Credit risk

T. Bielecki, M. Jeanblanc and M. Rutkowski

Complements and Exercises

Contents

Chapter 1

Two defaults

1.1 Two defaults, trivial reference filtration

We assume in this section that $r = 0$.

Let us first study the case with two random times τ_1, τ_2. We denote by $\tau_{(1)} = \inf(\tau_1, \tau_2)$ and $\tau_{(2)} = \sup(\tau_1, \tau_2)$, and we assume, for simplicity, that $\mathbb{P}(\tau_1 = \tau_2) = 0$. We denote by $(H^i_t, t \geq 0)$ the default process associated with $\tau_i, (i = 1, 2)$, and by $H_t = H^1_t + H^2_t$ the process associated with two defaults. As before, $\mathbf{H}^i$ is the filtration generated by the process H^i and $\mathbf{H}$ is the filtration generated by the process H. The σ-algebra $\mathcal{G}_t = \mathcal{H}^1_t \vee \mathcal{H}^2_t$ is equal to $\sigma(\tau_1 \wedge t) \vee \sigma(\tau_2 \wedge t)$. It is useful to note that $\mathcal{G}_t$ is strictly greater than $\mathcal{H}_t$. Exemple: assume that τ_1 and τ_2 are independent and identically distributed. Then, obviously, for $u < t$

$$P(\tau_1 < \tau_2 | \tau_{(1)} = u, \tau_{(2)} = t) = 1/2\,,$$

hence $\sigma(\tau_1, \tau_2) \neq \sigma(\tau_{(1)}, \tau_{(2)})$.

1.1.1 Computation of joint laws

A $\mathcal{H}^1_t \vee \mathcal{H}^2_t$-measurable random variable is equal to

- a constant on the set $t < \tau_{(1)}$,

- a $\sigma(\tau_{(1)})$-measurable random variable on the set $\tau_{(1)} \leq t < \tau_{(2)}$, i.e., a $\sigma(\tau_1)$-measurable random variable on the set $\tau_1 \leq t < \tau_2$, and a $\sigma(\tau_2)$-measurable random variable on the set $\tau_2 \leq t < \tau_1$

- a $\sigma(\tau_1, \tau_2)$-measurable random variable on the set $\tau_2 \leq t$.

We note G the survival probability of the pair (τ_1, τ_2), i.e.,

$$G(t, s) = \mathbb{P}(\tau_1 > t, \tau_2 > s)\,.$$

We shall also use the notation

$$g(s) = \frac{d}{ds} G(s, s) = \partial_1 G(s, s) + \partial_2 G(s, s)$$

where $\partial_1 G$ is the partial derivative of G with respect to the first variable.

• We present in a first step some computations of conditional laws.

$$\begin{aligned}
\mathbb{P}(\tau_{(1)} > s) &= \mathbb{P}(\tau_1 > s, \tau_2 > s) = G(s, s) \\
\mathbb{P}(\tau_{(2)} > t | \tau_{(1)} = s) &= \frac{1}{g(s)} \left(\partial_1 G(s, t) + \partial_2 G(t, s) \right), \text{ for } t > s
\end{aligned}$$

• We also compute conditional expectation in the filtration $\mathbf{G} = \mathbf{H}^1 \vee \mathbf{H}^2$: For $t < T$

$$\begin{aligned}
\mathbb{P}(T < \tau_{(1)} | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) &= \mathbb{1}_{t<\tau_{(1)}} \frac{\mathbb{P}(T < \tau_{(1)})}{\mathbb{P}(t < \tau_{(1)})} = \mathbb{1}_{t<\tau_{(1)}} \frac{G(T,T)}{G(t,t)} \\
\mathbb{P}(T < \tau_1 | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) &= \mathbb{1}_{t<\tau_1} \frac{\mathbb{P}(T < \tau_1 | \mathcal{H}_t^2)}{\mathbb{P}(t < \tau_1 | \mathcal{H}_t^2)} + \mathbb{1}_{\tau_1 < t} \\
&= \mathbb{1}_{t<\tau_1} \left(\mathbb{1}_{t<\tau_2} \frac{\mathbb{P}(T < \tau_1, t < \tau_2)}{\mathbb{P}(t < \tau_1, t < \tau_2)} + \mathbb{1}_{\tau_2 \le t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) + \mathbb{1}_{\tau_1 < t} \\
&= \mathbb{1}_{t<\tau_1} \left(\mathbb{1}_{t<\tau_2} \frac{G(T,t)}{G(t,t)} + \mathbb{1}_{\tau_2 < t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) + \mathbb{1}_{\tau_1 < t} \\
\mathbb{P}(\tau_{(2)} \le T | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) &= \mathbb{1}_{t<\tau_{(1)}} \frac{\mathbb{P}(t \le \tau_{(1)} < \tau_{(2)} < T)}{\mathbb{P}(t < \tau_{(1)})} + \mathbb{1}_{\tau_1 \le t < \tau_2} \frac{\mathbb{P}(t < \tau_2 < T | \tau_1)}{\mathbb{P}(t < \tau_2 | \tau_1)} \\
&\quad + \mathbb{1}_{\tau_2 \le t < \tau_1} \frac{\mathbb{P}(t < \tau_1 < T | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} + \mathbb{1}_{\tau_{(2)} < t} \,.
\end{aligned}$$

• The computation of $\mathbb{P}(T < \tau_1 | \tau_2)$ can be done as follows: the function h such that $\mathbb{P}(T < \tau_1 | \tau_2) = h(\tau_2)$ satisfies

$$\mathbb{E}(h(\tau_2)\varphi(\tau_2)\mathbb{1}_{\tau_2 < t}) = \mathbb{E}(\varphi(\tau_2)\mathbb{1}_{\tau_2 < t}\mathbb{1}_{T < \tau_1})$$

for any function φ. This implies that (assuming that the pair (τ_1, τ_2) has a density f)

$$\int_0^t dv h(v)\varphi(v) \int_0^\infty du f(u,v) = \int_0^t dv \varphi(v) \int_T^\infty du f(u,v)$$

or

$$\int_0^t dv h(v)\varphi(v)\partial_2 G(0,v) = \int_0^t dv \varphi(v) \partial_2 G(T,v)$$

hence, $h(v) = \frac{\partial_2 G(T,v)}{\partial_2 G(0,v)}$.
We can also write

$$\mathbb{P}(T < \tau_1 | \tau_2 = v) = \frac{\mathbb{P}(T < \tau_1, \tau_2 \in dv)}{\mathbb{P}(\tau_2 \in dv)} = -\frac{1}{\mathbb{P}(\tau_2 \in dv)} \frac{d}{dv} \mathbb{P}(\tau_1 > T, \tau_2 > v) = \frac{\partial_2 G(T,v)}{\partial_2 G(0,v)}$$

hence, on the set $\tau_2 < T$,

$$\mathbb{P}(T < \tau_1 | \tau_2) = h(\tau_2) = \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(0,\tau_2)}$$

• In the same way, for $T > t$

$$\mathbb{P}(\tau_1 \le T < \tau_2 | \mathcal{H}_t^1 \vee \mathcal{H}_t^2)\mathbb{1}_{\{\tau_1 \le t < \tau_2\}} = \mathbb{1}_{\{\tau_1 \le t < \tau_2\}} \Psi(\tau_1)$$

where Ψ satisfies

$$\mathbb{E}(\varphi(\tau_1)\mathbb{1}_{\tau_1 \le t < T < \tau_2}) = \mathbb{E}(\varphi(\tau_1)\Psi(\tau_1)\mathbb{1}_{\{\tau_1 \le t < \tau_2\}})$$

for any function φ. In other terms

$$\int_0^t du \varphi(u) \int_T^\infty dv f(u,v) = \int_0^t du \varphi(u)\Psi(u) \int_t^\infty dv f(u,v)$$

or

$$\int_0^t du \varphi(u) \partial_1 G(u,T) = \int_0^t du \varphi(u)\Psi(u)\partial_1 G(u,t) \,.$$

This implies that

$$\Psi(u) = \frac{\partial_1 G(u,T)}{\partial_1 G(u,t)}$$

$$\mathbb{P}(\tau_1 \le T < \tau_2 | \mathcal{H}_t^1 \vee \mathcal{H}_t^2)\mathbb{1}_{\{\tau_1 \le t < \tau_2\}} = \mathbb{1}_{\{\tau_1 \le t < \tau_2\}} \frac{\partial_1 G(\tau_1,T)}{\partial_1 G(\tau_1,t)} \,.$$

1.1.2 Value of credit derivatives

We introduce different credit derivatives

A defaultable zero-coupon related to the default times D^i delivers 1 monetary unit if τ_i is greater that T: $D^i(t,T) = \mathbb{E}(1\!\!1_{\{T<\tau_i\}}|\mathcal{H}^1_t \vee \mathcal{H}^2_t)$

A contract which pays R^1 is one default occurs before T and R_2 if the two default occur before T: $CD_t = \mathbb{E}(R_1 1\!\!1_{\{0<\tau_{(1)}\leq T\}} + R_2 1\!\!1_{\{0<\tau_{(2)}\leq T\}}|\mathcal{H}^1_t \vee \mathcal{H}^2_t)$

We obtain

$$D^1(t,T) = 1\!\!1_{\{\tau_1>t\}}\left(1\!\!1_{\{\tau_2\leq t\}}\frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} + 1\!\!1_{\{\tau_2>t\}}\frac{G(T,t)}{G(t,t)}\right) \tag{1.1}$$

$$D^2(t,T) = 1\!\!1_{\{\tau_2>t\}}\left(1\!\!1_{\{\tau_1\leq t\}}\frac{\partial_1 G(\tau_1,T)}{\partial_2 G(\tau_1,t)} + 1\!\!1_{\{\tau_1>t\}}\frac{G(t,T)}{G(t,t)}\right) \tag{1.2}$$

$$CD_t = R_1 1\!\!1_{\{\tau_{(1)}>t\}}\left(\frac{G(t,t)-G(T,T)}{G(t,t)}\right) + R_2 1\!\!1_{\{\tau_{(2)}\leq t\}} + R_1 1\!\!1_{\{\tau_{(1)}\leq t\}} \tag{1.3}$$

$$+R_2 1\!\!1_{\{\tau_{(2)}>t\}}\left\{I_t(0,1)\left(1-\frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)}\right) + I_t(1,0)\left(1-\frac{\partial_1 G(\tau_1,T)}{\partial_1 G(\tau_1,t)}\right)\right. \tag{1.4}$$

$$\left.+I_t(0,0)\left(1-\frac{G(t,T)+G(T,t)-G(T,T)}{G(t,t)}\right)\right\} \tag{1.5}$$

where by

$$I_t(1,1) = 1\!\!1_{\{\tau_1\leq t,\tau_2\leq t\}}\,, \qquad I_t(0,0) = 1\!\!1_{\{\tau_1>t,\tau_2>t\}}$$
$$I_t(1,0) = 1\!\!1_{\{\tau_1\leq t,\tau_2>t\}}\,, \qquad I_t(0,1) = 1\!\!1_{\{\tau_1>t,\tau_2\leq t\}}$$

More generally, some easy computation leads to

$$\mathbb{E}(h(\tau_1,\tau_2)|\mathcal{H}_t) = I_t(1,1)h(\tau_1,\tau_2) + I_t(1,0)\Psi_{1,0}(\tau_1) + I_t(0,1)\Psi_{0,1}(\tau_2) + I_t(0,0)\Psi_{0,0}$$

where

$$\Psi_{1,0}(u) = -\frac{1}{\partial_1 G(u,t)}\int_t^\infty h(u,v)\partial_1 G(u,dv)$$
$$\Psi_{0,1}(v) = -\frac{1}{\partial_2 G(t,v)}\int_t^\infty h(u,v)\partial_2 G(du,v)$$
$$\Psi_{0,0} = \frac{1}{G(t,t)}\int_t^\infty\int_t^\infty h(u,v)G(du,dv)$$

The next result deals with the valuation of a first-to-default claim in a bivariate set-up. Let us stress that the concept of the (tentative) price will be later supported by strict replication arguments. In this section, by a *pre-default price* associated with a $\mathbb{G}$-adapted price process π, we mean here the function $\widetilde{\pi}$ such that $\pi_t 1\!\!1_{\{\tau_{(1)}>t\}} = \widetilde{\pi}(t)1\!\!1_{\{\tau_{(1)}>t\}}$ for every $t\in[0,T]$. In other words, the pre-default price $\widetilde{\pi}$ and the price π coincide prior to the first default only.

Definition 1.1.1 *Let Z_i be two functions, and X a constant. A FtD claim pays $Z_1(\tau_1)$ at time τ_1 if $\tau_1<T, \tau_1<\tau_2$, pays $Z_2(\tau_2)$ at time τ_2 if $\tau_2<T,\tau_2<\tau_1$, and X at maturity if $\tau_1\wedge\tau_2>T$*

Proposition 1.1.1 *The pre-default price of a FtD claim $(X,0,Z,\tau_{(1)})$, where $Z=(Z_1,Z_2)$ and $X=c(T)$, equals*

$$\frac{1}{G(t,t)}\left(-\int_t^T Z_1(u)\,G(du,u) - \int_t^T Z_2(v)G(v,dv) + XG(T,T)\right).$$

PROOF: The price can be expressed as

$$\mathbb{E}_{\mathbb{Q}}(Z_1(\tau_1)\mathbb{1}_{\{\tau_1\le T,\tau_2>\tau_1\}}|\mathcal{H}_t)+\mathbb{E}_{\mathbb{Q}}(Z_2(\tau_2)\mathbb{1}_{\{\tau_2\le T,\tau_1>\tau_2\}}|\mathcal{H}_t)+\mathbb{E}_{\mathbb{Q}}(c(T)\mathbb{1}_{\{\tau_{(1)}>T\}}|\mathcal{H}_t).$$

The pricing formula now follows by evaluating the conditional expectation, using the joint distribution of default times under the martingale measure $\mathbb{Q}$. □

Comments 1.1.1 Same computations appear in Kurtz and Riboulet [?]

1.1.3 Martingales

We present the computation of the martingales associated to the times τ_i in different filtrations. In particular, we shall obtain the computation of the intensities in various filtrations.

We have established that, if $\mathbb{F}$ is a given reference filtration and $G_t=\mathbb{P}(\tau>t|\mathcal{F}_t)$ the Azéma supermartingale admitting a Doob-Meyer decomposition $G_t=Z_t-\int_0^t a_s ds$, then the process

$$H_t-\int_0^{t\wedge\tau}\frac{a_s}{G_{s-}}ds$$

is a $\mathbb{G}$-martingale, where $\mathbb{G}=\mathbb{F}\vee\mathbb{H}$ and $\mathcal{H}_t=\sigma(t\wedge tau)$.

• **Filtration $\mathbf{H}^i$** We study the decomposition of the semi-martingales H^i in the filtration $\mathbf{H}^i$. We set $F_i(s)=\mathbb{P}(\tau_i\le s)=\int_0^s f_i(u)du$. From our general result applied to the case where $\mathbb{F}$ is the trivial filtration, we obtain that for any $i=1,2$, the process

$$M_t^i=H_t^i-\int_0^{t\wedge\tau_i}\frac{f_i(s)}{1-F_i(s)}ds \tag{1.6}$$

is a $\mathbf{H}^i$-martingale.

• **Filtration G** We apply the general result to the case $\mathbb{F}=\mathbb{H}^2$ and $\mathbb{H}=\mathbb{H}^1$. Let

$$G_t^{1|2}=\mathbb{P}(\tau_1>t|\mathcal{H}_t^2)$$

be the Azéma supermartingale of τ_1 in the filtration $\mathbb{H}^2$. Then, the process

$$H_t^1-\int_0^{t\wedge\tau_1}\frac{a_s^{(1)}}{G_{s-}^{1|2}}ds$$

is a **G**-martingale with Doob-Meyer decomposition $G_t^{1|2}=Z_t^{1|2}-\int_0^t a_s^{(1)}ds$ where $Z^{1|2}$ is a $\mathbf{H}^2$-martingale. The process $A_t^{(1)}=\int_0^{t\wedge\tau_1}\frac{a_s^{(1)}}{G_{s-}^{1|2}}ds$ is the $\mathbf{H}^2$-adapted compensator of H^1. The same methodology can be applied for the compensator of H^2. In what follows, we assume that $G^{1|2}$ is continuous.

We now compute in an explicit form the compensator of H^1 in order to establish the proposition

Proposition 1.1.2 *The process*

$$H_t^1-\int_0^{t\wedge\tau_1}\frac{a_s^{(1)}}{G_s^{1|2}}ds$$

where $a_t^{(1)}=-H_t^2\,\partial_1 h^{(1)}(t,\tau_2)-(1-H_t^2)\frac{\partial_1 G(t,t)}{G(0,t)}$ *and*

$$h^{(1)}(t,s)=\frac{\partial_2 G(t,s)}{\partial_2 G(0,s)}\,.$$

is a $\mathbf{G}$*-martingale.*
The process

$$H_t^2 - \int_0^{t\wedge\tau_2} \frac{a_s^{(2)}}{G_s^{2|1}} ds$$

where $a_t^{(2)} = -H_t^1\, \partial_2 h^{(2)}(\tau_1, t) - (1 - H_t^1)\frac{\partial_2 G(t,t)}{G(t,0)}$ *and*

$$h^{(2)}(t,s) = \frac{\partial_1 G(t,s)}{\partial_1 G(t,0)}\,.$$

is a $\mathbf{G}$*-martingale.*

Proof: Some easy computation enables us to write

$$\begin{aligned} G_t^{1|2} &= H_t^2 \mathbb{P}(\tau_1 > t|\tau_2) + (1 - H_t^2)\frac{\mathbb{P}(\tau_1 > t, \tau_2 > t)}{\mathbb{P}(\tau_2 > t)} \\ &= H_t^2 h^{(1)}(t,\tau_2) + (1 - H_t^2)\frac{G(t,t)}{G(0,t)} = H_t^2 h^{(1)}(t,\tau_2) + (1 - H_t^2)\psi(t) \end{aligned} \tag{1.7}$$

where

$$h^{(1)}(t,v) = \frac{\partial_2 G(t,v)}{\partial_2 G(0,v)}\,; \psi(t) = G(t,t)/G(0,t).$$

Function $t \to \psi(t)$ and process $t \to h(t,\tau_2)$ are continuous and of finite variation, hence integration by parts rule leads to

$$\begin{aligned} dG_t^{1|2} &= h(t,\tau_2)dH_t^2 + H_t^2 \partial_1 h(t,\tau_2)dt + (1 - H_t^2)\psi'(t)dt - \psi(t)dH_t^2 \\ &= (h(t,\tau_2) - \psi(t))\, dH_t^2 + \left(H_t^2 \partial_1 h(t,\tau_2) + (1 - H_t^2)\psi'(t)\right) dt \\ &= \left(\frac{\partial_2 G(t,\tau_2)}{\partial_2 G(0,\tau_2)} - \frac{G(t,t)}{G(0,t)}\right) dH_t^2 + \left(H_t^2 \partial_1 h(t,\tau_2) + (1 - H_t^2)\psi'(t)\right) dt \end{aligned}$$

From the computation of the Stieljes integral, we can rewrite it as

$$\begin{aligned} \int_0^T \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,\tau_2)}{\partial_2 G(0,\tau_2)}\right) dH_t^2 &= \left(\frac{G(\tau_2,\tau_2)}{G(0,\tau_2)} - \frac{\partial_2 G(\tau_2,\tau_2)}{\partial_2 G(0,\tau_2)}\right) 1_{\{\tau_2 \le t\}} \\ &= \int_0^T \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right) dH_t^2 \end{aligned}$$

and substitute it in the expression of $dG^{1|2}$:

$$dG_t^{1|2} = \left(\frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} - \frac{G(t,t)}{G(0,t)}\right) dH_t^2 + \left(H_t^2 \partial_1 h(t,\tau_2) + (1 - H_t^2)\psi'(t)\right) dt$$

We now use that

$$dH_t^2 = dM_t^2 - \left(1 - H_t^2\right) \frac{\partial_2 G(0,t)}{G(0,t)} dt$$

where M^2 is a $\mathbb{H}^2$-martingale, and we get the $\mathbb{H}^2-$ Doob-Meyer decomposition of $G^{1|2}$:

$$\begin{aligned} dG_t^{1|2} &= \left(\frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} - \frac{G(t,t)}{G(0,t)}\right) dM_t^2 - \left(1 - H_t^2\right) \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right) \frac{\partial_2 G(0,t)}{G(0,t)} dt \\ &\quad + \left(H_t^2 \partial_1 h^{(1)}(t,\tau_2) + (1 - H_t^2)\psi'(t)\right) dt \end{aligned}$$

and from

$$\psi'(t) = \left(\frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} - \frac{G(t,t)}{G(0,t)}\right) \frac{\partial_2 G(0,t)}{G(0,t)} + \frac{\partial_1 G(t,t)}{G(0,t)}$$

we conclude

$$dG_t^{1|2} = \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} \right) dM_t^2 + \left(H_t^2 \partial_1 h^{(1)}(t,\tau_2) + (1-H_t^2) \frac{\partial_1 G(t,t)}{G(0,t)} \right) dt$$

From (1.7), the process $G^{1|2}$ has a single jump of size $\frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} - \frac{G(t,t)}{G(0,t)}$. From (1.7),

$$G^{1|2} = \frac{G(t,t)}{G(0,t)} = \psi(t)$$

on the set $\tau_2 > t$, and its bounded variation part is $\psi'(t)$. The hazard process has a non null martingale part, except if $\frac{G(t,t)}{G(0,t)} = \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}$ (this is the case if the default are independent). Hence, (H) hypothesis is not satisfied in a general setting between $\mathbf{H}^i$ and $\mathbf{G}$.

• **Filtration H** We reproduce now the result of Chou and Meyer [**?**], in order to obtain the martingales in the filtration $\mathbf{H}$, in case of two default times. Here, we denote by $\mathbb{H}$ the filtration generated by the process $H_t = H_t^1 + H_t^2$. This filtration is smaller than the filtration $\mathbb{G}$. We denote by $T_1 = \tau_1 \wedge \tau_2$ the infimum of the two default times and by $T_2 = \tau_1 \vee \tau_2$ the supremum. The filtration $\mathbf{H}$ is the filtration generated by $\sigma(T_1 \wedge t) \vee \sigma - t_2 \wedge t)$, up to completion with negligeable sets.
Let us denote by $G_1(t)$ the survival distribution function of T_1, i.e., $G_1(t) = \mathbb{P}(\tau_1 > t, \tau_2 > t) = G(t,t)$ and by $G_2(t;u)$ the survival conditional distribution function of T_2 with respect to T_1, i.e., for $t > u$,

$$G_2(u;t) = \mathbb{P}(T_2 > t | T_1 = u) = \frac{1}{g(u)} \left(\partial_1 G(u,t) + \partial_2 G(t,u) \right) ,$$

where $g(t) = \frac{d}{dt} G(t,t) = \frac{1}{dt} \mathbb{P}(T_1 \in dt)$. We shall also note

$$K(u;t) = \mathbb{P}(T_2 - T_1 > t | T_1 = u) = G_2(u; t+u)$$

The process $M_t \stackrel{def}{=} H_t - \Lambda_t$ is a $\mathbf{H}$-martingale, where

$$\Lambda_t = \Lambda_1(t) 1\!\!1_{t < T_1} + \left[\Lambda_1(T_1) + \Lambda_2(T_1, t - T_1) \right] 1\!\!1_{T_1 \le t < T_2}$$

with

$$\Lambda_1(t) = -\int_0^t \frac{dG_1(s)}{G_1(s)} = \int_0^t \frac{g(s)}{G(s,s)} ds = -\ln \frac{G(t,t)}{G(0,0)} = -\ln G(t,t)$$

and

$$\Lambda_2(s;t) = -\int_0^t \frac{d_u K(s;u)}{K(s,u)} = -\ln \frac{K(s;t)}{K(s;0)}$$

hence

$$\begin{aligned} \Lambda_2(T_1, t - T_1) &= -\ln \frac{K(T_1; t - T_1)}{K(T_1; 0)} = -\ln \frac{G_2(T_1; t)}{G_2(T_1; T_1)} \\ &= -\ln \frac{\partial_1 G(T_1, t) + \partial_2 G(t, T_1)}{\partial_1 G(T_1, T_1) + \partial_2 G(T_1, T_1)} \end{aligned}$$

It is proved in Chou-Meyer [**?**] that any $\mathbf{H}$-martingale is a stochastic integral with respect to M. This result admits an immediate extension to the case of n successive defaults.
This representation theorem has an interesting consequence: a single asset is enough to get a complete market. This asset with price M, and final payoff $H_T - \Lambda_T$. It corresponds to a swap with cumulative premium leg Λ_t

Remark 1.1.1 Note that

$$\begin{aligned} H_t^1 - \int_0^{t\wedge\tau_1} \frac{a_s^{(1)}}{G_s^{1|2}} ds &= H_t^1 - \int_0^{t\wedge\tau_1} \frac{H_s^2 \partial_1 h^{(1)}(s,\tau_2) - (1-H_s^2)\partial_1 G(s,s)/G(0,s)}{H_s^2 h^{(1)}(s,\tau_2) + (1-H_s^2)\psi(s)} ds \\ &= H_t^1 - \int_0^{t\wedge\tau_1} H_s^2 \frac{\partial_1 h^{(1)}(s,\tau_2)}{h^{(1)}(s,\tau_2)} - (1-H_s^2)\frac{\partial_1 G(s,s)/G(0,s)}{\psi(s)} ds \\ &= H_t^1 - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \frac{\partial_1 h^{(1)}(s,\tau_2)}{h^{(1)}(s,\tau_2)} ds - \int_0^{t\wedge\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)} ds \\ &= H_t^1 - \ln \frac{h^{(1)}(t\wedge\tau_1\wedge\tau_2,\tau_2)}{h^{(1)}(t\wedge\tau_1,\tau_2)} - \int_0^{t\wedge\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)} ds \end{aligned}$$

It follows that the intensity of τ_1 in the **G**-filtration is $\frac{\partial_1 G(s,s)}{G(s,s)}$ on the set $\{t < \tau_2 \wedge \tau_1\}$ and $\frac{\partial_1 h^{(1)}(s,\tau_2)}{h^{(1)}(s,\tau_2)}$ on the set $\{\tau_2 < t < \tau_1\}$. It can be proved that the intensity of $\tau_1 \wedge \tau_2$ is

$$\frac{\partial_1 G(s,s)}{G(s,s)} + \frac{\partial_2 G(s,s)}{G(s,s)} = \frac{g(t)}{G(t,t)}$$

where $g(t) = \frac{d}{dt} G(t,t)$

1.1.4 Application of Norros lemma for two defaults

Norros's lemma

Proposition 1.1.3 *Let $\tau_i, i = 1, \cdots, n$ be n finite-valued random times and $\mathcal{G}_t = \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^n$. Assume that*

$P(\tau_i = \tau_j) = 0, \forall i \neq j$

there exists continuous processes A^i such that $M_t^i = H_t^i - A_{t\wedge\tau_i}^i$ are **G***-martingales*

then, the r.v's $A_{\tau_i}^i$ are independent with exponential law.

Proof. For any $\mu_i > -1$ the processes $L_t^i = (1+\mu_i)^{H_t^i} e^{-\mu_i A_t^i}$, solution of

$$dL_t^i = L_{t-}^i \mu_i dM_t^i$$

are uniformly integrable martingales. Moreover, these martingales have no commun jumps, and are orthogonal. Hence $E(\prod_i (1+\mu_i) e^{-\mu_i A_\infty^i}) = 1$, which implies

$$E(\prod_i e^{-\mu_i A_\infty^i}) = \prod_i (1+\mu_i)^{-1}$$

hence the independence property. □

Application

In case of two defaults, this implies that U_1 and U_2 are independent, where

$$U_i = \int_0^{\tau_i} \frac{a_i(s)}{G_i^*(s)} ds$$

and

$$a_1(t) = -(1-H_t^2)\frac{\partial_1 G(t,t)}{G(0,t)} + H_t^2 \partial_1 h^{(1)}(t,\tau_2), \quad G_1^*(t) = H_t^2 h^{(1)}(t,\tau_2) + (1-H_t^2)\frac{G(t,t)}{G(0,t)},$$

$$a_2(t) = -(1-H^1_t)\frac{\partial_2 G(t,t)}{G(t,0)} + H^1_t \partial_2 h^{(2)}(\tau_1,t), \quad G^*_2(t) = H^1_t h^{(2)}(\tau_1,t) + (1-H^1_t)\frac{G(t,t)}{G(t,0)}$$

are independent. In a more explicit form,

$$\int_0^{\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)}ds + \ln\frac{h^{(1)}(\tau_1,\tau_2)}{h^{(1)}(\tau_1\wedge\tau_2,\tau_2)} = \int_0^{\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)}ds + \ln\frac{\partial_2 G(\tau_1,\tau_2)}{\partial_2 G(\tau_1\wedge\tau_2,\tau_2)}$$

is independent from

$$\int_0^{\tau_1\wedge\tau_2} \frac{\partial_2 G(s,s)}{G(s,s)}ds + \ln\frac{h^{(2)}(\tau_1,\tau_2)}{h^{(2)}(\tau_1,\tau_1\wedge\tau_2)} = \int_0^{\tau_1\wedge\tau_2} \frac{\partial_2 G(s,s)}{G(s,s)}ds + \ln\frac{\partial_1 G(\tau_1,\tau_2)}{\partial_1 G(\tau_1,\tau_1\wedge\tau_2)}$$

Example of Poisson process

In the case where τ_1 and τ_2 are the two first jumps of a Poisson process, we have

$$G(t,s) = \begin{cases} e^{-\lambda t} \text{ for } s<t \\ e^{-\lambda s}(1+\lambda(s-t) \text{ for } s>t \end{cases}$$

with partial derivatives

$$\partial_1 G(t,s) = \begin{cases} -\lambda e^{-\lambda t} \text{ for } t>s \\ -\lambda e^{-\lambda s} \text{ for } s>t \end{cases}, \quad \partial_2 G(t,s) = \begin{cases} 0 \text{ for } t>s \\ -\lambda^2 e^{-\lambda s}(s-t) \text{ for } s>t \end{cases}$$

and

$$h(t,s) = \begin{cases} 1 \text{ for } t>s \\ \frac{t}{s} \text{ for } s>t \end{cases}, \quad \partial_1 h(t,s) = \begin{cases} 0 \text{ for } t>s \\ \frac{1}{s} \text{ for } s>t \end{cases}$$

$$k(t,s) = \begin{cases} 0 \text{ for } t>s \\ 1-e^{-\lambda(s-t)} \text{ for } s>t \end{cases}, \quad \partial_2 k(t,s) = \begin{cases} 0 \text{ for } t>s \\ \lambda e^{-\lambda(s-t)} \text{ for } s>t \end{cases}$$

Then, one obtains $U_1 = \tau_1$ et $U_2 = \tau_2 - \tau_1$

1.2 Cox process modelling

We are now studying a financial market with null interest rate, and we work under the probability chosen by the market. We now assume that n non negative processes $\lambda_i, i = 1,\dots,n$, $\mathbb{F}$-adapted are given and we denote $\Lambda_{i,t} = \int_0^t \lambda_{i,s}ds$. We assume the existence of n r.v. $U_i, i = 1,\cdots,n$ with uniform law, independent and independent of $\mathcal{F}_\infty$ and we define

$$\tau_i = \inf\{t : U_i \geq \exp(-\Lambda_{i,t})\}.$$

We introduce the following different filtrations
• $\mathbb{H}_i$ generated by $H_{i,t} = 1\!\!1_{\tau_i\leq t}$
• the filtration $\mathbb{G}$ defined as

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_{1,t} \vee \cdots \vee \mathcal{H}_{i,t} \vee \cdots \mathcal{H}_{n,t}$$

• the filtration $\mathbb{G}_i$ as $\mathcal{G}_{i,t} = \mathcal{F}_t \vee \mathcal{H}_{i,t}$
• $\mathbb{H}_{(-i)}$ the filtration

$$\mathcal{H}_{(-i),t} = \mathcal{H}_{1,t} \vee \cdots \vee \mathcal{H}_{i-1,t} \vee \mathcal{H}_{i+1,t} \cdots \mathcal{H}_{n,t}$$

Note the obvious inclusions

$$\mathbb{F} \subset \mathbb{G}_i \subset \mathbb{G}, \quad \mathbb{H}_{(-i)} \subset \mathbb{G} = \mathbb{G}_i \vee \mathbb{H}_{(-i)}$$

We note $\ell_i(t,T)$ the loss process

$$\ell_i(t,T) = \mathbb{E}(1\!\!1_{\tau_i\leq T}|\mathcal{G}_t) = \mathbb{P}(\tau_i \leq T|\mathcal{G}_t) = \mathbb{E}(H_{i,T}|\mathcal{G}_t)$$

and $\widetilde{D}_i(t,T) = \mathbb{E}(\exp(\Lambda_{i,t} - \Lambda_{i,T})|\mathcal{F}_t)$ the predefault price if a DZC.

Lemma 1.2.1 *The following equalities holds*

$$\mathbb{P}(\tau_i \geq t_i, \forall i) = \mathbb{E}(\exp - \sum_i \Lambda_{t_i,i}) \tag{1.8}$$

$$\mathbb{P}(\tau_i \geq t_i, \forall i | \mathcal{F}_t) = \exp - \sum_i \Lambda_{t_i,i}, \ \forall t_i \leq t, \tag{1.9}$$

$$\mathbb{P}(\tau_i \geq t_i, \forall i | \mathcal{F}_t) = \prod_i \mathbb{P}(\tau_i \geq t | \mathcal{F}_t), \ \forall t_i \leq t, \ \forall i \tag{1.10}$$

$$\mathbb{P}(\tau_i \geq t_i, \forall i | \mathcal{F}_t) = \mathbb{E}(\exp - \sum_i \Lambda_{t_i,i} | \mathcal{F}_t), \ \forall t_i, \tag{1.11}$$

$$\mathbb{P}(\tau_i \geq t_i, \ \forall i | \mathcal{G}_t) = \frac{\mathbb{P}(\tau_i \geq t_i, \ \forall i | \mathcal{F}_t)}{\mathbb{P}(\tau_i \geq t, \ \forall i | \mathcal{F}_t)} \quad on \quad the \quad set \quad \tau_i \geq t_i, \forall i \tag{1.12}$$

PROOF: From the definition

$$\mathbb{P}(\tau_i \geq t_i, \forall i) = \mathbb{P}(\exp - \Lambda_{t_i,i} \geq U_i, \forall i) = \mathbb{E}(\exp - \sum_i \Lambda_{t_i,i})$$

where we have used that $\mathbb{P}(u_i \geq U_i) = u_i$ and $\mathbb{E}(\Psi(X,Y)) = \mathbb{E}(\psi(X))$ with $\psi(x) = \mathbb{E}(\Psi(x,Y))$ for independent r.v. X and Y.

In the same way,

$$\begin{aligned} \mathbb{P}(\tau_i \geq t_i, \forall i | \mathcal{F}_t) &= \mathbb{P}(\exp - \Lambda_{t_i,i} \geq U_i, \forall i | \mathcal{F}_t) \\ &= \exp - \sum_i \Lambda_{t_i,i} \end{aligned}$$

where we have used that $\mathbb{E}(\Psi(X,Y)|X)) = \psi(X)$ with $\psi(x) = \mathbb{E}(\Psi(x,Y)$ for independent r.v's X and Y, and that the $\Lambda_{t_i,i}$ are $\mathcal{F}_t$-measurable for $t_i \leq t$.

Lemma 1.2.2 *(a) Any bounded $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.*
(b) Any bounded $\mathbb{G}_i$-martingale is a $\mathbb{G}$-martingale

PROOF: (a) Using the caracterisation of conditional expectation, one has to check that

$$\mathbb{E}(\eta | \mathcal{F}_t) = \mathbb{E}(\eta | \mathcal{F}_\infty)$$

for any $\mathcal{G}_t$-measurable r.v. It suffices to prove the equality for

$$\eta = F_t h_1(t \wedge \tau_1) \cdots h_n(t \wedge \tau_n)$$

where $F_t \in \mathcal{F}_t$ and $h_i, i = 1, \cdots, n$ are bounded measurable functions. We can reduce attention to functions of the from $h_i(s) = \mathbb{1}_{[0,a_i]}(s)$. If $a_i > t$, $h_i(t \wedge \tau_i) = 1$, so we can pay attention to the case where all the a_i's are smaller than t. The equality is now equivalent to

$$\mathbb{E}(\tau_i \leq a_i, \forall i | \mathcal{F}_t) = \mathbb{E}(\tau_i \leq a_i, \forall i | \mathcal{F}_\infty)$$

By definition

$$\mathbb{E}(\tau_i \leq a_i, \ \forall i | \mathcal{F}_t) = \mathbb{E}(\exp - \Lambda_{i,a_i} < U_i, \ \forall i | \mathcal{F}_t) = \Psi(\Lambda_{i,t}; i = 1, \cdots, n)$$

with $\Psi(u_i; i = 1, \cdots, n) = \prod(1 - u_i)$. The same computation leads to

$$\mathbb{E}(\tau_i \leq a_i, \forall i | \mathcal{F}_\infty) = \Psi(\Lambda_{i,a_i}, i = 1, \cdots, n)$$

(b) Using the same methodology, we are reduced to prove that for any bounded $\mathcal{G}_t$-measurable r.v. η,

$$\mathbb{E}(\eta | \mathcal{G}_{i,t}) = \mathbb{E}(\eta | \mathcal{G}_{(i,\infty)})$$

or even only that

$$\mathbb{E}(\eta_1\eta_2|\mathcal{G}_{i,t}) = \mathbb{E}(\eta_1\eta_2|\mathcal{G}_{(i,\infty)})$$

for $\eta_1 \in \mathcal{G}_{i,t}$ and $\eta_2 \in \mathcal{H}_{(-i),t}$, that is

$$\mathbb{E}(\eta_2|\mathcal{G}_{i,t}) = \mathbb{E}(\eta_2|\mathcal{G}_{(i,\infty)})$$

To simplify, we assume that $i = 1$. Using the same elementary functions h as above, we have to prove that

$$\mathbb{E}(h_2(\tau_2 \wedge t) \cdots h_n(\tau_n \wedge a_n)|\mathcal{G}_{1,t}) = \mathbb{E}(h_2(\tau_2 \wedge t) \cdots h_n(\tau_n \wedge a_n)|\mathcal{G}_{(1,\infty)})$$

where $a_i < t$, that is

$$\mathbb{E}(\mathbb{1}_{\tau_2 \le a_2} \cdots \mathbb{1}_{\tau_n \le a_n}|\mathcal{G}_{1,t}) = \mathbb{E}(\mathbb{1}_{\tau_2 \le a_2} \cdots \mathbb{1}_{\tau_n \le a_n}|\mathcal{G}_{1,\infty})$$

Note that the vector $(U_2, \cdots, U_n)$ is independent from

$$\mathcal{G}_{1,\infty} = \mathcal{F}_\infty \vee \sigma(\tau_2) \vee \cdots \sigma(\tau_n) = \mathcal{F}_\infty \vee \sigma(U_2) \vee \cdots \sigma(U_n)$$

It follows that

$$\mathbb{E}(\mathbb{1}_{\tau_2 \le a_2} \cdots \mathbb{1}_{\tau_n \le a_n}|\mathcal{G}_{1,\infty}) = \mathbb{E}(\mathbb{1}_{\exp -\Lambda_{2,a_2} \le U_2} \cdots \mathbb{1}_{\exp -\Lambda_{n,a_n} \le U_n}|\mathcal{G}_{1,\infty}) = \prod_{i=2}^{n}(1 - \exp(-\Lambda_{i,a_i}))$$

Lemma 1.2.3 *The processes* $M_{i,t} \stackrel{def}{=} H_{i,t} - \int_0^t (1 - H_{i,s})\Lambda_{i,s} ds$ *are* $\mathbb{G}_i$*-martingales and* $\mathbb{G}$*-martingales*

PROOF: We have shown that $M_{i,t} \stackrel{def}{=} H_{i,t} - \int_0^t (1 - H_{i,s})\Lambda_{i,s} ds$ are $\mathbb{G}_i$-martingales. Now, from the lemma, $\mathbb{G}_i$ martingales are $\mathbb{G}$ martingales as well.

Lemma 1.2.4 *The processes* $\ell_i(t,T)$ *are* $\mathbb{G}$*-martingales and*

$$\ell_{i,t} = (1 - H_{i,t})(1 - \mathbb{E}(\exp(\Lambda_{i,t} - \Lambda_{i,T})|\mathcal{F}_t) + H_{i,t}$$

From the definition, the processes $\ell_i(t,T)$ are $\mathbb{G}$-martingales. From Lemma

$$\mathbb{P}(\tau_i \ge T,\ |\mathcal{G}_t) = \mathbb{1}_{t<\tau_i} \frac{\mathbb{P}(\tau_i \ge T|\mathcal{F}_t)}{\mathbb{P}(\tau_i \ge t,\ |\mathcal{F}_t)} = (1 - H_{i,t})\mathbb{E}(\exp -(\Lambda_{i,t} - \Lambda_{i,T})|\mathcal{F}_t)$$

hence $\ell_i(t,T) = H_{i,t} + (1 - H_{i,t})\mathbb{E}(1 - \exp -(\Lambda_{i,t} - \Lambda_{i,T})|\mathcal{F}_t)$

Chapter 2

Exercises

2.1 Toy Model

The proofs of the following exercises can be found in Osaka lecture notes.

Exercise 2.1.1 Prove that the payoff $1\!\!1_{T<\tau}$ can not be hedged with zero-coupon bonds.

Exercise 2.1.2 Prove that H is a submartingale.

Exercise 2.1.3 Assume that Γ is a continuous function. Then for any (bounded) Borel measurable function $h : I\!\!R_+ \to I\!\!R$, the process

$$M_t^h = 1\!\!1_{\{\tau \le t\}} h(\tau) - \int_0^{t\wedge\tau} h(u)\, d\Gamma(u) \tag{2.1}$$

is a $\mathbf{H}$-martingale.

Exercise 2.1.4 Let $\eta_t = \mathbb{E}_P(h(\tau)|\mathcal{H}_t)$. Prove that

$$\eta_t \quad = \quad \int_0^t h(s)dH_s + (1-H_t)g(t)$$

Prove that the martingale η admits a representation in terms of M as

$$\eta_t \quad = \quad 1 + \int_0^t \eta_{u-}(\frac{h(t)}{g(t)} - 1)dM_u$$

Exercise 2.1.5 Let $h : I\!\!R_+ \to I\!\!R$ be a (bounded) Borel measurable function. Then the process

$$\widetilde{M}_t^h = \exp\big(1\!\!1_{\{\tau \le t\}} h(\tau)\big) - \int_0^{t\wedge\tau} (e^{h(u)} - 1)\, d\Gamma(u) \tag{2.2}$$

is a $\mathbf{H}$-martingale.

Exercise 2.1.6 Assume that Γ is a continuous function. Let $h : I\!\!R_+ \to I\!\!R$ be a non-negative Borel measurable function such that the random variable $h(\tau)$ is integrable. Then the process

$$\widehat{M}_t = (1 + 1\!\!1_{\tau \le t} h(\tau)) \exp\Big(-\int_0^{t\wedge\tau} h(u)\, d\Gamma(u)\Big) \tag{2.3}$$

is a $\mathbf{H}$-martingale.

Exercise 2.1.7 In this exercise, F is only continuous on right, and $F(t-)$ is the left limit at point t. Prove that the process $(M_t, t \geq 0)$ defined as

$$M_t = H_t - \int_0^{\tau \wedge t} \frac{dF(s)}{1 - F(s-)} = H_t - \int_0^t (1 - H_{s-}) \frac{dF(s)}{1 - F(s-)}$$

is a **H**-martingale.

Exercise 2.1.8 If Γ is not continuous, prove that

$$\mathbb{E}(h(\tau)|\mathcal{H}_t) = \mathbb{E}(h(\tau)) - \int_0^{t \wedge \tau} e^{\Delta \Gamma(s)} (\widehat{h}(s) - h(s))\, dM_s\,.$$

The next result suggests that this martingale property uniquely characterizes the (continuous) hazard function of a random time.

Exercise 2.1.9 Suppose that an equivalent probability measure $\mathbb{P}^*$ is given by formula $\mathbb{P}^*(A) = \mathbb{E}_{\mathbb{P}}(1\!\!1_A h(\tau))$ for some function h. Let $\Lambda^* : I\!\!R_+ \to I\!\!R_+$ be an arbitrary continuous increasing function, with $\Lambda^*(0) = 0$. If the process $M_t^* := H_t - \Lambda^*(t \wedge \tau)$ follows a **H**-martingale under $\mathbb{P}^*$, then $\Lambda^*(t) = -\ln(1 - F^*(t))$

Exercise 2.1.10 Let M^1 and M^2 be arbitrary two $\mathbb{H}$-martingales under $\mathbb{Q}$. If for every $t \in [0, T]$ we have $1\!\!1_{\{t<\tau\}} M_t^1 = 1\!\!1_{\{t<\tau\}} M_t^2$ then $M_t^1 = M_t^2$ for every $t \in [0, T]$.

Exercise 2.1.11 The dynamics of the ex-dividend price $S_t(\kappa(s))$ on $[s, T]$ are also given as

$$dS_t(\kappa(s)) = -S_{t-}(\kappa(s))\, dM_t + (1 - H_t) \left(\frac{\int_t^T G(u)\, du}{G(t)}\, d_t \nu(t, s) - \nu(t, s)\, dt \right). \tag{2.4}$$

Exercise 2.1.12 Assume that

- the savings account $Y_t^0 = 1$
- a risky asset with risk-neutral dynamics

$$dY_t = Y_t \sigma dW_t$$

  where W is a Brownian motion
- a DZC of maturity T with price $D(t, T)$

are traded. The reference filtration is that of the BM W. We assume that $\mathbb{F}$ is immersed in $\mathbb{G}$.

Give the price of a defaultable call with payoff $1\!\!1_{T<\tau}(Y_T - K)^+$ and the associated hedging strategy

Solution: The price of the call is

$$\begin{aligned} C_t &= \mathbb{E}(1\!\!1_{T<\tau}(Y_T - K)^+|\mathcal{G}_t) = 1\!\!1_{t<\tau} e^{\Lambda_t} \mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+|\mathcal{F}_t) \\ &= L_t m_t^Y \end{aligned}$$

with $m_t^Y = \mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+|\mathcal{F}_t)$. hence

$$dC_t = L_t dm_t^Y - m_t^Y L_{t-} dM_t$$

In our model, λ is deterministic, hence

$$m_t^Y = e^{-\Lambda_T} E((Y_T - K)^+|\mathcal{F}_t) = e^{-\Lambda_T} C_t^Y$$

where C^Y is the price of a call in the Black Scholes model. This quantity is $C_t^Y = C^Y(t, Y_t)$ and satisfies $dC_t^Y = \Delta_t dY_t$ where $\Delta_t$is the Delta-hedge ($\Delta_t = \partial_y C^Y(t, Y_t)$).

$$C_t = 1\!\!1_{t<\tau} e^{\Lambda_t} e^{-\Lambda_T} C^Y(t, Y_t) = L_t e^{-\Lambda_T} C^Y(t, Y_t) = D(t, T) C^Y(t, Y_t)$$

From

$$C_t = D(t, T) C^Y(t, Y_t)$$

we deduce

$$\begin{aligned} dC_t &= e^{-\Lambda_T}(L_t dC^Y + C^Y dL_t) = e^{-\Lambda_T}(L_t \Delta_t dY_t - C^Y L_t dM_t) \\ &= e^{-\Lambda_T}(L_t \Delta_t dY_t - C^Y L_t dM_t) \end{aligned}$$

Therefore, using that $dD(t, T) = m_t dM_t = -e^{-\Lambda_T} L_t dM_t$ we get

$$dC_t = e^{-\Lambda_T} L_t \Delta_t dY_t - C^Y dD(t, T) = e^{-\Lambda_T} L_t \Delta_t dY_t + \frac{C_t}{D(t, T)} dD(t, T)$$

hence, an hedging strategy consists of holding $\frac{C_t}{D(t,T)}$ DZCs.

2.2 Hazard Process Approach

2.2.1 Application of Key lemma

Exercise 2.2.1 Assume that the process G is decreasing. Let $\widetilde{V}$ and R be $\mathbb{F}$-predictable processes. The process

$$V_t = \widetilde{V}_t 1\!\!1_{\{t<\tau\}} + R_\tau 1\!\!1_{\{\tau \le t\}}$$

is a $\mathbb{G}$-martingale if and only if the process

$$v_t \stackrel{def}{=} \widetilde{V}_t e^{-\Gamma_t} + \int_0^t R_u e^{-\Gamma_u} d\Gamma_u$$

is an $\mathbb{F}$-martingale

Proof: The direct part comes from the fact that

$$\mathbb{E}(V_t - V_s | \mathcal{G}_s) = 1\!\!1_{\tau>t} e^{\Gamma_t} \mathbb{E}(v_t - v_s | \mathcal{F}_s)\,.$$

□

Exercise 2.2.2 Let $\widetilde{V}$ and R be $\mathbb{F}$-predictable processes. The process

$$V_t = \widetilde{V}_t 1\!\!1_{\{t<\tau\}} + R_\tau 1\!\!1_{\{\tau \le t\}}$$

is a $\mathbb{G}$-martingale if and only if the process

$$v_t \stackrel{def}{=} \widetilde{V}_t e^{-\Gamma_t} + \int_0^t R_u dF_u$$

is an $\mathbb{F}$-martingale

Proof: The direct part comes from the fact that

$$\mathbb{E}(V_t - V_s | \mathcal{G}_s) = 1\!\!1_{\tau>t} e^{\Gamma_t} \mathbb{E}(v_t - v_s | \mathcal{F}_s)\,.$$

□

Exercise 2.2.3 Let P be the price process of a claim which delivers R_τ at default time and pays a cumulative coupon C till the default time, i.e. the discounted cum-dividend process

$$B_t^{-1}P_t + 1\!\!1_{\{\tau\leq t\}}B_\tau^{-1}R_\tau + \int_0^{t\wedge\tau} B_u^{-1}dC_u$$

is a $\mathbb{G}$-martingale. Let $\widetilde{P}_t$ be the predefault price of the process P, i.e., $\tilde{P}$ is $\mathbb{F}$-predictable and $P_t = 1\!\!1_{\{t<\tau\}}\widetilde{P}_t$. Let $\alpha_t = \beta_t e^{-\Gamma_t}$. Prove that the process

$$P_t^* = \alpha_t \widetilde{P}_t + \int_0^t \alpha_s dC_s + \int_0^t R_u \alpha_u \, d\Gamma_u$$

is an $\mathbb{F}$-martingale, where $\alpha_t = B_t^{-1}e^{-\Gamma_t}$.

Conversely, if $\widetilde{V}$ is an $\mathbb{F}$-predictable process such that the process $\alpha_t\widetilde{V}_t + \int_0^t \alpha_s dC_s + \int_0^t R_u\alpha_u \, d\Gamma_u$ is an $\mathbb{F}$-martingale, prove that (the discounted cum-dividend) process

$$B_t^{-1}\widetilde{V}_t 1\!\!1_{\{t<\tau\}} + 1\!\!1_{\{\tau\leq t\}}B_\tau^{-1}R_\tau + \int_0^{t\wedge\tau} B_u^{-1}dC_u$$

is a $\mathbb{G}$-martingale.

PROOF: This is an application of the Key Lemma. □

2.2.2 Stopping times

Exercise 2.2.4 Prove that, for any $\mathbb{F}$-stopping time θ, we have:

$$\mathbb{Q}(\tau > \theta \,|\, \mathcal{F}_\theta) = e^{-\Gamma_\theta}. \tag{2.5}$$

This lemma plays an important role while dealing with convertible bonds.

Exercise 2.2.5 Let us be given $t \in I\!\!R_+$ and θ an $\mathbb{F}$ stopping time, valued in $(t, T]$. Prove the following assertions
(i) For any bounded from below, $\mathcal{F}_\theta$-measurable random variable χ, we have:

$$\mathbb{E}_\mathbb{Q}(1\!\!1_{\{t<\tau\leq\theta\}}\chi \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\, \mathbb{E}_\mathbb{Q}\big((1-e^{\Gamma_t-\Gamma_\theta})\chi \,\big|\, \mathcal{F}_t\big)\ ,\ \mathbb{E}_\mathbb{Q}(1\!\!1_{\{\tau>\theta\}}\chi \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\, \mathbb{E}_\mathbb{Q}\big(e^{-\Gamma_\theta}\chi \,\big|\, \mathcal{F}_t\big)\ .$$

(ii) For any bounded from below, $\mathbb{F}$-predictable process Z, we have:

$$\mathbb{E}_\mathbb{Q}(Z_\tau 1\!\!1_{\{t<\tau\leq\theta\}}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}e^{\Gamma_t}\, \mathbb{E}_\mathbb{Q}\Big(\int_t^\theta Z_u e^{-\Gamma_u}\, d\Gamma_u \,\Big|\, \mathcal{F}_t\Big). \tag{2.6}$$

(iii) For any $\mathbb{F}$-predictable process process A with finite variation over $[0, T]$, we have:

$$\mathbb{E}_\mathbb{Q}\Big(\int_{t\wedge\tau}^{\theta\wedge\tau} dA_u \,\Big|\, \mathcal{G}_t\Big) = 1\!\!1_{\{t<\tau\}}e^{\Gamma_t}\, \mathbb{E}_\mathbb{Q}\Big(\int_t^\theta e^{-\Gamma_u}dA_u \,\Big|\, \mathcal{F}_t\Big)\ . \tag{2.7}$$

Proof:

(ii) If suffices to prove 2.6 for an elementary predictable process of the form $Z_s = 1\!\!1_{]u,v]}(s)A_u$ where $A_u \in \mathcal{F}_u$. For such a process, the result follows easily from part (i).

(iii) We have that

$$\int_{t\wedge\tau}^{\theta\wedge\tau} dQ_u = 1\!\!1_{\{t<\tau\}}\int_{t\wedge\tau}^{\theta\wedge\tau} dQ_u = 1\!\!1_{\{\theta<\tau\}}\int_t^\theta dQ_u + 1\!\!1_{\{t<\tau\leq\theta\}}\int_t^\tau dQ_u$$

where Q is $\mathbb{F}$-predictable. Using parts (i) and (ii), we obtain

$$\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\tau<\tau\}}\int_t^\theta dQ_u \,\Big|\, \mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau\}}\mathbb{E}_{\mathbb{Q}}\Big(e^{\Gamma_t-\Gamma_\theta}\int_t^\theta dQ_u \,\Big|\, \mathcal{F}_t\Big)$$

and

$$\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau\le\theta\}}\int_t^\tau dQ_u \,\Big|\, \mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau\}}\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\theta \Big(\int_t^s dQ_u\Big)e^{\Gamma_t-\Gamma_s}d\Gamma_s \,\Big|\, \mathcal{F}_t\Big),$$

where, by Fubini theorem,

$$\int_t^\theta \Big(\int_t^s dQ_u\Big)e^{\Gamma_t-\Gamma_s}\,d\Gamma_s = \int_t^\theta\int_t^s dQ_u e^{\Gamma_t-\Gamma_s}\,d\Gamma_s = \int_t^\theta e^{\Gamma_t-\Gamma_u}\,dQ_u - e^{\Gamma_t-\Gamma_\theta}\int_t^\theta \beta_u\,dQ_u.$$

Hence

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{t\wedge\tau}^{\theta\wedge\tau} dQ_u \,\Big|\, \mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau\}}\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\theta e^{\Gamma_t-\Gamma_u}\,dQ_u \,\Big|\, \mathcal{F}_t\Big),$$

and thus

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_{t\wedge\tau}^{\theta\wedge\tau} dQ_u \,\Big|\, \mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau\}}e^{\Gamma_t}\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\theta e^{-\Gamma_u}\,dQ_u \,\Big|\, \mathcal{F}_t\Big), \tag{2.8}$$

as expected.

2.2.3 Multiplicative decomposition

Exercise 2.2.6 Prove that the supermartingale $G = Z - A$ admits a multiplicative decomposition $G_t = C_t N_t$ where N is a martingale and C a decreasing process.

Proof: The supermartingale $G = Z - A$ admits a multiplicative decomposition $G_t = C_t N_t$ where N is a martingale and C a decreasing process satisfying

$$dN_t = -\frac{1}{C_t}dZ_t,\; dC_t = -C_t\frac{1}{G_t}dA_t\,.$$

Hence

$$C_t = \exp - \int_0^t \frac{1}{G_s}dA_s = \exp - \Lambda_t$$

and

$$e^{\Gamma_t}\mathbb{E}(e^{-\Gamma_T}X|\mathcal{F}_t) = \widehat{\mathbb{E}}(X\frac{C_T}{C_t}|\mathcal{F}_t) = \widehat{\mathbb{E}}(X\exp(-\int_t^T \lambda_s ds)|\mathcal{F}_t)$$

where

$$d\widehat{Q} = L_t d\mathbb{P}, \quad dL_t = -\exp(\Lambda_t)\,L_t\,dZ_t\,.$$

Exercise 2.2.7 Assume that $G_t = N_t e^{-\Lambda_t}$ where N is a continuous martingale. Prove that $H_t - \Lambda_{t\wedge\tau}$ is a $\mathbb{G}$-martingale.

Proof: The additive decomposition of G is

$$dG_t = e^{-\Lambda_t}dN_t - N_t e^{-\Lambda_t}d\Lambda_t$$

and the result follows

2.2.4 Immersion

Exercise 2.2.8 Let $\tau_1 < \tau_2$. Prove that $\mathbb{F}$ is immersed in $\mathbb{G}$ if and only if $\mathbb{F}$ is immersed in $\mathbb{F} \vee \mathbb{H}^1$ and $\mathbb{F} \vee \mathbb{H}^1$ immersed in $\mathbb{G}$.‘

Solution: The only fact to check is that if $\mathbb{F}$ is immersed in $\mathbb{G}$, then $\mathbb{F} \vee \mathbb{H}^1$ is immersed in $\mathbb{G}$, or that

$$\mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}_t^1) = \mathbb{P}(\tau_2 > t | \mathcal{F}_\infty \vee \mathcal{H}_\infty^1)$$

This is equivalent to, for any h, and any $A_\infty \in \mathcal{F}_\infty$

$$\mathbb{E}(A_\infty h(\tau_1) 1\!\!1_{\tau_2 > t}) = \mathbb{E}(A_\infty h(\tau_1) \mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}_t^1))$$

We spilt this equality in two parts. The first equality

$$\mathbb{E}(A_\infty h(\tau_1) 1\!\!1_{\tau_1 > t} 1\!\!1_{\tau_2 > t}) = \mathbb{E}(A_\infty h(\tau_1) 1\!\!1_{\tau_1 > t} \mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}_t^1))$$

is obvious since $1\!\!1_{\tau_1 > t} 1\!\!1_{\tau_2 > t} = 1\!\!1_{\tau_1 > t}$ and $1\!\!1_{\tau_1 > t} \mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}_t^1) = 1\!\!1_{\tau_1 > t}$. Now

$$\mathbb{E}(A_\infty h(\tau_1) 1\!\!1_{t \geq \tau_1} 1\!\!1_{\tau_2 > t}) = \mathbb{E}(\mathbb{E}(A_\infty | \mathcal{G}_t) h(\tau_1) 1\!\!1_{\tau_1 > t} \mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}_t^1))$$

Since $\mathbb{F}$ is immersed in $\mathbb{G}$, one has $\mathbb{E}(A_\infty | \mathcal{G}_t) = \mathbb{E}(A_\infty | \mathcal{F}_t)$ and it follows that $\mathbb{E}(A_\infty | \mathcal{G}_t) = \mathbb{E}(A_\infty | \mathcal{F}_t \vee \mathcal{H}_t^1)$, therefore

$$\begin{aligned} \mathbb{E}(A_\infty h(\tau_1) 1\!\!1_{t \geq \tau_1} 1\!\!1_{\tau_2 > t}) &= \mathbb{E}(\mathbb{E}(A_\infty | \mathcal{F}_t \vee \mathcal{H}_t^1) \mathbb{P}(h(\tau_1) 1\!\!1_{\tau_1 > t} 1\!\!1_{\tau_2 > t} | \mathcal{F}_t \vee \mathcal{H}_t^1)) \\ &= \mathbb{E}(A_\infty \mathbb{P}(h(\tau_1) 1\!\!1_{\tau_1 > t} 1\!\!1_{\tau_2 > t} | \mathcal{F}_t \vee \mathcal{H}_t^1)) \end{aligned}$$

Exercise 2.2.9 Prove that if λ is deterministic and $H_t - \int_0^t \lambda_u (1 - H_u)$ is a $\mathbb{G}$ martingale, then $\mathbb{P}(\tau > t) = e^{-\Lambda_t}$

Hint: $E(H_t) = \int_0^t \lambda(u)(1 - \mathbb{E}(H_u))$ leads to an ODE

Exercise 2.2.10 Prove that if $\mathbb{F}$ is immersed in $\mathbb{G}$ and $H_t - \int_0^t \lambda_u (1 - H_u)$ is a $\mathbb{G}$ martingale, then $\mathbb{P}(\tau > t | \mathcal{F}_t) = e^{-\Lambda_t}$

Hint: use the multiplicative decomposition of the supermartingale

2.2.5 Pricing

We work in a hazard process model with reference filtration $\mathbb{F}$. The pricing probability is denoted $\mathbb{P}$. The filtered probability space is $(\Omega, \mathbf{F}, \mathbb{P})$, τ is a strictly positive r.v., $H_t = 1\!\!1_{\tau \leq t}$, $\mathbf{H} = (\mathcal{H}_t, t \geq 0)$ is the natural filtration of H, (taken càd and complete), $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, and $G_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$. There exists λ such that $M_t := H_t - \int_0^t (1 - H_s) \lambda_s ds$ is a $\mathbb{G}$-martingale. The Doob-Meyer decomposition of G is denoted $G_t = Z_t - A_t$ where Z is an $\mathbb{F}$-martingale and A an $\mathbb{F}$-predictable non-decreasing process.

Exercise 2.2.11 Assume that λ be deterministic and that immersion property holds.

1. Prove that τ is independent of $\mathbf{F}$.
2. Let S an $\mathbf{F}$-adapted process which represents the price of some asset and assume that the interest rate $(r(s), s \geq 0)$ is deterministic. We note $\beta_t = \exp - \int_0^t r(s) ds$.
   (a) Compute the value V_t of an asset with payoff $\Phi = \varphi(S_T) 1\!\!1_{T < \tau}$.
   (b) Show that there is a relation between V_t and Φ_t, the price of an asset with payoff $\varphi(S_T)$.

(c) Compute the value $D(t,T)$ of the price of a defaultable zero-coupon (with null recovery). Determine the dynamics of $D(t,T)$?

(d) We recall that a self-financing portfolio with payoff ξ is a triple of **G**-adapted processes, π^1, π^2, π^3 such that, if $V_t = \pi_t^1 D(t,T) + \pi_t^2 S_t + \pi_t^3 S_t^0$, then

$$\begin{aligned} dV_t &= \pi_t^1 dD(t,T) + \pi_t^2 dS_t + \pi_t^3 S_t^0 r(t)dt \\ \xi &= \pi_T^1 D(T,T) + \pi_T^2 S_T + \pi_T^3 S_T^0 \end{aligned}$$

Prove that there exists a self-financing portfolio with payoff $\varphi(S_T)\mathbb{1}_{T<\tau}$. Compute π^1.

Exercise 2.2.12 Let Θ be a non-negative r.v. with cumulative distribution function F, independant of $\mathcal{F}_\infty$. Let $(\lambda_t, t \geq 0)$ be an **F**-adapted process, taking non-negative values and $\Lambda_t = \int_0^t \lambda_s ds$. We define

$$\tau = \inf\{t \,:\, \Lambda_t \geq \Theta\}\,.$$

We assume that the interest rate is null.

1. Check that τ is a **G**-stopping time.
2. Compute G_t in terms of Λ and F. Give the Doob-Meyer decomposition of G..
3. Let X be an $\mathcal{F}_T$-measurable, integrable r.v.. Compute $\mathbb{E}(X\mathbb{1}_{T<\tau}|\mathcal{G}_t)$ for $t < T$.
4. Prove that the process L defined as $L_t = (1-H_t)(1-F(\Lambda_t))^{-1}$ is a **G**-martingale.
5. Find the process γ such that the process $M_t = H_t - \int_0^{t\wedge\tau} \gamma_s ds$ is a **G**-martingale.
6. Let Z be an **F**-adapted process. A contingent claim pays Z_τ at time T, in the case $\tau \leq T$ (no payment if $\tau > T$. Compute the price at time t of this contingent claim and give the dynamics of this price
7. let $D(t,T) = \mathbb{E}(\mathbb{1}_{T<\tau}|\mathcal{G}_t)$ be the price at time t of a defaultable zero-coupon with maturity T. We assume that the folowwing assets are traded

   - an asset with price $Y_t^0 = 1$ (i.e., the savings account, with null interest rate),
   - an asset with price following the Black-Scholes dynamics

     $$dY_t = Y_t \sigma dW_t$$

     where W is a Brownoian motion
   - A DZC with price $D(t,T)$

   (a) Show that

   $$dD(t,T) = \mu_t dm_t + \varphi_t dM_t$$

   where m is a martingale that can be written as a conditional expectation and where μ and φ are given in a closed form. We shall assume that $dm_t = m_t \nu_t dW_t$.

   (b) Write the EDP evaluation formula for the price of an asset paying $\Phi(Y_T, H_T)$. What is the hedging portfolio?

Exercise 2.2.13 We assume that the interest rate is constant.

We assume that G is continuous and valued in $]0,1[$ and we define $\Gamma_t := -\ln G_t$. We assume that the process A in the Doob-Meyer decomposition of G is on the form $A_t = \int_0^t a_s ds$. We recall that

$$M_t := H_t - \int_0^{t\wedge\tau} \frac{a_s}{G_s} ds = H_t - \int_0^{t\wedge\tau} \lambda_s ds = H_t - \Lambda_{t\wedge\tau}$$

(where $\lambda_s = \frac{a_s}{G_s}$, $\Lambda_t = \int_0^t \lambda_s ds$) is a **G**-martingale. We recall that for any $\mathbb{F}$-predictable process h

$$\mathbb{E}(h_\tau \mathbb{1}_{\tau<T}|\mathcal{G}_t) = h_\tau \mathbb{1}_{\{\tau<t\}} + \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}\left(\int_t^T h_u dF_u | \mathcal{F}_t\right) .$$

1. We assume that G is non-increasing.
   (a) Prove that $L_t := (1 - H_t)(G_t)^{-1}$ is a martingale and that, for any $a > 0$, the process

   $$(1 + a)^{H_t} \exp\left(-a \int_0^t (1 - H_s)\lambda_s ds\right)$$

   is a martingale. Prove that $\mathbb{E}\big[(1 + a)\, e^{-a\Lambda_\tau}\big] = 1$. Compute the law of Λ_τ.
   (b) Let $\widetilde{V}$ and Z be $\mathbb{F}$-predictable processes. Prove that

   $$V_t := \widetilde{V}_t \mathbb{1}_{\{t<\tau\}} + Z_\tau \mathbb{1}_{\{\tau\le t\}}$$

   is a **G**-martingale if and only if

   $$\widetilde{V}_t e^{-\Gamma_t} + \int_0^t Z_u e^{-\Gamma_u} d\Gamma_u$$

   is an $\mathbb{F}$-martingale
2. Assume that $\tau := \inf\{t : C_t < U\}$ where U is a r.v. with uniform law on $[0, 1]$, independent of $\mathcal{F}_\infty$ and C an $\mathbb{F}$-adapted process, non-increasing of the form $C_t = \exp\left(-\int_0^t c_s ds\right)$ such that $C_0 = 1$ and $C_\infty = 0$.
   (a) Compute G_t in terms of C.
   (b) Compute the intensity of τ.
   (c) Let Z be an $\mathbb{F}$-predictable process and X an $\mathcal{F}_T$-mesurable i ntegrable r.v.. Compute the price at time t of an asset which delivers Z_τ at time τ if$\tau \le T$, and X at time T if $T < \tau$. Give the dynamics of this price.
   (d) On note $D(t, T) = \mathbb{E}(\mathbb{1}_{T<\tau}|\mathcal{G}_t)$ le prix à la date t d'un zéro coupon soumis au risque de défaut (DZC) de maturité T. On suppose que le marché comporte
   - un actif de prix $Y_t^0 = 1$ (le savings account, de taux r nul),
   - un actif de dynamique Black Scholes dont le prix suit, sous la probabilité risque neutre, la dynamique

     $$dY_t = Y_t \sigma dW_t$$

     où W est un mouvement Brownien; la filtration $\mathbb{F}$ est la filtration naturelle du mouvement Brownien W.
   - le DZC de prix $D(t, T)$

   i. Montrer que

   $$dD(t, T) = \mu_t dm_t + \varphi_t dM_t$$

   où m est une martingale que l'on caractérisera sous forme d'une espérance conditionnelle -sans expliciter le dm_t- et où μ et φ seront explicités. On supposera que $dm_t = m_t \nu_t dW_t$.

   ii. Ecrire l'EDP d'évaluation d'un produit de payoff $\Phi(Y_T, H_T)$. Quel est le portefeuille de couverture associé?

Exercise 2.2.14 Assume that (H) hypothesis holds and that the $\mathbb{F}$ martingales are continuous. Let M be a $\mathbb{F}$ martingale Lat a and b be $\mathbb{G}$ adapted processes such that $\int_0^t a_s dM_s$ and $\int_0^t b_s dM_s^d$ are martingales Let $Z_t = \int_0^t a_s dM_s + \int_0^t b_s dM_s^d$. Then $E(Z_t|\mathcal{F}_t) = \int_0^t E(a_s|\mathcal{F}_s) dM_s$

Exercise 2.2.15 Assume that H hypothesis holds and that $\mathcal{F}$ is continuous (or at least that F does not jump at time τ)

The process $M_t = H_t - \Gamma_{t\wedge\tau}$ is a martingale For any $\alpha \in I\!R$, the process $Z_t = \exp(\alpha H_t - (e^\alpha - 1)\Gamma_{t\wedge\tau})$ is a martingale Indeed

$$\begin{aligned} dZ_t &= e^{-(e^\alpha-1)\Gamma_{t\wedge\tau}} d(e^{\alpha H_t} - (e^\alpha - 1)Z_{t-}(1-H_{t-})d\Gamma_t \\ &= Z_{t-}(e^{\alpha(H_t - H_{t-})} dH_t - (e^\alpha - 1)Z_{t-}(1-H_{t-})d\Gamma_t \\ &= Z_{t-}(e^\alpha - 1)dH_t - (e^\alpha - 1)Z_{t-}(1-H_{t-})d\Gamma_t \end{aligned}$$

2.3 Multidefaults

2.3.1 Jarrow and Yu model

Let λ, α, β be given non negative numbers. Construct $\tau_i, i = 1, 2$ such that

$$M^1_t := H^1_t - \int_0^{t\wedge\tau_1} \lambda ds$$

is an $\mathbb{H}^1$ martingale and

$$M^2_t := H^2_t - \int_0^{t\wedge\tau_2} (\alpha + \beta H^1_s) ds$$

is an $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$ martingale. Prove that M^1 is an $\mathbb{H}$ martingale. Let L be the martingale

$$dL = L_{t-}\gamma H^2_{t-} dM^1_t$$

and set

$$d\mathbb{Q}|_{\mathcal{H}_t} = L_t d\mathbb{P}|_{\mathcal{H}_t}$$

Find the intensity of τ_1 under $\mathbb{Q}$. Compute the joint law of τ_1, τ_2 under $\mathbb{Q}$. Are various immersion properties satisfied?

2.3.2 Norros Lemma

Let τ_i be two default times, $\mathbb{F}$ a reference filtration. We introduce $(\mathcal{G}^i_t)_{t\geq 0}$ by $\mathcal{G}^i_t = \mathcal{F}_t \vee \mathcal{H}^i_t$, and $(\mathcal{G}_t)_{t\geq 0}$ by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}^1_t \vee \mathcal{H}^2_t$, for $t \geq 0$. It is further assumed that all the considered filtrations are right-continuous and completed by all the sets of P-measure zero. For any $i = 1, 2$, let $G^i = (G^i_t)_{t\geq 0}$ be the *conditional survival probability* process of the default time τ_i, defined by $G^i_t = P[\tau_i > t \,|\, \mathcal{F}_t]$, for all $t \geq 0$. There exists increasing predictable processes A^i such that $G^i + A^i$ are $\mathbb{F}$-martingales. Let us define the process $M^i = (M^i_t)_{t\geq 0}$ by:

$$M^i_t = H^i_t - \Lambda^i_{\tau_i \wedge t} \tag{2.9}$$

where the process $\Lambda^i = (\Lambda^i_t)_{t\geq 0}$ is given by:

$$\Lambda^i_t = \int_0^t \frac{dA^i_s}{G^i_s} \tag{2.10}$$

for all $t \geq 0$. The process M^i is a $(\mathcal{G}^i_t)_{t\geq 0}$-martingale and Λ^i is continuous.

Let the processes $G^i = (G^i_t)_{t\geq 0}$, $i = 1, 2$, be continuous and such that $G^i_0 = 1$, and assume that $P[\tau_1 = \tau_2] = 0$ is satisfied. Prove that

(i) the variable $\Lambda^i_{\tau_i}$, defined in (2.10), has standard exponential law (with parameter 1);

(ii) if $(\mathcal{F}_t)_{t\geq 0}$ is immersed in $(\mathcal{G}^i_t)_{t\geq 0}$, then the variable $\Lambda^i_{\tau_i}$ is independent of $\mathcal{F}_\infty$;

(iii) if $(\mathcal{G}^i_t)_{t\geq 0}$, $i=1,2$ are immersed in $(\mathcal{G}_t)_{t\geq 0}$, then the variables $\Lambda^i_{\tau_i}$, $i=1,2$, are independent;

(iv) if $(\mathcal{F}_t)_{t\geq 0}$ is immersed in $(\mathcal{G}^i_t)_{t\geq 0}$ and

$$P[\tau_i > t \,|\, \mathcal{F}_t] = P[\tau_i > t \,|\, \mathcal{G}^{3-i}_t] \tag{2.11}$$

hold for all $t \geq 0$, then the variables $\Lambda^i_{\tau_i}$, $i=1,2$, are conditionally independent with respect to $\mathcal{F}_\infty$.

2.3.3 Examples

Exercise 2.3.1 Let $\tau_1 < \tau_2$ be two random times and $\mathbb{F}$ a reference filtration. Prove that $\mathbb{F}$ is immersed in $\mathbb{F}\vee\mathbb{H}^1\vee\mathbb{H}^2$ if and only if $\mathbb{F}$ is immersed in $\mathbb{F}\vee\mathbb{H}^1$ and $\mathbb{F}\vee\mathbb{H}^1$ is immersed in $\mathbb{F}\vee\mathbb{H}^1\vee\mathbb{H}^2$.

Exercise 2.3.2 Let $\hat\tau_i$ be independent random times such that $\mathbb{P}(\hat\tau_i \geq t) = e^{-\hat q_i t}$ and set $\tau_i = \hat\tau_i \wedge \hat\tau_3$ for $i = 1,2$. Show that $H^1_t = 1\!\!1_{\tau_1\leq t}$ is a Markov process in its natural filtration and in $\mathbb{H} = \hat{\mathbb{H}}^1 \vee \hat{\mathbb{H}}^2 \vee \hat{\mathbb{H}}^3$
Setting $q_1 = \hat q_1 + \hat q_3$, prove that $H^1_t - \int_0^t (1-H^1_s) q_1 ds$ is a martingale in $\mathbb{H}^1$ and in $\mathbb{H}$.
Prove that (H^1, H^2) is a $\mathbb{H}$-Markov process

Exercise 2.3.3 Let T_1, T_2 the first and the second jump of a standard Poisson process, with intensity equal to 1, and $\mathbb{F}^N$ the natural filtration of the Poisson process.

1. Prove that one can write
$$\begin{aligned} T_1 &= \inf\{t \,:\, t \geq \Theta_1\} \\ T_2 &= \inf\{t \,:\, t \geq \Theta_1 + \Theta_2\} \end{aligned}$$
where Θ_i are independents r.v. with exponential law.
2. Prove that $H^1_t - \int_0^t (1-H^1_s) ds$ is a $\mathbb{H}^1$ martingale and a $\mathbb{F}^N$ martingale.
3. Prove that the cumulative function of $\Theta_1 + \Theta_2$ is $1 - e^{-x}(1+x)$.
4. Prove that the intensity λ_2 of T_2 in the filtration $\mathbb{H}^2$ (i.e. the process λ_2 such that $H^2_t - \int_0^t (1-H^2_s)\lambda^2_s ds$ is a $\mathbb{H}^2$ martingale) is $\lambda^2_s = \frac{s}{1+s}$.
5. Prove that
$$T_2 = \inf\{t \,:\, \int_0^t \gamma_s ds \geq \Theta\}$$
where $\gamma_s = 1\!\!1_{s>T_1}$ and Θ is an exponential law. Prove that $H^2_t - \int_0^t (1-H^2_s)\gamma_s ds$ is a $\mathbb{F}^N$ martingale.

Exercise 2.3.4

We assume that
$$\tau_i = \inf\{t \,:\, \Lambda^{(i)}_t \geq \Theta_i\}$$
where Θ_i are unit exponential r.vs, independent of $\mathcal{F}_\infty$, and $\Lambda^{(i)}_t = \int_0^t \lambda^{(i)}_s ds$ where the processes $(\lambda^{(i)}_t, t\geq 0)$ are non-negative and $\mathbf{F}$-adapted. A first to default claim pays some amount at time $\tau = \tau_1 \wedge \tau_2$.

1. We assume that Θ_i are independent. Let $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}^1_t \vee \mathcal{H}^2_t$ where $\mathbf{H}^i$ is the natural filtration of $H^i_t = 1\!\!1_{\tau_i\leq t}$. Let Z be an $\mathbf{F}$-adaped process. Compute $E(Z_\tau 1\!\!1_{\{\tau<T\}}|\mathcal{G}_t)$.

2. Assumpe that the joint law of Θ_i is known. Compute $E(Z_\tau \mathbb{1}_{\{\tau<T\}})$ in the case where $\lambda^{(i)}$ are deterministic and in the general case where $\lambda^{(i)}$ are processes.

3. let $D_i(t,T) = \mathbb{E}(\mathbb{1}_{T<\tau_i}|\mathcal{G}_t)$ be the price at time t of a defaultable zero-coupon bond with maturity T, on the default time i. Assuming that the r.vs Θ_i are independant, compute $D_i(t,T)$.

2.4 Density process

The random time τ admits a density process if there exists a family of non-negative processes $\alpha_t(u)$ such that

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \int_u^\infty \alpha_t(v)\eta(dv)$$

where η is the law of τ

Exercise 2.4.1 Compute the Doob-Meyer decomposition i=of the associated Azéma supermartingale. intensity of τ

Exercise 2.4.2 It is known that if X is an $\mathbb{F}$-martingale, then

$$X_t = \widehat{\mu}_t - \int_0^t \frac{d\left\langle X, \alpha^\theta \right\rangle_u}{\alpha^\theta_{u-}}\Bigg|_{\theta=\tau}$$

where $\widehat{\mu}$ is an $\mathbb{F}\vee\sigma(\tau)$-martingale. Prove that

$$X_t = \mu_t + \int_0^{t\wedge\tau} \frac{d\langle X, G\rangle_u}{G_u} - \int_{t\wedge\tau}^t \frac{d\left\langle X, \alpha^\theta \right\rangle_u}{\alpha^\theta_{u-}}\Bigg|_{\theta=\tau} \qquad (**)$$

where μ is an $\mathbb{G}$-martingale

Exercise 2.4.3 We assume that α_∞ exists. Let $\mathbb{Q}$ defined as

$$d\mathbb{Q} = \mathbb{E}_{\mathbb{P}}(1/\alpha^\tau_\infty|\mathcal{G}_t)/\mathbb{E}_{\mathbb{P}}(1/\alpha^\tau_\infty)d\mathbb{P}$$

Prove that, under $\mathbb{Q}$, τ is independent of $\mathcal{F}_\infty$