VALUATION OF BASKET CREDIT DERIVATIVES IN THE CREDIT MIGRATIONS ENVIRONMENT

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Contents

1 Introduction

The goal of this work is to present some methods and results related to the valuation and hedging of basket credit derivatives, as well as of portfolios of credits/loans, in the context of several possible credit ratings of underlying credit instruments. Thus, we are concerned with modeling dependent credit migrations and, in particular, with modeling dependent defaults. On the mathematical level, we are concerned with modeling dependence between random times and with evaluation of functionals of (dependent) random times; more generally, we are concerned with modeling dependence between random processes and with evaluation of functionals of (dependent) random processes. Modeling of dependent defaults and credit migrations was considered by several authors, who proposed several alternative approaches to this important issue. Since the detailed analysis of these methods is beyond the scope of this text, let us only mention that they can be roughly classified as follows:

- modeling correlated defaults in a static framework using copulae (Hull and White (2001), Gregory and Laurent (2004)),
- modeling dependent defaults in a "dynamic" framework using copulae (Schönbucher and Schubert (2001), Laurent and Gregory (2003), Giesecke (2004)),
- dynamic modelling of credit migrations and dependent defaults via proxies (Douady and Jeanblanc (2002), Chen and Filipovic (2003), Albanese et al. (2003), Albanese and Chen (2004a, 2004b)),
- factor approach (Jarrow and Yu (2001), Yu (2003), Frey and Backhaus (2004), Bielecki and Rutkowski (2003)),
- modeling dependent defaults using mixture models (Frey and McNeil (2003), Schmock and Seiler (2002)),
- modeling of the joint dynamics of credit ratings by a voter process (Giesecke and Weber (2002)),
- modeling dependent defaults by a dynamic approximation (Davis and Esparragoza (2004)).

The classification above is rather arbitrary and by no means exhaustive. In the next section, we shall briefly comment on some of the above-mentioned approaches. In this work, we propose a fairly general Markovian model that, in principle, nests several models previously studied in the literature. In particular, this model covers jump-diffusion dynamics, as well as some classes of Lévy processes. On the other hand, our model allows for incorporating several credit names, and thus it is suitable when dealing with valuation of basket credit products (such as, basket credit default swaps or collateralized debt obligations) in the multiple credit ratings environment. Another practically important feature of the model put forward in this paper is that it refers to market observables only. In contrast to most other papers in this field, we carefully analyze the issue of preservation of the Markovian structure of the model under equivalent changes of probability measures.

1.1 Conditional Expectations Associated with Credit Derivatives

We present here a few comments on evaluation of functionals of random times related to financial applications, so to put into perspective the approach that we present in this paper. In order to smoothly present the main ideas we shall keep technical details to a minimum.

Suppose that the underlying probability space is $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with some filtration $\mathbb{G}$ (see Section 2 for details). Let τ_l, $l = 1, 2, \ldots, L$ be a family of finite and strictly positive random times defined on this space. Let also real-valued random variables X and $\widetilde{X}$, as well as real-valued processes A (of finite variation) and Z be given. Next, consider an $\mathbb{R}^k_+$-valued random variable $\zeta := g(\tau_1, \tau_2, \ldots, \tau_L)$ where $g : \mathbb{R}^L_+ \to \mathbb{R}^k_+$ is some measurable function. In the context of valuation of credit derivatives, it is of interest to evaluate conditional expectations of the form

$$\mathbb{E}_{\mathbb{P}^\beta}\Big(\int_{]t,T]} \beta_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t\Big), \tag{1}$$

for some numeraire process β, where the *dividend process* D is given by the following generic formula:

$$D_t = (X\alpha_1(\zeta) + \widetilde{X}\alpha_2(\zeta))\mathbb{1}_{\{t\geq T\}} + \int_{]0,t]} \alpha_3(u;\zeta)\, dA_u + \int_{]0,t]} Z_u\, d\alpha_4(u;\zeta),$$

where the specification of α_is depends on a particular application. The probability measure $\mathbb{P}^\beta$, equivalent to $\mathbb{P}$, is the martingale measure associated with a numeraire β (see Section 4.2 below). We shall now illustrate this general set-up with four examples. In each case, it is easy to identify the processes A, Z as well as the α_is.

Example 1.1 Defaultable bond. We set $L = 1$ and $\tau = \tau_1$, and we interpret τ as a time of default of an issuer of a corporate bond (we set here $\zeta = \tau = \tau_1$). The face value of the bond (the promised payment) is a constant amount X that is paid to bondholder at maturity T, provided that there was no default by the time T. In addition, a coupon is paid continuously at the instantaneous rate c_t up to default time or bond's maturity, whichever comes first. In case default occurs by the time T, a recovery payment is paid, either as the lump sum $\widetilde{X}$ at bond's maturity, or as a time-dependent rebate Z_τ at the default time. In the former case, the dividend process of the bond equals

$$D_t = (X(1-H_T) + \widetilde{X}H_T)\mathbb{1}_{\{t\geq T\}} + \int_{]0,t]} (1-H_u)c_u\, du,$$

where $H_t = \mathbb{1}_{\{\tau\leq t\}}$, and in the latter case, we have that

$$D_t = X(1-H_T)\mathbb{1}_{\{t\geq T\}} + \int_{]0,t]} (1-H_u)c_u\, du + \int_{]0,t]} Z_u\, dH_u.$$

Example 1.2 Step-up corporate bonds. These are corporate coupon bonds for which the coupon payment depends on the issuer's credit quality: the coupon payment increases when the credit quality of the issuer declines. In practice, for such bonds, credit quality is reflected in credit ratings assigned to the issuer by at least one credit ratings agency (such as Moody's-KMV, Fitch Ratings or Standard & Poor's). Let X_t stand for some indicator of credit quality at time t. Assume that $t_i,\ i = 1, 2, \ldots, n$ are coupon payment dates and let $c_i = c(X_{t_{i-1}})$ be the coupons ($t_0 = 0$). The dividend process associated with the step-up bond equals

$$D_t = X(1-H_T)\mathbb{1}_{\{t\geq T\}} + \int_{]0,t]} (1-H_u)\, dA_u + \text{ possible recovery payment}$$

where τ, X and H are as in the previous example, and $A_t = \sum_{t_i\leq t} c_i$.

Example 1.3 Default payment leg of a CDO tranche. We consider a portfolio of L credit names. For each $l = 1, 2, \ldots, L$, the nominal payment is denoted by N_l, the corresponding default time by τ_l and the associated *loss given default* by $M_l = (1-\delta_l)N_l$, where δ_l is the *recovery rate* for the l^{th} credit name. We set $H_t^l = \mathbb{1}_{\{\tau_l\leq t\}}$ for every $l = 1, 2, \ldots, L$, and $\zeta = (\tau_1, \tau_2, \ldots, \tau_L)$. Thus, the *cumulative loss process* equals

$$L_t(\zeta) = \sum_{l=1}^{L} M_l H_t^l.$$

Similarly as in Laurent and Gregory (2003), we consider a cumulative default payments process on the mezzanine tranche of the CDO:

$$M_t(\zeta) = (L_t(\zeta) - a)\mathbb{1}_{[a,b]}(L_t(\zeta)) + (b-a)\mathbb{1}_{]b,N]}(L_t(\zeta)),$$

where a, b are some thresholds such that $0 \leq a \leq b \leq N := \sum_{l=1}^{L} N_l$. If we assume that $M_0 = 0$ then the dividend process corresponding to the default payment leg on the mezzanine tranche of the CDO is $D_t = \int_{]0,t]} dM_u(\zeta)$.

Example 1.4 Default payment leg of a k^{th}-to-default CDS. Consider a portfolio of L reference defaultable bonds. For each defaultable bond, the notional amount is taken to be deterministic and denoted as N_l; the corresponding recovery rate δ_l is also deterministic. We suppose that the maturities of the bonds are U_l and the maturity of the swap is $T < \min\{U_1, U_2, \dots, U_L\}$. Here, we set $\zeta = (\tau_1, \tau_2, \dots, \tau_L, \tau^{(k)})$, where $\tau^{(k)}$ is the k^{th} order statistics of the collection $\{\tau_1, \tau_2, \dots, \tau_L\}$.

A special case of the k^{th}-to-default-swap is the case when the protection buyer is protected against only the last default (i.e. the k^{th} default). In this case, the dividend process associated with the default payment leg is

$$D_t = (1 - \delta_{\iota^{(k)}}) N_{\iota^{(k)}} \mathbb{1}_{\{\tau^{(k)} \le T\}} H_t^{(k)},$$

where $H_t^{(k)} = \mathbb{1}_{\{\tau^{(k)} \le t\}}$ and $\iota^{(k)}$ stands for the identity of the k^{th} defaulting credit name. This can be also written as $D_t = \int_{]0,t]} dN_u(\zeta)$, where

$$N_t(\zeta) = \int_{]0,t]} \sum_{l=1}^{L} (1 - \delta_l) N_l \mathbb{1}_{\tau_l}(u) \, dH_u^{(k)}.$$

1.2 Existing Methods of Modelling Dependent Defaults

It is apparent that in order to evaluate the expectation in (1) one needs to know, among other things, the conditional distribution of ζ given $\mathcal{G}_t$. This, in general, requires the knowledge of conditional dependence structure between random times $\tau_l, \tau_2, \dots, \tau_L$, so that it is important to be able to appropriately model dependence between these random times. This is not an easy task, in general. Typically, the methodologies proposed in the literature so far handle well the task of evaluating the conditional expectation in (1) for $\zeta = \tau^{(1)} = \min\{\tau_1, \tau_2, \dots, \tau_L\}$, which, in practical applications, corresponds to *first-to-default* or *first-to-change* type credit derivatives. However, they suffer from more or less serious limitations when it comes to credit derivatives involving subsequent defaults or changes in credit quality, and not just the first default or the first change, unless restrictive assumptions are made, such as conditional independence between the random times in question. In consequence, the existing methodologies would not handle well computation of expectation in (1) with process D as in Examples 1.3 and 1.4, unless restrictive assumptions are made about the model. Likewise, the existing methodologies can't typically handle modeling dependence between credit migrations, so that they can't cope with basket derivatives whose payoffs explicitly depend on changes in credit ratings of the reference names.

Arguably, the best known and the most widespread among practitioners is the copula approach (cf. Li (2000), Schubert and Schönbucher (2001), and Laurent and Gregory (2003), for example). Although there are various versions of this approach, the unifying idea is to use a copula function so to model dependence between some auxiliary random variables, say $\upsilon_1, \upsilon_2, \dots, \upsilon_L$, which are supposed to be related in some way to $\tau_l, \tau_2, \dots, \tau_L$, and then to infer the dependence between the latter random variables from the dependence between the former.

It appears that the major deficiency of the copula approach, as it stands now, is its inability to compute certain important conditional distributions. Let us illustrate this point by means of a simple example. Suppose that $L = 2$ and consider the conditional probability $\mathbb{P}(\tau_2 > t + s \,|\, \mathcal{G}_t)$. Using the copula approach, one can typically compute the above probability (in terms of partial derivatives of the underlying copula function) on the set $\{\tau_1 = t_1\}$ for $t_1 \le t$, but not on the set $\{\tau_1 \le t_1\}$. This means, in particular, that the copula approach is not "Markovian", although this statement is rather vague without further qualifications. In addition, the copula approach, as it stands now, can't be applied to modeling dependence between changes in credit ratings, so that it can't be used in situations involving, for instance, baskets of corporate step-up bonds (cf. Example 1.2). In fact, this approach can't be applied to valuation and hedging of basket derivatives if one wants to account explicitly on credit ratings of the names in the basket. Modeling dependence between changes in credit ratings indeed requires modeling dependence between stochastic processes.

Another methodology that gained some popularity is a methodology of modeling dependence between random times in terms of some proxy processes, typically some Lévy processes (cf. Hull and White (2000), Albanese et al. (2002) and Chen and Filipović (2004), for example). The major problem with these approaches is that the proxy processes are latent processes whose states are unobservable virtual states. In addition, in this approach, when applied to modeling of credit quality, one can't model a succession of credit ratings, e.g., the joint evolution of the current and immediately preceding credit ratings (see Remark 2.1 (ii) below).

2 Notation and Preliminary Results

The underlying probability space containing all possible events over a finite time horizon is denoted by $(\Omega, \mathcal{G}, \mathbb{P})$, where $\mathbb{P}$ is a generic probability measure. Depending on the context, we shall consider various (mutually equivalent) probability measures on the space $(\Omega, \mathcal{G})$. The probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is endowed with a filtration $\mathbb{G} = \widetilde{\mathbb{H}} \vee \mathbb{F}$, where the filtration $\widetilde{\mathbb{H}}$ carries the information about evolution of credit events, such as changes in credit ratings of respective credit names, and where $\mathbb{F}$ is some *reference filtration*. We shall be more specific about both filtrations later on; at this point, we only postulate that they both satisfy the so-called "usual conditions".

The credit events of fundamental interest to us are changes in credit ratings, in particular – the default event. The evolution of credit ratings can be modeled in terms of an appropriate stochastic process defined on $(\Omega, \mathcal{G}, \mathbb{P})$. Various approaches to the choice of this process have been already proposed in the literature. We shall focus here on the Markov approach, in the sense explained in Section 2.1.1 below.

2.1 Credit Migrations

We consider L obligors (or credit names). We assume that current credit rating of the l^{th} reference entity can be classified to one of K_l different rating categories. We let $\mathcal{K}_l = \{1, 2, \ldots, K_l\}$ to denote the set of such categories. However, without a loss of generality, we assume that $\mathcal{K}_l = \mathcal{K} := \{1, 2, \ldots, K\}$ for every $l = 1, 2, \ldots, L$. By convention, the category K corresponds to default.

Let X^l, $l = 1, 2, \ldots, L$ be some processes on $(\Omega, \mathcal{G}, \mathbb{P})$ with values in $\mathcal{K}$. A process X^l represents the evolution of credit ratings of the l^{th} reference entity.

Let us write $X = (X^1, X^2, \ldots, X^L)$. The state space of X is $\mathcal{X} := \mathcal{K}^L$; the elements of $\mathcal{X}$ will be denoted by x. We call the process X the (joint) *migration process*. We assume that $X^l_0 \neq K$ for every $l = 1, 2, \ldots, L$, and we define the *default time* τ_l of the l^{th} reference entity by setting

$$\tau_l = \inf\{\, t > 0 \,:\, X^l_t = K\} \tag{2}$$

with the usual convention that $\inf \emptyset = \infty$. We assume that the default state K is absorbing, so that for each name the default event can only occur once. Put another way, for each l the process X^l is stopped at τ_l. Since we are considering a continuous time market then, without loss of practical generality, we assume that simultaneous defaults are not allowed. Specifically, the equality $\mathbb{P}(\tau_{l'} = \tau_l) = 0$ will hold for every $l' \neq l$ in our model.

Remarks. (i) In the special case when $K = 2$, only two categories are distinguished: pre-default ($j = 1$) and default ($j = 2$). We then have $X^l_t = H^l_t + 1$, where $H^l_t = \mathbb{1}_{\{\tau_l \le t\}}$.

(ii) Each credit rating j may include a "history" of transitions. For example, j may be a two-dimensional quantity, say $j = (j', j'')$, where j' represents the current credit rating, whereas j'' represents the immediately preceding credit rating.

2.1.1 Markovian Set-up

From now on, we set $\widetilde{\mathbb{H}} = \mathbb{F}^X$, so that the filtration $\widetilde{\mathbb{H}}$ is the natural filtration of the process X. Arguably, the most convenient set-up to work with is the one where the reference filtration $\mathbb{F}$ is the filtration $\mathbb{F}^Y$ generated by relevant (vector) factor process, say Y, and where the process (X, Y) is jointly Markov under $\mathbb{P}$ with respect to its natural filtration $\mathbb{G} = \mathbb{F}^X \vee \mathbb{F}^Y = \widetilde{\mathbb{H}} \vee \mathbb{F}$, so that we have, for every $0 \le t \le s$, $x \in \mathcal{X}$ and any set $\mathcal{Y}$ from the state space of Y,

$$\mathbb{P}(X_s = x, Y_s \in \mathcal{Y} \,|\, \mathcal{G}_t) = \mathbb{P}(X_s = x, Y_s \in \mathcal{Y} \,|\, X_t, Y_t). \tag{3}$$

This is the general framework adopted in the present paper. A specific Markov market model will be introduced in Section 3 below.

Of primary importance in this paper will be the k^{th} default time for an arbitrary $k = 1, 2, \ldots, L$. Let $\tau^{(1)} < \tau^{(2)} < \cdots < \tau^{(L)}$ be the ordering (for each ω) of the default times $\tau_1, \tau_2, \ldots, \tau_L$. By definition, the k^{th} default time is $\tau^{(k)}$.

It will be convenient to represent some probabilities associated with the k^{th} default time in terms of the *cumulative default process* H, defined as the increasing process

$$H_t = \sum_{l=1}^{L} H_t^l,$$

where $H_t^l = \mathbb{1}_{\{X_t^l = K\}} = \mathbb{1}_{\{\tau_l \le t\}}$ for every $t \in \mathbb{R}_+$. Evidently $\mathbb{H} \subseteq \widetilde{\mathbb{H}}$, where $\mathbb{H}$ is the filtration generated by the cumulative default process H. It is obvious that the process $S := (H, X, Y)$ has the Markov property under $\mathbb{P}$ with respect to the filtration $\mathbb{G}$. Also, it is useful to observe that we have $\{\tau^{(1)} > t\} = \{H_t = 0\}$, $\{\tau^{(k)} \le t\} = \{H_t \ge k\}$ and $\{\tau^{(k)} = \tau_l\} = \{H_{\tau_l} = k\}$ for every $l, k = 1, 2, \ldots, L$.

2.2 Conditional Expectations

Although we shall later focus on a Markovian set-up, in the sense of equality (3), we shall first derive some preliminary results in a more general set-up. To this end, it will be convenient to use the notation $\mathcal{F}^{X,t} = \sigma(X_s;\, s \ge t)$ and $\mathcal{F}^{Y,t} = \sigma(Y_s;\, s \ge t)$ for the information generated by the processes X and Y after time t. We postulate that for any random variable $Z \in \mathcal{F}^{X,t} \vee \mathcal{F}_\infty^Y$ and any bounded measurable function g, it holds that

$$\mathbb{E}_{\mathbb{P}}(g(Z) \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(g(Z) \,|\, \sigma(X_t) \vee \mathcal{F}_t^Y). \tag{4}$$

This implies, in particular, that the migration process X is *conditionally Markov* with regard to the reference filtration $\mathbb{F}^Y$, that is, for every $0 \le t \le s$ and $x \in \mathcal{X}$,

$$\mathbb{P}(X_s = x \,|\, \mathcal{G}_t) = \mathbb{P}(X_s = x \,|\, \sigma(X_t) \vee \mathcal{F}_t^Y). \tag{5}$$

Note that the Markov condition (3) is stronger than condition (4). We assume from now on that $t \ge 0$ and $x \in \mathcal{X}$ are such that $p_x(t) := \mathbb{P}(X_t = x \,|\, \mathcal{F}_t^Y) > 0$. We begin the analysis of conditional expectations with the following lemma.

Lemma 2.1 *Let $k \in \{1, 2, \ldots, L\}$, $x \in \mathcal{X}$, and let $Z \in \mathcal{F}^{X,t} \vee \mathcal{F}_\infty^Y$ be an integrable random variable. Then we have, for every $0 \le t \le s$,*

$$\mathbb{1}_{\{X_t = x\}}\, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{H_s < k\}} Z \,|\, \mathcal{G}_t) = \mathbb{1}_{\{H_t < k,\, X_t = x\}} \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{H_s < k,\, X_t = x\}} Z \,|\, \mathcal{F}_t^Y)}{p_x(t)}. \tag{6}$$

Consequently,

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{H_s < k\}} Z \,|\, \mathcal{G}_t) = \mathbb{1}_{\{H_t < k\}} \sum_{x \in \mathcal{X}} \mathbb{1}_{\{X_t = x\}} \frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{H_s < k,\, X_t = x\}} Z \,|\, \mathcal{F}_t^Y)}{p_x(t)}. \tag{7}$$

Proof. Let A_t be an arbitrary event from $\mathcal{G}_t$. We need to check that

$$\mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{A_t}\mathbb{1}_{\{X_t=x\}}\mathbb{1}_{\{H_s<k\}}Z\big) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{A_t}\mathbb{1}_{\{H_t<k,\,X_t=x\}}\,\frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{H_s<k,\,X_t=x\}}Z\,|\,\mathcal{F}^Y_t)}{p_x(t)}\right).$$

Since $\{H_s<k\}\subset\{H_t<k\}$ and the random variable $\tilde{Z}:=\mathbb{1}_{\{H_s<k,\,X_t=x\}}Z$ belongs to $\mathcal{F}^{X,t}\vee\mathcal{F}^Y_\infty$, the left-hand side is equal to

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{A_t}\mathbb{1}_{\{H_t<k,\,X_t=x\}}\mathbb{1}_{\{H_s<k\}}Z\,|\,\mathcal{G}_t)\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{A_t}\mathbb{1}_{\{H_t<k,\,X_t=x\}}\,\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{H_s<k,\,X_t=x\}}Z\,|\,\mathcal{G}_t)\big)\\
&\quad = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{A_t}\mathbb{1}_{\{H_t<k,\,X_t=x\}}\,\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{H_s<k,\,X_t=x\}}Z\,|\,\sigma(X_t)\vee\mathcal{F}^Y_t)\big)\\
&\quad = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{A_t}\mathbb{1}_{\{H_t<k,\,X_t=x\}}\,\frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{H_s<k,\,X_t=x\}}Z\,|\,\mathcal{F}^Y_t)}{p_x(t)}\right),
\end{aligned}$$

where the second equality is a consequence of (4), and the last one follows from the equality

$$\mathbb{1}_{\{X_t=x\}}\,\mathbb{E}_{\mathbb{P}}(\widehat{Z}\,|\,\sigma(X_t)\vee\mathcal{F}^Y_t) = \mathbb{1}_{\{X_t=x\}}\,\frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{X_t=x\}}\widehat{Z}\,|\,\mathcal{F}^Y_t)}{\mathbb{P}(X_t=x\,|\,\mathcal{F}^Y_t)},$$

which is valid for any integrable random variable $\widehat{Z}$. Equality (7) is an immediate consequence of (6). □

In the case of a single credit name, that is, in the case of $L=1$, we have for any $t\geq 0$ that $\{H_t<1\}=\{H_t\neq 1\}=\{X_t\neq K\}$. This leads to the following result.

Corollary 2.1 *Let $L=1$ and let $Z\in\mathcal{F}^{X,t}\vee\mathcal{F}^Y_\infty$ be an integrable random variable. Then we have, for any $0\leq t\leq s$,*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{X_s\neq K\}}Z\,|\,\mathcal{G}_t) = \sum_{x=1}^{K-1}\mathbb{1}_{\{X_t=x\}}\frac{\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{X_s\neq K,\,X_t=x\}}Z\,|\,\mathcal{F}^Y_t)}{p_x(t)}. \tag{8}$$

For any $0\leq t\leq s$, we write

$$\begin{aligned}
q_{k,x;t}(s) &= \mathbb{P}(H_s<k, X_t=x\,|\,\mathcal{F}^Y_t) = \mathbb{P}(\tau^{(k)}>s, X_t=x\,|\,\mathcal{F}^Y_t),\\
p_{k,x;t}(s) &= \mathbb{P}(H_s\geq k, X_t=x\,|\,\mathcal{F}^Y_t) = \mathbb{P}(\tau^{(k)}\leq s, X_t=x\,|\,\mathcal{F}^Y_t),
\end{aligned}$$

so that formally $dp_{k,x;t}(s)=\mathbb{P}(\tau^{(k)}\in ds, X_t=x\,|\,\mathcal{F}^Y_t)$. The following proposition extends Lemma 2.1.

Proposition 2.1 *Let $k\in\{1,2,\ldots,L\}$ and let Z be an integrable, $\mathbb{F}^Y$-predictable process. Then we have, for every $0\leq t\leq s$,*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau^{(k)}\leq s\}}Z_{\tau^{(k)}}\,|\,\mathcal{G}_t) = \mathbb{1}_{\{H_t<k\}}\sum_{x\in\mathcal{X}}\frac{\mathbb{1}_{\{X_t=x\}}}{p_x(t)}\;\mathbb{E}_{\mathbb{P}}\left(\int_{]t,s]}Z_u\,dp_{k,x;t}(u)\,\Big|\,\mathcal{F}^Y_t\right). \tag{9}$$

Proof. Let $t<\alpha<\beta<s$. Let us first establish (9) for a process Z of the form $Z_u=\mathbb{1}_{]\alpha,\beta]}(u)Z_\alpha$ where Z_α is a $\mathcal{F}^Y_\alpha$-measurable, integrable random variable. In this case, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau^{(k)}\leq s\}}Z_{\tau^{(k)}}\,|\,\mathcal{G}_t) &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\alpha<\tau^{(k)}\leq\beta\}}Z_\alpha\,|\,\mathcal{G}_t)\\
&= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{H_\alpha<k\}}Z_\alpha\,|\,\mathcal{G}_t) - \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{H_\beta<k\}}Z_\alpha\,|\,\mathcal{G}_t)\\
&= \mathbb{1}_{\{H_t<k\}}\sum_{x\in\mathcal{X}}\frac{\mathbb{1}_{\{X_t=x\}}}{p_x(t)}\;\mathbb{E}_{\mathbb{P}}\big(Z_\alpha[q_{k,x,t}(\alpha)-q_{k,x,t}(\beta)]\,|\,\mathcal{F}^Y_t\big)\\
&= \mathbb{1}_{\{H_t<k\}}\sum_{x\in\mathcal{X}}\frac{\mathbb{1}_{\{X_t=x\}}}{p_x(t)}\;\mathbb{E}_{\mathbb{P}}\left(\int_{]t,s]}Z_u\,dp_{k,x;t}(u)\,\Big|\,\mathcal{F}^Y_t\right),
\end{aligned}$$

where the third equality follows easily from (7) and the definitions of $q_{k,x;t}(s)$ and $p_{k,x;t}(s)$. The general case follows by standard approximation arguments. □

Corollary 2.2 *Let $L = 1$ and let Z be an integrable, $\mathbb{F}^Y$-predictable stochastic process. Then we have, for every $0 \le t \le s$,*

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau\le s\}} Z_\tau \,|\, \mathcal{G}_t) = \mathbb{1}_{\{X_t \neq K\}} \sum_{x=1}^{K-1} \frac{\mathbb{1}_{\{X_t = x\}}}{p_x(t)} \, \mathbb{E}_{\mathbb{P}} \left(\int_{]t,s]} Z_u \, dp_{1,k;t}(u) \, \Big| \, \mathcal{F}_t^Y \right). \tag{10}$$

For $K = 2$, Corollaries 2.1 and 2.2 coincide with Lemma 5.1.2 (i) and Proposition 5.1.1 (i) in Bielecki and Rutkowski (2002a), respectively.

2.2.1 Markovian Case

Let us now assume the Markovian set-up of Section 2.1.1. Let Z be a $\mathcal{G}^t = \mathcal{F}^{X,t} \vee \mathcal{F}^{Y,t}$-measurable, integrable random variable. Then formula (7) yields, for every $0 \le t \le s$,

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{H_s<k\}} Z \,|\, \mathcal{G}_t) = \mathbb{1}_{\{H_t<k\}} \sum_{x\in\mathcal{X}} \frac{\mathbb{1}_{\{X_t=x\}}}{\bar{p}_x(t)} \, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{H_s<k,\, X_t=x\}} Z \,|\, Y_t), \tag{11}$$

where $\bar{p}_x(t) = \mathbb{P}(X_t = x \,|\, Y_t)$, and formula (9) becomes

$$\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{t<\tau^{(k)}\le s\}} Z_{\tau^{(k)}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{H_t<k\}} \sum_{x\in\mathcal{X}} \frac{\mathbb{1}_{\{X_t=x\}}}{\bar{p}_x(t)} \, \mathbb{E}_{\mathbb{P}} \left(\int_{]t,s]} Z_u \, d\bar{p}_{k,x;t}(u) \, \Big| \, Y_t \right), \tag{12}$$

where

$$\bar{p}_{k,x;t}(u) = \mathbb{P}(H_u \ge k, X_t = x \,|\, Y_u) = \mathbb{P}(\tau^{(k)} \le u, X_t = x \,|\, Y_u).$$

3 Markovian Market Model

We assume that the factor process Y takes values in $\mathbb{R}^n$ so that the state space for the process $M = (X, Y)$ is $\mathcal{X} \times \mathbb{R}^n$. At the intuitive level, we wish to model the process $M = (X, Y)$ as a combination of a Markov chain X modulated by the Lévy-like process Y and a Lévy-like process Y modulated by a Markov chain X. To be more specific, we postulate that the *infinitesimal generator* $\mathbf{A}$ of M is given as

$$\begin{aligned} \mathbf{A}f(x,y) &= (1/2) \sum_{i,j=1}^n a_{ij}(x,y) \partial_i \partial_j f(x,y) + \sum_{i=1}^n b_i(x,y) \partial_i f(x,y) \\ &+ \gamma(x,y) \int_{\mathbb{R}^n} \big(f(x, y + g(x,y,y')) - f(x,y) \big) \Pi(x,y;dy') + \sum_{x'\in\mathcal{X}} \lambda(x,x';y) f(x',y), \end{aligned}$$

where $\lambda(x,x';y) \ge 0$ for every $x = (x^1, x^2, \ldots, x^L) \neq (x'^1, x'^2, \ldots, x'^L) = x'$, and

$$\lambda(x,x;y) = - \sum_{x'\in\mathcal{X},\, x'\neq x} \lambda(x,x';y).$$

Here ∂_i denotes the partial derivative with respect to the variable y^i. The existence and uniqueness of a Markov process M with the generator $\mathbf{A}$ will follow (under appropriate technical conditions) from the respective results regarding martingale problems. Specifically, one can refer to Theorems 4.1 and 5.4 in Chapter 4 of Ethier and Kurtz (1986).

We find it convenient to refer to X (Y, respectively) as the *Markov chain component* of M (the *jump-diffusion component* of M, respectively). At any time t, the intensity matrix of the Markov chain component is given as $\Lambda_t = [\lambda(x,x';Y_t)]_{x,x'\in\mathcal{X}}$. The jump-diffusion component satisfies the SDE:

$$dY_t = b(X_t, Y_t)\, dt + \sigma(X_t, Y_t)\, dW_t + \int_{\mathbb{R}^n} g(X_{t-}, Y_{t-}, y')\, \pi(X_{t-}, Y_{t-}; dy', dt),$$

where, for a fixed $(x, y) \in \mathcal{X} \times \mathbb{R}^n$, $\pi(x, y; dy', dt)$ is a Poisson measure with the intensity measure $\gamma(x, y)\Pi(x, y; dy')dt$, and where $\sigma(x, y)$ satisfies the equality $\sigma(x, y)\sigma(x, y)^{\mathsf{T}} = a(x, y)$.

Remarks. If we take $g(x, y, y') = y'$, and we suppose that the coefficients $\sigma = [\sigma_{ij}]$, $b = [b_i]$, γ, and the measure Π do not depend on x and y then the factor process Y is a Poisson-Lévy process with the characteristic triplet (a, b, ν), where the diffusion matrix is $a(x, y) = \sigma(x, y)\sigma(x, y)^{\mathsf{T}}$, the "drift" vector is $b(x, y)$, and the Lévy measure is $\nu(dy) = \gamma\Pi(dy)$. In this case, the migration process X is modulated by the factor process Y, but not vice versa. We shall not study here the "infinite activity" case, that is, the case when the jump measure π is not a Poisson measure, and the related Lévy measure is an infinite measure.

We shall provide with more structure the Markov chain part of the generator $\mathbf{A}$. Specifically, we make the following standing assumption.

Asumption (M). The infinitesimal generator of the process $M = (X, Y)$ takes the following form

$$\begin{aligned}\mathbf{A}f(x,y) = (1/2)&\sum_{i,j=1}^{n} a_{ij}(x,y)\partial_i\partial_j f(x,y) + \sum_{i=1}^{n} b_i(x,y)\partial_i f(x,y)\\ &+ \gamma(x,y)\int_{\mathbb{R}^n}\big(f(x, y+g(x,y,y')) - f(x,y)\big)\Pi(x,y;dy')\\ &+\sum_{l=1}^{L}\sum_{x^{l\prime}\in\mathcal{K}}\lambda^l(x,x_l';y)f(x_l',y),\end{aligned} \tag{13}$$

where we write $x_l' = (x^1, x^2, \ldots, x^{l-1}, x'^l, x^{l+1}, \ldots, x^L)$.

Note that x_l' is the vector $x = (x^1, x^2, \ldots, \ldots, x^L)$ with the l^{th} coordinate x^l replaced by x'^l. In the case of two obligors (i.e., for $L = 2$), the generator becomes

$$\begin{aligned}\mathbf{A}f(x,y) = (1/2)&\sum_{i,j=1}^{n} a_{ij}(x,y)\partial_i\partial_j f(x,y) + \sum_{i=1}^{n} b_i(x,y)\partial_i f(x,y)\\ &+ \gamma(x,y)\int_{\mathbb{R}^n}\big(f(x, y+g(x,y,y')) - f(x,y)\big)\Pi(x,y;dy')\\ &+\sum_{x'^1\in\mathcal{K}}\lambda^1(x,x_1';y)f(x_1',y) + \sum_{x'^2\in\mathcal{K}}\lambda^2(x,x_2';y)f(x_2',y),\end{aligned}$$

where $x = (x^1, x^2)$, $x_1' = (x'^1, x^2)$ and $x_2' = (x^1, x'^2)$. In this case, coming back to the general form, we have for $x = (x^1, x^2)$ and $x' = (x'^1, x'^2)$

$$\lambda(x,x';y) = \begin{cases}\lambda^1(x,x_1';y), & \text{if } x^2 = x'^2,\\ \lambda^2(x,x_2';y), & \text{if } x^1 = x'^1,\\ 0, & \text{otherwise.}\end{cases}$$

Similar expressions can be derived in the case of a general value of L. Note that the model specified by (13) does not allow for simultaneous jumps of the components X^l and $X^{l'}$ for $l \neq l'$. In other words, the ratings of different credit names may not change simultaneously. Nevertheless, this is not a serious lack of generality, as the ratings of both credit names may still change in an arbitrarily small time interval. The advantage is that, for the purpose of simulation of paths of process X, rather than dealing with $\mathcal{X} \times \mathcal{X}$ intensity matrix $[\lambda(x, x'; y)]$, we shall deal with L intensity matrices $[\lambda^l(x, x_l'; y)]$, each of dimension $\mathcal{K} \times \mathcal{K}$ (for any fixed y). The structure (13) is assumed in the rest of the paper. Let us stress that within the present set-up the current credit rating of the credit name l directly impacts the intensity of transition of the rating of the credit name l', and vice versa. This property, known as *frailty*, may contribute to default contagion.

Remarks. (i) It is clear that we can incorporate in the model the case when some – possibly all – components of the factor process Y follow Markov chains themselves. This feature is important, as factors such as economic cycles may be modeled as Markov chains. It is known that default rates are strongly related to business cycles.

(ii) Some of the factors $Y^1, Y^2, \ldots, Y^d$ may represent cumulative duration of visits of rating processes X^l in respective rating states. For example, we may set $Y^1_t = \int_0^t \mathbb{1}_{\{X^1_s=1\}}\, ds$. In this case, we have $b_1(x,y) = \mathbb{1}_{\{x^1=1\}}(x)$, and the corresponding components of coefficients σ and g equal zero.

(iii) In the area of *structural arbitrage*, so called *credit–to–equity* (C2E) models and/or *equity–to–credit* (E2C) models are studied. Our market model nests both types of interactions, that is C2E and E2C. For example, if one of the factors is the price process of the equity issued by a credit name, and if credit migration intensities depend on this factor (implicitly or explicitly) then we have a E2C type interaction. On the other hand, if credit ratings of a given obligor impact the equity dynamics (of this obligor and/or some other obligors), then we deal with a C2E type interaction.

As already mentioned, $S = (H, X, Y)$ is a Markov process on the state space $\{0, 1, \ldots, L\} \times \mathcal{X} \times \mathbb{R}^d$ with respect to its natural filtration. Given the form of the generator of the process (X, Y), we can easily describe the generator of the process (H, X, Y). It is enough to observe that the transition intensity at time t of the component H from the state H_t to the state $H_t + 1$ is equal to $\sum_{l=1}^L \lambda^l(X_t, K; X_t^{(l)}, Y_t)$, provided that $H_t < L$ (otherwise, the transition intensity equals zero), where we write $X_t^{(l)} = (X_t^1, \ldots, X_t^{l-1}, X_t^{l+1}, \ldots, X_t^L)$ and we set $\lambda^l(x^l, x'^l; x^{(l)}, y) = \lambda^l(x, x'_l; y)$.

3.1 Specification of Credit Ratings Transition Intensities

One always needs to find a compromise between realistic features of a financial model and its complexity. This issue frequently nests the issues of functional representation of a model, as well as its parameterization. We present here an example of a particular model for credit ratings transition rates, which is rather arbitrary, but is nevertheless relatively simple and should be easy to estimate/calibrate.

Let $\bar{X}_t$ be the average credit rating at time t, so that

$$\bar{X}_t = \frac{1}{L} \sum_{l=1}^L X_t^l.$$

Let $\mathcal{L} = \{l_1, l_2, \ldots, l_{\widehat{L}}\}$ be a subset of the set of all obligors, where $\widehat{L} < L$. We consider $\mathcal{L}$ to be a collection of "major players" whose economic situation, reflected by their credit ratings, effectively impacts all other credit names in the pool. The following exponential-linear "regression" model appears to be a plausible model for the rating transition intensities:

$$\begin{aligned} \ln \lambda^l(x, x'_l; y) = {} & \alpha_{l,0}(x^l, x'^l) + \sum_{j=1}^n \alpha_{l,j}(x^l, x'^l) y_j + \beta_{l,0}(x^l, x'^l) h \\ & + \sum_{i=1}^{\bar{L}} \beta_{l,i}(x^l, x'^l) x^i + \tilde{\beta}_l(x^l, x'^l)\bar{x} + \widehat{\beta}_l(x^l, x'^l)(x^l - x'^l), \end{aligned} \tag{14}$$

where h represents a generic value of H_t, so that $h = \sum_{l=1}^L \mathbb{1}_{\{K\}}(x^l)$, and $\bar{x}$ represents a generic value of $\bar{X}_t$, that is, $\bar{x} = \frac{1}{L}\sum_{l=1}^L x^l$.

The number of parameters involved in (14) can easily be controlled by the number of model variables, in particular – the number of factors and the number of credit ratings, as well as structure of the transition matrix (see Section 7.2 below). In addition, the reduction of the number of parameters can be obtained if the pool of all L obligors is partitioned into a (small) number of homogeneous sub-pools. All of this is a matter of practical implementation of the model. Assume, for instance, that there are $\widetilde{L} \ll L$ homogeneous sub-pools of obligors, and the parameters $\alpha, \beta, \tilde{\beta}$ and $\widehat{\beta}$ in (14) do not depend on x^l, x'^l. Then the migration intensities (14) are parameterized by $\widetilde{L}(n + \widehat{L} + 4)$ parameters.

3.2 Conditionally Independent Migrations

Suppose that the intensities $\lambda^l(x, x_l'; y)$ do not depend on $x^{(l)} = (x^1, x^2, \dots, x^{l-1}, x^{l+1}, \dots, x^L)$ for every $l = 1, 2, \dots, L$. In addition, assume that the dynamics of the factor process Y do not depend on the the migration process X. It turns out that in this case, given the structure of our generator as in (13), the migration processes $X^l,\ l = 1, 2, \dots, L$, are conditionally independent given the sample path of the process Y.

We shall illustrate this point in the case of only two credit names in the pool (i.e., for $L = 2$) and assuming that there is no factor process, so that conditional independence really means independence between migration processes X^1 and X^2. For this, suppose that X^1 and X^2 are independent Markov chains, each taking values in the state space $\mathcal{K}$, with infinitesimal generator matrices Λ^1 and Λ^2, respectively. It is clear that the joint process $X = (X^1, X^2)$ is a Markov chain on $\mathcal{K} \times \mathcal{K}$. An easy calculation reveals that the infinitesimal generator of the process X is given as

$$\Lambda = \Lambda^1 \otimes \mathrm{Id}_K + \mathrm{Id}_K \otimes \Lambda^2,$$

where Id_K is the identity matrix of order K and $\otimes$ denotes the matrix tensor product. This agrees with the structure (13) in the present case.

4 Changes of Measures and Markovian Numeraires

In financial applications, one frequently deals with various absolutely continuous probability measures. In order to exploit – for pricing applications – the Markovian structure of the market model introduced above, we need that the model is Markovian under a particular pricing measure corresponding to some particular numeraire process β that is convenient to use for some reasons. The model does not have to be Markovian under some other equivalent probability measures, such as the statistical probability, say $\mathbb{Q}$, or the spot martingale measure, say $\mathbb{Q}^*$. Nevertheless, it may be sometimes desirable that the Markovian structure of the market model is preserved under an equivalent change of a probability measure, for instance, when we change a numeraire from β to β'. In this section, we shall provide some discussion of the issue of preservation of the Markov property of the process M.

Let $T^* > 0$ be a fixed horizon date, and let η be a strictly positive, $\mathcal{G}_{T^*}$-measurable random variable such that $\mathbb{E}_{\mathbb{P}}\eta = 1$. We define an equivalent probability measure $\mathbb{P}^\eta$ on $(\Omega, \mathcal{G}_{T^*})$ by the equality

$$\frac{d\mathbb{P}^\eta}{d\mathbb{P}} = \eta, \quad \mathbb{P}\text{-a.s.}$$

4.1 Markovian Change of a Probability Measure

We place ourselves in the set-up of Section 2.1.1, and we follow Palmowski and Rolski (2002) in the presentation below. The standing assumption is that the process $M = (X, Y)$ has the Markov property under $\mathbb{P}$ with respect to the filtration $\mathbb{G}$. Let $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ be the *extended generator* of M. This means that the process

$$M_t^f = f(M_t) - \int_0^t \mathbf{A} f(M_s)\, ds \tag{15}$$

is a local $\mathbb{G}$-martingale for any function f in $\mathcal{D}(\mathbf{A})$.

For any strictly positive function $h \in \mathcal{D}(\mathbf{A})$, we define an auxiliary process η^h by setting

$$\eta_t^h = \frac{h(M_t)}{h(M_0)} \exp\left(- \int_0^t \frac{(\mathbf{A}h)(M_s)}{h(M_s)}\, ds \right), \quad t \in [0, T^*]. \tag{16}$$

Any function h for which the process η^h is a martingale is called a *good* function for $\mathbf{A}$. Observe that for any such function h, the equality $\mathbb{E}_{\mathbb{P}}(\eta_t^h) = 1$ holds for every $t \in [0, T^*]$. Note also that any constant function h is a good function for $\mathbf{A}$; in this case we have, of course, that $\eta^h \equiv 1$. The next lemma follows from results of Palmowski and Rolski (2002) (see Lemma 3.1 therein).

Lemma 4.1 *Let h be a good function for $\mathbf{A}$. Then the process η^h is given as Doléans exponential martingale*

$$\eta^h_t = \mathcal{E}_t(N^h),$$

where the local martingale N^h is given as

$$N^h_t = \int_0^t \kappa^h_{s-}\, dM^h_s$$

with $\kappa^h_t = 1/h(M_t)$. In other words, the process η^h satisfies the SDE

$$d\eta^h_t = \eta^h_{t-}\kappa^h_{t-}\, dM^h_t, \quad \eta^h_0 = 1. \tag{17}$$

Proof. An application of Itô's formula yields

$$d\eta^h_t = \frac{1}{h(M_0)} \exp\left(- \int_0^t \frac{(\mathbf{A}h)(M_s)}{h(M_s)}\, ds \right) dM^h_t,$$

where the local martingale M^h is given by (15). This proves formula (17). □

For any good function h for $\mathbf{A}$, we define an equivalent probability measure $\mathbb{P}^h$ on $(\Omega, \mathcal{G}_{T^*})$ by setting

$$\frac{d\mathbb{P}^h}{d\mathbb{P}} = \eta^h_{T^*}, \quad \mathbb{P}\text{-a.s.} \tag{18}$$

From Kunita and Watanabe (1963), we deduce that the process M preserves its Markov property with respect to the filtration $\mathbb{G}$ when the probability measure $\mathbb{P}$ is replaced by $\mathbb{P}^h$. In order to find the extended generator of M under $\mathbb{P}^h$, we set

$$\mathbf{A}^h f = \frac{1}{h}\left[\mathbf{A}(fh) - f\mathbf{A}(h)\right],$$

and we define the following two sets:

$$\mathcal{D}^h_{\mathbf{A}} = \left\{ f \in \mathcal{D}(\mathbf{A}) \,:\, fh \in \mathcal{D}(\mathbf{A}) \ \text{ and } \ \int_0^{T^*} |\mathbf{A}^h f(M_s)|\, ds < \infty,\ \mathbb{P}^h\text{-a.s.} \right\}$$

and

$$\mathcal{D}^{h^{-1}}_{\mathbf{A}^h} = \left\{ f \in \mathcal{D}(\mathbf{A}^h) \,:\, fh^{-1} \in \mathcal{D}(\mathbf{A}^h) \ \text{ and } \ \int_0^{T^*} |\mathbf{A} f(M_s)|\, ds < \infty,\ \mathbb{P}\text{-a.s.} \right\}.$$

Then the following result holds (see Theorem 4.2 in Palmowski and Rolski (2002)).

Theorem 4.1 *Suppose that $\mathcal{D}^h_{\mathbf{A}} = \mathcal{D}(\mathbf{A})$ and $\mathcal{D}^{h^{-1}}_{\mathbf{A}^h} = \mathcal{D}(\mathbf{A}^h)$. Then the process M is Markovian under $\mathbb{P}^h$ with the extended generator $\mathbf{A}^h$ and $\mathcal{D}(\mathbf{A}^h) = \mathcal{D}(\mathbf{A})$.*

We now apply the above theorem to our model. The domain of $\mathcal{D}(\mathbf{A})$ contains all functions $f(x,y)$ with compact support that are twice continuously differentiable with respect to y. Let h be a good function. Under mild assumptions on the coefficients of $\mathbf{A}$, the assumptions of Theorem 4.1 are satisfied. It follows that the generator of M under $\mathbb{P}^h$ is given as

$$\begin{aligned}\mathbf{A}^h f(x,y) &= (1/2) \sum_{i,j=1}^n a_{ij}(x,y)\partial_i\partial_j f(x,y) + \sum_{i=1}^n b^h_i(x,y)\partial_i f(x,y) \\ &\quad + \gamma(x,y) \int_{\mathbb{R}^n} \left(f(x, y + g(x,y,y')) - f(x,y)\right) \Pi^h(x,y;dy') + \sum_{x' \in \mathcal{X}} \lambda^h(x,x';y) f(x',y),\end{aligned}$$

where

$$
\begin{aligned}
b_i^h(x,y) &= b_i(x,y) + \frac{1}{h(x,y)} \sum_{i,j=1}^n a_{ij}(x,y)\partial_j h(x,y), \\
\Pi^h(x,y;dy') &= \frac{h(x,y+g(x,y,y'))}{h(x,y)}\, \Pi(x,y;dy'), \\
\lambda^h(x,x';y) &= \lambda(x,x';y)\frac{h(x',y)}{h(x,y)}, \ x \neq x', \quad \lambda^h(x,x;y) = -\sum_{x'\neq x} \lambda^h(x,x';y).
\end{aligned}
\tag{19}
$$

Before we proceed to the issue of valuation of credit derivatives, we state the following useful result, whose easy proof is omitted.

Lemma 4.2 *Let h and h' be two good functions for $\mathbf{A}$. Then $\phi(h,h') := h'/h$ is a good function for $\mathbf{A}^h$. Moreover, we have that*

$$
\frac{d\mathbb{P}^{h'}}{d\mathbb{P}^h} = \eta_{T^*}^{\phi(h,h')}, \quad \mathbb{P}^h\text{-a.s.} \tag{20}
$$

where the process $\eta^{\phi(h,h')}$ is defined in analogy to (16) *with $\mathbf{A}$ replaced with $\mathbf{A}^h$.*

4.2 Markovian Numeraires and Valuation Measures

Let us first consider a general set-up. We use here the notation and terminology of Jamshidian (2004). We fix the horizon date T^*, and we assume that $\mathcal{G} = \mathcal{G}_{T^*}$. Let us fix some ($\mathbb{G}$-adapted) *deflator process* ξ, that is, a strictly positive, integrable semimartingale, with $\xi_0 = 1$. Any $\mathcal{G}$-measurable random variable C such that $\xi_{T^*}C$ is integrable under $\mathbb{P}$ is called a *claim*. The *price process* C_t, $t \in [0,T^*]$, of a claim C is formally defined as

$$
C_t = \xi_t^{-1}\, \mathbb{E}_{\mathbb{P}}(\xi_{T^*}C \,|\, \mathcal{G}_t),
$$

so that, in particular, $C_{T^*} = C$. It is implicitly assumed here that the information carried by the filtration $\mathbb{G}$ is available to all trading agents.

Suppose that we are interested in providing valuation formulae for some financial products, and suppose that we find it convenient to use a particular *numeraire* (that is, a strictly positive claim), denoted by β. Let $\mathbb{P}^\beta$ be the corresponding *valuation measure*, defined on $(\Omega, \mathcal{G})$ as

$$
\frac{d\mathbb{P}^\beta}{d\mathbb{P}} = \frac{\xi_{T^*}\beta}{\beta_0} = \frac{\xi_{T^*}\beta_{T^*}}{\beta_0}, \quad \mathbb{P}\text{-a.s.} \tag{21}
$$

From the abstract Bayes rule, it follows that the price process C can be expressed as follows:

$$
C_t = \beta_t\, \mathbb{E}_{\mathbb{P}^\beta}(\beta_{T^*}^{-1}C \,|\, \mathcal{G}_t).
$$

As before, we assume that our market model M is Markovian under $\mathbb{P}$, where $\mathbb{P}$ might be the statistical probability measure $\mathbb{Q}$, the spot martingale measure $\mathbb{Q}^*$, or some other martingale measure. We want the process M to remain a time-homogeneous Markov process under $\mathbb{P}^\beta$.

Definition 4.1 A valuation measure $\mathbb{P}^\beta$ is said to be a *Markovian* if the process M remains a time-homogeneous Markov process under $\mathbb{P}^\beta$. Any numeraire process β such that the valuation measure $\mathbb{P}^\beta$ is Markovian is called a *Markovian numeraire.*

In view of results of Section 4.1, for a valuation measure $\mathbb{P}^\beta$ to be Markovian, it suffices that the Radon-Nikodým derivative process $\eta_t^\beta = \frac{d\mathbb{P}^\beta}{d\mathbb{P}}\big|_{\mathcal{G}_t}$ satisfies

$$
\eta_t^\beta = \frac{h^\beta(M_t)}{h^\beta(M_0)}\, \exp\left(-\int_0^t \frac{(\mathbf{A}h^\beta)(M_s)}{h^\beta(M_s)}\, ds \right), \quad t \in [0,T^*],
$$

for some good function h^β for $\mathbf{A}$. The corresponding deflator process is then given as $\xi^\beta_t = \beta_0 \beta_t^{-1} \eta^\beta_t$, that is, for any claim C we have that

$$C_t = \beta_t \, \mathbb{E}_{\mathbb{P}^\beta}(\beta^{-1}_{T^*} C \,|\, \mathcal{G}_t) = (\xi^\beta_t)^{-1} \, \mathbb{E}_{\mathbb{P}}(\xi^\beta_{T^*} C \,|\, \mathcal{G}_t).$$

If β and β' are two such numeraires, and h^β and $h^{\beta'}$ are the corresponding good functions then, in view of Lemma 4.2, we have

$$\frac{d\mathbb{P}^{\beta'}}{d\mathbb{P}^\beta} = \eta^{\phi(h^\beta, h^{\beta'})}_{T^*}, \quad \mathbb{P}^\beta\text{-a.s.} \tag{22}$$

An interesting question arises: under what conditions on ξ and β the probability measure $\mathbb{P}^\beta$ is a Markovian valuation measure? In order to partially address this question, we shall consider the case where the valuation measure $\mathbb{P}^\beta$ is a Markovian for any constant numeraire β, that is, for any $\beta \equiv \text{const} > 0$.

Proposition 4.1 *Assume that the deflator process satisfies $\xi = \eta^h$ for some good function h for $\mathbf{A}$. Then the following statements are true:*
(i) *for any constant numeraire β, the valuation measure $\mathbb{P}^\beta$ is Markovian,*
(ii) *if a numeraire β is such that $\beta = \beta_0 \eta^\chi_{T^*} / \eta^h_{T^*}$ for some good function χ for $\mathbf{A}$, then the valuation measure $\mathbb{P}^\beta$ is Markovian,*
(iii) *if numeraires β and β' are such that $\beta = \beta_0 \eta^\chi_{T^*} / \eta^h_{T^*}$ and $\beta' = \beta'_0 \eta^{\chi'}_{T^*} / \eta^h_{T^*}$ for some good functions χ and χ' for $\mathbf{A}$, then*

$$\frac{d\mathbb{P}^{\beta'}}{d\mathbb{P}^\beta} = \frac{\beta'/\beta'_0}{\beta/\beta_0} = \frac{\eta^{\chi'}_{T^*}}{\eta^{\chi}_{T^*}}, \quad \mathbb{P}^{\xi,\beta'}\text{-a.s.} \tag{23}$$

Proof. Let $\xi = \eta^h$ for some good function h, where η^h is given by (16). Then for any constant numeraire β we get $\mathbb{P}^\beta = \mathbb{P}^h$ and thus, by results of Kunita and Watanabe (1963), the process M is Markovian under the valuation measure $\mathbb{P}^\beta$. This proves part (i). To establish the second part, it suffices to note that

$$\frac{d\mathbb{P}^\beta}{d\mathbb{P}} = \frac{\xi_{T^*} \beta}{\beta_0} = \eta^\chi_{T^*},$$

and to use again the result of Kunita and Watanabe (1963). Formula (23) follows easily from (21) (it can also be seen as a special case of (22)). This completes the proof. □

4.3 Examples of Markov Market Models

We shall now present three pertinent examples of Markov market models. We assume here that a numaraire β is given; the choice of β depends on the problem at hand.

4.3.1 Markov Chain Migration Process

We assume here that there is no factor process Y. Thus, we only deal with a single migration process X. In this case, an attractive and efficient way to model credit migrations is to postulate that X is a *birth-and-death process* with absorption at state K. In this case, the intensity matrix Λ is tri-diagonal. To simplify the notation, we shall write $p_t(k, k') = \mathbb{P}^\beta(X_{s+t} = k' \,|\, X_s = k)$. The transition probabilities $p_t(k, k')$ satisfy the following system of ODEs, for $t \geq 0$ and $k' \in \{1, 2, \ldots, K\}$,

$$\frac{dp_t(1, k')}{dt} = -\lambda(1, 2) p_t(1, k') + \lambda(1, 2) p_t(2, k'),$$

$$\frac{dp_t(k, k')}{dt} = \lambda(k, k-1) p_t(k-1, k') - (\lambda(k, k-1) + \lambda(k, k+1)) p_t(k, k') + \lambda(k, k+1) p_t(k+1, k')$$

for $k = 2, 3, \ldots, K-1$, and

$$\frac{dp_t(K, k')}{dt} = 0,$$

with initial conditions $p_0(k,k') = \mathbb{1}_{\{k=k'\}}$. Once the transition intensities $\lambda(k,k')$ are specified, the above system can be easily solved. Note, in particular, that $p_t(K,k') = 0$ for every t if $k' \neq K$. The advantage of this representation is that the number of parameters can be kept small.

A slightly more flexible model is produced if we allow for jumps to the default state K from any other state. In this case, the master ODEs take the following form, for $t \geq 0$ and $k' \in \{1, 2, \ldots, K\}$,

$$\frac{dp_t(1,k')}{dt} = -(\lambda(1,2) + \lambda(1,K))p_t(1,k') + \lambda(1,2)p_t(2,k') + \lambda(1,K)p_t(K,k'),$$

$$\frac{dp_t(k,k')}{dt} = \lambda(k,k-1)p_t(k-1,k') - (\lambda(k,k-1) + \lambda(k,k+1) + \lambda(k,K))p_t(k,k') \\ +\lambda(k,k+1)p_t(k+1,k') + \lambda(k,K)p_t(K,k')$$

for $k = 2, 3, \ldots, K-1$, and

$$\frac{dp_t(K,k')}{dt} = 0,$$

with initial conditions $p_0(k,k') = \mathbb{1}_{\{k=k'\}}$. Some authors model migrations of credit ratings using a (proxy) diffusion, possibly with jumps to default. The birth-and-death process with jumps to default furnishes a Markov chain counterpart of such proxy diffusion models. The nice feature of the Markov chain model is that the credit ratings are (in principle) observable state variables – whereas in case of the proxy diffusion models they are not.

4.3.2 Diffusion-type Factor Process

We now add a factor process Y to the model. We postulate that the factor process is a diffusion process and that the generator of the process $M = (X, Y)$ takes the form

$$\mathbf{A}f(x,y) = (1/2)\sum_{i,j=1}^{n} a_{ij}(x,y)\partial_i\partial_j f(x,y) + \sum_{i=1}^{n} b_i(x,y)\partial_i f(x,y) \\ + \sum_{x'\in\mathcal{K},\, x'\neq x} \lambda(x,x';y)(f(x',y) - f(x,y)).$$

Let $\phi(t,x,y,x',y')$ be the transition probability of M. Formally,

$$\phi(t,x,y,x',y')\,dy' = \mathbb{P}^{\beta}(X_{s+t} = x', Y_{s+t} \in dy' \,|\, X_s = x, Y_s = y).$$

In order to determine the function ϕ, we need to study the following Kolmogorov equation

$$\frac{dv(s,x,y)}{ds} + \mathbf{A}v(s,x,y) = 0. \tag{24}$$

For the generator $\mathbf{A}$ of the present form, equation (24) is commonly known as the *reaction-diffusion equation*. Existence and uniqueness of classical solutions for such equations were recently studied by Becherer and Schweizer (2003). It is worth mentioning that a reaction-diffusion equation is a special case of a more general integro-partial-differential equation (IPDE). In a future work, we shall deal with issue of practical solving of equations of this kind.

4.3.3 CDS Spread Factor Model

Suppose now that the factor process $Y_t = \kappa^{(1)}(t,T^S,T^M)$ is the forward CDS spread (for the definition of $\kappa^{(1)}(t,T^S,T^M)$, see Section 5.3 below), and that the generator for (X,Y) is

$$\mathbf{A}f(x,y) = (1/2)y^2 a(x)\frac{d^2 f(x,y)}{dy^2} + \sum_{x'\in\mathcal{K},\, x'\neq x} \lambda(x,x')(f(x',y) - f(x,y)).$$

Thus, the credit spread satisfies the following SDE

$$d\kappa^{(1)}(t,T^S,T^M) = \kappa^{(1)}(t,T^S,T^M)\sigma(X_t)\,dW_t$$

for some Brownian motion process W, where $\sigma(x) = \sqrt{a(x)}$. Note that in this example $\kappa^{(1)}(t,T^S,T^M)$ is a conditionally log-Gaussian process given a sample path of the migration process X, so that we are in the position to make use of Proposition 5.1 below.

5 Valuation of Single Name Credit Derivatives

We maintain the Markovian set-up, so that $M = (X,Y)$ follows a Markov process with respect to $\mathbb{G}$ under $\mathbb{P}$. In this section, we only consider one underlying credit name, that is, we set $L = 1$. Basket credit derivatives will be studied in Section 6 below.

5.1 Survival Claims

Suppose that β is a Markovian numeraire, in the sense of Definition 4.1. Let us fix $t \in [0,T]$, and let us assume that a claim C and the random variable β_t/β_{T^*} are measurable with respect to $\mathcal{G}^t = \mathcal{F}^{X,t} \vee \mathcal{F}^{Y,t}$. Then we deduce easily that $C_t = \mathcal{V}_t^{\xi,\beta}(C)$, where

$$\mathcal{V}_t^{\xi,\beta}(C) = \mathbb{E}_{\mathbb{P}^\beta}\big(\beta_t\beta_{T^*}^{-1}C \,\big|\, M_t\big).$$

A claim C such that $C = 0$ on the set $\{\tau \le T\}$, so that

$$C = \mathbb{1}_{\{\tau > T\}}C = \mathbb{1}_{\{X_T \ne K\}}C = \mathbb{1}_{\{H_T < 1\}}C,$$

is termed a *T-survival claim*. For a survival claim, a more explicit expression for the price can be established. Since most standard credit derivatives can be seen as survival claims, the following simple result will prove useful in what follows.

Lemma 5.1 *Assume that a claim C and that the random variable β_t/β_{T^*} are measurable with respect to $\mathcal{G}^t$. If C is a T-survival claim then we have*

$$C_t = \mathbb{1}_{\{X_t \ne K\}}\,\mathcal{V}_t^{\xi,\beta}(C) = \sum_{x=1}^{K-1} \mathbb{1}_{\{X_t = x\}} \frac{\mathbb{E}_{\mathbb{P}^\beta}(\mathbb{1}_{\{X_t = x\}}\beta_t\beta_{T^*}^{-1}C \,|\, Y_t)}{\mathbb{P}^\beta(X_t = x \,|\, Y_t)}.$$

Proof. The first equality is clear. To derive the second equality above, it suffices to apply formula (11) with $L = 1, s = T, k = 1$ and $Z = C$. □

Remark. Assume that $K = 2$, so that only the pre-default state ($x = 1$) and the default state ($x = 2$) are recognized. Then we have, for any T-survival claim C,

$$C_t = \mathbb{1}_{\{X_t \ne 2\}}\,\mathcal{V}_t^{\xi,\beta}(C) = \frac{\mathbb{1}_{\{X_t = 1\}}V_t^{\xi,\beta}(C)}{\mathbb{P}^\beta(X_t = 1 \,|\, Y_t)},$$

where $V_t^{\xi,\beta}(C) = \mathbb{E}_{\mathbb{P}^\beta}(\beta_t\beta_{T^*}^{-1}C \,|\, Y_t)$. In the paper by Jamshidian (2004), the process $V_t^{\xi,\beta}(C)$ is termed the *preprice* of C.

5.2 Credit Default Swaps

The standing assumption is that β is a Markovian numeraire and β_t/β_s is $\mathcal{G}^t$-measurable for any $t \le s$. For simplicity, we shall discuss a vanilla *credit default swap* (CDS, for short) written on a corporate discount bond under the fractional recovery of par covenant. We suppose that the maturity of the reference bond is U, and the maturity of the swap is $T < U$.

5.2.1 Default Payment Leg

Let $N = 1$ be a notional amount of the bond, and let δ be a deterministic recovery rate in case of default. The recovery is paid at default, so that the cash flow associated with the *default payment leg* – also known as the *reference leg* – is given by $(1-\delta)\mathbb{1}_{\{\tau \le T\}}\mathbb{1}_\tau(t)$ per unit of a notional amount, where τ is the default time of a reference credit name. Consequently, the time-t value of the default payment leg is equal to

$$\begin{aligned} A_t^{(1)} &= (1-\delta)\,\mathbb{E}_{\mathbb{P}^\beta}\left(\mathbb{1}_{\{t<\tau\le T\}}\beta_t\beta_\tau^{-1}\,|\,M_t\right) \\ &= (1-\delta)\sum_{x=1}^{K-1}\mathbb{1}_{\{X_t=x\}}\frac{\mathbb{E}_{\mathbb{P}^\beta}\left(\mathbb{1}_{\{X_t=x,X_T=K\}}\beta_t\beta_\tau^{-1}\,|\,Y_t\right)}{\mathbb{P}^\beta(X_t=x\,|\,Y_t)}. \end{aligned}$$

The notation $A^{(1)}$ refers to the first default, which is formally the case here, since we currently deal with one name only. Since $L = 1$, the cumulative default process H takes values in the set $\{0, 1\}$, and we have that $\{H_t = 1\} = \{X_t = K\}$.

Since the process $S = (H, X, Y)$ is a Markov process under $\mathbb{P}^\beta$, and the transition intensity at time t of a jump from $H_t = 0$ to $H_t + 1$ is $\lambda(X_t, K; Y_t)$. Hence, it is easy to write down the form of the generator of the process S. Using the Chapman-Kolmogorov equation, we can thus compute the conditional probability (recall that conditioning on S_t is equivalent to conditioning on M_t)

$$\mathbb{P}^\beta(\tau \le s\,|\,S_t) = \mathbb{P}^\beta(\tau \le s\,|\,M_t).$$

Knowing the conditional density $\mathbb{P}^\beta(\tau \in ds\,|\,M_t)$, we can evaluate the conditional expectation

$$\mathbb{E}_{\mathbb{P}^\beta}\left(\mathbb{1}_{\{t<\tau\le T\}}\beta_t\beta_\tau^{-1}\,|\,M_t\right).$$

For example, if β is a deterministic function of time then we have

$$\mathbb{E}_{\mathbb{P}^\beta}\left(\mathbb{1}_{\{t<\tau\le T\}}\beta_t\beta_\tau^{-1}\,|\,M_t\right) = \beta_t\int_t^T \beta_s^{-1}\,\mathbb{P}^\beta(\tau\in ds\,|\,M_t).$$

5.2.2 Premium Payment Leg

Let $\mathcal{T} = \{T_1, T_2, \dots, T_J\}$ be the tenor of the *premium payment*, where $0 = T_0 < T_1 < \cdots < T_J < T$. If the premium accrual covenant is in force, then the cash flows associated with the premium payment leg are

$$\kappa\left(\sum_{j=1}^J \mathbb{1}_{\{T_j<\tau\}}\mathbb{1}_{T_j}(t) + \sum_{j=1}^J \mathbb{1}_{\{T_{j-1}<\tau\le T_j\}}\mathbb{1}_\tau(t)\frac{t-T_{j-1}}{T_j-T_{j-1}}\right),$$

where κ is the *CDS premium* (also known as the *CDS spread*). Thus, the time-t value of the premium payment leg equals $\kappa B_t^{(1)}$, where

$$B_t^{(1)} = \mathbb{E}_{\mathbb{P}^\beta}\left(\mathbb{1}_{\{t<\tau\}}\left[\sum_{j=j(t)}^J \frac{\beta_t}{\beta_{T_j}}\mathbb{1}_{\{T_j<\tau\}} + \sum_{j=j(t)}^J \frac{\beta_t}{\beta_\tau}\mathbb{1}_{\{T_{j-1}<\tau\le T_j\}}\frac{\tau-T_{j-1}}{T_j-T_{j-1}}\right]\,\middle|\,M_t\right),$$

where $j(t)$ is the smallest integer such that $T_{j(t)} > t$. Again, since we know the conditional density $\mathbb{P}^\beta(\tau \in ds\,|\,M_t)$, this expectation can be computed given our assumption about the numeraire β.

5.3 Forward CDS

As before, the reference claim is a defaultable bond maturing at time U. We now consider a *forward (start) CDS* with the maturity date $T^M < U$ and the start date $T^S < T^M$. If default occurs prior to or at time T^S the contract is terminated with no exchange of payments. Therefore, the two legs of this CDS are manifestly T^S-survival claims, and the valuation of a forward CDS is not much different from valuation a straight CDS discussed above.

5.3.1 Default Payment Leg

As before, we let $N = 1$ be the notional amount of the bond, and we let δ be a deterministic recovery rate in case of default. The recovery is paid at default, so that the cash flow associated with the default payment leg of the forward CDS can be represented as follows

$$(1-\delta)\mathbb{1}_{\{T^S<\tau\leq T^M\}}\mathbb{1}_\tau(t).$$

For any $t \leq T^S$, the time-t value of the default payment leg is equal to

$$A_t^{(1),T^S} = (1-\delta)\,\mathbb{E}_{\mathbb{P}^\beta}\big(\mathbb{1}_{\{T^S<\tau\leq T^M\}}\beta_t\beta_\tau^{-1}\,|\,M_t\big).$$

As explained above, we can compute this conditional expectation. If β is a deterministic function of time then simply

$$\mathbb{E}_{\mathbb{P}^\beta}\big(\mathbb{1}_{\{T^S<\tau\leq T^M\}}\beta_t\beta_\tau^{-1}\,|\,M_t\big) = \beta_t\int_{T^S}^{T^M}\beta_s^{-1}\,\mathbb{P}^\beta(\tau\in ds\,|\,M_t).$$

5.3.2 Premium Payment Leg

Let $\mathcal{T} = \{T_1, T_2, \dots, T_J\}$ be the tenor of premium payment, where $T^S < T_1 < \cdots < T_J < T^M$. As before, we assume that the premium accrual covenant is in force, so that the cash flows associated with the premium payment leg are

$$\kappa\left(\sum_{j=1}^J \mathbb{1}_{\{T_j<\tau\}}\mathbb{1}_{T_j}(t) + \sum_{j=1}^J \mathbb{1}_{\{T_{j-1}<\tau\leq T_j\}}\mathbb{1}_\tau(t)\frac{t-T_{j-1}}{T_j-T_{j-1}}\right).$$

Thus, for any $t \leq T^S$ the time-t value of the premium payment leg is $\kappa B_t^{(1),T^S}$, where

$$B_t^{(1),T^S} = \mathbb{E}_{\mathbb{P}^\beta}\left(\mathbb{1}_{\{T_S<\tau\}}\left[\sum_{j=1}^J\frac{\beta_t}{\beta_{T_j}}\mathbb{1}_{\{T_j<\tau\}} + \sum_{j=1}^J\frac{\beta_t}{\beta_\tau}\mathbb{1}_{\{T_{j-1}<\tau\leq T_j\}}\frac{\tau-T_{j-1}}{T_j-T_{j-1}}\right]\,\middle|\, M_t\right).$$

Again, knowing the conditional density $\mathbb{P}^\beta(\tau\in ds\,|\,M_t)$, we can evaluate this conditional expectation.

5.4 CDS Swaptions

We consider a forward CDS swap starting at T^S and maturing at $T^M > T^S$, as described in the previous section. We shall now value the corresponding *CDS swaption* with expiry date $T < T^S$. Let K be the strike CDS rate of the swaption. Then the swaption cash flow at expiry date T equals

$$\big(A_T^{(1),T^S} - KB_T^{(1),T^S}\big)^+,$$

so that the price of the swaption equals, for any $t \leq T$,

$$\mathbb{E}_{\mathbb{P}^\beta}\Big(\beta_t\beta_T^{-1}\big(A_T^{(1),T^S}-KB_T^{(1),T^S}\big)^+\,\Big|\,M_t\Big) = \mathbb{E}_{\mathbb{P}^\beta}\Big(\beta_t\beta_T^{-1}B_T^{(1),T^S}\big(\kappa^{(1)}(t,T^S,T^M)-K\big)^+\,\Big|\,M_t\Big),$$

where $\kappa^{(1)}(t,T^S,T^M) := A_t^{(1),T^S}/B_t^{(1),T^S}$ is the *forward CDS rate.* Note that the random variables $A_t^{(1),T^S}$ and $B_t^{(1),T^S}$ are strictly positive on the set $\{\tau > T\}$ for $t \leq T < T^S$, so that $\kappa^{(1)}(t,T^S,T^M)$ enjoys the same property.

5.4.1 Conditionally Gaussian Case

We shall now provide a more explicit representation for the value of a CDS swaption. To this end, we fix a Markovian numeraire β and we assume that the forward CDS swap rates $\kappa^{(1)}(t,T^S,T^M)$ are conditionally log-Gaussian under $\mathbb{P}^\beta$ for $t \leq T$ (for an example of such a model, see Section 4.3.3). Then we have the following result.

Proposition 5.1 *Suppose that, on the set $\{\tau > T\}$ and for arbitrary $t < t_1 < \cdots < t_n \leq T$, the conditional distribution*

$$\mathbb{P}^\beta\Big(\kappa^{(1)}(t_1,T^S,T^M) \leq k_1, \kappa^{(1)}(t_2,T^S,T^M) \leq k_2, \ldots, \kappa^{(1)}(t_n,T^S,T^M) \leq k_n \,\Big|\, \sigma(M_t) \vee \mathcal{F}_T^X\Big)$$

is $\mathbb{P}^\beta$-a.s. log-Gaussian. Let $\sigma(s,T^S,T^M)$, $s \in [t,T]$, denote the conditional volatility of the process $\kappa^{(1)}(s,T^S,T^M)$, $s \in [t,T]$, given the σ-field $\sigma(M_t) \vee \mathcal{F}_T^X$. Then the price of a CDS swaption equals, for $t < T$,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^\beta}\left(\beta_t\beta_T^{-1}\big(A_T^{(1),T^S} - KB_T^{(1),T^S}\big)^+ \,\Big|\, M_t\right) \\
&= \mathbb{E}_{\mathbb{P}^\beta}\left(\mathbb{1}_{\{\tau>T\}}\beta_t\beta_T^{-1}B_T^{(1),T^S}\left[\kappa^{(1)}(t,T^S,T^M)N\left(\frac{\ln\frac{\kappa^{(1)}(t,T^S,T^M)}{K}}{\upsilon_{t,T}} + \frac{\upsilon_{t,T}}{2}\right)\right.\right. \\
&\left.\left. - KN\left(\frac{\ln\frac{\kappa^{(1)}(t,T^S,T^M)}{K}}{\upsilon_{t,T}} - \frac{\upsilon_{t,T}}{2}\right)\right] \,\Bigg|\, M_t\right),
\end{aligned}$$

where

$$\upsilon_{t,T}^2 = \upsilon(t,T,T^S,T^M)^2 := \int_t^T \sigma(s,T^S,T^M)^2\,ds.$$

Proof. We have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^\beta}\left(\beta_t\beta_T^{-1}\big(A_T^{(1),T^S} - KB_T^{(1),T^S}\big)^+ \,\Big|\, M_t\right) = \mathbb{E}_{\mathbb{P}^\beta}\left(\mathbb{1}_{\{\tau>T\}}\beta_t\beta_T^{-1}\big(A_T^{(1),T^S} - KB_T^{(1),T^S}\big)^+ \,\Big|\, M_t\right) \\
&= \mathbb{E}_{\mathbb{P}^\beta}\left(\mathbb{1}_{\{\tau>T\}}\beta_t\beta_T^{-1}\mathbb{E}_{\mathbb{P}^\beta}\left(\big(A_T^{(1),T^S} - KB_T^{(1),T^S}\big)^+ \,|\, \sigma(M_t) \vee \mathcal{F}_T^X\right) \,\Big|\, M_t\right) \\
&= \mathbb{E}_{\mathbb{P}^\beta}\left(\mathbb{1}_{\{\tau>T\}}\beta_t\beta_T^{-1}B_T^{(1),T^S}\mathbb{E}_{\mathbb{P}^\beta}\left(\big(\kappa^{(1)}(T,T^S,T^M) - K\big)^+ \,|\, \sigma(M_t) \vee \mathcal{F}_T^X\right) \,\Big|\, M_t\right).
\end{aligned}$$

In view of our assumptions, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^\beta}\left(\big(\kappa^{(1)}(T,T^S,T^M) - K\big)^+ \,\Big|\, \sigma(M_t) \vee \mathcal{F}_T^X\right) \\
&= \kappa^{(1)}(t,T^S,T^M)N\left(\frac{\ln\frac{\kappa^{(1)}(t,T^S,T^M)}{K}}{\upsilon_{t,T}} + \frac{\upsilon_{t,T}}{2}\right) - KN\left(\frac{\ln\frac{\kappa^{(1)}(t,T^S,T^M)}{K}}{\upsilon_{t,T}} - \frac{\upsilon_{t,T}}{2}\right).
\end{aligned}$$

By combining the above equalities, we arrive at the stated formula. □

6 Valuation of Basket Credit Derivatives

In this section, we shall discuss the case of credit derivatives with several underlying credit names. Feasibility of closed-form calculations, such as analytic computation of relevant conditional expected values, depends to great extend on the type and amount of information one wants to utilize. Typically, in order to efficiently deal with exact calculations of conditional expectations, one will need to amend specifications of the underlying model so that information used in calculations is given by a coarser filtration, or perhaps by some proxy filtration.

6.1 k^{th}-to-default CDS

We shall now discuss the valuation of a generic k^{th}*-to-default credit default swap* relative to a portfolio of L reference defaultable bonds. The deterministic notional amount of the i^{th} bond is denoted as N_i, and the corresponding deterministic recovery rate equals δ_i. We suppose that the maturities of the bonds are $U_1, U_2, \dots, U_L$, and the maturity of the swap is $T < \min\{U_1, U_2, \dots, U_L\}$.

As before, we shall only discuss a vanilla basket CDS written on such a portfolio of corporate bonds under the fractional recovery of par covenant. Thus, in the event that $\tau^{(k)} < T$, the buyer of the protection is paid at time $\tau^{(k)}$ a cumulative compensation

$$\sum_{i \in \mathcal{L}_k} (1 - \delta_i) N_i,$$

where $\mathcal{L}_k$ is the (random) set of all reference credit names that defaulted in the time interval $]0, \tau^{(k)}]$. This means that the protection buyer is protected against the cumulative effect of the first k defaults. Recall that, in view of our model assumptions, the possibility of simultaneous defaults is excluded.

6.1.1 Default Payment Leg

The cash flow associated with the default payment leg is given by the expression

$$\sum_{i \in \mathcal{L}_k} (1 - \delta_i) N_i \mathbb{1}_{\{\tau^{(k)} \le T\}} \mathbb{1}_{\tau^{(k)}}(t),$$

so that the time-t value of the default payment leg is equal to

$$A_t^{(k)} = \mathbb{E}_{\mathbb{P}^\beta} \left(\mathbb{1}_{\{t < \tau^{(k)} \le T\}} \beta_t \beta_{\tau^{(k)}}^{-1} \sum_{i \in \mathcal{L}_k} (1 - \delta_i) N_i \,\Big|\, M_t \right).$$

In general, this expectation will need to be evaluated numerically by means of simulations.

A special case of a k^{th}-to-default-swap is when the protection buyer is protected against losses associated with the last default only. In the case of a *last-to-default credit default swap*, the cash flow associated with the default payment leg is given by the expression

$$(1 - \delta_{\iota^{(k)}}) N_{\iota^{(k)}} \mathbb{1}_{\{\tau^{(k)} \le T\}} \mathbb{1}_{\tau^{(k)}}(t) = \sum_{i=1}^{L} (1 - \delta_i) N_i \mathbb{1}_{\{H_{\tau_i} = k\}} \mathbb{1}_{\{\tau^{(i)} \le T\}} \mathbb{1}_{\tau^{(i)}}(t),$$

where $\iota^{(k)}$ stands for the identity of the k^{th} defaulting credit name. Assuming that the numeraire process β is deterministic, we can represent the value at time t of the default payment leg as follows:

$$\begin{aligned} A_t^{(k)} &= \sum_{i=1}^{L} \mathbb{E}_{\mathbb{P}^\beta} \left(\mathbb{1}_{\{t < \tau_i \le T\}} \mathbb{1}_{\{H_{\tau_i} = k\}} \beta_t \beta_{\tau_i}^{-1} (1 - \delta_i) N_i \,|\, M_t \right) \\ &= \sum_{i=1}^{L} \beta_t (1 - \delta_i) N_i \int_t^T \beta_s^{-1} \, \mathbb{P}^\beta(H_s = k \,|\, \tau_i = s, M_t) \, \mathbb{P}^\beta(\tau_i \in ds \,|\, M_t). \end{aligned}$$

Note that the conditional probability $\mathbb{P}^\beta(H_s = k \,|\, \tau_i = s, M_t)$ can be approximated as

$$\mathbb{P}^\beta(H_s = k \,|\, \tau_i = s, M_t) \approx \frac{\mathbb{P}^\beta(H_s = k, X_{s-\epsilon}^i \ne K, X_s^i = K \,|\, M_t)}{\mathbb{P}^\beta(X_{s-\epsilon}^i \ne K, X_s^i = K \,|\, M_t)}.$$

Hence, if the number L of credit names is small, so that the Kolmogorov equations for the conditional distribution of the process (H, X, Y) can be solved, the value of $A_t^{(k)}$ can be approximated analytically.

6.1.2 Premium Payment Leg

Let $\mathcal{T} = \{T_1, T_2, \dots, T_J\}$ denote the tenor of the premium payment, where $0 = T_0 < T_1 < \cdots < T_J < T$. If the premium accrual covenant is in force, then the cash flows associated with the premium payment leg admit the following representation:

$$\kappa^{(k)} \left(\sum_{j=1}^{J} \mathbb{1}_{\{T_j < \tau^{(k)}\}} \mathbb{1}_{T_j}(t) + \sum_{j=1}^{J} \mathbb{1}_{\{T_{j-1} < \tau^{(k)} \leq T_j\}} \mathbb{1}_{\tau^{(k)}}(t) \frac{t - T_{j-1}}{T_j - T_{j-1}} \right),$$

where $\kappa^{(k)}$ is the CDS premium. Thus, the time-t value of the premium payment leg is $\kappa^{(k)} B_t^{(k)}$, where

$$\begin{aligned} B_t^{(k)} &= \mathbb{E}_{\mathbb{P}^\beta} \left(\mathbb{1}_{\{t < \tau^{(k)}\}} \sum_{j=j(t)}^{N} \frac{\beta_t}{\beta_{T_j}} \mathbb{1}_{\{T_j < \tau^{(k)}\}} \,\middle|\, M_t \right) \\ &\quad + \mathbb{E}_{\mathbb{P}^\beta} \left(\mathbb{1}_{\{t < \tau^{(k)}\}} \sum_{j=j(t)}^{J} \frac{\beta_t}{\beta_{\tau^{(k)}}} \mathbb{1}_{\{T_{j-1} < \tau^{(k)}\} \leq T_j\}} \frac{\tau^{(k)} - T_{j-1}}{T_j - T_{j-1}} \,\middle|\, M_t \right), \end{aligned}$$

where $j(t)$ is the smallest integer such that $T_{j(t)} > t$. Again, in general, the above conditional expectation will need to be approximated by simulation. And again, for a small portfolio size L, if either exact or numerical solution of relevant Kolmogorov equations can be derived, then an analytical computation of the expectation can be done. This is left for a future study.

6.2 Forward k^{th}-to-default CDS

Forward k^{th}-to-default CDS is an analogous structure to the forward CDS. The notation used here is consistent with the notation used previously in Sections 5.3 and 6.1.

6.2.1 Default Payment Leg

The cash flow associated with the default payment leg can be expressed as follows

$$\sum_{i \in \mathcal{L}_k} (1 - \delta_i) N_i \mathbb{1}_{\{T^S < \tau^{(k)} \leq T^M\}} \mathbb{1}_{\tau^{(k)}}(t).$$

Consequently, the time-t value of the default payment leg equals, for every $t \leq T^S$,

$$A_t^{(k),T^S} = \mathbb{E}_{\mathbb{P}^\beta} \Big(\mathbb{1}_{\{T^S < \tau^{(k)} \leq T^M\}} \beta_t \beta_{\tau^{(k)}}^{-1} \sum_{i \in \mathcal{L}_k} (1 - \delta_i) N_i \,\Big|\, M_t \Big).$$

6.2.2 Premium Payment Leg

As before, let $\mathcal{T} = \{T_1, T_2, \dots, T_J\}$ be the tenor of a generic premium payment leg, where $T^S < T_1 < \cdots < T_J < T^M$. Under the premium accrual covenant, the cash flows associated with the premium payment leg are

$$\kappa^{(k)} \left(\sum_{j=1}^{J} \mathbb{1}_{\{T_j < \tau^{(k)}\}} \mathbb{1}_{T_j}(t) + \sum_{j=1}^{J} \mathbb{1}_{\{T_{j-1} < \tau^{(k)} \leq T_j\}} \mathbb{1}_{\tau^{(k)}}(t) \frac{t - T_{j-1}}{T_j - T_{j-1}} \right),$$

where $\kappa^{(k)}$ is the CDS premium. Thus, the time-t value of the premium payment leg is $\kappa^{(k)} B_t^{(k),T^S}$, where

$$B_t^{(k),T^S} = \mathbb{E}_{\mathbb{P}^\beta} \left(\mathbb{1}_{\{t < \tau^{(k)}\}} \left[\sum_{j=1}^{N} \frac{\beta_t}{\beta_{T_j}} \mathbb{1}_{\{T_j < \tau\}} + \sum_{j=1}^{J} \frac{\beta_t}{\beta_\tau} \mathbb{1}_{\{T_{j-1} < \tau^{(k)} \leq T_j\}} \frac{\tau - T_{j-1}}{T_j - T_{j-1}} \right] \,\middle|\, M_t \right).$$

7 Model Implementation

The last section is devoted to a brief discussion of issues related to the model implementation.

7.1 Curse of Dimensionality

When one deals with basket products involving multiple credit names, direct computations may not be feasible. The cardinality of the state space $\mathbf{K}$ for the migration process X is equal to K^L. Thus, for example, in case of $K = 18$ rating categories, as in Moody's ratings,[1] and in case of a portfolio of $L = 100$ credit names, the state space $\mathbf{K}$ has 18^{100} elements.[2] If one aims at closed-form expressions for conditional expectations, but K is large, then it will typically be infeasible to work directly with information provided by the state vector $(X, Y) = (X^1, X^2, \ldots, X^L, Y)$ and with the corresponding generator $\mathbf{A}$. A reduction in the amount of information that can be effectively used for analytical computations will be needed. Such reduction may be achieved by reducing the number of distinguished rating categories – this is typically done by considering only two categories: pre-default and default. However, this reduction may still not be sufficient enough, and further simplifying structural modifications to the model may need to be called for. Some types of additional modifications, such as *homogeneous grouping* of credit names and the *mean-field interactions* between credit names, are discussed in Frey and Backhaus (2004).[3]

7.2 Recursive Simulation Procedure

When closed-form computations are not feasible, but one does not want to give up on potentially available information, an alternative may be to carry approximate calculations by means of either approximating some involved formulae and/or by simulating sample paths of underlying random processes. This is the approach that we opt for.

In general, a simulation of the evolution of the process X will be infeasible, due to the curse of dimensionality. However, the structure of the generator $\mathbf{A}$ that we postulate (cf. (13)) makes it so that simulation of the evolution of process X reduces to recursive simulation of the evolution of processes X^l whose state spaces are only of size K each. In order to facilitate simulations even further, we also postulate that each migration process X^l behaves like a birth-and-death process with absorption at default, and with possible jumps to default from every intermediate state (cf. Section 4.3.1). Recall that $X^{(l)}_t = (X^1_t, \ldots, X^{l-1}_t, X^{l+1}_t, \ldots, X^L_t)$. Given the state $(x^{(l)}, y)$ of the process $(X^{(l)}, Y)$, the intensity matrix of the l^{th} migration process is sub-stochastic and is given as:

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \\ \vdots \\ K-1 \\ K \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & \cdots & K-1 & K \end{array} \\
\begin{pmatrix}
\lambda^l(1,1) & \lambda^l(1,2) & 0 & \cdots & 0 & \lambda^l(1,K) \\
\lambda^l(2,1) & \lambda^l(2,2) & \lambda^l(2,3) & \cdots & 0 & \lambda^l(2,K) \\
0 & \lambda^l(3,2) & \lambda^l(3,3) & \cdots & 0 & \lambda^l(3,K) \\
 & & & & & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda^l(K-1,K-1) & \lambda^l(K-1,K) \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\end{array},
$$

where we set $\lambda^l(x^l, x'^l) = \lambda^l(x, x'_l; y)$. Also, we find it convenient to write $\lambda^l(x^l, x'^l; x^{(l)}, y) = \lambda^l(x, x'_l; y)$ in what follows.

---

[1] We think here of the following Moody's rating categories: Aaa, Aa1, Aa2, Aa3, A1, A2, A3, Baa1, Baa2, Baa3, Ba1, Ba2, Ba3, B1, B2, B3, Caa, D(efault).

[2] The number known as *Googol* is equal to 10^{100}. It is believed that this number is greater than the number of atoms in the entire observed Universe.

[3] Homogeneous grouping was also introduced in Bielecki (2003).

Then the diagonal elements are specified as follows, for $x^l \neq K$,

$$\lambda^l(x,x;y) = -\lambda^l(x^l,x^l-1;x^{(l)},y) - \lambda^l(x^l,x^l+1;x^{(l)},y) - \lambda^l(x^l,K;x^{(l)},y) \\ - \sum_{i\neq l} \Big(\lambda^i(x^i,x^i-1;x^{(i)},y) + \lambda^i(x^i,x^i+1;x^{(i)},y) + \lambda^i(x^i,K;x^{(i)},y) \Big)$$

with the convention that $\lambda^l(1,0;x^{(l)},y) = 0$ for every $l = 1,2,\dots,L$.

It is implicit in the above description that $\lambda^l(K,x^l;x^{(l)},y) = 0$ for any $l = 1,2,\dots,L$ and $x^l = 1,2,\dots,K$. Suppose now that the current state of the process (X,Y) is (x,y). Then the intensity of a jump of the process X equals

$$\lambda(x,y) := -\sum_{l=1}^{L} \lambda^l(x,x;y).$$

Conditional on the occurrence of a jump of X, the probability distribution of a jump for the component X^l, $l = 1,2,\dots,L$, is given as follows:

- probability of a jump from x^l to $x^l - 1$ equals $p^l(x^l,x^l-1;x^{(l)},y) := \frac{\lambda^l(x^l,x^l-1;x^{(l)},y)}{\lambda(x,y)}$,
- probability of a jump from x^l to $x^l + 1$ equals $p^l(x^l,x^l+1;x^{(l)},y) := \frac{\lambda^l(x^l,x^l+1;x^{(l)},y)}{\lambda(x,y)}$,
- probability of a jump from x^l to K equals $p^l(x^l,K;x^{(l)},y) := \frac{\lambda^l(x^l,K;x^{(l)},y)}{\lambda(x,y)}$.

As expected, we have that

$$\sum_{l=1}^{L} \big(p^l(x^l,x^l-1;x^{(l)},y) + p^l(x^l,x^l+1;x^{(l)},y) + p^l(x^l,K;x^{(l)},y) \big) = 1.$$

For a generic state $x = (x^1,x^2,\dots,x^L)$ of the migration process X, we define the *jump space* $\mathcal{J}(x) = \bigcup_{l=1}^{L} \{(x^l-1,l),(x^l+1,l),(K,l)\}$ with the convention that $(K+1,l) = (K,l)$. The notation (a,l) refers to the l^{th} component of X. Given that the process (X,Y) is in the state (x,y), and conditional on the occurrence of a jump of X, the process X jumps to a point in the jump space $\mathcal{J}(x)$ according to the probability distribution denoted by $p(x,y)$ and determined by the probabilities p^l described above. Thus, if a random variable J has the distribution given by $p(x,y)$ then, for any $(x'^l,l) \in \mathcal{J}(x)$, we have that $Prob(J = (x'^l,l)) = p^l(x^l,x'^l;x^{(l)},y)$.

7.2.1 Simulation Algorithm: Special Case

We shall now present in detail the case when the dynamics of the factor process Y do not depend on the credit migrations process X. The general case appears to be much harder.

Under the assumption that the dynamics of the factor process Y do not depend on the process X, the simulation procedure splits into two steps. In Step 1, a sample path of the process Y is simulated; then, in Step 2, for a given sample path Y, a sample path of the process X is simulated. We consider here simulations of sample paths over some generic time interval, say $[t_1,t_2]$, where $0 \leq t_1 < t_2$. We assume that the number of defaulted names at time t_1 is less than k, that is $H_{t_1} < k$. We conduct the simulation until the k^{th} default occurs or until time t_2, whichever occurs first.

Step 1: The dynamics of the factor process are now given by the SDE

$$dY_t = b(Y_t)\,dt + \sigma(Y_t)\,dW_t + \int_{\mathbb{R}^n} g(Y_{t-},y)\pi(Y_{t-};dy,dt), \quad t \in [t_1,t_2].$$

Any standard procedure can be used to simulate a sample path of Y (the reader is referred, for example, to Kloeden and Platen (1995)). Let us denote by $\widehat{Y}$ the simulated sample path of Y.

Step 2: Once a sample path of Y has been simulated, simulate a sample path of X on the interval $[t_1, t_2]$ until the k^{th} default time.

We exploit the fact that, according to our assumptions about the infinitesimal generator $\mathbf{A}$, the components of the process X do not jump simultaneously. Thus, the following algorithm for simulating the evolution of X appears to be feasible:

Step 2.1: Set the counter $n = 1$ and simulate the first jump time of the process X in the time interval $[t_1, t_2]$. Towards this end, simulate first a value, say $\widehat{\eta}_1$, of a unit exponential random variable η_1. The simulated value of the first jump time, τ_1^X, is then given as

$$\widehat{\tau}_1^X = \inf\Big\{ t \in [t_1, t_2] \,:\, \int_{t_1}^{t} \lambda(X_{t_1}, \widehat{Y}_u)\, du \geq \widehat{\eta}_1 \Big\},$$

where by convention the infimum over an empty set is $+\infty$. If $\widehat{\tau}_1^X = +\infty$, set the simulated value of the k^{th} default time to be $\widehat{\tau}^{(k)} = +\infty$, stop the current run of the simulation procedure and go to Step 3. Otherwise, go to Step 2.2.

Step 2.2: Simulate the jump of X at time $\widehat{\tau}_1^X$ by drawing from the distribution $p(X_{t_1}, \widehat{Y}_{\widehat{\tau}_1^X -})$ (cf. discussion in Section 7.2). In this way, one obtains a simulated value $\widehat{X}_{\widehat{\tau}_1^X}$, as well as the simulated value of the number of defaults $\widehat{H}_{\widehat{\tau}_1^X}$. If $\widehat{H}_{\widehat{\tau}_1^X} < k$ then let $n := n + 1$ and go to Step 2.3; otherwise, set $\widehat{\tau}^{(k)} = \widehat{\tau}_1^X$ and go to Step 3.

Step 2.3: Simulate the n^{th} jump of process X. Towards this end, simulate a value, say $\widehat{\eta}_n$, of a unit exponential random variable η_n. The simulated value of the n^{th} jump time τ_n^X is obtained from the formula

$$\widehat{\tau}_n^X = \inf\Big\{ t \in [\widehat{\tau}_{n-1}^X, t_2] \,:\, \int_{\widehat{\tau}_{n-1}^X}^{t} \lambda(X_{\widehat{\tau}_{n-1}^X}, \widehat{Y}_u)\, du \geq \widehat{\eta}_n \Big\}.$$

In case $\widehat{\tau}_n^X = +\infty$, let the simulated value of the k^{th} default time to be $\widehat{\tau}^{(k)} = +\infty$; stop the current run of the simulation procedure, and go to Step 3. Otherwise, go to Step 2.4.

Step 2.4: Simulate the jump of X at time $\widehat{\tau}_n^X$ by drawing from the distribution $p(X_{\widehat{\tau}_{n-1}^X}, \widehat{Y}_{\widehat{\tau}_n^X -})$. In this way, produce a simulated value $\widehat{X}_{\widehat{\tau}_n^X}$, as well as the simulated value of the number of defaults $\widehat{H}_{\widehat{\tau}_n^X}$. If $\widehat{H}_{\widehat{\tau}_n^X} < k$, let $n := n + 1$ and go to Step 2.3; otherwise, set $\widehat{\tau}^{(k)} = \widehat{\tau}_n^X$ and go to Step 3.

Step 3: Calculate a simulated value of a relevant functional. For example, in case of the k^{th}-to-default CDS, compute

$$\widehat{A}_{t_1}^{(k)} = \mathbb{1}_{\{t_1 < \widehat{\tau}^{(k)} \leq T\}} \widehat{\beta}_{t_1} \widehat{\beta}_{\widehat{\tau}^{(k)}}^{-1} \sum_{i \in \widehat{\mathcal{L}}_k} (1 - \delta_i) N_i \tag{25}$$

and

$$\widehat{\beta}_{t_1}^{(k)} = \sum_{j=j(t_1)}^{N} \frac{\widehat{\beta}_{t_1}}{\widehat{\beta}_{T_j}} \mathbb{1}_{\{T_j < \widehat{\tau}^{(k)}\}} + \sum_{j=j(t_1)}^{J} \frac{\widehat{\beta}_{t_1}}{\widehat{\beta}_{\widehat{\tau}^{(k)}}} \mathbb{1}_{\{T_{j-1} < \widehat{\tau}^{(k)} \leq T_j\}} \frac{\widehat{\tau}^{(k)} - T_{j-1}}{T_j - T_{j-1}}, \tag{26}$$

where, as usual, the 'hat' indicates that we deal with simulated values.

7.3 Estimation and Calibration of the Model

Our market model (13) has the same structure under either the pricing probability measure or the statistical measure. The model parameters corresponding to the two measures (or any other two measures for that matter) are related via (19).

Estimation of the statistical parameters of the model, that is, the parameters corresponding to the statistical measure, can be split into two separate problems – the estimation of the dynamics of the factor process Y, and the estimation of the transition intensities of the process X. With regard to the former: typically, the estimation of parameters of the drift function and the estimation of parameters of the Poisson measure is not easy; the estimation of parameters of the volatility function $\sigma(x, y)$ is rather straightforward, as it can be done via estimation of the quadratic variation process of the diffusion component. Estimates of parameters involved in the transition intensities can, in principle, be obtained from the statistical estimates of transition probability matrices that are produced by major rating agencies.

Calibration of the pricing parameters of the model, that is, the parameters corresponding to the pricing measure, depends on the types of the market quotes data used for calibration. Since, in case of basket credit derivatives, we typically will not have access to closed-form pricing formulae, the calibration of the model parameters will need to be done via simulation. For example, if the model is calibrated to market quotes for the k^{th}-to-default basket swaps, in order to select the best fitted model, we shall use simulated averages of expressions (25) and (26) obtained for various parametric settings. Then, the market prices of credit risk can be obtained from estimated and calibrated values of the parameters and from formula (19). We shall deal with the issues of model estimation and calibration in a future work, which will be devoted to model implementation.

7.4 Portfolio Credit Risk

The issue of evaluating functionals associated with multiple credit migrations, defaults in particular, is also prominent with regard to portfolio credit risk. In some segments of the credit markets, only the deterioration of the value of a portfolio of debts (bonds or loans) due to defaults is typically considered. In fact, such is the situation regarding various tranches of (cash or synthetic) collateralized debt obligations (CDOs), as well as with various tranches of recently introduced CDS indices, such as, DJ CDX NA IG or DJ iTraxx Europe.[4] Nevertheless, it is rather apparent that a valuation model reflecting the possibility of intermediate credit migrations, and not only defaults, is called for in order to better account for changes in creditworthiness of the reference credit names. Likewise, for the purpose of managing risks of a debt portfolio, it is necessary to account for changes in value of the portfolio due to changes in credit ratings of the components of the portfolio.

The problem of valuation of tranches of a CDO (or tranches of a CDS index) is closely related to the problem of valuation of the k^{th}-to-default swap. In a future work, we shall focus on implementation of our model to all these problems. It is perhaps worth mentioning though that we have already done some numerical tests of our model so to see whether the model can reproduce so called market correlation skews. The picture below shows that the model performs very well in this regard[5]:

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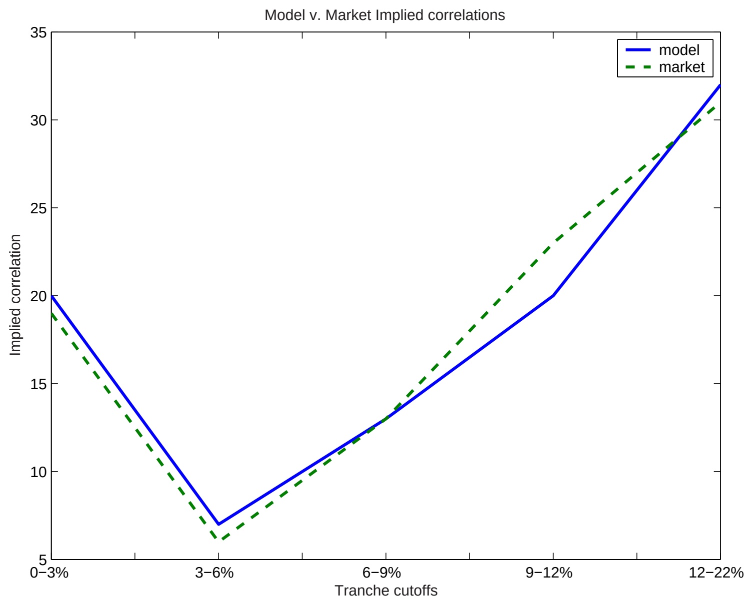
[4]See http://www.creditflux.com/public/publications/0409CFindexGuide.pdf.

[5]We thank Andrea and Luca Vidozzi from Applied Mathematics Department at the Illinois Institute of Technology for numerical implementation of the model and, in particular, for generating the picture.



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