Credit Risk V.

Summer School in Financial Mathematics

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Two Defaults

- **F** is a reference filtration,
- $\tau_i, i = 1, 2$ are two default times,
- $H_t^i = \mathbb{1}_{\tau_i \le t}$ are the default processes,
- $\mathbf{H}^i$ is the natural filtration of H^i,
- $\mathbf{G}^i$ is the filtration $\mathcal{G}_t^i = \mathcal{F}_t \vee \mathcal{H}_t^i$
- **G** is the filtration

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^1 \vee \mathcal{H}_t^2$$

We assume that the interest rate is null

Two Defaults, Trivial reference filtration

We assume that **F** is trivial (as in the toy model)

We introduce the *joint survival process* $G(u,v)$: for every $u, v \in \mathbb{R}_+$,

$$G(u,v) = \mathbb{Q}(\tau_1 > u, \tau_2 > v)$$

We write

$$\partial_1 G(u,v) = \frac{\partial G}{\partial u}(u,v), \quad \partial_{12} G(u,v) = \frac{\partial^2 G}{\partial u \partial v}(u,v).$$

We assume that the joint density $f(u,v) = \partial_{12} G(u,v)$ exists. In other words, we postulate that $G(u,v)$ can be represented as follows

$$G(u,v) = \int_u^\infty \Big(\int_v^\infty f(x,y)\, dy \Big)\, dx.$$

We compute conditional expectation in the filtration $\mathbf{G} = \mathbf{H}^1 \vee \mathbf{H}^2$:
For $t < T$

$$\mathbb{P}(T < \tau_1 | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) = \mathbb{1}_{t<\tau_1} \frac{\mathbb{P}(T < \tau_1 | \mathcal{H}_t^2)}{\mathbb{P}(t < \tau_1 | \mathcal{H}_t^2)}$$

$$\begin{aligned} &= \mathbb{1}_{t<\tau_1} \left(\mathbb{1}_{t<\tau_2} \frac{\mathbb{P}(T < \tau_1, t < \tau_2)}{\mathbb{P}(t < \tau_1, t < \tau_2)} + \mathbb{1}_{\tau_2 \le t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) \\ &= \mathbb{1}_{t<\tau_1} \left(\mathbb{1}_{t<\tau_2} \frac{G(T,t)}{G(t,t)} + \mathbb{1}_{\tau_2 \le t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) \end{aligned}$$

- The computation of $\mathbb{P}(T < \tau_1 | \tau_2)$ can be done as follows:

$$\mathbb{P}(T < \tau_1 | \tau_2 = v) = \frac{\mathbb{P}(T < \tau_1, \tau_2 \in dv)}{\mathbb{P}(\tau_2 \in dv)} = \frac{\partial_2 G(T, v)}{\partial_2 G(0, v)}$$

hence, on the set $\tau_2 < T$,

$$\mathbb{P}(T < \tau_1 | \tau_2) = \frac{\partial_2 G(T, \tau_2)}{\partial_2 G(0, \tau_2)}$$

Value of credit derivatives

We introduce different credit derivatives

A **defaultable zero-coupon** related to the default times τ_i delivers 1 monetary unit if τ_i is greater that T: $D^i(t,T) = \mathbb{E}(\mathbb{1}_{\{T<\tau_i\}} | \mathcal{H}_t^1 \vee \mathcal{H}_t^2)$

We obtain

$$D^1(t,T) = \mathbb{1}_{\{\tau_1>t\}} \left(\mathbb{1}_{\{\tau_2 \le t\}} \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} + \mathbb{1}_{\{\tau_2>t\}} \frac{G(T,t)}{G(t,t)} \right)$$

A contract which pays R^1 is one default occurs before T and R_2 if the two defaults occur before T:

$$\begin{aligned} CD_t &= \mathbb{E}(R_1 1\!\!1_{\{0<\tau_{(1)}\le T\}} + R_2 1\!\!1_{\{0<\tau_{(2)}\le T\}} | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) \\ &= R_1 1\!\!1_{\{\tau_{(1)}>t\}} \left(\frac{G(t,t) - G(T,T)}{G(t,t)} \right) + R_2 1\!\!1_{\{\tau_{(2)}\le t\}} + R_1 1\!\!1_{\{\tau_{(1)}\le t\}} \\ &\quad + R_2 1\!\!1_{\{\tau_{(2)}>t\}} \left\{ I_t(0,1) \left(1 - \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} \right) + I_t(1,0) \left(1 - \frac{\partial_1 G(\tau_1,T)}{\partial_1 G(\tau_1,t)} \right) \right. \\ &\qquad \left. + I_t(0,0) \left(1 - \frac{G(t,T) + G(T,t) - G(T,T)}{G(t,t)} \right) \right\} \end{aligned}$$

where by

$$I_t(1,1) = 1\!\!1_{\{\tau_1\le t,\tau_2\le t\}} \,, \qquad I_t(0,0) = 1\!\!1_{\{\tau_1>t,\tau_2>t\}}$$

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More generally, some easy computation leads to

$$\mathbb{E}(h(\tau_1,\tau_2)|\mathcal{H}_t) = I_t(1,1)h(\tau_1,\tau_2)+I_t(1,0)\Psi_{1,0}(\tau_1)+I_t(0,1)\Psi_{0,1}(\tau_2)+I_t(0,0)\Psi_{0,0}$$

where

$$\begin{aligned}
\Psi_{1,0}(u) &= -\frac{1}{\partial_1 G(u,t)}\int_t^\infty h(u,v)\partial_1 G(u,dv) \\
\Psi_{0,1}(v) &= -\frac{1}{\partial_2 G(t,v)}\int_t^\infty h(u,v)\partial_2 G(du,v) \\
\Psi_{0,0} &= \frac{1}{G(t,t)}\int_t^\infty \int_t^\infty h(u,v)G(du,dv)
\end{aligned}$$

Intensities

The process

$$M_t^1 := H_t^1 - \int_0^{t\wedge\tau_1\wedge\tau_2} \widetilde{\lambda}_u^1\,du - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \lambda^{1|2}(u,\tau_2)\,du,$$

is a **G**-martingale where

$$\widetilde{\lambda}_t^1 = -\frac{\partial_1 G(t,t)}{G(t,t)}, \quad \lambda^{1|2}(t,s) = -\frac{f(t,s)}{\partial_2 G(t,s)}$$

On the set $t < \tau_1 \wedge \tau_2$, the **G**-intensity of τ_1 is equal to

$$\lim_{h\to 0} \frac{1}{h} \frac{\mathbb{Q}(t < \tau_1 \le t+h, \tau_2 > t)}{\mathbb{Q}(\tau_1 > t, \tau_2 > t)} = -\frac{\partial_1 G(t,t)}{G(t,t)}$$

On the set $\tau_2 \le t < \tau_1$, the **G**-intensity of τ_1 is equal to

$$\lim_{h\to 0} \frac{1}{h}\mathbb{Q}(\tau_1 \in [t,t+h] | \tau_2) = -\frac{f(t,\tau_2)}{\partial_2 G(t,\tau_2)}$$

The process

$$\widehat{M}_t^1 := H_t^1 - \int_0^{t\wedge\tau_1} \frac{-\partial_1 G(s,0)}{G(s,0)} ds$$

is a $\mathbf{H}^1$-martingale. In a general setting, it is not a $\mathbf{G}$-martingale.

The Doob-Meyer decomposition of the $\mathbf{H}^2$-supermartingale

$$G_t^{1|2} = \mathbb{P}(\tau_1 > t|\mathcal{H}_t^2) = (1 - H_t^2)\frac{G(s,t)}{G(0,t)} + H_t^2\frac{\partial_2 G(s,\tau_2)}{\partial_2 G(0,\tau_2)}$$

is

$$dG_t^{1|2} = \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right) d\widehat{M}_t^2 + \left(H_t^2\frac{\partial_{1,2} G(t,\tau_2)}{\partial_2 G(0,\tau_2)} - (1 - H_t^2)\frac{\partial_1 G(t,t)}{G(0,t)}\right) dt$$

Valuation of a Defaultable claim

Let us now examine the valuation of a simple defaultable claim which delivers $\delta(\tau_1)$ at time τ_1 if $\tau_1 \leq T$, where δ is a deterministic function.

The value S of this claim, computed in the filtration $\mathbf{G}$, i.e., taking care on the information on the second default contained in that filtration, is

$$S_t = \mathbb{1}_{t<\tau_1} \mathbb{E} \left(\delta(\tau_1) \mathbb{1}_{\tau_1 \leq T} | \mathcal{G}_t \right)$$

Let us denote by $\tau_{(1)} = \tau_1 \wedge \tau_2$ the moment of the first default. Then, $\mathbb{1}_{\{t<\tau_{(1)}\}} S_t = \mathbb{1}_{\{t<\tau_{(1)}\}} \widetilde{S}_t$, where

$$\widetilde{S}_t \quad = \quad \mathbb{1}_{t<\tau_1} \frac{1}{G(t,t)} \mathbb{E}(\delta(\tau_1) \mathbb{1}_{\tau_1 \le T})$$

$$\widetilde{S}_t = \frac{1}{G(t,t)} \left(- \int_t^T \delta(u) \partial_1 G(u,t)\, du \right)$$

where $G(t,t) = \mathbb{Q}(\tau_{(1)} > t)$.

Hence the dynamics of the *pre-default ex-dividend price* $\widetilde{S}_t$ are

$$d\widetilde{S}_t = \Big(\big(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)\big)\widetilde{S}_t - \widetilde{\lambda}_1(t)\delta(t) - \widetilde{\lambda}_2(t)S_t^{1|2} \Big)\, dt,$$

where for $i = 1, 2$ the function $\widetilde{\lambda}_i(t)$ is the (deterministic) *pre-default intensity* of τ_i and $S_t^{1|2}$ is given by the expression

$$S_t^{1|2} = \frac{1}{\partial_2 G(t,t)} \left(- \int_t^T \delta(u) f(u,t)\, du \right).$$

In the financial interpretation, $S_t^{1|2}$ is the ex-dividend price at time t of the claim on the first credit name, under the assumption that the default τ_2 occurs at time t and the first name has not yet defaulted (recall that simultaneous defaults are excluded).

Let us now consider the event $\{\tau_2 \leq t < \tau_1\}$. The ex-dividend price of the claim equals

$$S_t = \frac{1}{\partial_2 G(t, \tau_2)} \left(- \int_t^T \delta(u) f(u, \tau_2) \, du \right).$$

Consequently, on the event $\{\tau_2 \leq t < \tau_1\}$ we obtain

$$dS_t = \lambda^{1|2}(t, \tau_2) \left(S_t - \delta(t) \right) \, dt$$

Dynamic of CDSs

We consider a CDS

- with a constant spread κ
- which delivers $\delta(\tau_1)$ at time τ_1 if $\tau_1 < T$, where δ is a deterministic function.

The value of the CDS takes the form

$$V_t(\kappa) = \widetilde{V}_t(\kappa) \mathbb{1}_{t < \tau_2 \wedge \tau_1} + \widehat{V}_t(\kappa) \mathbb{1}_{\tau_1 \wedge \tau_2 \leq t < \tau_1} \, .$$

First, we restrict our attention to the case $t < \tau_2 \wedge \tau_1$.

On the set $t < \tau_2 \wedge \tau_1$, the value of the CDS is

$$\widetilde{V}_t(\kappa) = \frac{1}{G(t,t)} \left(- \int_t^T \delta(u) \partial_1 G(u,t)\, du - \kappa \int_t^T G(u,t)\, du \right) .$$

Proof The value $V(\kappa)$ of this CDS, computed in the filtration $\mathbf{H}$, i.e., taking care on the information on the second default contained in that filtration, is

$$V_t(\kappa) = \mathbb{1}_{t<\tau_1} \mathbb{E}\left(\delta(\tau_1)\mathbb{1}_{\tau_1 \le T} - \kappa((T \wedge \tau_1) - t) | \mathcal{H}_t\right)$$

Let us denote by $\tau = \tau_1 \wedge \tau_2$ the first default time. Then, $\mathbb{1}_{\{t<\tau\}} V_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\kappa)$, where

$$\begin{aligned}
\widetilde{V}_t(\kappa) &= \frac{1}{\mathbb{Q}(\tau > t)} \mathbb{E}\left(\delta(\tau_1)\mathbb{1}_{\tau_1 \le T}\mathbb{1}_{t<\tau} - \kappa((T \wedge \tau_1) - t)\mathbb{1}_{t<\tau}\right) \\
&= \frac{1}{G(t,t)} \mathbb{E}\left(\delta(\tau_1)\mathbb{1}_{\tau_1 \le T}\mathbb{1}_{t<\tau} - \kappa((T \wedge \tau_1) - t)\mathbb{1}_{t<\tau}\right) \\
&= \frac{1}{G(t,t)} \Bigg(\int_t^T \delta(u)\mathbb{Q}(\tau_1 \in du, \tau_2 > t) \\
&\quad - \kappa \int_t^T (u-t)\mathbb{Q}(\tau_1 \in du, \tau_2 > t) - (T-t)\kappa \int_T^\infty \mathbb{Q}(\tau_1 \in du, \tau_2 > t) \Bigg)
\end{aligned}$$

In other terms, using integration by parts formula

$$\widetilde{V}_t(\kappa) = \frac{1}{G(t,t)} \left(- \int_t^T \delta(u) \partial_1 G(u,t)\, du - \kappa \int_t^T G(u,t)\, du \right)$$

On the event $\{\tau_2 \leq t < \tau_1\}$, the CDS price equals

$$\begin{aligned} V_t(\kappa) &= \hat{V}_t = \mathbb{1}_{t<\tau_1}\mathbb{E}\left(\delta(\tau_1)\mathbb{1}_{\tau_1 \leq T} - \kappa((T \wedge \tau_1) - t) | \sigma(\tau_2)\right) \\ &= \frac{1}{\partial_2 G(t,\tau_2)}\left(-\int_t^T \delta(u) f(u,\tau_2)\, du - \kappa \int_t^T \partial_2 G(u,\tau_2)\, du\right) := V_t^{1|2}(\tau_2) \end{aligned}$$

where

$$V_t^{1|2}(s) = \frac{1}{\partial_2 G(t,s)}\left(-\int_t^T \delta(u) f(u,s)\, du - \kappa \int_t^T \partial_2 G(u,s)\, du\right).$$

The price of a CDS is $V_t = \widetilde{V}_t \mathbb{1}_{t<\tau_2\wedge\tau_1} + \widehat{V}_t \mathbb{1}_{\tau_2\wedge\tau_1\leq t<\tau_1}$. Differentiating the deterministic function which gives the value of the CDS, we obtain

$$d\widetilde{V}_t(\kappa) = \left(\big(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)\big)\widetilde{V}_t(\kappa) + \kappa - \widetilde{\lambda}_1(t)\delta(t) - \widetilde{\lambda}_2(t)V_t^{1|2}(t)\right) dt,$$

$$d\widehat{V}_t(\kappa) = \left(\widetilde{\lambda}_t^{1|2}(\tau_2)\left(\widehat{V}_t(\kappa) - \delta(t)\right) + \kappa\right) dt$$

The price of a CDS follows

$$\begin{aligned} dV_t &= (1 - H_t^1)(1 - H_t^2)(\kappa - \delta(t)\widetilde{\lambda}^1(t))dt + (1 - H_t^1)H_t^2(\kappa - \delta(t)\widetilde{\lambda}_t^{1|2})dt \\ &\quad -V_{t-}dM_t^1 + (1 - H_t^1)(V_t^{1|2}(t) - V_{t-})dM_t^2 \end{aligned}$$

Proof: Differentiating $V_t = \widetilde{V}_t(1 - H_t^1)(1 - H_t^2) + \hat{V}_t(1 - H_t^1)H_t^2$ one obtains

$$\begin{aligned} dV_t &= (1 - H_t^1)(1 - H_t^2)d\widetilde{V}_t + (1 - H_t^1)H_t^2 d\widehat{V}_t - V_{t-}dH_t^1 \\ &\quad +(1 - H_t^1)(V_t^{1|2}(t) - \widetilde{V}_t)dH_t^2 \end{aligned}$$

which leads to the result after light computations △

Example: Jarrow and Yu's Model

Let $\tau_i = \inf\{t : \Lambda_i(t) \geq \Theta_i\}, i = 1, 2$ where $\Lambda_i(t) = \int_0^t \lambda_i(s)ds$ and Θ_i are independent random variables with exponential law of parameter 1. Jarrow and Yu study the case where λ_1 is a constant and

$$\lambda_2(t) = \lambda_2 + (\alpha_2 - \lambda_2)1\!\!1_{\{\tau_1 \leq t\}} = \lambda_2 1\!\!1_{\{t < \tau_1\}} + \alpha_2 1\!\!1_{\{\tau_1 \leq t\}}\,.$$

Assume for simplicity that $r = 0$ and compute the value of a defaultable zero-coupon with default time τ_i, with a rebate δ_i:

$$D_{i,d}(t, T) = \mathbb{E}(1\!\!1_{\{\tau_i > T\}} + \delta_i 1\!\!1_{\{\tau_i < T\}} | \mathcal{G}_t), \text{ for } \mathcal{G}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2\,.$$

Let $G(s, t) = \mathbb{P}(\tau_1 > s, \tau_2 > t)$

Case $t \leq s$ For $t < s < \tau_1$, one has $\lambda_2(t) = \lambda_2 t$. Hence, the following equality

$$\begin{aligned}\{\tau_1 > s\} \cap \{\tau_2 > t\} &= \{\tau_1 > s\} \cap \{\Lambda_2(t) < \Theta_2\} = \{\tau_1 > s\} \cap \{\lambda_2 t < \Theta_2\} \\ &= \{\lambda_1 s < \Theta_1\} \cap \{\lambda_2 t < \Theta_2\}\end{aligned}$$

leads to

$$\text{for } \; t < s, \; P(\tau_1 > s, \tau_2 > t) = e^{-\lambda_1 s} e^{-\lambda_2 t}$$

Case $t > s$

$$\{\tau_1 > s\} \cap \{\tau_2 > t\} = \{\{t > \tau_1 > s\} \cap \{\tau_2 > t\}\} \cup \{\cap\{\tau_1 > t\} \cap \{\tau_2 > t\}\}$$

$$\begin{aligned} \{t > \tau_1 > s\} \cap \{\tau_2 > t\} &= \{t > \tau_1 > s\} \cap \{\Lambda_2(t) < \Theta_2\} \\ &= \{t > \tau_1 > s\} \cap \{\lambda_2 \tau_1 + \alpha_2 (t - \tau_1) < \Theta_2\} \end{aligned}$$

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&= \{t > \tau_1 > s\} \cap \{\lambda_2 \tau_1 + \alpha_2(t - \tau_1) < \Theta_2\}
\end{aligned}$$

The independence between Θ_1 and Θ_2 implies that the r.v. τ_1 is independent from Θ_2, hence

$$\begin{aligned}
P(t > \tau_1 > s, \tau_2 > t) &= E\left(\mathbb{1}_{\{t > \tau_1 > s\}} e^{-(\lambda_2 \tau_1 + \alpha_2(t - \tau_1))}\right) \\
&= \int du\, \mathbb{1}_{\{t > u > s\}} e^{-(\lambda_2 u + \alpha_2(t-u))} \lambda_1 e^{-\lambda_1 u} \\
&= \frac{1}{\lambda_1 + \lambda_2 - \alpha_2} \lambda_1 e^{-\alpha_2 t} \left(e^{-s(\lambda_1 + \lambda_2 - \alpha_2)} - e^{-t(\lambda_1 + \lambda_2 - \alpha_2)}\right)
\end{aligned}$$

Setting $\Delta = \lambda_1 + \lambda_2 - \alpha_2$, it follows that

$$P(\tau_1 > s, \tau_2 > t) = \frac{1}{\Delta}\lambda_1 e^{-\alpha_2 t}\left(e^{-s\Delta} - e^{-t\Delta}\right) + e^{-\lambda_1 t} e^{-\lambda_2 t}.$$

In particular, for $s = 0$,

$$P(\tau_2 > t) = \frac{1}{\Delta}\left(\lambda_1\left(e^{-\alpha_2 t} - e^{-(\lambda_1+\lambda_2)t}\right) + \Delta e^{-\lambda_1 t}\right)$$

- The computation of $D_{1,d}$ reduces to that of

$$\mathbb{P}(\tau_1 > T|\mathcal{G}_t) = \mathbb{P}(\tau_1 > T|\mathcal{F}_t \vee \mathcal{H}_t^1)$$

where $\mathcal{F}_t = \mathcal{H}_t^2$. From the key lemma,

$$\mathbb{P}(\tau_1 > T|\mathcal{F}_t \vee \mathcal{H}_t^1) = 1\!\!1_{\{t<\tau_1\}} \frac{\mathbb{P}(\tau_1 > T|\mathcal{F}_t)}{\mathbb{P}(\tau_1 > t|\mathcal{F}_t)} .$$

Therefore,

$$P_{1,d}(t,T) = \delta_1 + 1\!\!1_{\{\tau_1>t\}}(1-\delta_1)e^{-\lambda_1(T-t)} .$$

One can also use that

$$\mathbb{P}(\tau_1 > T|\mathcal{G}_t) = 1 - DZC_t^1 \quad = \quad 1\!\!1_{\{\tau_1>t\}} \left(1\!\!1_{\{\tau_2\le t\}} \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} + 1\!\!1_{\{\tau_2>t\}} \frac{G(T,t)}{G(t,t)} \right)$$

- The computation of $D_{2,d}$ follows

$$\begin{aligned} D_{2,d}(t,T) &= \delta_2 + (1-\delta_2)\mathbb{1}_{\{\tau_2>t\}}\Big(\mathbb{1}_{\{\tau_1\le t\}}e^{-\alpha_2(T-t)} \\ &\quad +\mathbb{1}_{\{\tau_1>t\}}\frac{1}{\Delta}(\lambda_1 e^{-\alpha_2(T-t)} + (\lambda_2-\alpha_2)e^{-(\lambda_1+\lambda_2)(T-t)})\Big) \end{aligned}$$

Copula-Based Approaches

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(i) $C(1,\dots,1,v_i,1,\dots,1) = v_i$ for any i and any $v_i \in [0,1]$,

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(iii) For any $a, b \in [0,1]^n$ with $a \leq b$ (i.e., $a_i \leq b_i$, $\forall i$)

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\ldots+i_n} C(u_{1,i_1},\ldots,u_{n,i_n}) \geq 0\,,$$

where $u_{j,1} = a_j$, $u_{j,2} = b_j$.

Let us give few examples of copulas:

- Product copula: $\Pi(u_1, \ldots, u_n) = \Pi_{i=1}^n u_i$,
- Gumbel copula: for $\theta \in [1, \infty)$ we set

$$C(u_1, \ldots, u_n) = \exp\left(- \left[\sum_{i=1}^n (-\ln u_i)^\theta \right]^{1/\theta} \right),$$

- Gaussian copula:

$$C(u_1, \ldots, u_n) = N_\Sigma^n \left(N^{-1}(u_1), \ldots, N^{-1}(u_n) \right),$$

where N_Σ^n is the c.d.f for the n-variate central normal distribution with the linear correlation matrix Σ, and N^{-1} is the inverse of the c.d.f. for the univariate standard normal distribution.

Sklar Theorem:

For any cumulative distribution function F on $\mathbb{R}^n$ there exists a copula function C such that

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

where F_i is the i^{th} marginal cumulative distribution function.
If, in addition, F is continuous then C is unique.

Direct Application

Let F_i be the probability distribution for τ_i. A copula function C is chosen in order to introduce a dependence structure of the random vector $(\tau_1, \tau_2, \ldots, \tau_n)$. The joint distribution of the random vector $(\tau_1, \tau_2, \ldots, \tau_n)$ is derived by

$$P\{\tau_i \leq t_i,\, i = 1, 2, \ldots, n\} = C\big(F_1(t_1), \ldots, F_n(t_n)\big).$$

Indirect Application

Assume that the cumulative distribution function of $(\xi_1, \ldots, \xi_n)$ is given by an n-dimensional copula C, and that the univariate marginal laws are uniform on $[0, 1]$.

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Assume that the cumulative distribution function of $(\xi_1, \dots, \xi_n)$ is given by an n-dimensional copula C, and that the univariate marginal laws are uniform on $[0,1]$. We postulate that $(\xi_1, \dots, \xi_n)$ are independent of $\mathbf{F}$, and we set

$$\tau_i = \inf \{ t \,:\, \Gamma^i_t \geq -\ln \xi_i \}.$$

Then, $\{\tau_i > t_i\} = \{e^{-\Gamma^i_{t_i}} > \xi_i\}$.

Then:

- The case of default times conditionally independent with respect to $\mathbf{F}$ corresponds to the choice of the product copula Π. In this case, for $t_1, \ldots, t_n \leq T$ we have

$$P\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = \Pi(Z^1_{t_1}, \ldots, Z^n_{t_n}),$$

  where we set $Z^i_t = e^{-\Gamma^i_t}$.

- In general, for $t_1, \ldots, t_n \leq T$ we obtain

$$P\{\tau_1 > t_1, \ldots, \tau_n > t_n \,|\, \mathcal{F}_T\} = C(Z^1_{t_1}, \ldots, Z^n_{t_n}),$$

  where C is the copula used in the construction of $\xi_1, \ldots, \xi_n$.

An example

This example describes the use of one-factor Gaussian copula (Bank of International Settlements (BIS) standard).

Let q_i be a decreasing function taking values in $[0,1]$ with $q_i(0) = 1$.

$$\tau_i = \inf\{t : q_i(t) < U_i\}$$

Then, $q_i(t) = P(\tau_i > t) = 1 - p_i(t)$.

Correlation specification of the thresholds U_i: Let $Y_1, \cdots, Y_n$ and Y be independent random variables and $X_i = \rho_i Y + \sqrt{1-\rho_i^2} Y_i$. The default thresholds are defined by $U_i = 1 - F_i(X_i)$ where F_i is the cumulative distribution function of X_i. Then

$$\tau_i = \inf\{t : \rho_i Y + \sqrt{1-\rho_i^2} Y_i \le F_i^{-1}(1 - q_i(t))\}.$$

Conditioned on the common factor Y,

$$p^i(t|Y) = F_i^Y \left(\frac{F_i^{-1}(p_i(t)) - \rho_i Y}{\sqrt{1-\rho_i^2}} \right)$$

where F_i^Y is the cumulative distribution function of Y_i.

Let us consider the particular case where

$$X_i = \rho_i Y + \sqrt{1-\rho_i^2} Y_i,$$

where Y, $Y_i,\ i = 1, 2, \ldots, n$, are independent standard Gaussian variables. In that case, X_i is also a standard Gaussian law and

$$p^i(t|Y) = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(p_i(t)) - \rho_i Y}{\sqrt{1-\rho_i^2}}\right)$$

and

$$P(\tau_i \leq t_i, \forall i \leq n) = \int \prod_i \mathcal{N}\left(\frac{\mathcal{N}^{-1}(F_i(t_i)) - \rho_i y}{\sqrt{1-\rho_i^2}}\right) f(y) dy\,.$$

where f is the density of Y. The one-factor Gaussian copula model was proposed in the context of CDOs (Collateralized debt obligations) by Li (2000). It is now considered as the benchmark model. However, it does not fit well the market data.

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Recent alternative: **Lévy Copulae**

Let $X, Y^{(i)}$ be independent Lévy processes with same law and such that

$$\mathbb{E}(X_1) = 0, \mathrm{Var}(X_1) = 1$$

We set $X_i = X_\rho + Y^{(i)}_{1-\rho}$.

By properties of Lévy processes, X_i has the same law as X_1 and

$$\mathrm{Cor}(X_i, X_j) = \rho$$

CDO

- A CDO consists of a set of assets (its collateral portfolio) and a set of liabilities (the issued notes).
- A CDO cash flow structure allocates interest income and principal repayment from a collateral pool of different debt instruments to a prioritized collection of securities notes, which are commonly called tranches .
- A standard prioritizing structure is a simple subordination, i.e. senior CDO notes are paid before mezzanine and lower subordinated notes are paid, with any residual cash flow to an equity piece.
- The tranches are ordered so that losses in interest or principal to the collateral are absorbed first by the lowest level tranche and then in order to the next tranche and so on. The lowest tranche is the riskiest one, because has to respond immediately to the incurred losses, and it is called the equity tranche.

Let L_t be the accumulated loss on the credit portfolio at time t. Then we have the following natural expression for L_t.

$$L_t = \sum_{j=1}^{n} M_j H_t^j = \sum_{j=1}^{n} (1 - \delta_j) N_j \mathbb{1}_{\{\tau_j \le t\}}.$$

L is a pure jump process with jumps at defaults and the jump size equal to the non-recovered part of the credit, that is, $(1 - \delta_j) N_j$.

One-Factor Model

- Let us consider a one-factor model. Let V be the latent factor, such that conditionally on V the default times are independent. We assume that recoveries $\delta_1, \delta_2, \ldots, \delta_n$ are independent of V and $\tau_1, \tau_2, \ldots, \tau_n$.
- Let us denote the counting process associated with the number of defaults by $H_t = \sum_{j=1}^{n} H_t^j$.
- We first consider the probability generating function of H_t, which we will later use in the calculation of the characteristic function for the accumulated loss. We have

$$\begin{aligned}
\psi_{H_t}(u) &= \mathbb{E}_P\left(e^{uH_t}\right) \\
&= \mathbb{E}_P\left(e^{u\sum_{j=1}^{n} H_t^j}\right) \\
&= \mathbb{E}_P\left[\mathbb{E}_P\left(e^{u\sum_{j=1}^{n} H_t^j} \middle| V\right)\right].
\end{aligned}$$

- By noting that H_t^j is a Bernoulli random variable and by denoting

$$\mathbb{P}(\tau_j \leq t \,|\, V) = p_t^{j|V} \quad \text{and} \quad \mathbb{P}(\tau_j \geq t \,|\, V) = q_t^{j|V}$$

we can write

$$\mathbb{E}_P(e^{uH_t^j}|V) = q_t^{j|V} + p_t^{j|V} e^u.$$

- Recalling that $H_t^1, \ldots, H_t^n$ are conditionally independent given V, we obtain

$$\begin{aligned}
\psi_{H_t}(u) &= \mathbb{E}_P \left\{ \mathbb{E}_P \left[\exp\left(u \sum_{j=1}^{n} H_t^j \right) \Big| V \right] \right\} \\
&= \mathbb{E}_P \left\{ \mathbb{E}_P \left[e^{uH_t^1} e^{uH_t^2} \cdots e^{uH_t^n} |V \right] \right\} \\
&= \mathbb{E}_P \left[\mathbb{E}_P \left(e^{uH_t^1} |V \right) \mathbb{E}_P \left(e^{uH_t^2} |V \right) \cdots \mathbb{E}_P \left(e^{uH_t^n} |V \right) \right].
\end{aligned}$$

- Finally, we get the following representation

$$\begin{aligned}\psi_{H_t}(u) &= \mathbb{E}_P\left[\prod_{j=1}^{n}\mathbb{E}_P\left(e^{uH_t^j}|V\right)\right]\\ &= \mathbb{E}_P\left[\prod_{j=1}^{n} q_t^{j|V} + p_t^{j|V}e^{u}\right]\\ &= \int\prod_{j=1}^{n}\left(q_t^{j|V} + p_t^{j|V}e^{u}\right)f(v)\,dv\end{aligned}$$

where $f(v)$ is the density function of the factor V.

Characteristic Function of Loss Process

- We will now use this result to compute the characteristic function of loss process L_t, for different time horizons.
- The ultimate goal is to be able to find the distribution function of the loss process L_t, which is used in pricing CDO tranches. We have

$$
\begin{aligned}
\varphi_{L_t}(u) &= \mathbb{E}_P\left(\exp(iuL_t)\right) \\
&= \mathbb{E}_P\left[\mathbb{E}_P\left(\exp(iuL_t)|V\right)\right] \\
&= \mathbb{E}_P\Big\{\mathbb{E}_P\Big[\exp\Big(iu\sum_{j=1}^{n} M_j H_t^j\Big)\Big|V\Big]\Big\} \\
&= \mathbb{E}_P\Big\{\mathbb{E}_P\Big[\exp\Big(iu\sum_{j=1}^{n}(1-\delta_j)N_j H_t^j\Big)\Big|V\Big]\Big\} \\
&= \mathbb{E}_P\left\{\mathbb{E}_P\left[\left(e^{iu(1-\delta_1)N_1 H_t^1}\dots e^{iu(1-\delta_n)N_n H_t^n}\right)\Big|V\right]\right\}.
\end{aligned}
$$

- Recall that the random variable $H_t^1, \ldots, H_t^n$ are conditionally independent given V.
- Hence we obtain

$$\begin{aligned}\varphi_{L_t}(u) &= \mathbb{E}_P\left\{\mathbb{E}_P\left[e^{iu(1-\delta_1)N_1 H_t^1}|V\right]\cdots\mathbb{E}_P\left[e^{iu(1-\delta_n)N_n H_t^n}|V\right]\right\} \\ &= \mathbb{E}_P\left[\prod_{j=1}^{n}\mathbb{E}_P\left(e^{iu(1-\delta_j)N_j H_t^j}|V\right)\right] \\ &= \mathbb{E}_P\left[\prod_{j=1}^{n}\left(q_t^{j|V} + p_t^{j|V} e^{iu(1-\delta_j)N_j}\right)\right].\end{aligned}$$

- Note that we can write

$$e^{iu(1-\delta_j)N_j} = \varphi_{1-\delta_j}(uN_j).$$

- Let $f(v)$ be the density function of V. Then

$$\begin{aligned}\varphi_{L_t}(u) &= \mathbb{E}_P\left[\prod_{j=1}^{n}\left(q_t^{j|V} + p_t^{j|V}\varphi_{1-\delta_j}(uN_j)\right)\right] \\ &= \int_{\mathbb{R}}\prod_{j=1}^{n}\left(q_t^{j|V} + p_t^{j|V}\varphi_{1-\delta_j}(uN_j)\right) f(v)\,dv.\end{aligned}$$

- The last integral can be calculated through numerical integration over the distribution of the latent factor V.
- The distribution of the accumulated loss L_t can be obtained by some Fourier inversion techniques.
- Observe that the only input to the model are the conditional default and survival probabilities.

Pricing of the Mezzanine Tranche

- We can now examine the pricing of a particular leg of a CDO. The default payments on the different tranches of a CDO are obtained as functions of the accumulated losses L_t.
- Consider two thresholds A and B on the synthetic CDO where

$$0 < A < B < \sum_{j=1}^{n} N_j = C.$$

- Let the cumulative default payments on the mezzanine tranche be denoted by M_t. We know the mezzanine tranche only bears losses between A and B. Thus M_t equals

$$M_t = (L_t - A)\mathbb{1}_{[A,B]}(L_t) + (B - A)\mathbb{1}_{[B,C]}(L_t).$$

- We can write similar expressions for equity and senior tranches.

- More importantly we can now represent the discounted payoff corresponding to default payments as follows

$$\int_0^T \beta_t \, dM_t$$

where β_t is the discount factor for maturity t

$$\beta_t = \exp\Big(- \int_0^t r(u) \, du \Big).$$

- For simplicity, we assume deterministic interest rates.
- The integration by parts formula yields

$$\int_0^T \beta_t \, dM_t = \beta_T M_T + \int_0^T r(t) \beta_t M_t \, dt.$$

- Recall that in order to price the mezzanine tranche, we need to compute the following expectation

$$\mathbb{E}_P\Big(\int_0^T \beta_t\, dM_t\Big).$$

- Using Fubini's theorem, we obtain

$$\mathbb{E}_P\Big(\int_0^T \beta_t\, dM_t\Big) = \beta_T\, \mathbb{E}_P(M_T) + \int_0^T r(t)\beta_t\, \mathbb{E}_P(M_t)\, dt.$$

- Thus the pricing problem for the mezzanine tranche has been reduced to finding the expectation $\mathbb{E}_P(M_t)$.

- Using the expression for M_t derived before, we can now write

$$\begin{aligned} \mathbb{E}_P(M_t) &= \mathbb{E}_P\left[(L_t - A)\mathbb{1}_{[A,B]}(L_t)\right] + \mathbb{E}_P\left[(B - A)\mathbb{1}_{[B,C]}(L_t)\right] \\ &= \mathbb{E}_P\left[(L_t - A)\mathbb{1}_{[A,B]}(L_t)\right] + (B - A)\,\mathbb{E}_P\left[\mathbb{1}_{[B,C]}(L_t)\right] \\ &= \int_A^B (z - A)\,dF_{L_t}(z) + (B - A)\,\mathbb{P}(B < L_t \leq C) \\ &= \int_A^B (z - A)\,dF_{L_t}(z) + (B - A)(F_{L_t}(C) - F_{L_t}(B)) \end{aligned}$$

  where F_{L_t} is the cumulative distribution function of L_t.

- It can be checked that for the computation of the value of the fee leg of a CDO, we still only need the distribution of the accumulated losses L_t.

- For details, see the papers by Laurent and Gregory (2003) and Burtschell et al. (2005).

Credit Ratings

We consider n credit names and we assume that the credit quality of each reference entity falls to the set $\mathcal{K} = \{1, 2, \dots, K\}$ of K rating categories, where, by convention, the category K corresponds to default.

Let X^i, $i = 1, 2, \dots, n$ be some stochastic processes defined on $(\Omega, \mathcal{G}, \mathbb{Q})$ and taking values in the finite state space $\mathcal{K}$, where the process X^i represents the evolution of credit ratings of the ith underlying entity. Then we define the *default time* τ_i of the ith credit name by setting

$$\tau_i = \inf \{ t \in \mathbb{R}_+ : \, X^i_t = K \}.$$

We postulate that the default state K is absorbing, so that for each credit name the default event can only occur once.

Markov Chain Credit Ratings Process

Here, X is a *birth-and-death process* with absorption at state K.

The intensity matrix Λ is tri-diagonal.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ K-1 \\ K \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & \cdots & K-1 & K \end{array} \\
\left(\begin{array}{ccccc}
\lambda(1,1) & \lambda(1,2) & \cdots & 0 & 0 \\
\lambda(2,1) & \lambda(2,2) & \cdots & 0 & \\
0 & \lambda(3,2) & \cdots & 0 & \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda(K-1,K-1) & \lambda(K-1,K) \\
0 & 0 & \cdots & 0 & 0
\end{array}\right),
\end{array}
$$

Let $p_t(k,k') = \mathbb{Q}(X_{s+t} = k' \,|\, X_s = k)$.

The transition probabilities $p_t(k,k')$ satisfy the following system for $t \in \mathbb{R}_+$ and $k' = 1, 2, \ldots, K$,

$$\begin{aligned} \frac{dp_t(1,k')}{dt} &= -\lambda(1,2)p_t(1,k') + \lambda(1,2)p_t(2,k'), \\ \frac{dp_t(k,k')}{dt} &= \lambda(k,k-1)p_t(k-1,k') - (\lambda(k,k-1) + \lambda(k,k+1))p_t(k,k') \\ &\quad + \lambda(k,k+1)p_t(k+1,k') \end{aligned}$$

for $k = 2, 3, \ldots, K-1$, whereas for $k = K$ we simply have that

$$\frac{dp_t(K,k')}{dt} = 0,$$

with the initial conditions $p_0(k,k') = \mathbb{1}_{\{k=k'\}}$. Once the transition intensities $\lambda(k,k')$ are specified, the above system can be easily solved. Note, in particular, that $p_t(K,k') = 0$ for every t if $k' \neq K$. The advantage of this representation is that the number of parameters can be kept relatively small.

A more flexible credit ratings model is obtained if we allow for jumps to the default state K from any other state. In that case, the intensity matrix is no longer tri-diagonal and the ordinary differential equations for transition probabilities take the following form, for $t \in \mathbb{R}_+$ and $k' = 1, 2, \ldots, K$,

$$\begin{aligned} \frac{dp_t(1,k')}{dt} &= -(\lambda(1,2)+\lambda(1,K))p_t(1,k')+\lambda(1,2)p_t(2,k')+\lambda(1,K)p_t(K,k') \\ \frac{dp_t(k,k')}{dt}) &= \lambda(k,k-1)p_t(k-1,k') - (\lambda(k,k-1)+\lambda(k,k+1))p_t(k,k') \\ &\quad + \lambda(k,K)p_t(k,k')+\lambda(k,k+1)p_t(k+1,k')+\lambda(k,K)p_t(K,k') \end{aligned}$$

for $k = 2, 3, \ldots, K-1$ and for $k = K$

$$\frac{dp_t(K,k')}{dt} = 0,$$

with initial conditions $p_0(k,k') = \mathbb{1}_{\{k=k'\}}$.

Survival Intensities

For arbitrary $s \le t$ on the set $\{\tau_1 > s, \dots, \tau_n > s\} = \{\tau_{(1)} > s\}$ we have

$$P\{\tau_i > t \,|\, \mathcal{G}_s\} = \mathbb{E}_P \left(\frac{C(Z_s^1, \dots, Z_t^i, \dots, Z_s^n)}{C(Z_s^1, \dots, Z_s^n)} \,\Big|\, \mathcal{F}_s \right).$$

Survival Intensities

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Proof: The proof is straightforward, and follows from the key lemma

$$P\{\tau_i > t \,|\, \mathcal{G}_s\} \mathbb{1}_{\{\tau_{(1)} > s\}} = \mathbb{1}_{\{\tau_{(1)} > s\}} \frac{P(\tau_1 > s, \ldots, \tau_i > t, \ldots, \tau_n > s \,|\, \mathcal{F}_s)}{P(\tau_1 > s, \ldots, \tau_i > s, \ldots, \tau_n > s \,|\, \mathcal{F}_s)}$$

$\triangle$

Consequently, assuming that the derivatives $\gamma_t^i = \frac{d\Gamma_t^i}{dt}$ exist, the i-th intensity of survival equals, on the set $\{\tau_1 > t, \ldots, \tau_n > t\}$,

$$\lambda_t^i = \gamma_t^i \, Z_t^i \, \frac{\frac{\partial}{\partial v_i} C(Z_t^1, \ldots, Z_t^n)}{C(Z_t^1, \ldots, Z_t^n)} = \gamma_t^i \, Z_t^i \, \frac{\partial}{\partial v_i} \ln C(Z_t^1, \ldots, Z_t^n),$$

where λ_t^i is understood as the limit:

$$\lambda_t^i = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}\{t < \tau_i \leq t + h \,|\, \mathcal{F}_t, \tau_1 > t, \ldots, \tau_n > t\}.$$

It appears that, in general, the i-th intensity of survival jumps at time t, if the j-th entity defaults at time t for some $j \neq i$. In fact, it holds that

$$\lambda_t^{i,j} = \gamma_t^i \, Z_t^i \, \frac{\frac{\partial^2}{\partial v_i \partial v_j} C(Z_t^1, \ldots, Z_t^n)}{\frac{\partial}{\partial v_j} C(Z_t^1, \ldots, Z_t^n)} \, ,$$

where

$$\lambda_t^{i,j} = \lim_{h \downarrow 0} h^{-1} \, \mathbb{Q}\{t < \tau_i \leq t + h \,|\, \mathcal{F}_t, \tau_k > t, k \neq j, \tau_j = t\}.$$

Schönbucher and Schubert (2001) also examine the intensities of survival after the default times of some entities. Let us fix s, and let $t_i \leq s$ for $i = 1, 2, \ldots, k < n$, and $T_i \geq s$ for $i = k+1, k+2, \ldots, n$. Then,

$$\mathbb{Q}\Big\{\tau_i > T_i,\ i = k+1, k+2, \ldots, n \,\big|\, \mathcal{F}_s,\ \tau_j = t_j,\ j = 1, 2, \ldots, k,$$

$$\tau_i > s,\ i = k+1, k+2, \ldots, n\Big\}$$

$$= \frac{\mathbb{E}_{\mathbb{Q}}\left(\frac{\partial^k}{\partial v_1 \ldots \partial v_k} C(Z^1_{t_1}, \ldots, Z^k_{t_k}, Z^{k+1}_{T_{k+1}}, \ldots, Z^n_{T_n}) \,\Big|\, \mathcal{F}_s \right)}{\frac{\partial^k}{\partial v_1 \ldots \partial v_k} C(Z^1_{t_1}, \ldots, Z^k_{t_k}, Z^{k+1}_s, \ldots, Z^n_s)}.$$

Brownien reference filtration

Here $\mathbf{F}$ is a Brownian filtration. We work under the hypothesis that $\mathbf{F}$ is immersed in $\mathbf{G}$, i.e., $\mathbf{F}$ martingales are $\mathbf{G}$ martingales.

We introduce the **conditional joint survival process** $G(u, v; t)$

$$G(u, v; t) = \mathbb{Q}(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t).$$

We write

$$\partial_1 G(u, v; t) = \frac{\partial}{\partial u} G(u, v; t), \quad \partial_{12} G(u, v; t) = \frac{\partial^2}{\partial u \partial v} G(u, v; t) = f(u, v; t)$$

so that

$$\boxed{G(u, v; t) = \int_u^\infty \left(\int_v^\infty f(x, y; t)\, dy \right) dx}$$

where $(f(x, y; t), t \geq 0)$ is a family of $\mathbf{F}$-predictable processes (in fact $(\mathbf{F}, \mathbb{Q})$-martingales).

For any fixed $(u, v) \in \mathbb{R}^2_+$, the $\mathbf{F}$-martingale $G(u, v; t) = \mathbb{Q}(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t)$ admits the integral representation

$$G(u, v; t) = \mathbb{Q}(\tau_1 > u, \tau_2 > v) + \int_0^t g(u, v; s)\, dW_s$$

Valuation of Single-Name CDSs

Let us now examine the valuation of single-name CDSs.

We consider the CDS

- with the constant spread κ,
- which delivers $\delta(\tau_1)$ at time τ_1 if $\tau_1 \leq T$, where δ is a

deterministic function.

The value $S^1(\kappa)$ of this CDS, computed in the filtration $\mathbf{G}$, i.e., taking care on the information on the second default contained in that filtration,

$$\mathbb{E}(\delta(\tau_1)\mathbb{1}_{\tau_1 \leq T} - \kappa((T \wedge \tau_1) - t)|\mathcal{G}_t)$$

is computed in two successive steps.

On the set $t < \tau_{(1)}$, the ex-dividend price of the CDS is $S_t^1(\kappa) = \widetilde{S}_t^1(\kappa)$ where $\widetilde{S}^1(\kappa)$ is an $\mathbf{F}$-adapted process defined as

$$\begin{aligned}\widetilde{S}_t^1(\kappa) &= \frac{1}{\mathbb{P}(\tau_1 > t, \tau_2 > t|\mathcal{F}_t)}\mathbb{E}(\delta(\tau_1)\mathbb{1}_{\tau_1 \leq T} - \kappa((T \wedge \tau_1) - t)|\mathcal{F}_t) \\ &= \frac{1}{G(t,t;t)}\left(-\int_t^T \delta(u)\partial_1 G(u,t;t)\,du - \kappa\int_t^T G(u,t;t)\,du\right).\end{aligned}$$

On the event $\{\tau_2 \leq t < \tau_1\}$, we have that

$$
\begin{aligned}
S_t^1(\kappa) &= \frac{1}{\mathbb{P}(\tau_1 > t | \mathcal{F}_t \vee \sigma(\tau_2))} \mathbb{E}(\delta(\tau_1) \mathbb{1}_{\tau_1 \leq T} - \kappa((T \wedge \tau_1) - t) | \mathcal{F}_t \vee \sigma(\tau_2)) \\
&= \frac{1}{\partial_2 G(t, \tau_2; t)} \left(- \int_t^T \delta(u) f(u, \tau_2; t)\, du - \kappa \int_t^T \partial_2 G(u, \tau_2; t)\, du \right).
\end{aligned}
$$

Price Dynamics of Single-Name CDSs

By applying the Itô-Wentzell theorem, we get

$$
\begin{aligned}
G(u,t;t) &= G(u,0;0) + \int_0^t g(u,s;s)\,dW_s + \int_0^t \partial_2 G(u,s;s)\,ds \\
G(t,t;t) &= G(0,0;0) + \int_0^t g(s,s;s)\,dW_s + \int_0^t (\partial_1 G(s,s;s) + \partial_2 G(s,s;s))\,ds \\
&= G(0,0;0) + \int_0^t (\partial_1 G(s,s;s) + \partial_2 G(s,s;s))\,ds
\end{aligned}
$$

where the last equality is a consequence of the immersion hypothesis.

The process

$$M_t^1 = H_t^1 - \int_0^{t\wedge\tau_{(1)}} \widetilde{\lambda}_u^1 \, du - \int_{t\wedge\tau_{(1)}}^{t\wedge\tau_1} \lambda^{1|2}(u,\tau_2) du,$$

is a **G**-martingale. Here

$$\widetilde{\lambda}_t^i = -\frac{\partial_i G(t,t;t)}{G(t,t;t)}, i = 1,2 \quad \lambda^{1|2}(t,s) = -\frac{f(t,s;t)}{\partial_2 G(t,s;t)}$$

Note that $\widetilde{\lambda} = \widetilde{\lambda}^1 + \widetilde{\lambda}^2$ is the intensity of $\tau_{(1)} = \tau_1 \wedge \tau_2$: the process

$$\mathbb{1}_{\tau_{(1)} \le t} - \int_0^{t\wedge\tau_{(1)}} \widetilde{\lambda}_u du$$

is a **G**-martingale.

Define

$$S_t^{1|2}(\kappa_1) \quad = \quad \frac{1}{\partial_2 G(t,t;t)} \left(- \int_t^T \delta(u) f(u,t;t) \, du + \kappa \int_t^{T_1} \partial_2 G(u,t;t) \, du \right)$$

the time-t value of the CDS if default τ_2 occurs at time t.

The dynamics of the process $\widetilde{S}^1(\kappa)$ are

$$d\widetilde{S}^1_t(\kappa) = \Big(- \widetilde{\lambda}^1_t \delta(t) + \kappa + \widetilde{\lambda}_t \widetilde{S}^1_t(\kappa) - \widetilde{\lambda}^2_t S^{1|2}_t(\kappa) \Big) dt + \sigma^1(t,T)\, dW_t$$

where

$$\sigma^1(t,T) \quad = \quad -\frac{1}{G(t,t;t)} \int_t^T \big(\delta(u)\, \partial_1 g(u,t;t) + \kappa g(u,t;t)\big)\, du$$

The cumulative price

$$S^{\mathrm{cum},1}(\kappa) = S^1_t(\kappa) + B_t \int_{]0,t]} B_u^{-1}\, dD_u$$

where

$$D_t = D_t(\kappa, \delta, T, \tau_1) = \delta(\tau_1)1\!\!1_{\{\tau_1 \le t\}} - \kappa(t \wedge (T \wedge \tau_1))$$

satisfies, on $[0, T \wedge \tau_{(1)}]$,

$$dS_t^{\mathrm{cum},1}(\kappa) \quad = \quad (\delta(t) - \widetilde{S}^1_t(\kappa))\, dM^1_t + (S_t^{1|2}(\kappa) - \widetilde{S}^1_t(\kappa))\, dM^2_t + \sigma^1(t,T) dW_t\, .$$

On $\tau_1 > t > \tau_2$

$$dS_t^1 = \sigma_{1|2}(t,T)dW_t + (\kappa - \delta(t)\lambda_t^{1|2}(\tau_2) + S_t^1\lambda_t^{1|2}(\tau_2))dt$$

where

$$\begin{aligned}\sigma_{1|2}(t,T) &= -\int_t^T \delta_1(u)\partial_1\partial_2 g(u,\tau_2;t)du - \kappa_1\int_t^T \partial_2 g(u,\tau_2;t)du \\ \lambda^{1|2}(t,s) &= -\frac{f(t,s;t)}{\partial_2 G(t,s;t)}\end{aligned}$$

Replication of a First-to-Default Claim

A **first-to-default claim** with maturity T is a claim $(X, A, Z, \tau_{(1)})$ where

- X is an $\mathcal{F}_T$-measurable amount payable at maturity if no default occurs
- $A : [0, T] \rightarrow \mathbb{R}$ with $A_0 = 0$ represents the dividend stream up to $\tau_{(1)}$,
- $Z = (Z^1, Z^2, \ldots, Z^n)$ is the vector of $\mathbf{F}$-predictable, real-valued processes, where $Z^i_{\tau_{(1)}}$ specifies the recovery received at time $\tau_{(1)}$ if the ith name is the first defaulted name, that is, on the event $\{\tau_i = \tau_{(1)} \leq T\}$.
- We denote by $G_{(1)}(t; t) = G(t, \cdots, t; t)$

The cumulative price S^{cum} of the first to default claim is given by

$$dS_t^{cum} = \sum_{i=1}^{n} (Z_t^i - S_{t-})\, dM_t^i + (1 - H_t^{(1)})(G_{(1)}(t;t))^{-1}\, dm_t,$$

where the $\mathbf{F}$-martingale m is given by the formula $m_t =$

$$\mathbb{E}_{\mathbb{Q}}\left(G_{(1)}(T;T)X + \sum_{i=1}^{n} \int_0^T G_{(1)}(u;u) Z_u^i \widetilde{\lambda}_u^i\, du - \int_0^T G_{(1)}(u;u)\, dA_u \,\Big|\, \mathcal{F}_t \right).$$

Since **F** is generated by a Brownian motion, there exists an **F**-predictable process ζ such that

$$dS_t^{cum} = \sum_{i=1}^{n} (Z_t^i - S_{t-})\, dM_t^i + (1 - H_t^{(1)})(G_{(1)}(t;t))^{-1} \zeta_t \, dW_t.$$

We say that a self-financing strategy $\phi = (\phi^0, \phi^1, \dots, \phi^n)$ **replicates** a first-to-default claim $(X, A, Z, \tau_{(1)})$ if its wealth process $V(\phi)$ satisfies the equality $V_{t\wedge\tau_{(1)}}(\phi) = S_{t\wedge\tau_{(1)}}$ for any $t \in [0, T]$.

We have, for any $t \in [0, T]$,

$$dV_t(\phi) = \sum_{\ell=1}^{n} \phi_t^i \Big(\big(\delta_t^\ell - \widetilde{S}_t^\ell(\kappa_\ell)\big)\, dM_t^\ell + \sum_{j=1\,,j\neq\ell}^{n} \big(S_{t|j}^\ell - \widetilde{S}_t^\ell(\kappa_\ell)\big)\, dM_t^j$$
$$+ (1 - H_t)(G_{(1)}(t;t))^{-1}\, dn_t^\ell \Big)$$

where

$$n_t^\ell = \mathbb{E}_{\mathbb{Q}} \left(\int_0^{T_\ell} G(u,u;u) \Big(\delta_u^\ell \widetilde{\lambda}_u^i + \sum_{j=1, j\neq\ell}^{n} S_{u|j}^\ell \widetilde{\lambda}_u^j \Big)\, du - \kappa_\ell \int_0^{T_\ell} G_{(1)}(u;u)\, du \,\Big|\, \mathcal{F}_t \right).$$

Let $\widetilde{\phi}_t = (\widetilde{\phi}^1_t, \widetilde{\phi}^2_t, \dots, \widetilde{\phi}^n_t)$ be a solution to the following equations

$$\widetilde{\phi}^\ell_t \big(\delta^\ell_t - \widetilde{S}^\ell_t(\kappa_\ell)\big) + \sum_{j=1,\, j\neq\ell}^{n} \widetilde{\phi}^j_t \big(S^j_{t|\ell}(\kappa_j) - \widetilde{S}^j_t(\kappa_j)\big) = Z^\ell_t - \widetilde{S}_t$$

and $\sum_{\ell=1}^k \widetilde{\phi}^\ell_t \zeta^\ell_t = \zeta_t$.

Let us set $\phi^\ell_t = \widetilde{\phi}^\ell(\tau_{(1)} \wedge t)$ for $\ell = 1, 2, \dots, n$ and $t \in [0, T]$.

Then the self-financing trading strategy $\phi = ((V(\phi) - \phi \cdot S), \dots, \phi^k)$ replicates the first-to-default claim $(X, A, Z, \tau_{(1)})$.

Replication with Market CDSs

When considering trading strategies involving CDSs issued in the past, one encounters a practical difficulty regarding their liquidity.

Recall that for each maturity T_i by the CDS issued at time t we mean the CDS over $[t, T]$ with the spread $\kappa(t, T_i) = \kappa_i$.

We now define a **market CDS** — which at any time t has similar features as the T_i-maturity CDS issued at this date t, in particular, it has the ex-dividend price equal to zero.

A **T_i-maturity market CDS** has the dividend process equal to

$$^*D_t^i = \int_{]0,t]} B_u \, d(B_u^{-1} S_u^i(\kappa_i)) + D_t^i,$$

where $D^i = D(\kappa_i, \delta^i, T_i, \tau)$ for some fixed spread κ_i.

The ex-dividend price $^*S^i$ of the T_i-maturity market CDS equals zero for any $t \in [0, T_i]$.

Since market CDSs are traded on the ex-dividend basis, to describe the self-financing trading strategies in the savings account B and the market CDSs with ex-dividend prices ${}^*S^i$.

A strategy $\phi = (\phi^0, \dots, \phi^n)$ in the savings account B and the market CDSs with dividends ${}^*D^i$ is said to be *self-financing* if its wealth $V_t(\phi) = \phi^0_t B_t$ satisfies $V_t(\phi) = V_0(\phi) + G_t(\phi)$ for every $t \in [0, T]$, where the gains process $G(\phi)$ is defined as follows

$$G_t(\phi) = \int_{]0,t]} \phi^0_u \, dB_u + \sum_{i=1}^{n} \int_{]0,t]} \phi^i_u \, d\,{}^*D^i_u.$$

Let ϕ be a self-financing strategy in the savings account B and ex-dividend prices $S^i(\kappa_i)$, $i = 1, \ldots, n$.

Then the strategy $\psi = (\psi^0, \ldots, \psi^n)$ where $\psi^i = \phi^i$ for $i = 1, \ldots, n$ and $\psi_t^0 = B_t^{-1} V_t(\phi)$ is a self-financing strategy in the savings account B and the market CDSs with dividends ${}^*D^i$ and its wealth process satisfies $V(\psi) = V(\phi)$.

The cumulative price of the T_i-maturity market CDS satisfies

$$
\begin{aligned}
{}^*S_t^{c,i} &= {}^*S_t^i + B_t \int_{]0,t]} B_u^{-1}\, d\, {}^*D_u^i \\
&= \mathbb{1}_{\{t<\tau\}} (\kappa_t^i - \kappa_i) \widetilde{A}(t,T) - B_t S_0^i(\kappa_i) + B_t \int_{]0,t]} B_u^{-1}\, dD_u^i
\end{aligned}
$$

where

$$
\widetilde{A}(t,T) = \frac{B_t}{G_t}\, \mathbb{E}_{Q^*} \left(\int_t^{T\wedge\tau} B_u^{-1} du \,\Big|\, \mathcal{F}_t \right).
$$

If we choose $\kappa_i = \kappa_0^i$ then

$$ {}^*S_t^{c,i} = \mathbb{1}_{\{t<\tau\}}(\kappa_t^i - \kappa_0^i)\widetilde{A}(t,T) + B_t \int_{]0,t]} B_u^{-1}\, dD_u^i = S_t^{c,i}(\kappa_0^i). $$

Assume that there exist $\mathbf{F}$-predictable processes $\phi^1, \ldots, \phi^n$ satisfying the following conditions, for any $t \in [0, T]$,

$$\sum_{i=1}^{k} \phi_t^i \big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big) = Z_t - \widetilde{S}_t, \quad \sum_{i=1}^{k} \phi_t^i \zeta_t^i = \xi_t.$$

Let the process $V(\phi)$ be given by

$$dV_t(\phi) = \sum_{i=1}^{k} \phi_t^i \Big(\big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big)\, dM_t + (1 - H_t) B_t G_t^{-1} dn_t^i \Big)$$

with the initial condition $V_0(\phi) = Y_0$ and let ϕ^0 be given by, for $t \in [0, T]$,

$$\phi_t^0 = B_t^{-1} V_t(\phi).$$

Then the self-financing trading strategy $\phi = (\phi^0, \ldots, \phi^n)$ in the savings account B and market CDSs with dividends ${}^*D^i$, $i = 1, \ldots, n$ replicates the defaultable claim (X, A, Z, τ).