CDS with Counterparty Risk in a Markov Chain Copula Model with Joint Defaults

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Abstract

In this paper we study the counterparty risk on a payer CDS in a Markov chain model of two reference credits, the firm underlying the CDS and the protection seller in the CDS. We first state few preliminary results about pricing and CVA of a CDS with counterparty risk in a general set-up. We then introduce a Markov chain copula model in which wrong way risk is represented by the possibility of joint defaults between the counterpart and the firm underlying the CDS. In the set-up thus specified we have semi-explicit formulas for most quantities of interest with regard to CDS counterparty risk like price, CVA, EPE or hedging strategies. Model calibration is made simple by the copula property of the model. Numerical results show adequation of the behavior of EPE and CVA in the model with stylized features.

Contents

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1 Introduction

Since the sub-prime crisis, counterparty risk is a crucial issue in connection with valuation and risk management of credit derivatives. In general terms, counterparty risk is 'the risk that a party to an OTC derivatives contract may fail to perform on its contractual obligations, causing losses to the other party' (cf. Canabarro and Duffie [12]). A major issue in this regard is the so-called *wrong way risk*, namely the risk that the value of the contract be particularly high from the perspective of the other party at the moment of default of the counterparty. Classic examples of wrong way risk are selling a put option to a company on its own stock, or entering a forward contract in which oil is bought by an airline company (see Redon [20]).

Among papers dealing with counterparty risk in general, one can mention, apart from the above-mentioned references, Canabarro et al. [13], Zhu and Pykhtin [22], and the series of papers by Brigo et al. [8, 9, 10, 11]. From the point of view of measurement and management of counterparty risk, two important notions emerge:

• The Credit Value Adjustment process (CVA), which measures the depreciation of a contract due to counterparty risk. So, in rough terms, $\mathrm{CVA}_t = P_t - \Pi_t$, where Π and P denote the price process of a contract depending on whether one accounts or not for counterparty risk.

• The Expected Positive Exposure process (EPE), where EPE_t is the risk-neutral expectation of the loss on a contract conditional on a default of the counterparty occurring at time t, which can be related to the CVA by a suitable integral representations.

Note that the CVA process can be interpreted as an option price process, so that counterparty risk can, in principle, be dynamically managed.

1.1 Counterparty Credit Risk

Wrong way risk is particularly important in the case of *credit derivatives* transactions, at least from the perspective of a credit protection buyer, Indeed, via economic cycle and default contagion effects, the moment of default of a counterparty selling credit protection is typically a moment of higher value of credit protection.
In a first attempt to deal with counterparty credit risk, we consider in this paper the problem of valuing and hedging a ***Credit Default Swap with counterparty risk*** ('risky CDS' in the sequel, as opposed to 'risk-free CDS', without counterparty risk). Note that this problem already received a lot of attention in the literature. This admittedly specific problem can thus be considered as a benchmark problem of counterparty credit risk. To quote but a few:
• Huge and Lando [15] propose a rating-based approach,
• Hull and White [16] study this problem in the set-up of a static copula model,
• Jarrow and Yu [17] introduce an intensity contagion model, further considered in Leung and Kwok [18],
• Brigo and Chourdakis [8] work in the set-up of their Gaussian copula and CIR++ intensity model, extended to the issue of bilateral counterparty credit risk in Brigo and Capponi [7],
• Blanchet-Scalliet and Patras [6] or Lipton and Sepp [19] develop structural approaches.

1.2 A Markov Copula Approach

Here we consider a Markovian model of credit risk in which simultaneous defaults are possible Wrong way risk is thus represented in the model by the fact that at the time of default of the counterparty, there is a positive probability that the firm on which the CDS is written defaults too, in which case the loss due to counterparty risk is the loss given default of the firm, that is a very large amount. Of course, this simple model should not be taken too literally. We are not claiming here that simultaneous defaults can happen in actual practice. The rationale and financial interpretation of our model is rather that at the time of default of the counterparty, there is a positive probability of a high spreads environment, in which case, the value of the CDS for a protection buyer is close (if not equal) to the loss given default of the firm.

A nice feature of a Markovian set-up such as the one we use here is that it is possible to address in a dynamic and consistent ways the issues of valuing and hedging the CDS (and/or, if wished, the CVA, interpreted as an option as evoked above). More precisely, we shall be considering a four-state Markov Chain model of two obligors, so that all the computations are straightforward, either that there are explicit formulas for all the quantities of interest, or, in case less elementary parameterizations of the model are used, that these quantities can be easily and quickly computed by solving numerically the related Kolmogorov ODEs.

To make this even more practical, we shall work in a *Markovian copula* set-up (in the sense of Bielecki et al. [3]), in which calibration of the model marginals to the related CDS curves is straightforward (note that actual market CDS curves can be considered as 'risk-free CDS curves'). The

only really free model parameters are thus the few dependence parameters, which can be calibrated or estimated in ways that we shall explain in the paper.

1.3 Outline of the Paper

In Section 2 we first describe the mechanism and cash flows of a (payer) CDS with counterparty credit risk. We then state a few preliminary results about pricing and CVA of this CDS in a general set-up. In Section 3 we introduce our Markov chain copula model. In the set-up thus specified we are then able to derive explicit formulas for most quantities of interest in regard to a risky CDS, like price, EPE, CVA or hedging strategies. Section 4 is about implementation of the model. Alternative model parameterizations and related calibration or estimation procedures are proposed and analyzed. Numerical results are presented and discussed, showing good agreement of model's EPE and CVA with expected features. Finally Section 5 recapitulates our model's main properties and presents some directions for possible extensions of the previous results in terms of models (with, possibly, spread volatility) and/or products (in view of integrating the CDS-CVA tool of this paper into a real-life cross-products and markets CVA engine).

The ***main results*** are Proposition 3.2, yielding the pricing formula for a risky CDS in our set-up, and Proposition 3.4, yielding the formulas for the related EPE and CVA.

2 General Set-Up

2.1 Cash Flows

As is well known, a CDS contract involves three entities: A reference credit (firm), a buyer of default protection on the firm, and a seller of default protection on the firm. The issue of counterparty risk on a CDS is:

• Primarily, the fact that the seller of protection may fail to pay the protection cash flows to the buyer in case of a default of the firm;

• Also, the symmetric concern that the buyer may fail to pay the contractual CDS spread to the seller.

We shall focus in this paper on the so-called *unilateral counterparty credit risk* involved in a payer CDS contract, namely the risk corresponding to the first bullet point above; however it should be noted that the approach of this paper could be extended to the issue of bilateral credit risk, in a suitably enriched version of the model (see the Introduction for various approaches to unilateral or bilateral CDS counterparty credit risk). With this in mind we shall find convenient to refer to the buyer and the seller of protection on the firm as the (risk-free) *investor* and the (defaultable) *counterpart*, respectively. Indices 1 and 2 will refer to quantities related to the firm and to the counterpart, first of which, the related default times τ_1 and τ_2.

Under a risky CDS, the investor pays to the counterpart a stream of premia with spread κ, or *Fees Cash Flows*, from the inception date (time 0 henceforth) until the occurrence of a credit event (default of the counterpart or the firm) or the maturity T of the contract, whichever comes first.

Let us denote by R_1 and R_2 the recovery of the firm and the counterpart, supposed to be adapted to

the information available at time τ_1 and τ_2, respectively. If the firm defaults prior to the expiration of the contract, the *Protection Cash Flows* paid by the counterpart to the investor depends on the situation of the counterpart:
• If the counterpart is still alive, she can fully compensate the loss of investor, i.e., she pays $(1-R_1)$ times the face value of the CDS to the investor;
• If the counterpart defaults at the same time as the firm (note that it is important to take this case into account in the perspective of the model with simultaneous defaults to be introduced later in this paper), she will only be able to pay to the investor a fraction of this amount, namely $R_2(1-R_1)$ times the face value of the CDS.

Finally, there is a *Close-Out Cash Flow* which is associated to clearing the positions in the case of early default of the counterpart. As of today, CDSs are sold over-the-counter (OTC), meaning that the two parties have to negotiate and agree on the terms of the contract. In particular the two parties can agree on one of the following three possibilities to exit (unwind) a trade:
• *Termination:* The contract is stopped after a terminal cash flow χ (positive or negative) has been paid to the investor.
• *Offsetting:* The counterpart takes the opposite protection position. This new contract should have virtually the same terms as the original CDS except for the premium which is fixed at the prevailing market level, and for the tenor which is set at the remaining time to maturity of the original CDS. So the counterpart leaves the original transaction in place but effectively cancels out its economic effect.
• *Novation* (or *Assignment*): The original CDS is assigned to a new counterpart, settling the amount of gain or loss with him. In this assignment the original counterpart (transferor), the new counterpart (transferee) and the investor agree to transfer all the rights and obligations of the transferor to transferee. So the transferor thereby ends his involvement in the contract and the investor thereafter deals with the default risk of the transferee.
In this paper we shall focus on **termination**. The amount χ is supposed to be adapted to the information available at time τ_2. For emphasizing this feature it is denote henceforth $\chi_{(\tau_2)}$. More precisely, $\chi = \chi_{(\tau_2)}$ should be computable at time τ_2 according to a methodology specified in the CDS contract at inception. If the counterpart defaults in the life-time of the CDS while the firm is alive ($\tau_2 < \tau_1 \wedge T$), the value of $\chi_{(\tau_2)}$ is calculated. If it is positive for the counterpart, she will be paid fully by the investor and if it is negative for the counterpart, she will pay the investor just a portion R_2 of it.

Remark 2.1 A typical specification is $\chi_{(\tau_2)} = P_{\tau_2}$, where P_t is the value at time t of a risk-free CDS on the same reference name, with the same contractual maturity T and spread κ as the original (risky) CDS. The consistency of this rather standard way of specifying $\chi_{(\tau_2)}$ is, in a sense, questionable. Given a pricing model accounting for the major risks in the product at hand, including, if need be, counterparty credit risk, with related price process of the risky CDS denoted by Π, it could be argued that a more consistent specification would be $\chi_{(\tau_2)} = \Pi_{\tau_2}$ (or $\chi_{(\tau_2)} = \Pi_{\tau_2-}$, since $\Pi_{\tau_2} = 0$ in view of the usual conventions regarding the definition of ex-dividend prices). However we shall see henceforth that, at least in the specific model of this paper, adopting either convention makes little difference in practice.

2.2 Pricing

Let us be given a (risk-neutral) pricing model $(\Omega, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a given filtration making the τ_i's stopping times, and with a related (adapted) *discount factor* process β. So in particular $\chi_{(\tau_2)}$ is supposed to be an $\mathcal{F}_{\tau_2}$-measurable random variable. Let $\mathbb{E}_t$ stands for the conditional expectation under $\mathbb{P}$ given $\mathcal{F}_t$. We assume for simplicity that the face value of all the CDSs under consideration (risky or not) is equal to monetary unit and that the spreads are paid continuously in time. All the cash flows and prices are considered from the perspective of the investor. In accordance with the usual convention regarding the definition of *ex-dividend* prices, all the integrals are taken open on the left and closed on the right of the interval of integration. In view of the description of the cash-flows in section 2.1, one then has,

Definition 2.2 **(i)** The model (ex-dividend) *price* process of a risky CDS is given by $\Pi_t = \mathbb{E}_t\big[\pi_T(t)\big]$, where $\pi_T(t)$ corresponds to the *risky CDS cumulative discounted cash flows* on the time interval $(t, T]$, so,

$$\begin{aligned}\beta_t\pi_T(t) &= -\kappa\int_{t\wedge\tau_1\wedge\tau_2}^{T\wedge\tau_1\wedge\tau_2}\beta_s ds + \beta_{\tau_1}(1-R_1)\mathbb{1}_{t<\tau_1\le T}\big[\mathbb{1}_{\tau_1<\tau_2} + R_2\mathbb{1}_{\tau_1=\tau_2}\big] \\ &\quad +\beta_{\tau_2}\mathbb{1}_{t<\tau_2\le T}\mathbb{1}_{\tau_2<\tau_1}\big[R_2\chi^+_{(\tau_2)} - \chi^-_{(\tau_2)}\big]\ . \end{aligned} \tag{1}$$

(ii) The model (ex-dividend) *price* process of a risk-free CDS is given by $P_t = \mathbb{E}_t[p_T(t)]$, where $p_T(t)$ corresponds to the *risk-free CDS cumulative discounted cash flows* on the time interval $(t, T]$, so,

$$\beta_t p_T(t) = -\kappa\int_{t\wedge\tau_1}^{T\wedge\tau_1}\beta_s ds + (1-R_1)\beta_{\tau_1}\mathbb{1}_{t<\tau_1\le T}. \tag{2}$$

The first, second and third term on the right-hand side of (1) correspond to the fees, protection and close-out cash flows of a risky CDS, respectively. Note that there are no cash flows of any kind after time $\tau_1 \wedge \tau_2 \wedge T$ (in the case of the risky CDS) or $\tau_1 \wedge T$ (in the case of the risk-free CDS).

Remark 2.3 In these definitions it is implicitly assumed that, consistently with the usual theory of no-arbitrage (see, e.g., Delbaen and Schachermayer [14]), a primary market of financial instruments (along with the risk-free asset β^{-1}) has been defined, with price processes given as (locally bounded, say for simplicity) $(\Omega, \mathbb{F}, \mathbb{P})$ – local martingales. No-arbitrage on the extended market consisting of the primary assets and a further CDS then results in the previous definitions. Since the precise specification of the primary market is irrelevant until the question of hedging is dealt with, we leave it for section 3.3.

Let the $\mathcal{F}_{\tau_2}$-measurable random variable $\xi_{(\tau_2)}$, interpreted as the *loss incurred at τ_2 due to counterparty risk*, be given as

$$\xi_{(\tau_2)} = \begin{cases} (1-R_1)(1-R_2), & \tau_2 = \tau_1 < T, \\ \chi^+_{(\tau_2)}(1-R_2), & \tau_2 < \tau_1 \wedge T, \\ 0, & \text{otherwise}\ . \end{cases} \tag{3}$$

Definition 2.4 **(i)** The Credit Valuation Adjustment (CVA) is the $\mathbb{F}$-adapted process killed at $\tau_1 \wedge \tau_2 \wedge T$ defined by, for $t \in [0, T]$,

$$\beta_t \mathrm{CVA}_t = 1\!\!1_{\{t<\tau_1\wedge\tau_2\}} \mathbb{E}_t\big[\beta_{\tau_2}\xi_{(\tau_2)}\big] \ . \tag{4}$$

(ii) The *Expected Positive Exposure* (EPE) is the function of time defined by, for $t \in [0, T]$,

$$\mathrm{EPE}(t) = \mathbb{E}\left[\xi_{(\tau_2)}|\tau_2 = t\right] \ . \tag{5}$$

Remark 2.5 Again the subscript (τ_2) in $\xi_{(\tau_2)}$ emphasizes that $\xi_{(\tau_2)}$ is an $\mathbb{F}_{\tau_2}$-measurable random variable.

Let Π^0 denote Π in case $\chi_{(\tau_2)} = P_{\tau_2}$. The following Proposition justifies the name of Credit Valuation Adjustment which is used for the CVA process defined by (4). This is essentially the basic result that appears in the series of papers by Brigo et alii. Note that as opposed to Brigo et al. we do not exclude simultaneous defaults in our set-up, whence further terms in $1\!\!1_{t<\tau_1=\tau_2\leq T}$ in the proof of Proposition 2.1.

Proposition 2.1 *$CVA_t = P_t - \Pi^0_t$ on $\{t < \tau_2\}$.*

Proof. If $\{\tau_1 \leq t < \tau_2\}$, $\mathrm{CVA}_t = P_t - \Pi^0_t = 0$. Assume $t < \tau_1 \wedge \tau_2$. Subtracting $\pi_T(t)$ from $p_T(t)$ yields,

$$\begin{aligned}\beta_t\left(p_T(t) - \pi_T(t)\right) = -\kappa \int_{\tau_1\wedge\tau_2\wedge T}^{\tau_1\wedge T} \beta_s ds + \beta_{\tau_1}(1-R_1)1\!\!1_{t<\tau_1\leq T}1\!\!1_{\tau_1\geq\tau_2} \\ -\beta_{\tau_1}R_2(1-R_1)1\!\!1_{t<\tau_1\leq T}1\!\!1_{\tau_1=\tau_2} - \beta_{\tau_2}1\!\!1_{\tau_2<\tau_1}1\!\!1_{\tau_2\leq T}\big(R_2\chi^+_{(\tau_2)} - \chi^-_{(\tau_2)}\big) \ ,\end{aligned} \tag{6}$$

where the following identity was used in the second term on the right hand side of (6):

$$1\!\!1_{t<\tau_1\leq T}1\!\!1_{\tau_1\geq\tau_2} = 1\!\!1_{t<\tau_1\leq T}1\!\!1_{\tau_2<\tau_1} + 1\!\!1_{t<\tau_1=\tau_2\leq T}.$$

Moreover, in view of (2), one has,

$$\beta_{\tau_2}p_T(\tau_2)1\!\!1_{\tau_2<\tau_1}1\!\!1_{\tau_2\leq T} = -\kappa\int_{\tau_1\wedge\tau_2\wedge T}^{\tau_1\wedge T}\beta_s ds + (1-R_1)\beta_{\tau_1}1\!\!1_{\tau_2<\tau_1\leq T}. \tag{7}$$

Plugging (7) into (6) , it comes,

$$\begin{aligned}\beta_t\left(p_T(t) - \pi_T(t)\right) &= \beta_{\tau_2}1\!\!1_{\tau_2<\tau_1}1\!\!1_{\tau_2\leq T}p_T(\tau_2) \\ &\quad + \beta_{\tau_2}1\!\!1_{\tau_2=\tau_1}1\!\!1_{t<\tau_1\leq T}(1-R_2)(1-R_1) - \beta_{\tau_2}1\!\!1_{\tau_2<\tau_1}1\!\!1_{\tau_2\leq T}\big(R_2\chi^+_{(\tau_2)} - \chi^-_{(\tau_2)}\big).\end{aligned}$$

In case $\chi_{(\tau_2)} = P_{\tau_2}$ one can then proceed as follows:
• On the set $\{\tau_2 < \tau_1\}$,

$$\beta_t\left(p_T(t) - \pi_T(t)\right) = \beta_{\tau_2}p_T(\tau_2) - \beta_{\tau_2}\big(R_2P^+_{\tau_2} - P^-_{\tau_2}\big)$$

As $P_{\tau_2} = \mathbb{E}_{\tau_2}[p_T(\tau_2)]$, we have

$$\beta_t \mathbb{E}_{\tau_2}\big[p_T(t) - \pi_T(t)\big] = \beta_{\tau_2} P^+_{\tau_2}(1 - R_2)\ . \tag{8}$$

• On the set $\{\tau_1 = \tau_2\}$,

$$\beta_t\left(p_T(t) - \pi_T(t)\right) = \beta_{\tau_2}(1 - R_1)(1 - R_2)$$

and thus

$$\beta_t \mathbb{E}_{\tau_2}\big[p_T(t) - \pi_T(t)\big] = \mathbb{E}_{\tau_2}\big[\beta_{\tau_2}(1 - R_1)(1 - R_2)\big]. \tag{9}$$

One thus has on the set $t < \tau_2$, using the fact that $\tau_2 < \tau_1$ and $\tau_2 = \tau_1$ are $\mathcal{F}_{\tau_2}$-measurable,

$$\begin{aligned}
\beta_t P_t - \beta_t \Pi^0_t &= \beta_t \mathbb{E}_t[p_T(t)] - \beta_t \mathbb{E}_t[\pi_T(t)] = \beta_t \mathbb{E}_t\big[\mathbb{E}_{\tau_2}[p_T(t) - \pi_T(t)]\big] \\
&= \beta_t \mathbb{E}_t\Big[\mathbb{E}_{\tau_2}[p_T(t) - \pi_T(t)]\mathbb{1}_{\tau_2<\tau_1} + \mathbb{E}_{\tau_2}[p_T(t) - \pi_T(t)]\mathbb{1}_{\tau_2=\tau_1}\Big] \\
&= \mathbb{E}_t\big[\beta_{\tau_2}\chi_{(\tau_2)}\big] = \mathrm{CVA}_t.
\end{aligned}$$

□

2.3 Special Case $\mathbb{F} = \mathbb{H}$

The following proposition gathers a few useful results that can be established in the special case of a model filtration $\mathbb{F}$ given as $\mathbb{F} = \mathbb{F}^H =: \mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$, where $H = (H^1, H^2)$ denotes the pair of the default indicator processes of the firm and the counterpart.

Proposition 2.2 **(i)** *For $t \in [0, T]$, any $\mathcal{H}_t$-measurable random variable Y_t can be written as*

$$Y_t = y_0(t)\mathbb{1}_{t<\tau_1\wedge\tau_2} + y_1(t,\tau_1)\mathbb{1}_{\tau_1\le t<\tau_2} + y_2(t,\tau_2)\mathbb{1}_{\tau_2\le t<\tau_1} + y_3(t,\tau_1,\tau_2)\mathbb{1}_{\tau_2\vee\tau_1<t}$$

where $y_0(t)$, $y_1(t,u)$, $y_2(t,v)$, $y_3(t,u,v)$ are deterministic functions.
(ii) *For any random variable Z, one has,*

$$\mathbb{1}_{t<\tau_1\wedge\tau_2}\mathbb{E}(Z|\mathcal{F}_t) = \mathbb{1}_{t<\tau_1\wedge\tau_2}\frac{\mathbb{E}(Z\mathbb{1}_{t<\tau_1\wedge\tau_2})}{\mathbb{P}(t<\tau_1\wedge\tau_2)}\ . \tag{10}$$

(iii) $\Pi_t = \Pi(t, H_t)$, *for a* pricing function Π *defined on $R^+ \times E \times E$ with $E = \{0, 1\}$, such that $\Pi(t, e) = 0$ for $e \neq (0, 0)$. In particular, Π_t is given on the set $t < \tau_1 \wedge \tau_2$ by a deterministic function*

$$\Pi(t,0,0) = u(t) := \frac{\mathbb{E}\big[\pi_T(t)\big]}{\mathbb{P}(\tau_1\wedge\tau_2 > t)}\ . \tag{11}$$

(iv) *One has, for suitable functions $\tilde{\chi}(\cdot)$ and CVA$(\cdot)$,*

$$\mathbb{1}_{\{\tau_2<\tau_1\}}\chi_{(\tau_2)} = \mathbb{1}_{\{\tau_2<\tau_1\}}\tilde{\chi}(\tau_2) \tag{12}$$

$$\xi_{(\tau_2)} = \tilde{\xi}(\tau_1,\tau_2) := (1 - R_2)\big((1 - R_1)\mathbb{1}_{\tau_2=\tau_1<T} + \mathbb{1}_{\tau_2<\tau_1\wedge T}\tilde{\chi}^+(\tau_2)\big) \tag{13}$$

$$CVA_t = \mathbb{1}_{t<\tau_1\wedge\tau_2}CVA(t)\ . \tag{14}$$

(v) *One has, for* $t \in [0,T]$,

$$\beta_t CVA(t) = \int_t^T \beta_s EPE(s) \frac{\mathbb{P}(\tau_2 \in ds)}{\mathbb{P}(t < \tau_1 \wedge \tau_2)} \tag{15}$$

Proof. **(i)** and **(ii)** are standard (see, e.g., [5]; (ii) in particular is the so-called *Key Lemma*).
(iii) Since there are no cash flows of a risky CDS beyond the first default (cf. (1)), one has $\pi_T(t) = \pi_T(t)\mathbb{1}_{t<\tau_1\wedge\tau_2}$. The Key Lemma then yields,

$$\Pi_t = \mathbb{E}_t\big[\mathbb{1}_{t<\tau_1\wedge\tau_2}\pi_T(t)\big] = (1-H_t^1)(1-H_t^2)\frac{\mathbb{E}\big[\pi_T(t)\big]}{\mathbb{P}(\tau_1\wedge\tau_2 > t)} \cdot$$

Thus $\Pi_t = \Pi(t, H_t^1, H_t^2)$, for a pricing function Π defined by

$$\Pi(t, e_1, e_2) = (1-e_1)(1-e_2)u(t) \ ,$$

where $u(t)$ is defined by the right-hand-side in (11).
(iv) follows directly from part (i), given the definitions of the random variable $\chi_{(\tau_2)}$ and of the CVA process (see also (3) regarding (13)).
(v) By (iv), one has,

$$\begin{aligned}
\beta_t \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\text{CVA}(t) &= \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\mathbb{E}_t\big[\beta_{\tau_2}\xi_{(\tau_2)}\big] \\
&= \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\mathbb{E}_t\big[\beta_{\tau_2}\tilde{\xi}(\tau_1,\tau_2)\big] = \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\int_t^T \beta_s \mathbb{E}_t\big[\tilde{\xi}(\tau_1,s)\mathbb{1}_{\{\tau_2\in ds\}}\big] \\
&= \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\int_t^T \beta_s \frac{\mathbb{E}\big[\tilde{\xi}(\tau_1,s)\mathbb{1}_{\{\tau_2\in ds\}}\mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\big]}{\mathbb{P}(t<\tau_1\wedge\tau_2)} = \mathbb{1}_{\{t<\tau_1\wedge\tau_2\}}\int_t^T \beta_s \text{EPE}(s)\frac{\mathbb{P}(\tau_2\in ds)}{\mathbb{P}(t<\tau_1\wedge\tau_2)} \ ,
\end{aligned}$$

which is (15). □

3 Markov Copula Factor Set-Up

3.1 Factor Process Model

We shall now introduce a suitable *Markovian Copula Model* for the pair of default indicator processes $H = (H^1, H^2)$ of the firm and the counterpart. The name 'Markovian Copula' refers to the fact that the model will have prescribed marginals for the laws of H^1 and H^2, respectively (see Bielecki et al. [2, 3] for a general theory). The practical interest of a Markovian copula model is clear with respect to the task of model calibration, since the copula property allows one to decouple the calibration of the marginal and of the dependence parameters in the model (see section 4.1). More fundamentally, the opinion developed in this paper is that it is also a virtue for a model to 'take the right inputs to generate the right outputs', namely taking as basic inputs the individual default probabilities (individual CDS curves), which correspond to the more reliable information on the market, and are then 'coupled together' in a suitable way (see section 4.1).

An apparent shortcoming of the Markov copula approach is that it is not compatible with default contagion effects in the usual sense (default of a name impacting the default intensities of the other

ones). However, we shall be able to introduce dependence between τ_1 and τ_2 into the model by relaxing the standard assumption of no simultaneous defaults. As we shall see, allowing for simultaneous defaults is a powerful way of modeling defaults dependence.

Specifically, we model the pair $H = (H^1, H^2)$ as an inhomogeneous Markov chain relative to its own filtration $\mathbb{H}$ on $(\Omega, \mathcal{H}, \mathbb{P})$, with state space $E = \{(0,0), (1,0), (0,1), (1,1)\}$, and matrix-generator at time t given by the following 4×4 matrix $A(t)$, where the first to fourth rows (or columns) correspond to the four possible states $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ of H_t :

$$A(t) = \begin{bmatrix} -l(t) & l^1(t) & l^2(t) & l^3(t) \\ 0 & -q^2(t) & 0 & q^2(t) \\ 0 & 0 & -q^1(t) & q^1(t) \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{16}$$

In (16) the l's and q's denote deterministic functions of time integrable over $[0, T]$, with in particular $l(t) = l^1(t) + l^2(t) + l^3(t)$.

Remark 3.1 The intuitive meaning of '(16) being the matrix-generator of H' is the following (see, e.g., Rogers and Williams [21] for standard definitions and results on Markov Chains):
• *First line:* Conditional on the pair $H_t = (H_t^1, H_t^2)$ being in state $(0,0)$ (firm and counterpart still alive at time t), there is a probability $l^1(t)dt$, resp. $l^2(t)dt$, resp.resp. $l^3(t)dt$, of a default of the firm alone, resp. of the counterpart alone, resp.resp. of a simultaneous default of the firm and the counterpart, in the next infinitesimal time interval $(t, t + dt)$;
• *Second line:* Conditional on the pair $H_t = (H_t^1, H_t^2)$ being in state $(1,0)$ (firm defaulted but counterpart still alive at time t), there is a probability $q^2(t)dt$ of a further default of the counterpart in the time interval $(t, t + dt)$;
• *Third line:* Conditional on the pair $H_t = (H_t^1, H_t^2)$ being in state $(0,1)$ (firm still alive but counterpart i default at time t), there is a probability $q^1(t)dt$ of a further default of the firm in the time interval $(t, t + dt)$.
On each line the diagonal term is then set as minus the sum of the non-diagonal terms, so that the sum of the elements in each line be equal to zero, as should be for $A(t)$ to represent the generator of a Markov process.

Moreover, for the sake of the desired ***Markov copula property*** (Proposition 3.1(v) below), we impose the following relations between the l's and the q's.

Assumption 3.2 $q^1(t) = l^1(t) + l^3(t)$, $q^2(t) = l^2(t) + l^3(t)$.

Observe that in virtue of these relations,
• Conditional on H_t^1 being in state 0, and whatever the state of H_t^2 may be (that is, in the state $(0,0)$ as in the state $(0,1)$ for H_t), there is a probability $q^1(t)dt$ of a default of the firm (alone or jointly with the counterpart) in the next time interval $(t, t + dt)$;
• Conditional on H_t^2 being in state 0, and whatever the state of H_t^1 may be (that is, in the states $(0,0)$ or $(1,0)$ for H_t), there is a probability $q^2(t)dt$ of a default of the counterpart (alone or jointly with the firm) in the next time interval $(t, t + dt)$.

In mathematical terms, it is thus expected that the default indicator processes H^1 and H^2 be $\mathbb{H}$-Markov processes on the state space $E = \{0, 1\}$ with time t generators respectively given by

$$A^1(t) = \begin{bmatrix} -q^1(t) & q^1(t) \\ 0 & 0 \end{bmatrix} , \ A^2(t) = \begin{bmatrix} -q^2(t) & q^2(t) \\ 0 & 0 \end{bmatrix} . \tag{17}$$

To formalize the previous statements, and in view of the study of simultaneous jumps, let us further introduce the processes $H^{\{1\}}$, $H^{\{2\}}$ and $H^{\{1,2\}}$ standing for the indicator processes of a default of the firm alone, of the counterpart alone, and of a simultaneous default of the firm and the counterpart, respectively. So

$$H_t^{\{1\}} = \sum_{0<s\leq t} \mathbb{1}_{\Delta H_s=(1,0)} , \ H_t^{\{2\}} = \sum_{0<s\leq t} \mathbb{1}_{\Delta H_s=(0,1)} , \ H_t^{\{1,2\}} = \sum_{0<s\leq t} \mathbb{1}_{\Delta H_s=(1,1)} , \tag{18}$$

or, equivalently,

$$H_t^{\{1\}} = \mathbb{1}_{\tau_1\leq t, \tau_1\neq\tau_2} , \ H_t^{\{2\}} = \mathbb{1}_{\tau_2\leq t, \tau_1\neq\tau_2} , \ H_t^{\{1,2\}} = \mathbb{1}_{\tau_1=\tau_2\leq t} .$$

We denote $I = \{\{1\}, \{2\}, \{1, 2\}\}$. The proof of tThe following Proposition is deferred to Appendix A.

Proposition 3.1 **(i)** $H^1 = H^{\{1\}} + H^{\{1,2\}}$, $H^2 = H^{\{2\}} + H^{\{1,2\}}$,
(ii) *The natural filtration of* $(H^\iota)_{\iota\in I}$ *is equal to* $\mathbb{H}$,
(iii) *The* $\mathbb{H}$*-intensity of* H^ι *is of the form* $q^\iota(t, H_t)$ *for a suitable function* $q^\iota(t, e)$ *for every* $\iota \in I$, *namely,*

$$\begin{aligned} q^{\{1\}}(t, e) &= \mathbb{1}_{e_1=0} \left(\mathbb{1}_{e_2=0} l^1(t) + \mathbb{1}_{e_2=1} q^1(t)\right) \\ q^{\{2\}}(t, e) &= \mathbb{1}_{e_2=0} \left(\mathbb{1}_{e_1=0} l^2(t) + \mathbb{1}_{e_1=1} q^2(t)\right) \\ q^{\{1,2\}}(t, e) &= \mathbb{1}_{e=(0,0)} l^3(t) . \end{aligned}$$

Otherwise said, the processes M^ι *defined by, for every* $\iota \in I$,

$$M_t^\iota = H_t^\iota - \int_0^t q^\iota(s, H_s)ds , \tag{19}$$

with

$$\begin{aligned} q^{\{1\}}(t, H_t) &= (1 - H_t^1) \left((1 - H_t^2)l^1(t) + H_t^2 q^1(t)\right) \\ q^{\{2\}}(t, H_t) &= (1 - H_t^2) \left((1 - H_t^1)l^2(t) + H_t^1 q^2(t)\right) \\ q^{\{1,2\}}(t, H_t) &= (1 - H_t^1)(1 - H_t^2)l^3(t) , \end{aligned} \tag{20}$$

are $\mathbb{H}$*-martingales.*
(iv) *The* $\mathbb{H}$*-intensity process of* H^i *is given by* $(1 - H_t^i)q^i(t)$. *Otherwise said, the processes* M^i *defined by, for* $i = 1, 2$,

$$M_t^i = H_t^i - \int_0^t (1 - H_s^i)q^i(s)ds \tag{21}$$

are $\mathbb{H}$-martingales.
(v) *The processes H^1 and H^2 are $\mathbb{H}$-Markov processes with matrix-generator at time t given by $A^1(t)$ and $A^2(t)$ (cf. (17)).*
(vi) *One has, for $s < t$,*

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = e^{-\int_0^s l(u)du} e^{-\int_s^t q^2(u)du}\,,\ \mathbb{P}(\tau_1 > t, \tau_2 > s) = e^{-\int_0^s l(u)du} e^{-\int_s^t q^1(u)du} \tag{22}$$

and therefore

$$\begin{aligned}
&\mathbb{P}(\tau_1 > t) = e^{-\int_0^t q^1(u)du}\,,\ \mathbb{P}(\tau_2 > t) = e^{-\int_0^t q^2(u)du}\\
&\mathbb{P}(\tau_1 > s, \tau_2 \in dt) = q_t^2 e^{-\int_0^s l(u)du} e^{-\int_s^t q^2(u)du}\,,\ \mathbb{P}(\tau_1 \in dt, \tau_2 > s) = q_t^1 e^{-\int_0^s l(u)du} e^{-\int_s^t q^1(u)du}\\
&\mathbb{P}(\tau_1 > t, \tau_2 \in dt) = q_t^2 e^{-\int_0^t l(u)du}\,,\ \mathbb{P}(\tau_1 \in dt, \tau_2 > t) = q_t^1 e^{-\int_0^t l(u)du}\\
&\mathbb{P}(\tau_1 \wedge \tau_2 > t) = \exp\left(-\int_0^t l(u)du\right)\,.
\end{aligned} \tag{23}$$

(vii) *The correlation of H_t^1 and H_t^2 is*

$$\rho(t) = \frac{\exp\left(\int_0^t l^3(s)ds\right) - 1}{\sqrt{\left(\exp\left(\int_0^t q^1(s)ds\right) - 1\right)\left(\exp\left(\int_0^t q^2(s)ds\right) - 1\right)}}. \tag{24}$$

Remark 3.3 In the Markov copula [3] terminology, the so-called *consistency condition* is satisfied (H^1 and H^2 are $\mathbb{H}$-Markov processes, see the Proposition 5.1 of [3]). The bi-variate model H with generator A is thus a *Markovian copula model* with marginal generators A^1 and A^2.

3.2 Pricing

We use the notation of Proposition 2.2, which applies here since we are in the special case $\mathbb{F} = \mathbb{H}$. Recall in particular $\Pi_t = \Pi(t, H_t) = (1-H_t^1)(1-H_t^2)u(t)$, for a *pricing function* $\Pi(t, 0, 0) = u(t)$, as well as the identities (12), (14) (15).

We assume henceforth for simplicity that,
• The discount factor writes $\beta_t = \exp(-\int_0^t r(s)ds)$, for a deterministic *short-term interest-rate* function r,
• The recovery rates R_1 and R_2 are constant.

Proposition 3.2 *The pricing function $u(t)$ of the risky CDS solves the following pricing ODE,*

$$\begin{cases} u(T) = 0 \\ \frac{du}{dt}(t) - (r(t) + l(t))u(t) + \pi(t) = 0\,,\ t \in [0, T) \end{cases} \tag{25}$$

where we set

$$\pi(s) = (1 - R_1)\big[l^1(s) + R_2 l^3(s)\big] + l^2(s)\big[R_2\tilde{\chi}(s)^+ - \tilde{\chi}(s)^-\big] - \kappa\,. \tag{26}$$

So

$$\beta_t u(t) = \int_t^T \beta_s e^{-\int_t^s l(u)du} \pi(s) ds \ . \tag{27}$$

Proof. Recall (11):

$$u(t) = \frac{\mathbb{E}\big[\pi_T(t)\big]}{P(\tau_1 \wedge \tau_2 > t)} \ ,$$

where the denominator is given by Proposition 3.1(vi). For computing the numerator, let us rewrite the expressions for the cumulative discounted Fee, Protection and Close-out cash flows in terms of integrals with respect to $H^{\{1\}}$, $H^{\{2\}}$ and $H^{\{1,2\}}$, as follows:

$$\begin{aligned}
\text{Fees Cash Flow } &= \kappa \int_0^T \beta_s (1 - H_s^1)(1 - H_s^2) ds \\
\text{Protection Cash Flow } &= (1 - R_1) \int_0^T \beta_s (1 - H_{s-}^2) dH_s^{\{1\}} + R_2 (1 - R_1) \int_0^T \beta_s dH_s^{\{1,2\}} \\
&= (1 - R_1) \int_0^T \beta_s (1 - H_{s-}^2) dM_s^{\{1\}} + (1 - R_1) \int_0^T \beta_s (1 - H_s^2) q^{\{1\}}(s, H_s) ds \\
&+ R_2 (1 - R_1) \int_0^T \beta_s dM_s^{\{1,2\}} + R_2 (1 - R_1) \int_0^T \beta_s q^{\{1,2\}}(s, H_s) ds \\
\text{Close-out Cash Flow } &= \int_0^T \beta_s \big[R_2 \tilde{\chi}(s)^+ - \tilde{\chi}(s)^-\big] (1 - H_{s-}^1) dH_s^{\{2\}} \\
&= \int_0^T \beta_s \big[R_2 \tilde{\chi}(s)^+ - \tilde{\chi}(s)^-\big] (1 - H_{s-}^1) dM_s^{\{2\}} \\
&+ \int_0^T \beta_s \big[R_2 \tilde{\chi}(s)^+ - \tilde{\chi}(s)^-\big] (1 - H_s^1) q^{\{2\}}(s, H_s) ds
\end{aligned}$$

Taking care of the martingale property of $M^{\{1\}}$, $M^{\{2\}}$ and $M^{\{1,2\}}$ and of the fact that the integrals of bounded predictable processes with respect to these martingales are indeed martingales, it thus comes,

$$\mathbb{E}(\pi_T(t)) = \mathbb{E}(\tilde{\pi}_T(t)) \tag{28}$$

with

$$\begin{aligned}
\beta_t \tilde{\pi}_T(t) = &- \kappa \int_t^T \beta_s (1 - H_s^1)(1 - H_s^2) ds \\
&+ (1 - R_1) \int_t^T \beta_s (1 - H_s^2) q^{\{1\}}(s, H_s) ds + R_2 (1 - R_1) \int_t^T \beta_s q^{\{1,2\}}(s, H_s) ds \\
&+ \int_t^T \beta_s \big[R_2 \tilde{\chi}(s)^+ - \tilde{\chi}(s)^-\big] (1 - H_s^1) q^{\{2\}}(s, H_s) ds.
\end{aligned}$$

Moreover, in view of the expressions for $q^{\{1\}}$ and $q^{\{2\}}$ in (20), one has

$$\begin{aligned}(1-H_s^2)q^{\{1\}}(s,H_s) &= (1-H_s^1)(1-H_s^2)l^1(s)\ ,\\ (1-H_s^1)q^{\{2\}}(s,H_s) &= (1-H_s^1)(1-H_s^2)l^2(s)\ .\end{aligned} \tag{29}$$

Plugging this into (28) and using (23), it comes,

$$\begin{aligned}\beta_t\mathbb{E}[\pi_T(t)] &= \mathbb{E}\left[\int_t^T \beta_s(1-H_s^1)(1-H_s^2)\pi(s)ds\right]\\ &= \int_t^T \beta_s\mathbb{E}\left[(1-H_s^1)(1-H_s^2)\right]\pi(s)ds\Big]\\ &= \int_t^T \beta_s e^{-\int_0^s l(x)dx}\pi(s)ds\end{aligned}$$

where π is given by (26). One can now check by inspection that the function u satisfies the ODE (25). □

Remark 3.4 The pricing ODE (25) can also be interpreted as the Kolmogorov backward equation related to the valuation of a risky CDS in our set-up. This ODE can in fact be derived directly and independently by an application of the Itô formula to the martingale $\Pi(t,H_t^1,H_t^2)$, which results in an alternative proof of Proposition 3.2.

Remark 3.5 In case $\tilde{\chi}=u$, the negative and positive parts of u are sitting in the expression for π in (26). One thus deals with a non-linear valuation ODE (25), and the formula (27) is in fact a *fixed-point equation* to be solved in the unknown function u (which is 'hidden' in π in the right hand side of (27)).

From Proposition 2.2 the representation $\Pi_t=\Pi(t,H_t^1,H_t^2)$ is true in any model with filtration $\mathbb{H}$. It particular it does not depend on the Markov property of (H^1,H^2). On the contrary the Markov properties of the model (including the Consistency Condition which is satisfied in virtue of Proposition 3.1(v)) are key for deriving the analogous representation (30) regarding the *risk-free CDS* below.

Proposition 3.3 *The price of a risk-free CDS with spread κ on the firm admits the following representation:*

$$P_t = P(t,H_t^1)\ , \tag{30}$$

for a function P of the form $P(t,e_1)=(1-e_1)v(t)$. The pricing function v solves the following pricing ODE:

$$\begin{cases} v(T)=0\\ \frac{dv}{dt}(t)-(r(t)+q^1(t))v(t)+p(t)=0\,,\ \ t\in[0,T)\ .\end{cases}$$

where se set

$$p(t)=(1-R_1)q^1(s)-\kappa. \tag{31}$$

So, for $t \in [0,T]$,

$$\beta_t v(t) = \int_t^T \beta_s e^{-\int_t^s q^1(x)dx} p(s)ds \ .$$

Proof. One has,

$$\begin{aligned}\beta_t p_T(t) &= -\kappa \int_t^T \beta_s(1-H_s^1)ds + (1-R_1)\int_t^T \beta_s dH_s^1 \\ &= -\kappa \int_t^T \beta_s(1-H_s^1)ds + (1-R_1)\int_t^T \beta_s dM_s^1 + (1-R_1)\int_t^T \beta_s q^1(s)(1-H_s^1)ds.\end{aligned}$$

As M^1 is an $\mathbb{H}$-martingale,

$$\beta_t \mathbb{E}_t[p_T(t)] = \mathbb{E}_t\left[\int_t^T \beta_s(1-H_s^1)p(s)ds\right] = \int_t^T \beta_s \mathbb{E}_t[1-H_s^1]p(s)ds \ , \tag{32}$$

with $p(t)$ defined by (31), and where in virtue of Proposition 3.1(v) (Markov consistency) and of the Key Lemma:

$$\mathbb{E}_t[1-H_s^1] = \mathbb{E}[1-H_s^1|\mathcal{H}_t^1] = (1-H_t^1)\frac{\mathbb{P}(\tau_1 > s)}{\mathbb{P}(\tau_1 > t)} = (1-H_t^1)e^{-\int_t^s q^1(x)dx} \ .$$

□

Proposition 3.4 *One has (cf. (12), (14) (15)),*

$$EPE(t) = (1-R_2)\left((1-R_1)\frac{l^3(t)}{q^2(t)} \ + \ \tilde{\chi}^+(t)\frac{l^2(t)}{q^2(t)}\right)e^{-\int_0^t l^1(x)dx} \tag{33}$$

$$CVA(t) = \int_t^T (1-R_2)\beta_s\left((1-R_1)l^3(s) \ + \ \tilde{\chi}^+(s)l^2(s)\right)e^{-\int_t^s l(x)dx}ds \ . \tag{34}$$

Proof. Set

$$\Phi(\tau_2) = \mathbb{E}(\mathbb{1}_{\tau_1=\tau_2}|\tau_2) \ ,$$

which is characterized by, for every bounded Borel function ϕ,

$$\mathbb{E}(\Phi(\tau_2)\phi(\tau_2)) = \mathbb{E}(\phi(\tau_2)\mathbb{1}_{\tau_1=\tau_2<T}) \ . \tag{35}$$

Now, one has by application of the results of Proposition 3.1, in the present Markovian copula set-up,

$$\begin{aligned}\mathbb{E}(\Phi(\tau_2)\phi(\tau_2)) &= \int_0^\infty \Phi(t)\phi(t)q^2(t)e^{-\int_0^t q^2(s)ds}dt \\ \mathbb{E}(\phi(\tau_2)\mathbb{1}_{\tau_1=\tau_2<T}) &= \mathbb{E}(\int_0^T \phi(t)dH_t^{\{1,2\}}) \\ &= \int_0^T \phi(t)\mathbb{E}((1-H_t^1)(1-H_t^2))l^3(t)dt = \int_0^T \phi(t)e^{-\int_0^t l(t)dt}l^3(t)dt \ .\end{aligned}$$

Hence (35) can be rewritten as

$$\int_0^\infty \Phi(t)\phi(t)q^2(t)e^{-\int_0^t q^2(s)ds}dt = \int_0^T \phi(t)l^3(t)e^{-\int_0^t l(s)ds}dt \ ,$$

for etery bounded Borel function ϕ. So

$$\Phi(t) = \frac{l^3(t)e^{-\int_0^t l(s)ds}}{q^2(t)}e^{\int_0^t q^2(s)ds}\mathbb{1}_{t<T} \ ,$$

which gives the left term in (33).

As for the right term, one has likewise,

$$\mathbb{E}(\mathbb{1}_{\tau_2<\tau_1}|\tau_2) = \Psi(\tau_2) \ , \tag{36}$$

with for every bounded and measurable function ϕ,

$$\mathbb{E}(\Psi(\tau_2)\phi(\tau_2)) = \mathbb{E}(\phi(\tau_2)\mathbb{1}_{\tau_2<\tau_1}) \ ,$$

where by application of Proposition 3.1,

$$\begin{aligned}
\mathbb{E}(\Psi(\tau_2)\phi(\tau_2)) &= \int_0^\infty \Psi(t)\phi(t)q^2(t)e^{-\int_0^t q^2(s)ds}dt \\
\mathbb{E}(\phi(\tau_2)\mathbb{1}_{\tau_2<\tau_1\wedge T}) &= \mathbb{E}(\int_0^\infty \phi(t)\mathbb{1}_{t\leq\tau_1\wedge T}dH_t^{\{2\}}) = \mathbb{E}\left(\int_0^\infty \phi(t)\mathbb{1}_{t\leq\tau_1\wedge T}q^{\{2\}}(t)dt\right) \\
&= \mathbb{E}\left(\int_0^\infty \phi(t)\mathbb{1}_{t\leq T}(1-H^1(t))(1-H_t^2)l^2(t)dt\right) = \int_0^T \phi(t)e^{-\int_0^t l(s)}l^2(t)dt
\end{aligned}$$

where the second identity in the second line uses that $H^{\{2\}}$ does not jump at τ_1. So (36) can be rewritten as

$$\int_0^\infty \Psi(t)\phi(t)q^2(t)e^{-\int_0^t q^2(s)ds}dt = \int_0^\infty \phi(t)l^2(t)e^{-\int_0^t l(s)ds}ds \ ,$$

for every bounded Borel function ϕ. So

$$\Psi(t) = \frac{l^2(t)e^{-\int_0^t l(s)ds}}{q^2(t)}e^{\int_0^t q^2(s)ds} \ ,$$

and (33) follows.

Using (15), one thus has on $\{t < \tau_1 \wedge \tau_2\}$,

$$\begin{aligned}
\beta_t\text{CVA}(t) &= \int_t^T \beta_s\text{EPE}(s)e^{\int_0^s l(x)dx}e^{-\int_0^s q^2(x)dx}q^2(s)e^{-\int_t^s l(x)dx}ds \\
&= \int_t^T \beta_s\text{EPE}(s)e^{\int_0^s l^1(x)dx}q^2(s)e^{-\int_t^s l(x)dx}ds \ .
\end{aligned}$$

Hence (34) follows from (33). □

Remark 3.6 In view of the option-theoretical interpretation of the CVA, the CVA valuation formula (34) can also established 'directly' (without passing by the EPE), much like the formula (27) in Proposition 3.2 above (using a probabilistic computation, or resorting to the related Kolmogorov pricing ODE).

3.3 Hedging

We now give another perspective on the counterparty credit risk of the risky CDS, by assessing to which extent the risky CDS could, in principle, be hedged by the risk-free CDS (CDS with the same characteristics, except for the counterparty credit risk).

3.3.1 Price Dynamics

Let $\hat{\Pi}$ denote the discounted cum-dividend price of the risky CDS, that is, the local martingale

$$\hat{\Pi}_t = \beta_t \Pi_t + \pi_t(0).$$

The Itô formula applied to $\Pi_t = \Pi(t, H_t)$ yields, on $[0, \tau_1 \wedge \tau_2 \wedge T]$,

$$d\hat{\Pi}_t = \beta_t \big(\delta^{\{1\}}\Pi_t dM_t^{\{1\}} + \delta^{\{2\}}\Pi_t dM_t^{\{2\}} + \delta^{\{1,2\}}\Pi_t dM_t^{\{1,2\}}\big) \tag{37}$$

with

$$\delta^{\{1\}}\Pi_t = 1 - R_1 - u(t)\,,\ \ \delta^{\{2\}}\Pi_t = R_2\tilde{\chi}^+(t) - \tilde{\chi}^-(t) - u(t)\,,\ \ \delta^{\{1,2\}}\Pi_t = R_2(1-R_1) - u(t)\ .$$

Similarly, setting

$$\hat{P}_t = \beta_t P_t + p_t(0), \tag{38}$$

it comes

$$d\hat{P}_t = \beta_t \delta^1 P_t dM_t^1$$

with

$$\delta^1 P_t = 1 - R_1 - v(t)\ .$$

3.3.2 Min-Variance Hedging

Let us denote by ψ a (self-financing) strategy in the risk-free CDS with price process P (and the savings account β_t^{-1}) for tentatively hedging the risky CDS with price process Π.

Recall that $\mathbb{P}$ is the risk neutral probability chosen by market. So the discounted cum-dividend price process $\hat{P}$ is a $\mathbb{P}$-local martingale. As a result of the Galtchouk-Kunita-Watanabe decomposition, the min-variance hedging strategy ψ^{va} is given by

$$\psi_t^{va} = \frac{d\langle \hat{\Pi}, \hat{P}\rangle_t}{d\langle \hat{P}\rangle_t}\ .$$

In view of the price dynamics (37)-(38), one has, for $t < \tau_1 \wedge \tau_2$,

$$\begin{aligned}\frac{d\langle \hat{\Pi}, \hat{P}\rangle_t}{d\langle \hat{P}\rangle_t} &= \frac{q_t^{\{1\}}(\delta^{\{1\}}\Pi_t)(\delta^{\{1\}}P_t) + q_t^{\{2\}}(\delta^{\{2\}}\Pi_t)(\delta^{\{2\}}P_t) + q_t^{\{1,2\}}(\delta^{\{1,2\}}\Pi_t)(\delta^{\{1,2\}}P_t)}{q_t^{\{1\}}(\delta^{\{1\}}P_t)^2 + q_t^{\{2\}}(\delta^{\{2\}}P_t)^2 + q_t^{\{1,2\}}(\delta^{\{1,2\}}P_t)^2} \\ &= \frac{l^1(t)(1-R_1-u(t))(1-R_1-v(t)) + l^3(t)(R_2(1-R_1)-u(t))(1-R_1-v(t))}{q^1(t)(1-R_1-v(t))^2}\ .\end{aligned}$$

So

$$\psi_t^{va} = \frac{l^1(t)(1-R_1-u(t))}{q^1(t)(1-R_1-v(t))} + \frac{l^3(t)(R_2(1-R_1)-u(t))}{q^1(t)(1-R_1-v(t))}$$

on $[0, \tau^1 \wedge \tau^2 \wedge T]$ (and $\psi^{va} = 0$ on $(\tau^1 \wedge \tau^2 \wedge T, T]$).

Remark 3.7 This min-variance hedging strategy can be easily extended to multi-instrument hedging schemes. In case three non-redundant hedging instruments are available, then, in view of (37), the risky CDS can be perfectly replicated.

4 Implementation

4.1 Model Specification

The model primitives are the marginal pre-default intensity functions q^1 and q^2 as well as the 'dependence intensity function' l^3 in $A(t)$ (cf. (16)).

Let us specify (such an affine specification of intensities was already used by Bielecki et. al. [2] in a context of CDO modeling)

$$q^i(t) = a_i + b_i t \ . \tag{39}$$

We set further

$$l^3(t) = a_3 + b_3 t$$

with

$$a_3 = \alpha \min\{a_1, a_2\} \ , \ \ b_3 = \alpha \min\{b_1, b_2\} \ ,$$

for a *model dependence parameter* $\alpha \in [0, 1]$ (for the sake of Assumption 3.2).

It is immediate to check that under (39), the spread κ_i of a risk-free CDS on name i is given by (see Bielecki et. al. [2]),

$$\kappa_i = (1-R_i) \frac{\displaystyle\int_0^T \beta_t (a_i + b_i t) \exp(-a_i t - \frac{b_i}{2} t^2) dt}{\displaystyle\int_0^T \beta_t \exp(-a_i t - \frac{b_i}{2} t^2) dt} \ . \tag{40}$$

Also note that one has, by Proposition 3.1(vii),

$$\rho := \rho(T) = \frac{e^{a_3 T + b_3 T^2/2} - 1}{\sqrt{\left(e^{a_1 T + b_1 T^2/2} - 1\right)\left(e^{a_2 T + b_2 T^2/2} - 1\right)}} \ , \tag{41}$$

or, equivalently,

$$\alpha = \frac{\ln\left(1 + \rho\sqrt{\left(e^{a_1 T + b_1 T^2/2} - 1\right)\left(e^{a_2 T + b_2 T^2/2} - 1\right)}\right)}{aT + bT^2/2} \tag{42}$$

where $a = \min\{a_1, a_2\}$ and $b = \min\{b_1, b_2\}$.

4.1.1 Calibration Issues

Using (40), the a_i's and b_i's can be calibrated independently in a straightforward way to spreads of risk-free CDS's on the firm and the counterpart, respectively.

As for the model dependence parameter α, in case market prices of instruments sensitive to the dependence structure of default times (basket credit instrument on the firm and the counterpart) are available, these prices can be used to calibrate α. Admittedly however, this situation is an exception rather than the rule. It is thus important to devise a practical way of setting α in case such market data are not available. A possible procedure[1] thus consists in 'calibrating' α to a target value for the model probability $\mathbb{P}(H_T^1 = H_T^2 = 1)$ of joint default at the time horizon T. observed market prices of basket instruments on the firm and the counterpart). As a proxy for the target value for $\mathbb{P}(H_T^1 = H_T^2 = 1)$ one can use the value obtained by plugging a standard static Gaussian copula correlation $\hat{\rho}$ into a bivariate normal distribution function. Regulatory capital requirements being based on the Vasicek formula, such a static copula correlation $\hat{\rho}$ can be retrieved from the Basel II correlations per asset class (cf. [1, pages 63 to 66]).

4.1.2 Case of constant coefficients

We now look at a particular case in which $b_1 = b_2 = b_3 = 0$ and $r(t) = r$. This case will be referred to henceforth as the case of *constant intensities*, as opposed to the more general case of *affine intensities* introduced in subsection 4.1. In the case of constant intensities, one has,

$$q^1(t) = a_1 \ , \ q^2(t) = a_2 \ , \ l^3(t) = a_3.$$

The correlation coefficient ρ in (41) simplifies to

$$\rho = \frac{e^{a_3 T} - 1}{\sqrt{(e^{a_1 T} - 1)\,(e^{a_2 T} - 1)}}$$

from which a_3 can be calculated as

$$a_3 = \frac{1}{T} \ln \left(1 + \rho \sqrt{(e^{a_1 T} - 1)\,(e^{a_2 T} - 1)} \right).$$

As is well known, the price of a risk-free CDS in a constant intensity model is null, i.e., $v(t) \equiv 0$ when $b_1 = 0$. So the EPE formula (33) simplifies to

$$\text{EPE}(t) = \frac{(1 - R_1)(1 - R_2)}{a_2 T} \left[1 + \rho \sqrt{(e^{a_1 T} - 1)\,(e^{a_2 T} - 1)} \right] e^{-a_1 t}$$

Also in this case the pricing formula (27) for the risky CDS reduces to

$$u(t) = \frac{(1 - R_1)(a_1 + R_2 a_3) - \kappa}{r + a_1 + a_2 + a_3} \left(1 - e^{-(r + a_1 + a_2 + a_3)(T - t)} \right).$$

Finally, from Proposition 2.1, one gets

$$CVA(t) = -u(t) \ .$$

[1] We thank J.-P. Lardy for the suggestion of this procedure.

4.2 Numerical Results

Our aim is to assess by means of numerical experiments the impact of ρ (the correlation coefficient of H_T^1 and H_T^2) on one hand, and of κ_2 (risk-free CDS fair spread of the counterparty as of (40)) on the other hand, on the counterparty risk exposure of the investor.

Towards this end we fix the general data of Table 1 (case with affine intensities) or 3 (case with constant intensities, all b's equal to 0), and we further consider twelve alternative sets of values for the residual model parameters a_2, b_2, and ρ given in columns one, two and seven of Table 2 (case with affine intensities), resp. a_2 and ρ given in columns one and five and seven of Table 4 (case with constant intensities).

r	R_1	R_2	T	a_1	b_1	κ_1
5%	40%	40%	10 years	.0095	.0010	84 bp

Table 1: Fixed Data — Affine Intensities.

a_2	b_2	κ_2	a_3	b_3	α	ρ	CVA(0)
.0056	.0006	50 bp	.0008	.0001	.1368	10%	.0031
.0085	.0009	75 bp	.0009	.0001	.1124	10%	.0038
.0122	.0010	100 bp	.0011	.0001	.1170	10%	.0044
.0189	.0014	150 bp	.0014	.0001	.1466	10%	.0054
.0056	.0006	50 bp	.0030	.0003	.5380	40%	.0118
.0085	.0009	75 bp	.0037	.0004	.4400	40%	.0144
.0122	.0010	100 bp	.0043	.0005	.4566	40%	.0165
.0189	.0014	150 bp	.0054	.0006	.5684	40%	.0199
.0056	.0006	50 bp	.0052	.0006	.9257	70%	.0031
.0085	.0009	75 bp	.0064	.0007	.7543	70%	.0038
.0122	.0010	100 bp	.0074	.0008	.7803	70%	.0044
.0189	.0014	150 bp	.0092	.0010	.9660	70%	.0054

Table 2: Variable Data — Affine Intensities.

r	R_1	R_2	T	a_1	κ_1
5%	40%	40%	10 years	.0140	84 bp

Table 3: Fixed Data — Constant Intensities.

In the case of affine intensities the corresponding spreads κ_2 at time 0 for the risk-free CDS on the counterpart and the dependence parameter α are displayed respectively in the third and sixth column of Table 2, whereas the last column of Table 2 (which will be commented later in the text) gives the corresponding CVA's at time 0. The risky and risk-free CDS pricing functions u and v corresponding to each of our twelve sets of parameters are displayed in Figure 1. On each graph three curves are represented:

a_2	κ_2	a_3	α	ρ	CVA(0)
.0083	50 bp	.0008	.1366	10%	.0029
.0125	75 bp	.0009	.1124	10%	.0036
.0167	100 bp	.0011	.1171	10%	.0041
.0250	150 bp	.0014	.1461	10%	.0049
.0083	50 bp	.0045	.5374	40%	.0117
.0125	75 bp	.0055	.4403	40%	.0142
.0167	100 bp	.0064	.4572	40%	.0163
.0250	150 bp	.0079	.5671	40%	.0196
.0083	50 bp	.0077	.9253	70%	.0204
.0125	75 bp	.0094	.7553	70%	.0248
.0167	100 bp	.0109	.7811	70%	.0284
.0250	150 bp	.0135	.9648	70%	.0341

Table 4: Variable Data — Constant Intensities.

- $v(t)$ (dashed blue curve),
- $u(t)$ with $\tilde{\chi} = v$ therein, denoted by $u^0(t)$ (doted red curve),
- $u(t)$ with $\tilde{\chi} = u$ therein, denoted by $u^1(t)$ (black curve).

The analogous results in the case of constant intensities are displayed in Table 4 and Figure 2. Note that on each graph in Figure 2 the function v is then equal to 0, as must be in case of constant intensities.

In all the cases u^0 and u^1 are rather close to each other, and one can check numerically that using either one makes little difference regarding the related EPEs and CVAs. We present henceforth the results for $u = u^0$.

Figures 3, 4 and 5 show the graphs of the Expected Positive Exposure as a function of time, of the Credit valuation Adjustment as a function of time, and of the Credit valuation Adjustment at time 0 as a function of ρ, in the cases of affine (left graphs) and constant (right graphs) intensities.

One can thus see on Figure 3 the impact on the counterparty risk exposure of the investor of the default risk (as measured by the spread κ_2) of the counterpart. In each graph the correlation coefficient, ρ is fixed, with from up to down $\rho = 10\%$, 40% and 70%, respectively. The four curves on each graph of Figure 3 correspond to EPE(t) for $\kappa_2 = 50$, 75, 100 and 150bps, respectively. Observe that as κ_2 decreases the counterparty risk exposure increases. This is in line with the stylized features and the financial intuition regarding the EPE: EPE(t) is the expectation of the investor's loss, given the default of the counterpart at time t. A default of a counterpart with a lower spread is interpreted by the markets as a worse news than a default of a counterpart with a higher spread. The related EPE is thus larger.

Finally Figure 4 shows the graphs of the Credit valuation Adjustment as a function of time, for affine (left column) and constant (right column) intensities. One can thus see the impact of κ_2 on the CVA. In each graph the correlation coefficient ρ is fixed, with from top to down $\rho = 10\%$, 40% and 70%, respectively. The four curves on each graph of Figure 3 correspond to CVA(t) for $\kappa_2 = 50$, 75, 100 and 150bps, respectively. Observe that the CVA is increasing in κ_2, in line with stylized features.

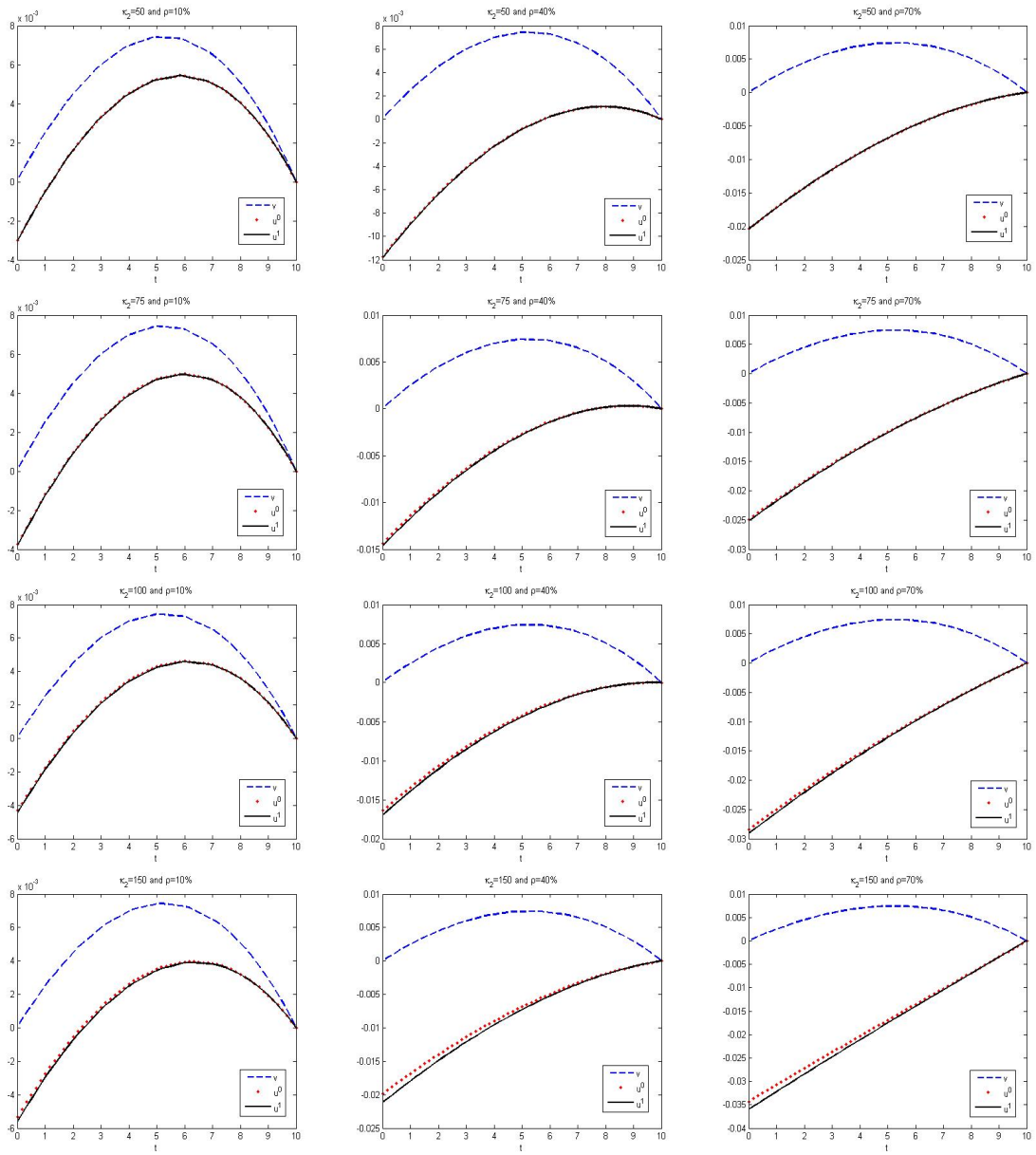

Figure 1: Pricing functions in the case of affine intensities — $v(t)$ (dashed blue curve), $u^0(t)$ (doted red curve) and $u^1(t)$ (black curve).

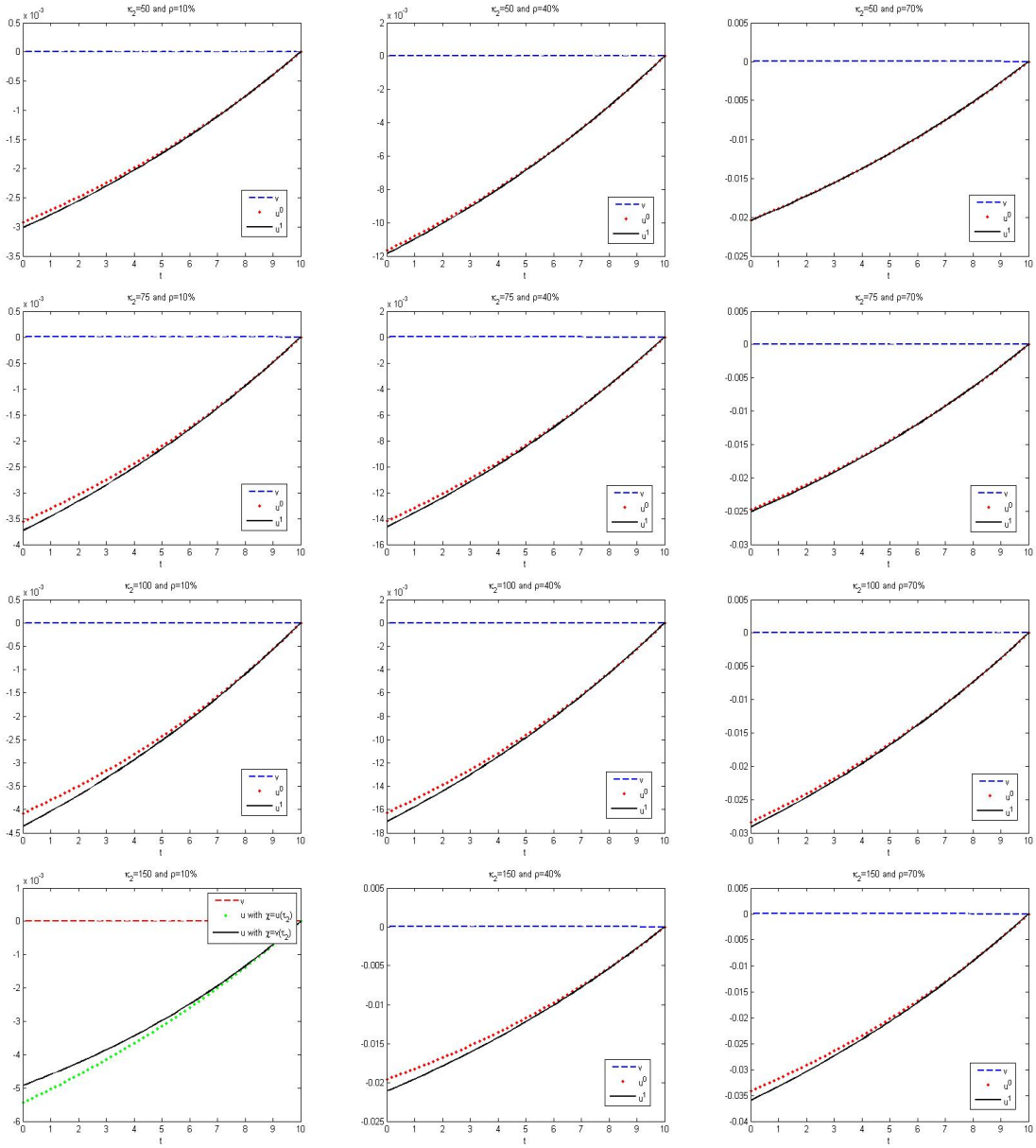

Figure 2: Pricing functions in the case of constant intensities — $v(t)$ (dashed blue curve), $u^0(t)$ (doted red curve) and $u^1(t)$ (black curve).

Also note that the CVA is a decreasing function of time, in accordance again with expected features: less time to maturity, less risk.

Finally Figure 5 represents the graphs of CVA(0) as a function of ρ for $\kappa_2 = 50$, 75, 100 and 150bps. One can see that CVA(0) grows essentially linearly in ρ.

5 Concluding Remarks and Perspectives

In this paper we propose a model of CDS with counterparty credit risk, with the following desirable properties:
• Adequation of the behavior of EPE and CVA in the model with expected features (see Section 4.2),
• Wrong way risk (via joint defaults, specifically),
• Simplicity, since the model is a four-state Markov Chain with automatically calibrated marginals (to the individual CDS curves),
• Fact (related to the previous one) that the model 'takes the right inputs to generate the right outputs', namely it takes as basic inputs the individual default probabilities (individual CDS curves), which correspond to the more reliable information on the market, which are then 'coupled' in a suitable way,
• Consistency, in the sense that it is a dynamic model with replication-based valuation and hedging arguments.

The work presented in this paper might be extended in at least three directions.

First, for certain applications, like in the case where the underlying reference contract is itself a CDS (see, e.g., Brigo and Capponi [7]), it is desirable to model credit spread volatility (as one would not like to model options without volatility in the underlying asset). One may thus want to enrich the model by adding a *reference filtration* $\tilde{\mathbb{F}}$ so that the model filtration $\mathbb{F}$ be given as $\tilde{\mathbb{F}} \vee \mathbb{H}$, and the intensities l, q are non-negative $\tilde{\mathbb{F}}$-adapted processes.

A second related issue is that of merging the CDS-CVA pricing tool of this paper into a more general, real-life CVA engine, including the following features:
• Netting, that is, aggregation in a suitable way of all the contracts (as opposed to only one CDS in this paper!) relative to a given counterpart,
• Market (other than credit) risk factors.
• Margin agreements.

Finally at the stage of implementation such real-life CVA engines pose interesting challenges from the numerical point of view of Monte Carlo simulations (see, e.g., Zhu and Pykhtin [22]).

A Proof of Proposition 3.1

We shall need the following (essentially classic) Lemma.

Lemma A.1 *Let X be a right-continuous process with a finite state space E and adapted to some filtration $\mathbb{G}$. Condition (i), (ii) or (iii) below are necessary and sufficient conditions for X to be a $\mathbb{G}$*

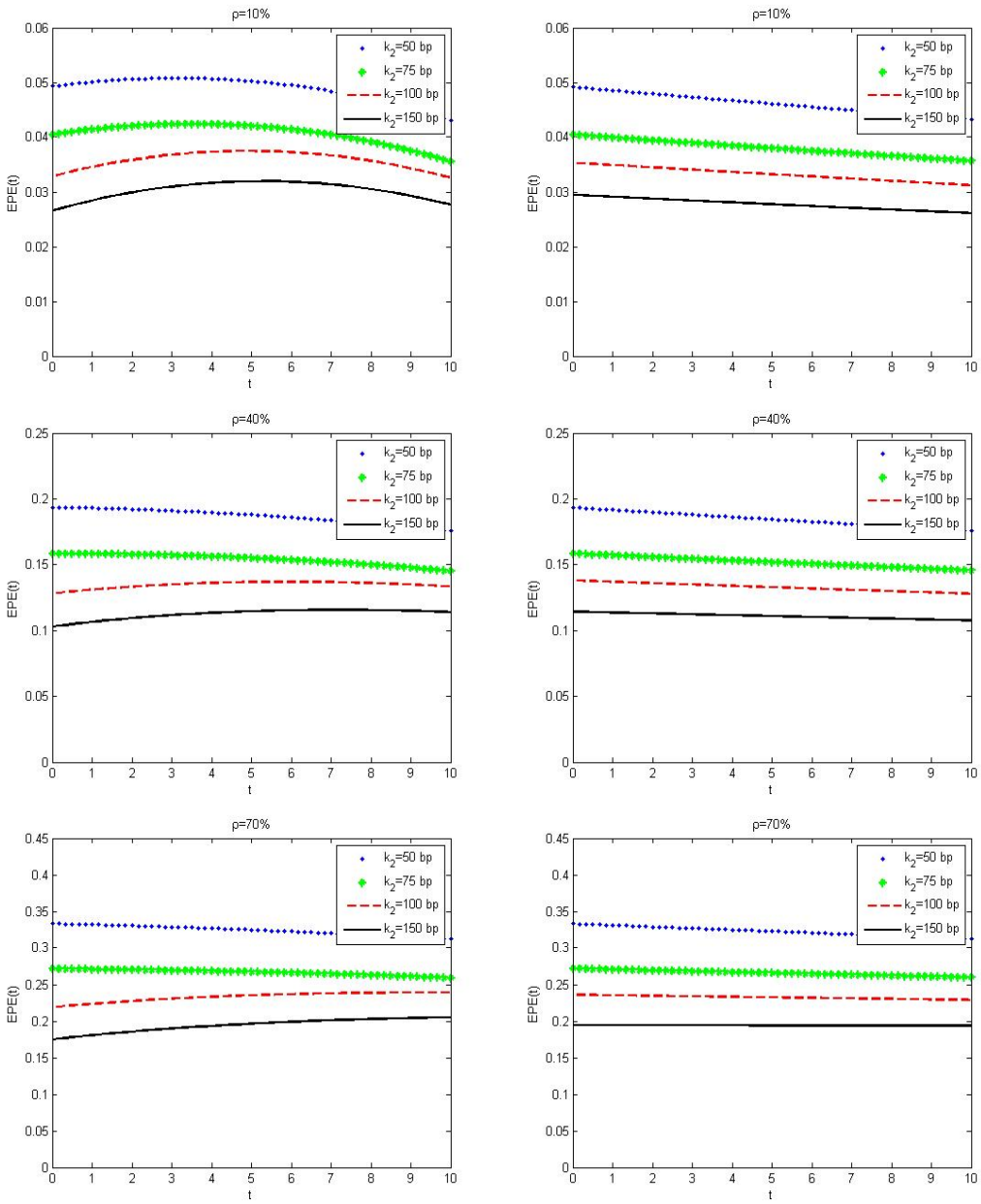

Figure 3: EPE(t) ($\tilde{\chi} = v$, $u = u^0$). In each graph ρ is fixed. From top to down $\rho = 10\%$, $\rho = 40\%$ and $\rho = 70\%$. Left column: affine intensities. Right column: constant intensities.

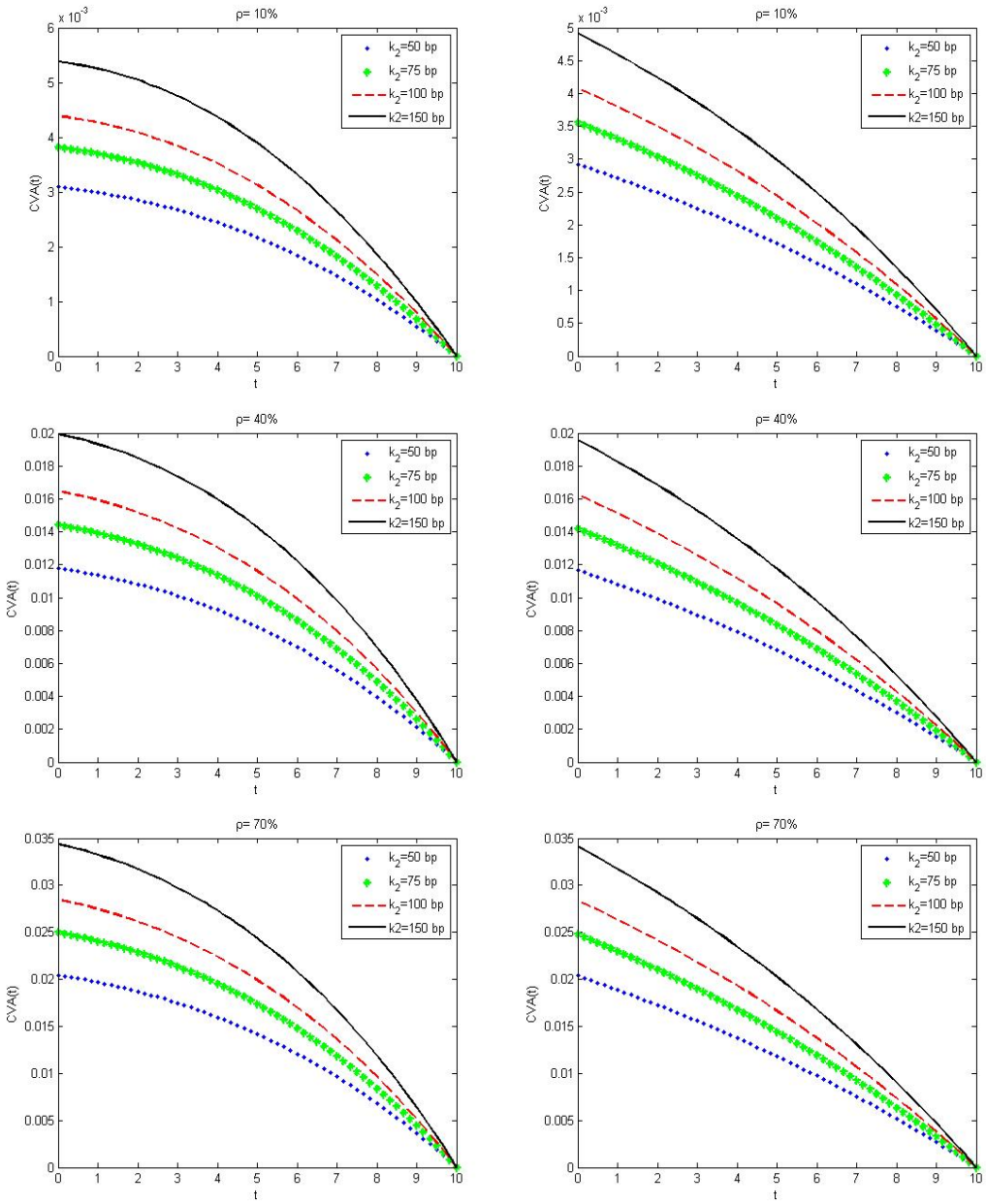

Figure 4: CVA(t) ($\tilde{\chi} = v$, $u = u^0$). In each graph ρ is fixed. From top to down $\rho = 10\%$, $\rho = 40\%$ and $\rho = 70\%$. Left column: affine intensities. Right column: constant intensities.

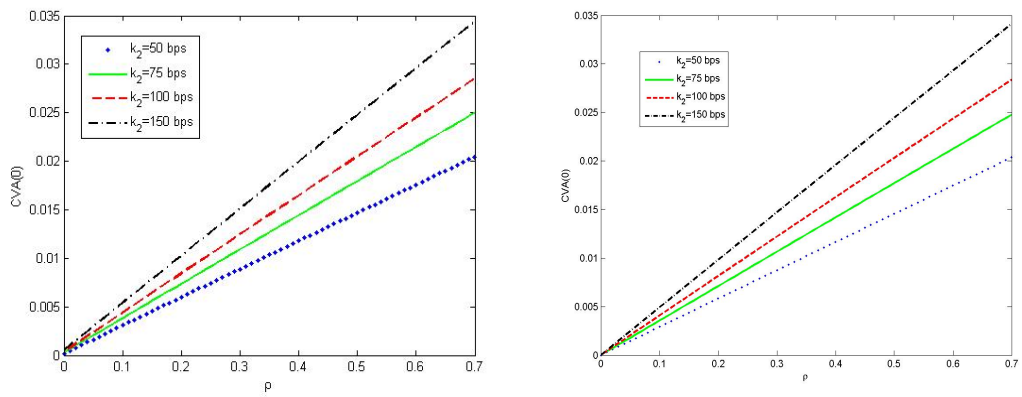

Figure 5: CVA(0) as a function of ρ for $\kappa_2 = 50$ bp, 75 bp, 100 bp and 150 bp ($\tilde{\chi} = v$, $u = u^0$). Left: Affine intensities. Right: Constant intensities.

– Markov chain with infinitesimal generator $a_t = [a_t^{i,j}]_{i,j\in E}$:
(i) *For every function h over E,*

$$m_t^h = h(X_t) - \int_0^t (a_s h)(X_s) ds \tag{43}$$

is a $\mathbb{G}$ – local martingale.
(ii) *For every $j \in E$, the process m^j defined by*

$$m_t^j = 1\!\!1_{X_t=j} - \int_0^t a_s^{X_s,j} ds$$

is a $\mathbb{G}$ – local martingale.
(iii) *For every $i, j \in E$ the process $m^{i,j}$ given by*

$$m_t^{i,j} = 1\!\!1_{X_{t-}=i, X_t=j} - \int_0^t 1\!\!1_{X_s=i} a_s^{i,j} ds$$

is a $\mathbb{G}$ – local martingale.

Proof. **(i)** is the usual local martingale characterization of markov Chains (see, e.g., Rogers and Williams [21]).
(ii) Since E is finite, the set of the indicator functions $1\!\!1_{\cdot=j}$ spans linearly the set of all functions over E. The condition of part (ii) is thus equivalent to that of (i).
(iii) Necessity follows by combination of Proposition 11.2.2 and Lemma 11.2.3 in [5]. As for sufficiency, note that the $m^{i,j}$'s being $\mathbb{G}$ – local martingales implies the same property for the m^j's in (ii), by summation over i. We thus conclude by the sufficiency in part (ii). □

Let us proceed with the proof of Proposition 3.1.
(i) From the definition of $H^{\{1\}}$ and $H^{\{1,2\}}$,

$$H_t^{\{1\}} + H_t^{\{1,2\}} = \sum_{0<s\leq t} 1\!\!1_{\Delta H_s=(1,0)} + \sum_{0<s\leq t} 1\!\!1_{\Delta H_s=(1,1)} = \sum_{0<s\leq t} 1\!\!1_{\Delta H_s^1=1} = H_t^1$$

and likewise $H^2 = H^{\{2\}} + H^{\{1,2\}}$.
(ii) By definition (cf. (18)), $\mathbb{F}^{\mathbb{H}} \supseteq \mathbb{F}^{\mathcal{H}}$, whereas the reverse inclusion follows from part (i).
(iii) Let us verify that the M^ι's in (19) is an $\mathbb{H}$-martingale. For $I = \{1, 2\}$, one has,

$$\begin{aligned} M^{\{1,2\}} &= H_t^{\{1,2\}} - \int_0^t q^{\{1,2\}}(s, H_s)ds \\ &= \sum_{0<s\leq t} \mathbb{1}_{\Delta H_s=(1,1)} - \int_0^t \mathbb{1}_{H_s=(0,0)} l^3(s)ds \\ &= \sum_{0<s\leq t} \mathbb{1}_{H_{s-}=(0,0),H_s=(1,1)} - \int_0^t \mathbb{1}_{H_s=(0,0)} l^3(s)ds \ . \end{aligned}$$

Thus Lemma A.1 with $i = (0, 0)$ and $j = (1, 1)$, guarantees the martingale property of $M^{\{1,2\}}$. As for $M^{\{1\}}$, one has,

$$\begin{aligned} M^{\{1\}} &= H_t^{\{1\}} - \int_0^t q^{\{1\}}(s, H_s)ds \\ &= \sum_{0<s\leq t} \mathbb{1}_{\Delta H_s=(1,0)} - \int_0^t \mathbb{1}_{H_s^1=0} \left[\mathbb{1}_{H_s^2=0} l^1(s) + \mathbb{1}_{H_s^2=1} q^1(s)\right] ds \\ &= \left\{ \sum_{0<s\leq t} \mathbb{1}_{H_{s-}=(0,0),H_s=(1,0)} - \int_0^t \mathbb{1}_{H_s=(0,0)} l^1(s)ds \right\} \\ &\quad + \left\{ \sum_{0<s\leq t} \mathbb{1}_{H_{s-}=(0,1),H_s=(1,1)} - \int_0^t \mathbb{1}_{H_s=(0,1)} q^1(s)ds \right\} \end{aligned}$$

Now we apply Lemma A.1 to the two terms in the last equation, with $i = (0, 0)$ and $j = (1, 0)$ for the first term and $i = (0, 1)$ and $j = (1, 1)$ for the second term. Thus $M^{\{1\}}$ being the sum of two $\mathbb{H}$-martingales is an $\mathbb{H}$-martingale. In the same way, $M^{\{2\}}$ is an $\mathbb{H}$-martingale.
(iv) As $q^i = l^i + l^3$ and $H^i = H^{\{i\}} + H^{\{1,2\}}$, one has $M^i = M^{\{i\}} + M^{\{1,2\}}$, so the M^i's are in turn $\mathbb{H}$-martingales.
(v) Since the M^i's are $\mathbb{H}$-martingales, this follows easily from the sufficiency in Lemma A.1(ii).
(vi) (23) follows directly from (22), in which we shall now show the first identity. Let us first compute, for $t > s$, the quantity

$$\mathbb{P}(\tau_1 > t|\mathcal{H}_s) = 1 - \Psi_s \ .$$

The $\mathbb{H}$-martingale property of H^1 implies

$$\Psi_t = \Psi_s + \int_s^t q^1(u)(1 - \Psi_u)du$$

whose solution is

$$\Psi_t = 1 + (H_s^1 - 1)e^{-\int_s^t q^1(u)du}$$

Thus

$$\begin{aligned}\mathbb{P}(\tau_1 > t, \tau_2 > s) &= \mathbb{E}(\mathbb{1}_{\tau_2>s}\mathbb{E}(\mathbb{1}_{\tau_1>t}|\mathcal{H}_s)) &(44)\\ &= \mathbb{E}\left\{(1-H_s^1)(1-H_s^2)e^{-\int_s^t q^1(u)du}\right\}, &(45)\end{aligned}$$

and the result follows.
(vii) Since H_t^i is a Bernoulli random variable with

$$\mathbb{P}(H_t^i = 1) = \mathbb{P}(\tau_i \leq t) = 1 - \exp(-\int_0^t q^i(s)ds),$$

one has

$$\mathbb{V}\mathrm{ar}(H_t^i) = \exp(-\int_0^t q^i(s)ds)\left(1 - \exp(-\int_0^t q^i(s)ds)\right).$$

Also

$$\begin{aligned}\mathbb{C}\mathrm{ov}(H_t^1, H_t^2) &= \mathbb{C}\mathrm{ov}(1-H_t^1, 1-H_t^2)\\ &= \mathbb{E}\left[(1-H_t^1)(1-H_t^2)\right] - \mathbb{E}(1-H_t^1)\mathbb{E}(1-H_t^2)\\ &= \mathbb{P}(\tau_1 > t, \tau_2 > t) - \mathbb{P}(\tau_1 > t)\mathbb{P}(\tau_2 > t)\\ &= \exp\left(-\int_0^t l(s)ds\right) - \exp\left(-\int_0^t q^1(s)ds\right)\exp\left(-\int_0^t q^2(s)ds\right).\end{aligned}$$

Thus, after some algebraic simplifications,

$$\rho(t) = \frac{\mathrm{Cov}(H_t^1, H_t^2)}{\sqrt{\mathrm{Var}(H_t^1)\mathrm{Var}(H_t^2)}} = \frac{\exp\left(\int_0^t l^3(s)ds\right) - 1}{\sqrt{\left(\exp\left(\int_0^t q^1(s)ds\right) - 1\right)\left(\exp\left(\int_0^t q^2(s)ds\right) - 1\right)}}.$$

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