Credit Risk, II.

Summer School in Financial Mathematics

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II. Cox Processes and Extensions

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2. Properties
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Default Time with a given Intensity

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F}$.

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We define the random time τ as the first time when the process $\Lambda_t = \int_0^t \lambda_s \, ds$ is above the random level Θ, i.e.,

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$$\tau = \inf\{t \geq 0 \, : \, \Lambda_t \geq \Theta\}.$$

In particular, $\{\tau > s\} = \{\Lambda_s < \Theta\}$.

Properties

Conditional Expectations

The conditional distribution function of τ *given the* σ*-field* $\mathcal{F}_t$ *is for* $t \geq s$

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \exp\left(-\Lambda_s\right).$$

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$$\mathbb{P}(\tau > s | \mathcal{F}_t) = \exp\left(-\Lambda_s\right).$$

Proof : For $s \leq t$,

$$\begin{aligned} \mathbb{P}(\tau > s | \mathcal{F}_t) &= \mathbb{P}(\Lambda_s < \Theta | \mathcal{F}_t) \\ &= \Psi(\Lambda_s) \end{aligned}$$

where $\Psi(x) = \mathbb{P}(x < \Theta)$.

Choice of filtration

We write as before $H_t = 1\!\!1_{\{\tau \le t\}}$ and $\mathcal{H}_t = \sigma(H_s : s \le t)$. We introduce the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, that is, the enlarged filtration generated by the underlying filtration $\mathbb{F}$ and the process H. (We denote by $\mathbb{F}$ the original **F**iltration and by $\mathbb{G}$ the enlar**G**ed one.)

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If $G_t \in \mathcal{G}_t$, then $G_t \cap \{\tau > t\} = B_t \cap \{\tau > t\}$ for some event $B_t \in \mathcal{F}_t$.

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If $G_t \in \mathcal{G}_t$, then $G_t \cap \{\tau > t\} = B_t \cap \{\tau > t\}$ for some event $B_t \in \mathcal{F}_t$.

Therefore any $\mathcal{G}_t$-measurable random variable Y_t satisfies $\mathbb{1}_{\{\tau > t\}} Y_t = \mathbb{1}_{\{\tau > t\}} y_t$, **where y_t is an $\mathcal{F}_t$-measurable random variable**.

Key lemma

Let Y be an integrable r.v. Then,

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}(Y|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}(Y\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}}e^{\Lambda_t}\mathbb{E}(Y\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t).$$

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If $X \in \mathcal{F}_T$

$$\boxed{\mathbb{E}(X\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, e^{\Lambda_t}\mathbb{E}(Xe^{-\Lambda_T}|\mathcal{F}_t).}$$

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The process λ is called **the intensity** of τ.

In particular, one an check that

(i) The process $L_t = \mathbb{1}_{t<\tau} e^{\Lambda_t} = (1 - H_t) e^{\Lambda_t}$ is a martingale

(ii) Let X be an $\mathcal{F}_\infty$-measurable r.v.. Then

$$\mathbb{E}(X|\mathcal{G}_t) = \mathbb{E}(X|\mathcal{F}_t)\,.$$

PROOF: (ii) Let X be an $\mathcal{F}_\infty$-measurable r.v. It suffices to check that

$$\mathbb{E}(B_t h(\tau \wedge t) X) = \mathbb{E}(B_t h(\tau \wedge t)\mathbb{E}(X|\mathcal{F}_t))$$

for any $B_t \in \mathcal{F}_t$ and any $h = \mathbb{1}_{[0,a]}$.
For $t \leq a$, the equality is obvious, since $h(\tau \wedge t) = 1$.
For $t > a$, we have $h(\tau \wedge t) = \mathbb{1}_{\{\tau \leq a\}}$ and

$$\begin{aligned}
\mathbb{E}(B_t \mathbb{1}_{\{\tau\leq a\}}\mathbb{E}(X|\mathcal{F}_t)) &= \mathbb{E}(B_t \mathbb{E}(X|\mathcal{F}_t)\mathbb{E}(\mathbb{1}_{\{\tau\leq a\}}|\mathcal{F}_\infty)) \\
&= \mathbb{E}(\mathbb{E}(B_t X|\mathcal{F}_t)\mathbb{E}(\mathbb{1}_{\{\tau\leq a\}}|\mathcal{F}_t)) \\
&= \mathbb{E}(X B_t \mathbb{E}(\mathbb{1}_{\{\tau\leq a\}}|\mathcal{F}_t)) = \mathbb{E}(B_t X \mathbb{1}_{\{\tau\leq a\}})
\end{aligned}$$

as expected. $\triangle$

We now compute the expectation of the value at time τ of a predictable process.

(i) If h is an $\mathbb{F}$-predictable (bounded) process then

$$\mathbb{E}(h_\tau|\mathcal{G}_t) = e^{\Lambda_t}\mathbb{E}\Big(\int_t^\infty h_u dF_u \,\Big|\, \mathcal{F}_t\Big)\mathbb{1}_{\{\tau>t\}} + h_\tau \mathbb{1}_{\{\tau\le t\}}$$

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In particular

$$\mathbb{E}(h_\tau) = \mathbb{E}\Big(\int_0^\infty h_u \lambda_u \exp\big(-\Lambda_u\big)\, du\Big)$$

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(iii) The martingale $L_t = 1\!\!1_{t<\tau} e^{\Lambda_t}$ satisfies $dL_t = -L_{t-} dM_t$.

Defaultable Assets

Let $B(t,T)$ be the price at time t of a default-free bond paying 1 at maturity T satisfies

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The market price $D(t,T)$ of a defaultable zero-coupon bond with maturity T is

$$\begin{aligned} D(t,T) &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{T<\tau\}} \exp\big(-\int_t^T r_s\, ds\big) \,\Big|\, \mathcal{G}_t \Big) \\ &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\Big(\exp\big(-\int_t^T [r_s + \lambda_s^{\mathbb{Q}}]\, ds\big) \,\Big|\, \mathcal{F}_t \Big). \end{aligned}$$

Promised payoff:

Let $X \in \mathcal{F}_T$

$$\mathbb{E}_{\mathbb{Q}}\left(X \mathbb{1}_{T<\tau} \exp - \int_t^T r_s ds | \mathcal{G}_t \right) = \mathbb{1}_{t<\tau} \mathbb{E}_{\mathbb{Q}}\left(X \exp - \int_t^T (r_s + \lambda_s) ds | \mathcal{F}_t \right)$$

λ is also called the **spread**.

Recovery paid at Maturity

We consider a contract which pays Z_τ at date T, if $\tau \leq T$ where Z is an $\mathbb{F}$-adapted process and no payment in the case $\tau > T$. We also assume that the interest rate is null.

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$$S_t \quad = \quad \mathbb{E}(Z_\tau 1\!\!1_{\tau \leq T} | \mathcal{G}_t) = Z_\tau 1\!\!1_{\tau \leq t} + 1\!\!1_{t < \tau} \mathbb{E}(Z_\tau 1\!\!1_{t < \tau \leq T} | \mathcal{G}_t)$$

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where $m_t^Z = \mathbb{E}(\int_0^T Z_u e^{-\Lambda_u} \lambda_u du | \mathcal{F}_t)$

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where $m_t^Z = \mathbb{E}(\int_0^T Z_u e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t)$ is an $\mathbb{F}$, hence a $\mathbb{G}$ martingale.

We assume here that $\mathbb{F}$-martingales are continuous. From $dL_t = -L_{t-}dM_t$ and integration by parts formula we deduce that

$$dS_t = Z_t(dH_t - \lambda_t(1-H_t)dt) - S_{t-}dM_t + L_t dm_t^Z$$

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In the case Z_t is a constant Z and with a deterministic interest rate

$$S_t = 1 - \mathbb{1}_{\{\tau > t\}}(Z - 1)\Big(1 - \mathbb{E}\Big(\exp - \int_t^T \lambda_s \, ds \,\Big|\, \mathcal{F}_t\Big)\Big).$$

Recovery paid at Default

If the payment Z is done at time τ

$$S_t = \mathbb{1}_{t<\tau}\mathbb{E}(Z_\tau \mathbb{1}_{t<\tau<T}|\mathcal{G}_t) = \mathbb{1}_{t<\tau}e^{\Lambda_t}\mathbb{E}(\int_t^T Z_u dF_u|\mathcal{F}_t)$$

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$$dS_t = -Z_t\lambda_t(1-H_t)dt - S_{t-}dM_t + L_t dm_t^Z\,.$$

The process $S_t + \int_0^t Z_s(1-H_s)\lambda_s ds$ is a martingale.

Price and Hedging a defaultable call

The savings account $Y_t^0 = 1$, a risky asset with risk-neutral dynamics $dY_t = Y_t \sigma dW_t$ and a DZC of maturity T with price $D(t, T)$ are traded.

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The savings account $Y_t^0 = 1$, a risky asset with risk-neutral dynamics $dY_t = Y_t \sigma dW_t$ and a DZC of maturity T with price $D(t, T)$ are traded. The reference filtration is that of the Brownian motion W.

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$$D(t,T) = L_t \mathbb{Q}(\tau > T | \mathcal{F}_t) = L_t m_t$$

with $m_t = \mathbb{Q}(\tau > T | \mathcal{F}_t) = \mathbb{E}(e^{-\Lambda_T} | \mathcal{F}_t)$.

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The price of a defaultable call with payoff $\mathbb{1}_{T<\tau}(Y_T - K)^+$ is

$$C_t = \mathbb{E}(\mathbb{1}_{T<\tau}(Y_T - K)^+|\mathcal{G}_t) = \mathbb{1}_{t<\tau} e^{\Lambda_t} \mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+|\mathcal{F}_t)$$

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The savings account $Y^0_t = 1$, a risky asset with risk-neutral dynamics $dY_t = Y_t \sigma dW_t$ and a DZC of maturity T with price $D(t,T)$ are traded. The reference filtration is that of the Brownian motion W. Then,

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with $m^Y_t = \mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+|\mathcal{F}_t)$, hence

$$dC_t = L_t dm^Y_t - m^Y_t L_{t-} dM_t$$

In the particular case where λ is deterministic, $m_t = e^{-\Lambda_T}$ and $dm_t = 0$. Therefore, $D(t,T) = m_t L_t = L_t e^{-\Lambda_T}$ and

$$dD(t,T) = m_t dL_t = -m_t L_{t-} dM_t = -e^{-\Lambda_T} L_{t-} dM_t \,.$$

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Furthermore,

$$m_t^Y = e^{-\Lambda_T} \mathbb{E}((Y_T - K)^+ | \mathcal{F}_t) = e^{-\Lambda_T} C_t^Y$$

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$$\begin{aligned} C_t &= L_t m_t^Y = \mathbb{1}_{t<\tau} e^{\Lambda_t} e^{-\Lambda_T} C^Y(t, Y_t) \\ &= L_t e^{-\Lambda_T} C^Y(t, Y_t) = D(t,T) C^Y(t, Y_t) \end{aligned}$$

From

$$C_t = D(t,T)C^Y(t,Y_t)$$

we deduce

$$\begin{aligned} dC_t &= e^{-\Lambda_T}(L_t dC_t^Y + C_t^Y dL_t) = e^{-\Lambda_T}(L_t \Delta_t dY_t - C_t^Y L_{t-} dM_t) \\ &= e^{-\Lambda_T}(L_t \Delta_t dY_t - C_t^Y L_{t-} dM_t) \end{aligned}$$

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$$dC_t = e^{-\Lambda_T} L_t \Delta_t dY_t + C_t^Y dD(t,T) = e^{-\Lambda_T} L_t \Delta_t dY_t + \frac{C_t}{D(t,T)} dD(t,T)$$

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hence, an hedging strategy consists of holding $\frac{C_{t-}}{D(t,T)}$ DZCs.

In the general case, one obtains

$$dC_t = \frac{C_{t-}}{D(t,T)} dD(t,T) - L_t \frac{m_t^Y}{m_t} dm_t + L_t dm_t^Y$$

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