Credit Risk, IV.

Summer School in Financial Mathematics

7-20 September 2009

Ljubljana

IV. Density Approach

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1. Density hypothesis
2. Computation of conditional expectation
3. Dynamic point of view
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Density Hypothesis

Let $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ be a filtered probability space.

A strictly positive and finite random variable τ (the default time) is given.

Our goals are

- to show how the information contained in the reference filtration $\mathbb{F}$ can be used to obtain information on the law of τ,
- to investigate the links between martingales in the different filtrations that will appear.

We assume the following **density hypothesis:** **there exists a non-atomic non-negative measure η such that, for any time t, there exists an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(\omega, \theta) \to \alpha_t(\omega, \theta)$ which satisfies**

$$\mathbb{P}(\tau \in d\theta | \mathcal{F}_t) = \alpha_t(\theta)\eta(d\theta), \qquad \mathbb{P} - a.s.$$

The conditional distribution of τ is characterized by the survival probability defined by

$$S_t(\theta) := \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty \alpha_t(u)\eta(du)$$

Let

$$S_t := S_t(t) = \mathbb{P}(\tau > t | \mathcal{F}_t) = \int_t^\infty \alpha_t(u)\eta(du)$$

Observe that the set $A_t := \{S_t > 0\}$ contains a.s. the event $\{\tau > t\}$.

The family $\alpha_t(.)$ is called the **conditional density** of τ w.r.t. η given $\mathcal{F}_t$.

Note that

- $S_t(\theta) = \mathbb{E}(S_\theta | \mathcal{F}_t)$ for any $\theta \geq t$
- the law of τ is $\mathbb{P}(\tau > \theta) = \int_\theta^\infty \alpha_0(u)\eta(du)$
- for any t, $\int_0^\infty \alpha_t(u)\eta(du) = 1$
- For an integrable $\mathcal{F}_T \otimes \sigma(\tau)$ r.v. $Y_T(\tau)$, one has, for $t \leq T$:

$$\mathbb{E}(Y_T(\tau)|\mathcal{F}_t) = \mathbb{E}(\int_0^\infty Y_T(u)\alpha_T(u)\eta(du)|\mathcal{F}_t)$$

- The default time τ avoids $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = \vartheta) = 0$ for every $\mathbb{F}$-stopping time ϑ.

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By using the density, we adopt an **additive** point of view to represent the conditional probability of τ

$$S_t(\theta) = \int_\theta^\infty \alpha_t(u)\eta(du)$$

In the default framework, the "intensity" point of view is often preferred, and one uses a **multiplicative** representation as

$$S_t(\theta) = \exp(-\int_0^\theta \lambda_t(u)\eta(du))$$

where $\lambda_t(u) = -\partial_u \ln S_t(u)$ is the "forward intensity".

Computation of conditional expectations

Let $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ be the smallest right-continuous filtration such that τ is a $\mathbb{D}$-stopping time, and let $\mathbb{G} = \mathbb{F} \vee \mathbb{D}$.

Any $\mathcal{G}_t$-measurable r.v. $H_t^{\mathbb{G}}$ may be represented as

$$H_t^{\mathbb{G}} = H_t^{\mathbb{F}} 1_{\{\tau > t\}} + H_t(\tau) 1_{\{\tau \leq t\}}$$

where $H_t^{\mathbb{F}}$ is an $\mathcal{F}_t$-measurable random variable and $H_t(\tau)$ is $\mathcal{F}_t \otimes \sigma(\tau)$ measurable

$$H_t^{\mathbb{F}} = \frac{\mathbb{E}[H_t^{\mathbb{G}} 1_{\{\tau > t\}} | \mathcal{F}_t]}{S_t} \quad a.s. \text{ on } A_t; \quad = 0 \quad \text{if not}.$$

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Immersion property

In the particular case where

$$\alpha_t(\theta) = \alpha_\theta(\theta), \quad \forall \theta \leq t$$

one has

$$S_t = 1 - \int_0^t \alpha_t(\theta)\eta(d\theta) = 1 - \int_0^t \alpha_T(\theta)\eta(d\theta) = \mathbb{P}(\tau > t|\mathcal{F}_T)\ a.s.$$

for any $T \geq t$ and $\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty)$. This last equality is equivalent to the immersion property (i.e. $\mathbb{F}$ martingales are $\mathbb{G}$-martingales).

Conversely, if immersion property holds, then

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty)$$

hence, the process S is decreasing and the conditional survival functions $S_t(\theta)$ are constant in time on $[\theta, \infty)$, i.e., $S_t(\theta) = S_\theta(\theta)$ for $t > \theta$.

Dynamic point of view and density process

Regular Version of Martingales

One of the major difficulties is to prove the existence of a universal càdlàg martingale version of this family of densities.

$\mathbb{F}$-decompositions of the survival process S

• The **Doob-Meyer decomposition** of the super-martingale S is given by

$$S_t = 1 + M_t^{\mathbb{F}} - \int_0^t \alpha_u(u)\eta(du)$$

where $M^{\mathbb{F}}$ is the càdlàg square-integrable martingale defined as

$$M_t^{\mathbb{F}} = -\int_0^t \big(\alpha_t(u) - \alpha_u(u)\big)\eta(du) = \mathbb{E}[\int_0^\infty \alpha_u(u)\eta(du)|\mathcal{F}_t] - 1.$$

• Let $\zeta^{\mathbb{F}} := \inf\{t : S_{t-} = 0\}$. We define $\lambda_t^{\mathbb{F}} := \frac{\alpha_t(t)}{S_{t-}}$ for any $t \leq \zeta^{\mathbb{F}}$ and let $\lambda_t^{\mathbb{F}} = \lambda_{t\wedge\zeta^{\mathbb{F}}}^{\mathbb{F}}$ for any $t > \zeta^{\mathbb{F}}$. The **multiplicative decomposition** of S is given by

$$S_t = L_t^{\mathbb{F}} e^{-\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} \qquad \text{where} \qquad dL_t^{\mathbb{F}} = e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} dM_t^{\mathbb{F}}.$$

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$$S_t = L_t^{\mathbb{F}} e^{-\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} \qquad \text{where} \qquad dL_t^{\mathbb{F}} = e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} dM_t^{\mathbb{F}}.$$

PROOF: 1) First notice that $(\int_0^t \alpha_u(u)\eta(du), t \geq 0)$ is an $\mathbb{F}$-adapted continuous increasing process. By the martingale property of $(\alpha_t(\theta), t \geq 0)$, for any fixed t,

$$S_t = \int_t^\infty \alpha_t(u)\eta(du) = \mathbb{E}[\int_t^\infty \alpha_u(u)\eta(du)|\mathcal{F}_t], \ a.s..$$

From the properties of the density, $1 - S_t = \int_0^t \alpha_t(u)\eta(du)$ and

$$M_t^{\mathbb{F}} := -\int_0^t (\alpha_t(u) - \alpha_u(u))\eta(du) = \mathbb{E}[\int_0^\infty \alpha_u(u)\eta(du)|\mathcal{F}_t] - 1.$$

2) By definition of $L_t^{\mathbb{F}}$ and 1), we have

$$dL_t^{\mathbb{F}} = e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} dS_t + e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} \lambda_t^{\mathbb{F}} S_t \eta(dt) = e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} dM_t^{\mathbb{F}},$$

which implies the result. $\triangle$

Relationship with the $\mathbb{G}$-intensity

Definition: *Let τ be a $\mathbb{G}$-stopping time. The $\mathbb{G}$-compensator $\Lambda^{\mathbb{G}}$ of τ is the $\mathbb{G}$-predictable increasing process such that $(1_{\{\tau \leq t\}} - \Lambda_t^{\mathbb{G}}, t \geq 0)$ is a $\mathbb{G}$-martingale. The $\mathbb{G}$-compensator is stopped at τ, i.e., $\Lambda_t^{\mathbb{G}} = \Lambda_{t\wedge\tau}^{\mathbb{G}}$.*
$\Lambda^{\mathbb{G}}$ coincides, on the set $\{\tau \geq t\}$, with an $\mathbb{F}$-predictable process $\Lambda^{\mathbb{F}}$, i.e. $\Lambda_t^{\mathbb{G}} 1_{\{\tau \geq t\}} = \Lambda_t^{\mathbb{F}} 1_{\{\tau \geq t\}}$.

- the $\mathbb{G}$-compensator $\Lambda^{\mathbb{G}}$ of τ admits a density given by

$$\lambda_t^{\mathbb{G}} = 1_{\{\tau > t\}} \lambda_t^{\mathbb{F}} = 1_{\{\tau > t\}} \frac{\alpha_t(t)}{S_{t-}}.$$

In particular, τ is a totally inaccessible $\mathbb{G}$-stopping time.

- For any $t < \zeta^{\mathbb{F}}$ and $T \geq t$, we have $\alpha_t(T) = \mathbb{E}[\lambda_T^{\mathbb{G}} | \mathcal{F}_t]$.

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- the $\mathbb{G}$-compensator $\Lambda^{\mathbb{G}}$ of τ is absolutely continuous w.r.t. η with

$$\lambda^{\mathbb{G}}_t = 1_{\{\tau > t\}} \lambda^{\mathbb{F}}_t = 1_{\{\tau > t\}} \frac{\alpha_t(t)}{S_{t-}}.$$

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- For any $t < \zeta^{\mathbb{F}}$ and $T \ge t$, we have $\alpha_t(T) = \mathbb{E}[\lambda^{\mathbb{G}}_T | \mathcal{F}_t]$.

Proof: 1) The $\mathbb{G}$-martingale property of $(1_{\{\tau\leq t\}} - \int_0^t \lambda_s^{\mathbb{G}}\eta(ds), t\geq 0)$ is equivalent to the $\mathbb{G}$-martingale property of

$$(1_{\{\tau>t\}}e^{\int_0^t \lambda_s^{\mathbb{G}}\eta(ds)} = 1_{\{\tau>t\}}e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)}, t\geq 0)$$

This follows from

$$\begin{aligned}\mathbb{E}[1_{\{\tau>t\}}e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)}|\mathcal{G}_s] &= 1_{\{\tau>s\}}\frac{\mathbb{E}[1_{\{\tau>t\}}e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)}|\mathcal{F}_s]}{S_s} \\ &= 1_{\{\tau>s\}}\frac{\mathbb{E}[S_t e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)}|\mathcal{F}_s]}{S_s} = 1_{\{\tau>s\}}\frac{L_s^{\mathbb{F}}}{S_s},\end{aligned}$$

where the last equality follows from the $\mathbb{F}$-local martingale property of $L^{\mathbb{F}}$. Moreover, the continuity of the compensator $\Lambda^{\mathbb{G}}$ implies that τ is totally inaccessible.

2) By the martingale property of density, for any $T \geq t$, $\alpha_t(T) = \mathbb{E}[\alpha_T(T)|\mathcal{F}_t]$. Applying 1), we obtain

$$\alpha_t(T) = \mathbb{E}\Big[\alpha_T(T)\frac{1_{\{\tau>T\}}}{S_{T-}}|\mathcal{F}_t\Big] = \mathbb{E}[\lambda_T^{\mathbb{G}}|\mathcal{F}_t], \quad \forall t < \zeta^{\mathbb{F}},$$

hence, the value of the density can be partially deduced from the intensity. △

$\mathbb{G}$-martingale characterization

A càdlàg process $Y^{\mathbb{G}}$ is a **$\mathbb{G}$-martingale** if and only if there exist an $\mathbb{F}$-adapted càdlàg process Y and an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-optional process $Y_t(.)$ such that

$$Y_t^{\mathbb{G}} = Y_t 1_{\{\tau > t\}} + Y_t(\tau) 1_{\{\tau \le t\}}$$

and that

- $(Y_t S_t + \int_0^t Y_s(s)\alpha_s(s)\eta(ds),\ t \ge 0)$ is an $\mathbb{F}$-local martingale;
- $(Y_t(\theta)\alpha_t(\theta), t \ge \theta)$ is an $\mathbb{F}$-martingale on $[\theta, \zeta^\theta)$.

Any $\mathbb{F}$-martingale $Y^{\mathbb{F}}$ is a $\mathbb{G}$-semimartingale. Moreover, it admits the decomposition $Y_t^{\mathbb{F}} = M_t^{Y,\mathbb{G}} + A_t^{Y,\mathbb{G}}$ where $M^{Y,\mathbb{G}}$ is a $\mathbb{G}$-martingale and $A^{Y,\mathbb{G}} := A_t 1_{\{\tau > t\}} + A_t(\tau) 1_{\{\tau \leq t\}}$ is an optional process with finite variation given by

$$A_t = \int_0^t \frac{d[Y^{\mathbb{F}}, S]_s}{S_{s-}} \quad \text{and} \quad A_t(\theta) = \int_\theta^t \frac{d[Y^{\mathbb{F}}, \alpha(\theta)]_s}{\alpha_s(\theta)}.$$

Girsanov theorem

Let $Z_t^{\mathbb{G}} = z_t 1_{\{\tau > t\}} + z_t(\tau) 1_{\{\tau \le t\}}$ be a positive $\mathbb{G}$-martingale with $Z_0^{\mathbb{G}} = 1$ and let $Z_t^{\mathbb{F}} = z_t S_t + \int_0^t z_t(u) \alpha_t(u) \eta(du)$ be its $\mathbb{F}$ projection.

Let $\mathbb{Q}$ be the probability measure defined on $\mathcal{G}_t$ by $d\mathbb{Q} = Z_t^{\mathbb{G}} d\mathbb{P}$.

Then, $\alpha_t^{\mathbb{Q}}(\theta) = \alpha_t(\theta) \frac{z_t(\theta)}{Z_t^{\mathbb{F}}}, \quad \forall t \in [\theta, \zeta^\theta)$;

and:

(i) the $\mathbb{Q}$-conditional survival process is defined on $[0, \zeta^{\mathbb{F}})$ by $S_t^{\mathbb{Q}} = S_t \dfrac{z_t}{Z_t^{\mathbb{F}}}$

(ii) the $(\mathbb{F}, \mathbb{Q})$-intensity process is $\lambda_t^{\mathbb{F},\mathbb{Q}} = \lambda_t^{\mathbb{F}} \dfrac{z_t(t)}{z_{t-}}$, $\eta(dt)$- a.s.;

(iii) $L^{\mathbb{F},\mathbb{Q}}$ is the $(\mathbb{F}, \mathbb{Q})$-local martingale

$$L_t^{\mathbb{F},\mathbb{Q}} = L_t^{\mathbb{F}} \frac{z_t}{Z_t^{\mathbb{F}}} \exp \int_0^t (\lambda_s^{\mathbb{F},\mathbb{Q}} - \lambda_s^{\mathbb{F}}) \eta(ds), t \in [0, \zeta^{\mathbb{F}})$$

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PROOF: For any $t \in [0, \zeta^{\mathbb{F}})$, the $\mathbb{Q}$-conditional probability can be calculated by

$$S_t^{\mathbb{Q}} = \mathbb{Q}(\tau > t|\mathcal{F}_t) = \frac{\mathbb{E}[1_{\{\tau > t\}} Z_t^{\mathbb{G}}|\mathcal{F}_t]}{Z_t^{\mathbb{F}}} = z_t \frac{S_t}{Z_t^{\mathbb{F}}}$$

and, for any $\theta \leq t$,

$$\mathbb{Q}(\tau \leq \theta|\mathcal{F}_t) = \frac{\mathbb{E}^{\mathbb{P}}[1_{\{\tau \leq \theta\}} Z_t^{\mathbb{G}}|\mathcal{F}_t]}{Z_t^{\mathbb{F}}} = \frac{\mathbb{E}^{\mathbb{P}}[1_{\{\tau \leq \theta\}} z_t(\tau)|\mathcal{F}_t]}{Z_t^{\mathbb{F}}} = \frac{\int_0^\theta z_t(u)\alpha_t(u)\eta(du)}{Z_t^{\mathbb{F}}}.$$

The density process is then obtained by taking derivatives. Finally, we use $\lambda_t^{\mathbb{F},\mathbb{Q}} = \alpha_t^{\mathbb{Q}}(t)/S_{t-}^{\mathbb{Q}}$ and $L_t^{\mathbb{F},\mathbb{Q}} = S_t^{\mathbb{Q}} e^{\int_0^t \lambda_s^{\mathbb{F},\mathbb{Q}}\eta(ds)}$.

Given a density process, it is possible to construct a random variable τ such that $\mathbb{P}(\tau > \theta | \mathcal{F}_t) = G_t(\theta)$ as we present now:

Let $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ and τ a random variable with law $\mathbb{P}(\tau > t) = G_0(t) = \int_t^\infty \eta(du)$, independent of $\mathcal{F}_\infty$, constructed on an extended probability space, and α the given density process.

Define

$$d\mathbb{Q}|_{\mathcal{G}_t} = Q_t^{\mathbb{G}} d\mathbb{P}|_{\mathcal{G}_t}$$

with

$$Q_t^{\mathbb{G}} = 1\!\!1_{t<\tau} \frac{1}{G_t} \int_t^\infty \alpha_t(u) \eta((du) + 1\!\!1_{\tau \le t} \alpha_t(\tau) \,.$$

Then, $\mathbb{Q}$ is a probability which coincides with $\mathbb{P}$ on $\mathcal{F}_t$ and under $\mathbb{Q}$, $\alpha^{\mathbb{Q}} = \alpha$.

(Note that this result was obtained by Grorud and Pontier in a finite horizon case)

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Define

$$d\mathbb{Q}|_{\mathcal{G}_t} = Z_t^{\mathbb{G}} d\mathbb{P}|_{\mathcal{G}_t}$$

with

$$Z_t^{\mathbb{G}} = 1\!\!1_{t<\tau} \frac{1}{G_t} \int_t^\infty \alpha_t(u)\eta(du) + 1\!\!1_{\tau \le t} \alpha_t(\tau)\,.$$

Then, $\mathbb{Q}$ is a probability which coincides with $\mathbb{P}$ on $\mathcal{F}_t$ and under $\mathbb{Q}$, $\alpha^{\mathbb{Q}} = \alpha$.

(Note that this result was obtained by Grorud and Pontier in a finite horizon case)

Given a density process, it is possible to construct a random variable τ such that $\mathbb{P}(\tau > \theta | \mathcal{F}_t) = G_t(\theta)$ as we present now:

Let $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ and τ a random variable with law $\mathbb{P}(\tau > t) = G_0(t) = \int_t^\infty \eta(du)$, independent of $\mathcal{F}_\infty$, constructed on an extended probability space, and α the given density process.

Define

$$d\mathbb{Q}|_{\mathcal{G}_t} = Z_t^{\mathbb{G}} d\mathbb{P}|_{\mathcal{G}_t}$$

with

$$Z_t^{\mathbb{G}} = \mathbb{1}_{t<\tau} \frac{1}{G_t} \int_t^\infty \alpha_t(u)\eta(du) + \mathbb{1}_{\tau \le t} \alpha_t(\tau) \, .$$

Then, $\mathbb{Q}$ is a probability which coincides with $\mathbb{P}$ on $\mathcal{F}_t$ and under $\mathbb{Q}$, $\alpha^{\mathbb{Q}} = \alpha$.

(Note that this result was obtained by Grorud and Pontier in a finite horizon case)

The change of probability measure generated by the two processes

$$z_t = (L_t^{\mathbb{F}})^{-1}, \qquad z_t(\theta) = \frac{\alpha_\theta(\theta)}{\alpha_t(\theta)}$$

provides a model where the immersion property holds true, and where the intensity processes does not change.
More generally, the only changes of probability measure for which the immersion property holds with the same intensity process are generated by a process z such that $(z_t L_t^{\mathbb{F}}, t \geq 0)$ is an uniformly integrable martingale.

Assume that immersion property holds under $\mathbb{P}$.

1) Let the Radon-Nikodým density $(Z_t^{\mathbb{G}}, t \geq 0)$ be a pure jump martingale with only one jump at time τ. Then, the $(\mathbb{F}, \mathbb{P})$-martingale $(Z_t^{\mathbb{F}}, t \geq 0)$ is the constant martingale equal to 1. Under $\mathbb{Q}$, the intensity process is $\lambda_t^{\mathbb{F},\mathbb{Q}} = \lambda_t^{\mathbb{F}} \dfrac{z_t(t)}{z_t}, \eta(dt)$-a.s., and the immersion property still holds.

2) The only changes of probability measure compatible with immersion property have Radon-Nikodým densities that are the product of a pure jump positive martingale with only one jump at time τ, and a positive $\mathbb{F}$-martingale.

Modelling of density process

HJM framework and short rate models

To model the family of density processes we make references to the classical interest rate models.

We suppose in what follows that $\mathbb{F}$ is a Brownian filtration.

See D.C. Brody and L. Hugston (2001) for related approach

Suppose that for any $\theta \geq 0$, the bounded martingale $(S_t(\theta), t \geq 0)$ satisfies

$$dS_t(\theta) = Z_t(\theta) dW_t$$

where $(Z_t(\theta), t \geq 0)$ is an $\mathbb{F}$-predictable process. If the process $z_t(\theta)$ such that $Z_t(\theta) = \int_0^\theta z_t(u)\eta(du)$ is bounded by an integrable process, then

1. $d\alpha_t(\theta) = -z_t(\theta) dW_t$.
2. The martingale part in the Doob-Meyer decomposition of S is given by $M_t^{\mathbb{F}} = 1 - \int_0^t Z_s(s) dW_s$.

PROOF: 1) is obvious by definition. 2) is obtained by using previous results and integration by part, in fact,

$$W_t^{\mathbb{F}} = 1 - \int_0^t \eta(du) \int_u^t z_s(u) dW_s = 1 - \int_0^t dW_s \Big(\int_0^s z_s(u) \eta(du) \Big).$$

Observe in addition that $Z_t(0) = 0$ since $S_t(0) = 1$ for any $t \geq 0$, which implies 2)

We can also consider $(S_t(\theta), t \geq 0)$ in the classical HJM models where its dynamics is given in multiplicative form. We also deduce the dynamics of the forward rate, in both forward and backward forms. The density can then be calculated as $\alpha_t(\theta) = \lambda_t(\theta) S_t(\theta)$.

See D.C. Brody and L. Hugston (2001) for related approach

For any $t, \theta \geq 0$, let $\Psi(t,\theta) = \frac{Z_t(\theta)}{S_t(\theta)}$ and define $\psi(t,\theta)$ by $\Psi(t,\theta) = \int_0^\theta \psi(t,u)\eta(du)$. Recall the forward rate $\lambda_t(\theta)$ of τ given by $\lambda_t(\theta) = -\partial_\theta \ln S_t(\theta)$. Then

1. $S_t(\theta) = S_0(\theta)\exp\left(\int_0^t \Psi(s,\theta)dW_s - \frac{1}{2}\int_0^t |\Psi(s,\theta)|^2 ds\right)$;
2. $\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi(s,\theta)dW_s + \int_0^t \psi(s,\theta)\Psi(s,\theta)^* ds$;
3. $S_t = \exp\left(-\int_0^t \lambda_s^{\mathbb{F}}\eta(ds) + \int_0^t \Psi(s,s)dW_s - \frac{1}{2}\int_0^t |\Psi(s,s)|^2 ds\right)$;

For any $t, \theta \geq 0$, let $\Psi(t,\theta) = \frac{Z_t(\theta)}{S_t(\theta)}$ and define $\psi(t,\theta)$ by $\Psi(t,\theta) = \int_0^\theta \psi(t,u)\eta(du)$. Recall the forward rate $\lambda_t(\theta)$ of τ given by $\lambda_t(\theta) = -\partial_\theta \ln S_t(\theta)$. Then

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3. $S_t = \exp\left(-\int_0^t \lambda_s^{\mathbb{F}}\eta(ds) + \int_0^t \Psi(s,s)dW_s - \frac{1}{2}\int_0^t |\Psi(s,s)|^2 d_s\right)$;

For any $t, \theta \geq 0$, let $\Psi(t,\theta) = \frac{Z_t(\theta)}{S_t(\theta)}$ and define $\psi(t,\theta)$ by $\Psi(t,\theta) = \int_0^\theta \psi(t,u)\eta(du)$. Recall the forward rate $\lambda_t(\theta)$ of τ given by $\lambda_t(\theta) = -\partial_\theta \ln S_t(\theta)$. Then

1. $S_t(\theta) = S_0(\theta) \exp\left(\int_0^t \Psi(s,\theta)dW_s - \frac{1}{2}\int_0^t |\Psi(s,\theta)|^2 ds\right)$;
2. $\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi(s,\theta)dW_s + \int_0^t \psi(s,\theta)\Psi(s,\theta)^* ds$;
3. $S_t = \exp\left(-\int_0^t \lambda_s^{\mathbb{F}}\eta(ds) + \int_0^t \Psi(s,s)dW_s - \frac{1}{2}\int_0^t |\Psi(s,s)|^2 ds\right)$;

For any $t, \theta \geq 0$, let $\Psi(t,\theta) = \frac{Z_t(\theta)}{S_t(\theta)}$ and define $\psi(t,\theta)$ by $\Psi(t,\theta) = \int_0^\theta \psi(t,u)\eta(du)$. Recall the forward rate $\lambda_t(\theta)$ of τ given by $\lambda_t(\theta) = -\partial_\theta \ln S_t(\theta)$. Then

1. $S_t(\theta) = S_0(\theta) \exp\left(\int_0^t \Psi(s,\theta)dW_s - \frac{1}{2}\int_0^t |\Psi(s,\theta)|^2 ds\right)$;
2. $\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi(s,\theta)dW_s + \int_0^t \psi(s,\theta)\Psi(s,\theta)^* ds$;
3. $S_t = \exp\left(-\int_0^t \lambda_s^{\mathbb{F}}\eta(ds) + \int_0^t \Psi(s,s)dW_s - \frac{1}{2}\int_0^t |\Psi(s,s)|^2 ds\right)$;

Proof: The process $S_t(\theta)$ is the solution of the equation

$$\frac{dS_t(\theta)}{S_t(\theta)} = \Psi(t,\theta)dW_t, \qquad \forall\, t,\theta \geq 0.$$

Hence 1), from which we deduce immediately 2) by differentiation w.r.t. θ.

Now, note that by 1),

$$\ln S_t = -\int_0^t \lambda_0(s)\eta(ds) + \int_0^t \Psi(s,t)dW_s - \frac{1}{2}\int_0^t |\Psi(s,t)|^2 ds$$

Woreover, we have by 2) that

$$\begin{aligned}\int_0^t \lambda_s(s)\eta(ds) &= \int_0^t \lambda_0(s)\eta(ds) - \int_0^t \eta(ds)\int_0^s \psi(u,s)dW_u \\ &\quad + \int_0^t \eta(ds)\int_0^s \psi(u,s)\Psi(u,s)^* du \\ &= \int_0^t \lambda_0(s)\eta(ds) - \int_0^t (\Psi(u,t)-\Psi(u,u))dW_u \\ &\quad + \frac{1}{2}\Big(\int_0^t |\Psi(u,t)|^2 - \int_0^t |\Psi(u,u)|^2\Big)du.\end{aligned}$$

Observe in addition that by definition of the forward rate $\lambda_t(\theta)$ and then, we have $\lambda_s(s) = \lambda_s^{\mathbb{F}}$, which implies 3). Finally, 4) is a direct result from 2).

As a conditional survival probability, $S_t(\theta)$ is decreasing on θ, which is equivalent to that $\lambda_t(\theta)$ is positive. This condition is similar as for the zero coupon bond prices.

In the second approach, we can borrow short rate models for the $\mathbb{F}$-intensity $\lambda^{\mathbb{F}}$ of τ, and then obtain the dynamics of conditional probability $S_t(T)$ for $T \geq t$. The monotonicity condition of $S_t(T)$ on T is equivalent to the positivity condition on $\lambda^{\mathbb{F}}$.

Example: The non-negativity of λ is satisfied if

- for any θ, the process $\psi(\theta)\Psi(\theta)$ is non negative, or if $\psi(\theta)$ is non negative;
- for any θ, the local martingale $\zeta_t(\theta) = \lambda_0(\theta) - \int_0^t \psi_s(\theta) dW_s$ is a Doléans-Dade exponential of some martingale, i.e., is solution of

$$\zeta_t(\theta) = \lambda_0(\theta) + \int_0^t \zeta_s(\theta) b_s(\theta) dW_s \, ,$$

that is, if $-\int_0^t \psi_s(\theta) dW_s = \int_0^t b_s(\theta)\zeta_s(\theta) dW_s$. Here the initial condition is a positive constant $\lambda_0(\theta)$.

Hence, we set

$$\psi_t(\theta) = -b_t(\theta)\zeta_t(\theta) = -b_t(\theta)\lambda_0(\theta)\exp\left(\int_0^t b_s(\theta)dW_s - \frac{1}{2}\int_0^t b_s^2(\theta)ds\right)$$

where λ_0 is a positive intensity function and $b(\theta)$ is a non-positive $\mathbb{F}$-adapted process. Then, the family

$$\alpha_t(\theta) = \lambda_t(\theta)\exp\left(-\int_0^\theta \lambda_t(v)\,dv\right),$$

where

$$\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi_s(\theta)\,dW_s + \int_0^t \psi_s(\theta)\Psi_s(\theta)\,ds$$

satisfies the required assumptions.

Other examples

Example: "Cox-like" construction. Here

- λ is a non-negative $\mathbb{F}$-adapted process, $\Lambda_t = \int_0^t \lambda_s ds$
- Θ is a given r.v. independent of $\mathcal{F}_\infty$ with unit exponential law
- V is a $\mathcal{F}_\infty$ -measurable non-negative random variable
- $\tau = \inf\{t : \Lambda_t \geq \Theta V\}$.

For any θ and t,

$$G_t(\theta) = \mathbb{P}(\tau > \theta|\mathcal{F}_t) = \mathbb{P}(\Lambda_\theta < \Theta V|\mathcal{F}_t) = \mathbb{P}\left(\exp -\frac{\Lambda_\theta}{V} \geq e^{-\Theta} \middle| \mathcal{F}_t\right).$$

Let us denote $\exp(-\Lambda_t/V) = 1 - \int_0^t \psi_s ds$, with

$$\psi_s = (\lambda_s/V) \exp - \int_0^s (\lambda_u/V)\, du,$$

and define $\gamma_t(s) = \mathbb{E}\left(\psi_s | \mathcal{F}_t\right)$. Then, $\alpha_t(s) = \gamma_t(s)/\gamma_0(s)$.

Backward construction of the density

Let $\varphi(\cdot, \alpha)$ be a family of densities on $\mathbb{R}^+$, depending of some parameter and $X \in \mathcal{F}_\infty$ a random variable. Then

$$\int_0^\infty \varphi(u, X) du = 1$$

and whe can chose

$$\alpha_t(u) = \mathbb{E}(\alpha_\infty(u)|\mathcal{F}_t) = \mathbb{E}(\varphi(u, X)|\mathcal{F}_t)$$

Multidefaults

Computation of prices in case of multidefaults is now easy

- In a first step, one orders the default
- Computation before the first default are done in the reference filtration
- Between the first and the second default, one takes as new reference filtration the filtration generated by the first default and the previous reference filtration, as explained previously for the "after default" computations
- and we continue till the end

Let $\tau = \tau_1 \wedge \tau_2$ and$\sigma = \tau_1 \vee \tau_2$ and assume that

$$S_t(\theta_1, \theta_2) := \mathbb{P}(\tau > \theta_1, \sigma > \theta_2 | \mathcal{F}_t) = \int_{\theta_1}^{\infty} \int_{\theta_2}^{\infty} \alpha_t(u, v) d\eta(u, v).$$

The $\mathbb{F}$-density of τ is given by

$$\alpha_t^{\tau|\mathbb{F}}(\theta_1) = \int_{\theta_1}^{\infty} \alpha_t(\theta_1, v) \eta_2(dv), \quad a.s..$$

For any $\theta_2, t \geq 0$, the $\mathbb{G}^{(1)}$-density of σ is given by

$$\alpha_t^{\sigma|\mathbb{G}^{(1)}}(\theta_2) = 1_{\{\tau > t\}} \frac{\int_t^{\infty} \alpha_t(u, \theta_2) \eta_1(du)}{S_t^{\tau|\mathbb{F}}(t)} + 1_{\{\tau \leq t\}} \frac{\alpha_t(\tau, \theta_2)}{\alpha_t^{\tau|\mathbb{F}}(\tau)}, \quad a.s..$$