Immersion Property and Credit Risk Modelling

Monique Jeanblanc[†], Yann Le Cam[‡],

[†,‡] Université d'Évry Val d'Essonne
91025 Évry Cedex, France, [*]
[†] Institut Europlace de Finance
[‡] French Treasury

November 18, 2008

Abstract

The goal of this paper is to study the immersion property through its links with credit risk modelling. The construction of a credit model by the enlargement of a reference filtration with the progressive knowledge of a credit event occurrence has become a standard for reduced form modelling. It is known that such a construction rises mathematical difficulties, mainly relied to the properties of the random time. Whereas the invariance of the property of semi-martingale in the enlargement is implied by the absence of arbitrage, we address in this paper the question of the invariance of the martingale property.

Introduction

Most of the literature on credit risk focuses on pricing problems and postulates the existence of a pricing measure, without questioning its features. The purpose of this paper is to propose a study of the set of equivalent martingale measures (e.m.ms) in the context of credit modelling. Within the reduced form approach and particularly under the filtration enlargement framework, such questions may be precisely studied, and lead to interesting properties. Only finite time horizon problems will be treated in this paper.

Three parts will be developed in the sequel, so that to present the issues relative to the discussion about the completeness of a market potentially exposed to a credit event.

- The first one presents the credit modelling framework and discusses the meaning of the options taken. We adopt a reduced form model, and specify the split of the full information $\mathbb{G}$ between

[*]This research benefited from the support of the "Chaire Risque de Crédit", Fédération Bancaire Française.

a “reference filtration” $\mathbb{F}$ and the credit event (generating a filtration $\mathbb{H}$) so that to benefit the methodology based on the hazard process[1]. A subsection is dedicated to the presentation of each filtration, and another one to the financial interpretation of that splitting.

We model the credit event by a random time belonging to the class of *initial times*[2]. The choice of initial times insures that the semi-martingales in the reference filtration remain semi-martingales in the full filtration $\mathbb{G}$.

- The second one is a mathematical part that aims at proving a representation theorem for martingales of the full filtration, as soon as the martingales of the reference market can be represented on a finite set of martingales, and when the credit event is an initial time.

  This theorem emphasizes the major role played by initial times in credit modelling. A corollary allows to describe the positive $\mathbb{G}$-martingales, hence the set of $\mathcal{G}_T$-equivalent probabilities once an ”historical probability” is given.

- The third one is a study of the special -and fundamental- case where the “reference market” is complete and arbitrage free. We describe thanks to the previous martingale representation theorem the set of the probabilities equivalent to the historical one on $\mathcal{G}_T$ under which the reference assets remain martingales. Then it is established under mild assumptions that the full market, where a credit event-sensitive asset is added to the collection of the reference assets, is also complete and without arbitrage opportunities.

  A following section presents the links between the completeness and the immersion property of the filtration enlargement: this property, often referred to as $(\mathcal{H})$-hypothesis, denotes the fact that the $\mathbb{F}$-martingales remain $\mathbb{G}$-martingales. We shall prove that immersion holds under any $\mathbb{G}$-e.m.m, and characterize the change of probability that allows to go from a ”reference neutral-risk” probability[3] to a neutral risk probability, under which immersion holds. We derive the important corollary that if the $\mathbb{F}$-martingale part of the survival process defined under a ”reference neutral-risk” probability $\mathbb{P}^*$ is not equal to zero, $\mathbb{P}^*$ is not neutral-risk probability of the full market.

- The last part is devoted to the case where the reference market is incomplete and a default sensitive asset is traded. Starting from a reference e.m.m. $\mathbb{P}^*$, we construct a unique e.m.m. in the full filtration that preserves the main properties of the reference market.

  Precisely, we prove that there exists a unique probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}^*$ such that the price process (composed of the reference assets and the default sensitive asset) is a $(\mathbb{G}, \mathbb{Q})$-martingale that preserves the“reference pricing”, i.e., such that $\mathbb{E}^{\mathbb{Q}}(X_T) = \mathbb{E}^*(X_T)$,

[1] See Jeanblanc and Le Cam [24] for a survey on reduced form modelling and hazard process.

[2] See the following section, Jiao [22], El Karoui et al. [12] or the paper of Jeanblanc and Le Cam [23] for a study of the properties of initial times and their application to progressive enlargement of filtrations.

[3] A probability under which the reference assets are $\mathbb{F}$-martingales.

for any $X_T \in L^2(\mathcal{F}_T)$. We establish that here again, immersion property of the filtration enlargement holds under this probability.

In this paper, all the processes are constructed on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where the probability measure $\mathbb{P}$ is referred to as the historical probability.

A financial market is represented in the sequel by a $n+2$-dimensional price process $\widetilde{S} = \left(S_t^0, ..., S_t^{n+1}; 0 \leq t \leq T\right)$, in which S^0 will denote the saving accounts, i.e., the risk-free asset[4], and its information by $\mathbb{G}$ assumed to be the natural (augmented) filtration generated by $\widetilde{S}$. We do not assume that $\mathcal{G}_T = \mathcal{A}$, and we emphasize that $\mathbb{P}$ is a probability measure defined on $\mathcal{A}$ (even if we shall be interested in the sequel in restrictions of the probabilities on sub-σ-algebras of $\mathcal{A}$).

We denote by $\Theta_{\mathbb{P}}^{\mathbb{G}}\left(\widetilde{S}\right)$ the set of $\mathbb{G}$-e.m.ms, i.e., the set of probability measures $\mathbb{Q}$ defined on $\mathcal{A}$, equivalent to $\mathbb{P}$ on $\mathcal{A}$, such that $\widetilde{S}/S^0 \in \mathcal{M}^{loc}(\mathbb{G}, \mathbb{Q})$, i.e., such that the discounted process $(\widetilde{S}_t/S_t^0, t \leq T)$ is a $(\mathbb{G}, \mathbb{Q})$-local martingale. In what follows, we assume for the sake of simplicity that $S^0 \equiv 1$.

It is well known that there are strong links between no-arbitrage hypothesis and the existence of an equivalent martingale measure (see Kabanov [26], Delbaen and Shachermayer [11]). In this paper we are interested with the property of $\Theta_{\mathbb{P}}^{\mathbb{G}}\left(\widetilde{S}\right)$ being not empty. This condition is equivalent to the No Free Lunch with Vanishing Risk condition (a condition slightly stronger than absence of arbitrage).

Recall that the market where the assets $S^i, i = 0, \ldots, n+1$ are traded is complete if any contingent claim is replicable: For any payoff $X_T \in L^2(\mathcal{G}_T)$ there exists a $\mathbb{G}$-predictable self-financed strategy with terminal value X_T. Our aim in this paper is neither to make a fine discussion on the best hypotheses on trading strategies, nor to use the most precise and efficient assumptions in that matter (the interested reader may refer for example to Delbaen and Schachermayer [11], Kabanov [26] or/and Protter [31]).

Given an e.m.m. $\mathbb{Q} \in \Theta_{\mathbb{P}}^{\mathbb{G}}\left(\widetilde{S}\right)$, we say that the $\mathbb{Q}$-local martingale $\widetilde{S}$ enjoys the predictable representation property[5] (under $\mathbb{Q}$), if every $(\mathbb{Q}, \mathbb{G})$-local martingale M can be written $M = M_0 + m \star \widetilde{S}$, where M_0 belongs to $\mathcal{G}_0$ and $m \star \widetilde{S}$ is the process $\int_0^t m_s d\widetilde{S}_s$ with m a $\mathbb{G}$ -predictable locally bounded process. If $\widetilde{S}$ enjoys the PRP, the market is complete. When only considering probabilities equivalent to a given one (in our case the historical probability $\mathbb{P}$), it is straightforward that if $\Theta_{\mathbb{P}}^{\mathbb{G}}\left(\widetilde{S}\right)$ restricted to $\mathcal{G}_T$ is a singleton, PRP holds under this martingale probability and the market is complete.

[4] S^{n+1} will represent in the sequel an asset on which may be read the occurrence of the credit event, at the opposite of the $n+1$ first ones.

[5] General presentations of PRP are available in Revuz and Yor [33], Jacod and Shiryaev [20] or Protter [32] .

1 Credit modelling framework

We work in this study within a progressive enlargement of filtration set-up, so that to study the pricing of derivatives written on underlyings sensitive to a credit event τ. We refer the reader to Elliott et al. [15] or to Jeanblanc and Rutkowski [25] for a detailed presentation of this approach, and to Jeanblanc and Le Cam [24] for the reasons that lead us to adopt it in this context.

In this framework, we shall split the information induced by the market into two components: a first one generated by what will be called the *default-free* assets, and a second by the knowledge of the occurrence of a credit event (the probability of occurrence of this event will depend on factors adapted to the first filtration). Precisely, two sources of information are introduced, the *reference filtration* and the *default-sensitive asset*.

1.1 The two information flows

The reference filtration. We consider an $n+1$-dimensional vector S of assets $S^0, \ldots, S^n$ and its natural filtration $\mathbb{F}$, that will be referred to as the reference filtration[6] in the sequel[7]. These assets may be shares, vanilla options, interest rates, change rates, etc.: All listed information that may be used by the market to build its anticipations on the probability of occurrence of the risk - and impact the bid-ask price of instruments written on τ). For example, if τ is the default time of a bond issued by a firm X, it is in general not a stopping time with respect to the filtration generated by the stock of X, or by interest rates (even it is far from being independent of such variables). The information flow $\mathbb{F}$ does not contain the information of the occurrence of the credit event, i.e., τ is not an $\mathbb{F}$-stopping time.

We denote by $\Theta_{\mathbb{P}}^{\mathbb{F}}(S)$ the set of $\mathbb{F}$-e.m.ms[8]. The next hypothesis will be systematically imposed on the model, and implies the absence of arbitrage in the reference market[9]: $\Theta_{\mathbb{P}}^{\mathbb{F}}(S)$ is not empty. This hypothesis will hold until the end of the article.

Default-sensitive asset. We introduce an asset S^{n+1}, that bears direct information on τ, i.e., that satisfies:

$$\mathcal{H}_t \subset \sigma\left(S_s^{n+1}, 0 \leq s \leq t\right) \subset \mathcal{H}_t \vee \mathcal{F}_t \text{ for any } t \geq 0, \tag{1}$$

where the notation $\mathbb{H} = (\mathcal{H}_t, t \geq 0)$ stands for the natural augmentation of the filtration generated by the process $H_t = \mathbb{1}_{\tau \leq t}$. This filtration models the knowledge of the occurrence of the credit event.

[6] In [4] Bélanger et al. refer to $\mathbb{F}$ as the *non firm specific information.* For us, this information flow must be considered as the "market risk" information, and can bear assets linked to the firm, for example its equity or even its directly its spread risk, see later.

[7] If needed, we set $\mathcal{F}_s = \mathcal{F}_T$ for $s > T$

[8] The set of probabilities $\mathbb{Q}$ defined on $\mathcal{A}$, equivalent to $\mathbb{P}$ on $\mathcal{A}$, such that $S \in \mathcal{M}^{loc}(\mathbb{F}, \mathbb{Q})$, i.e., such that the process $S = (S_t, t \leq T)$ is an $(\mathbb{F}, \mathbb{Q})$ -local martingale.

[9] Recall that we have assumed null interest rate to ease the presentation.

This relation explains the fact that the default can be read on the paths of S^{n+1}, and that this asset can be priced in terms of τ and $\mathbb{F}$ (think of a risky bond, a defaultable zero-coupon or a credit default swap[10]).

We denote by $\widetilde{S}$ the vector $\left(S^0, S^1, ..., S^{n+1}\right) = (S, S^{n+1})$, and by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, the natural augmentation of the filtration generated by $\widetilde{S}$ (the full information of the market).

1.2 Financial interpretation of this decomposition

In terms of risk analysis, S^{n+1} bears in general two types of risks (we consider in this survey only the single default case, hence do not enter in a discussion about the correlation risk):

- a "market risk" - typically a spread risk, i.e., the natural variation of the price of the asset when time goes on without default;
- a jump risk - the specific risk of default, due to the occurrence of τ.

This framework is based on the assumption that *the market risk can be hedged with* $\mathbb{F}$*-adapted instruments, and that the jump risk relies on* $\mathbb{H}$*-adapted instruments.* Two points of view can be considered to justify this assumption.

1− The first one - based on economic analysis - lies on the observation that the spread risk is mainly ruled out by the same noise sources as the assets that generate the reference filtration. For example:

- In the context of firm bonds pricing, Bélanger et al. link in [4] the spread risk of the defaultable zero-coupon to the stochastic interest rates. In such a modelling, the credit event is constructed as the hitting time of an independent random barrier by an increasing $\mathbb{F}$-adapted increasing process (the $\mathbb{F}$-intensity), where $\mathbb{F}$ is the filtration bearing the stochastic interest rates movements.

  The parameters of the intensity process may depend on the firm (see also Ehlers and Schönbucher [13], where the authors insist on the rôle of the systemic risk implied by the interest rates on a portfolio of credit risks).

- Moreover in a very close matter, Carr and Wu in [8] or Cremers et al. in [10] show that corporate CDS spreads covary with both the stock option implied volatilies and skewness.

  It confirms that the factors ruling out the movements of the spreads are linked to the variations of the interest rate and of the equity (and its volatility).

[10] CDS will refer to Credit Default Swap in the sequel.

- In the context of modelling CDS on debt issued by states (in their example Mexico and Brazil), Carr and Wu study in [9] the correlation between the currency options and the credit spreads. They prove that these quantities are deeply linked and propose a model in which the alea driving the intensity of the default is composed by the sum of a function of the alea of the stochastic volatility of the FX (see Heston [17]), and an independent noise (see also Ehlers and Schönbucher in [14]).

- More generally, this vision is shared by the supporters of structural modelling, in which the default time is triggered by a barrier reached by the equity value (see Merton [30] or Black and Cox [6] for example). In [2], Atlan and Leblanc model the credit time as the reaching time by the Equity of the firm of value zero, the stock following a CEV (see also Albanese and Chen [1] or Linetsky [29]).

2− The second one is based on the introduction of a new noise source, this alea driving the spread risk, considered as having its own evolution (both approaches can be combined). In this construction as well, the "market risk of the defaultable security" does not contain the default occurrence knowledge, and can be sorted in the $\mathbb{F}$-information with the other "market risks sensible" assets.

In practice, it is easy to synthesize an asset that is sensible to this spread risk and not to the jump risk. Take two instruments as S^{n+1} of different maturity for example, denoted by X^1 and X^2, and assume the market risk is modelled by a (risk-neutral) Brownian motion W. Assume that M is the compensated martingale associated to H, and that $dX^i_t = \beta^i_t dM_t + \delta^i_t dW_t$ under the e.m.m. Set up the self-financed portfolio Π that is long at any time of β^2_t of the asset X^1 and short of β^1_t of X^2 (and has a position in the savings account to stay self-financed). This portfolio has only sensitivity against the spread risk, and does not jump with τ. Indeed,

$$d\Pi_t = r\Pi_t dt + \beta^2_t dX^1_t - \beta^1_t dX^2_t = r\Pi_t dt + \left(\beta^2_t \delta^1_t - \beta^1_t \delta^2_t\right) dW_t.$$

Remark that with a δ-combination (instead of the β -combination), we can set up a portfolio only sensible of the jump risk (and that has no spread risk):

$$d\Pi'_t = r\Pi'_t dt + \delta^2_t dX^1_t - \delta^1_t dX^2_t = r\Pi'_t dt + \left(\beta^1_t \delta^2_t - \beta^2_t \delta^1_t\right) dM_t.$$

The two points of view (that need to be combined to achieve a maximum of precision in calibration procedures) converge on the idea that *splitting the information of the market in two sub-filtrations is finally quite natural.* Another nomenclature may consist in "market risk filtration" for $\mathbb{F}$, and "default risk filtration" for $\mathbb{H}$.

1.3 Absence of arbitrage

Starting from a reference market with no arbitrage, the absence of arbitrage of the full market is not automatic and deeply depends on the nature of τ. For it to hold, it is necessary to work in a mathematical set up where $\mathbb{F}$-semi-martingales remain $\mathbb{G}$-semi-martingales[11].

As developed in Jeanblanc and Le Cam [24], this property does not hold for any random time τ, and we choose to work in all the paper, under the following condition on τ.

Hypothesis $\mathbf{H}_1$: The credit event is an initial time, that is, there exists a family of processes $(\alpha^u, u \in \mathbb{R}^+)$ such that for any $u \geq 0$, the process $(\alpha_t^u, 0 \leq t \leq T)$ is an $\mathbb{F}$-martingale and that satisfies

$$\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty \alpha_t^u du$$

for any $\theta \geq 0$, and any $t \leq T$.

There exists a $\mathcal{O}(\mathbb{F} \otimes \mathcal{B})$-measurable version[12] of the mapping $(\omega, u, t) \to \alpha_t^u(\omega)$, right-continuous with left limits[13]. We shall consider this version in the sequel. In our setting, the law of τ admits a density w.r.t. Lebesgue measure, equal to α_0^u. In the general definition of initial times, $\alpha_0(u)du$ may be replaced by any probability measure on $\mathbb{R}^+$ $\nu(du)$.

We denote by G the Azéma super-martingale

$$G_t := \mathbb{P}(\tau > t | \mathcal{F}_t).$$

We write $G = Z - A$ the $\mathbb{F}$-Doob-Meyer decomposition of this super-martingale (of class (D)). From hypothesis $\mathbf{H}_1$, every $(\mathbb{F}, \mathbb{P})$-martingale X is a $(\mathbb{G}, \mathbb{P})$-semi-martingale, and if the $\mathbb{F}$-martingales are continuous:

$$X_t - \int_0^{t \wedge \tau} \frac{d \langle X, Z \rangle_u}{G_{u-}} - \int_{t \wedge \tau}^t \frac{d \langle X, \alpha^\theta \rangle_u}{\alpha_{u-}^\theta} \Bigg|_{\theta = \tau} \in \mathcal{M}(\mathbb{G}, \mathbb{P}). \tag{2}$$

(see Jeanblanc and Le Cam [23]).

Proposition 1 *When the law ν of the initial time τ has no atoms, for example under the hypothesis* $\mathbf{H}$*, it avoids the $\mathbb{F}$-stopping times, i.e.,*

$$\mathbb{P}(\tau = T) = 0, \ \forall T \text{ finite } \mathbb{F}\text{-stopping time.}$$

Proof: This result is a consequence of the lemma 2 of [23], that states that if τ is an initial time and if T is a finite $\mathbb{F}$-stopping time,

$$\mathbb{E}\left(1_{\{\tau = T\}} \middle| \mathcal{F}_T \right) = \alpha_T^T \nu(\{T\}) \text{ a.s.}$$

[11] So that $\Theta_{\mathbb{P}}^{\mathbb{G}}$ $\widetilde{S}$ be not empty.

[12] The σ-field $\mathcal{O}(\mathbb{F} \otimes \mathcal{B})$ is the optional σ-field on $\Omega \times \mathbb{R}^+ \times [0, T]$.

[13] See Jacod [19] for a presentation of the paths regularity of the martingale density family

It follows that if ν has no atoms,

$$\mathbb{P}(\tau = T) = \mathbb{E}\left(1_{\{\tau=T\}}\right) = 0,$$

hence τ avoids the $\mathbb{F}$-stopping times. □

As we shall focus on in this paper in change of probabilities, it is necessary to ensure that the initial property does not depend on the historical probability. It follows from:

Proposition 2 *If τ is an initial time under $\mathbb{P}$, and if $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$, then τ is a $\mathbb{Q}$-initial time.*

PROOF: The definition of τ being an initial time can be formulated in the following way: if $Q_t^{\mathbb{P}}$ denotes a regular version of the conditional law of τ: $Q_t(\omega,]\theta, \infty[) := \mathbb{P}(\tau > \theta|\mathcal{F}_t)(\omega)$, there exists a deterministic measure ν such that $Q_t^{\mathbb{P}}(\omega, d\theta) \ll \nu(d\theta)$ $\mathbb{P} - a.s.$ As two equivalent probabilities have the same null sets, if $\mathbb{Q} \sim \mathbb{P}$, $Q_t^{\mathbb{Q}}(\omega, d\theta) \ll \nu(d\theta)$ $\mathbb{Q} - a.s$, and the proposition is proved. The following alternative proof allows to derive the new martingale density.

Let $(\eta_t, t \geq 0)$ be the $\mathbb{G}$-martingale which is the Radon-Nikodym density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$

$$d\mathbb{Q}|_{\mathcal{G}_t} = \eta_t d\mathbb{P}|_{\mathcal{G}_t}.$$

Let T be fixed and $\theta, t < T$. Then, the Bayes rule implies:

$$\mathbb{Q}(\tau > \theta|\mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}((1 - H_\theta)|\mathcal{F}_t) = \frac{\mathbb{E}^{\mathbb{P}}((1 - H_\theta)\eta_T|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{P}}(\eta_T|\mathcal{F}_t)}$$

since $(1 - H_\theta)$ is $\mathcal{G}_\theta$ hence $\mathcal{G}_T$ -measurable. Assume in a first step that $\eta_T = \widetilde{\eta}_T h(\tau \wedge T)$ where $\widetilde{\eta}_T$ is a (bounded) $\mathcal{F}_T$-measurable random variable and h is a (bounded) deterministic function. We have:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}((1 - H_\theta)\eta_T|\mathcal{F}_t) &= \mathbb{E}^{\mathbb{P}}((1 - H_\theta)\widetilde{\eta}_T h(\tau \wedge T)|\mathcal{F}_t) = \mathbb{E}^{\mathbb{P}}(\widetilde{\eta}_T \mathbb{E}^{\mathbb{P}}((1 - H_\theta)h(\tau \wedge T)|\mathcal{F}_T)|\mathcal{F}_t) \\ &= \mathbb{E}^{\mathbb{P}}\left(\widetilde{\eta}_T \int_\theta^\infty h(u \wedge T)\alpha_T^u du \middle| \mathcal{F}_t\right) = \int_\theta^\infty \mathbb{E}^{\mathbb{P}}(\widetilde{\eta}_T \alpha_T^u|\mathcal{F}_t)h(u \wedge T)du \end{aligned}$$

It follows that

$$\mathbb{Q}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty \frac{\mathbb{E}^{\mathbb{P}}(\widetilde{\eta}_T \alpha_T^u|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{P}}(\eta_T|\mathcal{F}_t)} h(u \wedge T)du$$

Moreover, if μ_T denotes the $\mathcal{F}_T$-density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$, i.e., $d\mathbb{Q}|_{\mathcal{F}_T} = \mu_T d\mathbb{P}|_{\mathcal{F}_T}$, μ_T writes

$$\mu_T = \mathbb{E}^{\mathbb{P}}(\eta_T|\mathcal{F}_T) = \widetilde{\eta}_T \mathbb{E}^{\mathbb{P}}(h(\tau \wedge T)|\mathcal{F}_T) = \widetilde{\eta}_T \int_0^\infty h(u \wedge T)\alpha_T^u du := \widetilde{\eta}_T \phi_T.$$

We now introduce the family of $(\mathbb{F}, \mathbb{Q})$-martingales $\widehat{\alpha}_\cdot^u$, defined for any $u \geq 0$ by

$$\widehat{\alpha}_t^u := h(u \wedge T)\mathbb{E}^{\mathbb{Q}}(\alpha_T^u/\phi_T|\mathcal{F}_t).$$

Bayes rule implies:

$$\mathbb{E}^{\mathbb{Q}}(\alpha_T^u/\phi_T|\mathcal{F}_t) = \frac{\mathbb{E}^{\mathbb{P}}(\alpha_T^u \mu_T/\phi_T|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{P}}(\mu_T|\mathcal{F}_t)} = \frac{\mathbb{E}^{\mathbb{P}}(\alpha_T^u \widetilde{\eta}_T|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{P}}(\eta_T|\,\mathcal{F}_T|\mathcal{F}_t)} = \frac{\mathbb{E}^{\mathbb{P}}(\widetilde{\eta}_T \alpha_T^u|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{P}}(\eta_T|\mathcal{F}_t)},$$

and that, for any T, for any $t, \theta < T$,

$$\mathbb{Q}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty \widehat{\alpha}_t^u du,$$

which means that τ is an $\mathbb{F}$-initial time under $\mathbb{Q}$. The general case follows by application of the monotone class theorem. □

Remark 1 *When the credit event avoids the $\mathbb{F}$-stopping times, immersion property under the probability measure $\mathbb{P}$ -$\mathcal{M}(\mathbb{F},\mathbb{P}) \subset \mathcal{M}(\mathbb{G},\mathbb{P})$- is equivalent to the property that for any $u \geq 0$, the martingale α^u is constant after u (see Jeanblanc and Le Cam [23]), i.e.,*

$$\alpha_t^u = \alpha_{t\wedge u}^u, \text{ for any } (u,t) \geq 0 \tag{3}$$

That is the case under hypothesis **H**.

We recall that immersion property is not preserved by a change of probability (see for example Kusuoka [28]).

2 Representation theorem in the enlarged filtration

Under hypothesis $\mathbf{H}_1$ the $\mathbb{F}$-martingales are $\mathbb{G}$-semi-martingales and the initial time property is stable when changing the probability. We also assume, to ease the proofs the following condition:

Hypothesis $\mathbf{H}_2$: The process S is continuous.

This hypothesis will hold until the end of the paper. The aim of the following first subsection is to prove that under a progressive enlargement of a filtration by an initial time, if the reference filtration $\mathbb{F}$ enjoys a predictable representation property, the enlarged filtration $\mathbb{G}$ enjoys the same property. The goal of the second one is to apply this result to the description of all the $\mathbb{G}$-martingales and to parameterize the change of equivalent probabilities.

2.1 Representation of the $\mathbb{G}$-martingales

We starting this section by defining the:

Hypothesis $\mathbf{H}_3$: We assume that the $\mathbb{F}$-market is complete and arbitrage free, i.e., that the $(\mathbb{F},\mathbb{P}^*)$-local martingale S enjoys the PRP (with $\mathbb{P}^* \in \Theta_{\mathbb{P}}^{\mathbb{F}}(S)$).

As a consequence of Hypotheses $\mathbf{H}_2$ and $\mathbf{H}_3$, $\mathbb{F}$-martingales are continuous (in particular the densities). We start with a brief presentation of the two fundamental (local) martingales on which we shall prove that any $\mathbb{G}$-martingale can be represented.

The first one, M, is the martingale part of the Doob-Meyer decomposition of the increasing $\mathbb{G}$-adapted process H. It is well known (see for example, Bielecki and Rutkowski [5]) that with no particular condition on τ, the $\mathbb{G}$-compensator of H writes: $(1-H)\,dA/G_{-}$, where A is the $\mathbb{F}$-compensator of the $\mathbb{F}$-super-martingale G, hence

$$M_t = H_t - \int_0^{t\wedge\tau} \frac{dA_u}{G_{u-}}.$$

From hypothesis $\mathbf{H}_1$, the conditional survival probability writes

$$G_t^\theta := \mathbb{P}^*\left(\tau > \theta | \mathcal{F}_t\right) = \int_\theta^\infty \alpha_t^u du \tag{4}$$

that allows to explicitly compute A, from

$$G_t = G_t^t = \int_0^\infty \alpha_{t\wedge u}^u du - \int_0^t \alpha_u^u du \equiv Z_t - A_t, \tag{5}$$

and to conclude (using that G is continuous):

$$M_t = H_t - \int_0^t \frac{1-H_u}{G_u} \alpha_u^u du. \tag{6}$$

The second one is the local martingale part of the decomposition of the special $\mathbb{G}$-semi-martingale S ($\mathbf{H}_1$ implies that the $\mathbb{F}$-martingales remain $\mathbb{G}$-semi-martingales from (2), and we will see that the form of τ makes them special). From the PRP of $\mathbb{F}$, there exist:

- An n-dimensional $\mathbb{F}$-predictable process $z = \left(z^1, ..., z^n\right)$ such that Z (defined by (5)) writes

$$Z = Z_0 + z \star S,$$

  where $z \star S$ stands for the process $t \longmapsto \sum_{i\le n} \int_0^t z_s^i dS_s^i$,

- A family of n-dimensional $\mathbb{F}$-predictable processes a, such that for any $u \geq 0$ the $(\mathbb{F}, \mathbb{P}^*)$-martingale α^u writes

$$\alpha^u = \alpha_0^u + a^u \star S,$$

  where $a^u \star S$ stands for the process $t \longmapsto \sum_{i\le n} \int_0^t a_s^{u,i} dS_s^i$.

The quadratic covariations $\langle S, Z\rangle$ and $\left\langle S, \alpha^\theta \right\rangle$ are well defined, and from (2), the process

$$\widehat{S}_t := S_t - \int_0^t \frac{(1-H_s)}{G_S} d\left\langle S, Z\right\rangle_S + \frac{H_S}{\alpha_S^\theta} d\left\langle S, \alpha^\theta\right\rangle_s \bigg|_{\theta=\tau} \tag{7}$$

is a $(\mathbb{G}, \mathbb{P}^*)$-local martingale. It follows that for any $i \leq n$

$$\widehat{S}_t^i = S_t^i - \int_0^t \left(\frac{1-H_S}{G_S} z_S + \frac{H_S}{\alpha_s^\tau} a_s^\tau \right) \cdot d\left\langle S^i, S\right\rangle_s := S_t^i - C_t,$$

where $z \cdot d\left\langle S^i, S\right\rangle$ (resp. $a^\tau \cdot d\left\langle S^i, S\right\rangle$) stands for $d\left\langle S^i, z \star S\right\rangle$ (resp. $d\left\langle S^i, a^\tau \star S\right\rangle$).

Predictable representation. The next theorem establishes a predictable representation property for $\mathbb{G}$-local martingales under $\mathbb{P}^*$, as soon as the $\mathbb{F}$-market enjoys this predictable representation property. Indeed, any η which belongs to $\mathcal{M}^{loc}(\mathbb{G},\mathbb{P}^*)$ will write as the sum of an integral[14] with respect to the $(\mathbb{G},\mathbb{P}^*)$-martingale M and an integral with respect to the $(\mathbb{G},\mathbb{P}^*)$-local martingale $\widehat{S}$. This result extends the representation theorem by Kusuoka [28] to any complete reference market and to the case where immersion does not hold.

Theorem 2.1 *Assume that* $\mathbf{H}_1$ *to* $\mathbf{H}_3$ *hold. Denote by* $\widehat{S}$ *the local martingale part of the decomposition of* S *as* $\mathbb{G}$*-semi-martingale, given by* (7). *For every* $\eta \in \mathcal{M}^{loc}(\mathbb{G},\mathbb{P}^*)$*, there exist* $n+1$ $\mathbb{G}$*-predictable processes* β *and* γ *such that*

$$\eta_t = \eta_0 + (\beta \star M)_t + (\gamma \star \widehat{S})_t.$$

Proof: Without loss of generality we prove the theorem for $n = 1$ (considering only one component of the vector S), to ease the notations. The vectorial version of the proof is a straightforward generalization. By localization, we only consider martingales. Let $\eta \in \mathcal{M}(\mathbb{G},\mathbb{P}^*)$, and as we are only interested in finite time horizon, we write $\eta_t = \mathbb{E}^*(\eta_T|\mathcal{G}_t)$ for $t < T$. By a monotone class argument, we reduce ourself to the case where η_T writes $F_T h(\tau\wedge T)$, with $F_T \in \mathcal{F}_T$, assumed to be bounded and h is a (bounded) deterministic function. We split the problem in three parts:

$$\begin{aligned}
\eta_t &= \mathbb{E}^*\left(F_T h(T) 1_{\tau>T}\middle|\mathcal{G}_t\right) + \mathbb{E}^*\left(F_T h(\tau) 1_{\tau\le T}\middle|\mathcal{G}_t\right) = a_t + \mathbb{E}^*\left(F_T h(\tau) 1_{\tau\le T}\middle|\mathcal{G}_t\right)\\
&= \underbrace{L_t h(T)\,\mathbb{E}^*\left(F_T G_T\middle|\mathcal{F}_t\right)}_{a_t} + \underbrace{L_t\mathbb{E}^*\left(F_T h(\tau) 1_{t<\tau\le T}\middle|\mathcal{F}_t\right)}_{b_t} + \underbrace{H_t\mathbb{E}^*\left(F_T h(\tau) 1_{\tau\le t}\middle|\mathcal{F}_t\vee\sigma(\tau)\right)}_{c_t}.
\end{aligned}$$

with $L_t = (1-H_t)/G_t = D_t(1-H_t) \in \mathcal{M}(\mathbb{G},\mathbb{P}^*)$, where $D_t = G_t^{-1}$. From the decomposition (5), we have $dG_t = -\alpha_t^t dt + z_t dS_t$, and from Itô's formula, $dD_t = D_t^2\left(\alpha_t^t dt + z_t^2 D_t d\langle S\rangle_t\right) - D_t^2 z_t dS_t$.

Let us start by developing a : We first remark that the process a defined by $a_t := L_t h(T)\,\mathbb{E}^*\left(F_T G_T\middle|\mathcal{F}_t\right)$ is a $\mathbb{G}$-martingale, so one knows in advance that this particular semi-martingale has a null predictable bounded variation part; nevertheless, we keep all these terms in our computation. The process N where $N_t := \mathbb{E}^*\left(F_T G_T\middle|\mathcal{F}_t\right)$ is an $\mathbb{F}$-martingale, and writes by representation theorem $n + \int_0^t n_s dS_s$, with $(n_s, s\ge 0)$ predictable. Since S, D and N are continuous, $[S,H]=0$, $[D,N]=\langle D,N\rangle$, and, from $(h(T))^{-1}a_t = (1-H_t)D_tN_t$, one gets

$$\begin{aligned}
(h(T))^{-1}da_t &= -D_tN_tdH_t + (1-H_t)D_tdN_t + (1-H_t)N_tdD_t + (1-H_t)d\langle D,N\rangle_t\\
&= -D_tN_tdH_t + (1-H_t)D_tn_tdS_t + (1-H_t)N_tD_t^2\alpha_t^tdt\\
&\quad + (1-H_t)N_tz_t^2D_t^3d\langle S\rangle_t - (1-H_t)N_tD_t^2z_tdS_t - (1-H_t)D_t^2n_tz_td\langle S\rangle_t\\
&= -D_tN_tdM_t - (1-H_t)D_t^2N_t\alpha_t^tdt + (1-H_t)\left(D_tn_t - N_tD_t^2z_t\right)dS_t\\
&\quad + (1-H_t)N_tD_t^2\alpha_t^tdt + (1-H_t)(N_tz_tD_t - n_t)D_t^2z_td\langle S\rangle_t
\end{aligned}$$

[14]Recall for any $X \in \mathcal{H}_1$, one has $\mathbb{E}([X,M]_\infty) = \mathbb{E}X_\tau \le ||X||_{\mathcal{H}_1}$ hence M is a BMO (the dual of $\mathcal{H}_1$) martingale.

In the third equality, we have used that the $\mathbb{G}$-Doob-Meyer decomposition of the increasing process H writes $dH_t = dM_t + (1 - H_t) D_t \alpha_t^t dt$ (from $dA_t = \alpha_t^t dt$), with $M \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$ (see (6)). Moreover $S_t = \widehat{S}_t + C_t$ with $\widehat{S} \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$, and from (7) $(1 - H_t) dC_t = (1 - H_t) z_t D_t d\langle S\rangle_t$. It follows

$$\begin{aligned}(h(T))^{-1} da_t &= -D_t N_t dM_t + (1 - H_t) \left(D_t n_t - N_t D_t^2 z_t\right) d\widehat{S}_t \\ &\quad + (1 - H_t) \left(\left(D_t n_t - N_t D_t^2 z_t\right) z_t D_t + N_t z_t^2 D_t^3 - n_t D_t^2 z_t\right) d\langle S\rangle_t \\ &= -D_t N_t dM_t + (1 - H_t) \left(D_t n_t - N_t D_t^2 z_t\right) d\widehat{S}_t.\end{aligned}$$

To explicit the decomposition of the special $\mathbb{G}$ -semi-martingale b, where $b_t = L_t \mathbb{E}^* \left(F_T h(\tau) 1_{t<\tau\leq T} \middle| \mathcal{F}_t\right)$, we introduce for any $u \geq 0$ the $\mathbb{F}$ -martingale $N_t^u = \mathbb{E}^* \left(F_T \alpha_T^u \middle| \mathcal{F}_t\right)$ and its decomposition on S : $N_t^u = y^u + \int_0^t y_s^u dS_s$ provided by the martingale representation theorem on $\mathbb{F}$. By definition of initial times, it follows:

$$b_t = L_t \mathbb{E}^* \left(F_T \int_t^T h(u) \alpha_T^u du \middle| \mathcal{F}_t \right) = L_t \int_t^T h(u) N_t^u du$$

hence, one can differentiate using Itô Wentcell formula:

$$\begin{aligned}db_t &= -D_t \left(\int_t^T h(u) N_t^u du\right) dH_t + (1 - H_t) \left(\int_t^T h(u) N_t^u du\right) dD_t \\ &\quad - (1 - H_t) D_t h(t) N_t^t dt + (1 - H_t) D_t \left(\int_t^T h(u) y_t^u du\right) dS_t \\ &\quad - (1 - H_t) D_t^2 z_t \left(\int_t^T h(u) y_t^u du\right) d\langle S\rangle_t\end{aligned}$$

and, introducing the $\mathbb{G}$-decomposition of the semi-martingale S and the compensator of H, we obtain finally:

$$\begin{aligned}db_t &= -D_t \left(\int_t^T h(u) N_t^u du\right) dH_t + (1 - H_t) \left(\alpha_t^t D_t^2 \int_t^T h(u) N_t^u du - D_t h(t) N_t^t\right) dt \\ &\quad + (1 - H_t) \left(D_t \int_t^T h(u) \left(y_t^u - D_t N_t^u z_t\right) du\right) dS_t \\ &\quad - (1 - H_t) \left(D_t^2 z_t \int_t^T h(u) \left(y_t^u - D_t N_t^u z_t\right) du\right) d\langle S\rangle_t \\ &= -D_t \left(\int_t^T h(u) N_t^u du\right) dM_t + (1 - H_t) \left(D_t \int_t^T h(u) \left(y_t^u - D_t N_t^u z_t\right) du\right) d\widehat{S}_t \\ &\quad - (1 - H_t) D_t h(t) N_t^t dt\end{aligned}$$

Decomposition of c. We can write $c_t = H_t \mathbb{E}^* \left(F_T h(\tau) 1_{\tau\leq t} \middle| \mathcal{F}_t \vee \sigma(\tau)\right) = H_t F(t, \tau)$, where for each $u \geq 0$ the random variable $F(t, u)$ is $\mathcal{F}_t$-measurable and for any $t \geq 0$, $u \longmapsto F(t, u)$ is a

Borel function. Using the properties of initial times, we compute from last expression $F(t,u) = h(u)\, N_t^u/\alpha_t^u$. For any $u \geq 0$, the dynamics write (using $dN_t^u = y_t^u dS_t$ and $d\alpha_t^u = a_t^u dS_t$):

$$d_t F(t,u) = h(u)\left(\frac{y_t^u}{\alpha_t^u} - \frac{N_t^u a_t^u}{(\alpha_t^u)^2}\right) dS_t + h(u)\left(N_t^u \frac{(a_t^u)^2}{(\alpha_t^u)^3} - \frac{a_t^u y_t^u}{(\alpha_t^u)^2}\right) d\langle S\rangle_t\,.$$

It follows that, since

$$\int_0^t F(s,\tau)\, dH_s = F(\tau,\tau)\, \mathbb{1}_{\tau\leq t} = \int_0^t F(s,s)\, dH_s,$$

we can write the decomposition of c :

$$\begin{aligned}
dc_t &= F(t,\tau)dH_t + H_t h(\tau)\left(\frac{y_t^\tau}{\alpha_t^\tau} - \frac{N_t^\tau a_t^\tau}{(\alpha_t^\tau)^2}\right) dS_t + H_t h(\tau)\left(\frac{N_t^\tau (a_t^\tau)^2}{(\alpha_t^\tau)^3} - \frac{a_t^\tau y_t^\tau}{(\alpha_t^\tau)^2}\right) d\langle S\rangle_t \\
&= F(t,t)dM_t + H_t h(\tau)\left(\frac{y_t^\tau}{\alpha_t^\tau} - \frac{N_t^\tau a_t^\tau}{(\alpha_t^\tau)^2}\right) d\widehat{S}_t + (1-H_t)\, D_t F(t,t)\, \alpha_t^t dt \\
&\quad + H_t h(\tau)\left(\frac{y_t^\tau}{\alpha_t^\tau} - \frac{N_t^\tau a_t^\tau}{(\alpha_t^\tau)^2}\right) dC_t + H_t h(\tau)\left(\frac{N_t^\tau (a_t^\tau)^2}{(\alpha_t^\tau)^3} - \frac{a_t^\tau y_t^\tau}{(\alpha_t^\tau)^2}\right) d\langle S\rangle_t \\
&= F(t,t)dM_t + H_t h(\tau)\left(\frac{y_t^\tau}{\alpha_t^\tau} - \frac{N_t^\tau a_t^\tau}{(\alpha_t^\tau)^2}\right) d\widehat{S}_t + (1-H_t)D_t F(t,t)\alpha_t^t dt
\end{aligned}$$

where the last equality comes from the expression (7) of dC on $\{\tau \leq t\}$.

Conclusion. Adding the three parts a, b, and c, we conclude, since $F(t,t)\alpha_t^t = h(t)N_t^t$, that the $\mathbb{G}$ -martingale can be decomposed on the two martingales $\left(M, \widehat{S}\right)$ and writes:

$$d\eta_t = \gamma_t d\widehat{S}_t + \beta_t dM_t$$

where

$$\begin{aligned}
\gamma_t &= (1-H_t)D_t\left((n_t - N_t D_t z_t)\, h(T) + \int_t^T h(u)\,(y_t^u - N_t^u D_t z_t)\, du\right) + H_t\left(\frac{y_t^\tau}{\alpha_t^\tau} - \frac{N_t^\tau a_t^\tau}{(\alpha_t^\tau)^2}\right) \\
\beta_t &= F(t,t) - D_t\left(N_t h(T) + \int_t^T h(u) N_t^u du\right)
\end{aligned}$$

which concludes the proof. □

2.2 Change of probability

Once a probability $\mathbb{P}$ is given on $\mathcal{A}$, each probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ is fully described by its $\mathbb{G}$-martingale density w.r.t. $\mathbb{P}$. The representation theorem established in the last section allows to describe all the $(\mathbb{P},\mathbb{G})$-martingales, hence to describe all the probabilities equivalent to $\mathbb{P}$ on $\mathcal{G}_T$.

Applying Theorem 2.1 to the particular case of a strictly positive martingale -in particular the density of a change of probability- we derive the

Proposition 3 *If the filtration $\mathbb{F}$ is generated by the $(\mathbb{F},\mathbb{P})$-martingale S and enjoys the PRP, and if η is a strictly positive $(\mathbb{G},\mathbb{P})$-martingale, then, there exists a pair of predictable processes γ,β such that*

$$\frac{d\eta_t}{\eta_{t-}} = \gamma_t d\widehat{S}_t + \beta_t dM_t,$$

with $\beta > -1$, i.e., $\eta = \mathcal{E}\left(\gamma \star \widehat{S}\right)\mathcal{E}\left(\beta \star M\right)$, with

$$\begin{cases} \mathcal{E}\left(\gamma \star \widehat{S}\right)_t = \exp\left(\int_0^t \gamma_u d\widehat{S}_u - \frac{1}{2}\int_0^t \gamma_u^2 d\langle \widehat{S}\rangle_u\right) \\ \mathcal{E}\left(\beta \star M\right)_t = \exp\left(\int_0^t \ln\left(1+\beta_s\right) dH_s - \int_0^t \beta_s \frac{1-H_s}{G_s}\alpha_s^s ds\right) \end{cases}$$

3 Complete reference market

In this section, we still make the assumption that the reference market is arbitrage free and complete: For any $X_T \in L^2\left(\mathcal{F}_T\right)$, there exist a constant x and n $\mathbb{F}$-predictable processes φ^i such that $X_T = x + \int_0^T \sum_{1\leq i\leq n} \varphi_u^i dS_u^i$ (we recall that we assume null interest rate). As recalled, this property is equivalent to the fact that the restriction of $\Theta_{\mathbb{P}}^{\mathbb{F}}\left(S\right)$ on $\mathcal{F}_T$ is a singleton (Jacod and Yor theorem [?]).

This assertion does not imply that there exists a unique probability measure $\mathbb{Q}$ on $\mathcal{A}$ such that S is an $(\mathbb{F},\mathbb{Q})$ -martingale, but that if two probabilities $\mathbb{P}^*$ and $\mathbb{Q}^*$ belong to $\Theta_{\mathbb{P}}^{\mathbb{F}}\left(S\right)$, then, their restriction to $\mathcal{F}_T$ are equal: $\mathbb{P}^*|_{\mathcal{F}_T} = \mathbb{Q}^*|_{\mathcal{F}_T}$.

We shall prove in this section that the full market described by $\widetilde{S} = \left(S^0, ..., S^{n+1}\right)$ is also complete, and study the links of this property with immersion. Precisely,

- The first subsection describes the set $\Theta_{\mathbb{P}}^{\mathbb{G}}\left(S\right)$, i.e., the set of probabilities equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ such that the $n+1$-dimensional process S remains a local-martingale[15]);
- The second subsection describes the unique martingale measure on the full market, i.e., the unique element of $\Theta_{\mathbb{P}}^{\mathbb{G}}\left(\widetilde{S}\right)$;
- The third subsection presents the links of this construction with the immersion property.

3.1 Description of the $\mathbb{G}$-martingale probabilities

In this first section, we study the behavior of the reference assets in the full market, i.e., the properties of the $(\mathbb{F},\mathbb{P})$-martingale S viewed as a $\mathbb{G}$-adapted process. The goal of this part is to describe the set $\Theta_{\mathbb{P}}^{\mathbb{G}}\left(S\right)$ of probabilities under which this $\mathbb{G}$-semi-martingale[16] is a martingale

[15] We shall prove that the restrictions of $\Theta_{\mathbb{P}}^{\mathbb{F}}\left(S\right)$ and $\Theta_{\mathbb{P}}^{\mathbb{G}}\left(S\right)$ on $\mathcal{F}_T$ are the same.

[16] Recall that the initial property of τ ($\mathbf{H}_2$ holds) ensures that S remains a $\mathbb{G}$-semi-martingale.

Before stating the proposition, we start by a technical remark. The following proof is based on the martingale property of the process $\eta = \mathcal{E}(\ell) := \mathcal{E}\left(-\vartheta \star \widehat{S}\right)$ where

$$\vartheta_t = (1 - H_t)\frac{z_t}{G_t} + H_t \frac{a_t^\tau}{\alpha_t^\tau}. \tag{8}$$

As $\widehat{S}$ is a $(\mathbb{G}, \mathbb{P}^*)$-martingale, η is a local-martingale, and extra conditions have to be assumed so that it be a true martingale and might be used for changing the probability. However, the conditions on the process $-\vartheta \star \widehat{S}$ may be brought to $\mathbb{F}$-adapted processes, in the following way.

Define by R the process $\int_0^t dZ_u/G_u$, and $\Phi_t = \int_0^t (1 - H_u) \frac{z_u}{G_u} d\widehat{S}_u$. If $R \in \mathcal{M}(\mathbb{F}, \mathbb{P})$,

$$\begin{aligned} \widehat{R}_t &= R_t - \int_0^{t\wedge\tau} \frac{d\langle R, Z\rangle_u}{G_u} - \int_{t\wedge\tau}^t \frac{d\left\langle R, \alpha^\theta\right\rangle_u}{\alpha_u^\theta}\Bigg|_{\theta=\tau} \\ &= R_t - \int_0^t \vartheta_u \frac{z_u}{G_u} d\langle S\rangle_u \in \mathcal{M}(\mathbb{G}, \mathbb{P}) \end{aligned}$$

hence[17] $\Phi_t = \int_0^t (1 - H_u)\, d\widehat{R}_u \in \mathcal{M}(\mathbb{G}, \mathbb{P})$. For the same reasons, if we define the family of processes $R_t^x = \int_0^t d\alpha_u^x/\alpha_u^x$, and $\Phi_t^x = \int_0^t H_u a_u^x/\alpha_u^x d\widehat{S}_u$ for any $x \geq 0$, the condition $R^x \in \mathcal{M}(\mathbb{F}, \mathbb{P})$ implies $\Phi^x \in \mathcal{M}(\mathbb{G}, \mathbb{P})$. Indeed,

$$\begin{aligned} \widehat{R_t^x} &= R_t^x - \int_0^{t\wedge\tau} \frac{d\langle R^x, Z\rangle_u}{G_u} - \int_{t\wedge\tau}^t \frac{d\left\langle R^x, \alpha^\theta\right\rangle_u}{\alpha_u^\theta}\Bigg|_{\theta=\tau} \\ &= R_t^x - \int_0^t \vartheta_u \frac{a_u^x}{\alpha_u^x} d\langle S\rangle_u \in \mathcal{M}(\mathbb{G}, \mathbb{P}) \end{aligned}$$

and[18] $\Phi_t^x = \int_0^t H_u d\widehat{R_u^x} \in \mathcal{M}(\mathbb{G}, \mathbb{P})$.

It follows that if R and R^x (whose definitions only depends on the construction of the random time) are $\mathbb{F}$-martingales, ℓ is an $\mathbb{G}$-martingale. In the same way, if these processes satisfy a Novikov-type condition, ℓ also satisfy one and the density η is a true martingale. For example, if $\mathbb{E}(\exp 2\langle R\rangle_\infty) < \infty$ and $\mathbb{E}(\exp 2\langle R^x\rangle_\infty) < K$ for any $x \geq 0$, then $\ell = \Phi + \Phi^\tau$ satisfies Novikov criterion:

$$\begin{aligned} \mathbb{E}\exp\left(\frac{1}{2}\langle \ell\rangle_\infty\right) &\leq \mathbb{E}\exp\langle\Phi\rangle_\infty \exp\langle\Phi^\tau\rangle_\infty \\ &\leq \left(\mathbb{E}\exp 2\langle\Phi\rangle_\infty\right)^{1/2}\left(\mathbb{E}\exp 2\langle\Phi^\tau\rangle_\infty\right)^{1/2} \end{aligned}$$

and the result follows from

$$\begin{aligned} \langle\Phi\rangle_\infty &= \int_0^t (1 - H_u)\, d\langle\widehat{R}\rangle_u = \int_0^t (1 - H_u)\, d\langle R\rangle_u \leq \langle R\rangle_\infty \\ \langle\Phi^\tau\rangle_\infty &= \int_0^t H_u d\langle\widehat{R^\tau}\rangle_u = \int_0^t H_u d\langle R^\tau\rangle_u \leq \langle R^\tau\rangle_\infty. \end{aligned}$$

[17] $(1 - H_u)\, d\widehat{S}_u = (1 - H_u)\;\; dS_u - z_u D_u d\langle S\rangle_u \;\; \in \mathcal{M}(\mathbb{G}, \mathbb{P})$.

[18] $H_u d\widehat{S}_u = H_u \;\; dS_u - \frac{a_u^\theta}{\alpha_u^\theta}\Big|_{\theta=\tau} d\langle S\rangle_u \;\; \in \mathcal{M}(\mathbb{G}, \mathbb{P})$

Hypothesis $\mathbf{H}_4$: We assume that $\mathcal{E}(-\vartheta \star \widehat{S})$ is a true martingale.

As noticed in the last lines, this hypothesis holds for example if R and R^x satisfy a Novikov-type condition. Under this condition, we can prove the

Proposition 4 *Assume that $\mathbf{H}_1$ to $\mathbf{H}_4$ hold. Then, the set $\Theta^{\mathbb{G}}_{\mathbb{P}^*}(S)$ is not empty and we can fully describe it as*

$$\Theta^{\mathbb{G}}_{\mathbb{P}^*}(S) = \left\{ \mathbb{Q} : \left.\frac{d\mathbb{Q}}{d\mathbb{P}^*}\right|_{\mathcal{G}_t} = \mathcal{E}(-\vartheta \star \widehat{S})_t \mathcal{E}(\beta \star M)_t \right\},$$

with ϑ given by (8), and $\beta > -1$ a $\mathbb{G}$-predictable process.

Proof: If the $\mathbb{F}$-conditional survival process writes $G_t = \mathbb{P}^*(\tau > t|\mathcal{F}_t) = Z_t - A_t$, the $(\mathbb{G}, \mathbb{P}^*)$-dynamics of S follow the decomposition (7):

$$S_t = \widehat{S}_t + \int_0^t \vartheta_u d\langle S\rangle_u \text{ with } \widehat{S} \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*),$$

where ϑ was defined in (8). Hence $\mathbb{P}^*$ is not a $\mathbb{G}$-e.m.m. From Proposition 3, the set of $\mathbb{G}$-e.m.m can be perfectly described as:

$$\Theta^{\mathbb{G}}_{\mathbb{P}^*}(S) = \left\{ \mathbb{Q} : \left.\frac{d\mathbb{Q}}{d\mathbb{P}^*}\right|_{\mathcal{G}_t} = \mathcal{E}(-\vartheta \star \widehat{S})_t \mathcal{E}(\beta \star M)_t \right\}.$$

where β is a $\mathbb{G}$-predictable processes, taking values in $]-1,\infty[$. As a check, under such a probability $\mathbb{Q}$, as $\widehat{S} \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$, one has, setting $\eta_t = \mathbb{Q} : \frac{d\mathbb{Q}}{d\mathbb{P}^*}\big|_{\mathcal{G}_t}$

$$\begin{aligned} \widehat{S}_t - \int_0^t \frac{d\langle \widehat{S}, \eta\rangle_u}{\eta_u} &= \widehat{S}_t - \int_0^t d\left\langle \widehat{S}, -\vartheta \star \widehat{S} + \beta \star M \right\rangle_u \\ &= \widehat{S}_t + \int_0^t \vartheta_u d\langle S\rangle_u = S_t \end{aligned}$$

where the second equality comes from the fact that $\langle \widehat{S}\rangle = \langle S\rangle$ and $\langle \widehat{S}, M\rangle = 0$, since $\widehat{S}$ is continuous and M purely discontinuous. It follows from Girsanov's theorem that S is a $(\mathbb{G}, \mathbb{Q})$ -martingale. □

As a conclusion, there exists at least a probability $\mathbb{Q}$ such that S is a $(\mathbb{G}, \mathbb{Q})$-martingale. We shall see in the immersion part that the drift relative to the change of probability may be interpreted as a risk premium.

3.2 Completeness of the full market

When considering also a $n + 2^{th}$ asset S^{n+1} satisfying condition (1), we shall be able to select an e.m.m., in a unique way, and prove that the market defined by the price process $\widetilde{S}$ is complete.

3.2.1 The default-sensitive asset

We start this section by emphasizing that it is necessary to introduce the asset S^{n+1} to the collection $\left(S^0,...,S^n\right)$ when working on derivatives whose pay-off depends on τ : It is not possible to hedge the jumping risk with $\mathbb{F}$-adapted assets and without such a product.

Let us consider for example a credit default swap[19] (CDS). To ease the discussion we take a continuous tenor, with a proportional continuous premium κ , and a constant recovery fee δ (S_t^0 denotes the saving account, i.e., the value at t of one unit invested at 0, and $\mathbb{Q}$ a martingale measure associated to this numeraire - that exists by absence of arbitrage). The price of a CDS is the difference between the value of the protection leg and the premium leg: $CDS(t,\delta,\kappa,T) = \mathrm{Prot}_t - \mathrm{Prem}_t$ with

$$\begin{aligned}\mathrm{Prem}_t &= S_t^0\kappa\,\mathbb{E}\left(\int_t^T \frac{1-H_u}{S_u^0}du\middle|\mathcal{G}_t\right) = (1-H_t)\,\frac{S_t^0\kappa}{G_t}\int_t^T \mathbb{E}\left(\frac{1-H_u}{S_u^0}\middle|\mathcal{F}_t\right)du \text{ and}\\ \mathrm{Prot}_t &= S_t^0\delta\,\mathbb{E}\left(\int_t^T \frac{dH_u}{S_u^0}\middle|\mathcal{G}_t\right) = (1-H_t)\,\frac{S_t^0\delta}{G_t}\int_t^T \mathbb{E}\left(\frac{(1-H_u)\,\alpha_u^u}{S_u^0 G_u}\middle|\mathcal{F}_t\right)du.\end{aligned}$$

Both legs have a value whose variation may be due to two factors: (i) $\mathbb{F}$-events (through the $\mathbb{F}$-conditional expectation), that evolve according to the alea structure underlying the filtration $\mathbb{F}$, and (ii) $\mathbb{H}$ events, mainly the occurrence of default (through the expression $1-H$). Whereas it is reasonable to think that -under suitable assumptions- the "market/spread" variation of the value of the CDS (linked to the $\mathbb{F}$-events) may be hedged with $\mathbb{F}$-adapted instruments, the "jump" risk being not $\mathbb{F}$-adapted will not be hedgeable with assets of the reference market $\mathbb{F}$. It follows that a model containing only $\mathbb{F}$ -adapted assets S would not be able to remove the jump risk of defaultable portfolios.

For quoted instruments like CDS, a formula like above allows to calibrate the parameters involved in the construction of the default time. A natural class of assets for S^{n+1} would be the risky bonds associated to τ or a CDS.

3.2.2 The unique martingale probability

We introduce the asset S^{n+1} that is sensitive to the jump risk, i.e., for any t, the r.v. $\tau\wedge t$ is $\sigma\left(S_s^{n+1}, s\leq t\right)$ -mesurable[20]. *Our aim is to prove that if the $\mathbb{F}$-market is complete, the $\mathbb{G}$-market is complete as well, under weak assumption on S^{n+1}.*

From Hypothesis $\mathbf{H}_1$, the $\mathbb{F}$-compensator of G writes $dA_t = \alpha_t^t dt$. From

$$dM_t = dH_t - \frac{1-H_t}{G_t}\alpha_t^t dt, \tag{9}$$

[19] Contract in which the holder buys a protection in paying a premium at each date of a tenor to the seller until a predefined credit event occurs, and receives a recovery fee if default occurs.

[20] Precisely w.r.t. its natural augmented càd version.

we can derive the quadratic variation of process M :

$$[M]_t = [H]_t = \sum_{s \le t} \Delta H_s^2 = \sum_{s \le t} \Delta H_s = H_t,$$

since the two processes of the right-hand member of the equality (9) have finite variation paths and the second one is continuous. It follows that

$$[M]_t - \int_0^t \frac{1 - H_s}{G_s} \alpha_s^s ds = M_t \in \mathcal{M}(\mathbb{G}, \mathbb{P}),$$

hence that the angle brackets $\langle M \rangle$ exist and satisfy $d \langle M \rangle_t = (1 - H_t) \alpha_t^t / G_t dt$.

According to the last section, we consider the following framework for the full market:

1 - Reference assets: The "reference" assets are defined under the historical probability $\mathbb{P}$ by:

$$\begin{cases} dS_t = b_t dt + dS_t^* \\ S_0 = x \end{cases},$$

where $S^* \in \mathcal{M}(\mathbb{F}, \mathbb{P})$ continuous has a quadratic variation assumed to be absolutely continuous w.r.t. Lebesgue measure, $d\langle S^* \rangle_t = s_t dt$. Let $\widehat{S}$ be the $(\mathbb{G}, \mathbb{P})$-martingale part of the decomposition of S^* viewed as a $\mathbb{G}$-semi-martingale, which writes:

$$d\widehat{S}_t = dS_t^* - c_t dt,$$

with

$$c_u = (1 - H_u) z_u s_u / G_u + H_u a_u^\tau s_u / \alpha_u^\tau = s_u \vartheta_u,$$

with previous notations. It follows that the $\mathbb{G}$-decomposition of the semi-martingale S writes under the historical probability $\mathbb{P}$:

$$dS_t = (b_t + c_t) dt + d\widehat{S}_t := \nu_t dt + d\widehat{S}_t.$$

2 - Default sensitive asset. We postulate for the asset S^{n+1} the general form:

$$dS_t^{n+1} = \mu_t dt + \varepsilon_t d\widehat{S}_t + \zeta_t dM_t,$$

where μ_t is a drift term and the three processes μ, ε and ζ are $\mathbb{G}$-predictable, and where ζ_t does not vanish (such a decomposition is quite general, from Theorem 2.1, the only assumption being the absolute continuity of the drift w.r.t. Lebesgue measure).

We can state the most important result of this part.

Theorem 3.1 *Assume that the reference market $\mathbb{F}$ is complete. If $\mathbf{H}_1$ to $\mathbf{H}_5$ hold and if the quadratic variation of S and the drifts of the assets are absolutely continuous w.r.t. Lebesgue measure the full market composed of $\widetilde{S}$ is complete.*

PROOF: The set of martingale probabilities that make S a $\mathbb{G}$-martingale writes, by Proposition 3:

$$\Theta_{\mathbb{P}}^{\mathbb{G}}(S)=\left\{\mathbb{Q}\sim\mathbb{P},\ \exists\,\gamma,\beta\ \mathbb{G}\ \text{ predictable },\beta<1,\ \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}_t}=\eta_t=\mathcal{E}\left(-\gamma\star\widehat{S}\right)_t\mathcal{E}\left(-\beta\star M\right)_t\right\}.$$

It follows that the exists a unique $\mathbb{G}$-e.m.m, i.e., a unique probability that makes $\widetilde{S}=\left(1,S^1,...,S^n,S^{n+1}\right)$ a $\mathbb{G}$-martingale. It is defined by:

$$\gamma_t=\nu_t,\ \text{and}\ \beta_t=G_t\frac{\mu_t-\varepsilon_t\nu_t}{\alpha_t^t\zeta_t},$$

by application of Girsanov's theorem. □

Once this probability has been defined, it is possible to price and hedge the τ-sensitive claims with $\left(S^0,...,S^{n+1}\right)$, like for example CDS on τ (of shorter maturities if S^{n+1} is a CDS) or derivatives written on S^{n+1}.

3.3 Immersion property

We shall emphasize in this section the deep links between immersion and completeness. We start with some general results, precise them in the case where the credit event is an initial time and conclude with some considerations about the credit risk premium.

3.3.1 Immersion and completeness in an arbitrage free set up

Proposition 1 *Assume that* (i) *the reference market is complete and* (ii) *the full market is arbitrage free. Then:*

1. *The restrictions of all the* $\mathbb{G}$ *-e.m.m. on* $\mathcal{F}_T$ *are the same and*
2. *Immersion holds under every* $\mathbb{G}$ *-e.m.m.*

PROOF: *1 - The restrictions of all the* $\mathbb{G}$*-e.m.m. on* $\mathcal{F}_T$ *are the same.* For any $\mathbb{Q}\in\Theta_{\mathbb{P}}^{\mathbb{G}}(\widetilde{S})$, it is straightforward[21] that $\mathbb{Q}\in\Theta_{\mathbb{P}}^{\mathbb{F}}(S)$. As the reference market is complete and S is finite dimensional, the restriction of all the $\mathbb{Q}$'s to the σ-algebra $\mathcal{F}_T$ is unique: All the e.m.ms of the full market have the same restriction on $\mathcal{F}_T$.

2 - Immersion holds under every $\mathbb{G}$*-e.m.m.* Indeed if P_1 holds, let $\mathbb{Q}\in\Theta_{\mathbb{P}}^{\mathbb{G}}(S)$ and $X\in\mathcal{M}^2(\mathbb{F},\mathbb{Q})$, with $X_t=\mathbb{E}^{\mathbb{Q}}\left(X_T|\,\mathcal{F}_t\right)$. As recalled above, the completeness of the market implies the existence of a constant x and $\mathbb{F}$-predictable processes $(\varphi^i,i=1,\dots,n)$ such that $X_T=x+\int_0^T\sum_{1\le i\le n}\varphi_u^i dS_u^i$. Therefore,

$$\mathbb{E}^{\mathbb{Q}}\left(X_T|\,\mathcal{F}_t\right)=x+\sum_{1\le i\le n}\int_0^t\varphi_u^i dS_u^i+\mathbb{E}^{\mathbb{Q}}\left(\int_t^T\varphi_u^i dS_u^i\middle|\,\mathcal{F}_t\right)=x+\sum_{i\le n}\int_0^t\varphi_u^i dS_u^i,$$

[21] $\Theta_{\mathbb{P}}^{\mathbb{G}}(\widetilde{S})\subset\Theta_{\mathbb{P}}^{\mathbb{G}}(S)\subset\Theta_{\mathbb{P}}^{\mathbb{F}}(S)$, because S is $\mathbb{F}$-adapted

where the first equality comes from the fact that the random variable $\int_0^t \varphi_u^i dS_u^i$ is $\mathcal{F}_t$-measurable (the process φ^i is $\mathbb{F}$-predictable), and the second, from the fact that the process $\int_0^\cdot \varphi_u^i dS_u^i$ is a $(\mathbb{G},\mathbb{Q})$-martingale (φ is $\mathbb{G}$ -predictable and $\mathbb{Q} \in \Theta_{\mathbb{P}}^{\mathbb{G}}(S)$), and an $(\mathbb{F},\mathbb{Q})$-martingale since it is $\mathbb{F}$-adapted. Hence $\mathbb{E}^{\mathbb{Q}}\left(\int_t^T \varphi_u^i dS_u^i \middle| \mathcal{F}_t \right) = 0$ and $\mathbb{E}^{\mathbb{Q}}\left(\int_t^T \varphi_u^i dS_u^i \middle| \mathcal{G}_t \right) = 0$. Therefore

$$\mathbb{E}^{\mathbb{Q}}\left(X_T \middle| \mathcal{F}_t \right) = x + \sum_{i \leq n} \int_0^t \varphi_u^i dS_u^i + \mathbb{E}^{\mathbb{Q}}\left(\int_t^T \varphi_u^i dS_u^i \middle| \mathcal{G}_t \right) = \mathbb{E}^{\mathbb{Q}}\left(X_T \middle| \mathcal{G}_t \right),$$

hence $X \in \mathcal{M}^2(\mathbb{G},\mathbb{Q})$ and immersion holds under $\mathbb{Q}$. Such a result had already been pointed out by Blanchet-Scaillet and Jeanblanc in [7]. □

3.3.2 Case where the credit event is an initial time

The last result can be refined in the set up where the credit event is an initial time. Indeed we shall prove the

Proposition 5 *Assume that* $\mathbf{H}_1$ *to* $\mathbf{H}_4$ *hold. Then, immersion holds under any* $\mathbb{Q} \in \Theta_{\mathbb{P}^*}^{\mathbb{G}}(S)$.

This result implies that immersion holds under any $\mathbb{G}$-e.m.m[22] for S (it does not have to be a martingale probability of the full market).

PROOF: Let β be any predictable process such that $\beta_t > -1$, and $\mathbb{Q}^\beta$ be the corresponding e.m.m.:

- $d\mathbb{Q}^\beta\big|_{\mathcal{F}_\infty} = d\mathbb{P}^*|_{\mathcal{F}_\infty}$. Indeed, for any bounded $F_t \in \mathcal{F}_t$ with $\mathbb{P}^*$-null expectation, $F_t = \int_0^t f_s dS_s$ by PRP and

$$\begin{aligned}
\mathbb{E}^\beta(F_t) &= \mathbb{E}^*(F_t\eta_t) = \mathbb{E}^*\left(\int_0^t \eta_s f_s dS_s + \int_0^t F_s\, d\eta_s + \int_0^t f_s d\langle S,\eta\rangle_s \right) \\
&= \mathbb{E}^*\left(\int_0^t \eta_s f_s dS_s + \int_0^t F_s\, d\eta_s + \int_0^t f_s \vartheta_s \eta_s d\langle \widehat{S}\rangle_s \right) \\
&= \mathbb{E}^*\left(\int_0^t \eta_s f_s d\widehat{S}_s + \int_0^t F_s\, d\eta_s \right) = 0 = \mathbb{E}^*(F_t),
\end{aligned}$$

  where the first equality is obtained by the integration by parts formula (f is predictable), the second comes from the definition of the dynamics of the density η and the third from the definition of $\widehat{S}$, the expectation being null since $\widehat{S}$ and η belong to $\mathcal{M}(\mathbb{G},\mathbb{P}^*)$. It follows that $d\mathbb{Q}^\beta\big|_{\mathcal{F}_\infty} = d\mathbb{P}^*|_{\mathcal{F}_\infty}$.

- Let X be an $\left(\mathbb{F},\mathbb{Q}^\beta\right)$ -martingale. Then, it is a $(\mathbb{F},\mathbb{P}^*)$-martingale (from the first point) that writes $dX_t = x_t dS_t$. From the decomposition formula in the change of filtration

$$\widehat{X}_t := X_t - \int_0^t x_u \vartheta_u d\langle S\rangle_u \in \mathcal{M}(\mathbb{G},\mathbb{P}^*),$$

[22] The set of $\mathbb{G}$-e.m.m is infinite, parameterized by the predictable processes β.

and from the decomposition formula in the change of probability (Girsanov's theorem)

$$\widetilde{X}_t = \widehat{X}_t - \int_0^t \frac{d\langle X, \eta\rangle_u}{\eta_u} \in \mathcal{M}\left(\mathbb{G}, \mathbb{Q}^\beta\right).$$

It remains to note (by definition of η) that

$$\widetilde{X}_t = \widehat{X}_t + \int_0^t x_u \vartheta_u d\langle S\rangle_u = X_t,$$

hence that $X \in \mathcal{M}\left(\mathbb{G}, \mathbb{Q}^\beta\right)$. It follows that immersion holds under $\mathbb{Q}^\beta$. □

It follows the important

Corollary 1 *Assume that $\mathbb{F}$ is complete and that $\mathbb{P}^* \in \Theta_{\mathbb{P}}^{\mathbb{F}}(S)$. If the $(\mathbb{F}, \mathbb{P}^*)$ -conditional survival process G^* has a non constant martingale part, $\mathbb{P}^*$ is not a $\mathbb{G}$-e.m.m., i.e., $\mathbb{P}^* \notin \Theta_{\mathbb{P}}^{\mathbb{G}}(S)$.*

Proof: From (3), it follows that under immersion

$$G_t = \int_0^\infty \alpha_{t\wedge u}^u du - \int_0^t \alpha_u^u du = \int_0^\infty \alpha_t^u du - A_t = \mathbb{P}(\tau > 0|\mathcal{F}_t) - A_t = 1 - A_t$$

hence G is decreasing and predictable. □

A broader class of credit event may be reached through a definition where the martingale part of the survival process is not equal to zero (see [24]). Under such a framework, immersion does not hold, which means that the "reference neutral-risk probability" is not a neutral-risk probability. A change of measure has therefore to be performed to re-enter a neutral-risk framework. Next section points out the links of this remark with the credit risk premium.

3.3.3 Credit risk premium

The last corollary may be interpreted in the following way. The change from the historical probability $\mathbb{P}$ to a neutral-risk probability $\mathbb{P}^*$ aims at correcting the dynamics from the market risk premium. It is well known that to any financial market can be associated a risk premium. It characterizes the return over the risk-free return (the interest rate) an investor may expect for bearing the risk of taking a long position on a derivative written on this market.

Indeed if N is the martingale modelling the alea of this market, and if the asset's return writes:

$$\frac{dS_t}{S_{t-}} = \mu dt + \sigma dN_t,$$

the dynamics of any derivatives written on S sold for a price P_t at a date t would write $dP_t/P_{t-} = \kappa dt + \alpha dN_t$. A risk-free portfolio can be set-up in buying a quantity $S\sigma$ of the derivative, for a total value of $S\sigma P$ and selling a quantity $P\alpha$ of the asset (completing by ς_t of money market S_t^0 to remain self financed). The value of this portfolio is $\Pi_t = \varsigma_t S_t^0 + S_t \sigma P_t - P_t \alpha S_t$ and the self-financing condition yields

$$\begin{aligned} d\Pi_t &= \varsigma_t r S_t^0 dt + S_{t-} \sigma dP_t - P_{t-} \alpha dS_t \\ &= \varsigma_t r S_t^0 dt + \sigma\kappa S_t P_t dt + \alpha\sigma S_{t-} P_{t-} dN_t - \alpha P_t S_t \mu dt - \alpha\sigma P_{t-} S_{t-} dN_t \\ &= \left(\left(\Pi_t - (\sigma - \alpha) P_t S_t \right) r + (\sigma\kappa - \alpha\mu) P_t S_t \right) dt\,. \end{aligned}$$

By absence of arbitrage, the return of this risk-free portfolio must be equal to r, so that:

$$(\sigma\kappa - \alpha\mu) P_t S_t = r (\sigma - \alpha) P_t S_t \Longleftrightarrow \frac{\kappa - r}{\alpha} = \frac{\mu - r}{\sigma} = \lambda_S.$$

On the reference market, the e.m.m. $\mathbb{P}^*$ corrects the historical probability $\mathbb{P}$ from the market risk premium: *If immersion does not hold under $\mathbb{P}^*$, it means the market risk premium does not take into account the jump risk premium, and it is necessary to change to a $\mathbb{G}$-e.m.m $\mathbb{Q}$ under which S remains a martingale.*

4 Incomplete markets

The question addressed in this last part is the adaptation of the above results in the case where the reference market is incomplete.

When an incomplete model is chosen - in general for its ability to well reproduce a given class of (calibration) instruments and its dynamics property (regarding to the products to price) - one is often conduced to focus on one particular martingale probability, even if the set of $\mathbb{F}$-e.m.m. is not unique. The selection of the e.m.m. is performed by the calibration procedure. The law of the price process is then uniquely determined, and a change of probability within the set of $\mathbb{F}$ -e.m.m. will change the price of the selected options or break the imposed constraints.

We assume therefore in this section that an $\mathbb{F}$-e.m.m has been chosen (for pricing the default-free derivatives), hence we restrict our attention on a given $\mathbb{F}$ e.m.m $\mathbb{P}^*$ defined on $\mathcal{A}$ and equivalent to the historical probability $\mathbb{P}$, such that $S \in \mathcal{M}(\mathbb{F}, \mathbb{P}^*)$.

The purpose of this section is:

- To prove that there exists a unique probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}^*$ such that $\widetilde{S}$ (defined in the last part as the $n+2$-uplet composed of the reference asset S and S^{n+1}) is a $(\mathbb{G}, \mathbb{Q})$-martingale, and that preserves the"reference pricing", i.e.,

$$\mathbb{E}^{\mathbb{Q}} (X_T) = \mathbb{E}^* (X_T)\,, \text{ for any } X_T \in L^2 (\mathcal{F}_T)\,.$$

- To prove that immersion property holds under this measure.

Recall that, from hypothesis $\mathbf{H}_1$ and $\mathbf{H}_2$, the random time τ is an initial time that avoids the $\mathbb{F}$-stopping times.

Recall that if X is a real valued local-martingale and Y a $\mathbb{R}^d$-valued local-martingale, a Galtchouck Kunita Watanabe decomposition of X is a decomposition of the form $X = X_0 + H * Y + L$ with $H \in L^2_{loc}(Y)$ and L a local-martingale such that $L_0 = 0$ and strongly orthogonal[23] to Y. It is classical that if Y is continuous or locally square integrable, X admits a Galtchouck-Kunita-Watanabe decomposition (Kunita and Watanabe [27], Galtchouck [16], Jacod [18]).

4.1 The risk-neutral probabilities of the full market

Reference assets and $\Theta^{\mathbb{G}}_{\mathbb{P}^*}(S)$. As in the complete case, we assume that S is continuous and that its quadratic variation process is continuous w.r.t. Lebesgue measure, $d\langle S\rangle_t = s_t dt$.

As S is continuous, there exist:

- for any $u \geq 0$, an $\mathbb{F}$-predictable process a^u and a square integrable $\mathbb{F}$ -martingale N^u strongly orthogonal to S and
- an $\mathbb{F}$-predictable process z and a square integrable $\mathbb{F}$ -martingale $N^Z \in \mathcal{M}(\mathbb{F}, \mathbb{P}^*)$ strongly orthogonal to S

such that the following Galtchouck-Kunita-Watanabe decompositions exist:

$$\begin{cases} \alpha^u &= \alpha^u_0 + a^u \star S + N^u \\ Z &= Z_0 + z \star S + N^Z \end{cases} \tag{10}$$

where $Z = G + A$. Moreover there exist optional versions of the functions a^u and N^u.

We have seen in (7) that the decomposition of the $(\mathbb{G}, \mathbb{P}^*)$-semi-martingale S writes $S = \widehat{S} + C$ with:

$$dC_t = \left(\frac{1 - H_t}{G_t} z_t + \frac{H_t}{\alpha^\tau_t} a^\tau_t\right) d\langle S\rangle_t =: \vartheta_t d\langle S\rangle_t,$$

and $\widehat{S}$ continuous. It follows that any $\mathbb{G}$-martingale l admits a Galtchouck-Kunita-Watanabe decomposition of the form

$$l = l_0 + \varphi \star \widehat{S} + h$$

with h a local martingale strongly orthogonal to l. Recall that the martingale M is strongly orthogonal to $\widehat{S}$ (it is purely discontinuous) and is a locally square martingale, hence h admits a Galtchouck-Kunita-Watanabe decomposition:

$$h = \psi \star M + N^\perp$$

[23] Recall that two local-martingales are strongly orthogonal if their product is a local martingale.

with $N^{\perp}$ strongly orthogonal to $\left(M, \widehat{S}\right)$. It follows that the set of probabilities $\mathbb{G}$-equivalent to $\mathbb{P}^*$ writes:

$$\left\{ \mathbb{Q}, \left.\frac{d\mathbb{Q}}{d\mathbb{P}^*}\right|_{\mathcal{G}_t} = \eta_t = \mathcal{E}\left(\varphi \star \widehat{S}\right)_t \mathcal{E}\left(\psi \star M\right)_t \mathcal{E}\left(N^{\perp}\right)_t \right\},$$

with (φ, ψ) a pair of $\mathbb{F}$-predictable processes and $N^{\perp}$ a $\mathbb{G}$-martingale that is strongly orthogonal to $\widehat{S}$ and M.

Finally, the set of $\mathbb{G}$ -e.m.m $\Theta^{\mathbb{G}}_{\mathbb{P}^*}(S)$ writes:

$$\Theta^{\mathbb{G}}_{\mathbb{P}^*}(S) = \left\{ \mathbb{Q}, \left.\frac{d\mathbb{Q}}{d\mathbb{P}^*}\right|_{\mathcal{G}_t} = \eta_t = \mathcal{E}\left(-\vartheta \star \widehat{S}\right)_t \mathcal{E}\left(\psi \star M\right)_t \mathcal{E}\left(N^{\perp}\right)_t \right\}$$

with $N^{\perp} \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$ strongly orthogonal to the pair $\left(\widehat{S}, M\right)$ and $-1 < \psi \in \mathcal{P}(\mathbb{G})$. Indeed the "Girsanov's drift" under each such probability writes $\left\langle -\vartheta \star \widehat{S}, \widehat{S} \right\rangle$ because of the *strong* orthogonality of the other terms w.r.t. $\widehat{S}$. This set is parameterized by the pair $(\psi, N^{\perp})$.

Default sensitive asset. If μ_t denotes the drift of the default sensitive asset S^{n+1}, the general dynamics of the price process can be written as:

$$dS^{n+1}_t = \mu_t dt + \varepsilon_t d\widehat{S}_t + \zeta_t dM_t + dN^{n+1}_t,$$

with $N^{n+1} \in \mathcal{M}^2(\mathbb{G}, \mathbb{P}^*)$ strongly orthogonal to the pair $\left(\widehat{S}, M\right)$ (same argument as before). As before, we assume $d\left\langle S^{n+1}\right\rangle_t << dt$.

Finally, it follows that the set $\Theta^{\mathbb{G}}_{\mathbb{P}^*}(\widetilde{S})$ writes:

$$\Theta^{\mathbb{G}}_{\mathbb{P}^*}\left(\widetilde{S}\right) = \left\{ \mathbb{Q}, \left.\frac{d\mathbb{Q}}{d\mathbb{P}^*}\right|_{\mathcal{G}_t} = \eta_t = \mathcal{E}\left(-\vartheta \star \widehat{S}\right)_t \mathcal{E}\left(\psi \star M\right)_t \mathcal{E}\left(N^{\perp}\right)_t \right\}$$

with $N^{\perp} \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$ strongly orthogonal to the pair $\left(\widehat{S}, M\right)$ and $\psi \in \mathcal{P}(\mathbb{G})$, defined[24] by

$$\psi_t \zeta_t \alpha^t_t dt = \mu_t dt + \varepsilon_t \vartheta_t s_t dt - d\left\langle N^{n+1}, N^{\perp}\right\rangle_t. \tag{11}$$

4.2 Default-free pricing invariance

In both cases, the martingale probability is going to be uniquely defined thanks to the constraint of $\mathbb{F}$-pricing invariance.

Let $X_T \in L^2(\mathcal{F}_T)$, such that $\mathbb{E}^*(X_T) = 0$. The $(\mathbb{F}, \mathbb{P}^*)$ -martingale X can be decomposed as $X = x \star S + N$, with $(S, N) \in \mathcal{M}(\mathbb{F}, \mathbb{P}^*)$ strongly orthogonal. The decomposition of this $(\mathbb{G}, \mathbb{P}^*)$-semi-martingale writes

$$X = x \star \widehat{S} + x \star C + \widehat{N} + K,$$

[24]Notice that $d \; N^{n+1}, N^{\perp} \;_t << dt$, from Kunita Watanabe ($d \; S^{n+1} \;_t << dt$ implies $d \; N^{n+1} \;_t << dt$).

with $(\widehat{S},\widehat{N})\in\mathcal{M}(\mathbb{G},\mathbb{P}^*)$, $dC_t=\vartheta_t d\langle S\rangle_t=\vartheta_t s_t dt$ and:

$$\begin{aligned} dK_t &= \frac{1-H_t}{G_t}d\langle N,G\rangle_t+\frac{H_t}{\alpha_t^u}d\langle N,\alpha^u\rangle_t\Big|_{u=\tau} \\ &= \frac{1-H_t}{G_t}d\left\langle N,N^Z\right\rangle_t+\frac{H_t}{\alpha_t^u}d\langle N,N^u\rangle_t\Big|_{u=\tau}, \end{aligned}$$

where N^u and N^Z are defined in (10). It follows that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(X_T) &= \mathbb{E}^*(X_T\eta_T)=\mathbb{E}^*\left(\int_0^T\eta_t dX_t+[X,\eta]_T\right) \\ &= \mathbb{E}^*\left(\int_0^T\eta_t(x_t dC_t+dK_t)+[X,\eta]_T\right) \\ &= \mathbb{E}^*\left(\int_0^T\eta_t\left(x_t\vartheta_t d\langle\widehat{S}\rangle_t+dK_t\right)+[X,\eta]_T\right). \end{aligned}$$

Moreover we can write:

$$\mathbb{E}^*([X,\eta]_T)=\mathbb{E}^*\left(\left[x\star\widehat{S}+\widehat{N},\mathcal{E}\left(-\vartheta\star\widehat{S}\right)\mathcal{E}(\psi\star M)\mathcal{E}\left(N^{\perp}\right)\right]_T\right)$$

since $x\star C+K$ is $\mathbb{G}$-predictable with finite variation,

$$\begin{aligned} \mathbb{E}^*([X,\eta]_T) &= \mathbb{E}^*\left(\left[x\star\widehat{S},\mathcal{E}\left(-\vartheta\star\widehat{S}\right)\mathcal{E}(\psi\star M)\mathcal{E}\left(N^{\perp}\right)\right]_T\right) \\ &\quad+\mathbb{E}^*\left(\left[\widehat{N},\mathcal{E}\left(-\vartheta\star\widehat{S}\right)\mathcal{E}(\psi\star M)\mathcal{E}\left(N^{\perp}\right)\right]_T\right) \\ &= \mathbb{E}^*\left(\int_0^T-\eta_t\vartheta_t x_t d\left[\widehat{S}\right]_t\right)+\mathbb{E}^*\left(\int_0^T\eta_t\psi_t d\left[\widehat{N},M\right]_t\right) \\ &\quad+\mathbb{E}^*\left(\int_0^T\eta_t d\left[\widehat{N},N^{\perp}\right]_t\right) \end{aligned}$$

and since K is $\mathbb{G}$-predictable with finite variation, $[K,M]=0$, and $\left[\widehat{N},M\right]=[N,M]=0$ (recall τ avoids the $\mathbb{F}$-stopping times - M is purely discontinuous and jumps at τ). It follows

$$\mathbb{E}^*([X,\eta]_T)=\mathbb{E}^*\left(\int_0^T-\eta_t\vartheta_t x_t d\left\langle\widehat{S}\right\rangle_t+\int_0^T\eta_t d\left[\widehat{N},N^{\perp}\right]_t\right)$$

hence

$$\mathbb{E}^{\mathbb{Q}}(X_T)=\mathbb{E}^*\int_0^T\eta_t d\left(K_t+\left[\widehat{N},N^{\perp}\right]_t\right). \tag{12}$$

We shall prove that there exists a unique $N^{\perp}$ such that $\mathbb{E}^{\mathbb{Q}}(X_T)=0$ for any $X_T\in L^2(\mathcal{F}_T)$, such that $\mathbb{E}^*(X_T)=0$.

Let us define the $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}^+)$-optional process

$$R_t^x=-\int_0^t 1_{\{x\le u\}}\frac{dN_u^x}{\alpha_{u-}^x},$$

which is for any $x\ge 0$ an $\mathbb{F}$-martingale. We start with the following

Lemma 1 *If X^x is an $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}^+)$-optional family of $\mathbb{F}$-semi-martingales, $X=(X_t^{\tau\wedge t},t\geq 0)$ is a $\mathbb{G}$-semi-martingale.*

Proof: 1. The process X is $\mathbb{G}$-adapted since $\tau\wedge t\in\mathcal{G}_t$, and the function X is $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}^+)$-optional.

2. We shall prove that for any sequence of càg piecewise constant $\mathbb{F}$-adapted processes K^n that tends to zero uniformly in (t,ω), $K^n\star S$ tends to zero in probability[25].

For any subdivision $(t_1,...,t_n)$ of $[0,T]$, and any $(K_1^n,...,K_n^n)$ such that for any $i\leq n$ $K_i^n\in\mathcal{G}_{t_i}$, and $K_i^n=F_i^nh_i^n(t_i\wedge\tau)$ with h_i^n Borel measurable and $F_i^n\in\mathcal{F}_{t_i}$, we introduce

$$\begin{aligned}K^n\star X_T &= \sum_i K_i^n\left(X_{i+1}-X_i\right)=\sum_i K_i^n\left(X_{t_{i+1}}^{\tau\wedge t_{i+1}}-X_{t_i}^{\tau\wedge t_i}\right)\\ &= \sum_i F_i^nh_i^n\left(t_i\wedge\tau\right)\left(X_{t_{i+1}}^{\tau\wedge t_{i+1}}-X_{t_i}^{\tau\wedge t_i}\right).\end{aligned}$$

It is classical that a sequence of random variables I_n tends to zero in probability iff the real sequence $\mathbb{E}\left(1\wedge|I_n|\right)$ tends to zero. Following this trail, we have

$$\begin{aligned}\mathbb{E}\left(1\wedge|K^n\star X_T|\right) &= \mathbb{E}\left(1\wedge\left|\sum_i F_i^nh_i^n\left(t_i\wedge\tau\right)\left(X_{t_{i+1}}^{\tau\wedge t_{i+1}}-X_{t_i}^{\tau\wedge t_i}\right)\right|\right)\\ &= \int_0^\infty\mathbb{E}\left(1\wedge\left|\sum_i F_i^nh_i^n\left(t_i\wedge u\right)\left(X_{t_{i+1}}^{u\wedge t_{i+1}}-X_{t_i}^{u\wedge t_i}\right)\right|\alpha_T^u\right)du\end{aligned}$$

and from Lebesgue's theorem, since

$$\mathbb{E}\left(1\wedge\left|\sum_i F_i^nh_i^n\left(t_i\wedge u\right)\left(X_{t_{i+1}}^{u\wedge t_{i+1}}-X_{t_i}^{u\wedge t_i}\right)\right|\alpha_T^u\right)\leq\mathbb{E}\left(1*\alpha_T^u\right)\leq 1,$$

we can write

$$\lim_n\mathbb{E}\left(1\wedge|K^n\star X_T|\right)=\int_0^\infty\lim_n\mathbb{E}\left(1\wedge\left|\sum_i F_i^nh_i^n\left(t_i\wedge u\right)\left(X_{t_{i+1}}^{u\wedge t_{i+1}}-X_{t_i}^{u\wedge t_i}\right)\right|\alpha_T^u\right)du.$$

Moreover, if the probability $\mathbb{Q}$ is defined on $\mathcal{F}_\infty$ by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}=\frac{\alpha_t^u}{\alpha_0^u},$$

we have

$$\begin{aligned}&\lim_n\mathbb{E}\left(1\wedge\left|\sum_i F_i^nh_i^n\left(t_i\wedge u\right)\left(X_{t_{i+1}}^{u\wedge t_{i+1}}-X_{t_i}^{u\wedge t_i}\right)\right|\alpha_T^u\right)\\ =\;&\lim_n\mathbb{E}^{\mathbb{Q}}\left(1\wedge\left|\sum_i F_i^nh_i^n\left(t_i\wedge u\right)\left(X_{t_{i+1}}^{u\wedge t_{i+1}}-X_{t_i}^{u\wedge t_i}\right)\right|\right)=0\end{aligned}$$

[25] Recall that an $(\mathbb{F},\mathbb{P})$-semi-martingale S can wether be define as the sum of an $\mathbb{F}$-adapted process of finite variation process and an $(\mathbb{F},\mathbb{P})$-local-martingale, or as an $\mathbb{F}$-adapted processes such that for any sequence of càg piecewise constant $\mathbb{F}$-adapted processes K_n that tends to zero uniformly in (t,ω), $K_n\star S$ tends to zero in probability (Bichteler-Dellacherie theorem). We use this second definition in the proof.

since X^u is an $\mathbb{F}$-martingale, hence an $\mathbb{F}$-semi-martingale. It follows by a class monotone argument that X is a $\mathbb{G}$-semi-martingale. $\square$

We deduce from this lemma that the process $X = HR^\tau$ is a $\mathbb{G}$-semi-martingale, since $X = X^{\tau\wedge t}$ with $X_t^x = 1_{x\le t} R_t^x$ (on $\{\tau \le t\}$, $\tau = \tau \wedge t$). So that to be able to develop the equality (12), we need to prove the

Lemma 2 *If Y^x is an $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}^+)$-optional family of $\mathbb{G}$-semi-martingales, and $Y = (Y_t^{\tau\wedge t}, t \ge 0)$ we have*

$$\left([Y, HR^\tau]_t - \int_0^t \frac{H_u}{\alpha_{u-}^x} d\left\langle Y^x, N^x \right\rangle_u \bigg|_{x=\tau}, t \ge 0 \right) \in \mathcal{M}(\mathbb{G}).$$

PROOF: 1. Let us first mention that the angle bracket $\langle Y^x, N^x\rangle$ is well defined as

$$\langle Y^x, N^x\rangle = \langle Y^x, a^u \star S\rangle - \langle Y^x, \alpha^x\rangle,$$

the first bracket being defined since S is continuous and the second from [23]. Moreover, $[Y, HR^\tau]$ exists from the last lemma, since both processes are semi-martingales.

2. Let F_t be an $\mathcal{F}_t$-measurable variable and h be Borel measurable. We have:

$$\begin{aligned}
\mathbb{E}\left(F_t h(t\wedge\tau)[Y, HR^\tau]_t^T\right) &= \int_0^\infty \mathbb{E}\left(F_t h(t\wedge x)[Y^x, HR^x]_T \alpha_T^x\right) dx \\
&\quad - \int_0^\infty \mathbb{E}\left(F_t h(t\wedge x)[Y^x, HR^x]_t \alpha_t^x\right) dx \\
&= \int_0^\infty \mathbb{E}\left(F_t h(t\wedge x)\int_t^T \alpha_{u-}^x d[Y^x, HR^x]_u\right) dx
\end{aligned}$$

since by Itô's rule (the last integral being an $\mathbb{F}$-martingale):

$$[Y^x, HR^x]_T \alpha_T^x - [Y^x, HR^x]_t \alpha_t^x = \int_t^T \alpha_{u-}^x d[Y^x, HR^x]_u + \int_t^T [Y^x, HR^x]_{u-} d\alpha_u^x,$$

hence finally, by definition of R^x:

$$\begin{aligned}
\mathbb{E}\left(F_t h(t\wedge\tau)[Y, HR^\tau]_t^T\right) &= \int_0^\infty \mathbb{E}\left(\int_t^T F_t h(t\wedge x)\alpha_{u-}^x \frac{1_{x\le u}}{\alpha_{u-}^x} d[Y^x, N^x]_u\right) dx \\
&= \int_0^\infty \mathbb{E}\left(\int_t^T F_t h(t\wedge x) 1_{x\le u} d\langle Y^x, N^x\rangle_u\right) dx.
\end{aligned}$$

Moreover

$$\begin{aligned}
\mathbb{E}\left(F_t h(t\wedge\tau)\int_t^T \frac{1_{x\le u}}{\alpha_{u-}^x} d\langle Y^x, N^x\rangle_u \bigg|_{x=\tau}\right) &= \int_0^\infty \mathbb{E}\left(\int_t^T F_t h(t\wedge x)\frac{1_{x\le u}}{\alpha_{u-}^x} d\langle Y^x, N^x\rangle_u \alpha_T^x\right) dx \\
&= \int_0^\infty \mathbb{E}\left(\int_t^T F_t h(t\wedge x) 1_{x\le u} d\langle Y^x, N^x\rangle_u\right) \eta dx,
\end{aligned}$$

since $\mathbb{E}\left(\alpha_T^x \middle| F_{u-}\right) = \alpha_{u-}^x$, which concludes the proof, by a class monotone argument. $\square$

We deduce from this lemma the two following points:

- The $\mathbb{G}$-semi-martingale HR^{τ} is special. Indeed, the lemma applied to $Y^x = HR^x$ leads to the existence of the angle bracket $\langle HR^{\tau}\rangle$.

- The lemma applied to the $\mathbb{F}$-martingale N leads to

$$\left([N, HR^{\tau}]_t - \int_0^t \frac{H_u}{\alpha^x_{u-}} d\langle N, N^x\rangle_u \bigg|_{x=\tau}, t \geq 0 \right) \in \mathcal{M}(\mathbb{G}).$$

We introduce the $\mathbb{G}$-semi-martingale

$$\Gamma_t = \int_0^t \frac{H_u - 1}{G_u} dN^Z_u - H_t R^{\tau}_t.$$

As the first integral is special (N^Z is an $\mathbb{F}$-martingale, hence a special $\mathbb{G}$-semi-martingale, from 2), Γ is a special $\mathbb{G}$-semi-martingale. We can conclude this part with the

Proposition 6 *There exists a unique $\mathbb{G}$-e.m.m. $\mathbb{Q} \in \Theta^{\mathbb{G}}_{\mathbb{P}^*}\left(\widetilde{S}\right)$, that preserves $\mathcal{F}_\infty$, i.e., such that*

$$\mathbb{E}^{\mathbb{Q}}(X_T) = \mathbb{E}^*(X_T), \text{ for any } X_T \in L^2(\mathcal{F}_T).$$

Proof: 1. We have seen that for $X_T \in L^2(\mathcal{F}_T)$ with $\mathbb{E}^*(X_T) = 0$,

$$\mathbb{E}^{\mathbb{Q}}(X_T) = \mathbb{E}^{\mathbb{Q}}(X_T) = \mathbb{E}^* \int_0^T \eta_t d\left(K_t + \left[\widehat{N}, N^{\perp}\right]_t\right).$$

If we choose for $N^{\perp}$ the (unique) $\mathbb{G}$-martingale part of Γ (in its special decomposition[26]), we have

$$\begin{aligned} \mathbb{E}^* \int_0^T \eta_t d\left[\widehat{N}, N^{\perp}\right]_t &= \mathbb{E}^* \int_0^T \eta_t \frac{H_u - 1}{G_u} d\left\langle N, N^Z_u\right\rangle_t - \mathbb{E}^* \int_0^T \eta_t \frac{H_t}{\alpha^u_t} d\langle N, N^u\rangle_t \bigg|_{u=\tau} \\ &= -\mathbb{E}^* \int_0^T \eta_t dK_t, \end{aligned}$$

where the first equality comes from the last lemma. It follows that $\mathbb{E}^{\mathbb{Q}}(X_T) = 0$.

2. For any probability $\mathbb{Q}$ defined with another martingale $\widetilde{N}^{\perp} = N^{\perp} + \mu$ where $\mu \in \mathcal{M}^2(\mathbb{G}, \mathbb{P}^*)$ is strongly orthogonal to $N^{\perp}$ non constant, when computing $\mathbb{E}^{\mathbb{Q}}(X_T)$ for a variable such that $\widehat{N} = \mu$,

$$\mathbb{E}^* \int_0^T \eta_t d\left(n_t + \left\langle \widehat{N}, \widetilde{N}^{\perp}\right\rangle_t\right) = \mathbb{E}^* \int_0^T \eta_t d\left\langle \widehat{N}, \mu\right\rangle_t = \mathbb{E}^* \int_0^T \eta_t d\langle \mu\rangle_t \neq 0.$$

It follows that this probability measure is unique. □

4.3 Immersion property

We have seen that given a reference risk-neutral probability there exists a unique risk-neutral probability on the full market that preserve the "reference pricing". We conclude this survey by the important

[26] The decomposition martingale plus predictable process with finite variation paths.

Proposition 7 *Under the $\mathbb{G}$-e.m.m. $\mathbb{Q} \in \Theta^{\mathbb{G}}_{\mathbb{P}^*}\left(\widetilde{S}\right)$ that preserves $\mathcal{F}_\infty$, immersion holds.*

Proof: Let $X \in \mathcal{M}(\mathbb{F}, \mathbb{Q})$, $X_t = \mathbb{E}^{\mathbb{Q}}\left(X_T \middle| \mathcal{F}_t\right)$. As $\mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}^*|_{\mathcal{F}_\infty}$, $X_t = \mathbb{E}^*\left(X_T \middle| \mathcal{F}_t\right)$. Indeed, for $F_t \in \mathcal{F}_t$,

$$\mathbb{E}^*\left(X_T F_t\right) = \mathbb{E}^{\mathbb{Q}}\left(X_T F_t\right) = \mathbb{E}^{\mathbb{Q}}\left(X_t F_t\right) = \mathbb{E}^*\left(X_t F_t\right).$$

Moreover, if $X = X_0 + x \star S + N$ with $N \in \mathcal{M}(\mathbb{F}, \mathbb{P}^*)$ strongly orthogonal to S, the $(\mathbb{G}, \mathbb{P}^*)$ -decomposition of this process writes:

$$X = X_0 + x \star \widehat{S} + x \star C + \widehat{N} + K.$$

Under $\mathbb{Q}$, from Girsanov's theorem:

$$X = X_0 + x \star \widetilde{S} + x \star \left\langle \widehat{S}, \log \eta \right\rangle + x \star C + \widetilde{N} + \left\langle \widehat{N}, \log \eta \right\rangle + K,$$

and by definition of $\mathbb{Q}$, $\left\langle \widehat{S}, \log \eta \right\rangle = -C$ and $\left\langle \widehat{N}, \log \eta \right\rangle = -K$ (see above), hence

$$X = x \star \widetilde{S} + \widetilde{N} \in \mathcal{M}(\mathbb{G}, \mathbb{Q}),$$

which concludes the proof. □

5 Conclusion

In this paper, we have given some arguments that show that it is natural to assume that immersion hypothesis holds for a study of a single default, and proved its deep link with completeness and martingale decomposition.

However, it is well known that it is usually impossible to assume this hypothesis in case of (non-ordered) multi-defaults,and that the martingale parts of the survival probabilities reflects the correlation between the different default times.

Acknowledgement: The authors wish to thank Stéphane Crépey for pointing out some mistakes in a preliminary version and the anonymous referee for all his/her accurate remarks.

References

[1] C. Albanese, O. Chen, *Pricing equity default swap* , Risk 6, 2005.

[2] M. Atlan, B. Leblanc, *Hybrid equity-credit modelling*, RISK, August. 2005, 61–66.

[3] R. Bättig: *Completeness of securities market models - an operator point of view*, The Annals of Applied Probability, 9, 529-566, 1999.

[4] A. Bélanger, S.E. Shreve, D. Wong: *A general framework for pricing credit risk,* Mathematical Finance, 14: 317-350, 2004.

[5] T. Bielecki, M. Rutkowski: *Credit Risk: modeling, Valuation and Hedging.* Springer-Verlag, Berlin Heidelberg New York, 2002.

[6] F. Black, J.C. Cox: *Valuing corporate securities: Some effects of bond indenture provisions.* Journal of Finance 31, 351-367, 1976.

[7] C. Blanchet-Scalliet, M. Jeanblanc: *Hazard rate for credit risk and hedging defaultable contingent claims.* Finance and Stochastics,8: 145-159, 2004.

[8] P. Carr, L. Wu: *Stock options and credit default swaps: a joint framework for valuation and estimation.* working paper, 2005.

[9] P. Carr and L. Wu: *Theory and evidence on the dynamic interactions between sovereign credit default swaps and currency options.* Journal of banking and finance, 31, 2383-2403, 2007.

[10] M. Cremers, M. Driessen, J. Maenhout, and P.J. Weinbaum: *Individual stock options and credit spreads*, working paper, Yale University.

[11] F. Delbaen and W. Schachermayer: *A general version of the fundamental theorem of asset pricing.* Math. Annal, 300:463-520, 1994.

[12] N. El Karoui, M. Jeanblanc, Y. Jiao: *Density model for single default.* Working paper 2008.

[13] P. Ehlers, P. Schonbucher: *Pricing interest rate sensitive portfolio derivatives.* Working paper 2007.

[14] P. Ehlers, P. Schonbucher: *The influence of FX risk on credit spreads.* Working paper 2007.

[15] R.J. Elliott, M. Jeanblanc, M. Yor: *On models of default risk.* Mathematical Finance 10, 179-195, 2000.

[16] L.I. Galtchouk *The structure of a class of martingales.* Proceedings seminar on random processes, Drusininkai, Academy of Sciences of Lithuanian SSP I, 7-32 1975.

[17] S.I. Heston: *A closed-form solution for options with stochastic volatility with applications to bond and currency options.* The Review of Financial Studies, 6, 327-343, 1993.

[18] J. Jacod: *Calcul stochastique et problèmes de martingales.* Lecture notes in mathematics 714, Springer, 1979.

[19] J. Jacod. *A general theorem of representation for matingales.* Proc. AMS Pro. Symp. Urbana 1976, 37-53.

[20] J. Jacod, A.N. Shiryaev: *Limit theorems for stochastic Processes.* Springer Verlag, Berlin, second edition, (2003).

[21] J. Jacod, M. Yor. *Etude des solutions extrémales et représentation intégrale des solutions pour certains problèmes de martingales.* Z. Wahr. Verw. Gebiete, 38:83-125, 1977.

[22] Y. Jiao: *Le risque de crédit: la modèlisation et la simulation numérique*, thèse de doctorat, Ecole Polytechnique, 2006.

[23] M. Jeanblanc, Y. Le Cam: *Progressive enlargement of filtrations with Initial times.* Preprint, Evry University, Submitted, 2007.

[24] M. Jeanblanc, Y. Le Cam: *Reduced form modeling.* Preprint, Evry University, Submitted, 2007.

[25] M. Jeanblanc, M. Rutkowski: *Default risk and hazard processes. In: Mathematical Finance - Bachelier Congress 2000.* H. Geman, D. Madan, S.R. Pliska and T. Vorst, eds., Springer-Verlag, Berlin Heidelberg New York, pp. 281-312.

[26] Yu. Kabanov: *Arbitrage theory* in Option pricing, Interest rates and risk management, Jouini, E. and Cvitanić, J. and Musiela, M.eds, 3-42, Cambridge University Press, 2001.

[27] H. Kunita, S. Watanabe: *On square integrable martingales,* Nagoya mathematical Journal, 30: 29-245, 1967.

[28] S. Kusuoka: *A remark on default risk models*, Adv. Math. Econ., 1, 69-82, 1999.

[29] V. Linetsky: *Pricing equity derivatives subject to bankrupcy.* Mathematical finance, 16, 255-282, 2005.

[30] R. Merton: *On the pricing of corporate debt: The risk structure of interest rates,* J. Finance 29, 449–470, 1974.

[31] P.E. Protter: *A partial introduction to financial asset pricing theory.* Stochastic Processes and Appl., 91:169, 2001.

[32] P.E. Protter: *Stochastic integration and differential equations*, Springer. Second edition, 2003.

[33] D. Revuz, M. Yor: *Continuous Martingales and Brownian Motion.* Springer Verlag, Berlin, third edition, 1999.