Big| \, \mathcal{F}_t \Big) \tag{3.51}$$

and thus the cumulative price of a CDS equals, for every $t \in [0, T]$,

$$S_t^c(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \, \mathbb{E}_{\mathbb{Q}} \Big(\int_t^T B_u^{-1} G_u (\delta_u \lambda_u - \kappa) \, du \, \Big| \, \mathcal{F}_t \Big) + B_t \int_{]0,t]} B_u^{-1} \, dD_u.$$

The next result is a direct consequence of Proposition 3.7.2 and Corollary 3.7.2.

Corollary 3.7.4. *The dynamics of the ex-dividend price $S(\kappa)$ are*

$$dS_t(\kappa) = -S_{t-}(\kappa)\,dM_t + (1-H_t)(r_tS_t + \kappa - \lambda_t\delta_t)\,dt \qquad (3.52)$$
$$+ (1-H_t)G_t^{-1}(B_t\,dn_t - S_t\,d\mu_t) + (1-H_t)G_t^{-2}\big(S_t\,d\langle\mu\rangle_t - B_t\,d\langle\mu,n\rangle_t\big)$$

with the $\mathbb{F}$-martingale n given by the formula

$$n_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T B_u^{-1}G_u(\delta_u\lambda_u - \kappa)\,du\,\Big|\,\mathcal{F}_t\Big). \qquad (3.53)$$

The cumulative price $S^c(\kappa)$ satisfies, for every $t\in[0,T]$,

$$dS^c_t(\kappa) = r_tS^c_t(\kappa)\,dt + \big(\delta_t - S_{t-}(\kappa)\big)\,dM_t + (1-H_t)G_t^{-1}\big(B_t\,dn_t - S_t(\kappa)\,d\mu_t\big)$$
$$+ (1-H_t)G_t^{-2}\big(S_t(\kappa)\,d\langle\mu\rangle_t - B_t\,d\langle\mu,n\rangle_t\big)$$

or, equivalently,

$$dS^c_t(\kappa) = r_tS^c_t(\kappa)\,dt + \big(\delta_t - S_{t-}(\kappa)\big)\,dM_t + G_t^{-1}\big(B_t\,d\check{n}_t - S_t(\kappa)\,d\check{\mu}_t\big), \qquad (3.54)$$

where the $\mathbb{G}$-martingales $\check{n}$ and $\check{\mu}$ are given by (3.40) and (3.41) respectively.

Dynamics under the hypothesis (H). If the immersion property of $\mathbb{F}$ and $\mathbb{G}$ holds, the martingale μ is null and thus (3.52) reduces to

$$dS_t(\kappa) = -\widetilde{S}_t(\kappa)\,dM_t + (1-H_t)\big(r_tS_t(\kappa) + \kappa - \lambda_t\delta_t\big)\,dt + (1-H_t)B_tG_t^{-1}\,dn_t$$

since the process $\widetilde{S}_t(\kappa)$, $t\in[0,T]$, is continuous and satisfies (cf. (3.44))

$$d\widetilde{S}_t(\kappa) = \big((\lambda_t + r_t)\widetilde{S}_t(\kappa) + \kappa - \lambda_t\delta_t\big)\,dt + B_tG_t^{-1}\,dn_t.$$

Let us note that the quantity $\kappa - \lambda_t\delta_t$ has the intuitive interpretation as the *pre-default dividend rate* of a CDS.

Similarly, we obtain from (3.54)

$$dS^c_t(\kappa) = r_tS^c_t(\kappa)\,dt + \big(\delta_t - \widetilde{S}_t(\kappa)\big)\,dM_t + (1-H_t)B_tG_t^{-1}\,dn_t \qquad (3.55)$$

and

$$d\widetilde{S}^c_t(\kappa) = r_t\widetilde{S}^c_t(\kappa)\,dt + \lambda_t\big(\widetilde{S}_t(\kappa) - \delta_t\big)\,dt + B_tG_t^{-1}\,dn_t.$$

3.7.5 Dynamics of the Market CDS Spread

Let us now introduce the notion of the market CDS spread. It reflects the real-world feature that for any date s the CDS issued at this time has the fixed spread chosen in such a way that the CDS is worthless at its inception. Note that the protection process $(\delta_t,\,t\in[0,T])$ is fixed throughout. We fix the maturity date T and we assume that credit default swaps with different inception dates have a common protection process δ.

Definition 3.7.6. The *T-maturity market CDS spread* $\kappa(s,T)$ at any date $s \in [0,T]$ is the level of the CDS spread that makes the values of the two legs of a CDS equal to each other at time s.

It should be noted that CDSs are quoted in terms of spreads. At any date t, one can take at no cost a long or short position in the CDS issued at this date with the fixed spread equal to the actual value of the market CDS spread for a given maturity and a given reference credit name.

Let us stress that the market CDS spread $\kappa(s,T)$ is not defined neither at the moment of default nor after this date, so that we shall deal in fact with the pre-default value of the market CDS spread. Observe that $\kappa(s,T)$ is represented by an $\mathcal{F}_s$-measurable random variable. In fact, it follows immediately from (3.51) that $\kappa(s,T)$ admits the following representation, for every $s \in [0,T]$,

$$\kappa(s,T) = \frac{\widetilde{P}(s,T)}{\widetilde{A}(s,T)} = \frac{\mathbb{E}_{\mathbb{Q}}\Big(\int_s^T B_u^{-1} G_u \delta_u \lambda_u \, du \,\Big|\, \mathcal{F}_s \Big)}{\mathbb{E}_{\mathbb{Q}}\Big(\int_s^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_s \Big)} = \frac{K_s^1}{K_s^2},$$

where we denote

$$K_s^1 = \mathbb{E}_{\mathbb{Q}}\Big(\int_s^T B_u^{-1} G_u \delta_u \lambda_u \, du \,\Big|\, \mathcal{F}_s \Big)$$

and

$$K_s^2 = \mathbb{E}_{\mathbb{Q}}\Big(\int_s^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_s \Big).$$

In what follows, we shall write briefly κ_s instead of $\kappa(s,T)$. The next result furnishes a convenient representation for the price at time t of a CDS issued at some date $s \leq t$, that is, the marked-to-market value of a CDS that exists already for some time (recall that the market value of the just issued CDS is null).

Proposition 3.7.3. *The ex-dividend price $S(\kappa_s)$ of a T-maturity market CDS initiated at time s equals, for every $t \in [s,T]$,*

$$S_t(\kappa_s) = \mathbb{1}_{\{t<\tau\}} \, (\kappa_t - \kappa_s) \, \widetilde{A}(t,T) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa_s), \qquad (3.56)$$

where $\widetilde{S}_t(\kappa_s)$ is the pre-default ex-dividend price at time t.

Proof. To establish (3.56), it suffices to observe that $S_t(\kappa_s) = S_t(\kappa_s) - S_t(\kappa_t)$ since $S_t(\kappa_t) = 0$. Therefore, in order to conclude it suffices to use (3.50) with $\kappa = \kappa_t$ and $\kappa = \kappa_s$. □

Let us now derive the dynamics of the market CDS spread. We define the $\mathbb{F}$-martingales

$$m_s^1 = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T B_u^{-1} G_u \delta_u \lambda_u \, du \,\Big|\, \mathcal{F}_s \Big) = K_s^1 + \int_0^s B_u^{-1} G_u \delta_u \lambda_u \, du$$

and

$$m_s^2 = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_s \Big) = K_s^2 + \int_0^s B_u^{-1} G_u \, du.$$

Under Assumption 3.7.2, the $\mathbb{F}$-martingales m^1 and m^2 are continuous. Therefore, using the Itô formula, we find easily that the semimartingale decomposition of the market spread process reads

$$\begin{aligned} d\kappa_s = {} & \frac{1}{K_s^2}\Big(B_s^{-1} G_s (\kappa_s - \delta_s \lambda_s) \, ds + \frac{\kappa_s}{K_s^2} \, d\langle m^2 \rangle_s - \frac{1}{K_s^2} \, d\langle m^1, m^2 \rangle_s \Big) \\ & + \frac{1}{K_s^2} \big(dm_s^1 - \kappa_s \, dm_s^2 \big). \end{aligned}$$

3.7.6 Trading Strategies in the CDS Market

We assume from now that k credit default swaps with certain maturities $T_i \geq T$, spreads κ_i and protection payments δ^i for $i = 1, 2, \ldots, k$ are traded over the time interval $[0, T]$. All these contracts are supposed to refer to the same underlying credit name and thus they necessarily refer to a common default time τ.

More formally, this family of credit default swaps is represented by the corresponding dividend processes $D^i = D(\kappa_i, \delta^i, T_i, \tau)$, which are given by formula (3.49). For brevity, the ex-dividend price of the ith traded CDS will be denoted as $S^i(\kappa_i)$, rather than $S(\kappa_i, \delta^i, T_i, \tau)$. Similarly, $S^{c,i}(\kappa_i)$ will stand for the cumulative price process of the ith traded CDS. The 0th traded asset is the savings account B.

Our goal is to examine replicating strategies for a defaultable claim (X, A, Z, τ). As expected, as primary traded assets we take the family of k credit default swaps and the savings account. Therefore, we consider trading strategies $\phi = (\phi^0, \ldots, \phi^k)$ where ϕ^0 is a $\mathbb{G}$-adapted process and the processes $\phi^1, \ldots, \phi^k$ are $\mathbb{G}$-predictable.

In the present setup, we consider trading strategies that are self-financing in the standard sense, as recalled in the following definition.

Definition 3.7.7. The *wealth process* $V(\phi)$ of a strategy $\phi = (\phi^0, \ldots, \phi^k)$ in the savings account B and ex-dividend CDS prices $S^i(\kappa_i)$, $i = 1, 2, \ldots, k$ equals, for any $t \in [0, T]$,

$$V_t(\phi) = \phi_t^0 B_t + \sum_{i=1}^k \phi_t^i S_t^i(\kappa_i).$$

Definition 3.7.8. A trading strategy ϕ is said to be *self-financing* if $V_t(\phi) = V_0(\phi) + G_t(\phi)$ for every $t \in [0,T]$, where the gains process $G(\phi)$ is defined as follows

$$G_t(\phi) = \int_{]0,t]} \phi^0_u \, dB_u + \sum_{i=1}^{k} \int_{]0,t]} \phi^i_u \, d(S^i_u(\kappa_i) + D^i_u),$$

where $D^i = D(\kappa_i, \delta^i, T_i, \tau)$ is the dividend process of the ith CDS (see formula (3.49)).

The following lemma is fairly general; in particular, it is independent of the choice of the underlying model. Indeed, in the proof of this result we only use the obvious relationships $dB_t = r_t B_t \, dt$ and the relationship (cf. (3.38))

$$S^{c,i}_t(\kappa_i) = S^i_t(\kappa_i) + B_t \int_{]0,t]} B^{-1}_u \, dD^i_u. \tag{3.57}$$

Let $V^*(\phi) = B^{-1}V(\phi)$ stand for the discounted wealth process and let $S^{c,i,*}(\kappa_i) = B^{-1}S^{c,i}(\kappa_i)$ be the discounted cumulative price.

Lemma 3.7.2. *Let $\phi = (\phi^0, \dots, \phi^k)$ be a self-financing trading strategy in the savings account B and ex-dividend prices $S^i(\kappa_i)$, $i = 1, 2, \dots, k$. Then the discounted wealth process $V^* = B^{-1}V(\phi)$ satisfies, for $t \in [0,T]$*

$$dV^*_t(\phi) = \sum_{i=1}^{k} \phi^i_t \, dS^{c,i,*}_t(\kappa_i). \tag{3.58}$$

Proof. We have

$$\begin{aligned}
dV^*_t(\phi) &= B^{-1}_t \, dV_t(\phi) - r_t B^{-1}_t V_t(\phi) \, dt = B^{-1}_t \big(dV_t(\phi) - r_t V_t(\phi) \, dt \big) \\
&= B^{-1}_t \Big[\phi^0_t r_t B_t \, dt + \sum_{i=1}^{k} \phi^i_t \big(dS^i_t(\kappa_i) + dD^i_t \big) - r_t V_t(\phi) \, dt \Big] \\
&= B^{-1}_t \Big[\Big(V_t(\phi) - \sum_{i=1}^{k} \phi^i_t S^i_t(\kappa_i) \Big) r_t \, dt + \sum_{i=1}^{k} \phi^i_t \big(dS^i_t(\kappa_i) + dD^i_t \big) \Big] \\
&\quad - r_t B^{-1}_t V_t(\phi) \, dt \\
&= B^{-1}_t \sum_{i=1}^{k} \phi^i_t \Big(dS^i_t(\kappa_i) - r_t S^i_t(\kappa_i) \, dt + dD^i_t \Big) \\
&= \sum_{i=1}^{k} \phi^i_t \, \Big(d(B^{-1}_t S^i_t(\kappa_i)) + B^{-1}_t dD^i_t \Big).
\end{aligned}$$

By comparing the last formula with (3.57), we see that (3.58) holds. □

3.7.7 Replication with Ex-Dividend Prices of CDSs

Recall that the cumulative price of a defaultable claim (X, A, Z, τ) is denoted as S^c. We adopt the following, quite natural, definition of replication of a defaultable claim. Note that the set of traded assets is not explicitly specified in this definition. Hence this definition can be used for any choice of primary traded assets.

Definition 3.7.9. We say that a self-financing strategy $\phi = (\phi^0, \dots, \phi^k)$ *replicates* a defaultable claim (X, A, Z, τ) if its wealth process $V(\phi)$ satisfies $V_t(\phi) = S^c_t$ for every $t \in [0, T]$. In particular, the equality $V_{t\wedge\tau}(\phi) = S^c_{t\wedge\tau}$ holds for every $t \in [0, T]$.

In the remaining part of this section we assume that the hypothesis (H) holds. Hence the hazard process Γ of default time is increasing and thus, by Assumption 3.7.1, we have that, for any $t \in [0, T]$,

$$\Gamma_t = \Lambda_t = \int_0^t \lambda_u \, du.$$

The discounted cumulative price $S^{c,i,*}(\kappa_i)$ of the ith CDS is governed by (cf. (3.55))

$$dS^{c,i,*}_t(\kappa_i) = B_t^{-1}\big(\delta^i_t - \widetilde{S}^i_t(\kappa_i)\big)\, dM_t + (1 - H_t)G_t^{-1}\, dn^i_t, \tag{3.59}$$

where (cf. (3.53))

$$n^i_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^{T_i} B_u^{-1} G_u(\delta^i_u \lambda_u - \kappa_i)\, du \,\Big|\, \mathcal{F}_t\Big). \tag{3.60}$$

The next lemma yields the dynamics of the wealth process $V(\phi)$ for a self-financing strategy ϕ.

Lemma 3.7.3. *The discounted wealth process $V^*(\phi) = B^{-1}V(\phi)$ of any self-financing trading strategy ϕ satisfies, for any $t \in [0, T]$,*

$$dV^*_t(\phi) = \sum_{i=1}^k \phi^i_t \Big(B_t^{-1}\big(\delta^i_t - \widetilde{S}^i_t(\kappa_i)\big)\, dM_t + (1 - H_t)G_t^{-1} dn^i_t\Big). \tag{3.61}$$

Proof. It suffices to combine (3.58) with (3.59). □

It is clear from the lemma that it is enough to search for the components $\phi^1, \dots, \phi^k$ of a strategy ϕ. The same remark applies to self-financing strategies introduced in Definition 3.7.8. It is worth stressing that in what follows, we shall only consider *admissible* trading strategies, that is, strategies for which the discounted wealth process $V^*(\phi) = B^{-1}V(\phi)$ is a $\mathbb{G}$-martingale under $\mathbb{Q}$. The market model in which only admissible trading strategies are allowed is *arbitrage-free*, that is, arbitrage opportunities are ruled out. Admissibility of a replicating strategy will be ensured by the equality $V(\phi) = S^c$ and the fact that the discounted cumulative price $S^{c*} = B^{-1}S^c$ of a defaultable claim is a $\mathbb{G}$-martingale under $\mathbb{Q}$.

We work throughout under the standing Assumptions 3.7.1 and 3.7.2 and the following postulate.

Assumption 3.7.3. The filtration $\mathbb{F}$ is generated by a d-dimensional Brownian motion W under $\mathbb{Q}$.

Since the hypothesis (H) is assumed to hold, the process W is also a Brownian motion with respect to the enlarged filtration $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. Recall that any (local) martingales with respect to a Brownian filtration is necessarily continuous. Hence Assumption 3.7.2 is obviously satisfied under Assumption 3.7.3.

The crucial observation is that, by the predictable representation property of a Brownian motion, there exist $\mathbb{F}$-predictable, $\mathbb{R}^d$-valued processes ξ and ζ^i, $i = 1, 2, \ldots, k$ such that $dm_t = \xi_t \, dW_t$ and $dn^i_t = \zeta^i_t \, dW_t$, where the $\mathbb{F}$-martingales m and n^i are given by (3.43) and (3.60), respectively.

We are now in a position to state the hedging result for a defaultable claim in the single-name setup. We consider a defaultable claim (X, A, Z, τ) satisfying the natural integrability conditions under $\mathbb{Q}$, which ensure the cumulative price process S^c for this claim is well defined.

Theorem 3.7.1. *Assume that there exist $\mathbb{F}$-predictable processes $\phi^1, \ldots, \phi^k$ satisfying the following conditions, for any $t \in [0, T]$,*

$$\sum_{i=1}^{k} \phi^i_t \big(\delta^i_t - \widetilde{S}^i_t(\kappa_i)\big) = Z_t - \widetilde{S}_t, \quad \sum_{i=1}^{k} \phi^i_t \zeta^i_t = \xi_t. \tag{3.62}$$

Let the process $V(\phi)$ be given by (3.61) with the initial condition $V_0(\phi) = S^c_0$ and let ϕ^0 be given by, for $t \in [0, T]$,

$$\phi^0_t = B^{-1}_t \Big(V_t(\phi) - \sum_{i=1}^{k} \phi^i_t S^i_t(\kappa_i) \Big).$$

Then the self-financing trading strategy $\phi = (\phi^0, \ldots, \phi^k)$ in the savings account B and the assets $S^i(\kappa_i)$, $i = 1, 2, \ldots, k$ replicates the defaultable claim (X, A, Z, τ).

Proof. From Lemma 3.7.3, we know that the discounted wealth process satisfies

$$dV_t^*(\phi) = \sum_{i=1}^{k} \phi_t^i \Big(B_t^{-1}(\delta_t^i - \widetilde{S}_t^i(\kappa_i))\, dM_t + (1-H_t)G_t^{-1} dn_t^i \Big). \tag{3.63}$$

Recall also that the discounted cumulative price S^{c*} of a defaultable claim is governed by

$$dS_t^{c*} = B_t^{-1}(Z_t - \widetilde{S}_t)\, dM_t + (1-H_t)G_t^{-1} dm_t. \tag{3.64}$$

We will show that if the two conditions in (3.62) are satisfied for any $t \in [0,T]$, then the equality $V_t(\phi) = S_t^c$ holds for any $t \in [0,T]$.

Let $\widetilde{V}^*(\phi) = B^{-1}\widetilde{V}(\phi)$ stand for the *discounted pre-default wealth*, where $\widetilde{V}(\phi)$ is the unique $\mathbb{F}$-adapted process such that $\mathbb{1}_{\{t<\tau\}} V_t(\phi) = \mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\phi)$ for every $t \in [0,T]$. On the one hand, using (3.62), we obtain

$$\begin{aligned} d\widetilde{V}_t^*(\phi) &= \sum_{i=1}^{k} \phi_t^i \Big(\lambda_t B_t^{-1}(\widetilde{S}_t^i(\kappa_i) - \delta_t^i)\, dt + G_t^{-1}\, \zeta_t^i\, dW_t \Big) \\ &= \lambda_t B_t^{-1}(\widetilde{S}_t - Z_t)\, dt + G_t^{-1}\xi_t\, dW_t. \end{aligned}$$

On the other hand, the discounted pre-default cumulative price $\widetilde{S}^{c*}$ satisfies

$$d\widetilde{S}_t^{c*} = \lambda_t B_t^{-1}(\widetilde{S}_t - Z_t)\, dt + G_t^{-1}\xi_t\, dW_t.$$

Since by assumption

$$\widetilde{V}_0^*(\phi) = V_0(\phi) = S_0^c = \widetilde{S}_0^{c*},$$

it is clear that $\widetilde{V}_t^*(\phi) = \widetilde{S}_t^{c*}$ for every $t \in [0,T]$. We thus conclude that the pre-default wealth $\widetilde{V}(\phi)$ of ϕ and the pre-default cumulative price $\widetilde{S}^c$ of the claim coincide. Note that the first equality in (3.62) is in fact only essential for those values of $t \in [0,T]$ for which $\lambda_t \neq 0$.

To complete the proof, we need to check what happens when default occurs prior to or at maturity T. To this end, it suffices to compare the jumps of S^c and $V(\phi)$ at time τ. In view of (3.62), (3.63) and (3.64), we obtain

$$\Delta V_\tau(\phi) = Z_\tau - \widetilde{S}_\tau = \Delta S_\tau^c$$

and thus $V_{t\wedge\tau}(\phi) = S_{t\wedge\tau}^c$ for any $t \in [0,T]$. After default, we have

$$dV_t(\phi) = r_t V_t(\phi)\, dt, \quad dS_t^c = r_t S_t^c\, dt,$$

so that we conclude that the desired equality $V_t(\phi) = S_t^c$ is indeed satisfied for every $t \in [0,T]$. □

3.8 Multi-Name CDS Market

In this section, we shall deal with a market model driven by a Brownian filtration in which a finite family of CDSs with different underlying names is traded.

3.8.1 Valuation of a First-to-Default Claim

Our first goal is to extend the pricing results of Section 3.7.1 to the case of a multi-name credit risk model with stochastic default intensities.

Joint Survival Process

We assume that we are given n strictly positive random times $\tau_1, \dots, \tau_n$, defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, and referred to as default times of n credit names. We postulate that this space is endowed with a *reference filtration* $\mathbb{F}$, which satisfies Assumption 3.7.2.

In order to describe dynamic joint behavior of default times, we introduce the *conditional joint survival process* $G(u_1, \dots, u_n; t)$ by setting, for every $u_1, \dots, u_n, t \in \mathbb{R}_+$,

$$G(u_1, \dots, u_n; t) = \mathbb{Q}(\tau_1 > u_1, \dots, \tau_n > u_n \,|\, \mathcal{F}_t).$$

Let us set $\tau_{(1)} = \tau_1 \wedge \dots \wedge \tau_n$ and let us define the process $G_{(1)}(t;t)$, $t \in \mathbb{R}_+$ by setting

$$G_{(1)}(t;t) = G(t, \dots, t; t) = \mathbb{Q}(\tau_1 > t, \dots, \tau_n > t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau_{(1)} > t \,|\, \mathcal{F}_t).$$

It is easy to check that $G_{(1)}$ is a bounded supermartingale and thus it admits the unique Doob-Meyer decomposition $G_{(1)} = \mu - C$. We shall work throughout under the following extension of Assumption 3.7.1.

Assumption 3.8.1. We assume that the process $G_{(1)}$ is continuous and the increasing process C is absolutely continuous with respect to the Lebesgue measure, so that $dC_t = c_t \, dt$ for some $\mathbb{F}$-progressively measurable, non-negative process c. We denote by $\widetilde{\lambda}$ the $\mathbb{F}$-progressively measurable process defined as $\widetilde{\lambda}_t = G_{(1)}^{-1}(t;t) c_t$. The process $\widetilde{\lambda}$ is hereafter referred to as the *first-to-default intensity*.

We denote $H_t^i = \mathbb{1}_{\{t \geq \tau_i\}}$ and we introduce the filtrations $\mathbb{H}^i, \mathbb{H}$ and $\mathbb{G}$ with the corresponding σ-fields $\mathcal{H}_t^i, \mathcal{H}_t$ and $\mathcal{G}_t$ defined as follows:

$$\mathcal{H}_t^i = \sigma(H_s^i;\, s \in [0,t]), \quad \mathcal{H}_t = \mathcal{H}_t^1 \vee \dots \vee \mathcal{H}_t^n, \quad \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t,$$

We assume that the usual conditions of completeness and right-continuity are satisfied by these filtrations. Arguing as in Section 3.7.1, we see that the process

$$\widehat{M}_t = H_t^{(1)} - \widetilde{\Lambda}_{t\wedge\tau_{(1)}} = H_t^{(1)} - \int_0^{t\wedge\tau_{(1)}} \widetilde{\lambda}_u\,du = H_t^{(1)} - \int_0^t (1 - H_u^{(1)})\widetilde{\lambda}_u\,du,$$

is a $\mathbb{G}$-martingale, where we denote $H_t^{(1)} = \mathbb{1}_{\{t\geq\tau_{(1)}\}}$ and $\widetilde{\Lambda}_t = \int_0^t \widetilde{\lambda}_u\,du$. Note that the first-to-default intensity $\widetilde{\lambda}$ satisfies

$$\widetilde{\lambda}_t = \lim_{h\downarrow 0}\frac{1}{h}\,\frac{\mathbb{Q}(t<\tau_{(1)}\leq t+h\,|\,\mathcal{F}_t)}{\mathbb{Q}(\tau_{(1)}>t\,|\,\mathcal{F}_t)} = \frac{1}{G_{(1)}(t;t)}\lim_{h\downarrow 0}\frac{1}{h}\,(C_{t+h}-C_t).$$

We make an additional assumption, in which we introduce the *first-to-default intensity* $\widetilde{\lambda}^i$ and the associated martingale $\widehat{M}^i$ for each credit name $i=1,\dots,n$.

Assumption 3.8.2. For any $i=1,2,\dots,n$, the process $\widetilde{\lambda}^i$ given by

$$\widetilde{\lambda}_t^i = \lim_{h\downarrow 0}\frac{1}{h}\,\frac{\mathbb{Q}(t<\tau_i\leq t+h,\,\tau_{(1)}>t\,|\,\mathcal{F}_t)}{\mathbb{Q}(\tau_{(1)}>t\,|\,\mathcal{F}_t)}$$

is well defined and the process $\widehat{M}^i$, given by the formula

$$\widehat{M}_t^i = H_{t\wedge\tau_{(1)}}^i - \int_0^{t\wedge\tau_{(1)}} \widetilde{\lambda}_u^i\,du, \tag{3.65}$$

is a $\mathbb{G}$-martingale.

It is worth noting that the equalities $\sum_{i=1}^n \widetilde{\lambda}^i = \widetilde{\lambda}$ and $\widehat{M} = \sum_{i=1}^n \widehat{M}^i$ are valid.

Special Case

Let $\widehat{\Gamma}^i$, $i=1,2,\dots,n$ be a given family of $\mathbb{F}$-adapted, increasing, continuous processes, defined on a filtered probability space $(\widetilde{\Omega},\mathbb{F},\mathbb{P})$. We postulate that $\widehat{\Gamma}_0^i=0$ and $\lim_{t\to\infty}\widehat{\Gamma}_t^i=\infty$. For the construction of default times satisfying Assumptions 3.8.1 and 3.8.2, we postulate that $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{P}})$ is an auxiliary probability space endowed with a family ξ_i, $i=1,2,\dots,n$ of random variables uniformly distributed on $[0,1]$ and such that their joint probability distribution is given by an n-dimensional copula function C (see Section 5.4). We then define, for every $i=1,2,\dots,n$,

$$\tau_i(\widetilde{\omega},\widehat{\omega}) = \inf\{\,t\in\mathbb{R}_+ : \widehat{\Gamma}_t^i(\widetilde{\omega}) \geq -\ln\xi_i(\widehat{\omega})\}.$$

We endow the space $(\Omega, \mathcal{G}, \mathbb{Q})$ with the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n$, where the filtration $\mathbb{H}^i$ is generated by the process $H^i_t = \mathbb{1}_{\{t \geq \tau_i\}}$ for every $i = 1, 2, \ldots, n$.

We have that, for any $T > 0$ and arbitrary $t_1, \ldots, t_n \leq T$,

$$\mathbb{Q}(\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T) = C(K^1_{t_1}, \ldots, K^n_{t_n}),$$

where we denote $K^i_t = e^{-\widehat{\Gamma}^i_t}$.

Schönbucher and Schubert [161] show that the following equality holds, for arbitrary $t \leq s$,

$$\mathbb{Q}(\tau_i > s \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t < \tau_{(1)}\}} \, \mathbb{E}_{\mathbb{Q}}\bigg(\frac{C(K^1_t, \ldots, K^i_s, \ldots, K^n_t)}{C(K^1_t, \ldots, K^n_t)} \,\Big|\, \mathcal{F}_t \bigg).$$

Consequently, assuming that $\widehat{\Gamma}^i_t = \int_0^t \widehat{\gamma}^i_u \, du$, the ith survival intensity equals, on the event $\{t < \tau_{(1)}\}$,

$$\widetilde{\lambda}^i_t = \widehat{\gamma}^i_t K^i_t \, \frac{\frac{\partial}{\partial v_i} C(K^1_t, \ldots, K^n_t)}{C(K^1_t, \ldots, K^n_t)} = \widehat{\gamma}^i_t K^i_t \, \frac{\partial}{\partial v_i} \ln C(K^1_t, \ldots, K^n_t).$$

One can now easily show that the process $\widehat{M}^i$, which is given by formula (3.65), is a $\mathbb{G}$-martingale. This indeed follows from Aven's lemma [8].

3.8.2 Price Dynamics of a First-to-Default Claim

We will now analyze the risk-neutral valuation of first-to-default claims on a basket of n credit names. As before, $\tau_1, \ldots, \tau_n$ are respective default times and $\tau_{(1)} = \tau_1 \wedge \cdots \wedge \tau_n$ stands for the moment of the first default.

Definition 3.8.1. A *first-to-default claim* with maturity T associated with $\tau_1, \ldots, \tau_n$ is a defaultable claim $(X, A, Z, \tau_{(1)})$, where X is an $\mathcal{F}_T$-measurable amount payable at maturity T if no default occurs prior to or at T and an $\mathbb{F}$-adapted, continuous process of finite variation $A : [0, T] \to \mathbb{R}$ with $A_0 = 0$ represents the dividend stream up to $\tau_{(1)}$. Finally, $Z = (Z^1, \ldots, Z^n)$ is the vector of $\mathbb{F}$-predictable, real-valued processes, where $Z^i_{\tau_{(1)}}$ specifies the recovery received at time $\tau_{(1)}$ if default occurs prior to or at T and the ith name is the first defaulted name, that is, on the event $\{\tau_i = \tau_{(1)} \leq T\}$.

The next definition extends Definition 3.7.2 to the case of a first-to-default claim. Recall that we denote $H^{(1)}_t = \mathbb{1}_{\{t \geq \tau_{(1)}\}}$ for every $t \in [0, T]$.

Definition 3.8.2. The *dividend process* $(D_t,\, t \in \mathbb{R}_+)$ of a first-to-default claim maturing at T equals, for every $t \in \mathbb{R}_+$,

$$D_t = X\mathbb{1}_{\{T<\tau_{(1)}\}}\mathbb{1}_{[T,\infty[}(t) + \int_0^{t\wedge T} (1 - H^{(1)}_u)\, dA_u + \int_{]0,t\wedge T]} \sum_{i=1}^n \mathbb{1}_{\{\tau_{(1)}=\tau_i\}} Z^i_u\, dH^{(1)}_u.$$

We are in a position to examine the prices of the first-to-default claim. Note that

$$\mathbb{1}_{\{t<\tau_{(1)}\}} S^c_t = \mathbb{1}_{\{t<\tau_{(1)}\}} \widetilde{S}^c_t, \quad \mathbb{1}_{\{t<\tau_{(1)}\}} S_t = \mathbb{1}_{\{t<\tau_{(1)}\}} \widetilde{S}_t,$$

where $\widetilde{S}^c$ and $\widetilde{S}$ are pre-default values of S^c and S, where the price processes S^c and S are given by Definitions 3.7.3 and 3.7.4, respectively. We postulate that $\mathbb{E}_{\mathbb{Q}}|B_T^{-1}X| < \infty$,

$$\mathbb{E}_{\mathbb{Q}}\Big| \int_0^T B_u^{-1}(1 - H^{(1)}_u)\, dA_u \Big| < \infty,$$

and for $i = 1, 2, \ldots, n$

$$\mathbb{E}_{\mathbb{Q}}|B^{-1}_{\tau_{(1)}\wedge T} Z^i_{\tau_{(1)}\wedge T}| < \infty,$$

so that the ex-dividend price S_t (and thus also cumulative price S^c) is well defined for any $t \in [0, T]$. In the next auxiliary result, we denote $Y^i = B^{-1}Z^i$. Hence Y^i is a real-valued, $\mathbb{F}$-predictable process such that the inequality $\mathbb{E}_{\mathbb{Q}}|Y^i_{\tau_{(1)}\wedge T}| < \infty$ is satisfied.

Lemma 3.8.1. *We have that*

$$B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\sum_{i=1}^n \mathbb{1}_{\{t<\tau_{(1)}=\tau_i\le T\}} Y^i_{\tau_{(1)}} \,\Big|\, \mathcal{G}_t \Big) = \mathbb{1}_{\{t<\tau_{(1)}\}} \frac{B_t}{G_{(1)}(t;t)}\, \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \sum_{i=1}^n Y^i_u \widetilde{\lambda}^i_u G_{(1)}(u;u)\, du \,\Big|\, \mathcal{F}_t \Big).$$

Proof. Let us fix i and let us consider the process $Y^i_u = \mathbb{1}_A \mathbb{1}_{]s,v]}(u)$ for some fixed date $t \le s < v \le T$ and some event $A \in \mathcal{F}_s$. We note that

$$\mathbb{1}_{\{s<\tau_{(1)}=\tau_i\le v\}} = H^i_{v\wedge\tau_{(1)}} - H^i_{s\wedge\tau_{(1)}} = \widehat{M}^i_v - \widehat{M}^i_s + \int_{s\wedge\tau_{(1)}}^{v\wedge\tau_{(1)}} \widetilde{\lambda}^i_u\, du.$$

Using Assumption 3.8.2, we thus obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\leq T\}}Y^i_{\tau_{(1)}}\,\Big|\,\mathcal{G}_t\Big) &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_A\mathbb{1}_{\{s<\tau_{(1)}=\tau_i\leq v\}}\,\Big|\,\mathcal{G}_t\Big)\\
&= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_A\Big(\widehat{M}^i_v-\widehat{M}^i_s+\int_{s\wedge\tau_{(1)}}^{v\wedge\tau_{(1)}}\widetilde{\lambda}^i_u\,du\Big)\,\Big|\,\mathcal{G}_t\Big)\\
&= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_A\,\mathbb{E}_{\mathbb{Q}}\Big(\widehat{M}^i_v-\widehat{M}^i_s+\int_{s\wedge\tau_{(1)}}^{v\wedge\tau_{(1)}}\widetilde{\lambda}^i_u\,du\,\Big|\,\mathcal{G}_s\Big)\,\Big|\,\mathcal{G}_t\Big)\\
&= \mathbb{E}_{\mathbb{Q}}\Big(\int_{t\wedge\tau_{(1)}}^{T\wedge\tau_{(1)}}Y^i_u\widetilde{\lambda}^i_u\,du\,\Big|\,\mathcal{G}_t\Big)\\
&= \mathbb{1}_{\{t<\tau_{(1)}\}}\,\frac{1}{G_{(1)}(t;t)}\,\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T Y^i_u\widetilde{\lambda}^i_u G_{(1)}(u;u)\,du\,\Big|\,\mathcal{F}_t\Big),
\end{aligned}$$

where the last equality follows from the formula

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{t\wedge\tau_{(1)}}^{T\wedge\tau_{(1)}}R_u\,du\,\Big|\,\mathcal{G}_t\Big)=\mathbb{1}_{\{t<\tau_{(1)}\}}\,\frac{1}{G_{(1)}(t;t)}\,\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T R_uG_{(1)}(u;u)\,du\,\Big|\,\mathcal{F}_t\Big),$$

which holds for any $\mathbb{F}$-predictable process R such that the right-hand side is well defined. □

Given Lemma 3.8.1, the proof of the next result is very much similar to that of Proposition 3.7.1 and thus is omitted.

Proposition 3.8.1. *The pre-default ex-dividend price $\widetilde{S}$ of a first-to-default claim $(X,A,Z,\tau_{(1)})$ satisfies*

$$\begin{aligned}
\widetilde{S}_t &= \frac{B_t}{G_{(1)}(t;t)}\,\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T B_u^{-1}G_{(1)}(u;u)\Big(\sum_{i=1}^n Z^i_u\widetilde{\lambda}^i_u\,du+dA_u\Big)\,\Big|\,\mathcal{F}_t\Big).\\
&\quad+\frac{B_t}{G_{(1)}(t;t)}\,\mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1}G_{(1)}(T;T)X\mathbb{1}_{\{t<T\}}\,\Big|\,\mathcal{F}_t\Big).
\end{aligned}$$

By proceeding as in the proof of Proposition 3.7.2, one can also establish the following result, which gives dynamics of price processes $\widetilde{S}$ and S^c of a first-to-default claim.

Recall that μ is the continuous martingale arising in the Doob-Meyer decomposition of the process $G_{(1)}$ (see Assumption 3.8.1).

Proposition 3.8.2. *The dynamics of the pre-default ex-dividend price $\widetilde{S}$ of a first-to-default claim $(X,A,Z,\tau_{(1)})$ on $[0,\tau_{(1)}\wedge T]$ are*

$$\begin{aligned}
d\widetilde{S}_t &= (r_t+\widetilde{\lambda}_t)\widetilde{S}_t\,dt-\sum_{i=1}^n\widetilde{\lambda}^i_tZ^i_t\,dt-dA_t+G^{-1}_{(1)}(t;t)\big(B_t\,dm_t-\widetilde{S}_t\,d\mu_t\big)\\
&\quad+G^{-2}_{(1)}(t;t)\big(\widetilde{S}_t\,d\langle\mu\rangle_t-B_t\,d\langle\mu,m\rangle_t\big),
\end{aligned}$$

where the continuous $\mathbb{F}$-martingale m is given by the formula

$$
\begin{aligned}
m_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T B_u^{-1} G_{(1)}(u;u) \Big(\sum_{i=1}^n Z_u^i \widetilde{\lambda}_u^i \, du + dA_u \Big) \,\Big|\, \mathcal{F}_t \Big) \\
+ \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} G_{(1)}(T;T) X \,\Big|\, \mathcal{F}_t \Big).
\end{aligned}
$$

The dynamics of the cumulative price S^c on $[0, \tau_{(1)} \wedge T]$ are

$$
\begin{aligned}
dS_t^c &= \sum_{i=1}^n (Z_t^i - \widetilde{S}_{t-}) \, dM_t^i + \Big(r_t \widetilde{S}_t - \sum_{i=1}^n \widetilde{\lambda}_t^i Z_t^i \Big) \, dt - dA_t \\
&\quad + G_{(1)}^{-1}(t;t) \big(B_t \, dm_t - \widetilde{S}_t \, d\mu_t \big) + G_{(1)}^{-2}(t;t) \big(\widetilde{S}_t \, d\langle \mu \rangle_t - B_t \, d\langle \mu, m \rangle_t \big).
\end{aligned}
$$

Hypothesis (H)

As in the single-name case, the most explicit results can be derived under an additional assumption of the immersion property of filtrations $\mathbb{F}$ and $\mathbb{G}$.

Assumption 3.8.3. Any $\mathbb{F}$-martingale under $\mathbb{Q}$ is a $\mathbb{G}$-martingale under $\mathbb{Q}$. This also implies that the hypothesis (H) holds between $\mathbb{F}$ and $\mathbb{G}$. In particular, any $\mathbb{F}$-martingale is also a $\mathbb{G}^i$-martingale for $i = 1, 2, \ldots, n$, that is, the hypothesis (H) holds between $\mathbb{F}$ and $\mathbb{G}^i$ for $i = 1, 2, \ldots, n$.

It is worth stressing that, in general, there is no reason to expect that any $\mathbb{G}^i$-martingale is necessarily a $\mathbb{G}$-martingale. We shall argue that even when the reference filtration $\mathbb{F}$ is trivial this is not the case, in general (except for some special cases, for instance, under the independence assumption).

Example 3.8.1. Let us take $n = 2$ and let us denote $G_t^{1|2} = \mathbb{Q}(\tau_1 > t \,|\, \mathcal{H}_t^2)$ and $G(u, v) = \mathbb{Q}(\tau_1 > u, \tau_2 > v)$. It is then easy to prove that

$$
\begin{aligned}
dG_t^{1|2} &= \left(\frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} - \frac{G(t,t)}{G(0,t)} \right) dM_t^2 \\
&\quad + \left(H_t^2 \partial_1 h(t, \tau_2) + (1 - H_t^2) \frac{\partial_1 G(t,t)}{G(0,t)} \right) dt,
\end{aligned}
$$

where $h(t, u) = \frac{\partial_2 G(t,u)}{\partial_2 G(0,u)}$ and M^2 is the $\mathbb{H}^2$-martingale given by

$$
M_t^2 = H_t^2 + \int_0^{t \wedge \tau_2} \frac{\partial_2 G(0,u)}{G(0,u)} \, du.
$$

If the hypothesis (H) holds between the filtrations $\mathbb{H}^2$ and $\mathbb{H}^1 \vee \mathbb{H}^2$ then the martingale part in the Doob-Meyer decomposition of the process $G^{1|2}$

vanishes. We thus see that the hypothesis (H) is not always valid, since clearly the quantity

$$\frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} - \frac{G(t,t)}{G(0,t)}$$

does not vanish, in general. One can also note that in the special case when the inequality $\tau_2 < \tau_1$ is satisfied, the martingale part in the above-mentioned decomposition disappears and thus the hypothesis (H) holds between the filtrations $\mathbb{H}^2$ and $\mathbb{H}^1 \vee \mathbb{H}^2$.

From now on, we shall work under Assumption 3.8.3. In that case, the dynamics of price processes obtained in Proposition 3.8.1 simplify, as the following result shows.

Corollary 3.8.1. *The pre-default ex-dividend price $\widetilde{S}$ of a first-to-default claim $(X, A, Z, \tau_{(1)})$ satisfies*

$$d\widetilde{S}_t = (r_t + \widetilde{\lambda}_t)\widetilde{S}_t\,dt - \sum_{i=1}^{n} \widetilde{\lambda}^i_t Z^i_t\,dt - dA_t + B_t G^{-1}_{(1)}(t;t)\,dm_t$$

where m is the continuous $\mathbb{F}$-martingale defined in Proposition 3.8.2. The cumulative price S^c of a first-to-default claim $(X, A, Z, \tau_{(1)})$ is given by the expression, for $t \in [0, T \wedge \tau_{(1)}]$,

$$dS^c_t = r_t S^c_t\,dt + \sum_{i=1}^{n} (Z^i_t - \widetilde{S}_t)\,d\widehat{M}^i_t + B_t G^{-1}_{(1)}(t;t)\,dm_t.$$

Equivalently, for every $t \in [0, T \wedge \tau_{(1)}]$,

$$dS^c_t = r_t S^c_t\,dt + \sum_{i=1}^{n} (Z^i_t - \widetilde{S}_t)\,d\widehat{M}^i_t + B_t G^{-1}_{(1)}(t;t)\,d\check{m}_t,$$

where $\check{m}$ is a $\mathbb{G}$-martingale given by $\check{m}_t = m_{t\wedge\tau_{(1)}}$ for every $t \in [0, T]$.

Let us assume, in addition, that the reference filtration $\mathbb{F}$ is generated by the d-dimensional standard Brownian motion W. Then there exists an $\mathbb{R}^d$-valued, $\mathbb{F}$-predictable process ξ for which $dm_t = \xi_t\,dW_t$ and thus the last formula in Corollary 3.8.1 yields the following result.

Corollary 3.8.2. *The discounted cumulative price of a first-to-default claim $(X, A, Z, \tau_{(1)})$ satisfies, for every $t \in [0, T \wedge \tau_{(1)}]$,*

$$dS^{c*}_t = \sum_{i=1}^{n} B^{-1}_t (Z^i_t - \widetilde{S}_t)\,d\widehat{M}^i_t + G^{-1}_{(1)}(t;t)\xi_t\,dW_t.$$

3.8.3 Price Dynamics of a CDS

Let us first introduce a family of traded single-name credit default swaps.

Definition 3.8.3. By the *ith CDS* we mean the credit default swap written on the ith reference credit name, with the maturity date T_i, the constant spread κ_i and the protection process δ^i, as specified by Definition 3.7.5.

Let $S^i_{t|j}(\kappa_i)$ stand for the ex-dividend price at time t of the ith CDS on the event $\tau_{(1)} = \tau_j = t$ for some $j \neq i$. This value can be represented using a suitable extension of Proposition 3.8.1, but we decided to omit the derivation of this pricing formula.

Assume that we have already computed $S^i_{t|j}(\kappa_i)$ for $t \in [0, T_i]$. Then the ith CDS can be seen, on the random interval $[0, T_i \wedge \tau_{(1)}]$, as a first-to-default claim $(X, A, Z, \tau_{(1)})$ with $X = 0$,

$$Z = (S^i_{t|1}(\kappa_i), \dots, \delta^i, \dots, S^i_{t|n}(\kappa_i))$$

and $A_t = -\kappa_i t$. The last observation applies also to the random interval $[0, T \wedge \tau_{(1)}]$ for any fixed date $T \leq T_i$. Let us denote by n^i the following $\mathbb{F}$-martingale

$$n^i_t = \mathbb{E}_{\mathbb{Q}}\Big(\sum_{i=1}^{n} \int_0^{T_i} B_u^{-1} G_{(1)}(u;u) \Big(\delta^i_u \widetilde{\lambda}^i_u + \sum_{j=1\,, j\neq i}^{n} S^i_{u|j}(\kappa_i) \widetilde{\lambda}^j_u - \kappa_i \Big)\, du \,\Big|\, \mathcal{F}_t \Big).$$

The following result can be easily deduced from Proposition 3.8.1.

Corollary 3.8.3. *The cumulative price of the ith CDS satisfies, for every* $t \in [0, T_i \wedge \tau_{(1)}]$,

$$\begin{aligned} dS^{c,i}_t(\kappa_i) &= r_t S^{c,i}_t(\kappa_i)\, dt + (\delta^i_t - \widetilde{S}^i_t(\kappa_i))\, d\widehat{M}^i_t \\ &\quad + \sum_{j=1,\, j\neq i}^{n} (S^i_{t|j}(\kappa_i) - \widetilde{S}^i_t(\kappa_i))\, d\widehat{M}^j_t + B_t G^{-1}_{(1)}(t;t)\, dn^i_t. \end{aligned}$$

If, in addition, the reference filtration $\mathbb{F}$ *is generated by the* d*-dimensional standard Brownian motion* W *then the discounted cumulative price of the ith CDS satisfies, for every* $t \in [0, T_i \wedge \tau_{(1)}]$,

$$\begin{aligned} dS^{c,i,*}_t(\kappa_i) &= B_t^{-1}(\delta^i_t - \widetilde{S}^i_t(\kappa_i))\, d\widehat{M}^i_t + B_t^{-1} \sum_{j=1,\, j\neq i}^{n} (S^i_{t|j}(\kappa_i) - \widetilde{S}^i_t(\kappa_i))\, d\widehat{M}^j_t \\ &\quad + G^{-1}_{(1)}(t;t) \zeta^i_t\, dW_t, \end{aligned}$$

where ζ^i *is the* $\mathbb{R}^d$*-valued,* $\mathbb{F}$*-predictable process such that* $dn^i_t = \zeta^i_t\, dW_t$.

3.8.4 Replication of a First-to-Default Claim

Our final goal is to extend Theorem 3.7.1 of Section 2.2.8 to the case of several credit names in a hazard process model in which credit spreads are driven by a multi-dimensional Brownian motion. We consider a self-financing trading strategy $\phi = (\phi^0, \dots, \phi^k)$ with $\mathbb{G}$-predictable components, as defined in Section 3.7.6. The 0th traded asset is thus the savings account; the remaining k primary assets are single-name CDSs with different underlying credit names and/or maturities.

As before, for any $l = 1, 2, \dots, k$ we will use the shorthand notation $S^l(\kappa_l)$ and $S^{c,l}(\kappa_l)$ to denote the ex-dividend and cumulative prices of CDSs with respective dividend processes $D(\kappa_l, \delta^l, T_l, \tilde{\tau}_l)$ given by a suitable version of formula (3.49). Note that here $\tilde{\tau}_l = \tau_j$ for some $j = 1, 2, \dots, n$. We will thus write $\tilde{\tau}_l = \tau_{j_l}$ in what follows.

Remark 3.8.1. Note that, typically, we will have $k = n + d$ so that the number of traded assets will be equal to $n + d + 1$.

Recall that we denote by S^c the cumulative price of a first-to-default claim $(X, A, Z, \tau_{(1)})$, where the recovery process Z is n-dimensional, specifically, $Z = (Z^1, \dots, Z^n)$. We already know that if the hypothesis (H) is satisfied by the filtrations $\mathbb{F}$ and $\mathbb{G}$ then the dynamics of S^c under the risk-neutral measure $\mathbb{Q}$ are (see Corollary 3.8.1)

$$dS^c_t = r_t S^c_t\, dt + \sum_{i=1}^n (Z^i_t - \widetilde{S}_t)\, d\widehat{M}^i_t + B_t G^{-1}_{(1)}(t;t)\, dm_t,$$

where the continuous $\mathbb{F}$-martingale m under $\mathbb{Q}$ is given by the formula (see Proposition 3.8.2)

$$\begin{aligned} m_t = {} & \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T B_u^{-1} G_{(1)}(u;u) \Big(\sum_{i=1}^n Z^i_u \widetilde{\lambda}^i_u\, du + dA_u \Big) \,\Big|\, \mathcal{F}_t \Big) \\ & + \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} G_{(1)}(T;T) X \,\Big|\, \mathcal{F}_t \Big). \end{aligned}$$

We adopt the following natural definition of replication of a first-to-default claim.

Definition 3.8.4. We say that a self-financing strategy $\phi = (\phi^0, \dots, \phi^k)$ *replicates* a first-to-default claim $(X, A, Z, \tau_{(1)})$ if its wealth process $V(\phi)$ satisfies the equality $V_{t\wedge\tau_{(1)}}(\phi) = S^c_{t\wedge\tau_{(1)}}$ for any $t \in [0, T]$.

When dealing with replicating strategies in the sense of Definition 3.8.4, we may and do assume, without loss of generality, that the components of the process ϕ are $\mathbb{F}$-predictable processes. More formally, we will search for

$\mathbb{F}$-predictable processes $(\widetilde{\phi}^0, \dots, \widetilde{\phi}^k)$ representing the pre-default components of a trading strategy.

Remark 3.8.2. The property that it is enough to consider $\mathbb{F}$-predictable trading strategies is rather clear from the mathematical point of view, since it is well known that prior to default any $\mathbb{G}$-predictable process is equal to the unique $\mathbb{F}$-predictable process. From the practical point of view, this property is supported by the common intuition that the observations of default times should not be used in the construction of the replicating strategy for a first-to-default claim.

The following auxiliary result is a direct counterpart of Lemma 3.7.3.

Lemma 3.8.2. *We have, for any* $t \in [0, T \wedge \tau_{(1)}]$,

$$dV_t^*(\phi) = \sum_{l=1}^{k} \phi_t^l B_t^{-1} \big(\delta_t^l - \widetilde{S}_t^l(\kappa_l)\big)\, d\widehat{M}_t^{j_l} + \sum_{l=1}^{k} \Big(\sum_{j=1\,, j \neq j_l}^{n} B_t^{-1} \big(S_{t|j}^l(\kappa_l) - \widetilde{S}_t^l(\kappa_l)\big)\, d\widehat{M}_t^j + G_{(1)}^{-1}(t;t)\, dn_t^l \Big),$$

where

$$n_t^l = \mathbb{E}_{\mathbb{Q}} \Big(\int_0^{T_l} B_u^{-1} G_{(1)}(u;u) \Big(\delta_u^l \widetilde{\lambda}_u^{j_l} + \sum_{j=1\,, j \neq j_l}^{n} S_{u|j}^l(\kappa_l) \widetilde{\lambda}_u^j - \kappa_l \Big)\, du \,\Big|\, \mathcal{F}_t \Big).$$

Proof. The proof of the lemma easily follows from Lemma 3.7.2 combined with Corollary 3.8.3. The details are left to the reader. □

We are now in a position to extend Theorem 3.7.1 to the case of a first-to-default claim on a basket of n credit names. At the same time, Theorem 3.8.1 is also a generalization of Theorem 2.5.1 to the case of a non-trivial reference filtration $\mathbb{F}$.

Before we state the main result of this section, we need to introduce some auxiliary notation. Let $\mathbb{F}$ be generated by a Brownian motion W and let ξ and ζ^l, $l = 1, 2, \dots, k$ be the $\mathbb{R}^d$-valued, $\mathbb{F}$-predictable processes such that the following representations are valid

$$dm_t = \xi_t \, dW_t$$

and

$$dn_t^l = \zeta_t^l \, dW_t.$$

The existence of processes ξ and ζ^l for $l = 1, 2, \dots, k$ is an immediate consequence of the classic predictable representation theorem for the Brownian filtration. Needless to say that these processes are rarely explicitly known.

Theorem 3.8.1. *Assume that the processes* $\widetilde{\phi}^1, \ldots, \widetilde{\phi}^n$ *satisfy, for* $t \in [0, T]$ *and* $i = 1, 2, \ldots, n$

$$\sum_{l=1,\, j_l = i}^{k} \widetilde{\phi}^l_t \big(\delta^l_t - \widetilde{S}^l_t(\kappa_l)\big) + \sum_{l=1,\, j_l \neq i}^{k} \widetilde{\phi}^l_t \big(S^l_{t|i}(\kappa_l) - \widetilde{S}^l_t(\kappa_l)\big) = Z^i_t - \widetilde{S}_t$$

and

$$\sum_{l=1}^{k} \widetilde{\phi}^l_t \zeta^l_t = \xi_t.$$

Let us set $\phi^i_t = \widetilde{\phi}^i(t \wedge \tau_{(1)})$ *for* $i = 1, 2, \ldots, k$ *and* $t \in [0, T]$. *Let the process* $V(\phi)$ *be given by Lemma 3.8.2 with the initial condition* $V_0(\phi) = S^c_0$ *and let* ϕ^0 *be given by*

$$V_t(\phi) = \phi^0_t B_t + \sum_{l=1}^{k} \phi^l_t S^l_t(\kappa_l).$$

Then the self-financing strategy $\phi = (\phi^0, \ldots, \phi^k)$ *replicates the first-to-default claim* $(X, A, Z, \tau_{(1)})$.

Proof. The proof goes along the similar lines as the proof of Theorem 3.7.1. It suffices to examine replicating strategy on the random interval $[0, T \wedge \tau_{(1)}]$. On the one hand, in view of Lemma 3.8.2, the wealth process of a self-financing strategy ϕ satisfies on $[0, T \wedge \tau_{(1)}]$

$$\begin{aligned} dV^*_t(\phi) &= \sum_{l=1}^{k} \widetilde{\phi}^l_t B^{-1}_t \big(\delta^l_t - \widetilde{S}^l_t(\kappa_l)\big)\, d\widehat{M}^{j_l}_t \\ &\quad + \sum_{l=1}^{k} \Big(\sum_{j=1\,, j \neq j_l}^{n} B^{-1}_t \big(S^l_{t|j}(\kappa_l) - \widetilde{S}^l_t(\kappa_l)\big)\, d\widehat{M}^j_t + G^{-1}_{(1)}(t;t) \zeta^l_t \, dW_t \Big). \end{aligned}$$

On the other hand, the discounted cumulative price of a first-to-default claim $(X, A, Z, \tau_{(1)})$ satisfies on the interval $[0, T \wedge \tau_{(1)}]$

$$dS^{c*}_t = \sum_{i=1}^{n} B^{-1}_t (Z^i_t - S_{t-})\, d\widehat{M}^i_t + (1 - H^{(1)}_t) G^{-1}_{(1)}(t;t) \xi_t \, dW_t.$$

A comparison of the last two formulae leads directly to the stated conditions. To complete the proof, it suffices to verify that the strategy $\phi = (\phi^0, \ldots, \phi^k)$ introduced in the statement of the theorem replicates a first-to-default claim, in the sense of Definition 3.8.4. Since this verification is rather standard, we leave the details to the reader. □

Chapter 4

Hedging of Defaultable Claims

In this chapter, we study hedging strategies for credit derivatives under the assumption that certain primary assets are traded. We follow here Bielecki et al. [15, 17] and we put special emphasis on the PDE approach in a Markovian setup. For related methods and results, the interested reader is referred to Arvanitis and Laurent [7], Blanchet-Scalliet and Jeanblanc [30], Collin-Dufresne and Hugonnier [59], Greenfield [94], Laurent et al. [130], Laurent [131], Petrelli et al. [152], Rutkowski and Yousiph [157], Vaillant [168], and Vellekoop et al. [169].

4.1 Semimartingale Market Model

We assume that we are given a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a (one- or multi-dimensional) standard Brownian motion W and a random time τ, which admits the $\mathbb{F}$-intensity γ under $\mathbb{P}$, where $\mathbb{F}$ is the filtration generated by the process W. Since the default time is assumed to admit the $\mathbb{F}$-intensity, it is not an $\mathbb{F}$-stopping time. Indeed, it is well known that any stopping time with respect to a Brownian filtration is predictable, and thus does not admit an $\mathbb{F}$-intensity.

4.1.1 Dynamics of Asset Prices

We interpret τ as the common default time for all defaultable assets in our model. In what follows, we fix a finite horizon date $T > 0$. For simplicity, we assume that only three primary assets are traded in the market and the dynamics under the historical probability $\mathbb{P}$ of their prices are, for $i = 1, 2, 3$ and every $t \in [0, T]$,

$$dY_t^i = Y_{t-}^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t + \kappa_{i,t}\, dM_t\big), \tag{4.1}$$

where $M_t = H_t - \int_0^{t\wedge\tau} \gamma_u\, du$ is a martingale or, equivalently,

$$dY_t^i = Y_{t-}^i \big((\mu_{i,t} - \kappa_{i,t}\gamma_t \mathbb{1}_{\{t<\tau\}})\, dt + \sigma_{i,t}\, dW_t + \kappa_{i,t}\, dH_t\big). \tag{4.2}$$

The processes $(\mu_i, \sigma_i, \kappa_i) = (\mu_{i,t}, \sigma_{i,t}, \kappa_{i,t},\, t \in \mathbb{R}_+)$, $i = 1,2,3$, are assumed to be $\mathbb{G}$-adapted, where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. In addition, we assume that $Y_0^i > 0$ and $\kappa_i \geq -1$ for any $i = 1,2,3$, so that Y^i are non-negative processes and they are strictly positive prior to τ. In the case of constant coefficients, we have

$$Y_t^i = Y_0^i e^{\mu_i t} e^{\sigma_i W_t - \sigma_i^2 t/2} e^{-\kappa_i \gamma_i (t \wedge \tau)} (1+\kappa_i)^{H_t}.$$

According to Definition 4.1.2 below, replication refers to the behavior of the wealth process $V(\phi)$ on the random interval $[\![0, \tau \wedge T]\!]$ only. Therefore, for the purpose of replication of defaultable claims of the form (X, Z, τ), it is sufficient to consider prices of primary assets stopped at $\tau \wedge T$. This implies that instead of dealing with $\mathbb{G}$-adapted coefficients in (4.1), it suffices to focus on $\mathbb{F}$-adapted coefficients for the price processes stopped at $\tau \wedge T$. However, for the sake of completeness, we will also deal with a T-maturity claim of the form $Y = G(Y_T^1, Y_T^2, Y_T^3, H_T)$ (see Section 4.4 below).

4.1.2 Pre-Default Values

As will become clear in what follows, when dealing with defaultable claims of the form (X, Z, τ), we will be mainly concerned with the pre-default prices. The *pre-default price* $\widetilde{Y}^i$ of the ith asset is an $\mathbb{F}$-adapted, continuous process, given by the equation, for $i = 1,2,3$ and $t \in [0, T]$,

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i \big((\mu_{i,t} - \kappa_{i,t}\gamma_t)\, dt + \sigma_{i,t}\, dW_t \big) \tag{4.3}$$

with $\widetilde{Y}_0^i = Y_0^i$. Put another way, $\widetilde{Y}^i$ is the unique $\mathbb{F}$-predictable process such that the equality

$$\widetilde{Y}_t^i \mathbb{1}_{\{t \leq \tau\}} = Y_t^i \mathbb{1}_{\{t \leq \tau\}}$$

holds for every $t \in \mathbb{R}_+$. When dealing with the pre-default prices, we may and do assume, without loss of generality, that the processes μ_i, σ_i and κ_i are $\mathbb{F}$-predictable.

Let us stress that the historically observed drift coefficient is $\mu_{i,t} - \kappa_{i,t}\gamma_t$, which appears in (4.2), rather than the drift $\mu_{i,t}$, which appears (4.1). The drift coefficient $\mu_{i,t}$ is already credit-risk adjusted in the sense of our model and it is not directly observed. This convention was chosen here for the sake of simplicity of notation. It also lends itself to the following intuitive interpretation: if ϕ^i is the number of units of the ith asset held in our portfolio at time t then the gains/losses from trades in this asset, prior to default time, can be represented by the differential

$$\phi_t^i\, d\widetilde{Y}_t^i = \phi_t^i \widetilde{Y}_t^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t \big) - \phi_t^i \widetilde{Y}_t^i \kappa_{i,t} \gamma_t\, dt.$$

The last term in the formula above may be formally treated as an effect of dividends that are paid continuously at the random dividend rate $\kappa_{i,t}\gamma_t$.

This nice interpretation is not necessarily useful in practice, since the quantity $\kappa_{i,t}\gamma_t$ cannot be observed directly and, as is well known, a reliable estimation of the drift coefficient in dynamics (4.3) is extremely difficult anyway. Moreover, it is a delicate issue how to disentangle in practice the two components of the drift coefficient in (4.3). Still, if this formal interpretation is adopted, it is sometimes possible to make use of the standard results concerning the valuation of derivatives of dividend-paying assets.

We shall argue below that, although there is formally nothing wrong with the dividend-based approach, a more pertinent and simpler approach to hedging of defaultable claims hinges on the assumption that only the *effective drift*, which is given by the expression

$$\widehat{\mu}_{i,t} = \mu_{i,t} - \kappa_{i,t}\gamma_t,$$

is observable. Moreover, in practical approach to hedging, the values of drift coefficients in dynamics of asset prices will play no essential role, so that we will not postulate that they are among market observables.

4.1.3 Market Observables

To summarize, we assume throughout that the *market observables* are: the pre-default market prices of primary assets, their volatilities and correlations, as well as the jump coefficients $\kappa_{i,t}$ (the financial interpretation of jump coefficients is examined in the next subsection). To summarize, we postulate that under the statistical probability $\mathbb{P}$ the processes Y^i, $i = 1, 2, 3$ satisfy

$$dY^i_t = Y^i_{t-}\big(\widetilde{\mu}_{i,t}\,dt + \sigma_{i,t}\,dW_t + \kappa_{i,t}\,dH_t\big)$$

where the drift terms $\widetilde{\mu}_{i,t}$ are not observed, but we can observe the volatilities $\sigma_{i,t}$ (and thus the asset correlations) and we have an a priori assessment of jump coefficients $\kappa_{i,t}$. In this general setup, the most natural assumption is that the dimension of a driving Brownian motion W coincide with the number of tradable assets. However, for the sake of simplicity of presentation, we will frequently assume that the process W is one-dimensional.

One of our goals will be to establish closed-form expressions for replicating strategies for derivative securities in terms of market observables only (whenever replication of a given claim is actually feasible). To achieve this goal, we shall combine a general theory of hedging defaultable claims within a continuous semimartingale setup, with a judicious specification of particular models with deterministic volatilities and correlations.

4.1.4 Recovery Schemes

It is clear that the sample paths of price processes Y^i are continuous, except for a possible discontinuity at time τ. Specifically, we have that

$$\Delta Y^i_\tau := Y^i_\tau - Y^i_{\tau-} = \kappa_{i,\tau} Y^i_{\tau-},$$

so that the value of Y^i at τ is given by

$$Y^i_\tau = Y^i_{\tau-}(1+\kappa_{i,\tau}) = \widetilde{Y}^i_{\tau-}(1+\kappa_{i,\tau}).$$

A primary asset Y^i is termed a *default-free asset* (*defaultable asset*, respectively) if $\kappa_i = 0$ ($\kappa_i \neq 0$, respectively). In the special case when $\kappa_i = -1$, we say that a defaultable asset Y^i is subject to the *zero recovery* scheme, since its price drops to zero at time τ and remains null at any later date. Such an asset ceases to exist after default, in the sense that it is no longer traded after default. This feature makes the case of a zero recovery essentially different from other cases, as we shall see in the sequel.

In the market practice, it is much more common for a credit derivative to deliver a positive recovery if default event occurs during the contract's lifetime (for instance, a *protection payment* of a credit default swap).

Formally, the value of recovery at default is given as the value of some predetermined stochastic process, that is, it is equal to the value at time τ of some $\mathbb{F}$-adapted *recovery process* Z.

For instance, the recovery process Z can be equal to δ, where δ is a constant, or to $g(t,\delta Y_t)$ where g is a deterministic function and $(Y_t,\, t\in\mathbb{R}_+)$ is the price process of some default-free asset. Typically, the recovery is paid at default time, but it is sometimes postponed to the maturity date.

Let us observe that the case where a defaultable asset Y^i pays a predetermined recovery at default is covered by our setup defined in (4.1). For example, the case of a constant recovery payoff $\delta_i \geq 0$ at default time τ corresponds to the process $\kappa_{i,t} = \delta_i (Y^i_{t-})^{-1} - 1$. Under this convention, the price Y^i is governed under $\mathbb{P}$ by the SDE

$$dY^i_t = Y^i_{t-}\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW_t + (\delta_i (Y^i_{t-})^{-1} - 1)\,dM_t\big).$$

If the recovery is proportional to the pre-default value $Y^i_{\tau-}$ and it is paid at default time τ (this scheme is known as the *fractional recovery of market value*), we set $\kappa_{i,t} = \delta_i - 1$ and thus

$$dY^i_t = Y^i_{t-}\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW_t + (\delta_i - 1)\,dM_t\big).$$

4.1.5 Defaultable Claims

For the purpose of this chapter, it will be enough to define a generic defaultable claim as follows (note that, formally, it suffices to set $A = 0$ in Definition 3.7.1 of a defaultable claim with promised dividends).

Definition 4.1.1. A *defaultable claim* with maturity date T is represented by a triplet (X, Z, τ), where:
(i) the *default time* τ specifies the random time of default, and thus also the default events $\{t \geq \tau\}$ for every $t \in [0, T]$,
(ii) the *promised payoff* $X \in \mathcal{F}_T$ represents the random payoff received by the owner of the claim at time T, provided that there was no default prior to or at time T; the actual payoff at time T associated with X thus equals $X\mathbb{1}_{\{T<\tau\}}$,
(iii) the $\mathbb{F}$-adapted *recovery process* $(Z_t,\, t \in [0, T])$ specifies the recovery payoff Z_τ received by the owner of a claim at time of default (or at maturity), provided that the default occurred prior to or at maturity date T.

In practice, hedging of a credit derivative after default time is usually of minor interest. Also, in a model with a single default time, hedging after default reduces to replication of a non-defaultable claim. It is thus natural to define the replication of a defaultable claim in the following way.

Definition 4.1.2. We say that a self-financing strategy ϕ replicates a defaultable claim (X, Z, τ) if its wealth process $(V_t(\phi),\, t \in [0, T])$ satisfies $V_T(\phi)\mathbb{1}_{\{T<\tau\}} = X\mathbb{1}_{\{T<\tau\}}$ and $V_\tau(\phi)\mathbb{1}_{\{T\geq\tau\}} = Z_\tau\mathbb{1}_{\{T\geq\tau\}}$.

When dealing with replicating strategies, in the sense of Definition 4.1.2, we will always assume, without loss of generality, that the components of the process ϕ are $\mathbb{F}$-predictable processes, rather than $\mathbb{G}$-predictable.

4.2 Trading Strategies

In this section, we consider a fairly general setup. In particular, processes $(Y^i_t,\, t \in [0, T])$ for $i = 1, 2, 3$ are assumed to be non-negative semimartingales on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with some filtration $\mathbb{G}$.

We assume that Y^1, Y^2 and Y^3 represent spot prices of traded assets in our model of the financial market. Neither the existence of a savings account nor the market completeness are postulated, in general. We will restrict here our attention to the special case where only three primary assets are traded. The general case of k traded primary assets with semimartingale prices was examined in papers by Bielecki et al. [16, 18].

Our goal is to characterize contingent claims which are *attainable*, in the sense that they can be replicated by continuously rebalanced portfolios consisting of primary assets. Here, by a contingent claim we mean an arbitrary $\mathcal{G}_T$-measurable random variable. We will work throughout under the standard assumptions of a frictionless market (no transaction costs or taxes, no restrictions on the short sale of assets, perfect liquidity, etc.)

4.2.1 Unconstrained Strategies

Let $\phi = (\phi^1, \phi^2, \phi^3)$ be a trading strategy; in particular, each process ϕ^i is predictable with respect to the filtration $\mathbb{G}$. The corresponding wealth process $(V_t(\phi),\ t \in [0,T])$ is defined by the formula, for every $t \in [0,T]$,

$$V_t(\phi) = \sum_{i=1}^{3} \phi^i_t Y^i_t.$$

A trading strategy ϕ is said to be *self-financing* if the wealth process satisfies, for every $t \in [0,T]$,

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{3} \int_{]0,t]} \phi^i_u \, dY^i_u.$$

Let Φ stand for the class of all self-financing trading strategies. We shall first prove that a self-financing strategy is determined by its initial wealth and the two components ϕ^2, ϕ^3. To this end, we postulate that the price of Y^1 follows a strictly positive process and we choose Y^1 as a numéraire asset. We shall now analyze the relative values, V^1 and $Y^{i,1}$, which are given by

$$V^1_t(\phi) = V_t(\phi)(Y^1_t)^{-1}, \quad Y^{i,1}_t = Y^i_t (Y^1_t)^{-1}.$$

Lemma 4.2.1. *(i) For any $\phi \in \Phi$, we have, for every $t \in [0,T]$,*

$$V^1_t(\phi) = V^1_0(\phi) + \sum_{i=2}^{3} \int_{]0,t]} \phi^i_u \, dY^{i,1}_u.$$

(ii) Conversely, let X be a $\mathcal{G}_T$-measurable random variable, and let us assume that there exists $x \in \mathbb{R}$ and $\mathbb{G}$-predictable processes ϕ^i, $i = 2, 3$ such that

$$X = Y^1_T \Big(x + \sum_{i=2}^{3} \int_{]0,T]} \phi^i_u \, dY^{i,1}_u \Big). \tag{4.4}$$

Then there exists a $\mathbb{G}$-predictable process ϕ^1 such that the trading strategy $\phi = (\phi^1, \phi^2, \phi^3)$ is self-financing and replicates X. Moreover, the wealth process of ϕ (that is, the price of X at time t) satisfies $V_t(\phi) = V^1_t Y^1_t$, where, for every $t \in [0,T]$,

$$V^1_t = x + \sum_{i=2}^{3} \int_{]0,t]} \phi^i_u \, dY^{i,1}_u. \tag{4.5}$$

Proof. In the case of continuous semimartingales, the result is well known; the demonstration for discontinuous semimartingales is not much different. Nevertheless, for the reader's convenience, we provide a detailed proof.

Let us first introduce some notation. As usual, $[X,Y]$ stands for the *quadratic covariation* (the *bracket*) of the two semimartingales X and Y, as formally defined by the Itô integration by parts formula

$$X_tY_t = X_0Y_0 + \int_{]0,t]} X_{u-}\,dY_u + \int_{]0,t]} Y_{u-}\,dX_u + [X,Y]_t.$$

For any càdlàg process Y, we denote by $\Delta Y_t = Y_t - Y_{t-}$ the size of the jump at time t. Let $V = V(\phi)$ be the value of a self-financing strategy and let $V^1 = V^1(\phi) = V(\phi)(Y^1)^{-1}$ be its value relative to the numéraire Y^1. The integration by parts formula yields

$$dV_t^1 = V_{t-}\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}dV_t + d[(Y^1)^{-1},V]_t.$$

From the self-financing condition, we have $dV_t = \sum_{i=1}^3 \phi_t^i\,dY_t^i$. Hence, using elementary rules to compute the quadratic covariation $[X,Y]$ of the two semimartingales X,Y, we obtain

$$\begin{aligned}
dV_t^1 &= \phi_t^1Y_{t-}^1\,d(Y_t^1)^{-1} + \phi_t^2Y_{t-}^2\,d(Y_t^1)^{-1} + \phi_t^3Y_{t-}^3\,d(Y_t^1)^{-1}\\
&\quad + (Y_{t-}^1)^{-1}\phi_t^1\,dY_t^1 + (Y_{t-}^1)^{-1}\phi_t^2\,dY_t^1 + (Y_{t-}^1)^{-1}\phi_t^3\,dY_t^1\\
&\quad + \phi_t^1\,d[(Y^1)^{-1},Y^1]_t + \phi_t^2\,d[(Y^1)^{-1},Y^2]_t + \phi_t^3\,d[(Y^1)^{-1},Y^1]_t\\
&= \phi_t^1\big(Y_{t-}^1\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_t^1 + d[(Y^1)^{-1},Y^1]_t\big)\\
&\quad + \phi_t^2\big(Y_{t-}^2\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_{t-}^1 + d[(Y^1)^{-1},Y^2]_t\big)\\
&\quad + \phi_t^3\big(Y_{t-}^3\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_{t-}^1 + d[(Y^1)^{-1},Y^3]_t\big).
\end{aligned}$$

We now observe that

$$Y_{t-}^1\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_t^1 + d[(Y^1)^{-1},Y^1]_t = d(Y_t^1(Y_t^1)^{-1}) = 0$$

and

$$Y_{t-}^i\,d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1}\,dY_t^i + d[(Y^1)^{-1},Y^i]_t = d((Y_t^1)^{-1}Y_t^i).$$

Consequently,

$$dV_t^1 = \phi_t^2\,dY_t^{2,1} + \phi_t^3\,dY_t^{3,1},$$

as was claimed in part (i). We now proceed to the proof of part (ii). We assume that (4.4) holds for some constant x and processes ϕ^2,ϕ^3 and we define the process V^1 by setting, for every $t\in[0,T]$ (cf. (4.5)),

$$V_t^1 = x + \sum_{i=2}^3 \int_{]0,t]} \phi_u^i\,dY_u^{i,1}.$$

Next, we define the process ϕ^1 as follows

$$\phi^1_t = V^1_t - \sum_{i=2}^3 \phi^i_t Y^{i,1}_t = (Y^1_t)^{-1}\Big(V_t - \sum_{i=2}^3 \phi^i_t Y^i_t\Big),$$

where we set $V_t = V^1_t Y^1_t$ for $t \in [0,T]$. Since

$$dV^1_t = \sum_{i=2}^3 \phi^i_t \, dY^{i,1}_t,$$

for the process V we obtain

$$\begin{aligned} dV_t &= d(V^1_t Y^1_t) = V^1_{t-} dY^1_t + Y^1_{t-} dV^1_t + d[Y^1, V^1]_t \\ &= V^1_{t-} dY^1_t + \sum_{i=2}^3 \phi^i_t \big(Y^1_{t-}\, dY^{i,1}_t + d[Y^1, Y^{i,1}]_t\big). \end{aligned}$$

From the Itô integration by parts formula, we obtain

$$dY^i_t = d(Y^{i,1}_t Y^1_t) = Y^{i,1}_{t-}\, dY^1_t + Y^1_{t-}\, dY^{i,1}_t + d[Y^1, Y^{i,1}]_t,$$

and thus

$$\begin{aligned} dV_t &= V^1_{t-}\, dY^1_t + \sum_{i=2}^3 \phi^i_t \big(dY^i_t - Y^{i,1}_{t-}\, dY^1_t\big) \\ &= \Big(V^1_{t-} - \sum_{i=2}^3 \phi^i_t Y^{i,1}_{t-}\Big) dY^1_t + \sum_{i=2}^3 \phi^i_t \, dY^i_t. \end{aligned}$$

Our aim was to prove that $dV_t = \sum_{i=1}^3 \phi^i_t \, dY^i_t$. The last equality is indeed satisfied if

$$\phi^1_t = V^1_t - \sum_{i=2}^3 \phi^i_t Y^{i,1}_t = V^1_{t-} - \sum_{i=2}^3 \phi^i_t Y^{i,1}_{t-}, \tag{4.6}$$

that is, provided that

$$\Delta V^1_t = \sum_{i=2}^3 \phi^i_t \Delta Y^{i,1}_t,$$

which is satisfied, in view of definition (4.5) of V^1. Note also that, from the second equality in (4.6), we deduce that the process ϕ^1 is $\mathbb{G}$-predictable. Finally, the wealth process of ϕ satisfies $V_t(\phi) = V^1_t Y^1_t$ for every $t \in [0,T]$ and thus $V_T(\phi) = X$. □

We say that a self-financing strategy ϕ *replicates* a claim $X \in \mathcal{G}_T$ if

$$X = \sum_{i=1}^{3} \phi_T^i Y_T^i = V_T(\phi)$$

or, equivalently,

$$X = V_0(\phi) + \sum_{i=1}^{3} \int_{]0,T]} \phi_t^i \, dY_t^i.$$

Suppose that there exists an EMM for some choice of a numéraire asset, and let us restrict our attention to the class of all *admissible* trading strategies, so that our model is arbitrage-free.

Assume that a claim X can be replicated by some admissible trading strategy, so that it is *attainable* (or *hedgeable*). Then, by definition, the *arbitrage price* at time t of X, denoted as $\pi_t(X)$, equals $V_t(\phi)$ for any admissible trading strategy ϕ that replicates X.

In the context of Lemma 4.2.1, it is natural to choose as an EMM a probability measure $\mathbb{Q}^1$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ and such that the prices $Y^{i,1}$, $i = 2, 3$, are $\mathbb{G}$-martingales under $\mathbb{Q}^1$. If a contingent claim X is attainable then its arbitrage price satisfies

$$\pi_t(X) = Y_t^1 \, \mathbb{E}_{\mathbb{Q}^1}(X(Y_T^1)^{-1} \,|\, \mathcal{G}_t). \tag{4.7}$$

We emphasize that even when an EMM $\mathbb{Q}^1$ is not unique, the price of any attainable claim X is given by the conditional expectation above. Put another way, in the case of an attainable claim, the conditional expectations (4.7) under various equivalent martingale measures coincide.

4.2.2 Constrained Strategies

In this section, we make an additional assumption that the price process Y^3 is strictly positive. Let $\phi = (\phi^1, \phi^2, \phi^3)$ be a self-financing trading strategy satisfying the following constraint

$$\sum_{i=1}^{2} \phi_t^i Y_{t-}^i = Z_t, \quad \forall \, t \in [0, T], \tag{4.8}$$

for a predetermined, $\mathbb{G}$-predictable process Z. In the financial interpretation, equality (4.8) means that a portfolio ϕ is rebalanced in such a way that the total wealth invested in assets Y^1, Y^2 matches a predetermined stochastic process Z. For this reason, the constraint given by (4.8) is referred to as the *balance condition*.

Our first goal is to extend part (i) in Lemma 4.2.1 to the case of constrained strategies. Let $\Phi(Z)$ stand for the class of all (admissible) self-financing trading strategies that satisfy the balance condition (4.8). They will be sometimes referred to as *constrained strategies*. Since any strategy $\phi \in \Phi(Z)$ is self-financing, from $dV_t(\phi) = \sum_{i=1}^3 \phi_t^i \, dY_t^i$, we obtain

$$\Delta V_t(\phi) = \sum_{i=1}^3 \phi_t^i \Delta Y_t^i = V_t(\phi) - \sum_{i=1}^3 \phi_t^i Y_{t-}^i.$$

By combining this equality with (4.8), we deduce that

$$V_{t-}(\phi) = \sum_{i=1}^3 \phi_t^i Y_{t-}^i = Z_t + \phi_t^3 Y_{t-}^i.$$

Let us write

$$Y_t^{i,3} = Y_t^i (Y_t^3)^{-1}, \quad Z_t^3 = Z_t (Y_t^3)^{-1}.$$

The following result extends Lemma 1.7 in Bielecki et al. [13] from the case of continuous semimartingales to the general case (see also [16, 18]). It is apparent from Proposition 4.2.1 that the wealth process $V(\phi)$ of a strategy $\phi \in \Phi(Z)$ depends only on a single component of ϕ, namely, ϕ^2.

Proposition 4.2.1. *The relative wealth $V_t^3(\phi) = V_t(\phi)(Y_t^3)^{-1}$ of any trading strategy $\phi \in \Phi(Z)$ satisfies*

$$V_t^3(\phi) = V_0^3(\phi) + \int_{]0,t]} \phi_u^2 \left(dY_u^{2,3} - \frac{Y_{u-}^{2,3}}{Y_{u-}^{1,3}} \, dY_u^{1,3} \right) + \int_{]0,t]} \frac{Z_u^3}{Y_{u-}^{1,3}} \, dY_u^{1,3}. \quad (4.9)$$

Proof. Let us consider discounted values of price processes Y^1, Y^2, Y^3, with Y^3 taken as a numéraire asset. By virtue of part (i) in Lemma 4.2.1, we thus have

$$V_t^3(\phi) = V_0^3(\phi) + \sum_{i=1}^2 \int_{]0,t]} \phi_u^i \, dY_u^{i,3}. \quad (4.10)$$

The balance condition (4.8) implies that

$$\sum_{i=1}^2 \phi_t^i Y_{t-}^{i,3} = Z_t^3,$$

and thus

$$\phi_t^1 = (Y_{t-}^{1,3})^{-1} \Big(Z_t^3 - \phi_t^2 Y_{t-}^{2,3} \Big). \quad (4.11)$$

By inserting (4.11) into (4.10), we arrive at the asserted formula (4.9). □

The next result will prove particularly useful for deriving replicating strategies for defaultable claims.

Proposition 4.2.2. *Let a $\mathcal{G}_T$-measurable random variable X represent a contingent claim that settles at time T. We set*

$$dY_t^* = dY_t^{2,3} - \frac{Y_{t-}^{2,3}}{Y_{t-}^{1,3}}\, dY_t^{1,3} = dY_t^{2,3} - Y_{t-}^{2,1}\, dY_t^{1,3}, \tag{4.12}$$

where, by convention, the initial value is $Y_0^* = 0$. Assume that there exists a $\mathbb{G}$-predictable process ϕ^2, such that

$$X = Y_T^3 \left(x + \int_{]0,T]} \phi_t^2\, dY_t^* + \int_{]0,T]} \frac{Z_t^3}{Y_{t-}^{1,3}}\, dY_t^{1,3} \right). \tag{4.13}$$

Then there exist $\mathbb{G}$-predictable processes ϕ^1 and ϕ^3 such that the strategy $\phi = (\phi^1, \phi^2, \phi^3)$ belongs to $\Phi(Z)$ and replicates X. The wealth process of ϕ equals, for every $t \in [0,T]$,

$$V_t(\phi) = Y_t^3 \left(x + \int_{]0,t]} \phi_u^2\, dY_u^* + \int_{]0,t]} \frac{Z_u^3}{Y_{u-}^{1,3}}\, dY_u^{1,3} \right).$$

Proof. As expected, we first set (note that the component ϕ^1 follows a $\mathbb{G}$-predictable process)

$$\phi_t^1 = \frac{1}{Y_{t-}^1} \Big(Z_t - \phi_t^2 Y_{t-}^2 \Big) \tag{4.14}$$

and

$$V_t^3 = x + \int_{]0,t]} \phi_u^2\, dY_u^* + \int_{]0,t]} \frac{Z_u^3}{Y_{u-}^{1,3}}\, dY_u^{1,3}.$$

Arguing along the same lines as in the proof of Proposition 4.2.1, we obtain

$$V_t^3 = V_0^3 + \sum_{i=1}^{2} \int_{]0,t]} \phi_u^i\, dY_u^{i,3}.$$

Now, we define

$$\phi_t^3 = V_t^3 - \sum_{i=1}^{2} \phi_t^i Y_t^{i,3} = (Y_t^3)^{-1} \Big(V_t - \sum_{i=1}^{2} \phi_t^i Y_t^i \Big),$$

where $V_t = V_t^3 Y_t^3$. As in the proof of Lemma 4.2.1, we check that

$$\phi_t^3 = V_{t-}^3 - \sum_{i=1}^{2} \phi_t^i Y_{t-}^{i,3},$$

and thus the process ϕ^3 is $\mathbb{G}$-predictable. It is clear that the strategy $\phi = (\phi^1, \phi^2, \phi^3)$ is self-financing and its wealth process satisfies $V_t(\phi) = V_t$ for every $t \in [0,T]$. In particular, $V_T(\phi) = X$, so that ϕ replicates X. Finally, equality (4.14) implies (4.8) and thus ϕ belongs to the class $\Phi(Z)$. □

Note that equality (4.13) is a necessary (by Lemma 4.2.1) and sufficient (by Proposition 4.2.2) condition for the existence of a constrained strategy that replicates a given contingent claim X.

4.2.3 Synthetic Asset

Let us take $Z = 0$ so that $\phi \in \Phi(0)$. Then the balance condition becomes $\sum_{i=1}^2 \phi_t^i Y_{t-}^i = 0$ and formula (4.9) reduces to

$$dV_t^3(\phi) = \phi_t^2 \left(dY_t^{2,3} - \frac{Y_{t-}^{2,3}}{Y_{t-}^{1,3}}\, dY_t^{1,3} \right). \tag{4.15}$$

The process $\bar{Y}^2 = Y^3 Y^*$, where Y^* is defined in (4.12) is called a *synthetic asset*. It corresponds to a particular self-financing portfolio, with the long position in Y^2, the short position of $Y_{t-}^{2,1}$ number of shares of Y^1, and suitably re-balanced positions in the third asset, so that the portfolio is self-financing, as in Lemma 4.2.1.

It is not difficult to show (see Bielecki et al. [16, 18]) that trading in primary assets Y^1, Y^2, Y^3 is formally equivalent to trading in assets $Y^1, \bar{Y}^2, Y^3$. This observation supports the name synthetic asset attributed to the process $\bar{Y}^2$. It is worth noting, however, that the synthetic asset process may take negative values, so that it is unsuitable as a numéraire, in general.

Case of Continuous Asset Prices

In the case of continuous asset prices, the relative price $Y^* = \bar{Y}^2 (Y^3)^{-1}$ of the synthetic asset can be given an alternative representation, as the following result shows. Recall that the *predictable bracket* of the two continuous semimartingales X and Y, denoted as $\langle X, Y \rangle$, coincides with their quadratic covariation $[X, Y]$.

Proposition 4.2.3. *Assume that the price processes Y^1 and Y^2 are continuous. Then the relative price of the synthetic asset satisfies*

$$Y_t^* = \int_0^t (Y_u^{3,1})^{-1} e^{\alpha_u}\, d\widehat{Y}_u,$$

where we denote $\widehat{Y}_t = Y_t^{2,1} e^{-\alpha_t}$ and

$$\alpha_t = \langle \ln Y^{2,1}, \ln Y^{3,1} \rangle_t = \int_0^t (Y_u^{2,1})^{-1} (Y_u^{3,1})^{-1}\, d\langle Y^{2,1}, Y^{3,1} \rangle_u. \tag{4.16}$$

In terms of the auxiliary process $\widehat{Y}$, formula (4.9) becomes

$$V_t^3(\phi) = V_0^3(\phi) + \int_0^t \widehat{\phi}_u \, d\widehat{Y}_u + \int_0^t \frac{Z_u^3}{Y_{u-}^{1,3}} \, dY_u^{1,3},$$

where $\widehat{\phi}_t = \phi_t^2 (Y_t^{3,1})^{-1} e^{\alpha_t}$.

Proof. It suffices to give the proof for $Z = 0$. The proof relies on the integration by parts formula stating that we have, for any two continuous semimartingales, say X and Y,

$$Y_t^{-1}\big(dX_t - Y_t^{-1} d\langle X, Y\rangle_t\big) = d(X_t Y_t^{-1}) - X_t \, dY_t^{-1},$$

provided that Y is strictly positive. By applying this formula to processes $X = Y^{2,1}$ and $Y = Y^{3,1}$, we obtain

$$(Y_t^{3,1})^{-1}(dY_t^{2,1} - (Y_t^{3,1})^{-1} d\langle Y^{2,1}, Y^{3,1}\rangle_t) = d(Y_t^{2,1}(Y_t^{3,1})^{-1}) - Y_t^{2,1} \, d(Y^{3,1})_t^{-1}.$$

The relative wealth $V_t^3(\phi) = V_t(\phi)(Y_t^3)^{-1}$ of a strategy $\phi \in \Phi(0)$ satisfies

$$\begin{aligned} V_t^3(\phi) &= V_0^3(\phi) + \int_0^t \phi_u^2 \, dY_u^* \\ &= V_0^3(\phi) + \int_0^t \phi_u^2 (Y_u^{3,1})^{-1} e^{\alpha_u} \, d\widehat{Y}_u, \\ &= V_0^3(\phi) + \int_0^t \widehat{\phi}_u \, d\widehat{Y}_u \end{aligned}$$

where we denote $\widehat{\phi}_t = \phi_t^2 (Y_t^{3,1})^{-1} e^{\alpha_t}$. □

Remark 4.2.1. The financial interpretation of the auxiliary process $\widehat{Y}$ will be studied below. Let us only observe here that if Y^* is a local martingale under some probability $\mathbb{Q}$ then $\widehat{Y}$ is a $\mathbb{Q}$-local martingale (and vice versa, if $\widehat{Y}$ is a $\widehat{\mathbb{Q}}$-local martingale under some probability $\widehat{\mathbb{Q}}$ then Y^* is a $\widehat{\mathbb{Q}}$-local martingale). Nevertheless, for the reader's convenience, we shall use two symbols $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$, since this equivalence holds for continuous processes only.

Remark 4.2.2. It is thus worth stressing that we will apply Proposition 4.2.3 to pre-default values of assets, rather than directly to asset prices, within the setup of a semimartingale model with a common default, as described in Section 4.1.1. In this model, the asset prices may have discontinuities, but their pre-default values follow continuous processes.

4.3 Martingale Approach

Our goal is to derive quasi-explicit conditions for replicating strategies for a defaultable claim in a fairly general setup introduced in Section 4.1.1. In this section, we only deal with trading strategies based on the reference filtration $\mathbb{F}$ and the underlying price processes (that is, prices of default-free assets and pre-default values of defaultable assets) are assumed to be continuous. Therefore, our arguments will hinge on Proposition 4.2.3, rather than on a more general Proposition 4.2.1. We shall also adapt Proposition 4.2.2 to our current purposes.

To simplify the presentation, we make the standing assumption that all coefficient processes are such that the SDEs, which appear in what follows, admit unique strong solutions and the Doléans exponentials (the Radon-Nikodým derivatives) are true martingales under respective probabilities.

4.3.1 Defaultable Asset with Zero Recovery

We will first examine in some detail a particular model where the two assets, Y^1 and Y^2, are default-free and satisfy, for $i = 1, 2$,

$$dY_t^i = Y_t^i \big(\mu_{i,t} \, dt + \sigma_{i,t} \, dW_t \big),$$

where W is a one-dimensional Brownian motion. The third asset is a defaultable asset with zero recovery, so that

$$dY_t^3 = Y_{t-}^3 \big(\mu_{3,t} \, dt + \sigma_{3,t} \, dW_t - dM_t \big).$$

Since we will be interested in replicating strategies in the sense of Definition 4.1.2, we may and do assume, without loss of generality, that the coefficients $\mu_{i,t}$, $\sigma_{i,t}$, $i = 1, 2$, are $\mathbb{F}$-predictable, rather than $\mathbb{G}$-predictable. Recall that, in general, there exist $\mathbb{F}$-predictable processes $\widetilde{\mu}_3$ and $\widetilde{\sigma}_3$ such that

$$\widetilde{\mu}_{3,t} \mathbb{1}_{\{t \le \tau\}} = \mu_{3,t} \mathbb{1}_{\{t \le \tau\}}, \quad \widetilde{\sigma}_{3,t} \mathbb{1}_{\{t \le \tau\}} = \sigma_{3,t} \mathbb{1}_{\{t \le \tau\}}.$$

We assume throughout that $Y_0^i > 0$ for every i, so that the price processes Y^1, Y^2 are strictly positive and the process Y^3 is non-negative and has strictly positive pre-default value.

4.3.2 Default-Free Market

It is natural to postulate that the default-free market with two traded assets, Y^1 and Y^2, is arbitrage-free. To be more specific, we choose Y^1 as a numéraire and we require the existence of a probability measure $\mathbb{P}^1$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, and such that the process $Y^{2,1}$ is a $\mathbb{P}^1$-martingale.

It is easy to check that the dynamics of processes $(Y^1)^{-1}$ and $Y^{2,1}$ are

$$d(Y_t^1)^{-1} = (Y_t^1)^{-1}\big((\sigma_{1,t}^2 - \mu_{1,t})\,dt - \sigma_{1,t}\,dW_t\big), \tag{4.17}$$

and

$$dY_t^{2,1} = Y_t^{2,1}\big((\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t}))\,dt + (\sigma_{2,t} - \sigma_{1,t})\,dW_t\big),$$

respectively. Hence the necessary condition for the existence of an EMM $\mathbb{P}^1$ is the inclusion $A \subseteq B$, where $A = \{(t,\omega) \in [0,T] \times \Omega : \sigma_{1,t}(\omega) = \sigma_{2,t}(\omega)\}$ and $B = \{(t,\omega) \in [0,T] \times \Omega : \mu_{1,t}(\omega) = \mu_{2,t}(\omega)\}$. The necessary and sufficient condition for the existence and uniqueness of an EMM $\mathbb{P}^1$ reads

$$\mathbb{E}_{\mathbb{P}}\left\{\mathcal{E}_T\left(\int_0^{\cdot} \theta_u\,dW_u\right)\right\} = 1 \tag{4.18}$$

where the process θ is given by the formula, for every $t \in [0,T]$,

$$\theta_t = \sigma_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}}, \tag{4.19}$$

where, by convention, $0/0 = 0$. Note that in the case of constant coefficients, if $\sigma_1 = \sigma_2$ then the considered model is arbitrage-free only in the trivial case when $\mu_2 = \mu_1$.

Remark 4.3.1. Since the martingale measure $\mathbb{P}^1$ is unique, the default-free model (Y^1, Y^2) is complete. However, this assumption is not necessary and thus it can be relaxed. As we shall see in what follows, it is typically more natural to assume that the driving Brownian motion W is multi-dimensional.

4.3.3 Arbitrage-Free Property

Let us now consider also a defaultable asset Y^3. Our goal is now to find a martingale measure $\mathbb{Q}^1$ (if it exists) for relative prices $Y^{2,1}$ and $Y^{3,1}$. Recall that we postulate that the hypothesis (H) holds under $\mathbb{P}$ for filtrations $\mathbb{F}$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. The dynamics of $Y^{3,1}$ under $\mathbb{P}$ are

$$dY_t^{3,1} = Y_{t-}^{3,1}\Big\{\big(\mu_{3,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{3,t})\big)dt + (\sigma_{3,t} - \sigma_{1,t})\,dW_t - dM_t\Big\}.$$

Let $\mathbb{Q}^1$ be any probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ and let η be the associated Radon-Nikodým density process, so that

$$d\mathbb{Q}^1\,|\,_{\mathcal{G}_t} = \eta_t\,d\mathbb{P}\,|\,_{\mathcal{G}_t}, \tag{4.20}$$

where the process η is a $\mathbb{G}$-martingale under $\mathbb{P}$ and satisfies

$$d\eta_t = \eta_{t-}(\theta_t\,dW_t + \zeta_t\,dM_t) \tag{4.21}$$

for some $\mathbb{G}$-predictable processes θ and ζ.

From the Girsanov theorem (cf. Theorem 3.4.1), the processes $\widehat{W}$ and $\widehat{M}$, which are given by the expressions

$$\widehat{W}_t = W_t - \int_0^t \theta_u \, du \tag{4.22}$$

and

$$\widehat{M}_t = M_t - \int_0^{t\wedge\tau} \gamma_u \zeta_u \, du, \tag{4.23}$$

are $\mathbb{G}$-martingales under $\mathbb{Q}^1$.

To ensure that $Y^{2,1}$ is a $\mathbb{Q}^1$-martingale, we postulate that conditions (4.18) and (4.19) are satisfied. Consequently, for the process $Y^{3,1}$ to be a $\mathbb{Q}^1$-martingale, it is necessary and sufficient that a process ζ satisfies

$$\gamma_t \zeta_t = \mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}} (\sigma_{3,t} - \sigma_{1,t}).$$

To ensure that $\mathbb{Q}^1$ is a probability measure equivalent to $\mathbb{P}$, we require that the inequality $\zeta_t > -1$ is valid. Then the unique martingale measure $\mathbb{Q}^1$ is given by formula (4.20) where η solves (4.21), so that

$$\eta_t = \mathcal{E}_t \left(\int_0^{\cdot} \theta_u \, dW_u \right) \mathcal{E}_t \left(\int_{]0,\,\cdot\,]} \zeta_u \, dM_u \right).$$

We are in a position to formulate the following result.

Proposition 4.3.1. *Assume that the process θ given by (4.19) satisfies (4.18) and*

$$\zeta_t = \frac{1}{\gamma_t} \left(\mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}} (\sigma_{3,t} - \sigma_{1,t}) \right) > -1. \tag{4.24}$$

Then the market model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is arbitrage-free and complete. The dynamics of relative prices under the unique martingale measure $\mathbb{Q}^1$ are

$$\begin{aligned} dY_t^{2,1} &= Y_t^{2,1} (\sigma_{2,t} - \sigma_{1,t}) \, d\widehat{W}_t, \\ dY_t^{3,1} &= Y_{t-}^{3,1} \big((\sigma_{3,t} - \sigma_{1,t}) \, d\widehat{W}_t - d\widehat{M}_t \big). \end{aligned}$$

Since the coefficients $\mu_{i,t}$, $\sigma_{i,t}$, $i = 1, 2$, are $\mathbb{F}$-adapted, the process $\widehat{W}$ is an $\mathbb{F}$-martingale (hence, a Brownian motion) under $\mathbb{Q}^1$. Therefore, by virtue of Proposition 3.5.1, the hypothesis (H) holds under $\mathbb{Q}^1$, and the $\mathbb{F}$-intensity of default under $\mathbb{Q}^1$ equals

$$\widehat{\gamma}_t = \gamma_t (1 + \zeta_t) = \gamma_t + \left(\mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}} (\sigma_{3,t} - \sigma_{1,t}) \right).$$

Example 4.3.1. We present an example where the condition (4.24) does not hold and thus arbitrage opportunities arise. Assume that the coefficients are constant and satisfy $\mu_1 = \mu_2 = \sigma_1 = 0$, $\mu_3 < -\gamma$ for a constant default intensity $\gamma > 0$. Then

$$\begin{aligned} Y_t^3 &= \mathbb{1}_{\{t<\tau\}} Y_0^3 \exp\left(\sigma_3 W_t - \frac{1}{2}\sigma_3^2 t + (\mu_3 + \gamma)t\right) \\ &\leq Y_0^3 \exp\left(\sigma_3 W_t - \frac{1}{2}\sigma_3^2 t\right) = V_t(\phi), \end{aligned}$$

where $V(\phi)$ represents the wealth of a self-financing strategy $(\phi^1, \phi^2, 0)$ with $\phi^2 = \frac{\sigma_3}{\sigma_2}$. Hence the arbitrage strategy would be to sell the asset Y^3 and to follow the strategy ϕ.

Remark 4.3.2. Let us stress once again, that the existence of an EMM is a necessary condition for the model viability, but the uniqueness of an EMM is not always a natural condition to be imposed. In fact, when constructing a model, we should be mostly concerned with its flexibility and ability to reflect the pertinent risk factors, rather than with its mathematical completeness. In the present context, it would be natural to postulate that the dimension of the underlying Brownian motion coincides with the number of traded risky assets.

4.3.4 Hedging a Survival Claim

We first focus on replication of a *survival claim* $(X, 0, \tau)$, that is, a defaultable claim represented by the terminal payoff $X\mathbb{1}_{\{T<\tau\}}$, where X is an $\mathcal{F}_T$-measurable random variable. For the moment, we maintain the simplifying assumption that W is one-dimensional. As we shall see in what follows, it may lead to certain pathological features of a model. If, on the contrary, the driving noise is multi-dimensional, most of the analysis remains valid, except that the model completeness is no longer ensured, in general.

Recall that $\widetilde{Y}^3$ stands for the pre-default price of Y^3, defined as follows (see (4.3))

$$d\widetilde{Y}_t^3 = \widetilde{Y}_t^3 \big((\widetilde{\mu}_{3,t} + \gamma_t)\, dt + \widetilde{\sigma}_{3,t}\, dW_t \big)$$

with $\widetilde{Y}_0^3 = Y_0^3$. This strictly positive, continuous, $\mathbb{F}$-adapted process enjoys the property that $Y_t^3 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^3$. Let us denote the pre-default values relative to the numéraire $\widetilde{Y}^3$ by $\widetilde{Y}_t^{i,3} = Y_t^i (\widetilde{Y}_t^3)^{-1}$ for $i = 1, 2$ and let us introduce the pre-default relative price $\widetilde{Y}^*$ of the synthetic asset $\bar{Y}^2$ by setting

$$\begin{aligned} d\widetilde{Y}_t^* &= d\widetilde{Y}_t^{2,3} - \frac{\widetilde{Y}_t^{2,3}}{\widetilde{Y}_t^{1,3}}\, d\widetilde{Y}_t^{1,3} \\ &= \widetilde{Y}_t^{2,3}\Big(\big(\mu_{2,t} - \mu_{1,t} + \widetilde{\sigma}_{3,t}(\sigma_{1,t} - \sigma_{2,t})\big)\, dt + (\sigma_{2,t} - \sigma_{1,t})\, dW_t \Big). \end{aligned}$$

We postulate that $\sigma_{1,t} - \sigma_{2,t} \neq 0$. It is useful to note that the process $\widehat{Y}$ defined in Proposition 4.2.3 satisfies

$$d\widehat{Y}_t = \widehat{Y}_t \Big(\big(\mu_{2,t} - \mu_{1,t} + \widetilde{\sigma}_{3,t}(\sigma_{1,t} - \sigma_{2,t}) \big) \, dt + (\sigma_{2,t} - \sigma_{1,t}) \, dW_t \Big).$$

We are going to show that when α given by (4.16) is deterministic, the process $\widehat{Y}$ has the financial interpretation as the *credit-risk adjusted forward price* of Y^2 relative to Y^1. Therefore, it is more convenient to work with the process $\widetilde{Y}^*$ when dealing with the general case, but to use instead the process $\widehat{Y}$ when analyzing a model with deterministic volatilities.

Consider an $\mathbb{F}$-predictable self-financing strategy ϕ satisfying the balance condition $\phi^1_t Y^1_t + \phi^2_t Y^2_t = 0$, and the corresponding wealth process

$$V_t(\phi) := \sum_{i=1}^{3} \phi^i_t Y^i_t = \phi^3_t Y^3_t.$$

Let us set $\widetilde{V}_t(\phi) := \phi^3_t \widetilde{Y}^3_t$. Since the process $\widetilde{V}(\phi)$ is $\mathbb{F}$-adapted, it is rather clear that it represents the *pre-default price* process of the portfolio ϕ, in the sense that the equality $\mathbb{1}_{\{t<\tau\}} V_t(\phi) = \mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\phi)$ is valid for every $t \in [0, T]$. We shall call the process $\widetilde{V}_t(\phi)$ the *pre-default wealth* of ϕ. Consequently, the process $\widetilde{V}^3_t(\phi) := \widetilde{V}_t(\phi)(\widetilde{Y}^3_t)^{-1} = \phi^3_t$ is termed the relative pre-default wealth.

Using Proposition 4.2.1, with a suitably adjusted notation, we find that the $\mathbb{F}$-adapted process $\widetilde{V}^3(\phi)$ satisfies, for every $t \in [0, T]$,

$$\widetilde{V}^3_t(\phi) = \widetilde{V}^3_0(\phi) + \int_0^t \phi^2_u \, d\widetilde{Y}^*_u.$$

Let us define an equivalent probability measure $\mathbb{Q}^*$ on $(\Omega, \mathcal{F}_T)$ by setting

$$d\mathbb{Q}^* = \eta^*_T \, d\mathbb{P},$$

where $d\eta^*_t = \eta^*_t \theta^*_t \, dW_t$ and

$$\theta^*_t = \frac{\mu_{2,t} - \mu_{1,t} + \widetilde{\sigma}_{3,t}(\sigma_{1,t} - \sigma_{2,t})}{\sigma_{1,t} - \sigma_{2,t}}. \tag{4.25}$$

The process $(\widetilde{Y}^*_t, \, t \in [0, T])$ is a (local) martingale under $\mathbb{Q}^*$ driven by a Brownian motion. We shall require that this process is in fact a true martingale; a sufficient condition for this is that

$$\int_0^T \mathbb{E}_{\mathbb{Q}^*} \Big(\widetilde{Y}^{2,3}_t (\sigma_{2,t} - \sigma_{1,t}) \Big)^2 dt < \infty.$$

From the predictable representation theorem for the Brownian filtration, it follows that for any random variable $X \in \mathcal{F}_T$, such that the random variable $X(\widetilde{Y}^3_T)^{-1}$ is square-integrable under $\mathbb{Q}^*$, there exists a constant x and an $\mathbb{F}$-predictable process ϕ^2 such that

$$X = \widetilde{Y}^3_T \bigg(x + \int_{]0,T]} \phi^2_u \, d\widetilde{Y}^*_u \bigg). \tag{4.26}$$

We now deduce from Proposition 4.2.2 that there exists a self-financing strategy ϕ with the pre-default wealth $\widetilde{V}_t(\phi) = \widetilde{Y}^3_t \widetilde{V}^3_t$ for every $t \in [0,T]$, where we set

$$\widetilde{V}^3_t = x + \int_0^t \phi^2_u \, d\widetilde{Y}^*_u. \tag{4.27}$$

Moreover, the balance condition $\phi^1_t Y^1_t + \phi^2_t Y^2_t = 0$ is satisfied for every $t \in [0,T]$. Since, clearly, $\widetilde{V}_T(\phi) = X$, we have that

$$V_T(\phi) = \phi^3_T Y^3_T = \mathbb{1}_{\{T<\tau\}} \phi^3_T \widetilde{Y}^3_T = \mathbb{1}_{\{T<\tau\}} \widetilde{V}_T(\phi) = \mathbb{1}_{\{T<\tau\}} X.$$

We conclude that the strategy ϕ replicates the survival claim $(X,0,\tau)$. In particular, we have that $V_t(\phi) = 0$ on the random interval $[\![\tau, T \wedge \tau]\!]$.

Definition 4.3.1. We say that a survival claim $(X,0,\tau)$ is *attainable* if the process $\widetilde{V}^3$ given by (4.27) is a martingale under $\mathbb{Q}^*$.

The following result is an immediate consequence of (4.26) and (4.27).

Corollary 4.3.1. *Let $X \in \mathcal{F}_T$ be such that $X(\widetilde{Y}^3_T)^{-1}$ is square-integrable under $\mathbb{Q}^*$. Then the survival claim $(X,0,\tau)$ is attainable. Moreover, the pre-default price $\widetilde{\pi}_t(X,0,\tau)$ of the survival claim $(X,0,\tau)$ is given by the following conditional expectation, for every $t \in [0,T]$,*

$$\widetilde{\pi}_t(X,0,\tau) = \widetilde{Y}^3_t \, \mathbb{E}_{\mathbb{Q}^*}(X(\widetilde{Y}^3_T)^{-1} \,|\, \mathcal{F}_t). \tag{4.28}$$

The process $\widetilde{\pi}(X,0,\tau)(\widetilde{Y}^3)^{-1}$ is an $\mathbb{F}$-martingale under $\mathbb{Q}^*$.

Proof. Since $X(\widetilde{Y}^3_T)^{-1}$ is square-integrable under $\mathbb{Q}^*$, we know from the predictable representation theorem for the Brownian filtration that the process ϕ^2 in formula (4.26) is such that

$$\mathbb{E}_{\mathbb{Q}^*} \bigg(\int_0^T (\phi^2_t)^2 \, d\langle \widetilde{Y}^* \rangle_t \bigg) < \infty.$$

Therefore, the process $\widetilde{V}^3$ given by (4.27) is a true martingale under $\mathbb{Q}^*$. We conclude that the survival claim $(X,0,\tau)$ is attainable.

Now, let us denote by $\pi_t(X,0,\tau)$ the price at time t of the survival claim $(X,0,\tau)$. Since ϕ is a replicating strategy for the claim $(X,0,\tau)$, we have that $V_t(\phi)=\pi_t(X,0,\tau)$ for every $t\in[0,T]$. Consequently, for every $t\in[0,T]$,

$$\begin{aligned}\mathbb{1}_{\{t<\tau\}}\widetilde{\pi}_t(X,0,\tau) &= \mathbb{1}_{\{t<\tau\}}\widetilde{V}_t(\phi)=\mathbb{1}_{\{t<\tau\}}\widetilde{Y}^3_t\,\mathbb{E}_{\mathbb{Q}^*}(\widetilde{V}^3_T\,|\,\mathcal{F}_t)\\ &= \mathbb{1}_{\{t<\tau\}}\widetilde{Y}^3_t\,\mathbb{E}_{\mathbb{Q}^*}(X(\widetilde{Y}^3_T)^{-1}\,|\,\mathcal{F}_t).\end{aligned}$$

This proves equality (4.28). □

In view of the last result, it is justified to refer to $\mathbb{Q}^*$ as the *pricing measure relative to* Y^3 for attainable survival claims.

Remark 4.3.3. It can be proved that there exists a unique absolutely continuous probability measure $\bar{\mathbb{Q}}$ on $(\Omega,\mathcal{G}_T)$ such that we have

$$Y^3_t\,\mathbb{E}_{\bar{\mathbb{Q}}}\bigg(\frac{\mathbb{1}_{\{T<\tau\}}X}{Y^3_T}\,\Big|\,\mathcal{G}_t\bigg)=\mathbb{1}_{\{t<\tau\}}\widetilde{Y}^3_t\,\mathbb{E}_{\mathbb{Q}^*}\bigg(\frac{X}{\widetilde{Y}^3_T}\,\Big|\,\mathcal{F}_t\bigg).$$

However, this probability measure is manifestly not equivalent to $\mathbb{Q}^*$, since its Radon-Nikodým density process vanishes after τ. For a related result, the interested reader is referred to the paper by Collin-Dufresne et al. [58].

Example 4.3.2. We provide here an explicit calculation of the pre-default price of a survival claim. For simplicity, we assume that $X=1$, so that the claim represents a defaultable zero-coupon bond. Also, we set $\gamma_t=\gamma=const$, $\mu_{i,t}=0$, and $\sigma_{i,t}=\sigma_i$, $i=1,2,3$. Straightforward calculations yield the following pricing formula

$$\widetilde{\pi}_0(1,0,\tau)=Y^3_0e^{-(\gamma+\frac{1}{2}\sigma_3^2)T}.$$

We see that here the pre-default price $\widetilde{\pi}_0(1,0,\tau)$ depends explicitly on the intensity γ, or rather on the drift term in dynamics of the pre-default value of a defaultable asset. Indeed, from the practical viewpoint, the interpretation of the drift coefficient in dynamics of Y^2 as the real-world default intensity is questionable, since, within the present setup, the default intensity never appears as an independent variable; indeed, it is merely one component of the drift term in dynamics of the pre-default value of Y^3.

Note also that we deal here with a model in which three traded assets are driven by a common one-dimensional Brownian motion. No wonder that this model enjoys the nice property of market completeness, but, at the same time, it also exhibits an undesirable property that the pre-default values of all three assets are perfectly correlated.

As we shall see later, if traded primary assets are judiciously chosen then, typically, the pre-default price (and hence the price) of a survival claim will not depend in an explicit way on the default intensity process.

Remark 4.3.4. From the practical perspective, it seems natural to consider a given market model as an *acceptable model* if its implementation does not require estimation of drift parameters of pre-default prices, at least for the purpose of hedging and valuation of a sufficiently large class of defaultable contingent claims of interest. It is worth recalling that we do not postulate that the drift coefficients are market observables. Since the default intensity can formally be interpreted as a component of the drift term in dynamics of pre-default prices, in an acceptable model there should be no need to estimate this quantity. From this perspective, the model considered in Example 4.3.2 may serve as an example of an 'unacceptable' model, since its implementation would require the knowledge of the drift parameter in dynamics of Y^3.

Let us stress that we do not claim that it is always possible to hedge derivative assets without using the drift coefficients in dynamics of traded assets; we merely argue that one should strive to develop market models in which this knowledge is not essential.

4.3.5 Hedging a Recovery Process

Let us now briefly study the situation where the promised payoff equals zero and the recovery payoff is paid at time τ and equals Z_τ for some $\mathbb{F}$-adapted process Z. Put another way, we consider a defaultable claim of the form $(0, Z, \tau)$. Once again, we make use of Propositions 4.2.1 and 4.2.2. In view of (4.13), we need to find a constant x and an $\mathbb{F}$-predictable process ϕ^2 such that

$$\psi_T := -\int_0^T \frac{Z_t}{Y_t^1}\, d\widetilde{Y}_t^{1,3} = x + \int_0^T \phi_t^2\, d\widetilde{Y}_t^*.$$

Similarly as before, we conclude that, under suitable integrability conditions on ψ_T, there exists ϕ^2 such that $d\psi_t = \phi_t^2\, dY_t^*$, where $\psi_t = \mathbb{E}_{\mathbb{Q}^*}(\psi_T \,|\, \mathcal{F}_t)$. We now set

$$\widetilde{V}_t^3 = x + \int_0^t \phi_u^2\, dY_u^* + \int_0^T \frac{\widetilde{Z}_u^3}{\widetilde{Y}_u^{1,3}}\, d\widetilde{Y}_u^{1,3},$$

so that, in particular, $\widetilde{V}_T^3 = 0$. Then it is possible to find processes ϕ^1 and ϕ^3 such that the strategy ϕ is self-financing and it satisfies: $\widetilde{V}_t(\phi) = \widetilde{V}_t^3 \widetilde{Y}_t^3$ and $V_t(\phi) = Z_t + \phi_t^3 Y_t^3$ for every $t \in [0, T]$. It is thus clear that $V_\tau(\phi) = Z_\tau$ on the event $\{T \geq \tau\}$ and $V_T(\phi) = 0$ on the event $\{T < \tau\}$.

4.3.6 Hedging with a Defaultable Bond

Of course, an abstract semimartingale model considered until now furnishes only a generic framework for a construction of acceptable models for hedging of default risk. A choice of traded assets and specification of their dynamics need to be examined on a case-by-case basis, rather than in an abstract semimartingale setup. We shall address these important issues by examining a few practically appealing examples of defaultable claims and the corresponding models.

For the sake of concreteness, we postulate throughout this section that $Y^1_t = B(t,T)$ is the price of a default-free ZCB with maturity T, whereas $Y^3_t = D^0(t,T)$ is the price of a defaultable ZCB with zero recovery, that is, a defaultable asset with the terminal payoff $Y^3_T = \mathbb{1}_{\{T<\tau\}}$ at maturity T.

We postulate that the dynamics under $\mathbb{P}$ of the default-free ZCB are

$$dB(t,T) = B(t,T)\big(\mu(t,T)\,dt + b(t,T)\,dW_t\big)$$

for some $\mathbb{F}$-predictable processes $\mu(t,T)$ and $b(t,T)$ and we select the process $Y^1_t = B(t,T)$ as a numéraire. Since the prices of the other two assets are not given a priori, we may take any probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ to play the role of $\mathbb{Q}^1$.

In such a case, the probability measure $\mathbb{Q}^1$ is commonly referred to as the *forward martingale measure* for the date T and is denoted by $\mathbb{Q}_T$. Hence the Radon-Nikodým density of $\mathbb{Q}_T$ with respect to $\mathbb{P}$ is given by (4.21) for some $\mathbb{F}$-predictable processes θ and ζ, and the process

$$W^T_t = W_t - \int_0^t \theta_u\,du, \quad \forall\, t \in [0,T],$$

is a Brownian motion under $\mathbb{Q}_T$. Under $\mathbb{Q}_T$ the default-free ZCB is governed by

$$dB(t,T) = B(t,T)\big(\widehat{\mu}(t,T)\,dt + b(t,T)\,dW^T_t\big)$$

where $\widehat{\mu}(t,T) = \mu(t,T) + \theta_t b(t,T)$.

Let now $\widehat{\Gamma}$ stand for the $\mathbb{F}$-hazard process of default time τ under $\mathbb{Q}_T$, so that $\widehat{\Gamma}_t = -\ln(1-\widehat{F}_t)$, where $\widehat{F}_t = \mathbb{Q}_T(\tau \le t \,|\, \mathcal{F}_t)$. Assume that the hypothesis (H) is valid under $\mathbb{Q}_T$ so that, in particular, the process $\widehat{\Gamma}$ is increasing. We define the price process of the defaultable ZCB with zero recovery by the formula

$$D^0(t,T) := B(t,T)\,\mathbb{E}_{\mathbb{Q}_T}(\mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} B(t,T)\,\mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T} \,\big|\, \mathcal{F}_t\big).$$

It is then easily seen that $Y^{3,1}_t = D^0(t,T)(B(t,T))^{-1}$ is a $\mathbb{Q}_T$-martingale and the pre-default price $\widetilde{D}^0(t,T)$ equals

$$\widetilde{D}^0(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T} \,\big|\, \mathcal{F}_t\big).$$

The next result examines the basic properties of the auxiliary process $\widehat{\Gamma}(t,T)$, which is given as, for every $t \in [0,T]$,

$$\widehat{\Gamma}(t,T) = \widetilde{Y}^{3,1}_t = \widetilde{D}^0(t,T)(B(t,T))^{-1} = \mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T} \,\big|\, \mathcal{F}_t\big).$$

The quantity $\widehat{\Gamma}(t,T)$ can be interpreted as the conditional probability under $\mathbb{Q}_T$ that default will not occur prior to the maturity date T, given that we observe $\mathcal{F}_t$ and we know that the default has not yet happened. We will be interested in its *volatility process* $\beta(t,T)$, which is implicitly defined by the following result.

Lemma 4.3.1. *Assume that the $\mathbb{F}$-hazard process $\widehat{\Gamma}$ of τ under $\mathbb{Q}_T$ is continuous. Then the process $\widehat{\Gamma}(t,T)$, $t \in [0,T]$, is a continuous $\mathbb{F}$-submartingale and*

$$d\widehat{\Gamma}(t,T) = \widehat{\Gamma}(t,T)\big(d\widehat{\Gamma}_t + \beta(t,T)\,dW_t^T\big) \tag{4.29}$$

for some $\mathbb{F}$-predictable process $\beta(t,T)$. The process $\widehat{\Gamma}(t,T)$ is of finite variation if and only if the hazard process $\widehat{\Gamma}$ is deterministic. In this case, we have $\widehat{\Gamma}(t,T) = e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T}$.

Proof. We have

$$\widehat{\Gamma}(t,T) = \mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T} \,|\, \mathcal{F}_t\big) = e^{\widehat{\Gamma}_t} L_t,$$

where we set $L_t = \mathbb{E}_{\mathbb{Q}_T}\big(e^{-\widehat{\Gamma}_T} \,|\, \mathcal{F}_t\big)$. Hence $\widehat{\Gamma}(t,T)$ is equal to the product of a strictly positive, increasing, right-continuous, $\mathbb{F}$-adapted process $e^{\widehat{\Gamma}_t}$ and a strictly positive, continuous $\mathbb{F}$-martingale L. Furthermore, there exists an $\mathbb{F}$-predictable process $\widehat{\beta}(t,T)$ such that L satisfies

$$dL_t = L_t \widehat{\beta}(t,T)\,dW_t^T$$

with the initial condition $L_0 = \mathbb{E}_{\mathbb{Q}_T}\big(e^{-\widehat{\Gamma}_T}\big)$. Formula (4.29) now follows by an application of Itô's formula and by setting $\beta(t,T) = e^{-\widehat{\Gamma}_t}\widehat{\beta}(t,T)$. To complete the proof, it suffices to recall that a continuous martingale is never a process of finite variation, unless it is a constant process. □

Remark 4.3.5. It can be checked that $\beta(t,T)$ is also the volatility of the process

$$\Gamma(t,T) = \mathbb{E}_{\mathbb{P}}\big(e^{\Gamma_t - \Gamma_T} \,\big|\, \mathcal{F}_t\big).$$

Assume that $\widehat{\Gamma}_t = \int_0^t \widehat{\gamma}_u\,du$ for some $\mathbb{F}$-predictable, non-negative default intensity process $\widehat{\gamma}$ under $\mathbb{Q}_T$. Then we have the following auxiliary result, which yields, in particular, the volatility process of the defaultable ZCB.

Corollary 4.3.2. *The dynamics under $\mathbb{Q}_T$ of the pre-default price $\widetilde{D}^0(t,T)$ are*

$$\begin{aligned} d\widetilde{D}^0(t,T) = {} & \widetilde{D}^0(t,T)\big(\widehat{\mu}(t,T) + b(t,T)\beta(t,T) + \widehat{\gamma}_t\big)\,dt \\ & + \widetilde{D}^0(t,T)\big(b(t,T) + \beta(t,T)\big)\widetilde{d}(t,T)\,dW_t^T. \end{aligned}$$

Equivalently, the price $D^0(t,T)$ *of the defaultable ZCB satisfies under* $\mathbb{Q}_T$

$$dD^0(t,T) = D^0(t,T)\Big(\big(\widehat{\mu}(t,T) + b(t,T)\beta(t,T)\big)dt + \widetilde{d}(t,T)\,dW_t^T - dM_t\Big)$$

where we write $\widetilde{d}(t,T) = b(t,T) + \beta(t,T)$.

It is worth noting that the process $\beta(t,T)$ can be expressed in terms of market observables, in the sense, that it can be represented as the difference of volatilities $\widetilde{d}(t,T)$ and $b(t,T)$ of pre-default prices of traded assets.

4.3.7 Credit-Risk-Adjusted Forward Price

Assume that the price Y^2 satisfies under the statistical probability $\mathbb{P}$

$$dY_t^2 = Y_t^2\big(\mu_{2,t}\,dt + \sigma_t\,dW_t\big) \tag{4.30}$$

with $\mathbb{F}$-predictable coefficients μ and σ. Let $F_{Y^2}(t,T) = Y_t^2(B(t,T))^{-1}$ be the forward price of Y_T^2. For an appropriate choice of θ (see 4.25), we shall have that

$$dF_{Y^2}(t,T) = F_{Y^2}(t,T)\big(\sigma_t - b(t,T)\big)\,dW_t^T.$$

Therefore, the dynamics of the pre-default synthetic asset $\widetilde{Y}_t^*$ under $\mathbb{Q}^T$ are

$$d\widetilde{Y}_t^* = \widetilde{Y}_t^{2,3}\big(\sigma_t - b(t,T)\big)\,\big(dW_t^T - \beta(t,T)\,dt\big),$$

and the process $\widehat{Y}_t = Y_t^{2,1}e^{-\alpha_t}$ (see Proposition 4.2.3 for the definition of α) satisfies

$$d\widehat{Y}_t = \widehat{Y}_t\big(\sigma_t - b(t,T)\big)\,\big(dW_t^T - \beta(t,T)\,dt\big).$$

Let $\widehat{\mathbb{Q}}$ be an equivalent probability measure on $(\Omega, \mathcal{G}_T)$ such that $\widehat{Y}$ (or, equivalently, $\widetilde{Y}^*$) is a $\widehat{\mathbb{Q}}$-martingale. By virtue of the Girsanov theorem, the process $\widehat{W}$ given by the formula, for $t \in [0,T]$,

$$\widehat{W}_t = W_t^T - \int_0^t \beta(u,T)\,du,$$

is a Brownian motion under $\widehat{\mathbb{Q}}$. Thus, the forward price $F_{Y^2}(t,T)$ satisfies under $\widehat{\mathbb{Q}}$

$$dF_{Y^2}(t,T) = F_{Y^2}(t,T)\big(\sigma_t - b(t,T)\big)\big(d\widehat{W}_t + \beta(t,T)\,dt\big). \tag{4.31}$$

It appears that the valuation results are easier to interpret when they are expressed in terms of forward prices associated with vulnerable forward contracts, rather than in terms of spot prices of primary assets. For this reason, we shall now examine credit-risk-adjusted forward prices of default-free and defaultable assets.

Definition 4.3.2. Let Y be a $\mathcal{G}_T$-measurable claim. An $\mathcal{F}_t$-measurable random variable K is called the *credit-risk-adjusted forward price* of Y if the pre-default value at time t of the vulnerable forward contract represented by the claim $\mathbb{1}_{\{T<\tau\}}(Y-K)$ equals 0.

Lemma 4.3.2. *The credit-risk-adjusted forward price $\widehat{F}_Y(t,T)$ of an attainable survival claim $(X,0,\tau)$, which is represented by a $\mathcal{G}_T$-measurable claim $Y = X\mathbb{1}_{\{T<\tau\}}$, equals $\widetilde{\pi}_t(X,0,\tau)(\widetilde{D}^0(t,T))^{-1}$, where $\widetilde{\pi}_t(X,0,\tau)$ is the pre-default price of $(X,0,\tau)$. The process $\widehat{F}_Y(t,T)$, $t\in[0,T]$, is an $\mathbb{F}$-martingale under $\widehat{\mathbb{Q}}$.*

Proof. The forward price is defined as an $\mathcal{F}_t$-measurable random variable K such that the claim

$$\mathbb{1}_{\{T<\tau\}}(X\mathbb{1}_{\{T<\tau\}} - K) = X\mathbb{1}_{\{T<\tau\}} - KD^0(T,T)$$

is worthless at time t on the event $\{t<\tau\}$. It is clear that the pre-default value at time t of this claim equals $\widetilde{\pi}_t(X,0,\tau) - K\widetilde{D}^0(t,T)$. Consequently, we obtain $\widetilde{F}_Y(t,T) = \widetilde{\pi}_t(X,0,\tau)(\widetilde{D}^0(t,T))^{-1}$. □

Let us now focus on default-free assets. It is clear that the credit-risk-adjusted forward price of the bond $B(t,T)$ equals 1. To find the credit-risk-adjusted forward price of Y^2, let us write

$$\widehat{F}_{Y^2}(t,T) := F_{Y^2}(t,T)\, e^{\alpha_T-\alpha_t} = Y^{2,1}_t e^{\alpha_T-\alpha_t}, \tag{4.32}$$

where α is given by (see (4.16))

$$\begin{aligned}\alpha_t &= \int_0^t \big(\sigma_u - b(u,T)\big)\beta(u,T)\,du \qquad (4.33)\\ &= \int_0^t \big(\sigma_u - b(u,T)\big)\big(\widetilde{d}(u,T) - b(u,T)\big)\,du.\end{aligned}$$

Lemma 4.3.3. *Assume that α given by (4.33) is a deterministic function. Then the credit-risk-adjusted forward price of Y^2, denoted as $\widehat{F}_{Y^2}(t,T)$, is given by (4.32) for every $t\in[0,T]$.*

Proof. According to Definition 4.3.2, the price $\widehat{F}_{Y^2}(t,T)$ is an $\mathcal{F}_t$-measurable random variable K, which makes the forward contract represented by the claim $D^0(T,T)(Y^2_T - K)$ worthless on the set $\{t<\tau\}$. Assume that the claim $Y^2_T - K$ is attainable. Since $\widetilde{D}^0(T,T)=1$, from equation (4.28) it follows that the pre-default value of this claim is given by the conditional expectation

$$\widetilde{D}^0(t,T)\,\mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y^2_T - K \,\big|\, \mathcal{F}_t\big).$$

Therefore,

$$\widehat{F}_{Y^2}(t,T) = \mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y_T^2 \,\big|\, \mathcal{F}_t\big) = \mathbb{E}_{\widehat{\mathbb{Q}}}\big(F_{Y^2}(T,T) \,\big|\, \mathcal{F}_t\big) = F_{Y^2}(t,T)\, e^{\alpha_T - \alpha_t},$$

as was claimed. □

It is worth noting that the process $\widehat{F}_{Y^2}(t,T)$ is a (local) martingale under the pricing measure $\widehat{\mathbb{Q}}$, since it satisfies

$$d\widehat{F}_{Y^2}(t,T) = \widehat{F}_{Y^2}(t,T)(\sigma_t - b(t,T))\, d\widehat{W}_t. \tag{4.34}$$

Under the present assumptions, the auxiliary process $\widehat{Y}$ introduced in Proposition 4.2.3 and the credit-risk-adjusted forward price $\widehat{F}_{Y^2}(t,T)$ are closely related to each other. Indeed, we have $\widehat{F}_{Y^2}(t,T) = \widehat{Y}_t e^{\alpha_T}$, so that the two processes are proportional.

4.3.8 Vulnerable Option on a Default-Free Asset

We shall now analyze a vulnerable call option with the payoff

$$C_T^d = \mathbb{1}_{\{T<\tau\}}(Y_T^2 - K)^+$$

for a constant strike K. Our goal is to find a replicating strategy for this claim, which is interpreted as a survival claim $(X, 0, \tau)$ with the promised payoff $X = C_T = (Y_T^2 - K)^+$, where C_T is the payoff of an equivalent non-vulnerable option. The method presented below is quite general, however, so that it can be applied to any survival claim with the promised payoff $X = G(Y_T^2)$ for some function $G : \mathbb{R} \to \mathbb{R}$ satisfying mild integrability assumptions.

We assume that $Y_t^1 = B(t,T)$, $Y_t^3 = D^0(t,T)$ and the price of a default-free asset Y^2 is governed by (4.30). Then

$$C_T^d = \mathbb{1}_{\{T<\tau\}}(Y_T^2 - K)^+ = \mathbb{1}_{\{T<\tau\}}(Y_T^2 - KY_T^1)^+.$$

We are going to apply Proposition 4.2.3. In the present setup, we have $Y_t^{2,1} = F_{Y^2}(t,T)$ and $\widehat{Y}_t = F_{Y^2}(t,T)e^{-\alpha_t}$. Since a vulnerable option is an example of a survival claim, in view of Lemma 4.3.2, its credit-risk-adjusted forward price satisfies $\widehat{F}_{C^d}(t,T) = \widetilde{C}_t^d(\widetilde{D}^0(t,T))^{-1}$.

Proposition 4.3.2. *Suppose that the volatilities σ, b and β are deterministic functions. Then the credit-risk-adjusted forward price of a vulnerable call option written on a default-free asset Y^2 equals*

$$\widehat{F}_{C^d}(t,T) = \widehat{F}_{Y^2}(t,T)N(d_+(\widehat{F}_{Y^2}(t,T),t,T)) - KN(d_-(\widehat{F}_{Y^2}(t,T),t,T))$$

where

$$d_{\pm}(z,t,T) = \frac{\ln z - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_u - b(u,T))^2\, du.$$

The replicating strategy ϕ *in the spot market satisfies,for every* $t \in [0,T]$ *on the event* $\{t < \tau\}$,

$$\begin{aligned}
&\phi^1_t B(t,T) = -\phi^2_t Y^2_t,\\
&\phi^2_t = \widetilde{D}^0(t,T)(B(t,T))^{-1} N(d_+(t,T)) e^{\alpha_T - \alpha_t},\\
&\phi^3_t \widetilde{D}^0(t,T) = \widetilde{C}^d_t,
\end{aligned}$$

where $d_+(t,T) = d_+(\widehat{F}_{Y^2}(t,T), t, T)$.

Proof. In the first step, we establish the valuation formula. Assume for the moment that the option is attainable. Then the pre-default value of the option equals, for every $t \in [0,T]$,

$$\begin{aligned}
\widetilde{C}^d_t &= \widetilde{D}^0(t,T)\, \mathbb{E}_{\widehat{\mathbb{Q}}}\big((F_{Y^2}(T,T) - K)^+ \,\big|\, \mathcal{F}_t\big)\\
&= \widetilde{D}^0(t,T)\, \mathbb{E}_{\widehat{\mathbb{Q}}}\big((\widehat{F}_{Y^2}(T,T) - K)^+ \,\big|\, \mathcal{F}_t\big).
\end{aligned}$$

In view of (4.34), this conditional expectation can be evaluated explicitly, yielding the stated valuation formula.

To find the replicating strategy and establish attainability of the option, we consider the Itô differential $d\widehat{F}_{C^d}(t,T)$ and we identify terms in (4.27). It appears that

$$\begin{aligned}
d\widehat{F}_{C^d}(t,T) &= N(d_+(t,T))\, d\widehat{F}_{Y^2}(t,T) = N(d_+(t,T)) e^{\alpha_T}\, d\widehat{Y}_t \qquad (4.35)\\
&= N(d_+(t,T)) \widetilde{Y}^{3,1}_t e^{\alpha_T - \alpha_t}\, d\widetilde{Y}^*_t,
\end{aligned}$$

so that the process ϕ^2 in (4.26) equals

$$\phi^2_t = \widetilde{Y}^{3,1}_t N(d_+(t,T)) e^{\alpha_T - \alpha_t}.$$

Moreover, ϕ^1 is such that $\phi^1_t B(t,T) + \phi^2_t Y^2_t = 0$ and $\phi^3_t = \widetilde{C}^d_t (\widetilde{D}^0(t,T))^{-1}$. It is easily seen that this proves also the attainability of the option. □

Let us now examine the financial interpretation of Proposition 4.3.2.

First, equality (4.35) shows that it is easy to replicate the option using vulnerable forward contracts. Indeed, we have

$$\widehat{F}_{C^d}(T,T) = X = \frac{\widetilde{C}^d_0}{\widetilde{D}^0(0,T)} + \int_0^T N(d_+(t,T))\, d\widehat{F}_{Y^2}(t,T)$$

so that it is enough to invest the premium $\widetilde{C}_0^d = C_0^d$ in defaultable ZCBs of maturity T and take, at any instant t prior to default, $N(d_+(t,T))$ positions in vulnerable forward contracts. It is apparent that if default occurs prior to T, all outstanding vulnerable forward contracts become void.

Second, it is worth stressing that neither the arbitrage price, nor the replicating strategy for a vulnerable option, depend explicitly on the default intensity. This remarkable feature is due to the fact that the default risk of the writer of the option can be completely eliminated by trading in defaultable zero-coupon bond with the same exposure to credit risk as a vulnerable option.

In fact, since the volatility β is invariant with respect to an equivalent change of a probability measure, and so are the volatilities σ and $b(t,T)$, the formulae of Proposition 4.3.2 are valid for any choice of a forward measure $\mathbb{Q}_T$ equivalent to $\mathbb{P}$ (and, of course, they are valid under $\mathbb{P}$ as well). The only way in which the choice of a forward measure $\mathbb{Q}_T$ impacts these results is through the pre-default value of a defaultable ZCB.

We conclude that we deal here with the volatility based relative pricing a defaultable claim. This should be contrasted with more popular intensity-based risk-neutral pricing, which is commonly used to produce an arbitrage-free model of traded defaultable assets. Recall, however, that if traded assets are not chosen carefully for a given class of survival claims, then both hedging strategy and pre-default price may depend explicitly on values of drift parameters that appear in our market model and which, in turn, can be linked to the default intensity (see Example 4.3.2).

Remark 4.3.6. Assume that the promised payoff $X = G(Y_T^2)$ for some function $G : \mathbb{R} \to \mathbb{R}$. The pricing formula of Proposition 4.3.2 leads to the conjecture that the credit-risk-adjusted forward price $\widehat{F}_Y(t,T)$ of the survival claim $Y = \mathbb{1}_{\{T<\tau\}} G(Y_T^2)$ satisfies the equality

$$\widehat{F}_Y(t,T) = w(t, \widehat{F}_{Y^2}(t,T)),$$

where the pricing function w solves the PDE

$$\partial_t w(t,z) + \tfrac{1}{2}(\sigma_t - b(t,T))^2 z^2 \partial_{zz} w(t,z) = 0$$

with the terminal condition $w(T,z) = G(z)$. Let us mention that the PDE approach is studied in some detail in Section 4.4 below.

Remark 4.3.7. Proposition 4.3.2 is still valid if the driving Brownian motion is two-dimensional, rather than one-dimensional. In an extended model, the volatilities $\sigma_t, b(t,T)$ and $\beta(t,T)$ take values in $\mathbb{R}^2$ and the respective products are interpreted as inner products in $\mathbb{R}^3$. Equivalently, one may prefer to deal with real-valued volatilities, but with correlated one-dimensional Brownian motions.

4.3.9 Abstract Vulnerable Swaption

In this section, we relax the assumption that Y^1 is the price of a default-free bond. We now let Y^1 and Y^2 to be arbitrary default-free assets, with dynamics

$$dY_t^i = Y_t^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t \big), \quad i = 1, 2. \tag{4.36}$$

We still take the defaultable zero-coupon bond with zero recovery and the price process $Y_t^3 = D^0(t,T)$ to be the third traded asset.

We maintain the assumption that the model is arbitrage-free, but we no longer postulate that it is complete. In other words, we postulate the existence an EMM $\mathbb{Q}^1$, as defined in subsection on the arbitrage-free property, but not the uniqueness of $\mathbb{Q}^1$.

We take the first asset as the numéraire, so that all prices are expressed in units of Y^1. In particular, $Y_t^{1,1} = 1$ for every $t \in \mathbb{R}_+$, and the relative prices $Y^{2,1}$ and $Y^{3,1}$ satisfy under $\mathbb{Q}^1$ (cf. Proposition 4.3.1)

$$\begin{aligned} dY_t^{2,1} &= Y_t^{2,1} (\sigma_{2,t} - \sigma_{1,t})\, d\widehat{W}_t, \\ dY_t^{3,1} &= Y_{t-}^{3,1} \big((\sigma_{3,t} - \sigma_{1,t})\, d\widehat{W}_t - d\widehat{M}_t \big). \end{aligned}$$

It is natural to postulate that the driving Brownian noise is two-dimensional. In such a case, we may represent the joint dynamics of relative prices $Y^{2,1}$ and $Y^{3,1}$ under $\mathbb{Q}^1$ as follows

$$\begin{aligned} dY_t^{2,1} &= Y_t^{2,1} (\sigma_{2,t} - \sigma_{1,t})\, dW_t^1, \\ dY_t^{3,1} &= Y_{t-}^{3,1} \big((\sigma_{3,t} - \sigma_{1,t})\, dW_t^2 - d\widehat{M}_t \big), \end{aligned}$$

where W^1, W^2 are one-dimensional Brownian motions under $\mathbb{Q}^1$, such that $d\langle W^1, W^2 \rangle_t = \rho_t\, dt$ for a deterministic instantaneous correlation coefficient ρ taking values in $[-1, 1]$.

We assume from now on that the volatilities σ_i, $i = 1, 2, 3$ are deterministic. Let us set

$$\alpha_t = \langle \ln \widetilde{Y}^{2,1}, \ln \widetilde{Y}^{3,1} \rangle_t = \int_0^t \rho_u (\sigma_{2,u} - \sigma_{1,u})(\sigma_{3,u} - \sigma_{1,u})\, du, \tag{4.37}$$

and let $\widehat{\mathbb{Q}}$ be an equivalent probability measure on $(\Omega, \mathcal{G}_T)$ such that the process $\widehat{Y}_t = Y_t^{2,1} e^{-\alpha_t}$ is a $\widehat{\mathbb{Q}}$-martingale. To clarify the financial interpretation of the auxiliary process $\widehat{Y}$ in the present context, we introduce the concept of credit-risk-adjusted forward price relative to the numéraire Y^1.

Definition 4.3.3. Let Y be a $\mathcal{G}_T$-measurable claim. An $\mathcal{F}_t$-measurable random variable K is called the time-t *credit-risk-adjusted* Y^1*-forward price* of Y if the pre-default value at time t of a vulnerable forward contract, represented by the claim

$$\mathbb{1}_{\{T<\tau\}}(Y_T^1)^{-1}(Y - KY_T^1) = \mathbb{1}_{\{T<\tau\}}(Y(Y_T^1)^{-1} - K),$$

equals 0.

The credit-risk-adjusted Y^1-forward price of Y is denoted by $\widehat{F}_{Y|Y^1}(t,T)$ and it is also interpreted as an abstract *defaultable swap rate*. The following auxiliary results are easy to establish, by arguing along the same lines as in Lemmas 4.3.2 and 4.3.3.

Lemma 4.3.4. *The credit-risk-adjusted Y^1-forward price of a survival claim $Y = (X, 0, \tau)$ equals*

$$\widehat{F}_{Y|Y^1}(t,T) = \widetilde{\pi}_t(X^1, 0, \tau)(\widetilde{D}^0(t,T))^{-1},$$

where $X^1 = X(Y_T^1)^{-1}$ is the price of X in the numéraire Y^1 and $\widetilde{\pi}_t(X^1, 0, \tau)$ is the pre-default value of a survival claim with the promised payoff X^1.

Proof. It suffices to note that for $Y = \mathbb{1}_{\{T<\tau\}}X$ we have

$$\mathbb{1}_{\{T<\tau\}}(Y(Y_T^1)^{-1} - K) = \mathbb{1}_{\{T<\tau\}}X^1 - KD^0(T,T),$$

where $X^1 = X(Y_T^1)^{-1}$, and to consider the pre-default values. □

Lemma 4.3.5. *The credit-risk-adjusted Y^1-forward price of the asset Y^2 equals*

$$\widehat{F}_{Y^2|Y^1}(t,T) = Y_t^{2,1}\, e^{\alpha_T - \alpha_t} = \widehat{Y}_t e^{\alpha_T},$$

where α, assumed here to be deterministic, is given by formula (4.37).

Proof. It suffices to find an $\mathcal{F}_t$-measurable random variable K for which

$$\widetilde{D}^0(t,T)\, \mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y_T^2(Y_T^1)^{-1} - K \,\big|\, \mathcal{F}_t\big) = 0.$$

From the last equality, we obtain $K = \widehat{F}_{Y^2|Y^1}(t,T)$, where

$$\widehat{F}_{Y^2|Y^1}(t,T) = \mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y_T^{2,1} \,\big|\, \mathcal{F}_t\big) = Y_t^{2,1}\, e^{\alpha_T - \alpha_t} = \widehat{Y}_t\, e^{\alpha_T}.$$

We have used here the facts that $\widehat{Y}_t = Y_t^{2,1} e^{-\alpha_t}$ is a $\widehat{\mathbb{Q}}$-martingale and α is deterministic. □

We are in a position to examine a vulnerable option to exchange default-free assets with the payoff

$$C_T^d = \mathbb{1}_{\{T<\tau\}}(Y_T^1)^{-1}(Y_T^2 - KY_T^1)^+ = \mathbb{1}_{\{T<\tau\}}(Y_T^{2,1} - K)^+. \tag{4.38}$$

The last expression shows that the option can be interpreted as a vulnerable swaption associated with the assets Y^1 and Y^2. It is useful to observe that

$$\frac{C_T^d}{Y_T^1} = \frac{\mathbb{1}_{\{T<\tau\}}}{Y_T^1}\left(\frac{Y_T^2}{Y_T^1} - K\right)^+,$$

so that, when expressed in units of the numéraire Y^1, the payoff becomes

$$C_T^{d,1} = D^{0,1}(T,T)(Y_T^{2,1} - K)^+,$$

where $C_t^{d,1} = C_t^d(Y_t^1)^{-1}$ and $D^{0,1}(t,T) = D^0(t,T)(Y_t^1)^{-1}$ stand for the prices relative to the numéraire Y^1.

It is clear that we deal here with a model analogous to the model examined in previous subsections in which, however, all prices are expressed in units of the numéraire asset Y^1. This observation allows us to directly deduce the valuation formula from Proposition 4.3.2.

Proposition 4.3.3. *Let us consider the market model (4.36) with a two-dimensional Brownian motion W and deterministic volatilities σ_i, $i = 1,2,3$. The credit-risk-adjusted Y^1-forward price of a vulnerable call option, with the terminal payoff given by (4.38), equals*

$$\widehat{F}_{C^d|Y^1}(t,T) = \widehat{F}_t N\big(d_+(\widehat{F}_t,t,T)\big) - KN\big(d_-(\widehat{F}_t,t,T)\big),$$

where we write $\widehat{F}_t = \widehat{F}_{Y^2|Y^1}(t,T)$ and

$$d_\pm(z,t,T) = \frac{\ln z - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

with

$$v^2(t,T) = \int_t^T (\sigma_{2,u} - \sigma_{1,u})^2\,du.$$

The replicating strategy ϕ in the spot market satisfies, on the event $\{t<\tau\}$,

$$\phi_t^1 Y_t^1 = -\phi_t^2 Y_t^2,\ \phi_t^2 = \widetilde{D}^0(t,T)(Y_t^1)^{-1}N(d_+(t,T))e^{\alpha_T-\alpha_t},\ \phi_t^3\widetilde{D}^0(t,T) = \widetilde{C}_t^d,$$

where $d_+(t,T) = d_+\big(\widehat{F}_{Y^2|Y^1}(t,T),t,T\big)$.

Proof. The proof is analogous to that of Proposition 4.3.2 and thus it is omitted. □

It is worth noting that the payoff (4.38) was judiciously chosen. Suppose instead that the option payoff is not defined by (4.38), but it is given by an apparently simpler expression

$$C_T^d = \mathbb{1}_{\{T<\tau\}}(Y_T^2 - KY_T^1)^+.$$

Since the payoff C^d_T can be represented as follows

$$C^d_T = \widehat{G}(Y^1_T, Y^2_T, Y^3_T) = Y^3_T(Y^2_T - KY^1_T)^+,$$

where $\widehat{G}(y_1, y_2, y_3) = y_3(y_2 - Ky_1)^+$, we deal with an option to exchange the second asset for K units of the first asset, but with the payoff expressed in units of the defaultable asset Y^3. When expressed in relative prices, the payoff becomes

$$C^{d,1}_T = \mathbb{1}_{\{T<\tau\}}(Y^{2,1}_T - K)^+.$$

where $\mathbb{1}_{\{T<\tau\}} = D^{0,1}(T,T)Y^1_T$. It is thus rather clear that it is not longer possible to apply the same method as in the proof of Proposition 4.3.2.

4.3.10 Defaultable Asset with Non-Zero Recovery

In this subsection, we still postulate that Y^1 and Y^2 are default-free assets with price processes

$$dY^i_t = Y^i_t\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW_t\big),$$

where W is a one-dimensional Brownian motion, but we now assume that

$$dY^3_t = Y^3_{t-}(\mu_3\,dt + \sigma_3\,dW_t + \kappa_3\,dM_t)$$

with $\kappa_3 > -1$ and $\kappa_3 \neq 0$. We assume that $Y^3_0 > 0$, so that $Y^3_t > 0$ for every $t \in \mathbb{R}_+$. We shall briefly describe the same steps as in the case of a defaultable asset with zero recovery.

Arbitrage-Free Property

As usual, we need first to impose specific constraints on model coefficients, so that the model is arbitrage-free. In the case of constant coefficients, an EMM $\mathbb{Q}^1$ exists if there exists a pair (θ, ζ) such that, for $i = 2, 3$,

$$\theta_t(\sigma_i - \sigma_1) + \zeta_t\xi_t\frac{\kappa_i - \kappa_1}{1+\kappa_1} = \mu_1 - \mu_i + \sigma_1(\sigma_i - \sigma_1) + \xi_t(\kappa_i - \kappa_1)\frac{\kappa_1}{1+\kappa_1}.$$

To ensure the existence of a solution (θ, ζ) on the event $\{\tau < t\}$ under the present assumptions, we impose the condition

$$\sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3},$$

that is,

$$\mu_1(\sigma_3 - \sigma_2) + \mu_2(\sigma_1 - \sigma_3) + \mu_3(\sigma_2 - \sigma_1) = 0.$$

Since $\kappa_1 = \kappa_2 = 0$, on the event $\{\tau \geq t\}$, we have to solve the following equations

$$\theta_t(\sigma_2 - \sigma_1) = \mu_1 - \mu_2 + \sigma_1(\sigma_2 - \sigma_1),$$
$$\theta_t(\sigma_3 - \sigma_1) + \zeta_t \gamma \kappa_3 = \mu_1 - \mu_3 + \sigma_1(\sigma_3 - \sigma_1).$$

If, in addition, $(\sigma_2 - \sigma_1)\kappa_3 \neq 0$, we obtain the unique solution

$$\theta = \sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3},$$
$$\zeta = 0 > -1,$$

so that the martingale measure $\mathbb{Q}^1$ exists and is unique.

Observe that, since $\zeta = 0$, the default intensity under $\mathbb{Q}^1$ coincides here with the default intensity under the real-life probability $\mathbb{Q}$. It is interesting to note that, in a more general situation when all three assets are defaultable with non-zero recovery, the default intensity under $\mathbb{Q}^1$ coincides with the default intensity under the real-life probability $\mathbb{Q}$ if and only if the process Y^1 is continuous. For more details, the interested reader is referred to Bielecki et al. [15], where the general case is studied.

4.3.11 Two Defaultable Assets with Zero Recovery

In the remaining part of Section 4.3, we assume that we have only two assets and both are defaultable assets with zero recovery. This case was recently examined by Carr [49], who studied an imperfect hedging of digital options. Note that we present here results for replication, that is, perfect hedging.

We shall briefly outline the analysis of hedging of a survival claim. Under the present assumptions, we have, for $i = 1, 2$,

$$dY_t^i = Y_{t-}^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t - dM_t\big), \tag{4.39}$$

where W is a one-dimensional Brownian motion, so that

$$Y_t^1 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^1, \quad Y_t^2 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^2,$$

with the pre-default prices governed by the SDEs

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i \big((\mu_{i,t} + \gamma_t)\, dt + \sigma_{i,t}\, dW_t\big). \tag{4.40}$$

The wealth process V associated with the self-financing trading strategy (ϕ^1, ϕ^2) satisfies, for every $t \in [0, T]$,

$$V_t = Y_t^1 \left(V_0^1 + \int_0^t \phi_u^2\, d\widetilde{Y}_u^{2,1} \right),$$

where $\widetilde{Y}^{2,1}_t = \widetilde{Y}^2_t / \widetilde{Y}^1_t$. Since both primary traded assets are subject to zero recovery, it is clear that the present model is incomplete, in the sense, that not all defaultable claims can be replicated.

We shall check in what follows that, under the assumption that the driving Brownian motion W is one-dimensional, all survival claims satisfying mild technical conditions are attainable, however. In the more realistic case of a two-dimensional driving noise, we will still be able to replicate a fairly large class of survival claims, including options written on a defaultable asset, as well as options to exchange one defaultable asset for another.

4.3.12 Hedging a Survival Claim

For the sake of expositional simplicity, we assume in this subsection that the driving Brownian motion W is one-dimensional. Arguably, this is not the right choice, since we deal here with two risky assets, so that they will be perfectly correlated. However, this assumption is convenient for the expositional purposes, since it ensures the model completeness with respect to survival claims. We will later relax this temporary assumption so it is fair to say that this assumption is not crucial.

We shall now argue that in a market model with two defaultable assets that are subject to zero recovery, the replication of a survival claim $(X, 0, \tau)$ is in fact equivalent to replication of an associated promised payoff X using the pre-default price processes.

Lemma 4.3.6. *If a trading strategy ϕ^i, $i = 1, 2$, based on pre-default values $\widetilde{Y}^i$, $i = 1, 2$, is a replicating strategy for an $\mathcal{F}_T$-measurable claim X, that is, if ϕ is such that the process*

$$\widetilde{V}_t(\phi) = \phi^1_t \widetilde{Y}^1_t + \phi^2_t \widetilde{Y}^2_t$$

satisfies, for every $t \in [0, T]$,

$$\begin{aligned} d\widetilde{V}_t(\phi) &= \phi^1_t \, d\widetilde{Y}^1_t + \phi^2_t \, d\widetilde{Y}^2_t, \\ \widetilde{V}_T(\phi) &= X, \end{aligned}$$

then for the process $V_t(\phi) = \phi^1_t Y^1_t + \phi^2_t Y^2_t$ we have, for every $t \in [0, T]$,

$$\begin{aligned} dV_t(\phi) &= \phi^1_t \, dY^1_t + \phi^2_t \, dY^2_t, \\ V_T(\phi) &= \mathbb{1}_{\{T<\tau\}} X. \end{aligned}$$

This means that the strategy ϕ replicates the survival claim $(X, 0, \tau)$.

Proof. It is clear that $V_t(\phi) = \mathbb{1}_{\{t<\tau\}} V_t(\phi) = \mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\phi)$. From the equality

$$\phi^1_t \, dY^1_t + \phi^2_t \, dY^2_t = -(\phi^1_t \widetilde{Y}^1_t + \phi^2_t \widetilde{Y}^2_t) \, dH_t + (1 - H_{t-})(\phi^1_t \, d\widetilde{Y}^1_t + \phi^2_t \, d\widetilde{Y}^2_t),$$

it follows that

$$\phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2 = -\widetilde{V}_t(\phi) \, dH_t + (1 - H_{t-}) d\widetilde{V}_t(\phi),$$

that is,

$$\phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2 = d(\mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\phi)) = dV_t(\phi).$$

It is also easily seen that the equality $V_T(\phi) = X\mathbb{1}_{\{T<\tau\}}$ holds. □

Combining the last result with Lemma 4.2.1, we see that a strategy (ϕ^1, ϕ^2) replicates a survival claim $(X, 0, \tau)$ whenever we have

$$\widetilde{Y}_T^1 \Big(x + \int_0^T \phi_t^2 \, d\widetilde{Y}_t^{2,1} \Big) = X$$

for some constant x and some $\mathbb{F}$-predictable process ϕ^2, where, in view of (4.40),

$$d\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^{2,1} \Big(\big(\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t}) \big) \, dt + (\sigma_{2,t} - \sigma_{1,t}) \, dW_t \Big).$$

We introduce a probability measure $\widetilde{\mathbb{Q}}$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, and such that $\widetilde{Y}^{2,1}$ is an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$. It is easily seen that the Radon-Nikodým density η satisfies, for $t \in [0, T]$,

$$d\widetilde{\mathbb{Q}} \,|\, _{\mathcal{G}_t} = \eta_t \, d\mathbb{P} \,|\, _{\mathcal{G}_t} = \mathcal{E}_t \left(\int_0^{\cdot} \theta_s \, dW_s \right) \, d\mathbb{P} \,|\, _{\mathcal{G}_t}$$

with

$$\theta_t = \frac{\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t})}{\sigma_{1,t} - \sigma_{2,t}},$$

provided, of course, that the process θ is well defined and satisfies suitable integrability conditions. We shall show that a survival claim is attainable if the random variable $X(\widetilde{Y}_T^1)^{-1}$ is $\widetilde{\mathbb{Q}}$-integrable. Indeed, the pre-default value $\widetilde{V}_t$ at time t of a survival claim equals

$$\widetilde{V}_t = \widetilde{Y}_t^1 \, \mathbb{E}_{\widetilde{\mathbb{Q}}} \big(X (\widetilde{Y}_T^1)^{-1} \,|\, \mathcal{F}_t \big)$$

and, from the predictable representation theorem, we deduce that there exists a process ϕ^2 such that

$$\mathbb{E}_{\widetilde{\mathbb{Q}}} \big(X (\widetilde{Y}_T^1)^{-1} \,|\, \mathcal{F}_t \big) = \mathbb{E}_{\widetilde{\mathbb{Q}}} \big(X (\widetilde{Y}_T^1)^{-1} \big) + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^{2,1}.$$

The component ϕ^1 of the self-financing trading strategy $\phi = (\phi^1, \phi^2)$ is then chosen in such a way that, for every $t \in [0, T]$,

$$\phi_t^1 \widetilde{Y}_t^1 + \phi_t^2 \widetilde{Y}_t^2 = \widetilde{V}_t.$$

To conclude, by focusing on pre-default values, we have shown that the replication of survival claims can be reduced here to classic results on replication of (non-defaultable) contingent claims in a default-free market model.

4.3.13 Option on a Defaultable Asset

In order to get a complete model with respect to survival claims, we postulated in the preceding subsection that the driving Brownian motion in dynamics (4.39) is one-dimensional. This assumption is questionable, since it clearly implies the perfect correlation between risky assets. However, we may relax this restriction and work instead with the two correlated one-dimensional Brownian motions. The model will no longer be complete, but options on a defaultable asset will still be attainable.

The payoff of a (non-vulnerable) call option written on the defaultable asset Y^2 equals

$$C_T = (Y_T^2 - K)^+ = \mathbb{1}_{\{T<\tau\}}(\widetilde{Y}_T^2 - K)^+,$$

so that it is natural to interpret this contract as a survival claim with the promised payoff $X = (\widetilde{Y}_T^2 - K)^+$.

To deal with this option in an efficient way, we consider a model in which

$$dY_t^i = Y_{t-}^i\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW_t^i - dM_t\big),$$

where W^1 and W^2 are two one-dimensional correlated Brownian motions with the instantaneous correlation coefficient ρ_t. More specifically, we assume that $Y_t^1 = D^0(t,T) = \mathbb{1}_{\{t<\tau\}}\widetilde{D}^0(t,T)$ represents a defaultable ZCB with zero recovery, and $Y_t^2 = \mathbb{1}_{\{t<\tau\}}\widetilde{Y}_t^2$ is a generic defaultable asset with zero recovery. Within the present setup, the payoff can also be represented as follows

$$C_T = (Y_T^2 - KY_T^1)^+ = g(Y_T^1, Y_T^2),$$

where $g(y_1,y_2) = (y_2 - Ky_1)^+$, and thus it can also be seen as an option to exchange the second asset for K units of the first asset.

The requirement that the process $\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^2(\widetilde{Y}_t^1)^{-1}$ is an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$ implies that

$$d\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^{2,1}\Big(\big(\sigma_{2,t}\rho_t - \sigma_{1,t}\big)\,d\widetilde{W}_t^1 + \sigma_{2,t}\sqrt{1-\rho_t^2}\,d\widetilde{W}_t^2\Big),$$

where $\widetilde{W} = (\widetilde{W}^1, \widetilde{W}^2)$ follows a two-dimensional Brownian motion under $\widetilde{\mathbb{Q}}$. Since $\widetilde{Y}_T^1 = 1$, a replication of the option reduces to finding a constant x and an $\mathbb{F}$-predictable process ϕ^2 satisfying

$$x + \int_0^T \phi_t^2\,d\widetilde{Y}_t^{2,1} = (\widetilde{Y}_T^2 - K)^+.$$

To obtain closed-form expressions for the option price and replicating strategy, we postulate that the volatilities σ_1, σ_2 and the correlation coefficient ρ are deterministic. Let

$$\widehat{F}_{Y^2}(t,T) = \widetilde{Y}_t^2(\widetilde{D}^0(t,T))^{-1}$$

and

$$\widehat{F}_C(t,T) = \widetilde{C}_t(\widetilde{D}^0(t,T))^{-1}$$

stand for the credit-risk-adjusted forward price of the second asset and of the option, respectively. The proof of the following valuation result is fairly standard and thus it is omitted.

Proposition 4.3.4. *Assume that σ_1, σ_2 and ρ are deterministic. Let Y^1 be a defaultable zero-coupon bond with zero recovery. Then the credit-risk-adjusted forward price of the option written on a defaultable asset Y^2 equals*

$$\widehat{F}_C(t,T) = \widehat{F}_{Y^2}(t,T)N\big(d_+(\widehat{F}_{Y^2}(t,T),t,T)\big) - KN\big(d_-(\widehat{F}_{Y^2}(t,T),t,T)\big).$$

Equivalently, the pre-default price of the option equals

$$\widetilde{C}_t = \widetilde{Y}^2_t N\big(d_+(\widehat{F}_{Y^2}(t,T),t,T)\big) - K\widetilde{D}^0(t,T)N\big(d_-(\widehat{F}_{Y^2}(t,T),t,T)\big),$$

where

$$d_\pm(z,t,T) = \frac{\ln z - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma^2_{1,u} + \sigma^2_{2,u} - 2\rho_u\sigma_{1,u}\sigma_{2,u})\,du.$$

Moreover the replicating strategy ϕ in the spot market satisfies, for every $t \in [0,T]$ on the event $\{t < \tau\}$,

$$\phi^1_t = -KN\big(d_-(\widehat{F}_{Y^2}(t,T),t,T)\big),\ \ \phi^2_t = N\big(d_+(\widehat{F}_{Y^2}(t,T),t,T)\big).$$

4.4 PDE Approach

In the remaining part of this chapter, in which we follow Bielecki et al. [15] (see also Vellekoop et al. [169] and Rutkowski and Yousiph [157]), we take a different perspective. We assume that trading occurs on the time interval $[0,T]$ and we consider a contingent claim settling at time T of the form

$$Y = G(Y^1_T, Y^2_T, Y^3_T, H_T) = \mathbb{1}_{\{T\geq\tau\}}g_1(Y^1_T, Y^2_T, Y^3_T) + \mathbb{1}_{\{T<\tau\}}g_0(Y^1_T, Y^2_T, Y^3_T).$$

We do not need to assume here that the coefficients in the dynamics of primary assets are $\mathbb{F}$-predictable. Since our goal is to develop the PDE approach, it will be essential to postulate a Markovian character of a model. For the sake of simplicity, we use the notation with constant coefficients, so that we write, for $i = 1,2,3$,

$$dY^i_t = Y^i_{t-}\big(\mu_i\,dt + \sigma_i\,dW_t + \kappa_i\,dM_t\big).$$

The assumption of constant coefficients is rarely, if ever, satisfied in practically relevant models of credit risk. It is thus important to stress that it is postulated here mainly for the sake of notational convenience and the results established in this section cover also the non-homogeneous Markov case in which $\mu_{i,t} = \mu_i(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-}, H_{t-})$, $\sigma_{i,t} = \sigma_i(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-}, H_{t-})$, etc.

4.4.1 Defaultable Asset with Zero Recovery

We first assume that Y^1 and Y^2 are default-free, so that $\kappa_1 = \kappa_2 = 0$, and the third asset is subject to total default, that is, $\kappa_3 = -1$ and thus

$$dY^3_t = Y^3_{t-}\big(\mu_3\, dt + \sigma_3\, dW_t - dM_t\big).$$

We work throughout under the assumptions of Proposition 4.3.1. This means that any $\mathbb{Q}^1$-integrable contingent claim $Y = G(Y^1_T, Y^2_T, Y^3_T; H_T)$ is attainable and its arbitrage price equals, for every $t \in [0,T]$,

$$\pi_t(Y) = Y^1_t\, \mathbb{E}_{\mathbb{Q}^1}(Y (Y^1_T)^{-1} \,|\, \mathcal{G}_t). \tag{4.41}$$

The following auxiliary result is thus rather obvious.

Lemma 4.4.1. *The process (Y^1, Y^2, Y^3, H) has the Markov property with respect to the filtration $\mathbb{G}$ under the martingale measure $\mathbb{Q}^1$. Consequently, for any attainable claim $Y = G(Y^1_T, Y^2_T, Y^3_T; H_T)$ there exists a pricing function $v : [0,T] \times \mathbb{R}^3 \times \{0,1\} \to \mathbb{R}$ such that $\pi_t(Y) = v(t, Y^1_t, Y^2_t, Y^3_t; H_t)$.*

We introduce the *pre-default* pricing function $v(\cdot\,;0) = v(t, y_1, y_2, y_3; 0)$ and the *post-default* pricing function $v(\cdot\,;1) = v(t, y_1, y_2, y_3; 1)$.

In fact, since we manifestly have that $Y^3_t = 0$ if $H_t = 1$, it suffices to study the post-default function $v(t, y_1, y_2; 1) = v(t, y_1, y_2, 0; 1)$. We denote

$$\alpha_i = \mu_i - \sigma_i \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2},\ b = (\mu_3 - \mu_1)(\sigma_1 - \sigma_2) - (\mu_1 - \mu_3)(\sigma_1 - \sigma_3).$$

Let $\gamma > 0$ be the default intensity under $\mathbb{P}$ and let $\zeta > -1$ be given by (4.24).

Proposition 4.4.1. *Assume that the functions $v(\cdot\,;0)$ and $v(\cdot\,;1)$ belong to the class $\mathrm{C}^{1,2}([0,T] \times \mathbb{R}^3_+, \mathbb{R})$. Then $v(t, y_1, y_2, y_3; 0)$ satisfies the PDE*

$$\begin{aligned}\partial_t v(\cdot\,;0) + \sum_{i=1}^{2} \alpha_i y_i \partial_i v(\cdot\,;0) + (\alpha_3 + \zeta) y_3 \partial_3 v(\cdot\,;0) + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} v(\cdot\,;0) \\ - \alpha_1 v(\cdot\,;0) + \Big(\gamma - \frac{b}{\sigma_1 - \sigma_2}\Big)\big[v(t, y_1, y_2; 1) - v(t, y_1, y_2, y_3; 0)\big] = 0\end{aligned}$$

with the terminal condition $v(T, y_1, y_2, y_3; 0) = G(y_1, y_2, y_3; 0)$. *Furthermore, the function* $v(t, y_1, y_2; 1)$ *satisfies the PDE*

$$\partial_t v(\cdot\,;1) + \sum_{i=1}^{2} \alpha_i y_i \partial_i v(\cdot\,;1) + \frac{1}{2} \sum_{i,j=1}^{2} \sigma_i \sigma_j y_i y_j \partial_{ij} v(\cdot\,;1) - \alpha_1 v(\cdot\,;1) = 0$$

with the terminal condition $v(T, y_1, y_2; 1) = G(y_1, y_2, 0; 1)$.

Proof. For simplicity, we write $C_t = \pi_t(Y)$. Let us define

$$\Delta v(t, y_1, y_2, y_3) = v(t, y_1, y_2; 1) - v(t, y_1, y_2, y_3; 0).$$

Then the jump $\Delta C_t = C_t - C_{t-}$ can also be represented as follows

$$\mathbb{1}_{\{\tau=t\}} \big(v(t, Y_t^1, Y_t^2; 1) - v(t, Y_t^1, Y_t^2, Y_{t-}^3; 0) \big) = \mathbb{1}_{\{\tau=t\}} \, \Delta v(t, Y_t^1, Y_t^2, Y_{t-}^3).$$

We write ∂_i to denote the partial derivative with respect to the variable y_i and we typically omit the variables $(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3, H_{t-})$ in expressions $\partial_t v$, $\partial_i v$, Δv, etc. We shall also make use of the fact that for any Borel measurable function g we have

$$\int_0^t g(u, Y_u^2, Y_{u-}^3)\, du = \int_0^t g(u, Y_u^2, Y_u^3)\, du$$

since Y_u^3 and Y_{u-}^3 differ only for at most one value of u (for each ω). Let $\xi_t = \mathbb{1}_{\{t<\tau\}}\gamma$. An application of Itô's formula yields

$$\begin{aligned} dC_t &= \partial_t v\, dt + \sum_{i=1}^{3} \partial_i v\, dY_t^i + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v\, dt \\ &\quad + \Big(\Delta v + Y_{t-}^3 \partial_3 v \Big)\, dH_t \\ &= \partial_t v\, dt + \sum_{i=1}^{3} \partial_i v\, dY_t^i + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} v\, dt \\ &\quad + \Big(\Delta v + Y_{t-}^3 \partial_3 v \Big) \big(dM_t + \xi_t\, dt \big), \end{aligned}$$

and this in turn implies that

$$
\begin{aligned}
dC_t &= \partial_t v\,dt + \sum_{i=1}^{3} Y^i_{t-}\partial_i v\big(\mu_i\,dt + \sigma_i\,dW_t\big) + \frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v\,dt \\
&\quad + \Delta v\,dM_t + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t\,dt \\
&= \Big\{\partial_t v + \sum_{i=1}^{3}\mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v\Big\}\,dt \\
&\qquad \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t\,dt + \Big(\sum_{i=1}^{3}\sigma_i Y^i_{t-}\partial_i v\Big)\,dW_t + \Delta v\,dM_t.
\end{aligned}
$$

The Itô integration by parts formula and (4.17) yield for $\widehat{C}_t = C_t(Y^1_t)^{-1}$

$$
\begin{aligned}
d\widehat{C}_t &= \widehat{C}_{t-}\Big((-\mu_1 + \sigma_1^2)\,dt - \sigma_1\,dW_t\Big) + (Y^1_{t-})^{-1}\Big(\partial_t v + \sum_{i=1}^{3}\mu_i Y^i_{t-}\partial_i v\Big)\,dt \\
&+ (Y^1_{t-})^{-1}\Big\{\frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t\Big\}\,dt \\
&+ (Y^1_{t-})^{-1}\sum_{i=1}^{3}\sigma_i Y^i_{t-}\partial_i v\,dW_t + (Y^1_{t-})^{-1}\Delta v\,dM_t \\
&- (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^{3}\sigma_i Y^i_{t-}\partial_i v\,dt.
\end{aligned}
$$

Using (4.22)–(4.23), we obtain

$$
\begin{aligned}
d\widehat{C}_t &= \widehat{C}_{t-}\Big(\big(-\mu_1 + \sigma_1^2 - \sigma_1\theta\big)\,dt - \sigma_1\,d\widehat{W}_t\Big) + (Y^1_{t-})^{-1}\sum_{i=1}^{3}\mu_i Y^i_{t-}\partial_i v\,dt \\
&+ (Y^1_{t-})^{-1}\Big\{\partial_t v + \frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t\Big\}\,dt \\
&+ (Y^1_{t-})^{-1}\sum_{i=1}^{3}\sigma_i Y^i_{t-}\partial_i v\,d\widehat{W}_t + (Y^1_{t-})^{-1}\sum_{i=1}^{3}\sigma_i Y^i_{t-}\theta\partial_i v\,dt \\
&+ (Y^1_{t-})^{-1}\Delta v\,d\widehat{M}_t + (Y^1_{t-})^{-1}\zeta\xi_t\Delta v\,dt - (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^{3}\sigma^i Y^i_{t-}\partial_i v\,dt.
\end{aligned}
$$

Hence the process $\widehat{C}$ admits the following decomposition under $\mathbb{Q}^1$

$$\begin{aligned}
d\widehat{C}_t &= \widehat{C}_{t-}\big(-\mu_1+\sigma_1^2-\sigma_1\theta\big)\,dt + (Y^1_{t-})^{-1}\sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v\,dt \\
&+ (Y^1_{t-})^{-1}\Big\{\partial_t v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t\Big\}\,dt \\
&+ (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i v\,dt + (Y^1_{t-})^{-1}\zeta\xi_t\Delta v\,dt \\
&- (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v\,dt + \text{a } \mathbb{Q}^1\text{-martingale.}
\end{aligned}$$

From (4.41), it follows that the process $\widehat{C}$ is a martingale under $\mathbb{Q}^1$. Therefore, the continuous finite variation part in the above decomposition necessarily vanishes, and thus we get

$$\begin{aligned}
0 &= C_{t-}(Y^1_{t-})^{-1}\big(-\mu_1+\sigma_1^2-\sigma_1\theta\big) + (Y^1_{t-})^{-1}\sum_{i=1}^3 \mu_i Y^i_t\partial_i v \\
&+ (Y^1_{t-})^{-1}\Big\{\partial_t v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_t Y^j_t\partial_{ij}v + \big(\Delta v + Y^3_t\partial_3 v\big)\,\xi_t\Big\} \\
&+ (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i v + (Y^1_{t-})^{-1}\zeta\xi_t\Delta v - (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v.
\end{aligned}$$

Consequently, we have that

$$\begin{aligned}
0 &= C_{t-}\big(-\mu_1+\sigma_1^2-\sigma_1\theta\big) \\
&+ \partial_t v + \sum_{i=1}^3 \mu_i Y^i_t\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_t Y^j_t\partial_{ij}v + \big(\Delta v + Y^3_t\partial_3 v\big)\,\xi_t \\
&+ \sum_{i=1}^3 \sigma_i Y^i_t\theta\partial_i v + \zeta\xi_t\Delta v - \sigma_1\sum_{i=1}^3 \sigma_i Y^i_t\partial_i v.
\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}
&\partial_t v + \sum_{i=1}^2 \alpha_i Y^i_t\partial_i v + (\alpha_3+\xi_t)\,Y^3_t\partial_3 v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_t Y^j_t\partial_{ij}v \\
&\quad - \alpha_1 C_{t-} + (1+\zeta)\xi_t\Delta v = 0.
\end{aligned}$$

Recall that $\xi_t = \mathbb{1}_{\{t<\tau\}}\gamma$. It is thus clear that the functions $v(\cdot, 0)$ and $v(\cdot; 1)$ satisfy the PDEs given in the statement of the proposition. □

It should be stressed that in what follows we only examine the form of a replicating strategy prior to default time.

Proposition 4.4.2. *The replicating strategy ϕ for the claim Y is given by formulae*

$$\begin{aligned}\phi^3_t Y^3_{t-} &= -\Delta v(t, Y^1_t, Y^2_t, Y^3_{t-}) = v(t, Y^1_t, Y^2_t, Y^3_{t-}; 0) - v(t, Y^1_t, Y^2_t; 1),\\ \phi^2_t Y^2_t(\sigma_2 - \sigma_1) &= -(\sigma_1 - \sigma_3)\Delta v - \sigma_1 v + \sum_{i=1}^{3} Y^i_{t-}\sigma_i \partial_i v,\\ \phi^1_t Y^1_t &= v - \phi^2_t Y^2_t - \phi^3_t Y^3_t.\end{aligned}$$

Proof. Let us sketch the proof. As a by-product of our computations, we obtain

$$d\widehat{C}_t = -(Y^1_t)^{-1}\sigma_1 v\, d\widehat{W}_t + (Y^1_t)^{-1}\sum_{i=1}^{3}\sigma_i Y^i_{t-}\partial_i v\, d\widehat{W}_t + (Y^1_t)^{-1}\Delta v\, d\widehat{M}_t.$$

The self-financing strategy that replicates Y is determined by two components ϕ^2, ϕ^3 and the following relationship

$$\begin{aligned}d\widehat{C}_t &= \phi^2_t\, dY^{2,1}_t + \phi^3_t\, dY^{3,1}_t\\ &= \phi^2_t Y^{2,1}_t(\sigma_2 - \sigma_1)\, d\widehat{W}_t + \phi^3_t Y^{3,1}_{t-}\Big((\sigma_3 - \sigma_1)\, d\widehat{W}_t - d\widehat{M}_t\Big).\end{aligned}$$

By identification, we thus obtain $\phi^3_t Y^{3,1}_{t-} = (Y^1_t)^{-1}\Delta v$ and

$$\phi^2_t Y^2_t(\sigma_2 - \sigma_1) - (\sigma_3 - \sigma_1)\Delta v = -\sigma_1 C_t + \sum_{i=1}^{3} Y^i_{t-}\sigma_i \partial_i v.$$

This yields the required formulae. □

Corollary 4.4.1. *In the case of a defaultable claim with zero recovery, the hedging strategy satisfies the balance condition $\phi^1_t Y^1_t + \phi^2_t Y^2_t = 0$ for every $t \in [0, T]$.*

Proof. A zero recovery corresponds to the equality $G(y_1, y_2, y_3, 1) = 0$. We now have $v(t, y_1, y_2; 1) = 0$ and thus necessarily

$$\phi^3_t Y^3_{t-} = v(t, Y^1_t, Y^2_t, Y^3_{t-}; 0)$$

for every $t \in [0, T]$. Hence the equality $\phi^1_t Y^1_t + \phi^2_t Y^2_t = 0$ holds for every $t \in [0, T]$. The last equality is the balance condition for $Z = 0$; it ensures that the wealth of a replicating portfolio jumps to zero at default time. □

Hedging with the Savings Account

Let us now study the particular case where Y^1 is the savings account, i.e.,

$$dY_t^1 = rY_t^1\,dt, \quad Y_0^1 = 1.$$

Of course, this corresponds to $\mu_1 = r$ and $\sigma_1 = 0$. Let $\widehat{r} = r + \widehat{\gamma}$, where $\widehat{\gamma}$, which equals

$$\widehat{\gamma} = \gamma(1+\zeta) = \gamma + \mu_3 - r + \frac{\sigma_3}{\sigma_2}(r - \mu_2),$$

represents the default intensity under the martingale measure $\mathbb{Q}^1$. The quantity $\widehat{r}$ defined above has a rather natural interpretation as the risk-neutral *credit-risk adjusted* short-term interest rate. Straightforward calculations yield the following corollary to Proposition 4.4.1.

Corollary 4.4.2. *Assume that* $\sigma_2 \neq 0$ *and*

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\big(\mu_2\,dt + \sigma_2\,dW_t\big), \\ dY_t^3 &= Y_{t-}^3\big(\mu_3\,dt + \sigma_3\,dW_t - dM_t\big). \end{aligned}$$

Then the function $v(\cdot\,;0)$ *satisfies*

$$\begin{aligned} &\partial_t v(t,y_2,y_3;0) + ry_2\partial_2 v(t,y_2,y_3;0) + \widehat{r}y_3\partial_3 v(t,y_2,y_3;0) - \widehat{r}v(t,y_2,y_3;0) \\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;0) + \widehat{\gamma}v(t,y_2;1) = 0 \end{aligned}$$

with $v(T,y_2,y_3;0) = G(y_2,y_3;0)$ *and the function* $v(\cdot\,;1)$ *satisfies*

$$\partial_t v(t,y_2;1) + ry_2\partial_2 v(t,y_2;1) + \tfrac{1}{2}\sigma_2^2 y_2^2 \partial_{22} v(t,y_2;1) - rv(t,y_2;1) = 0$$

with $v(T,y_2;1) = G(y_2,0;1)$.

In the special case of a survival claim, the function $v(\cdot\,;1)$ vanishes identically since the value of the claim after default is obviously zero, and thus the following result can be established.

Corollary 4.4.3. *The pre-default pricing function* $v(\cdot\,;0)$ *of a survival claim* $Y = \mathbb{1}_{\{T<\tau\}}G(Y_T^2, Y_T^3)$ *is a solution of the following PDE*

$$\begin{aligned} &\partial_t v(t,y_2,y_3;0) + ry_2\partial_2 v(t,y_2,y_3;0) + \widehat{r}y_3\partial_3 v(t,y_2,y_3;0) \\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;0) - \widehat{r}v(t,y_2,y_3;0) = 0 \end{aligned}$$

with the terminal condition $v(T,y_2,y_3;0) = G(y_2,y_3)$.

The replicating strategy ϕ satisfies, on the event $\{t < \tau\}$,

$$\begin{aligned}
\phi_t^2 &= \frac{1}{\sigma_2 Y_t^2} \sum_{i=2}^{3} \sigma_i Y_{t-}^i \partial_i v(t, Y_t^2, Y_{t-}^3; 0) + \sigma_3 v(t, Y_t^2, Y_{t-}^3; 0), \\
\phi_t^3 &= (Y_{t-}^3)^{-1} v(t, Y_t^2, Y_{t-}^3; 0), \\
\phi_t^1 &= e^{-rt} \big(C_t - \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 \big),
\end{aligned}$$

where C is the price of Y, that is, $C_t = e^{-r(T-t)}\, \mathbb{E}_{\mathbb{Q}^1}(Y \,|\, \mathcal{G}_t)$.

Example 4.4.1. Consider a survival claim $Y = \mathbb{1}_{\{T<\tau\}} g(Y_T^2)$, that is, a vulnerable claim with a default-free underlying asset. Its pre-default pricing function $v(\cdot\,; 0)$ does not depend on y_3 and satisfies the following PDE

$$\partial_t v(t, y_2; 0) + r y_2 \partial_2 v(t, y_2; 0) + \tfrac{1}{2} \sigma_2^2 y_2^2 \partial_{22} v(t, y_2; 0) - \widehat{r} v(t, y_2; 0) = 0$$

with the terminal condition $v(T, y_2; 0) = g(y_2)$. One can check that the solution to this PDE can be represented as follows

$$v(t, y_2) = e^{-(\widehat{r}-r)(T-t)}\, v_g^{r,\sigma_2}(t, y_2) = e^{-\widehat{\gamma}(T-t)}\, v_g^{r,\sigma_2}(t, y_2),$$

where the function $v_g^{r,\sigma_2}(t, y_2)$ is the price of the default-free claim $g(Y_T^2)$ when the dynamics of price processes (Y^1, Y^2) are given by the Black-Scholes model with the interest rate r and the volatility parameter σ_2.

4.4.2 Defaultable Asset with Non-Zero Recovery

We now assume that the price of a defaultable asset is governed by the SDE

$$dY_t^3 = Y_{t-}^3 (\mu_3\, dt + \sigma_3\, dW_t + \kappa_3\, dM_t)$$

with $\kappa_3 > -1$ and $\kappa_3 \neq 0$. We assume that $Y_0^3 > 0$, so that the inequality $Y_t^3 > 0$ is valid for every $t \in \mathbb{R}_+$. We shall briefly describe the same steps as in the case of a defaultable asset with zero recovery.

Arbitrage-Free Property

Assume that the prices Y^1, Y^2, Y^3 of traded assets are governed by the following equations

$$\begin{aligned}
dY_t^1 &= r Y_t^1\, dt, \\
dY_t^2 &= Y_t^2 \big(\mu_2\, dt + \sigma_2\, dW_t \big), \\
dY_t^3 &= Y_{t-}^3 \big(\mu_3\, dt + \sigma_3\, dW_t + \kappa_3\, dM_t \big),
\end{aligned}$$

where we postulate that $\sigma_2 \neq 0$ and $\sigma_3 \neq 0$.

The existence of an EMM for this model was examined in Section 4.3.10. Recall that in order to ensure the existence of an EMM, on the event $\{t > \tau\}$, we need to impose the following condition

$$\frac{r-\mu_2}{\sigma_2} = \frac{r-\mu_3}{\sigma_3},$$

that is,

$$r(\sigma_3 - \sigma_2) - \mu_2\sigma_3 + \mu_3\sigma_2 = 0.$$

Furthermore, on the event $\{t \le \tau\}$, we obtain the following equations

$$\begin{aligned} \theta_t\sigma_2 &= r - \mu_2, \\ \theta_t\sigma_3 + \zeta_t\gamma\kappa_3 &= r - \mu_3 + \sigma_1. \end{aligned}$$

If, in addition, $(\sigma_2 - \sigma_1)\kappa_3 \neq 0$, we obtain the unique solution

$$\begin{aligned} &\theta = \frac{r-\mu_2}{\sigma_2} = \frac{r-\mu_3}{\sigma_3}, \\ &\zeta = 0 > -1, \end{aligned}$$

so that the martingale measure $\mathbb{Q}^1$ for $Y^{2,1}$ and $Y^{3,1}$ exists and is unique.

Pricing PDE and Replicating Strategy

We are in a position to derive the pricing PDEs. For the sake of simplicity, we assume that Y^1 is the savings account, so that the foregoing result is a counterpart of Corollary 4.4.2. For the proof of Proposition 4.4.3, the interested reader is referred to Bielecki et al. [15].

Proposition 4.4.3. *Let $\sigma_2 \neq 0$ and let the price processes Y^1, Y^2, Y^3 satisfy*

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\big(\mu_2\,dt + \sigma_2\,dW_t\big), \\ dY_t^3 &= Y_{t-}^3\big(\mu_3\,dt + \sigma_3\,dW_t + \kappa_3\,dM_t\big). \end{aligned}$$

Assume, in addition, that $\sigma_2(r-\mu_3) = \sigma_3(r-\mu_2)$ and $\kappa_3 \neq 0$, $\kappa_3 > -1$. Then the price of a contingent claim $Y = G(Y_T^2, Y_T^3, H_T)$ can be represented as $\pi_t(Y) = v(t, Y_t^2, Y_t^3, H_t)$, where the pricing functions $v(\cdot\,;0)$ and $v(\cdot\,;1)$ satisfy the following PDEs

$$\begin{aligned} &\partial_t v(t,y_2,y_3;0) + ry_2\partial_2 v(t,y_2,y_3;0) + y_3\,(r-\kappa_3\gamma)\,\partial_3 v(t,y_2,y_3;0) \\ &- rv(t,y_2,y_3;0) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;0) \\ &+ \gamma\big(v(t,y_2,y_3(1+\kappa_3);1) - v(t,y_2,y_3;0)\big) = 0 \end{aligned}$$

and

$$\partial_t v(t,y_2,y_3;1) + ry_2\partial_2 v(t,y_2,y_3;1) + ry_3\partial_3 v(t,y_2,y_3;1) - rv(t,y_2,y_3;1) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;1) = 0$$

subject to the terminal conditions

$$v(T,y_2,y_3;0) = G(y_2,y_3;0), \quad v(T,y_2,y_3;1) = G(y_2,y_3;1).$$

The replicating strategy ϕ *satisfies, on the event* $\{t<\tau\}$,

$$\begin{aligned}\phi_t^2 &= \frac{1}{\sigma_2 Y_t^2}\sum_{i=2}^{3}\sigma_i y_i \partial_i v(t,Y_t^2,Y_{t-}^3,H_{t-}) \\ &\quad - \frac{\sigma_3}{\sigma_2\kappa_3 Y_t^2}\big(v(t,Y_t^2,Y_{t-}^3(1+\kappa_3);1) - v(t,Y_t^2,Y_{t-}^3;0)\big), \\ \phi_t^3 &= \frac{1}{\kappa_3 Y_{t-}^3}\big(v(t,Y_t^2,Y_{t-}^3(1+\kappa_3);1) - v(t,Y_t^2,Y_{t-}^3;0)\big), \\ \phi_t^1 &= e^{-rt}\big(C_t - \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3\big),\end{aligned}$$

where C *is the price of* Y, *that is,* $C_t = e^{-r(T-t)}\,\mathbb{E}_{\mathbb{Q}^1}(Y\,|\,\mathcal{G}_t)$ *for* $t\in[0,T]$.

Hedging of a Survival Claim

We shall now illustrate Proposition 4.4.3 by means of examples. As a first example, we will examine hedging of a survival claim Y of the form

$$Y = G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T<\tau\}}\, g(Y_T^3).$$

Then the post-default pricing function $v(\cdot\,;1)$ vanishes identically and the pre-default pricing function $v(\cdot\,;0)$ solves the PDE

$$\partial_t v(t,y_2,y_3;0) + ry_2\partial_2 v(t,y_2,y_3;0) + y_3\,(r-\kappa_3\gamma)\,\partial_3 v(t,y_2,y_3;0) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j\partial_{ij} v(t,y_2,y_3;0) - (r+\gamma)v(t,y_2,y_3;0) = 0$$

with the terminal condition $v(T,y_2,y_3;0) = g(y_3)$. Let us denote $\alpha = r-\kappa_3\gamma$ and $\beta = \gamma(1+\kappa_3)$. It is not difficult to check that the function

$$v(t,y_2,y_3;0) = e^{\beta(T-t)} v_g^{\alpha,\sigma_3}(t,y_3)$$

is a solution of the above equation, where the function $w(t,y_3) = v_g^{\alpha,\sigma_3}(t,y_3)$ is the solution of the following version of the Black-Scholes PDE

$$\partial_t w + \alpha y_3 \partial_{y_3} w + \tfrac{1}{2}\sigma_3^2 y_3^2 \partial_{y_3 y_3} w - \alpha w = 0$$

with the terminal condition $v_g^{\alpha,\sigma_3}(T, y_3) = g(y_3)$, that is, the price of the default-free claim $g(Y_T^3)$ when the dynamics of (Y^1, Y^3) are given by the Black-Scholes model with the interest rate $r = \alpha$ and the volatility σ_3.

Let C_t be the current value of the contingent claim Y, so that

$$C_t = \mathbb{1}_{\{t<\tau\}} e^{\beta(T-t)} v_g^{\alpha,\sigma_3}(t, Y_t^3).$$

The hedging strategy for the survival claim Y satisfies, on the event $\{t < \tau\}$,

$$\begin{aligned}
\phi_t^3 Y_t^3 &= -\frac{1}{\kappa_3} e^{-\beta(T-t)} v_g^{\alpha,\sigma_3}(t, Y_t^3) = -\frac{C_t}{\kappa_3}, \\
\phi_t^2 Y_t^2 &= \frac{\sigma_3}{\sigma_2} \left(Y_t^3 e^{-\beta(T-t)} \partial_y v_g^{\alpha,\sigma_3}(t, Y_t^3) - \phi_t^3 Y_t^3 \right), \\
\phi_t^1 Y_t^1 &= C_t - \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3.
\end{aligned}$$

Hedging of a Recovery Payoff

As another illustration of Proposition 4.4.3, we shall now consider the claim $G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T \geq \tau\}} g(Y_T^2)$, that is, we assume that recovery is paid at maturity and equals $g(Y_T^2)$. We argue that the post-default pricing function $v(\cdot\,; 1)$ is independent of y_3. Indeed, the post-default pricing PDE

$$\partial_t v(t, y_2, y_3; 1) + ry\partial_2 v(t, y_2, y_3; 1) + \tfrac{1}{2}\sigma_2^2 y^2 \partial_{22} v(t, y_2, y_3; 1) - rv(t, y_2, y_3; 1) = 0$$

with the terminal condition $v(T, y_2, y_3; 1) = g(y_2)$, admits a unique solution $v_g^{r,\sigma_2}(t, y_2)$, which is the price of $g(Y_T^2)$ in the Black-Scholes model with the interest rate r and the volatility σ_2. Prior to default, the price of the claim can be found by solving the following PDE

$$\begin{aligned}
&\partial_t v(t, y_2, y_3; 0) + ry_2 \partial_2 v(t, y_2, y_3; 0) + y_3(r - \kappa_3 \gamma) \partial_3 v(t, y_2, y_3; 0) \\
&\quad + \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} v(t, y_2, y_3; 0) - (r + \gamma) v(t, y_2, y_3; 0) = -\gamma v_g^{r,\sigma_2}(t, y_2)
\end{aligned}$$

with the terminal condition $v(T, y_2, y_3; 0) = 0$. It is not difficult to check that

$$v(t, y_2, y_3; 0) = (1 - e^{-\gamma(T-t)}) v_g^{r,\sigma_2}(t, y_2).$$

It could be instructive to compare this result with Example 4.4.1.

4.4.3 Two Defaultable Assets with Zero Recovery

We shall now assume that only two primary assets are traded, and they are defaultable assets with zero recovery. We postulate that, for $i = 1, 2$,

$$dY_t^i = Y_{t-}^i \big(\mu_i \, dt + \sigma_i \, dW_t - dM_t \big).$$

This means that $Y_t^i = \mathbb{1}_{\{t<\tau\}}\widetilde{Y}_t^i$, $i = 1, 2$, with the pre-default prices governed by the SDEs, for $i = 1, 2$,

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i\big((\mu_i + \gamma)\,dt + \sigma_i\,dW_t\big).$$

In the case where the promised payoff X of a survival claim $Y = X\mathbb{1}_{\{T<\tau\}}$ is path-independent, so that

$$Y = X\mathbb{1}_{\{T<\tau\}} = G(Y_T^1, Y_T^2)\mathbb{1}_{\{T<\tau\}} = G(\widetilde{Y}_T^1, \widetilde{Y}_T^2)\mathbb{1}_{\{T<\tau\}}$$

for some function G, it is possible to use the PDE approach in order to value and replicate a survival claim prior to default. Under the present assumptions, we need not to examine the balance condition, since, if default event occurs prior to the maturity date of the claim, the wealth of the portfolio will fall to zero, as it should in view of the equality $Z = 0$.

From the martingale approach presented in Section 4.3.11, we already know that hedging of a survival claim $Y = X\mathbb{1}_{\{T<\tau\}}$ is formally equivalent in the present framework to replication of the promised payoff $X = G(\widetilde{Y}_T^1, \widetilde{Y}_T^2)$ using the pre-default values $\widetilde{Y}^1$ and $\widetilde{Y}^2$ of traded assets.

We shall find the pre-default pricing function $v(t, y_1, y_2; 0)$, which is required to satisfy the terminal condition

$$v(T, y_1, y_2; 0) = G(y_1, y_2),$$

as well as the replicating strategy (ϕ^1, ϕ^2) for a survival claim. The replicating strategy ϕ is such that for the pre-default value $\widetilde{C}$ of the considered claim Y we have

$$\widetilde{C}_t := v(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2; 0) = \phi_t^1\widetilde{Y}_t^1 + \phi_t^2\widetilde{Y}_t^2,$$

and

$$d\widetilde{C}_t = \phi_t^1\,d\widetilde{Y}_t^1 + \phi_t^2\,d\widetilde{Y}_t^2. \tag{4.42}$$

The following result furnishes the pre-default pricing PDE and an explicit formulae for the replication strategy for a survival claim.

Proposition 4.4.4. *Assume that $\sigma_1 \neq \sigma_2$. Then the pre-default pricing function $v = v(t, y_1, y_2; 0)$ satisfies the PDE*

$$\begin{aligned}\partial_t v + y_1\Big(\mu_1 + \gamma - \sigma_1\frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}\Big)\partial_1 v + y_2\Big(\mu_2 + \gamma - \sigma_2\frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}\Big)\partial_2 v \\ + \frac{1}{2}\Big(y_1^2\sigma_1^2\partial_{11}v + y_2^2\sigma_2^2\partial_{22}v + 2y_1y_2\sigma_1\sigma_2\partial_{12}v\Big) = \Big(\mu_1 + \gamma - \sigma_1\frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}\Big)v\end{aligned}$$

with the terminal condition $v(T, y_1, y_2) = G(y_1, y_2)$. The replicating strategy satisfies

$$\phi_t^1\widetilde{Y}_t^1 + \phi_t^2\widetilde{Y}_t^2 = v(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2)$$

and

$$\phi_t^2 \widetilde{Y}_t^2 = \frac{\widetilde{Y}_t^1 \sigma_1 \partial_1 v(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2) + \widetilde{Y}_t^2 \sigma_2 \partial_2 v(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2) - \sigma_1 v(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2)}{\sigma_2 - \sigma_1}.$$

Proof. Let us sketch the derivation of the pricing PDE and the replicating strategy. By applying the Itô formula to $v(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2)$ and comparing the diffusion terms in (4.42) and in the Itô differential $dv(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2)$, we find that

$$y_1 \sigma_1 \partial_1 v + y_2 \sigma_2 \partial_2 v = \phi^1 y_1 \sigma_1 + \phi^2 y_2 \sigma_2, \tag{4.43}$$

where $\phi^i = \phi^i(t, y_1, y_2)$, $i = 1, 2$ is a replicating strategy. Since we have

$$\phi^1 y_1 = v(t, y_1, y_2) - \phi^2 y_2, \tag{4.44}$$

we deduce from (4.43) that

$$y_1 \sigma_1 \partial_1 v + y_2 \sigma_2 \partial_2 v = v \sigma_1 + \phi^2 y_2 (\sigma_2 - \sigma_1),$$

and thus the function ϕ^2 equals

$$\phi^2 y_2 = \frac{y_1 \sigma_1 \partial_1 v + y_2 \sigma_2 \partial_2 v - v \sigma_1}{\sigma_2 - \sigma_1}. \tag{4.45}$$

Furthermore, by identification of drift terms in (4.43), we obtain

$$\begin{aligned} &\partial_t v + y_1(\mu_1 + \gamma)\partial_1 v + y_2(\mu_2 + \gamma)\partial_2 v \\ &+ \frac{1}{2}\Big(y_1^2 \sigma_1^2 \partial_{11} v + y_2^2 \sigma_2^2 \partial_{22} v + 2 y_1 y_2 \sigma_1 \sigma_2 \partial_{12} v\Big) \\ &= \phi^1 y_1 (\mu_1 + \gamma) + \phi^2 y_2 (\mu_2 + \gamma). \end{aligned}$$

Upon elimination of ϕ^1 and ϕ^2, we arrive at the stated PDE. Formulae (4.44) and (4.45) yield the claimed equalities for the replicating strategy. □

Recall that the historically observed drift terms in dynamics of traded assets are $\widehat{\mu}_i = \mu_i + \gamma$, rather than μ_i. The pre-default pricing PDE derived in Proposition 4.4.4 can thus be represented as follows

$$\begin{aligned} \partial_t v &+ y_1 \left(\widehat{\mu}_1 - \sigma_1 \frac{\widehat{\mu}_2 - \widehat{\mu}_1}{\sigma_2 - \sigma_1} \right) \partial_1 v + y_2 \left(\widehat{\mu}_2 - \sigma_2 \frac{\widehat{\mu}_2 - \widehat{\mu}_1}{\sigma_2 - \sigma_1} \right) \partial_2 v \\ &+ \frac{1}{2}\Big(y_1^2 \sigma_1^2 \partial_{11} v + y_2^2 \sigma_2^2 \partial_{22} v + 2 y_1 y_2 \sigma_1 \sigma_2 \partial_{12} v\Big) = v \left(\widehat{\mu}_1 - \sigma_1 \frac{\widehat{\mu}_2 - \widehat{\mu}_1}{\sigma_2 - \sigma_1} \right). \end{aligned}$$

It is worth noting that the pre-default pricing function v does not depend on the default intensity.

In order to further simplify the pre-default pricing PDE for a survival claim, we will make an additional assumption about the corresponding payoff function G. Specifically, we suppose, in addition, that the payoff function G of our claim is such that

$$G(y_1, y_2) = y_1 g(y_2/y_1)$$

for a certain function $g : \mathbb{R}_+ \to \mathbb{R}$ or, equivalently, that the equality

$$G(y_1, y_2) = y_2 h(y_1/y_2)$$

holds for some function $h : \mathbb{R}_+ \to \mathbb{R}$. In that case, it is enough to focus on the relative pre-default prices defined as follows

$$\widehat{C}_t = \widetilde{C}_t(\widetilde{Y}_t^1)^{-1}, \quad \widetilde{Y}_t^{2,1} = \widetilde{Y}_t^2(\widetilde{Y}_t^1)^{-1}.$$

The corresponding pre-default pricing function $\widehat{v}(t, z)$, which is defined as the function such that the equality $\widehat{C}_t = \widehat{v}(t, \widetilde{Y}_t^{2,1})$ holds for every $t \in [0, T]$, satisfies the following PDE

$$\partial_t \widehat{v} + \frac{1}{2}(\sigma_2 - \sigma_1)^2 z^2 \partial_{zz} \widehat{v} = 0$$

with the terminal condition $\widehat{v}(T, z) = g(z)$.

We conclude that the pre-default price $\widetilde{C}_t = \widetilde{Y}_t^1 \widehat{v}(t, \widetilde{Y}_t^{2,1})$ does not depend directly on the drift coefficients $\widehat{\mu}_1$ and $\widehat{\mu}_2$. It is thus natural to conjecture that one should always be able to derive an expression for the arbitrage price of a defaultable claim in terms of market observables only, that is, the prices of the underlying assets, their volatilities and the correlation coefficient. Put another way, it natural to expect that neither the default intensity nor the drift coefficients of the underlying assets will appear explicitly as parameters in the formula for the pre-default pricing function.

Let us conclude this chapter by mentioning that we decided to present here only some special cases of semimartingale market models and pricing partial differential equations that were analyzed in papers by Bielecki et al. [15] and Rutkowski and Yousiph [157]. It is thus worth stressing that an extension of the PDE approach presented in this chapter to the case of any finite number of primary traded assets and several default times presents no essential technical difficulties. Note, however, that the study of the completeness of such a general semimartingale model with several default times requires a detailed specification of properties of traded assets and their dependencies and a thorough examination of the existence and uniqueness of an equivalent martingale measure.

Chapter 5

Modeling Dependent Defaults

The issue of modeling dependent defaults is one of the most important and challenging research areas in credit risk modeling and thus it attracted attention of numerous researchers in recent years. In this chapter, we describe the case of conditionally independent default times, the industry standard copula-based approach, as well as the Jarrow and Yu [107] approach to the modeling of contagious defaults through dependent stochastic intensities. We conclude by presenting one of many alternative approaches that were recently developed for the purpose of modeling joint credit ratings migrations for several firms. Let us observe that the valuation of basket credit derivatives covers, in particular, the following instruments:

- classic *first-to-default swaps*, which are aimed to offer protection against the first default in a reference basket of credit names (e.g., Duffie [71] or Kijima and Muromachi [122])
- general kth-to-default contracts, which give protection against the first k defaults in a reference basket of credit names (e.g., Bielecki and Rutkowski [24] or Brasch [32]).

Modeling issues arising in the context of portfolio credit derivatives include:

- correlated defaults in the structural framework (Zhou [175]),
- conditionally independent default times (Kijima and Muromachi [122]),
- simulation of correlated defaults (Duffie and Singleton [76]),
- modeling of infectious defaults (Davis and Lo [64]),
- asymmetric default intensities (Jarrow and Yu [107]),
- copulae (Laurent and Gregory [133], Schönbucher and Schubert [161]),
- dependent credit ratings (Lando [127], Bielecki and Rutkowski [23]),
- dependent credit migrations (Kijima et al. [120]),
- modeling defaults via the Marshall-Olkin copula (Elouerkhaoui [81]),
- modeling of losses for a large portfolio (Frey and McNeil [88]).

5.1 Basket Credit Derivatives

Basket credit derivatives are credit derivatives deriving their cash flows (and thus their values) from credit risks of several reference entities (or prespecified credit events).

Standing assumptions. We assume that:

- we are given a collection of default times $\tau_1, \tau_2, \ldots, \tau_n$ defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{Q})$,
- $\mathbb{Q}(\tau_i = 0) = 0$ and $\mathbb{Q}(\tau_i > t) > 0$ for every i and t,
- $\mathbb{Q}(\tau_i = \tau_j) = 0$ for arbitrary $i \neq j$ (in a continuous time setup).

For a given collection $\tau_1, \tau_2, \ldots, \tau_n$ of default times, we define the ordered sequence $\tau_{(1)} < \tau_{(2)} < \cdots < \tau_{(n)}$, where $\tau_{(k)}$ stands for the random time of the kth default. Formally, we set

$$\tau_{(1)} = \min\{\tau_1, \tau_2, \ldots, \tau_n\}$$

and, recursively, for $k = 2, 3, \ldots, n$

$$\tau_{(k)} = \min\left\{\tau_i : i = 1, 2, \ldots, n,\ \tau_i > \tau_{(k-1)}\right\}.$$

In particular, $\tau_{(n)}$ represents the moment of the last default, that is,

$$\tau_{(n)} = \max\{\tau_1, \tau_2, \ldots, \tau_n\}.$$

5.1.1 The kth-to-Default Contingent Claims

We set $H_t^i = \mathbb{1}_{\{t \geq \tau_i\}}$ and we denote by $\mathbb{H}^i$ the filtration generated by the process H^i, that is, by the observations of the default time τ_i. In addition, we are given a reference filtration $\mathbb{F}$ on the space $(\Omega, \mathcal{G}, \mathbb{Q})$. The filtration $\mathbb{F}$ is related to some other market risks, for instance, to the interest rate risk. Finally, we introduce the enlarged filtration $\mathbb{G}$ by setting

$$\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2 \vee \cdots \vee \mathbb{H}^n.$$

Note that the σ-field $\mathcal{G}_t$ models the total information available to market participants at time t.

A general kth-to-default contingent claim, which matures at time T, is formally specified by the following covenants:

- if $\tau_{(k)} = \tau_i \leq T$ for some $i = 1, 2, \ldots, n$ then the claim pays at time $\tau_{(k)}$ the amount $Z^i_{\tau_{(k)}}$, where Z^i is an $\mathbb{F}$-predictable recovery process,
- if $\tau_{(k)} > T$ then the claim pays at time T an $\mathcal{F}_T$-measurable promised amount X.

5.1.2 Case of Two Credit Names

For the sake of notational simplicity, we shall frequently consider the case of two reference credit names. In that case, the cash flows of considered contracts can be described as follows.

Cash flows of a *first-to-default claim* (FTDC):

- if $\tau_{(1)} = \min\{\tau_1, \tau_2\} = \tau_i \le T$ for $i = 1, 2$, the claim pays at time τ_i the amount $Z^i_{\tau_i}$,
- if $\min\{\tau_1, \tau_2\} > T$, it pays at time T the amount X.

Cash flows of a *last-to-default claim* (LTDC):

- if $\tau_{(2)} = \max\{\tau_1, \tau_2\} = \tau_i \le T$ for $i = 1, 2$, the claim pays at time τ_i the amount $Z^i_{\tau_i}$,
- if $\max\{\tau_1, \tau_2\} > T$, it pays at time T the amount X.

We recall that the savings account B equals

$$B_t = \exp\Big(\int_0^t r_u\, du\Big),$$

and the probability measure $\mathbb{Q}$ is interpreted as a martingale measure for our model of the financial market, which is assumed to include defaultable securities. Consequently, the price $B(t,T)$ of a zero-coupon default-free bond maturing at T equals, for every $t \in [0,T]$,

$$B(t,T) = B_t\, \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} \,|\, \mathcal{G}_t\big).$$

Pricing of FTDC and LTDC

In general, the ex-dividend price at time t of a defaultable claim (X, Z, τ) is given by the *risk-neutral valuation formula*

$$S_t = B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t\Big)$$

where D is the *dividend process,* which describes all cash flows associated with a given defaultable claim. Consequently, the ex-dividend price at any date $t \in [0,T]$ of an FTDC is given by the expression

$$\begin{aligned} S_t^{(1)} &= B_t\, \mathbb{E}_{\mathbb{Q}}\Big(B_{\tau_1}^{-1} Z^1_{\tau_1} \mathbb{1}_{\{\tau_1 < \tau_2,\, t < \tau_1 \le T\}} \,\Big|\, \mathcal{G}_t\Big) \\ &\quad + B_t\, \mathbb{E}_{\mathbb{Q}}\Big(B_{\tau_2}^{-1} Z^2_{\tau_2} \mathbb{1}_{\{\tau_2 < \tau_1,\, t < \tau_2 \le T\}} \,\Big|\, \mathcal{G}_t\Big) \\ &\quad + B_t\, \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} X \mathbb{1}_{\{T < \tau_{(1)}\}} \,\Big|\, \mathcal{G}_t\Big). \end{aligned}$$

Similarly, the ex-dividend price of an LTDC equals, for every $t \in [0, T]$,

$$\begin{aligned} S_t^{(2)} &= B_t \, \mathbb{E}_{\mathbb{Q}} \Big(B_{\tau_1}^{-1} Z_{\tau_1}^1 \mathbb{1}_{\{\tau_2 < \tau_1 ,\, t < \tau_1 \le T\}} \, \Big| \, \mathcal{G}_t \Big) \\ &\quad + B_t \, \mathbb{E}_{\mathbb{Q}} \Big(B_{\tau_2}^{-1} Z_{\tau_2}^2 \mathbb{1}_{\{\tau_1 < \tau_2 ,\, t < \tau_2 \le T\}} \, \Big| \, \mathcal{G}_t \Big) \\ &\quad + B_t \, \mathbb{E}_{\mathbb{Q}} \Big(B_T^{-1} X \mathbb{1}_{\{T < \tau_{(2)}\}} \, \Big| \, \mathcal{G}_t \Big). \end{aligned}$$

Both expressions above are merely special cases of a general formula. The goal is to either derive more explicit representations under various assumptions about τ_1 and τ_2 or to provide ways of efficient calculation of involved expected values by means of Monte Carlo simulation (using perhaps an equivalent probability measure).

5.2 Conditionally Independent Defaults

The concept of conditional independence of default times with respect to a reference filtration $\mathbb{F}$ is defined as follows.

Definition 5.2.1. The random times τ_i, $i = 1, 2, \ldots, n$ are said to be *conditionally independent* with respect to $\mathbb{F}$ under $\mathbb{Q}$ if we have, for any $T > 0$ and any $t_1, \ldots, t_n \in [0, T]$,

$$\mathbb{Q}(\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T) = \prod_{i=1}^{n} \mathbb{Q}(\tau_i > t_i \,|\, \mathcal{F}_T).$$

Let us comment briefly on Definition 5.2.1.

- Conditional independence has the following intuitive interpretation: the reference credits (credit names) are subject to common risk factors that may trigger credit (default) events. In addition, each credit name is subject to idiosyncratic risks that are specific for this name.
- Conditional independence of default times means that once the common risk factors are fixed then the idiosyncratic risk factors are independent of each other. This means that most computations can be done similarly as in the case of independent default times.
- It is worth stressing that the property of conditional independence is not invariant with respect to an equivalent change of a probability measure (for a suitable counter-example, see Section 5.7).

5.2.1 Canonical Construction

Let Γ^i, $i = 1, 2, \ldots, n$ be a given family of $\mathbb{F}$-adapted, increasing, continuous processes, defined on a probability space $(\tilde{\Omega}, \mathbb{F}, \tilde{\mathbb{P}})$. We make the standard assumptions that $\Gamma_0^i = 0$ and $\Gamma_\infty^i = \infty$. Let $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ be an auxiliary probability space with a sequence ξ_i, $i = 1, 2, \ldots, n$ of independent random variables uniformly distributed on $[0, 1]$. We define $\tau_1, \ldots, \tau_n$ by setting

$$\tau_i(\tilde{\omega}, \widehat{\omega}) = \inf \{ t \in \mathbb{R}_+ : \Gamma_t^i(\tilde{\omega}) \geq -\ln \xi_i(\widehat{\omega}) \}$$

for every elementary event $(\tilde{\omega}, \widehat{\omega})$ belonging to the product probability space $(\Omega, \mathcal{G}, \mathbb{Q}) = (\tilde{\Omega} \times \widehat{\Omega}, \mathcal{F}_\infty \otimes \widehat{\mathcal{F}}, \tilde{\mathbb{P}} \otimes \widehat{\mathbb{P}})$. We endow the space $(\Omega, \mathcal{G}, \mathbb{Q})$ with the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n$.

Proposition 5.2.1. *Let $\xi_1, \ldots, \xi_n$ be independent random variables uniformly distributed on $[0, 1]$. Then Γ^i is the $\mathbb{F}$-hazard process of τ_i and thus, for any $s \geq t$,*

$$\mathbb{Q}(\tau_i > s \,|\, \mathcal{F}_t \vee \mathcal{H}_t^i) = \mathbb{1}_{\{t < \tau_i\}} \, \mathbb{E}_{\mathbb{Q}}\big(e^{\Gamma_t^i - \Gamma_s^i} \,|\, \mathcal{F}_t\big).$$

We have that $\mathbb{Q}(\tau_i = \tau_j) = 0$ for every $i \neq j$ and the default times $\tau_1, \ldots, \tau_n$ are conditionally independent with respect to $\mathbb{F}$ under $\mathbb{Q}$.

Proof. It suffices to note that, for $t_i < T$,

$$\begin{aligned} \mathbb{Q}(\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T) &= \mathbb{Q}(\Gamma_{t_1}^1 \geq -\ln \xi_1, \ldots, \Gamma_{t_n}^n \geq -\ln \xi_n \,|\, \mathcal{F}_T) \\ &= \prod_{i=1}^n e^{-\Gamma_{t_i}^i}. \end{aligned}$$

The details are left to the reader. □

Recall that if $\Gamma_t^i = \int_0^t \gamma_u^i \, du$ then γ^i is the $\mathbb{F}$-intensity of τ_i. Intuitively,

$$\mathbb{Q}(\tau_i \in [t, t + dt] \,|\, \mathcal{F}_t \vee \mathcal{H}_t^i) \approx \mathbb{1}_{\{t < \tau_i\}} \gamma_t^i \, dt.$$

5.2.2 Hypothesis (H)

If the hypothesis (H) holds between the filtrations $\mathbb{F}$ and $\mathbb{G}$ then it also holds between the filtrations $\mathbb{F}$ and $\mathbb{F} \vee \mathbb{H}^{i_1} \vee \cdots \vee \mathbb{H}^{i_k}$ for any $i_1, \ldots, i_k$. However, there is no reason for the hypothesis (H) to hold between $\mathbb{F} \vee \mathbb{H}^{i_1}$ and $\mathbb{G}$. Note that, if the hypothesis (H) holds then one has, for every $t_1 \leq \cdots \leq t_n \leq T$,

$$\mathbb{Q}(\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T) = \mathbb{Q}(\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_\infty).$$

It is not difficult to check that the hypothesis (H) holds when the random times $\tau_1, \ldots, \tau_n$ are given by the canonical construction of Section 5.2.1.

5.2.3 Independent Default Times

We shall first examine the case of default times $\tau_1, \ldots, \tau_n$ that are independent under $\mathbb{Q}$. Suppose that for every $i = 1, 2, \ldots, n$ we know the cumulative distribution function $F_i(t) = \mathbb{Q}(\tau_i \le t)$ of the default time of the ith reference entity. The cumulative distribution functions of $\tau_{(1)}$ and $\tau_{(n)}$ are

$$F_{(1)}(t) = \mathbb{Q}(\tau_{(1)} \le t) = 1 - \prod_{i=1}^{n} (1 - F_i(t))$$

and

$$F_{(n)}(t) = \mathbb{Q}(\tau_{(n)} \le t) = \prod_{i=1}^{n} F_i(t).$$

More generally, we have that, for any $i = 1, 2, \ldots, n$,

$$F_{(i)}(t) = \mathbb{Q}(\tau_{(i)} \le t) = \sum_{m=i}^{n} \sum_{\pi \in \Pi^m} \prod_{j \in \pi} F_{k_j}(t) \prod_{l \notin \pi} (1 - F_{k_l}(t)),$$

where Π^m denote the family of all subsets of $\{1, 2, \ldots, n\}$ consisting of m elements. Suppose, in addition, that the default times $\tau_1, \ldots, \tau_n$ admit the corresponding intensity functions $\gamma_1(t), \ldots, \gamma_n(t)$, so that the processes

$$H_t^i - \int_0^{t \wedge \tau_i} \gamma_i(u)\, du$$

are known to be $\mathbb{H}^i$-martingales. Recall that $\mathbb{Q}(\tau_i > t) = e^{-\int_0^t \gamma_i(u)\, du}$. It is then easily seen that, for every $t \in \mathbb{R}_+$,

$$\mathbb{Q}(\tau_{(1)} > t) = \prod_{i=1}^{n} \mathbb{Q}(\tau_i > t) = e^{-\int_0^t \gamma_{(1)}(u)\, du},$$

where $\gamma_{(1)}(t) = \gamma_1(t) + \cdots + \gamma_n(t)$ for every $t \in \mathbb{R}_+$. Therefore, the process

$$H_t^{(1)} - \int_0^{t \wedge \tau_{(1)}} \gamma_{(1)}(u)\, du$$

follows an $\mathbb{H}^{(1)}$-martingale, where the filtration $\mathbb{H}^{(1)}$ is generated by the process $\mathcal{H}_t^{(1)} = \sigma(\tau_{(1)} \wedge t)$. By similar computations, it is also possible to find the intensity function $\gamma_{(k)}$ of the random time $\tau_{(k)}$ of the kth default for every $k = 2, 3, \ldots, n$.

Let us consider, for instance, a *digital default put* of basket type. To be more specific, we postulate that a contract pays a fixed amount (e.g., one unit of cash) at the moment $\tau_{(k)}$ of the kth default provided that $\tau_{(k)} \leq T$. If the interest rate is non-random then its value at time 0 equals

$$S_0 = \mathbb{E}_{\mathbb{Q}}\big(B_\tau^{-1} \mathbb{1}_{\{\tau_{(k)} \leq T\}}\big) = \int_{]0,T]} B_u^{-1}\, dF_{(k)}(u).$$

If the default times $\tau_1, \ldots, \tau_n$ have intensity functions $\gamma_1, \ldots, \gamma_n$ then

$$S_0 = \int_0^T B_u^{-1}\, dF_{(k)}(u) = \int_0^T B_u^{-1} \gamma_{(k)}(u) e^{-\int_0^u \gamma_{(k)}(v)\, dv}\, du.$$

5.2.4 Signed Intensities

Some authors (see, e.g., Kijima and Muromachi [122]) examine credit risk models in which the negative values of "default intensities" are allowed. In that case, the process chosen to model the "default intensity" does not play the role of the actual default intensity, in particular, the process

$$M_t = H_t - \int_0^{t \wedge \tau} \gamma_t\, dt$$

is not necessarily a martingale. Negative values of the "default intensity" process clearly contradict the usual interpretation of the intensity as the conditional default probability over an infinitesimal time interval.

Nevertheless, for a given collection Γ^i, $i = 1, 2, \ldots, n$ of $\mathbb{F}$-adapted, continuous processes, with $\Gamma_0^i = 0$, which are defined on $(\tilde{\Omega}, \mathbb{F}, \tilde{\mathbb{P}})$, one can construct random times τ_i, $i = 1, 2, \ldots, n$ on the enlarged probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ by setting

$$\tau_i = \inf\{\, t \in \mathbb{R}_+ : \Gamma_t^i \geq -\ln \xi_i \,\}. \tag{5.1}$$

Let us denote $\widehat{\Gamma}_t^i = \max_{u \leq t} \Gamma_u^i$. Observe that if the process Γ^i is absolutely continuous then so is the process $\widehat{\Gamma}^i$. In that case, the actual intensity of τ_i is obtained as the derivative of $\widehat{\Gamma}^i$ with respect to the time variable. The following result examines the case of signed intensities.

Lemma 5.2.1. *Random times τ_i, $i = 1, 2, \ldots, n$ given by (5.1) are conditionally independent with respect to $\mathbb{F}$ under $\mathbb{Q}$. In particular, for any $T > 0$ and every $t_1, \ldots, t_n \leq T$,*

$$\mathbb{Q}(\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T) = \prod_{i=1}^n e^{-\widehat{\Gamma}_{t_i}^i} = e^{-\sum_{i=1}^n \widehat{\Gamma}_{t_i}^i}.$$

5.3 Valuation of FTDC and LTDC

Pricing of a first-to-default claim or a last-to-default claim is straightforward under the assumption of conditional independence of default times as manifested by the following result in which, for notational simplicity, we consider only the case of two credit names. As usual, we do not state explicitly integrability conditions that should be imposed on a recovery process Z and a terminal payoff X.

Proposition 5.3.1. *Let the default times τ_j, $j = 1,2$ be $\mathbb{F}$-conditionally independent. Assume that the recovery $Z = Z^1 = Z^2$ is an $\mathbb{F}$-predictable process and the terminal payoff X is $\mathcal{F}_T$-measurable.*
(i) If the hypothesis (H) holds between $\mathbb{F}$ and $\mathbb{G}$ and processes F^i, $i = 1,2$ are continuous then the price at time $t = 0$ of the first-to-default claim with $Z^1 = Z^2 = Z$ equals

$$S_0^{(1)} = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T B_u^{-1} Z_u e^{-(\Gamma_u^1+\Gamma_u^2)}\, d(\Gamma_u^1+\Gamma_u^2) + B_T^{-1} X G_T^{(1)}\Big), \tag{5.2}$$

where we denote

$$G_T^{(1)} = \mathbb{Q}(\tau_{(1)} > T \,|\, \mathcal{F}_T) = \mathbb{Q}(\tau_1 > T \,|\, \mathcal{F}_T)\, \mathbb{Q}(\tau_2 > T \,|\, \mathcal{F}_T) = e^{-(\Gamma_T^1+\Gamma_T^2)}.$$

(ii) In the general case, let

$$F_t^i = \mathbb{Q}(\tau_i \le t \,|\, \mathcal{F}_t) = N_t^i + C_t^i = N_t^i + \int_0^t c_u^i\, du,$$

where N^i is a continuous $\mathbb{F}$-martingale. Then we have

$$S_0^{(1)} = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T B_u^{-1} Z_u \big(e^{-(\Gamma_u^1+\Gamma_u^2)}(\lambda_u^1+\lambda_u^2)\, du + d\langle N^1, N^2\rangle_u\big) + B_T^{-1} X G_T^{(1)}\Big)$$

where $\lambda_u^i = c_u^i(1-F_u^i)^{-1}$.

Proof. To simplify the notation, we will only consider the case where $B = 1$. A computation of the expectation $\mathbb{E}_{\mathbb{Q}}(X \mathbb{1}_{\{\tau_{(1)} > T\}})$ is straightforward. Thus, let us focus on the evaluation of the expected value

$$\mathbb{E}_{\mathbb{Q}}(Z_\tau \mathbb{1}_{\{\tau \le T\}}),$$

where, for brevity, we denote $\tau = \tau_{(1)} = \tau_1 \wedge \tau_2$.

From Lemma 3.1.3, we know that if Z is $\mathbb{F}$-predictable then

$$\mathbb{E}_{\mathbb{Q}}\big(Z_\tau \mathbb{1}_{\{\tau \le T\}}\big) = \mathbb{E}_{\mathbb{Q}}\Big(\int_{]0,T]} Z_u\, dF_u\Big),$$

where $F_u = \mathbb{Q}(\tau \le u \,|\, \mathcal{F}_u)$.

For $\tau = \tau_1 \wedge \tau_2$, the conditional independence assumption yields

$$\begin{aligned} 1 - F_u &= \mathbb{Q}(\tau_1 > u,\, \tau_2 > u \,|\, \mathcal{F}_u) = \mathbb{Q}(\tau_1 > u \,|\, \mathcal{F}_u)\, \mathbb{Q}(\tau_2 > u \,|\, \mathcal{F}_u) \\ &= (1 - F_u^1)(1 - F_u^2). \end{aligned}$$

Case (i). Under the assumption that the hypothesis (H) holds between filtrations $\mathbb{F}$ and $\mathbb{G}^i$ for $i = 1, 2$, the processes F^i are continuous and increasing. Consequently,

$$dF_u = e^{-\Gamma_u^1}\, dF_u^2 + e^{-\Gamma_u^2}\, dF_u^1 = e^{-(\Gamma_u^1 + \Gamma_u^2)}\, d(\Gamma_u^1 + \Gamma_u^2),$$

and this in turn yields

$$\mathbb{E}_{\mathbb{Q}}\big(Z_{\tau_1 \wedge \tau_2} \mathbb{1}_{\{\tau_1 \wedge \tau_2 < T\}}\big) = \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T Z_u e^{-(\Gamma_u^1 + \Gamma_u^2)}\, d(\Gamma_u^1 + \Gamma_u^2)\Big).$$

Case (ii). In the general case, the Doob-Meyer decomposition of the process F^i is $F^i = N^i + C^i$ and, under our assumptions, the process

$$H_t^i - \int_0^{t \wedge \tau_i} \lambda_u^i\, du$$

is a $\mathbb{G}^i$-martingale, where we write $\lambda_u^i = c_u^i (1 - F_u^i)^{-1}$. We now have

$$dF_u = e^{-\Gamma_u^1}\, dF_u^2 + e^{-\Gamma_u^2}\, dF_u^1 + d\langle N^1, N^2 \rangle_u.$$

Since N^1 and N^2 are martingales, it follows that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\big(Z_{\tau_1 \wedge \tau_2} \mathbb{1}_{\{\tau_1 \wedge \tau_2 < T\}}\big) &= \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T Z_u (e^{-\Gamma_u^1}\, dC_u^2 + e^{-\Gamma_u^2}\, dC_u^1 + d\langle N^1, N^2 \rangle_u)\Big) \\ &= \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T Z_u \big(e^{-(\Gamma_u^1 + \Gamma_u^2)} (\lambda_u^1 + \lambda_u^2)\, du + d\langle N^1, N^2 \rangle_u\big)\Big), \end{aligned}$$

as required. □

The valuation formula (5.2) can be easily extended to the case of an arbitrary date $t \in [0, T]$. This is left as an exercise for the reader.

5.4 Copula-Based Approaches

As already mentioned in Section 2.6, the classic concept of a *copula function* provides a convenient tool for producing multivariate probability distributions with predetermined univariate marginal distributions.

Definition 5.4.1. A function $C : [0,1]^n \to [0,1]$ is called a *copula function* if the following conditions are satisfied:
(i) $C(1,\dots,1,v_i,1,\dots,1) = v_i$ for any $i = 1,2,\dots,n$ and any $v_i \in [0,1]$,
(ii) C is an n-dimensional cumulative distribution function.

The following well known theorem, due to Sklar, underpins the theory of copula functions. For the proof of this result and further properties of copula functions, the interested reader is referred to Nelsen [147].

Theorem 5.4.1. *For any cumulative distribution function F on $\mathbb{R}^n$ there exists a copula function C such that $F(x_1,\dots,x_n) = C(F_1(x_1),\dots,F_n(x_n))$ for every $x_1,\dots,x_n \in \mathbb{R}$, where F_i is the ith marginal cumulative distribution function. If, in addition, the function F is continuous then C is unique.*

Let us first give a few examples of copula functions:
(i) the *product copula* $C(v_1,\dots,v_n) = \Pi_{i=1}^n v_i$, which corresponds to the independence,
(ii) the *Gumbel copula*, which is given by the formula, for $\theta \in [1,\infty)$,

$$C(v_1,\dots,v_n) = \exp\bigg(- \Big[\sum_{i=1}^n (-\ln v_i)^\theta \Big]^{1/\theta} \bigg),$$

(iii) the *Gaussian copula*, which is given by the expression

$$C(v_1,\dots,v_n) = N_\Sigma^n \left(N^{-1}(v_1),\dots,N^{-1}(v_n) \right),$$

where N_Σ^n is the cumulative distribution function for the n-variate central Gaussian distribution with the linear correlation matrix Σ and N^{-1} is the inverse of the cumulative distribution function for the univariate standard Gaussian distribution.
(iv) the *Student t-copula*, defined as

$$C(v_1,\dots,v_n) = \Theta_{\nu,\Sigma}^n \left(t_\nu^{-1}(v_1),\dots,t_\nu^{-1}(v_n) \right),$$

where $\Theta_{\nu,\Sigma}^n$ stands for the cumulative distribution function of the n-variate t-distribution with ν degrees of freedom and with the linear correlation matrix Σ and t_ν^{-1} is the inverse of the cumulative distribution function of the univariate Student t-distribution with ν degrees of freedom.

5.4.1 Direct Approach

In the direct approach, we first postulate that a (univariate marginal) cumulative distribution function F_i for each random variable τ_i, $i = 1,2,\dots,n$ is given. A particular copula function C is then chosen in order to introduce an appropriate dependence structure of the random vector $(\tau_1, \dots, \tau_n)$. The joint probability distribution of the random vector $(\tau_1, \dots, \tau_n)$ is thus given as

$$\mathbb{Q}(\tau_1 \le t_1, \dots, \tau_n \le t_n) = C\big(F_1(t_1), \dots, F_n(t_n)\big).$$

The direct copula-based approach has an apparent shortcoming of being essentially a static approach, since it makes no account of the flow of market information, which can be represented by some reference filtration.

5.4.2 Indirect Approach

A less straightforward application of copula functions relies on an extension of the canonical construction of conditionally independent default times. In the approach described below, the dependence between the default times is enforced both through the dependence between the *marginal hazard processes* $\widehat{\Gamma}^i$, $i = 1, 2, \dots, n$ and through the choice of a copula function C. For this reason, it is sometimes referred to as the *double correlation* case.

Assume that the joint probability distribution of $(\xi_1, \dots, \xi_n)$ in the canonical construction is given by an n-dimensional copula function C. Similarly as in Section 5.2.1, we postulate that the random vector $(\xi_1, \dots, \xi_n)$ is independent of $\mathbb{F}$ and we set

$$\tau_i(\widetilde{\omega}, \widehat{\omega}) = \inf \{ t \in \mathbb{R}_+ : \widehat{\Gamma}^i_t(\widetilde{\omega}) \ge - \ln \xi_i(\widehat{\omega}) \}.$$

We have that, for any $T > 0$ and arbitrary $t_1, \dots, t_n \le T$,

$$\mathbb{Q}(\tau_1 > t_1, \dots, \tau_n > t_n \,|\, \mathcal{F}_T) = C(K^1_{t_1}, \dots, K^n_{t_n}),$$

where we denote $K^i_t = e^{-\widehat{\Gamma}^i_t}$. Schönbucher and Schubert [161] show that the following equality holds, for arbitrary $s \le t$ on the event $\{\tau_1 > s, \dots, \tau_n > s\}$,

$$\mathbb{Q}(\tau_i > t \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{Q}} \left(\frac{C(K^1_s, \dots, K^i_t, \dots, K^n_s)}{C(K^1_s, \dots, K^n_s)} \,\Big|\, \mathcal{F}_s \right).$$

Consequently, assuming that the derivatives $\widehat{\gamma}^i_t = \frac{d\widehat{\Gamma}^i_t}{dt}$ exist, the ith survival intensity equals, on the event $\{\tau_1 > t, \dots, \tau_n > t\}$,

$$\lambda^i_t = \widehat{\gamma}^i_t K^i_t \, \frac{\frac{\partial}{\partial v_i} C(K^1_t, \dots, K^n_t)}{C(K^1_t, \dots, K^n_t)} = \widehat{\gamma}^i_t K^i_t \, \frac{\partial}{\partial v_i} \ln C(K^1_t, \dots, K^n_t),$$

where λ^i_t is understood as the following limit

$$\lambda^i_t = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}(t < \tau_i \le t + h \,|\, \mathcal{F}_t, \tau_1 > t, \dots, \tau_n > t).$$

It appears that, in general, the survival intensity of the ith name jumps at time t if the jth name defaults at time t for some $j \neq i$. In fact, it can be shown that

$$\lambda_t^{i,j} = \widehat{\gamma}_t^i \, K_t^i \, \frac{\frac{\partial^2}{\partial v_i \partial v_j} \, C(K_t^1, \ldots, K_t^n)}{\frac{\partial}{\partial v_j} \, C(K_t^1, \ldots, K_t^n)} \,,$$

where $\lambda_t^{i,j}$ is defined as follows

$$\lambda_t^{i,j} = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}(t < \tau_i \leq t + h \,|\, \mathcal{F}_t, \tau_k > t, k \neq j, \tau_j = t).$$

Schönbucher and Schubert [161] examine the behavior of survival intensities after defaults of some names. Let us fix s, and let $t_i \leq s$ for $i = 1, 2, \ldots, k < n$ and $t_i \geq s$ for $i = k+1, k+2, \ldots, n$. They show that

$$\mathbb{Q}\big(\tau_i > t_i, i = k+1, \ldots, n \,|\, \mathcal{F}_s, \tau_j = t_j, j = 1, \ldots, k, \tau_i > s, i = k+1, \ldots, n\big)$$
$$= \frac{\mathbb{E}_{\mathbb{Q}}\Big(\frac{\partial^k}{\partial v_1 \ldots \partial v_k} \, C(K_{t_1}^1, \ldots, K_{t_k}^k, K_{t_{k+1}}^{k+1}, \ldots, K_{t_n}^n) \,\Big|\, \mathcal{F}_s\Big)}{\frac{\partial^k}{\partial v_1 \ldots \partial v_k} \, C(K_{t_1}^1, \ldots, K_{t_k}^k, K_s^{k+1}, \ldots, K_s^n)} \,.$$

Unfortunately, in this approach it is difficult to control the jumps of intensities, otherwise than by a judicious choice of the copula function C.

5.5 One-factor Gaussian Copula Model

Laurent and Gregory [133] examine a simplified version of Schönbucher and Schubert [161] approach, corresponding to the trivial reference filtration $\mathbb{F}$ (we thus deal here with the direct approach). The *marginal default intensities* $\widehat{\gamma}^i$ are deterministic functions and the marginal distributions of defaults are given by the expression

$$\mathbb{Q}(\tau_i > t) = 1 - F_i(t) = e^{-\int_0^t \widehat{\gamma}^i(u) \, du}.$$

They derive closed-form expressions for certain conditional default intensities by making specific assumptions regarding the choice of a copula C.

Let us describe the *one-factor Gaussian copula* model, proposed by Li [137]. It is worth noting that this model was widely adopted by the financial industry as a benchmark model for valuing traded and bespoke tranches of collateralized debt obligations (see Section 5.8.2). Let us set

$$X_i = \rho V + \sqrt{1 - \rho^2} V_i,$$

where V and $V_i, \ i = 1, 2, \ldots, n$ are independent, standard Gaussian variables under $\mathbb{Q}$ and the correlation parameter ρ belongs to $(-1, 1)$.

Let C be the copula function corresponding to the distribution of the vector $(X_1,\dots,X_n)$, that is, let C be given by the expression, for every $v_1,\dots,v_n\in[0,1]$,

$$C(v_1,\dots,v_n)=\mathbb{Q}\big(X_1<N^{-1}(v_1),\dots,X_n<N^{-1}(v_n)\big).$$

We define the default times $\tau_i,\ i=1,2,\dots,n$ by the formula

$$\tau_i=\inf\Big\{\,t\in\mathbb{R}_+:\ \int_0^t\widehat{\gamma}^i(u)\,du>-\ln\xi_i\Big\}$$

or, equivalently,

$$\tau_i=\inf\{\,t\in\mathbb{R}_+:\ 1-F_i(t)<\xi_i\},$$

where the uniformly distributed random barriers are defined by the equality $\xi_i=1-N(X_i)$. It is worth noting that the random vectors $(X_1,\dots,X_n)$, $(\xi_1,\dots,\xi_n)$ and $(\tau_1,\dots,\tau_n)$ share a common Gaussian copula function C; this follows from the monotonicity of the transformations involved.

Moreover, the following equality is valid, for every $i=1,2,\dots,n$ and every $t\in\mathbb{R}_+$,

$$\{\tau_i\le t\}=\{\xi_i\ge 1-F_i(t)\}=\left\{V_i\le\frac{N^{-1}(F_i(t))-\rho V}{\sqrt{1-\rho^2}}\right\}.$$

By the conditional independence of $X_1,\dots,X_n$ with respect to the common factor V, which represents the *market-wide* (or *systematic*) credit risk, we thus obtain, for every $t_1,\dots,t_n\in\mathbb{R}_+$,

$$\mathbb{Q}(\tau_1\le t_1,\dots,\tau_n\le t_n)=\int_{\mathbb{R}}\prod_{i=1}^n N\left(\frac{N^{-1}(F_i(t_i))-\rho v}{\sqrt{1-\rho^2}}\right)n(v)\,dv,$$

where n is the probability density function of V. It is worth noting that the components V_i are aimed to represent the *firm-specific* (or *idiosyncratic*) part of the credit risk for individual names in a credit portfolio. For numerical issues arising in implementations of the Li model, see Amman and Brommundt [3], Joshi and Kainth [116], and Chen and Glasserman [54].

5.6 Jarrow and Yu Model

Jarrow and Yu [107] (see also Yu [173]) approach can be considered as another attempt to develop a dynamic approach to dependence between default times by modeling directly the *contagion effect.* For a given finite family of reference credit names, Jarrow and Yu [107] propose to make a distinction between *primary* and *secondary* firms.

At the intuitive level, the rationale for their approach can be summarized as follows:

- the class of primary firms encompasses these entities whose probabilities of default are influenced by macroeconomic conditions, but not by the credit risk of counterparties; the pricing of bonds and other defaultable securities issued by primary firms is feasible through the standard intensity-based methodology,
- it is thus sufficient to focus on defaultable securities issued by a secondary firm, that is, a firm for which the intensity of default depends explicitly on the status of some other firms.

Let $\{1, 2, \ldots, n\}$ represent the set of all firms in our model and let $\mathbb{F}$ stand for some reference filtration. Jarrow and Yu [107] formally postulate that:

- for any firm from the set $\{1, 2, \ldots, k\}$ of primary firms, the 'default intensity' depends only on a reference filtration $\mathbb{F}$,
- the 'default intensity' for any credit name that belongs to the class $\{k+1, k+2, \ldots, n\}$ of secondary firms may depend not only on the filtration $\mathbb{F}$, but also on the status (default or no-default) of the primary firms.

5.6.1 Construction of Default Times

To construct default times $\tau_1, \ldots, \tau_n$, we proceed in two steps.

First step. We first construct default times for all primary firms. To this end, we assume that we are given a family of $\mathbb{F}$-adapted 'intensity processes' $\lambda^1, \ldots, \lambda^k$ and we produce a collection $\tau_1, \ldots, \tau_k$ of $\mathbb{F}$-conditionally independent random times through the canonical method, that is, we set

$$\tau_i = \inf \left\{ t \in \mathbb{R}_+ : \int_0^t \lambda_u^i \, du \geq -\ln \xi_i \right\}$$

where ξ_i, $i = 1, 2, \ldots, k$ are mutually independent and identically distributed random variables with the uniform distribution on $[0, 1]$ under the martingale measure $\mathbb{Q}$.

Second step. In the second step, we are going to construct default times corresponding to secondary firms. To this end, we assume that:

- the probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ is large enough to support a family ξ_i, $i = k+1, k+2, \ldots, n$ of mutually independent random variables, with uniform distribution on $[0, 1]$,
- these random variables are independent not only of the filtration $\mathbb{F}$, but also of the already constructed in the first step default times $\tau_1, \ldots, \tau_k$ of primary firms.

The default times τ_i, $i = k+1, k+2, \dots, n$ are also defined by means of the standard formula

$$\tau_i = \inf \Big\{ t \in \mathbb{R}_+ : \int_0^t \lambda^i_u \, du \geq -\ln \xi_i \Big\}.$$

However, the 'intensity processes' λ^i for $i = k+1, k+2, \dots, n$ are now given by the following expression

$$\lambda^i_t = \mu^i_t + \sum_{l=1}^{k} \nu^{i,l}_t \mathbb{1}_{\{t \geq \tau_l\}},$$

where μ^i and $\nu^{i,l}$ are $\mathbb{F}$-adapted stochastic processes. In case where the default of the jth primary firm does not affect the 'default intensity' of the ith secondary firm, we set $\nu^{i,j} = 0$.

Let $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \dots \vee \mathbb{H}^n$ stand for the enlarged filtration and let $\widehat{\mathbb{F}} = \mathbb{F} \vee \mathbb{H}^{k+1} \vee \dots \vee \mathbb{H}^n$ be the filtration generated by the reference filtration $\mathbb{F}$ and the observations of defaults of secondary firms. Then:

- the default times $\tau_1, \dots, \tau_k$ of primary firms are conditionally independent with respect to $\mathbb{F}$,
- the default times $\tau_1, \dots, \tau_k$ of primary firms are no longer conditionally independent when we replace the filtration $\mathbb{F}$ by $\widehat{\mathbb{F}}$,
- in general, the default intensity of a primary firm with respect to the filtration $\widehat{\mathbb{F}}$ differs from the intensity λ^i with respect to $\mathbb{F}$.

5.6.2 Case of Two Credit Names

To illustrate the credit contagion effect, we will now consider the case of only two credit names, A and B say, and we postulate that A is a primary firm, whereas B is a secondary firm.

Let the constant process $\lambda^1_t = \lambda_1$ represent the $\mathbb{F}$-intensity of default time for firm A, so that

$$\tau_1 = \inf \Big\{ t \in \mathbb{R}_+ : \int_0^t \lambda^1_u \, du = \lambda_1 t \geq -\ln \xi_1 \Big\},$$

where ξ_1 is a random variable independent of $\mathbb{F}$ with the uniform distribution on $[0,1]$. For the second firm, the 'default intensity' is assumed to satisfy

$$\lambda^2_t = \lambda_2 \mathbb{1}_{\{t < \tau_1\}} + \alpha_2 \mathbb{1}_{\{t \geq \tau_1\}}$$

for some positive constants λ_2 and α_2. We set

$$\tau_2 = \inf \Big\{ t \in \mathbb{R}_+ : \int_0^t \lambda^2_u \, du \geq -\ln \xi_2 \Big\},$$

where ξ_2 is a random variable with the uniform probability distribution, independent of $\mathbb{F}$, and such that ξ_1 and ξ_2 are mutually independent. The following result summarizes properties of processes Λ^1 and Λ^2.

Lemma 5.6.1. *The following properties hold:*
(i) the process Λ^1 is the hazard process of τ_1 with respect to $\mathbb{F}$,
(ii) the process Λ^2 is the hazard process of τ_2 with respect to $\mathbb{F} \vee \mathbb{H}^1$,
(iii) the process Λ^1 is not the hazard process of τ_1 with respect to $\mathbb{F} \vee \mathbb{H}^2$ if the inequality $\lambda_2 \neq \alpha_2$ holds.

Assume for simplicity that $r = 0$. We wish to price a defaultable zero-coupon bond with the default time τ_i and with constant recovery payoff δ_i. We thus need to compute the following conditional expectation, for $i = 1, 2$,

$$D_i^{\delta_i}(t,T) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau_i > T\}} + \delta_i \mathbb{1}_{\{\tau_i \leq T\}} \,|\, \mathcal{G}_t), \tag{5.3}$$

where $\mathcal{G}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2$. To this end, we will first find the joint probability distribution of the pair (τ_1, τ_2). Let us denote $G(s,t) = \mathbb{Q}(\tau_1 > s, \tau_2 > t)$. We write $\Delta = \lambda_1 + \lambda_2 - \alpha_2$ and we assume that $\Delta \neq 0$.

Lemma 5.6.2. *The joint distribution of (τ_1, τ_2) under $\mathbb{Q}$ is given by, for every $0 \leq t \leq s$,*

$$\mathbb{Q}(\tau_1 > s, \tau_2 > t) = e^{-\lambda_1 s - \lambda_2 t}$$

and, for every $0 \leq s < t$,

$$\mathbb{Q}(\tau_1 > s, \tau_2 > t) = \frac{1}{\Delta} \lambda_1 e^{-\alpha_2 t} \left(e^{-s\Delta} - e^{-t\Delta} \right) + e^{-(\lambda_1 + \lambda_2)t}.$$

Proof. Let $\psi_i = -\ln \xi_i$. For $t < s$, we have $\lambda_t^2 = \lambda_2 t$ on the set $\{s < \tau_1\}$. The equalities

$$\{\tau_1 > s\} \cap \{\tau_2 > t\} = \{\Lambda_s^1 < \psi_1\} \cap \{\Lambda_t^2 < \psi_2\} = \{\lambda_1 s < \psi_1\} \cap \{\lambda_2 t < \psi_2\}$$

and the independence of ψ_1 and ψ_2 lead to

$$\mathbb{Q}(\tau_1 > s, \tau_2 > t) = e^{-\lambda_1 s - \lambda_2 t}.$$

In particular, by setting $t = 0$, we obtain the equality $\mathbb{Q}(\tau_1 > s) = e^{-\lambda_1 s}$ for every $s \in \mathbb{R}_+$.

For $t > s$, we have that

$$\{\tau_1 > s\} \cap \{\tau_2 > t\} = \{\{t > \tau_1 > s\} \cap \{\tau_2 > t\}\} \cup \{\{\tau_1 > t\} \cap \{\tau_2 > t\}\}$$

and

$$\begin{aligned} \{t > \tau_1 > s\} \cap \{\tau_2 > t\} &= \{t > \tau_1 > s\} \cap \{\Lambda_t^2 < \psi_2\} \\ &= \{t > \tau_1 > s\} \cap \{\lambda_2 \tau_1 + \alpha_2 (t - \tau_1) < \psi_2\}. \end{aligned}$$

The independence between ψ_1 and ψ_2 implies that the random variable τ_1 is independent of ψ_2 (note that $\tau_1 = (\lambda_1)^{-1}\psi_1$). Consequently,

$$\begin{aligned}\mathbb{Q}(t > \tau_1 > s, \tau_2 > t) &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t>\tau_1>s\}} e^{-(\lambda_2\tau_1 + \alpha_2(t-\tau_1))}\Big) \\ &= \int_s^t e^{-(\lambda_2 u + \alpha_2(t-u))}\lambda_1 e^{-\lambda_1 u}\, du \\ &= \frac{1}{\lambda_1+\lambda_2-\alpha_2}\lambda_1 e^{-\alpha_2 t}\left(e^{-(\lambda_1+\lambda_2-\alpha_2)s} - e^{-(\lambda_1+\lambda_2-\alpha_2)t}\right).\end{aligned}$$

Denoting $\Delta = \lambda_1+\lambda_2-\alpha_2$, it follows that

$$\mathbb{Q}(\tau_1 > s, \tau_2 > t) = \frac{1}{\Delta}\lambda_1 e^{-\alpha_2 t}\left(e^{-\Delta s} - e^{-\Delta t}\right) + e^{-(\lambda_1+\lambda_2)t}.$$

In particular, for $s = 0$, we obtain

$$\mathbb{Q}(\tau_2 > t) = \frac{1}{\Delta}\left(\lambda_1\left(e^{-\alpha_2 t} - e^{-(\lambda_1+\lambda_2)t}\right) + \Delta e^{-(\lambda_1+\lambda_2)t}\right).$$

This completes the proof. □

Bonds with Non-Zero Recovery

In view of (5.3), to find the price $D_1^{\delta_1}(t,T)$, it suffices to compute

$$\mathbb{Q}(\tau_1 > T \,|\, \mathcal{G}_t) = \mathbb{Q}(\tau_1 > T \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2) = \mathbb{1}_{\{t<\tau_1\}}\frac{\mathbb{Q}(\tau_1 > T \,|\, \mathcal{H}_t^2)}{\mathbb{Q}(\tau_1 > t \,|\, \mathcal{H}_t^2)}.$$

Observe that

$$\mathbb{Q}(\tau_1 > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau_1\}}\left(\mathbb{1}_{\{t\geq\tau_2\}}\frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} + \mathbb{1}_{\{t<\tau_2\}}\frac{G(T,t)}{G(t,t)}\right).$$

Similarly, the valuation of $D_2^{\delta_2}(t,T)$ follows from the computation of

$$\mathbb{Q}(\tau_2 > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau_2\}}\frac{\mathbb{Q}(\tau_2 > T \,|\, \mathcal{H}_t^1)}{\mathbb{Q}(\tau_2 > t \,|\, \mathcal{H}_t^1)},$$

where, by symmetry, we have that

$$\mathbb{Q}(\tau_2 > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau_2\}}\left(\mathbb{1}_{\{t\geq\tau_1\}}\frac{\partial_1 G(\tau_1,T)}{\partial_1 G(\tau_1,t)} + \mathbb{1}_{\{t<\tau_1\}}\frac{G(t,T)}{G(t,t)}\right).$$

By straightforward computations, we obtain the following pricing result for defaultable bonds with non-zero recovery.

Corollary 5.6.1. *The prices of defaultable bonds equal, for every* $t \in [0, T]$

$$D_1^{\delta_1}(t,T) = \mathbb{1}_{\{t \geq \tau_1\}}\delta_1 + \mathbb{1}_{\{t<\tau_1\}}\big(e^{-\lambda_1(T-t)} + \delta_1(1 - e^{-\lambda_1(T-t)})\big)$$

and

$$D_2^{\delta_2}(t,T) = \delta_2 + (1-\delta_2)\mathbb{1}_{\{t<\tau_2\}}\bigg\{\mathbb{1}_{\{t\geq\tau_1\}}e^{-\alpha_2(T-t)} + \mathbb{1}_{\{t<\tau_1\}}\frac{1}{\lambda_1+\lambda_2-\alpha_2}\Big(\lambda_1 e^{-\alpha_2(T-t)} + (\lambda_2-\alpha_2)e^{-(\lambda_1+\lambda_2)(T-t)}\Big)\bigg\}.$$

Bonds with Zero Recovery

Assume that $\lambda_1 + \lambda_2 - \alpha_2 \neq 0$ and that the bond is subject to the zero recovery scheme. We maintain the assumption that $r = 0$ so that $B(t,T) = 1$ for $t \leq T$. Therefore, we have $D_2^0(t,T) = \mathbb{Q}(\tau_2 > T \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2)$ and thus the general formula yields

$$D_2^0(t,T) = \mathbb{1}_{\{t<\tau_2\}}\frac{\mathbb{Q}(\tau_2 > T \,|\, \mathcal{H}_t^1)}{\mathbb{Q}(\tau_2 > t \,|\, \mathcal{H}_t^1)}.$$

The following pricing result is an immediate consequence of Corollary 5.6.1.

Corollary 5.6.2. *Assume that the recovery* $\delta_2 = 0$*. Then* $D_2(t,T) = 0$ *on the event* $\{t \geq \tau_2\}$*. On the event* $\{t < \tau_2\}$ *we have*

$$D_2^0(t,T) = \mathbb{1}_{\{t<\tau_1\}}\frac{1}{\lambda_1+\lambda_2-\alpha_2}\Big(\lambda_1 e^{-\alpha_2(T-t)} + (\lambda_2-\alpha_2)e^{-(\lambda_1+\lambda_2)(T-t)}\Big) + \mathbb{1}_{\{t\geq\tau_1\}}\, e^{-\alpha_2(T-t)}.$$

5.7 Kusuoka's Model

Following Kusuoka [125] (see also Bielecki and Rutkowski [23]), we will argue that the assumption that some firms are classified as primary, while some other are considered to be secondary, is of no relevance from the point of view of modeling. For simplicity, we make the following standing assumptions:

- we set $n = 2$, that is, we consider the case of two credit names,
- the interest rate r equals zero, so that $B(t,T) = 1$ for every $t \leq T$,
- the reference filtration $\mathbb{F}$ is trivial,
- all corporate bonds are subject to the zero recovery scheme.

In view of the model symmetry, it suffices to analyze a bond issued by the first firm. By definition, the price of this bond at time $t \in [0, T]$ equals

$$D_1^0(t,T) = \mathbb{Q}(\tau_1 > T \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2).$$

Of course, this value is based on the complete information, as modeled by the full filtration $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$. For the sake of comparison, we will also evaluate the corresponding values, which are based on the assumption that only a partial observation is available; specifically, we will compute

$$\widehat{D}_1^0(t,T) = \mathbb{Q}(\tau_1 > T \,|\, \mathcal{H}_t^1), \quad \bar{D}_1^0(t,T) = \mathbb{Q}(\tau_1 > T \,|\, \mathcal{H}_t^2).$$

5.7.1 Model Specification

We follow here Kusuoka [125]. Under the original probability measure $\mathbb{P}$ the random times τ_i, $i = 1, 2$ are assumed to be mutually independent random variables with exponential laws with parameters λ_1 and λ_2, respectively.

For a fixed $T > 0$, we define a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$ by setting

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \eta_T, \quad \mathbb{P}\text{-a.s.},$$

where the Radon-Nikodým density process $(\eta_t,\, t \in [0,T])$ satisfies

$$\eta_t = 1 + \sum_{i=1}^{2} \int_{]0,t]} \eta_{u-} \kappa_u^i \, dM_u^i,$$

where in turn the processes M^1 and M^2 are given by

$$M_t^i = H_t^i - \int_0^{t \wedge \tau_i} \lambda_i \, du = H_t^i - (t \wedge \tau_i)\lambda_i,$$

where we write, as usual, $H_t^i = \mathbb{1}_{\{t \geq \tau_i\}}$, and the $\mathbb{G}$-predictable processes κ^1 and κ^2 are given by the following expressions

$$\kappa_t^1 = \mathbb{1}_{\{t > \tau_2\}} \Big(\frac{\alpha_1}{\lambda_1} - 1 \Big)$$

and

$$\kappa_t^2 = \mathbb{1}_{\{t > \tau_1\}} \Big(\frac{\alpha_2}{\lambda_2} - 1 \Big)$$

for some constants $\alpha_i > 0$ for $i = 1, 2$. Note that the inequality $\kappa_t^i > -1$ holds for $i = 1, 2$ and every $t \in [0, T]$. It is not difficult to check, using the Girsanov theorem, that the $\mathbb{G}$-intensities (cf. Section 3.6) of τ_1 and τ_2 under $\mathbb{Q}$ are given by the expressions

$$\lambda_t^1 = \lambda_1 \mathbb{1}_{\{t < \tau_2\}} + \alpha_1 \mathbb{1}_{\{t \geq \tau_2\}}$$

and

$$\lambda_t^2 = \lambda_2 \mathbb{1}_{\{t<\tau_1\}} + \alpha_2 \mathbb{1}_{\{t\geq\tau_1\}}.$$

We focus on τ_1 and we denote $\Lambda_t^1 = \int_0^t \lambda_u^1 \, du$. Let us make few observations. First, we note that the process λ^1 is $\mathbb{H}^2$-predictable and the process

$$M_t^1 = H_t^1 - \int_0^{t\wedge\tau_1} \lambda_u^1 \, du = H_t^1 - \Lambda_{t\wedge\tau_1}^1$$

is a $\mathbb{G}$-martingale under $\mathbb{Q}$, so that the process λ^1 is a version of $\mathbb{G}$-intensity of τ under $\mathbb{Q}$. In general, the process λ^1 is not the $\mathbb{H}^2$-intensity of τ_1 under $\mathbb{Q}$, since we have

$$\mathbb{Q}(\tau_1 > s \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2) \neq \mathbb{1}_{\{t<\tau_1\}} \, \mathbb{E}_{\mathbb{Q}}\big(e^{\Lambda_t^1 - \Lambda_s^1} \,|\, \mathcal{H}_t^2\big).$$

It is also interesting to observe that the process λ^1 is the $\mathbb{H}^2$-intensity of τ_1 under a probability measure $\widetilde{\mathbb{Q}}$, which is equivalent to $\mathbb{P}$ and is given by

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} = \widetilde{\eta}_T, \quad \mathbb{P}\text{-a.s.},$$

where the Radon-Nikodým density process $(\widetilde{\eta}_t, \, t \in [0,T])$ satisfies

$$\widetilde{\eta}_t = 1 + \int_{]0,t]} \widetilde{\eta}_{u-} \kappa_u^2 \, dM_u^2.$$

It can be checked that the following equality is satisfied, for every $s > t$,

$$\widetilde{\mathbb{Q}}(\tau_1 > s \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2) = \mathbb{1}_{\{t<\tau_1\}} \, \mathbb{E}_{\widetilde{\mathbb{Q}}}\big(e^{\Lambda_t^1 - \Lambda_s^1} \,|\, \mathcal{H}_t^2\big).$$

Recall that the processes λ^1 and λ^2 have jumps provided that $\alpha_i \neq \lambda_i$.

The next result shows that the processes λ^1 and λ^2 specify the *transition intensities*, so that the model can be dealt with as a two-dimensional Markov chain (for related results and applications, see Herbertsson [95], Lando [127], and Shaked and Shanthikumar [163]).

Proposition 5.7.1. *For $i = 1, 2$ and every $t \in \mathbb{R}_+$ we have*

$$\begin{aligned}
\lambda_i &= \lim_{h\downarrow 0} h^{-1} \, \mathbb{Q}(t < \tau_i \leq t + h \,|\, \tau_1 > t, \, \tau_2 > t), \\
\alpha_1 &= \lim_{h\downarrow 0} h^{-1} \, \mathbb{Q}(t < \tau_1 \leq t + h \,|\, \tau_1 > t, \, \tau_2 \leq t), \\
\alpha_2 &= \lim_{h\downarrow 0} h^{-1} \, \mathbb{Q}(t < \tau_2 \leq t + h \,|\, \tau_2 > t, \, \tau_1 \leq t).
\end{aligned}$$

5.7.2 Bonds with Zero Recovery

We now present the bond valuation result in Kusuoka's [125] model. We focus on the bond price $D_1^0(t,T)$; an analogous formula is valid for $D_2^0(t,T)$ as well. Recall that we consider corporate bonds with zero recovery.

Proposition 5.7.2. *The price $D_1^0(t,T)$ equals, on the event $\{t < \tau_1\}$,*

$$\begin{aligned} D_1^0(t,T) &= \mathbb{1}_{\{t<\tau_2\}} \frac{1}{\lambda - \alpha_1} \Big(\lambda_2 e^{-\alpha_1(T-t)} + (\lambda_1 - \alpha_1) e^{-\lambda(T-t)} \Big) \\ &\quad + \mathbb{1}_{\{t \geq \tau_2\}} \, e^{-\alpha_1(T-t)}, \end{aligned}$$

where $\lambda = \lambda_1 + \lambda_2$. Furthermore,

$$\widehat{D}_1^0(t,T) = \mathbb{1}_{\{t<\tau_1\}} \frac{\lambda_2 e^{-\alpha_1 T} + (\lambda_1 - \alpha_1) e^{-\lambda T}}{\lambda_2 e^{-\alpha_1 t} + (\lambda_1 - \alpha_1) e^{-\lambda t}}$$

and

$$\begin{aligned} \bar{D}_1^0(t,T) &= \mathbb{1}_{\{t<\tau_2\}} \frac{\lambda - \alpha_2}{\lambda - \alpha_1} \frac{(\lambda_1 - \alpha_1) e^{-\lambda(T-t)} + \lambda_2 e^{-\alpha_1(T-t)}}{\lambda_1 e^{-(\lambda - \alpha_2)t} + \lambda_2 - \alpha_2} \\ &+ \mathbb{1}_{\{t \geq \tau_2\}} \frac{(\lambda - \alpha_2)\lambda_2 e^{-\alpha_1(T-\tau_2)}}{\lambda_1 \alpha_2 e^{(\lambda-\alpha_2)\tau_2} + \lambda(\lambda_2 - \alpha_2)}. \end{aligned}$$

It is worth noting that:

- the prices $D_1^0(t,T)$ and $D_2^0(t,T)$ correspond to the Jarrow and Yu price of the bond issued by a secondary firm (cf. Corollary 5.6.2),
- the processes $D_1^0(t,T)$ and $\widehat{D}_1^0(t,T)$ represent ex-dividend prices of the bond issued by the first firm, so they vanish after the default time τ_1,
- the second remark does not apply to the process $\bar{D}_1^0(t,T)$.

5.8 Basket Credit Derivatives

We will now describe the mainstream basket credit derivatives, focusing on the recently developed standardized instruments: *credit default swap indices* and *collateralized debt obligations*. For various methods of valuation and hedging of basket credit products, we refer to Andersen and Sidenius [5], Cont and Minca [61], Cont and Kan [60], Brasch [32], Brigo [37], Brigo and Alfonsi [39], Brigo and El-Bachir [40], Brigo and Morini [42, 145], Brigo et al. [43], Burtschell et al. [45, 46], Di Graziano and Rogers [70], Duffie and Gârleanu [72], Frey and Backhaus [86, 87], Giesecke and Goldberg [93], Herbertsson [96], Ho and Wu [98], Hull and White [100, 101, 102], Laurent et al. [132], Laurent and Gregory [133], Pedersen [151], Rutkowski and Armstrong [156], Sidenius et al. [167], Wu [171] and Zheng [174].

5.8.1 Credit Default Index Swaps

A *credit default index swap* (CDIS), also known as a *CDS index*, is a static portfolio of n equally weighted credit default swaps with standard maturities, typically five or ten years. Standard examples of a CDIS are iTraxx and CDX. A credit default index swap usually matures few months before the underlying CDSs. The CDSs in the pool are selected from among those with highest trading volume in the respective industry sector. Credit default index swaps are issued by a pool of licensed financial institutions, which are called the *market makers*. At time of issuance of a CDIS, the market makers determine an annual rate, known as the *index spread*, to be paid out to investors on a periodic basis. The index spread, denoted by κ_0, is constant over the lifetime of a CDIS. Let us summarize the main provisions of a CDIS.

- We assume that the face value of each reference entity is one. Thus the total notional of a CDIS equals n. The notional on which the market maker pays the spread, henceforth referred to as *residual protection*, is reduced by 1 after each default. For instance, after the first default, the residual protection is revised from the original value n to $n-1$.
- By purchasing a CDIS, an investor assumes the role of a protection seller and she agrees to absorb all losses due to defaults in the reference portfolio, occurring between the time of inception 0 and the maturity T. In case of default of a reference entity, an investor makes the protection payment to a market maker in the amount of $1-\delta$, where $\delta \in [0,1]$ is a constant recovery rate, which is pre-determined in a given CDIS (typically, it equals to 40%).
- In exchange, the protection seller receives from a market maker a periodic fixed premium on the residual protection at the annual rate of κ_0, which equals the *fair CDIS spread* at the inception date.
- A CDIS is also traded after its issuance date. Recall that whenever one of reference entities defaults, its weight in the index is set to zero. Therefore, by purchasing one unit of an index at time t, an investor owes protection only on those names that have not yet defaulted prior to time t. If the quotation of the *market CDIS spread* at time t differs from the index spread fixed at issuance, i.e., $\kappa_t \neq \kappa_0$, the credit-risky present value of the difference is settled through an upfront payment.

The provisions of a single-name CDS correspond to the CDIS with $n=1$, except for the fact that, by the market convention, a buyer of a single-name CDS is the protection buyer, rather than the protection seller.

We denote by τ_i the default time of the ith credit name in the index portfolio and by H^i the default indicator process defined as $H^i_t = \mathbb{1}_{\{t \geq \tau_i\}}$ for every $i = 1, 2, \dots, n$.

Furthermore, we set $N_0 = n$ and we write

$$N_t = N_0 - \sum_{i=1}^{n} H_t^i \tag{5.4}$$

to denote the *residual protection* (or the *reduced nominal*) at time $t \in [0, T]$.

Let $t_j, j = 0, 1, \ldots, J$ with $t_0 = 0$ and $t_J = T$ denote the tenor of the premium leg payments dates. The discounted cumulative cash flows associated with a CDIS are as follows

$$\text{Premium leg} = \kappa_0 \sum_{j=1}^{J} \frac{B_0}{B_{t_j}} \Big(N_0 - \sum_{i=1}^{n} H_{t_j}^i \Big) = \kappa_0 \sum_{j=1}^{J} \frac{B_0}{B_{t_j}} N_{t_j}$$

and

$$\text{Protection leg} = (1 - \delta) \sum_{i=1}^{n} \frac{B_0}{B_{\tau_i}} H_T^i.$$

5.8.2 Collateralized Debt Obligations

Collateralized debt obligations (CDO) are credit derivatives backed by portfolios of assets. If the underlying portfolio is made up of bonds, loans or other securitised receivables, the collateralized debt obligation is known as the *cash* CDO. Alternatively, the underlying portfolio may consist of credit derivatives referencing a pool of debt obligations. In the latter case, a CDO is said to be *synthetic*.

Because of their recently acquired popularity, we focus our discussion on standardized synthetic CDO contracts backed by CDS indices. We begin with an overview of the covenants of a typical synthetic collateralized debt obligation.

- The time of issuance of the contract is 0 and its maturity is T. The notional of the CDO contract at any date t after issuance is equal to the residual protection N_t of the reference CDS index (cf. formula (5.4)).
- The credit risk (that is, the potential loss due to credit events) borne by the reference pool is layered into various standardized risk levels, with the range in between two adjacent risk levels called a *CDO tranche*. The lower bound of a tranche is usually referred to as *attachment point* and the upper bound as *detachment point*. The credit risk is originally sold in these tranches to protection sellers. For instance, in a typical CDO contract on iTraxx, the credit risk is split into the *equity tranche* ($0 - 3\%$ of the total losses), four *mezzanine tranches* (corresponding to $3 - 6\%, 6 - 9\%, 9 - 12\%$ and $12 - 22\%$ of the total losses respectively), and the *senior tranche* (over 22% of the total losses). At issuance, the

notional value of each tranche is equal to the CDO notional weighted by the respective *tranche width*.

- The tranche buyer sells partial protection to the pool owner, by agreeing to absorb the pool's losses comprised in between the tranche attachment and detachment point. This is better understood by an example. Assume, for instance, that at time 0 the protection seller purchases the $6-9\%$ tranche with a given notional value. One year later, consequently to a default event, the cumulative loss breaks through the attachment point, reaching 8%. The protection seller then fulfills his obligation by disbursing two thirds $(=\frac{8\%-6\%}{9\%-6\%})$ of a currency unit. The tranche notional is then reduced to one third of its pre-default event value. We refer to the remaining tranche notional as *residual tranche protection*.
- In exchange, as of time t and up to time T, the CDO issuer (protection buyer) makes periodic payments to the tranche buyer according to a predetermined rate – termed *tranche spread* – on the residual tranche protection. Returning to our example, after the loss reaches 8%, premium payments are made on $\frac{1}{3}$ $(=\frac{9\%-8\%}{9\%-6\%})$ of the tranche notional, until the next credit event occurs or the contract matures.

We denote by L_l and U_l the lower and upper attachment points for the lth tranche and by κ_0^l the corresponding spread. It is convenient to introduce the *percentage loss* process

$$Q_t = \frac{1-\delta}{n}\sum_{i=1}^{n} H_t^i = (1-\delta)\frac{N_0 - N_t}{N_0},$$

where $N_0 = n$ is the number of credit names in the reference portfolio and the residual protection N_t is given by (5.4). Finally, denote by $C_l = U_l - L_l$ the width of the lth tranche; in particular, for the first (i.e., equity) tranche we have $C_1 = U_1$ since $L_1 = 0$.

Purchasing one unit of the lth tranche at time 0 generates the following discounted cash flows

$$\text{Premium leg} = \kappa_0^l \sum_{j=1}^{J} \frac{B_0}{B_{t_j}} N_{t_j}^l,$$

where N_t^l is the *residual tranche protection* at time t, that is,

$$N_t^l = N_0\Big(C_l - \min\big(C_l,\, \max\left(Q_t - L_l, 0\right)\big)\Big).$$

The discounted cash flows of the protection leg are

$$\text{Protection leg} = (1-\delta)\sum_{i=1}^{n} \frac{B_0}{B_{\tau_i}} H_T^i \mathbb{1}_{\{L_l < Q_{\tau_i} \le U_l\}}.$$

The equity tranche of the CDO on iTraxx or CDX is quoted differently; specifically, it is quoted in terms if an upfront rate, say κ_0^1, on the total tranche notional, in addition to 500 basis points (5% rate) paid annually on the residual tranche nominal. The discounted premium leg cash flows of the equity tranche are thus given by the expression

$$\kappa_0^1 N_0 C_0 + .05 \sum_{j=1}^{J} \frac{B_0}{B_{t_j}} N_{t_j}^0$$

or, more explicitly,

$$\kappa_0^1 n C_0 + .05 \sum_{j=1}^{J} \frac{B_0}{B_{t_j}} n\big(C_0 - \min(C_0, Q_{t_j})\big).$$

Additionally to standard traded tranches of a CDO, some non-standard tranches – commonly referred to as *bespoke tranches* – are traded over-the-counter. Typically, a credit risk model is first calibrated to market quotes for standard tranches, and subsequently it is used to value bespoke tranches.

5.8.3 First-to-Default Swaps

A kth-to-default swap is a basket credit instrument backed by a portfolio of single-name CDSs. Due to the rapid growth in popularity of credit default swap indices and the associated derivatives, the kth-to-default swaps have become rather illiquid. Currently, such products are typically customized contracts between a bank and its customer, and hence they are relatively bespoke to the customer's credit portfolio.

For this reason, in the sequel we focus our attention on *first-to-default swaps* issued on the iTraxx index, which are the only ones with a certain degree of liquidity. Standardized first-to-default swaps are now issued on each of the iTraxx sector sub-indices. Each first-to-default swap is backed by an equally weighted portfolio of five single-name CDSs in the relative sub-index, chosen according to some liquidity criteria. Let us describe the main provisions of a first-to-default swap (FTDS).

- The time of issuance of the contract is 0 and the maturity is T.
- By investing in a first-to-default swap, the protection seller agrees to absorb the loss produced by the first default in the reference credit portfolio.
- In exchange, the protection seller is paid a periodic premium, known as FTDS spread, up to maturity T or the moment of the first default, whichever comes first. We denote the FTDS spread at time 0 by κ_0.

Recall that by $t_j, j = 0, 1, \ldots, J$ with $t_0 = 0$ and $t_J = T$ we denote the tenor of the premium leg payments dates. As usual, we denote by $\tau_{(1)}$ the random time of the first default in the pool. The discounted cumulative cash flows associated with a first-to-default swap are as follows

$$\text{Premium leg} = \kappa_0 \sum_{j=1}^{J} \frac{B_0}{B_{t_j}} \mathbb{1}_{\{t_j \le \tau_{(1)}\}}$$

and

$$\text{Protection leg} = (1-\delta)\frac{B_0}{B_{\tau_{(1)}}} \mathbb{1}_{\{\tau_{(1)} \le T\}}.$$

It is worth stressing that the market convention stipulates that the notional corresponding to each credit name in the reference credit portfolio is equal. Moreover, the recovery rate is assumed to be constant, that is, the recovery rate does not depend on a particular credit name.

5.8.4 Step-up Corporate Bonds

As of now, step-up corporate bonds are not traded in baskets; however, they are of our interest since they offer protection against credit events other than defaults, for instance, the downgrade of the rating of the reference name.

Step-up corporate bonds are coupon-bearing bond issues for which the amounts of coupon payments depend on the credit quality of the bond's issuer. As the name of the bond suggests, the coupon payment increases when the credit quality of the issuer declines.

In practice, the above-mentioned *credit quality* is reflected by a credit rating assigned to an issuer by at least one specialized ratings agency (such as: Moody's KMV, Fitch, or Standard & Poor's). The provisions linking the cash flows of the step-up bonds to the credit rating of an issuer have different step amounts and different rating event triggers. In some cases, a step-up of the coupon requires a downgrade to the trigger level by both rating agencies. In other cases, there are step-up triggers for actions of each rating agency. Under this specification, a downgrade by one of agencies will trigger an increase in the coupon regardless of the rating from the other agency.

Provisions also vary with respect to *step-down* features which, as the name suggests, trigger a lowering of the coupon if the company regains its original rating after a downgrade. In general, there is no step-down below the initial coupon for ratings exceeding the initial rating.

Let X_t stand for some indicator of the issuer's credit quality at time t. Assume that t_j, $j = 1, 2, \ldots, J$ are coupon payment dates and denote by $c_j = c(X_{t_{j-1}})$ the coupon amount at time t_j. The time t discounted cumulative cash flows of a step-up bond are given by the expression

$$(1 - H_T)\frac{B_t}{B_T} + \int_{]t,T]} (1 - H_u)\frac{B_t}{B_u}\, dC_u + \text{ recovery payment}$$

where we denote by C the process given by the expression $C_t = \sum_{t_j \le t} c_j$.

5.8.5 Valuation of Basket Credit Derivatives

Computation of the *fair spread* at time t for a basket credit derivative involves evaluating the conditional expectation under the martingale measure $\mathbb{Q}$ of the associated discounted cash flows. In the case of CDS indices, CDOs and FTDSs, the fair spread at time t is such that the value of the contract at time t is exactly zero, i.e., the risk-neutral conditional expectations of discounted cumulative cash flows of the premium and protection legs are identical.

The following expressions for fair spreads or values at time $t \in [0, T]$ can be easily derived from the discounted cumulative cash flows given in the preceding subsections (note, however, that $\sum_{j=1}^{J}$ now stands for $\sum_{j=1,\, t_j \ge t}^{J}$ and we assume that the CDO tranches were issued at time 0):

- the fair spread of a single-name CDS on the ith credit name

$$\kappa_t^i = \frac{(1 - \delta_i)\, \mathbb{E}_{\mathbb{Q}}\Big(\frac{B_t}{B_{\tau_i}} (H_T^i - H_t^i) \,\Big|\, \mathcal{G}_t \Big)}{\mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^{J} \frac{B_t}{B_{t_j}} (1 - H_{t_j}^i) \,\Big|\, \mathcal{G}_t \Big)},$$

- the fair spread of a CDIS

$$\kappa_t = \frac{(1 - \delta)\, \mathbb{E}_{\mathbb{Q}}\Big(\sum_{i=1}^{n} \frac{B_t}{B_{\tau_i}} (H_T^i - H_t^i) \,\Big|\, \mathcal{G}_t \Big)}{\mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^{J} \frac{B_t}{B_{t_j}} \Big(n - \sum_{i=1}^{n} H_{t_j}^i \Big) \,\Big|\, \mathcal{G}_t \Big)},$$

- the fair spread of the lth tranche of a CDO

$$\kappa_t^l = \frac{(1 - \delta)\, \mathbb{E}_{\mathbb{Q}}\Big(\sum_{i=1}^{n} \frac{B_t}{B_{\tau_i}} (H_T^i - H_t^i) \mathbb{1}_{\{L_l \le Q_{\tau_i} \le U_l\}} \,\Big|\, \mathcal{G}_t \Big)}{\mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^{J} \frac{B_t}{B_{t_j}} n \Big(C_l - \min\big(C_l,\, \max\,(Q_{t_j} - L_l, 0)\big)\Big) \,\Big|\, \mathcal{G}_t \Big)},$$

- the fair upfront rate of the equity tranche of a CDO

$$\kappa_t^1 = \frac{1}{nC_0}\, \mathbb{E}_{\mathbb{Q}}\Big((1 - \delta) \sum_{i=1}^{n} \frac{B_t}{B_{\tau_i}} (H_T^i - H_t^i) \mathbb{1}_{\{Q_{\tau_i} \le U_0\}} \,\Big|\, \mathcal{G}_t \Big)$$

$$- \frac{.05}{nC_0}\, \mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^{J} \frac{B_t}{B_{t_j}}\, n\big(C_0 - \min\,(C_0, Q_{t_j})\big) \,\Big|\, \mathcal{G}_t \Big),$$

- the fair spread of a first-to-default swap

$$\kappa_t = \frac{(1-\delta)\,\mathbb{E}_{\mathbb{Q}}\Big(\frac{B_t}{B_{\tau_{(1)}}}\mathbb{1}_{\{\tau_{(1)}\le T\}}\,\Big|\,\mathcal{G}_t\Big)}{\mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^J \frac{B_t}{B_{t_j}}\mathbb{1}_{\{t_j\le \tau_{(1)}\}}\,\Big|\,\mathcal{G}_t\Big)},$$

- the fair value of a step-up corporate bond

$$\mathbb{E}_{\mathbb{Q}}\Big((1-H_T)\frac{B_t}{B_T} + \int_{]t,T]}(1-H_u)\frac{B_t}{B_u}\,dC_u + \text{ recovery payment}\,\Big|\,\mathcal{G}_t\Big).$$

Depending on the dimensionality of the problem, the above conditional expectations will be evaluated either by means of Monte Carlo simulation or through some other numerical method.

5.9 Modeling of Credit Ratings

We will now give a brief description of a generic Markovian market model that can be efficiently used for valuation and hedging basket credit instruments. The model presented below is a special case of a general approach examined in Bielecki et al. [12]. Some preliminary empirical studies of this model and its extensions are reported in Bielecki et al. [25, 26].

For related methods and models, the interested reader is referred to, e.g., Albanese and Chen [1], Chen and Filipović [52], Frey and Backhaus [86, 87], Jarrow et al. [105], Kijima and Komoribayashi [119], and Kijima et al. [120].

Let the underlying probability space be denoted by $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, where $\mathbb{Q}$ is a risk-neutral measure inferred from the market via calibration and $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ is a filtration containing all information available to market agents. The filtration $\mathbb{H}$ carries information about evolution of credit events, such as changes in credit ratings or defaults of respective credit names. An additional filtration $\mathbb{F}$ is called a *reference filtration*; it is meant to contain the information pertaining to the evolution of relevant macroeconomic variables.

We consider n credit names and we assume that the credit quality of each reference entity falls to the set $\mathcal{K} = \{1, 2, \dots, K\}$ of K rating categories, where, by convention, the category K corresponds to default.

Let X^i, $i = 1, 2, \dots, n$ be some stochastic processes defined on $(\Omega, \mathcal{G}, \mathbb{Q})$ and taking values in the finite state space $\mathcal{K}$, where the process X^i represents the evolution of credit ratings of the ith underlying entity. Then we define the *default time* τ_i of the ith credit name by setting

$$\tau_i = \inf\{\, t \in \mathbb{R}_+ : X^i_t = K\}.$$

We postulate that the default state K is absorbing, so that for each credit name the default event can only occur once.

We denote by $X = (X^1, X^2, \dots, X^n)$ the joint *credit ratings process* for a given portfolio of n credit names. The state space of X is thus $\mathcal{X} = \mathcal{K}^n$ and the elements of $\mathcal{X}$ will be denoted by x. We postulate that the filtration $\mathbb{H}$ is the natural filtration of the process X, whereas the reference filtration $\mathbb{F}$ is generated by an $\mathbb{R}^d$-valued *factor process* Y, which represents the evolution of other relevant economic variables, like interest rates or equity prices.

5.9.1 Infinitesimal Generator

Under the standing assumption that the factor process Y is $\mathbb{R}^d$-valued, the state space for the process $M = (X, Y)$ equals $\mathcal{X} \times \mathbb{R}^d$. At the intuitive level, we wish to model the process $M = (X, Y)$ as a combination of a Markov chain X modulated by a Lévy-type process Y and a Lévy-type process Y modulated by a Markov chain X.

For this purpose, we start by making a general postulate that the *infinitesimal generator* $\mathbf{A}$ of M is given by the expression

$$
\begin{aligned}
\mathbf{A}f(x,y) &= \frac{1}{2}\sum_{l,m=1}^{d} a_{lm}(x,y)\partial_l\partial_m f(x,y) + \sum_{l=1}^{d} b_l(x,y)\partial_l f(x,y) \\
&\quad + \gamma(x,y)\int_{\mathbb{R}^d} \big(f(x, y + g(x,y,y')) - f(x,y)\big)\Pi(x,y;dy') \\
&\quad + \sum_{x' \in \mathcal{X}} \lambda(x,x';y) f(x',y),
\end{aligned}
$$

where $\lambda(x,x';y) \geq 0$ for every $x = (x_1, x_2, \dots, x_n) \neq (x'_1, x'_2, \dots, x'_n) = x'$ and

$$
\lambda(x,x;y) = -\sum_{x' \in \mathcal{X},\, x' \neq x} \lambda(x,x';y).
$$

Here ∂_l denotes the partial derivative with respect to the variable y_l. The existence and uniqueness of a Markov process M with the generator $\mathbf{A}$ will follow (under appropriate technical conditions) from the classic results regarding solutions to martingale problems.

We find it convenient to refer to X (Y, respectively) as the *Markov chain component* of M (the *jump-diffusion component* of M, respectively). At any time t, the intensity matrix of the Markov chain component is given as $\Lambda_t = [\lambda(x,x';Y_t)]_{x,x' \in \mathcal{X}}$. The jump-diffusion component satisfies the SDE

$$
dY_t = b(X_t, Y_t)\,dt + \sigma(X_t, Y_t)\,dW_t + \int_{\mathbb{R}^d} g(X_{t-}, Y_{t-}, y')\,\pi(X_{t-}, Y_{t-}; dy', dt),
$$

where, for any fixed $(x,y) \in \mathcal{X} \times \mathbb{R}^d$, $\pi(x,y;dy',dt)$ is a Poisson measure with the intensity measure $\gamma(x,y)\Pi(x,y;dy')dt$ and $\sigma(x,y)$ satisfies the equality $\sigma(x,y)\sigma(x,y)^{\mathsf{T}} = a(x,y)$.

Remarks 5.9.1. If we take $g(x,y,y') = y'$ and we suppose that the coefficients $\sigma = [\sigma_{ij}], b = [b_i], \gamma$ and the measure Π do not depend on x and y then the factor process Y is a Poisson-Lévy process with the characteristic triplet (a,b,ν), where the diffusion matrix is $a(x,y) = \sigma(x,y)\sigma(x,y)^{\mathsf{T}}$, the drift vector equals $b(x,y)$ and the Lévy measure ν satisfies $\nu(dy) = \gamma\Pi(dy)$.

In order to proceed further with the analysis of the model, we need to provide with more structure the Markov chain component of the infinitesimal generator $\mathbf{A}$. To this end, we make the following standing assumption.

Assumption (M). The infinitesimal generator of the process $M = (X,Y)$ has the following form

$$\begin{aligned}\mathbf{A}f(x,y) &= \frac{1}{2}\sum_{l,m=1}^{d} a_{lm}(x,y)\partial_l\partial_m f(x,y) + \sum_{l=1}^{d} b_l(x,y)\partial_l f(x,y)\\ &\quad + \gamma(x,y)\int_{\mathbb{R}^d}\big(f(x,y+g(x,y,y')) - f(x,y)\big)\Pi(x,y;dy') \qquad (5.5)\\ &\quad + \sum_{i=1}^{n}\sum_{x^{i\prime}\in\mathcal{K}} \lambda^i(x,x_i';y)f(x_i',y),\end{aligned}$$

where we use the shorthand notation $x'^i = (x_1,x_2,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_n)$. Note that x'^i is simply the vector $x = (x_1,x_2,\ldots,x_n)$ with the ith coordinate x_i replaced by x_i'.

In the case of two reference credit entities (that is, when $n = 2$), the infinitesimal generator $\mathbf{A}$ becomes

$$\begin{aligned}\mathbf{A}f(x,y) &= \frac{1}{2}\sum_{l,m=1}^{d} a_{lm}(x,y)\partial_l\partial_m f(x,y) + \sum_{l=1}^{d} b_l(x,y)\partial_l f(x,y)\\ &\quad + \gamma(x,y)\int_{\mathbb{R}^d}\big(f(x,y+g(x,y,y')) - f(x,y)\big)\Pi(x,y;dy')\\ &\quad + \sum_{x'^1\in\mathcal{K}} \lambda^1(x,x_1';y)f(x_1',y) + \sum_{x'^2\in\mathcal{K}} \lambda^2(x,x_2';y)f(x_2',y),\end{aligned}$$

where $x = (x_1,x_2)$, $x'^1 = (x_1',x_2)$ and $x'^2 = (x_1,x_2')$. Returning to the general form, we have that, for $x = (x_1,x_2)$ and $x' = (x_1',x_2')$,

$$\lambda(x,x';y) = \begin{cases} \lambda^1(x,x'^1;y), & \text{if } x_2 = x_2',\\ \lambda^2(x,x'^2;y), & \text{if } x_1 = x_1',\\ 0, & \text{otherwise.}\end{cases}$$

Similar expressions can be derived for the case of an arbitrary number of underlying credit names. Note that the model specified by (5.5) does not allow for simultaneous jumps of credit ratings X^i and $X^{i'}$ for $i \neq i'$. This is not a serious lack of generality, however, since the ratings of both credit names may still change in an arbitrarily small time interval. The advantage is that, for the purpose of simulation of paths of process X, rather than dealing with $K^n \times K^n$ intensity matrix $[\lambda(x, x'; y)]$, it will be sufficient to deal with n intensity matrices $[\lambda^i(x, x'^i; y)]$ of dimension $K \times K$ (for any fixed y). Within the present setup, the current credit rating of the credit name i has a direct influence on the level of the transition intensity for the current rating of the credit name i', and vice versa. This property, known as *frailty*, is likely to contribute to the default contagion effect.

Remarks 5.9.2. (i) It is clear that we can incorporate in the model the case when at least some components of the factor process Y follow Markov chains themselves. This feature is important, as factors such as economic cycles may be modeled as Markov chains. It is known that default rates are strongly related to business cycles.
(ii) Some of the factors $Y^1, Y^2, \dots, Y^d$ may represent cumulative duration of visits of processes X^i in particular rating states. For example, we may set

$$Y_t^1 = \int_0^t \mathbb{1}_{\{X_u^1 = 1\}}\, du.$$

so that $b_1(x, y) = \mathbb{1}_{\{x^1=1\}}(x)$ and the corresponding components of coefficients σ and g equal zero.
(iii) In the area of *structural arbitrage*, the so-called *credit–to–equity* models and/or *equity–to–credit* models are studied. The market model presented in this section nests both types of interactions. For example, if one of the factors is the price process of the equity issued by a credit name, and if credit migration intensities depend on this factor (either implicitly or explicitly), then we have a equity–to–credit type interaction. If the credit rating of a given name impacts the equity dynamics for this name (and/or some other names), then we deal with a credit–to–equity type interaction.

Let $H_t^i = \mathbb{1}_{\{t \geq \tau_i\}}$ for every $i = 1, 2, \dots, n$ and let the process H be defined as $H_t = \sum_{i=1}^n H_t^i$. It can be observed that the process $S = (H, X, Y)$ is a Markov process on the state space $\{0, 1, \dots, n\} \times \mathcal{X} \times \mathbb{R}^d$ with respect to its natural filtration. Given the form of the infinitesimal generator of the process (X, Y), we can easily describe the infinitesimal generator of the process (H, X, Y). To this end, it is enough to observe that the transition intensity at time t of the component H from the state H_t to the state $H_t + 1$ is equal to $\sum_{i=1}^n \lambda^i(X_t, K; X_t^{(i)}, Y_t)$, provided that $H_t < n$ (otherwise, the transition intensity equals zero), where we write $X_t^{(i)} = (X_t^1, \ldots, X_t^{i-1}, X_t^{i+1}, \ldots, X_t^n)$ and we set $\lambda^i(x_i, x_i'; x^{(i)}, y) = \lambda^i(x, x'^i; y)$.

5.9.2 Transition Intensities for Credit Ratings

One should always strive to find a right balance between the realistic features of a financial model and its complexity. This issue frequently nests the issues of functional representation of a model, as well as its parameterization. In what follows, we present an example of a particular model for credit ratings, which is rather arbitrary, but is nevertheless relatively simple, and thus it should be easy to estimate and/or calibrate.

Let $\bar{X}_t$ be the average credit rating at time t, so that

$$\bar{X}_t = \frac{1}{n} \sum_{i=1}^{n} X_t^i.$$

Let $\mathcal{L} = \{i_1, i_2, \ldots, i_{\widehat{n}}\}$ be a subset of the set of all credit names, where $\widehat{n} < n$. We consider $\mathcal{L}$ to be a collection of "major players" whose economic situation, reflected by their credit ratings, effectively impacts all other credit names in the pool. The following exponential-linear regression model appears to be a plausible model for the ratings transitions intensities

$$\begin{aligned} \ln \lambda^i(x, x'^i; y) &= \alpha_{i,0}(x_i, x_i') + \sum_{l=1}^{d} \alpha_{i,l}(x_i, x_i') y_l + \beta_{i,0}(x_i, x_i') h \\ &+ \sum_{k=1}^{\widehat{n}} \beta_{i,k}(x_i, x_i') x_k + \tilde{\beta}_i(x_i, x_i') \bar{x} + \widehat{\beta}_i(x_i, x_i')(x_i - x_i'), \end{aligned} \tag{5.6}$$

where h represents a generic value of H_t, so that $h = \sum_{i=1}^{n} \mathbb{1}_{\{K\}}(x_i)$. Similarly, $\bar{x}$ stands for a generic value of $\bar{X}_t$, that is, $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$.

The number of parameters involved in (5.6) can easily be controlled by the number of model variables, in particular, the number of factors and the number of credit ratings, as well as structure of the transition matrix (see Section 5.9.9 below). In addition, the reduction of the number of parameters can be obtained if the pool of all n credit names is partitioned into a (small) number of homogeneous sub-pools. All of this is a matter of a practical implementation of a specific Markovian model of credit ratings.

Assume, for instance, that there are $\widetilde{n} \ll n$ homogeneous sub-pools of credit names, and the parameters $\alpha, \beta, \widetilde{\beta}$ and $\widehat{\beta}$ in (5.6) do not depend on x_i, x_i'. Then the migration intensities (5.6) are parameterized by $\widetilde{n}(d+\widehat{n}+4)$ parameters.

5.9.3 Conditionally Independent Credit Migrations

Suppose that the transition intensities $\lambda^i(x, x'^i; y)$ do not depend on the vector $x^{(i)} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ for every $i = 1, 2, \ldots, n$. In addition, assume that the dynamics of the factor process Y do not depend on the migration process X. It turns out that in this case, given the structure of our generator as in (5.5), the credit ratings processes X^i, $i = 1, 2, \ldots, n$, are conditionally independent given the sample path of the factor process Y.

We shall illustrate this point in the case of only two credit names in the pool (i.e., for $n = 2$) and assuming that there is no factor process, so that conditional independence really means independence between migration processes X^1 and X^2. For this, suppose that X^1 and X^2 are independent Markov chains, each taking values in the state space $\mathcal{K}$ and with the infinitesimal generator matrices Λ^1 and Λ^2, respectively. It is clear that the joint process $X = (X^1, X^2)$ is a Markov chain on $\mathcal{K} \times \mathcal{K}$. An easy calculation reveals that the infinitesimal generator of the process X is given as

$$\Lambda = \Lambda^1 \otimes \mathrm{Id}_K + \mathrm{Id}_K \otimes \Lambda^2,$$

where Id_K is the identity matrix of size K and $\otimes$ denotes the matrix tensor product. This result is consistent with structure (5.5) in the present case.

5.9.4 Examples of Markovian Models

We will now present three pertinent examples of Markovian market models.

Markov Chain Credit Ratings Process

In the first example, we assume that there is no factor process Y and thus we only deal with a ratings migration process X. In this situation, an attractive and efficient way to model credit ratings is to postulate that X is a *birth-and-death process* with absorption at state K. The intensity matrix Λ is here tri-diagonal. Let us write $p_t(k, k') = \mathbb{Q}(X_{s+t} = k' \,|\, X_s = k)$.

The transition probabilities $p_t(k, k')$ are known to satisfy the following system of ordinary differential equations, for $t \in \mathbb{R}_+$ and $k' = 1, 2, \ldots, K$,

$$\begin{aligned}
\frac{dp_t(1, k')}{dt} &= -\lambda(1,2)p_t(1,k') + \lambda(1,2)p_t(2,k'), \\
\frac{dp_t(k, k')}{dt} &= \lambda(k,k-1)p_t(k-1,k') - (\lambda(k,k-1) + \lambda(k,k+1))p_t(k,k') \\
&\quad + \lambda(k,k+1)p_t(k+1,k')
\end{aligned}$$

for $k = 2, 3, \ldots, K-1$, whereas for $k = K$ we simply have that

$$\frac{dp_t(K, k')}{dt} = 0,$$

with the initial conditions $p_0(k,k') = \mathbb{1}_{\{k=k'\}}$. Once the transition intensities $\lambda(k,k')$ are specified, the above system can be easily solved. Note, in particular, that $p_t(K,k') = 0$ for every t if $k' \neq K$. The advantage of this representation is that the number of parameters can be kept relatively small.

A more flexible credit ratings model is obtained if we allow for jumps to the default state K from any other state. In that case, the intensity matrix is no longer tri-diagonal and the ordinary differential equations for transition probabilities take the following form, for $t \in \mathbb{R}_+$ and $k' = 1,2,\ldots,K$,

$$\frac{dp_t(1,k')}{dt} = -(\lambda(1,2)+\lambda(1,K))p_t(1,k')+\lambda(1,2)p_t(2,k')+\lambda(1,K)p_t(K,k')$$

$$\begin{aligned}\frac{dp_t(k,k')}{dt} &= \lambda(k,k-1)p_t(k-1,k') - (\lambda(k,k-1)+\lambda(k,k+1))p_t(k,k') \\ &\quad + \lambda(k,K)p_t(k,k')+\lambda(k,k+1)p_t(k+1,k')+\lambda(k,K)p_t(K,k')\end{aligned}$$

for $k = 2,3,\ldots,K-1$ and for $k = K$

$$\frac{dp_t(K,k')}{dt} = 0,$$

with initial conditions $p_0(k,k') = \mathbb{1}_{\{k=k'\}}$.

Remark 5.9.1. Some authors model migrations of credit ratings using a proxy diffusion, possibly with a jump to default. The birth-and-death process with jumps to default furnishes a Markov chain counterpart of such proxy diffusion models. The nice feature of the Markov chain model is that, at least in principle, the credit ratings are here observable state variables, whereas in the case of a proxy diffusion model they are not directly observable.

Diffusion-type Factor Process

We will now extend the Markov chain process by adding a factor process Y. We may postulate, for instance, that the factor process follows a diffusion process and that the generator of the Markov process $M = (X,Y)$ takes the following form

$$\begin{aligned}\mathbf{A}f(x,y) = \frac{1}{2}\sum_{l,m=1}^{d} a_{lm}(x,y)\partial_l\partial_m f(x,y) + \sum_{l=1}^{d} b_l(x,y)\partial_l f(x,y) \\ + \sum_{x'\in\mathcal{K},\, x'\neq x} \lambda(x,x';y)(f(x',y)-f(x,y)).\end{aligned}$$

Let $\phi(t,x,y,x',y')$ be the transition probability of M, specifically,

$$\phi(t,x,y,x',y')\,dy' = \mathbb{Q}(X_{s+t} = x', Y_{s+t} \in dy' \mid X_s = x, Y_s = y).$$

In order to determine the function ϕ, one needs to examine the Kolmogorov equation of the form

$$\frac{dv(s,x,y)}{ds} + \mathbf{A}v(s,x,y) = 0. \tag{5.7}$$

For the generator $\mathbf{A}$ of the present form, the corresponding equation (5.7) is commonly known as the *reaction-diffusion equation* (see, for instance, Becherer and Schweizer [10]). Let us mention that a reaction-diffusion equation is a special case of a more general integro-partial-differential equation, which was extensively studied in the mathematical literature.

Forward CDS Spread Model

Suppose now that the factor process $Y_t = \kappa(t, T_S, T_M)$ is the *forward CDS spread* (for the definition of $\kappa(t, T_S, T_M)$, see Section 5.9.6 below). We now postulate that the generator of $M = (X, Y)$ is

$$\mathbf{A}f(x,y) = \frac{1}{2}y^2 a(x)\frac{d^2 f(x,y)}{dy^2} + \sum_{x' \in \mathcal{K},\, x' \neq x} \lambda(x,x')(f(x',y) - f(x,y)),$$

so that the forward CDS spread process satisfies the following SDE

$$d\kappa(t, T_S, T_M) = \kappa(t, T_S, T_M)\sigma(X_t)\, dW_t$$

for some Brownian motion process W, where $\sigma(x) = \sqrt{a(x)}$. Note that in this example $\kappa(t, T_S, T_M)$ is a conditionally log-Gaussian process given a particular sample path of the credit ratings process X. Therefore, we are in a position to make use of Proposition 5.9.1 below to value a credit default swaption.

5.9.5 Forward Credit Default Swap

Let us first examine two examples of a single-name credit derivative. We assume that the reference asset is a corporate bond maturing at time U and we consider a *forward CDS* with the maturity date $T_M < U$ and the start date $T_S < T_M$. If default occurs prior to or at time T_S the contract is terminated with no exchange of payments. Therefore, the two legs of this CDS are manifestly T_S-survival claims and thus the valuation of a forward CDS is not much different from valuation of a spot CDS.

Protection Leg

Assume that the notional amount of the bond equals 1 and denote by δ a deterministic recovery rate in case of default. Under the assumption that the recovery is paid at default time τ of the reference credit name, the value at time t of the protection leg of a forward CDS is equal to, for every $t \le T_S$,

$$P_t = P(t, T_S, T_M) = (1-\delta) B_t \, \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{T_S < \tau \le T_M\}} B_\tau^{-1} \,|\, M_t\big).$$

The valuation of the protection leg relies on computation of this conditional expectation for a given term structure model. In particular, if the savings account B is a deterministic function of time then the computation reduces to the following integration

$$B_t \, \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{T_S < \tau \le T_M\}} B_\tau^{-1} \,|\, M_t\right) = B_t \int_{T_S}^{T_M} B_u^{-1} \, \mathbb{Q}(\tau \in du \,|\, M_t).$$

Premium Leg

Let us denote by $t_1 < t_2 < \cdots < t_J$ the tenor of premium payments, where $T_S < t_1 < \cdots < t_J \le T_M$. We assume that the premium accrual covenant is in force, so that the cash flows associated with the premium leg are

$$\kappa \Big(\sum_{j=1}^{J} \mathbb{1}_{\{t_j < \tau\}} \mathbb{1}_{t_j}(t) + \sum_{j=1}^{J} \mathbb{1}_{\{t_{j-1} < \tau \le t_j\}} \mathbb{1}_{\tau}(t) \frac{t - t_{j-1}}{t_j - t_{j-1}} \Big).$$

where κ is the fixed CDS spread. Consequently, the value at time $t \in [0, T_S]$ of the premium leg equals κA_t, where $A_t = A(t, T_S, T_M)$ equals

$$A_t = \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{T_S < \tau\}} \sum_{j=1}^{J} \Big[\frac{B_t}{B_{t_j}} \mathbb{1}_{\{t_j < \tau\}} + \frac{B_t}{B_\tau} \mathbb{1}_{\{t_{j-1} < \tau \le t_j\}} \frac{\tau - t_{j-1}}{t_j - t_{j-1}} \Big] \Big| \, M_t \Big).$$

Under the assumption that B is deterministic and the conditional distribution $\mathbb{Q}(\tau \le s \,|\, M_t)$ is known, this conditional expectation can be evaluated.

5.9.6 Credit Default Swaptions

We consider a forward credit default swap starting at T_S and maturing at $T_M > T_S$, as described in the previous section. Our next goal is to examine valuation of the corresponding *credit default swaption* with expiry date $T < T_S$ and the strike spread K. The swaption's payoff at its expiry date T equals

$$\big(P_T - K A_T\big)^+,$$

and thus the swaption's price equals, for every $t \in [0, T]$,

$$\begin{aligned} B_t \, \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} \big(P_T - K A_T\big)^+ \,\Big|\, M_t \Big) \\ = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(B_T^{-1} A_T \big(\kappa(T, T_S, T_M) - K\big)^+ \,\Big|\, M_t \Big), \end{aligned}$$

where the process $\kappa(t, T_S, T_M) = P_t/A_t$, $t \in [0, T_S]$, represents the *forward CDS spread.*

Note that the random variables P_t and A_t are strictly positive on the event $\{\tau > t\}$ for $t \le T < T_S$ and thus the forward CDS spread $\kappa(t, T_S, T_M)$ enjoys this property as well.

Conditionally Gaussian Case

In order to provide a more explicit representation for the value of a credit default swaption, we assume that B is deterministic and the forward CDS spread is conditionally log-Gaussian under $\mathbb{Q}$. It is worth recalling that an example of such a model was presented in Section 5.9.4.

Proposition 5.9.1. *Suppose that, on the event $\{\tau > t\}$ and for arbitrary $t < t_1 < \cdots < t_k \le T$, the conditional distribution*

$$\mathbb{Q}\Big(\kappa(t_m, T_S, T_M) \le k_m,\ m = 1, 2, \ldots, k \,\Big|\, \sigma(M_t) \vee \mathcal{F}_T^X\Big)$$

is lognormal, $\mathbb{Q}$-a.s. Let us denote by $\sigma(u, T_S, T_M)$, $u \in [t, T]$, the conditional volatility of the process $\kappa(u, T_S, T_M)$, $u \in [t, T]$, with respect to the σ-field $\sigma(M_t) \vee \mathcal{F}_T^X$. Then the price at time t of the credit default swaption is given by the expression

$$\begin{aligned} &B_t\, \mathbb{E}_{\mathbb{Q}}\left(B_T^{-1}(P_T - KA_T)^+ \,\Big|\, M_t\right) \\ &= B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\tau > T\}} A_T B_T^{-1}\Big[\kappa_t N(d_+(t,T)) - KN(d_-(t,T))\Big] \,\Big|\, M_t\Big), \end{aligned}$$

where, for brevity, we write $\kappa_t = \kappa(t, T_S, T_M)$. Moreover, we denote

$$d_\pm(t, T) = \frac{\ln \frac{\kappa_t}{K}}{\upsilon_{t,T}} \pm \frac{\upsilon_{t,T}}{2},$$

and

$$\upsilon_{t,T}^2 = \upsilon(t, T, T_S, T_M)^2 = \int_t^T \sigma(u, T_S, T_M)^2\, du.$$

Proof. We start by noting that

$$\begin{aligned} &B_t\, \mathbb{E}_{\mathbb{Q}}\left(B_T^{-1}(P_T - KA_T)^+ \,\Big|\, M_t\right) \\ &= B_t\, \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\tau > T\}} B_T^{-1}(P_T - KA_T)^+ \,\Big|\, M_t\right) \\ &= B_t\, \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\tau > T\}} B_T^{-1} \mathbb{E}_{\mathbb{Q}}\left((P_T - KA_T)^+ \,|\, \sigma(M_t) \vee \mathcal{F}_T^X\right) \,\Big|\, M_t\right) \\ &= B_t\, \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\tau > T\}} A_T B_T^{-1} \mathbb{E}_{\mathbb{Q}}\left((\kappa_T - K)^+ \,|\, \sigma(M_t) \vee \mathcal{F}_T^X\right) \,\Big|\, M_t\right). \end{aligned}$$

In view of the present assumptions, we also have that

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left(\left(\kappa_T-K\right)^+\,\Big|\,\sigma(M_t)\vee\mathcal{F}_T^X\right)\\&\qquad=\kappa_t N\left(\frac{\ln\frac{\kappa_t}{K}}{\upsilon_{t,T}}+\frac{\upsilon_{t,T}}{2}\right)-KN\left(\frac{\ln\frac{\kappa_t}{K}}{\upsilon_{t,T}}-\frac{\upsilon_{t,T}}{2}\right).\end{aligned}$$

By combining the above equalities, we arrive at the stated formula. □

5.9.7 Spot kth-to-Default Credit Swap

Let us now examine the valuation of credit derivatives with several underlying credit names within the present framework. Feasibility of closed-form computations of relevant conditional expectations depends to a large extent on the type and amount of information one wishes to utilize. Typically, in order to efficiently deal with exact calculations of conditional expectations, one will need to amend specifications of the underlying model so that information used in calculations is given by a coarser filtration, or perhaps by some proxy filtration.

In this subsection, we will discuss the valuation of a generic *kth-to-default credit swap* relative to a portfolio of n reference corporate bonds. The deterministic notional value of the ith constituent bond is denoted by N_i and the corresponding deterministic recovery rate equals δ_i.

The maturities of the bonds in the portfolio are $T_1, T_2, \ldots, T_n$, whereas the maturity of the swap is $T_M < \min\{T_1, T_2, \ldots, T_n\}$. Let us consider, for instance, a plain-vanilla basket CDS written on such a portfolio of corporate bonds under the convention of the fractional recovery of par value.

This means that, on the event $\{\tau_{(k)} < T_M\}$, the protection buyer receives at time $\tau_{(k)}$ the cumulative compensation

$$\sum_{i\in\mathcal{L}_k}(1-\delta_i)N_i,$$

where $\mathcal{L}_k$ is the (random) set of all constituent credit names that defaulted in the time interval $]0,\tau_{(k)}]$. This means that the protection buyer is protected against the cumulative effect of the first k defaults. Recall that, in view of the model assumptions, the possibility of simultaneous defaults is excluded.

Protection Leg

The cash flows of the protection leg are given by the expression

$$\sum_{i\in\mathcal{L}_k}(1-\delta_i)N_i\mathbb{1}_{\{\tau_{(k)}\le T_M\}}\mathbb{1}_{\tau_{(k)}}(t).$$

Hence the value at time t of the protection leg is equal to

$$P_t^{(k)} = P^{(k)}(t, T_M) = B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_{(k)}\le T_M\}} B_{\tau_{(k)}}^{-1} \sum_{i\in\mathcal{L}_k} (1-\delta_i) N_i \,\Big|\, M_t \Big).$$

In general, this conditional expectation will need to be evaluated numerically by means of a Monte Carlo simulation.

A special case of a kth-to-default credit swap is when the protection buyer is protected against losses associated with the kth default only. In that case, the cash flow associated with the default protection leg is given by the expression

$$(1-\delta_{\iota^{(k)}}) N_{\iota^{(k)}} \mathbb{1}_{\{\tau_{(k)}\le T_M\}} \mathbb{1}_{\tau_{(k)}}(t) = \sum_{i=1}^{n} (1-\delta_i) N_i \mathbb{1}_{\{H_{\tau_i}=k\}} \mathbb{1}_{\{\tau_i \le T_M\}} \mathbb{1}_{\tau_i}(t),$$

where $\iota^{(k)}$ stands for the identity of the kth-to-default credit name. Under the assumption that the numéraire process B is deterministic, it is possible to represent the value at time t of the protection leg as the following conditional expectation

$$\begin{aligned} P_t^{(k)} &= \sum_{i=1}^{n} B_t \, \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_i\le T_M\}} \mathbb{1}_{\{H_{\tau_i}=k\}} B_{\tau_i}^{-1} (1-\delta_i) N_i \,\Big|\, M_t \Big) \\ &= \sum_{i=1}^{n} B_t (1-\delta_i) N_i \int_t^{T_M} B_u^{-1} \, \mathbb{Q}(H_u = k \,|\, \tau_i = u, M_t) \, \mathbb{Q}(\tau_i \in du \,|\, M_t). \end{aligned}$$

Note also that the conditional probability $\mathbb{Q}(H_u = k \,|\, \tau_i = u, M_t)$ can be approximated by the following expression

$$\mathbb{Q}(H_u = k \,|\, \tau_i = u, M_t) \approx \frac{\mathbb{Q}(H_u = k, X_{u-\epsilon}^i \ne K, X_u^i = K \,|\, M_t)}{\mathbb{Q}(X_{u-\epsilon}^i \ne K, X_u^i = K \,|\, M_t)}.$$

Therefore, if the number n of credit names is small, so that the Kolmogorov equations for the conditional distribution of the process (H, X, Y) can be solved, the value of $P_t^{(k)}$ can be approximated analytically.

Premium Leg

Let $t_1 < t_2 < \cdots < t_J$ denote the tenor of premium payments, where $0 = t_0 < t_1 < \cdots < t_J < T_M$. Under the assumption that the premium accrual covenant is in force, the cash flows associated with the premium leg of the kth-to-default CDS admit the following representation

$$\kappa^{(k)} \Big(\sum_{j=1}^{J} \mathbb{1}_{\{t_j<\tau_{(k)}\}} \mathbb{1}_{t_j}(t) + \sum_{j=1}^{J} \mathbb{1}_{\{t_{j-1}<\tau_{(k)}\le t_j\}} \mathbb{1}_{\tau_{(k)}}(t) \frac{t - t_{j-1}}{t_j - t_{j-1}} \Big),$$

where $\kappa^{(k)}$ is the fixed spread of the kth-to-default CDS.

Consequently, the value at time t of the premium leg equals $\kappa^{(k)} A_t^{(k)}$, where

$$A_t^{(k)} = A^{(k)}(t, T_M) = \mathbb{E}_{\mathbb{Q}}\bigg(\mathbb{1}_{\{t<\tau_{(k)}\}} \sum_{j=j(t)}^{J} \frac{B_t}{B_{t_j}} \mathbb{1}_{\{t_j<\tau_{(k)}\}} \,\Big|\, M_t \bigg)$$
$$+ \mathbb{E}_{\mathbb{Q}}\bigg(\mathbb{1}_{\{t<\tau_{(k)}\}} \sum_{j=j(t)}^{J} \frac{B_t}{B_{\tau_{(k)}}} \mathbb{1}_{\{t_{j-1}<\tau_{(k)}\}\le t_j\}} \frac{\tau_{(k)} - t_{j-1}}{t_j - t_{j-1}} \,\Big|\, M_t \bigg),$$

where $j(t)$ is the smallest integer such that $t_{j(t)} > t$. Again, in general, the above conditional expectation will need to be approximated by simulation. And again, for a small portfolio size n, if either exact or a numerical solution of relevant Kolmogorov equations can be derived, then an analytical computation of the expectation can be done, at least in principle.

5.9.8 Forward kth-to-Default Credit Swap

A forward kth-to-default credit swap has an analogous structure to a forward CDS. The notation used here is consistent with the notation that was introduced in Sections 5.9.5 and 5.9.7.

Protection Leg

The cash flow associated with the protection leg of a forward kth-to-default credit swap can be expressed as follows

$$\sum_{i\in\mathcal{L}_k} (1-\delta_i) N_i \mathbb{1}_{\{T_S<\tau_{(k)}\le T_M\}} \mathbb{1}_{\tau_{(k)}}(t).$$

Consequently, the value of the protection leg equals, for every $t \in [0, T_S]$,

$$P_t^{(k)} = P^{(k)}(t, T_S, T_M) = B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{T_S<\tau_{(k)}\le T_M\}} B_{\tau_{(k)}}^{-1} \sum_{i\in\mathcal{L}_k} (1-\delta_i) N_i \,\Big|\, M_t \Big).$$

Premium Leg

As before, let $t_1 < t_2 < \cdots < t_J$ be the tenor of premium payments, where $T_S < t_1 < \cdots < t_J < T_M$. Under the premium accrual covenant, the cash flows associated with the premium leg are

$$\kappa^{(k)} \bigg(\sum_{j=1}^{J} \mathbb{1}_{\{t_j<\tau_{(k)}\}} \mathbb{1}_{t_j}(t) + \sum_{j=1}^{J} \mathbb{1}_{\{t_{j-1}<\tau_{(k)}\le t_j\}} \mathbb{1}_{\tau_{(k)}}(t) \frac{t - t_{j-1}}{t_j - t_{j-1}} \bigg),$$

where $\kappa^{(k)}$ is the fixed spread.

Therefore, the value at time t of the premium leg is $\kappa^{(k)} A_t^{(k)}$, where the random variable $A_t^{(k)} = A^{(k)}(t, T_S, T_M)$ is given by the expression

$$\mathbb{E}_{\mathbb{Q}}\bigg(\mathbb{1}_{\{t<\tau_{(k)}\}} \Big[\sum_{j=1}^{J} \frac{B_t}{B_{t_j}} \mathbb{1}_{\{t_j<\tau\}} + \sum_{j=1}^{J} \frac{B_t}{B_\tau} \mathbb{1}_{\{t_{j-1}<\tau_{(k)}\le t_j\}} \frac{\tau - t_{j-1}}{t_j - t_{j-1}} \Big] \,\Big|\, M_t \bigg).$$

We have only presented here two examples of credit derivatives with several reference credit names. Computations of arbitrage prices and fair spreads for other examples of basket credit derivative involve evaluating the conditional expectations presented in Section 5.8.5.

It is worth stressing that the choice of a particular model for the valuation of a given class of basket credit derivatives should be motivated by arguments regarding its practical relevance as well as its mathematical tractability. In the remaining part of this section, we will examine some issues arising in this context.

5.9.9 Model Implementation

Let us now briefly discuss some practical problems related to the model implementation. As already mentioned, when one deals with basket products involving several reference credit names, direct computations may not be feasible, since the cardinality of the state space $\mathbf{K}$ for the migration process X is equal to K^n. Thus, for example, in case of $K = 18$ rating categories, as in Moody's ratings,[1] and in case of a portfolio of $n = 100$ credit names, the cardinality of the state space $\mathbf{K}$ equals 18^{100}.

If one aims to derive closed-form expressions for conditional expectations, but K is large, then, typically, it will be infeasible to work directly with information provided by the state vector $(X, Y) = (X^1, X^2, \ldots, X^n, Y)$ and with the corresponding infinitesimal generator $\mathbf{A}$. An essential reduction in the amount of information that can be effectively used for analytical computations will be required. This goal can be achieved by reducing the number of rating categories; this is typically done by considering only two categories: pre-default and default.

This reduction may still not be sufficient enough in some circumstances, however, and thus further simplifying structural modifications to the model may need to be called for. Some types of additional modifications – such as: *homogeneous grouping* of credit names and *mean-field interactions* between credit names – were proposed in the financial literature to address this important issue. The interested reader is referred, for instance, to the paper by Frey and Backhaus [86].

[1] We refer here to the following rating categories attributed by Moody's: Aaa, Aa1, Aa2, Aa3, A1, A2, A3, Baa1, Baa2, Baa3, Ba1, Ba2, Ba3, B1, B2, B3, Caa, D(efault).

Recursive Simulation Procedure

When closed-form computations are not feasible, but one does not want to give up on potentially available information, an alternative may be to carry approximate calculations by means of either approximating some involved formulae and/or by simulating sample paths of underlying random processes. We will briefly examine the Monte Carlo simulations approach.

In general, a simulation of the evolution of the process X will be infeasible, due to the curse of dimensionality. However, by virtue of the postulated structure of the infinitesimal generator $\mathbf{A}$ (see (5.5)), a simulation of the evolution of the process X reduces to a recursive simulation of the evolution of processes X^i, whose state spaces are only of size K each. To facilitate simulations even further, we also postulate that each migration process X^i behaves like a birth-and-death process with absorption at default and with possible jumps to default from every intermediate state (see Section 5.9.4).

Recall that we denote $X_t^{(i)} = (X_t^1, \ldots, X_t^{i-1}, X_t^{i+1}, \ldots, X_t^n)$.

Given the state $(x^{(i)}, y)$ of the process $(X^{(i)}, Y)$, the intensity matrix of the ith migration process is sub-stochastic and is given as

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ K-1 \\ K \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & \cdots & K-1 & K \end{array} \\
\begin{pmatrix}
\lambda^i(1,1) & \lambda^i(1,2) & \cdots & 0 & \lambda^i(1,K) \\
\lambda^i(2,1) & \lambda^i(2,2) & \cdots & 0 & \lambda^i(2,K) \\
0 & \lambda^i(3,2) & \cdots & 0 & \lambda^i(3,K) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda^i(K-1,K-1) & \lambda^i(K-1,K) \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix} \end{array},$$

where we use the shorthand notation $\lambda^i(x_i, x_i') = \lambda^i(x, x'^i; y)$.

Also, we find it convenient to write $\lambda^i(x_i, x_i'; x^{(i)}, y) = \lambda^i(x, x'^i; y)$ in what follows. Then the diagonal elements are given as follows, for $x_i \neq K$,

$$\begin{aligned}
\lambda^i(x,x;y) &= -\lambda^i(x_i, x_i - 1; x^{(i)}, y) - \lambda^i(x_i, x_i + 1; x^{(i)}, y) - \lambda^i(x_i, K; x^{(i)}, y) \\
&\quad - \sum_{l \neq i} \Big(\lambda^l(x_l, x_l - 1; x^{(l)}, y) + \lambda^l(x_l, x_l + 1; x^{(l)}, y) + \lambda^l(x_l, K; x^{(l)}, y) \Big)
\end{aligned}$$

with the convention that $\lambda^i(1, 0; x^{(i)}, y) = 0$ for every $i = 1, 2, \ldots, n$.

It is implicit in the above description that $\lambda^i(K, x_i; x^{(i)}, y) = 0$ for any $i = 1, 2, \ldots, n$ and $x_i = 1, 2, \ldots, K$. Suppose now that the current state of the process (X, Y) is (x, y). Then the intensity of a jump of the process X equals

$$\lambda(x, y) := - \sum_{i=1}^{n} \lambda^i(x, x; y).$$

Conditional on the occurrence of a jump of X, the probability distribution of a jump for the component X^i, $i = 1, 2, \ldots, n$, is given as follows:

- the probability of a jump from x_i to $x_i - 1$ equals

$$p_i(x_i, x_i - 1; x^{(i)}, y) = \frac{\lambda^i(x_i, x_i - 1; x^{(i)}, y)}{\lambda(x, y)},$$

- the probability of a jump from x_i to $x_i + 1$ equals

$$p_i(x_i, x_i + 1; x^{(i)}, y) = \frac{\lambda^i(x_i, x_i + 1; x^{(i)}, y)}{\lambda(x, y)},$$

- the probability of a jump from x_i to K equals

$$p_i(x_i, K; x^{(i)}, y) = \frac{\lambda^i(x_i, K; x^{(i)}, y)}{\lambda(x, y)}.$$

As expected, the following equality is valid

$$\sum_{i=1}^{n} \big(p_i(x_i, x_i - 1; x^{(i)}, y) + p_i(x_i, x_i + 1; x^{(i)}, y) + p_i(x_i, K; x^{(i)}, y) \big) = 1.$$

For a generic state $x = (x_1, x_2, \ldots, x_n)$ of the migration process X, we define the *jump space*

$$\mathcal{J}(x) = \bigcup_{i=1}^{n} \{(x_i - 1, i), (x_i + 1, i), (K, i)\}$$

with the convention that $(K + 1, i) = (K, i)$, where the shorthand notation (a, i) refers to the ith component of X. Given that the process (X, Y) is in the state (x, y) and conditional on the occurrence of a jump of X, the process X jumps to a point in the space $\mathcal{J}(x)$ according to the probability distribution denoted by $p(x, y)$ and determined by the probabilities p_i described above. Thus, if a random variable ζ has the distribution given by $p(x, y)$ then we have that, for any $(x'_i, i) \in \mathcal{J}(x)$,

$$\mathbb{Q}(\zeta = (x'_i, i)) = p_i(x_i, x'_i; x^{(i)}, y).$$

Simulation Algorithm

We conclude this section by presenting in some detail the simulation algorithm for the case when the dynamics of the factor process Y do not depend on the credit ratings process X. The general case appears to be much harder.

Under the assumption that the dynamics of the factor process Y do not depend on the process X, the simulation procedure splits into two steps. In Step 1, a sample path of the process Y is simulated; then, in Step 2, for a given sample path Y, a sample path of the process X is simulated.

We consider here simulations of sample paths over some generic time interval, say $[t_1, t_2]$, where $0 \leq t_1 < t_2$. We assume that the number of defaulted names at time t_1 is less than k, that is $H_{t_1} < k$. We conduct the simulation either until the kth default occurs or until time t_2, depending on whichever occurs first.

Step 1: The dynamics of the factor process are now given by the SDE

$$dY_t = b(Y_t)\,dt + \sigma(Y_t)\,dW_t + \int_{\mathbb{R}^d} g(Y_{t-}, y)\pi(Y_{t-}; dy, dt), \quad t \in [t_1, t_2].$$

Any standard procedure can be used to simulate a sample path of Y. Let us denote by $\widehat{Y}$ the simulated sample path of Y.

Step 2: Once a sample path of Y has been simulated, simulate a sample path of X on the interval $[t_1, t_2]$ until the kth default time.

We exploit the fact that, according to our assumptions about the infinitesimal generator $\mathbf{A}$, the components of the credit ratings process X do not have simultaneous jumps. Therefore, the following algorithm for simulating the evolution of X appears to be feasible:

Step 2.1: Set the counter $m = 1$ and simulate the first jump time of the process X in the time interval $[t_1, t_2]$. Towards this end, simulate first a value, say $\widehat{\eta}_1$, of a unit exponential random variable η_1. The simulated value of the first jump time, τ_1^X, is then given as

$$\widehat{\tau}_1^X = \inf\Big\{\, t \in [t_1, t_2] \,:\, \int_{t_1}^{t} \lambda(X_{t_1}, \widehat{Y}_u)\,du \geq \widehat{\eta}_1 \Big\},$$

where by convention the infimum over an empty set is $+\infty$. If $\widehat{\tau}_1^X = +\infty$, set the simulated value of the kth default time to be $\widehat{\tau}_{(k)} = +\infty$, stop the current run of the simulation procedure and go to Step 3. Otherwise, go to Step 2.2.

Step 2.2: Simulate the jump of X at time $\widehat{\tau}_1^X$ by drawing from the distribution $p(X_{t_1}, \widehat{Y}_{\widehat{\tau}_1^X-})$ (see the discussion in Section 5.9.9). In this way, one obtains a simulated value $\widehat{X}_{\widehat{\tau}_1^X}$, as well as the simulated value of the number of defaults $\widehat{H}_{\widehat{\tau}_1^X}$. If $\widehat{H}_{\widehat{\tau}_1^X} < k$ then let $m := m + 1$ and go to Step 2.3; otherwise, set $\widehat{\tau}_{(k)} = \widehat{\tau}_1^X$ and go to Step 3.

Step 2.3: Simulate the mth jump of process X. Towards this end, simulate a value, say $\widehat{\eta}_m$, of a unit exponential random variable η_m. The simulated value of the mth jump time τ_m^X is obtained from the formula

$$\widehat{\tau}_m^X = \inf \Big\{ t \in [\widehat{\tau}_{m-1}^X, t_2] \, : \, \int_{\widehat{\tau}_{m-1}^X}^t \lambda(X_{\widehat{\tau}_{m-1}^X}, \widehat{Y}_u) \, du \geq \widehat{\eta}_m \Big\}.$$

In case $\widehat{\tau}_m^X = +\infty$, let the simulated value of the kth default time to be $\widehat{\tau}_{(k)} = +\infty$; stop the current run of the simulation procedure and go to Step 3. Otherwise, go to Step 2.4.

Step 2.4: Simulate the jump of X at time $\widehat{\tau}_m^X$ by drawing from the distribution $p(X_{\widehat{\tau}_{m-1}^X}, \widehat{Y}_{\widehat{\tau}_m^X -})$. In this way, produce a simulated value $\widehat{X}_{\widehat{\tau}_m^X}$, as well as the simulated value of the number of defaults $\widehat{H}_{\widehat{\tau}_m^X}$. If $\widehat{H}_{\widehat{\tau}_m^X} < k$, let $m := m + 1$ and go to Step 2.3; otherwise, set $\widehat{\tau}_{(k)} = \widehat{\tau}_m^X$ and go to Step 3.

Step 3: Calculate a simulated value of a relevant functional. For example, in case of the kth-to-default CDS, compute

$$\widehat{P}_{t_1}^{(k)} = \mathbb{1}_{\{t_1 < \widehat{\tau}_{(k)} \leq T\}} \widehat{B}_{t_1} \widehat{B}_{\widehat{\tau}_{(k)}}^{-1} \sum_{i \in \widehat{\mathcal{L}}_k} (1 - \delta_i) N_i$$

and

$$\widehat{A}_{t_1}^{(k)} = \sum_{j=j(t_1)}^{J} \frac{\widehat{B}_{t_1}}{\widehat{B}_{t_j}} \mathbb{1}_{\{t_j < \widehat{\tau}_{(k)}\}} + \sum_{j=j(t_1)}^{J} \frac{\widehat{B}_{t_1}}{\widehat{B}_{\widehat{\tau}_{(k)}}} \mathbb{1}_{\{t_{j-1} < \widehat{\tau}_{(k)} \leq t_j\}} \frac{\widehat{\tau}_{(k)} - t_{j-1}}{t_j - t_{j-1}},$$

where, as before, the 'hat' indicates that we deal with simulated values.

Concluding Remarks

The issue of evaluating functionals associated with multiple credit migrations is prominent with regard to measuring and managing of portfolio credit risk. In some segments of the credit derivatives market, only the deterioration of the value of a portfolio of debts (bonds or loans) due to defaults is essential. For instance, such is the situation regarding the tranches of both cash and synthetic collateralized debt obligations, as well as the tranches of traded credit default swap indices, such as CDX and iTraxx.

It is rather clear, however, that a valuation model reflecting the possibility of intermediate credit migrations through other ratings classes, and not only defaults, is called for in order to better account for changes in creditworthiness of the reference credit entities. Likewise, for the purpose of managing risks of a debt portfolio, it is necessary to account for changes in value of the portfolio due to variations in credit ratings of constituent credit names.

Appendix A

Complements

In some credit risk models, the need to model a sequence of successive defaults may arise. This can be achieved by utilizing the $\mathbb{F}$-*conditional Poisson process,* which is also known as the *doubly stochastic Poisson process.* The general idea is quite similar to the *canonical construction* of a single random time (cf. Section 3.2.2). We start by assuming that we are given a stochastic process Λ, to be interpreted as the *hazard process,* and we construct a jump process, with unit jump size, such that the probabilistic features of consecutive jump times are governed by the hazard process Λ.

A.1 Standard Poisson Process

We start be recalling the definition of the standard Poisson process.

Definition A.1.1. A process N defined on a probability space $(\Omega, \mathbb{G}, \mathbb{P})$ is called the (standard) *Poisson process* with a constant intensity λ with respect to the filtration $\mathbb{G}$ if $N_0 = 0$ and for any $0 \leq s < t$ the following two conditions are satisfied:
(i) the increment $N_t - N_s$ is independent of the σ-field $\mathcal{G}_s$,
(ii) the increment $N_t - N_s$ has the Poisson law with parameter $\lambda(t-s)$; specifically, for any $k = 0, 1, \ldots$ we have

$$\mathbb{P}(N_t - N_s = k \,|\, \mathcal{G}_s) = \mathbb{P}(N_t - N_s = k) = \frac{\lambda^k (t-s)^k}{k!}\, e^{-\lambda(t-s)}.$$

The Poisson process of Definition A.1.1 is termed *time-homogeneous,* since the probability law of the increment $N_{t+h} - N_{s+h}$ is invariant with respect to the shift $h \geq -s$. In particular, for arbitrary $s < t$ the probability law of the increment $N_t - N_s$ coincides with the law of the random variable N_{t-s}. Let us finally observe that, for every $0 \leq s < t$,

$$\mathbb{E}_{\mathbb{P}}(N_t - N_s \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(N_t - N_s) = \lambda(t-s). \tag{A.1}$$

It is standard to take a version of the Poisson process whose sample paths are, with probability 1, right-continuous stepwise functions with all jumps of size 1.

Let us set $\tau_0 = 0$ and let us denote by $\tau_1, \tau_2, \dots$ the $\mathbb{G}$-stopping times given as the random moments of the successive jumps of N. For any $k = 0, 1, \dots$

$$\tau_{k+1} = \inf \{ t > \tau_k : N_t \neq N_{\tau_k} \} = \inf \{ t > \tau_k : N_t - N_{\tau_k} = 1 \}.$$

One shows without difficulties that $\mathbb{P}(\lim_{k\to\infty} \tau_k = \infty) = 1$. It is convenient to introduce the sequence $(\xi_k,\ k \in \mathbb{N})$ of non-negative random variables, where $\xi_k = \tau_k - \tau_{k-1}$ for every $k \in \mathbb{N}$. Let us quote the following well known result.

Proposition A.1.1. *The random variables ξ_k, $k \in \mathbb{N}$ are mutually independent and identically distributed, with the exponential law with parameter λ, that is, for any $k \in \mathbb{N}$ we have, for every $t \in \mathbb{R}_+$,*

$$\mathbb{P}(\xi_k \le t) = \mathbb{P}(\tau_k - \tau_{k-1} \le t) = 1 - e^{-\lambda t}.$$

Proposition A.1.1 suggests a simple construction of a process N, which follows a time-homogeneous Poisson process with respect to its natural filtration $\mathbb{F}^N$. Suppose that the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is large enough to support a family of mutually independent random variables ξ_k, $k \in \mathbb{N}$ with the common exponential law with parameter $\lambda > 0$. We define the process N on $(\Omega, \mathcal{G}, \mathbb{P})$ by setting $N_t = 0$ if $\{t < \xi_1\}$ and, for any natural k,

$$N_t = k \quad \text{if and only if} \quad \sum_{i=1}^{k} \xi_i \le t < \sum_{i=1}^{k+1} \xi_i.$$

It can checked that the process N defined in this way is indeed a Poisson process with parameter λ, with respect to its natural filtration $\mathbb{F}^N$. The jump times of N are, of course, the random times $\tau_k = \sum_{i=1}^{k} \xi_i$, $k \in \mathbb{N}$.

Let us recall some useful equalities that are not hard to establish through elementary calculations involving the Poisson law. For any $a \in \mathbb{R}$ and every $0 \le s < t$ we have

$$\mathbb{E}_{\mathbb{P}}\big(e^{ia(N_t - N_s)} \,\big|\, \mathcal{G}_s\big) = \mathbb{E}_{\mathbb{P}}\big(e^{ia(N_t - N_s)}\big) = e^{\lambda(t-s)(e^{ia}-1)}$$

and

$$\mathbb{E}_{\mathbb{P}}\big(e^{a(N_t - N_s)} \,\big|\, \mathcal{G}_s\big) = \mathbb{E}_{\mathbb{P}}\big(e^{a(N_t - N_s)}\big) = e^{\lambda(t-s)(e^{a}-1)}.$$

The next result is an easy consequence of (A.1) and the above formulae. The proof of the proposition is thus left to the reader.

Proposition A.1.2. *The following stochastic processes are $\mathbb{G}$-martingales.*
(i) The compensated Poisson process $\widehat{N}$ defined as

$$\widehat{N}_t = N_t - \lambda t.$$

(ii) For any $k \in \mathbb{N}$, the compensated Poisson process stopped at τ_k

$$\widehat{M}_t^k = N_{t\wedge\tau_k} - \lambda(t \wedge \tau_k).$$

(iii) For any $a \in \mathbb{R}$, the exponential martingale M^a given by the formula

$$M_t^a = e^{aN_t - \lambda t(e^a-1)} = e^{a\widehat{N}_t - \lambda t(e^a - a - 1)}.$$

(iv) For any fixed $a \in \mathbb{R}$, the exponential martingale K^a given by the formula

$$K_t^a = e^{iaN_t - \lambda t(e^{ia}-1)} = e^{ia\widehat{N}_t - \lambda t(e^{ia} - ia - 1)}.$$

Remark A.1.1. (i) For any $\mathbb{G}$-martingale M, defined on some filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$, and an arbitrary $\mathbb{G}$-stopping time τ, the stopped process $M_t^\tau = M_{t\wedge\tau}$ is necessarily a $\mathbb{G}$-martingale. Thus, the second statement of the proposition is an immediate consequence of the first, combined with the simple observation that each jump time τ_k is a $\mathbb{G}$-stopping time.
(ii) Consider the random time $\tau = \tau_1$, where τ_1 is the time of the first jump of the Poisson process N. Then $N_{t\wedge\tau} = N_{t\wedge\tau_1} = H_t$, so that the process $\widehat{M}^1$ introduced in part (ii) of the proposition coincides with the martingale $\widehat{M}$ associated with τ.
(iii) The property described in part (iii) of Proposition A.1.2 characterizes the Poisson process in the following sense: if $N_0 = 0$ and for every $a \in \mathbb{R}$ the process M^a is a $\mathbb{G}$-martingale, then N is the Poisson process with parameter λ. Indeed, the martingale property of M^a yields, for every $0 \le s < t$,

$$\mathbb{E}_{\mathbb{P}}\big(e^{a(N_t - N_s)} \,\big|\, \mathcal{G}_s\big) = e^{\lambda(t-s)(e^a-1)}.$$

By standard arguments, this implies that the random variable $N_t - N_s$ is independent of the σ-field $\mathcal{G}_s$ and has the Poisson law with parameter $\lambda(t-s)$. A similar remark applies to property (iv) in Proposition A.1.2.

Let us consider the case of a Brownian motion W and a Poisson process N that are defined on a common filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$. In particular, for every $0 \le s < t$, the increment $W_t - W_s$ is independent of the σ-field $\mathcal{G}_s$ and has the Gaussian law $N(0, t-s)$.

It might be useful to recall that for any real number b the following processes follow martingales with respect to $\mathbb{G}$:

$$\widehat{W}_t = W_t - t, \quad m_t^b = e^{bW_t - \frac{1}{2}b^2 t}, \quad k_t^b = e^{ibW_t + \frac{1}{2}b^2 t}.$$

Proposition A.1.3. *Let a Brownian motion W with respect to $\mathbb{G}$ and a Poisson process N with respect to $\mathbb{G}$ be defined on a common probability space $(\Omega, \mathbb{G}, \mathbb{P})$. Then the processes W and N are mutually independent.*

Proof. In the first step of the proof, we will show that for any $t \in \mathbb{R}_+$ the random variables W_t and N_t are mutually independent under $\mathbb{P}$. On the one hand, for any fixed $a \in \mathbb{R}$ and every $t > 0$, we have

$$\begin{aligned} e^{iaN_t} &= 1 + \sum_{0<u\le t} (e^{iaN_t} - e^{iaN_{t-}}) = 1 + \int_{]0,t]} (e^{ia} - 1) e^{iaN_{u-}} \, dN_u, \\ &= 1 + \int_{]0,t]} (e^{ia} - 1) e^{iaN_{u-}} \, d\widehat{N}_u + \lambda \int_0^t (e^{ia} - 1) e^{iaN_{u-}} \, du. \end{aligned}$$

On the other hand, for any $b \in \mathbb{R}$, an application of the Itô formula yields the following equality

$$e^{ibW_t} = 1 + ib \int_0^t e^{ibW_u} \, dW_u - \frac{1}{2} b^2 \int_0^t e^{ibW_u} \, du.$$

The continuous martingale part of the compensated Poisson process $\widehat{N}$ is identically equal to 0 (since $\widehat{N}$ is a process of finite variation), and obviously the processes $\widehat{N}$ and W have no common jumps. Therefore, using the Itô product rule for semimartingales (see, for instance, Elliott [78] or Protter [153]), we obtain

$$\begin{aligned} e^{i(aN_t + bW_t)} &= 1 + ib \int_0^t e^{i(aN_u + bW_u)} \, dW_u - \frac{1}{2} b^2 \int_0^t e^{i(aN_u + bW_u)} \, du \\ &+ \int_{]0,t]} (e^{ia} - 1) e^{i(aN_{u-} + bW_u)} \, d\widehat{N}_u + \lambda \int_0^t (e^{ia} - 1) e^{i(aN_u + bW_u)} \, du. \end{aligned}$$

Let us denote $f_{a,b}(t) = \mathbb{E}_{\mathbb{P}}(e^{i(aN_t + bW_t)})$. By taking the expectations of both sides of the last equality, we get

$$f_{a,b}(t) = 1 + \lambda \int_0^t (e^{ia} - 1) f_{a,b}(u) \, du - \frac{1}{2} b^2 \int_0^t f_{a,b}(u) \, du.$$

By solving the last equation, we obtain, for arbitrary $a, b \in \mathbb{R}$,

$$\mathbb{E}_{\mathbb{P}}\big(e^{i(aN_t + bW_t)}\big) = f_{a,b}(t) = e^{\lambda t(e^{ia} - 1)} e^{-\frac{1}{2} b^2 t} = \mathbb{E}_{\mathbb{P}}\big(e^{iaN_t}\big) \mathbb{E}_{\mathbb{P}}\big(e^{ibW_t}\big).$$

In view of the last equality, we conclude that, for any fixed $t \in \mathbb{R}_+$, the random variables W_t and N_t are mutually independent under $\mathbb{P}$.

In the second step, we fix $0 < t < s$ and we consider the following expectation, for arbitrary real numbers a_1, a_2, b_1 and b_2,

$$f(t,s) := \mathbb{E}_{\mathbb{P}}\big(e^{i(a_1 N_t + a_2 N_s + b_1 W_t + b_2 W_s)}\big).$$

Let us denote $\tilde{a}_1 = a_1 + a_2$ and $\tilde{b}_1 = b_1 + b_2$. By standard computations, we obtain the following chain of equalities

$$\begin{aligned} f(t,s) &= \mathbb{E}_{\mathbb{P}}\big(e^{i(a_1 N_t + a_2 N_s + b_1 W_t + b_2 W_s)}\big) \\ &= \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}\big(e^{i(\tilde{a}_1 N_t + a_2(N_s - N_t) + \tilde{b}_1 W_t + b_2(W_s - W_t))} \,\big|\, \mathcal{G}_t\big)\big) \\ &= \mathbb{E}_{\mathbb{P}}\big(e^{i(\tilde{a}_1 N_t + \tilde{b}_1 W_t)} \mathbb{E}_{\mathbb{P}}\big(e^{i(a_2(N_s - N_t) + b_2(W_s - W_t))} \,\big|\, \mathcal{G}_t\big)\big) \\ &= \mathbb{E}_{\mathbb{P}}\big(e^{i(\tilde{a}_1 N_t + \tilde{b}_1 W_t)} \mathbb{E}_{\mathbb{P}}\big(e^{i(a_2 N_{s-t} + b_2 W_{s-t})}\big)\big) \\ &= f_{a_1,b_1}(s-t)\, \mathbb{E}_{\mathbb{P}}\big(e^{i(\tilde{a}_1 N_t + \tilde{b}_1 W_t)}\big) \\ &= f_{a_1,b_1}(s-t) f_{\tilde{a}_1,\tilde{b}_1}(t), \end{aligned}$$

where we have used, in particular, the independence of the increments $N_s - N_t$ and $W_s - W_t$ of the σ-field $\mathcal{G}_t$ and the time-homogeneity of the Poisson process N and the Brownian motion W.

By setting $b_1 = b_2 = 0$ in the last formula, we obtain

$$\mathbb{E}_{\mathbb{P}}\big(e^{i(a_1 N_t + a_2 N_s)}\big) = f_{a_1,0}(s-t) f_{\tilde{a}_1,0}(t),$$

whereas the choice of $a_1 = a_2 = 0$ yields

$$\mathbb{E}_{\mathbb{P}}\big(e^{i(b_1 W_t + b_2 W_s)}\big) = f_{0,b_1}(s-t) f_{0,\tilde{b}_1}(t).$$

It is not difficult to check that

$$f_{a_1,b_1}(s-t) f_{\tilde{a}_1,\tilde{b}_1}(t) = f_{a_1,0}(s-t) f_{\tilde{a}_1,0}(t) f_{0,b_1}(s-t) f_{0,\tilde{b}_1}(t).$$

We conclude that for any $0 \le t < s$ and arbitrary $a_1, a_2, b_1, b_2 \in \mathbb{R}$:

$$\mathbb{E}_{\mathbb{P}}\big(e^{i(a_1 N_t + a_2 N_s + b_1 W_t + b_2 W_s)}\big) = \mathbb{E}_{\mathbb{P}}\big(e^{i(a_1 N_t + a_2 N_s)}\big) \mathbb{E}_{\mathbb{P}}\big(e^{i(b_1 W_t + b_2 W_s)}\big).$$

This means that the random variables (N_t, N_s) and (W_t, W_s) are mutually independent. By proceeding along the same lines, one may check that the random variables $(N_{t_1}, \ldots, N_{t_n})$ and $(W_{t_1}, \ldots, W_{t_n})$ are mutually independent for any $n \in \mathbb{N}$ and for any choice of $0 \le t_1 < \cdots < t_n$. □

Let us now examine the behavior of the Poisson process under a specific equivalent change of the underlying probability measure. For a fixed $T > 0$, we introduce a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ by setting

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}_T} = \eta_T, \quad \mathbb{P}\text{-a.s.}, \tag{A.2}$$

where the Radon-Nikodým density process $(\eta_t,\, t \in [0,T])$ satisfies, for some constant $\kappa > -1$,

$$d\eta_t = \eta_{t-} \kappa \, d\widehat{N}_t, \quad \eta_0 = 1, \tag{A.3}$$

Since $Y := \kappa\widehat{N}$ is a process of finite variation, (A.3) admits a unique solution, denoted as $\mathcal{E}_t(Y)$ or $\mathcal{E}_t(\kappa\widehat{N})$. Clearly, this solution can be seen as a special case of the Doléans (or stochastic) exponential. By solving (A.3) in the path-by-path manner, we obtain

$$\eta_t = \mathcal{E}_t(\kappa\widehat{N}) = e^{Y_t} \prod_{0<u\leq t} (1+\Delta Y_u)e^{-\Delta Y_u} = e^{Y_t^c} \prod_{0<u\leq t} (1+\Delta Y_u),$$

where $Y_t^c := Y_t - \sum_{0<u\leq t} \Delta Y_u$ is the path-by-path continuous part of Y. Direct calculations show that

$$\eta_t = e^{-\kappa\lambda t} \prod_{0<u\leq t} (1+\kappa\Delta N_u) = e^{-\kappa\lambda t}(1+\kappa)^{N_t} = e^{N_t \ln(1+\kappa)-\kappa\lambda t},$$

where the last equality is valid provided that $\kappa > -1$. Upon setting $a = \ln(1+\kappa)$ in part (iii) of Proposition A.1.2, we obtain $\eta = M^a$; this confirms that the process η is a $\mathbb{G}$-martingale under $\mathbb{P}$. We have thus proved the following result.

Lemma A.1.1. *Assume that $\kappa > -1$. The unique solution η to the SDE (A.3) is an exponential $\mathbb{G}$-martingale under $\mathbb{P}$. Specifically,*

$$\eta_t = e^{N_t \ln(1+\kappa)-\kappa\lambda t} = e^{\widehat{N}_t \ln(1+\kappa)-\lambda t(\kappa-\ln(1+\kappa))} = M_t^a, \tag{A.4}$$

where $a = \ln(1+\kappa)$. In particular, the random variable η_T is strictly positive, $\mathbb{P}$-a.s. and $\mathbb{E}_{\mathbb{P}}(\eta_T) = 1$. Furthermore, the process M^a solves the following SDE

$$dM_t^a = M_{t-}^a (e^a - 1)\, d\widehat{N}_t, \quad M_0^a = 1.$$

We are in a position to establish the well-known result, which states that the process $(N_t,\, t \in [0,T])$ is a Poisson process with the constant intensity $\lambda^* = (1+\kappa)\lambda$ under $\mathbb{Q}$.

Proposition A.1.4. *Assume that under $\mathbb{P}$ a process N is a Poisson process with intensity λ with respect to the filtration $\mathbb{G}$. Suppose that the probability measure $\mathbb{Q}$ is defined on $(\Omega, \mathcal{G}_T)$ through (A.2) and (A.3) for some $\kappa > -1$.*
(i) The process $(N_t,\, t \in [0,T])$ is a Poisson process under $\mathbb{Q}$ with respect to $\mathbb{G}$ with the constant intensity $\lambda^* = (1+\kappa)\lambda$.
(ii) The compensated process $(N_t^*,\, t \in [0,T])$ defined as

$$N_t^* = N_t - \lambda^* t = N_t - (1+\kappa)\lambda t = \widehat{N}_t - \kappa\lambda t,$$

is a $\mathbb{G}$-martingale under $\mathbb{Q}$.

Proof. From Remark A.1.1(iii), we know that it suffices to find λ^* such that, for any fixed $b \in \mathbb{R}$, the process $\widetilde{M}^b$, given as

$$\widetilde{M}^b_t := e^{bN_t - \lambda^* t(e^b - 1)}, \quad \forall\, t \in [0,T], \tag{A.5}$$

is a $\mathbb{G}$-martingale under $\mathbb{Q}$. By standard arguments, the process $\widetilde{M}^b$ is a $\mathbb{Q}$-martingale if and only if the product $\widetilde{M}^b\eta$ is a martingale under the original probability measure $\mathbb{P}$. But in view of (A.4), we have

$$\widetilde{M}^b_t \eta_t = \exp\Big(N_t\big(b + \ln(1+\kappa)\big) - t\big(\kappa\lambda + \lambda^*(e^b - 1)\big)\Big).$$

Let us write $a = b + \ln(1+\kappa)$. Since b is an arbitrary real number, so is a. Then, by virtue of part (iii) in Proposition A.1.2, we necessarily have

$$\kappa\lambda + \lambda^*(e^b - 1) = \lambda(e^a - 1).$$

After simplifications, we conclude that, for any fixed real number b, the process $\widetilde{M}^b$ defined by (A.5) is a $\mathbb{G}$-martingale under $\mathbb{Q}$ if and only if $\lambda^* = (1+\kappa)\lambda$. In other words, the intensity λ^* of N under $\mathbb{Q}$ satisfies $\lambda^* = (1+\kappa)\lambda$. Also the second statement is clear. □

Assume that W is a Brownian motion and N is a Poisson process under $\mathbb{P}$ with respect to $\mathbb{G}$. Let η satisfy

$$d\eta_t = \eta_{t-}\big(\theta_t\, dW_t + \kappa\, d\widehat{N}_t\big), \quad \eta_0 = 1, \tag{A.6}$$

for some $\mathbb{G}$-predictable stochastic process θ and some constant $\kappa > -1$. A simple application of the Itô's product rule shows that if processes η^1 and η^2 satisfy the SDEs $d\eta^1_t = \eta^1_{t-}\theta_t\, dW_t$ and $d\eta^2_t = \eta^2_{t-}\kappa\, d\widehat{N}_t$ then their product $\eta_t = \eta^1_t\eta^2_t$ satisfies (A.6).

Taking the uniqueness of solutions to the linear SDE (A.6) for granted, we conclude that the unique solution to this SDE is given by the expression:

$$\eta_t = \exp\Big(\int_0^t \theta_u\, dW_u - \frac{1}{2}\int_0^t \theta_u^2\, du\Big) \exp\big(N_t \ln(1+\kappa) - \kappa\lambda t\big). \tag{A.7}$$

We leave the proof of the next result as an exercise for the reader.

Proposition A.1.5. *Let the probability $\mathbb{Q}$ be given by (A.2) and (A.7) for some constant $\kappa > -1$ and a $\mathbb{G}$-predictable process θ such that $\mathbb{E}_{\mathbb{P}}(\eta_T) = 1$.*
(i) The process $\big(W^*_t = W_t - \int_0^t \theta_u\, du,\ t \in [0,T]\big)$ is a Brownian motion under $\mathbb{Q}$ with respect to the filtration $\mathbb{G}$.
(ii) The process $(N_t,\ t \in [0,T])$ is a Poisson process with the constant intensity $\lambda^* = (1+\kappa)\lambda$ under $\mathbb{Q}$ with respect to the filtration $\mathbb{G}$.
(iii) Processes W^* and N are mutually independent under $\mathbb{Q}$.

A.2 Inhomogeneous Poisson Process

Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be any non-negative, locally integrable function satisfying the equality $\int_0^\infty \lambda(u)\, du = \infty$.

Definition A.2.1. A process N (with $N_0 = 0$) is called the *(inhomogeneous) Poisson process with intensity function* λ if for every $0 \le s < t$ the increment $N_t - N_s$ is independent of the σ-field $\mathcal{G}_s$ and has the Poisson law with parameter $\Lambda(t) - \Lambda(s)$, where the *hazard function* Λ equals $\Lambda(t) = \int_0^t \lambda(u)\, du$.

More generally, let $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a right-continuous, increasing function with $\Lambda(0) = 0$ and $\Lambda(\infty) = \infty$. The Poisson process with the hazard function Λ satisfies, for every $0 \le s < t$ and every $k = 0, 1, \ldots$

$$\mathbb{P}(N_t - N_s = k \,|\, \mathcal{G}_s) = \mathbb{P}(N_t - N_s = k) = \frac{(\Lambda(t) - \Lambda(s))^k}{k!}\, e^{-(\Lambda(t)-\Lambda(s))}.$$

Example A.2.1. The most convenient, and thus widely used, method of constructing a Poisson process with a hazard function Λ runs as follows: we take a Poisson process $\widetilde{N}$ with the constant intensity $\lambda = 1$ with respect to some filtration $\widetilde{\mathbb{G}}$ and we define the time-changed process $N_t := \widetilde{N}_{\Lambda(t)}$. The process N is easily seen to follow a Poisson process with the hazard function Λ with respect to the time-changed filtration $\mathbb{G}$, where $\mathcal{G}_t = \widetilde{\mathcal{G}}_{\Lambda(t)}$ for every $t \in \mathbb{R}_+$.

Since for arbitrary $0 \le s < t$

$$\mathbb{E}_{\mathbb{P}}(N_t - N_s \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(N_t - N_s) = \Lambda(t) - \Lambda(s),$$

it is clear that the compensated Poisson process $\widehat{N}_t = N_t - \Lambda(t)$ is a $\mathbb{G}$-martingale under $\mathbb{P}$. A suitable generalization of Proposition A.1.3 shows that a Poisson process with the hazard function Λ and a Brownian motion with respect to $\mathbb{G}$ follow mutually independent processes under $\mathbb{P}$. The proof of the next lemma relies on a direct application of the Itô formula and so it is omitted.

Lemma A.2.1. *Let Z be an arbitrary bounded, $\mathbb{G}$-predictable process. Then the process M^Z, given by the formula*

$$M_t^Z = \exp\Big(\int_{]0,t]} Z_u\, dN_u - \int_0^t (e^{Z_u} - 1)\, d\Lambda(u) \Big),$$

is a $\mathbb{G}$-martingale under $\mathbb{P}$. Moreover, M^Z is the unique solution to the SDE

$$dM_t^Z = M_{t-}^Z (e^{Z_t} - 1)\, d\widehat{N}_t, \quad M_0^Z = 1.$$

In the case of an inhomogeneous Poisson process with intensity function λ, it can be easily deduced from Lemma A.2.1 that for any Borel measurable function $\kappa : \mathbb{R}_+ \to]-1, \infty[$ the process

$$\zeta_t = \exp\Big(\int_{]0,t]} \ln(1 + \kappa(u))\, dN_u - \int_0^t \kappa(u)\lambda(u)\, du \Big)$$

is the unique solution to the SDE $d\zeta_t = \zeta_{t-}\kappa(t)\, d\widehat{N}_t$ with $\eta_0 = 1$. Using similar arguments as in the case of a constant κ, one can show that the unique solution to the SDE

$$d\eta_t = \eta_{t-}\big(\theta_t\, dW_t + \kappa(t)\, d\widehat{N}_t\big), \quad \eta_0 = 1,$$

is given by the following expression

$$\eta_t = \zeta_t \exp\Big(\int_0^t \theta_u\, dW_u - \frac{1}{2}\int_0^t \theta_u^2\, du \Big). \tag{A.8}$$

The proof of the following version of the Girsanov theorem is to the reader. Note that this next result extends Proposition A.1.5.

Proposition A.2.1. *Let $\mathbb{Q}$ be a probability measure, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, such that the density process η in (A.2) is given by (A.8). Then under $\mathbb{Q}$ and with respect to $\mathbb{G}$ we have that:*
(i) the process $\big(W_t^* = W_t - \int_0^t \theta_u\, du,\ t \in [0,T]\big)$ is a Brownian motion,
(ii) the process $(N_t,\ t \in [0,T])$ is a Poisson process with the intensity function λ^* given by $\lambda^*(t) = 1 + \kappa(t)\lambda(t)$,
(iii) the processes W^* and N are mutually independent under $\mathbb{Q}$.

A.3 Conditional Poisson Process

We start by assuming that we are given a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$ and a certain sub-filtration $\mathbb{F}$ of $\mathbb{G}$. Let Λ be an $\mathbb{F}$-adapted, right-continuous, increasing process with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$. We refer to Λ as the *hazard process*. In some cases, we have $\Lambda_t = \int_0^t \lambda_u\, du$ for some $\mathbb{F}$-progressively measurable process λ with locally integrable sample paths. Then the process λ is called the $\mathbb{F}$*-intensity process.*

We are in a position to state the definition of the $\mathbb{F}$-conditional Poisson process, which is also sometimes referred to as the *doubly stochastic Poisson process*. A slightly different, but essentially equivalent, definition of a conditional Poisson process can be found in monographs by Brémaud [33] and Last and Brandt [129].

Definition A.3.1. A process N defined on a probability space $(\Omega, \mathbb{G}, \mathbb{P})$ is called the $\mathbb{F}$-*conditional Poisson process* with respect to $\mathbb{G}$, associated with the hazard process Λ, if for any $0 \le s < t$ and every $k = 0, 1, \dots$

$$\mathbb{P}(N_t - N_s = k \,|\, \mathcal{G}_s \vee \mathcal{F}_\infty) = \frac{(\Lambda_t - \Lambda_s)^k}{k!}\, e^{-(\Lambda_t - \Lambda_s)}, \tag{A.9}$$

where $\mathcal{F}_\infty = \sigma(\mathcal{F}_u : u \in \mathbb{R}_+)$.

At the intuitive level, if a particular sample path $\Lambda_{\cdot}(\omega)$ of the hazard process is known, the process N has exactly the same probabilistic properties as the Poisson process with respect to $\mathbb{G}$ with the hazard function $\Lambda_{\cdot}(\omega)$. In particular, it follows from (A.9) that

$$\mathbb{P}(N_t - N_s = k \,|\, \mathcal{G}_s \vee \mathcal{F}_\infty) = \mathbb{P}(N_t - N_s = k \,|\, \mathcal{F}_\infty),$$

i.e., conditionally on the σ-field $\mathcal{F}_\infty$ the increment $N_t - N_s$ is independent of the σ-field $\mathcal{G}_s$ for any $0 \le s < t$. Similarly, for any $0 \le s < t \le u$ and every $k = 0, 1, \dots$, we have

$$\mathbb{P}(N_t - N_s = k \,|\, \mathcal{G}_s \vee \mathcal{F}_u) = \frac{(\Lambda_t - \Lambda_s)^k}{k!}\, e^{-(\Lambda_t - \Lambda_s)}. \tag{A.10}$$

In other words, conditionally on the σ-field $\mathcal{F}_u$, the process $(N_t,\, t \in [0, u])$ behaves like a Poisson process with the hazard function $\Lambda_{\cdot}(\omega)$.

Consequently, for any $n \in \mathbb{N}$, any non-negative integers $k_1, k_2, \dots, k_n$, and arbitrary non-negative real numbers $s_1 < t_1 \le s_2 < t_2 \le \dots \le s_n < t_n$, we have that

$$\mathbb{P}\Big(\bigcap_{i=1}^n \{N_{t_i} - N_{s_i} = k_i\} \Big) = \mathbb{E}_{\mathbb{P}}\Big(\prod_{i=1}^n \frac{\big(\Lambda_{t_i} - \Lambda_{s_i}\big)^{k_i}}{k_i!}\, e^{-(\Lambda_{t_i} - \Lambda_{s_i})} \Big).$$

Let us observe that in all conditional expectations above, the reference filtration $\mathbb{F}$ can be replaced by the filtration $\mathbb{F}^\Lambda$ generated by the hazard process. In fact, the $\mathbb{F}$-conditional Poisson process with respect to $\mathbb{G}$ is also the conditional Poisson process with respect to the filtrations $\mathbb{F}^N \vee \mathbb{F}$ and $\mathbb{F}^N \vee \mathbb{F}^\Lambda$ with the same hazard process.

We shall henceforth postulate that $\mathbb{E}_{\mathbb{P}}(\Lambda_t) < \infty$ for every $t \in \mathbb{R}_+$.

Lemma A.3.1. *The compensated process* $\widehat{N}_t = N_t - \Lambda_t$ *is a martingale with respect to* $\mathbb{G}$.

Proof. It is enough to notice that, for arbitrary $0 \le s < t$,

$$\mathbb{E}_{\mathbb{P}}(\widehat{N}_t \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(N_t - \Lambda_t \,|\, \mathcal{G}_s \vee \mathcal{F}_\infty) \,|\, \mathcal{G}_s) = \mathbb{E}_{\mathbb{P}}(N_s - \Lambda_s \,|\, \mathcal{G}_s) = \widehat{N}_s,$$

where, in the second equality, we have used the property of a Poisson process with a deterministic hazard function. □

Given the two filtrations $\mathbb{F}$ and $\mathbb{G}$ and the hazard process Λ, it is not obvious whether we may find a process N satisfying Definition A.3.1. To provide a simple construction of a conditional Poisson process, we assume that the underlying probability space $(\Omega, \mathcal{G}, \mathbb{P})$, endowed with a reference filtration $\mathbb{F}$, is sufficiently large to accommodate for the following stochastic processes: a Poisson process $\widetilde{N}$ with the constant intensity equal to 1 and an $\mathbb{F}$-adapted hazard process Λ. In addition, we postulate that the Poisson process $\widetilde{N}$ is independent of the filtration $\mathbb{F}$

Remark A.3.1. Given a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, it is always possible to enlarge this space in such a way that there exists a Poisson process $\widetilde{N}$, which is defined on the enlarged space, has the constant intensity equal to 1 and is independent of the filtration $\mathbb{F}$.

Under the present assumptions, we have that, for every $0 \leq s < t$ and $u \in \mathbb{R}_+$, and any non-negative integer k,

$$\mathbb{P}(\widetilde{N}_t - \widetilde{N}_s = k \,|\, \mathcal{F}_\infty) = \mathbb{P}(\widetilde{N}_t - \widetilde{N}_s = k \,|\, \mathcal{F}_u) = \mathbb{P}(\widetilde{N}_t - \widetilde{N}_s = k)$$

and

$$\mathbb{P}(\widetilde{N}_t - \widetilde{N}_s = k \,|\, \mathcal{F}_s^{\widetilde{N}} \vee \mathcal{F}_s) = \mathbb{P}(\widetilde{N}_t - \widetilde{N}_s = k) = \frac{(t-s)^k}{k!}\, e^{-(t-s)}.$$

The next result describes an explicit construction of a conditional Poisson process. This construction is based on a random time change associated with the increasing process Λ.

Proposition A.3.1. *Let $\widetilde{N}$ be a Poisson process with the constant intensity equal to 1 such that $\widetilde{N}$ is independent of a reference filtration $\mathbb{F}$. Let Λ be an $\mathbb{F}$-adapted, right-continuous, increasing process with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$. Then the process $N_t = \widetilde{N}_{\Lambda_t}$, $t \in \mathbb{R}_+$, is the $\mathbb{F}$-conditional Poisson process with the hazard process Λ with respect to the filtration $\mathbb{G} = \mathbb{F}^N \vee \mathbb{F}$.*

Proof. Since $\mathcal{G}_s \vee \mathcal{F}_\infty = \mathcal{F}_s^N \vee \mathcal{F}_\infty$, it suffices to check that

$$\mathbb{P}(N_t - N_s = k \,|\, \mathcal{F}_s^N \vee \mathcal{F}_\infty) = \frac{(\Lambda_t - \Lambda_s)^k}{k!}\, e^{-(\Lambda_t - \Lambda_s)}$$

or, equivalently,

$$\mathbb{P}(\widetilde{N}_{\Lambda_t} - \widetilde{N}_{\Lambda_s} = k \,|\, \mathcal{F}_{\Lambda_s}^{\widetilde{N}} \vee \mathcal{F}_\infty) = \frac{(\Lambda_t - \Lambda_s)^k}{k!}\, e^{-(\Lambda_t - \Lambda_s)}.$$

The last equality follows from the assumed independence of $\widetilde{N}$ and $\mathbb{F}$. $\square$

Remark A.3.2. Within the setup of Proposition A.3.1, any $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale, so that the hypothesis (H) is satisfied.

Example A.3.1. *Cox process.* In some applications, it is natural to consider a special case of an $\mathbb{F}$-conditional Poisson process, with the filtration $\mathbb{F}$ generated by a certain stochastic process, representing the *state variables.* To be more specific, on considers a conditional Poisson process with the intensity process λ given as $\lambda_t = g(t, Y_t)$, where Y is an $\mathbb{R}^d$-valued stochastic process independent of the Poisson process $\widetilde{N}$ and $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ is some function. The reference filtration $\mathbb{F}$ is typically chosen to be the natural filtration of the process Y; that is, we set $\mathbb{F} = \mathbb{F}^Y$. In that case, the resulting $\mathbb{F}$-conditional Poisson process is referred to as the *Cox process* associated with the state-variables process Y and the intensity map g.

Our last goal is to examine the behavior of an $\mathbb{F}$-conditional Poisson process N under an equivalent change of a probability measure. For the sake of simplicity, we assume that the hazard process Λ is continuous, and the reference filtration $\mathbb{F}$ is generated by a process W, which is a Brownian motion with respect to $\mathbb{G}$. For a fixed $T > 0$, we define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ by setting

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\,\Big|\, \mathcal{G}_T = \eta_T, \quad \mathbb{P}\text{-a.s.}, \tag{A.11}$$

where the Radon-Nikodým density process $(\eta_t,\, t \in [0, T])$ solves the SDE

$$d\eta_t = \eta_{t-}\big(\theta_t\, dW_t + \kappa_t\, d\widehat{N}_t\big), \quad \eta_0 = 1, \tag{A.12}$$

for some $\mathbb{G}$-predictable processes θ and κ such that $\kappa > -1$ and $\mathbb{E}_{\mathbb{P}}(\eta_T) = 1$. An application of the Itô product rule shows that the unique solution to (A.12) is equal to the product $\nu\zeta$, where ν and ζ are solutions to SDEs

$$d\nu_t = \nu_t \theta_t\, dW_t \tag{A.13}$$

and

$$d\zeta_t = \zeta_{t-}\kappa_t\, d\widehat{N}_t \tag{A.14}$$

with the initial values $\nu_0 = \zeta_0 = 1$. It is well know that the unique solution to the SDE (A.13) is given by the expression

$$\nu_t = \exp\Big(\int_0^t \theta_u\, dW_u - \frac{1}{2}\int_0^t \theta_u^2\, du\Big),$$

whereas unique solution to the SDE (A.14), which can be solved pathwise, is given by the formula

$$\zeta_t = \exp(U_t) \prod_{0<u\le t} (1 + \Delta U_u) \exp(-\Delta U_u),$$

where in turn we denote $U_t = \int_{]0,t]} \kappa_u \, d\widehat{N}_u$. Observe that the process ζ admits also the following equivalent representations

$$\zeta_t = \exp\Big(-\int_0^t \kappa_u \, d\Lambda_u \Big) \prod_{0<u\leq t} (1 + \kappa_u \Delta N_u)$$

and

$$\zeta_t = \exp\Big(\int_{]0,t]} \ln(1+\kappa_u) \, dN_u - \int_0^t \kappa_u \, d\Lambda_u \Big).$$

Proposition A.3.2. *Let the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$ be given by (A.11)–(A.12). Then the process W^* defined by, for $t \in [0,T]$,*

$$W_t^* = W_t - \int_0^t \theta_u \, du,$$

is a Brownian motion with respect to $\mathbb{G}$ under $\mathbb{Q}$ and the process N^* given by, for $t \in [0,T]$,

$$N_t^* = \widehat{N}_t - \int_0^t \kappa_u \, d\Lambda_u = N_t - \int_0^t (1+\kappa_u) \, d\Lambda_u,$$

is a $\mathbb{G}$-martingale under $\mathbb{Q}$. If, in addition, the process κ is $\mathbb{F}$-adapted then the process N is under $\mathbb{Q}$ the $\mathbb{F}$-conditional Poisson process with respect to $\mathbb{G}$ and the hazard process of N under $\mathbb{Q}$ equals

$$\Lambda_t^* = \int_0^t (1+\kappa_u) \, d\Lambda_u.$$

A.4 The Doléans Exponential

We now recall some well-known results from stochastic analysis, regarding the so-called *Doléans exponential*, which is also known as the *stochastic exponential*. For the theory of Itô stochastic integration and stochastic differential equations, the reader is referred to monographs by Elliott [78], Jeanblanc, Yor and Chesney [114], Karatzas and Shreve [117], Klebaner [123], Kuo [124], Øksendal [150], Protter [153], or Revuz and Yor [154].

A.4.1 Exponential of a Process of Finite Variation

Let us first examine a particular case of the Doléans exponential for a process of finite variation. Let A be a real-valued, càdlàg process of finite variation defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$. Consider the following linear stochastic differential equation

$$dZ_t = Z_{t-} \, dA_t,$$

with the initial condition $Z_0 = 1$.

Equivalently, the process Z satisfies, for every $t \in \mathbb{R}_+$,

$$Z_t = 1 + \int_{]0,t]} Z_{u-} \, dA_u, \tag{A.15}$$

where the integral is the pathwise Stieltjes integral.

Definition A.4.1. The unique solution $Z = \mathcal{E}(A)$ to the stochastic differential equation (A.15) is called the *Doléans exponential* of A,

The next result gives an explicit representation for $\mathcal{E}(A)$.

Proposition A.4.1. *Let A be a real-valued, càdlàg process of finite variation. Then the Doléans exponential of A is given by the following expression, for every $t \in \mathbb{R}_+$,*

$$\mathcal{E}_t(A) = e^{A_t} \prod_{0<u\leq t} (1 + \Delta A_u) e^{-\Delta A_u} = e^{A_t^c} \prod_{0<u\leq t} (1 + \Delta A_u), \tag{A.16}$$

where A^c is the path-by-path continuous part of A, that is, the continuous process of finite variation given by the formula, for every $t \in \mathbb{R}_+$,

$$A_t^c = A_t - \sum_{0<u\leq t} \Delta A_u.$$

A.4.2 Exponential of a Special Semimartingale

Let Y be a real-valued, càdlàg, *special semimartingale* defined on $(\Omega, \mathbb{F}, \mathbb{P})$.

Definition A.4.2. The *Doléans exponential* of Y, denoted as $\mathcal{E}(Y)$, is the unique solution Z to the linear stochastic differential equation

$$dZ_t = Z_{t-} \, dY_t, \tag{A.17}$$

with the initial condition $Z_0 = 1$.

Of course, formula (A.17) is merely a shorthand notation for the integral equation

$$Z_t = 1 + \int_{]0,t]} Z_{u-} \, dY_u, \tag{A.18}$$

where the integral should now be interpreted as the Itô stochastic integral.

Recall that the process of *quadratic variation* of an arbitrary semimartingale Y is defined by the formula, for every $t \in \mathbb{R}_+$,

$$[Y]_t = Y_t^2 - Y_0^2 - 2 \int_{]0,t]} Y_{u-} \, dY_u.$$

The next result furnishes an extension of Proposition A.4.1 to the case of a process Y that follows a special semimartingale.

Proposition A.4.2. *Assume that Y is a special semimartingale. Then the Doléans exponential of Y, that is, the unique solution to linear stochastic differential equation* (A.17), *is given by the formula*

$$\mathcal{E}_t(Y) = \exp\Big(Y_t - Y_0 - \frac{1}{2}\,[Y]^c_t\Big) \prod_{0<u\leq t} (1+\Delta Y_u)\exp(-\Delta Y_u),$$

where $\Delta Y_u = Y_u - Y_{u-}$ and the process $[Y]^c$ is defined as the path-by-path continuous part of the quadratic variation process $[Y]$.

Recall that any special semimartingale Y admits the unique decomposition

$$Y_t = Y_0 + M^c_t + M^d_t + A_t, \quad \forall\, t \in \mathbb{R}_+,$$

where M^c is a continuous local martingale, M^d is a purely discontinuous local martingale, and A is a predictable process of finite variation, with the initial values $M^c_0 = M^d_0 = A_0 = 0$. This decomposition is referred to as the *canonical decomposition* of a special semimartingale Y. It is well known that $[Y]^c = \langle M^c \rangle$, where M^c is the continuous martingale part of a special semimartingale Y.

Let us state the following immediate corollary to Proposition A.4.2.

Corollary A.4.1. *The Doléans exponential of Y is a strictly positive process if and only if the jumps of Y satisfy $\Delta Y_t > -1$ for every $t \in \mathbb{R}_+$.*

The following result summarizes the properties of the Doléans exponential that prove useful in the context of the Girsanov theorem.

Proposition A.4.3. *Assume that Y is a local martingale such that the jumps of Y satisfy the inequality $\Delta Y_t > -1$ for every $t \in \mathbb{R}_+$.*
(i) The Doléans exponential $\mathcal{E}(Y)$ is a strictly positive local martingale and thus a supermartingale. Hence the process $\mathcal{E}(Y)$ is a martingale whenever $\mathbb{E}_{\mathbb{P}}(\mathcal{E}_t(Y)) = 1$ for every $t \in \mathbb{R}_+$.
(ii) The Doléans exponential $\mathcal{E}(Y)$ is a uniformly integrable martingale whenever $\mathbb{E}_{\mathbb{P}}(\mathcal{E}_\infty(Y)) = 1$, where

$$\mathcal{E}_\infty(Y) := \lim_{t\to\infty} \mathcal{E}_t(Y)$$

and the limit in the right-hand side of the last formula is known to exist.

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