Partial Information, Default Hazard Process, and Default-Risky Bonds*

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Abstract

This paper studies in some examples the role of information in a default-risk framework. We examine three types of information for a firm's unlevered asset value to the secondary bond market: the classical case of continuous and perfect information, observation of past and contemporaneous asset values at selected discrete times, and observation of contemporaneous asset value at discrete times. The third information filtration is contained in the second, which– in turn, is contained in the first. We investigate the changes of the distributional properties of the default time and the properties of bond prices with the reductions of the information sets. Consistently with the observed market prices, model bond prices with partial information have surprise jumps prior to default. Credit spreads are increasing with the reductions of the information sets and with the increases of the observation lags. The two information constrained models admit reduced form representations, in which the time of default is a totally inaccessible time with default arrival intensities, but it is better avoiding the intensity approach to valuation since the hazard process approach is more efficient.

Key words: credit risk, information, hazard process of a default time, intensity, default-risky bonds

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1 Introduction

The structural or firm-value models of default are based on specifications of the firm's unlevered asset value process. In the first models of this type (Black and Scholes [2]; Merton [27]), default can take place only at the maturity date of the debt if asset value is lower than debt face value. This is in contrast to the actual times of defaults on corporate debt to the extent that firms frequently default prior to their debts' maturities. To obtain a more realistic model for the default time, Black and Cox [3] reformulated the valuation problem as a first passage time problem of asset value to a deterministic boundary. In this model, a poor performance is captured by a low value of the firm's productive assets relative to its stipulated debt payments and default occurs when asset value falls to some exogenously specified barrier, referred to as the lower reorganization boundary.

In the standard first passage models, based on diffusions, the time of default is a predictable stopping time in the filtration generated by asset value and its properties are not interesting from a mathematical point of view. Zhou [30] puts forth a jump-diffision process for asset value, in which default can come by surprise. Even though this models yield more realistic shapes of the term structure of credit spreads, as pointed out by this author, explicit solutions for first passage times in such setting are not known and the price of a defaultable bond is obtained by numerical methods. The standard geometric Brownian models are based on the assumption of perfect and continuous observation of the firm's asset value. While there have been many generalizations and extensions of the basic first passage approach, such as to stochastic interest rates (Longstaff and Schwartz [25]; Rutkowski [29]) and endogenous bankruptcy (Leland [22]; Leland and Toft [23]; Leland [24]), the effects of incomplete information have been considered only recently.

The first systematic study of the consequences of the incomplete information in the structural credit risk models is Duffie and Lando [10]. These authors noticed that frequently the standard assumption of continuous observation of asset value in the first-passage models is not satisfied in practice. While bond investors in a publicly traded firm can estimate the level of its assets by observing the prices of its equity and debt, the bondholders in a privately held firm have to rely on periodic accounting reports received at discrete dates. Therefore, the information flows to bondholders in such a firm can be more realistically modeled by filtrations generated by discrete observations of asset value. As noted by Duffie and Lando, for bondholders having such information, the default time is a totally inaccessible time and admits intensity. In this way, the authors reconcile the structural and reduced form approaches to the valuation of defaultable bonds and obtain higher and more realistic short-term credit spreads.

Examples of reduced form credit risk models are [15], [9], [21] and [11]. The reduced form approach does not look at the structure of a firm's asset and liabilities directly, but assumes that the time of default τ is the first jump time of a point process. In the classical case, in which the compensator of the point process is absolutely continuous, the compensator is called generalized intensity and the integrand in its integral representation is known as default arrival intensity. Consequently, the reduced form models relate the credit spread of a defaultable bond to the intensity of default. Intensity is usually taken to be a function of state variables that are in the information set of the market and are relevant for predicting the likelihood of default of the issuer. It can be also viewed as the instantaneous default probability. It is well known from survival analysis that the survival probability (in the present context, the probability of a bond surviving without default) equals the exponential of the negative of the hazard process. It is important to note that the hazard rate is the integrand in the integral representation of the hazard process. It is equal to intensity only in special cases.

Kusuoka [20] and Nakagawa [28] investigate cases of continuous but imperfect observation of asset value. These authors assume observation of smooth functions of asset value and solve the related filtering problems. With such information, default time admits default arrival intensity of the same form as in [10]. The paper of Cetin et al. [5] examines another case of continuous but incomplete information where default occurs when the firm's cash balances are negative for some pre-specified period and hit a default threshold level after that. These authors assume that the managers observe continuously the cash balances of the firm, while the market has a continuous subfiltration of the continuous manager's information filtration. This subfiltration consists of information on whether the firm is in financial distress, the duration of the distress and whether the default threshold has been reached. CreditMetrics (Gupton et al. [13]) and Giesecke [12] approach the incomplete information problem from another perspective. In these models, bond investors have incomplete information about the default-triggering barrier instead of the asset process. Actually, the first document assumes multiple uncertain barriers for the different rating classes and uses information from the rating transition probability matrix to solve for the barriers. Both studies provide solutions for the joint default probability and default correlation.

This paper focuses on the evaluation of conditional expectations with respect to three different types of asset value information in a diffusion setting. Our view is toward examining how the distributional properties of the default time and bond prices change with the changes of the information sets of the secondary bond market. We model the total information of the secondary bond market as information generated by asset value and observations of the default time. We start with the definition of the information subfiltration generated by asset value and compute the conditional default probability and the hazard process in this filtration. Following the hazard process approach (Elliott at al. [11]), they can be used to value different contingent claims with such information. This valuation approach is more general than the intensity based valuation, to the extent that it does not require the compensator of the default process and the computations are independent of the properties of the compensator. We then obtain valuations of defaultable bonds when the asset value subfiltration is enlarged with observations of the default time. We examine three different asset value subfiltrations.

First, we investigate the classical case, in which asset value is continuously and perfectly observed and the information is represented by a Brownian filtration. We refer to this case as full information since the time of default is a stopping time in this filtration and there is no need the filtration to be enlarged with observations of it. Second, we examine the case when bondholders observe the contemporaneous and all past asset values on a sequence of selected discrete times. If all the claims against the firm's assets are publicly traded, then asset value is observed since the value of the firm's assets is equal to that of its liabilities. The problem is that firms frequently have multiple liabilities, some of which are not publicly and continuously traded. For example, a firm may have multiple classes of common and preferred stocks. Also, the defaultable bonds of small firms are not traded frequently and their prices are often not continuously available. In many cases, bondholders receive price data on some of the firm's contingent claims with an observation lag and the first type of information that we introduce models such information flows. Third, as in Duffie and Lando [10], we examine the case when the secondary bond market observes only the contemporaneous asset values at selected discrete times. However, we assume perfect observation of asset value at these times instead of noisy observations as do these authors. The two partial information subfiltrations are jump-discontinuous, incorporating surprises at the observation dates.

The time of default changes from a predictable stopping time in the full information filtration to a totally inaccessible time in the two partial information subfiltrations. We obtain the semimartingale representations of the conditional default probabilities in the two subfiltrations, and it turns out that their martingale components are pure discontinuous processes. We show that structural models with realistic informational assumptions (incomplete, discretely arriving information) for the secondary bond market admit reduced form representations. However, with the two subfiltrations, we have two specific real-world examples where the hazard process and generalized intensity (see, e.g., Jeanblanc and Rutkowski [16], [17]) are different. The default time admits risk-neutral default arrival intensities, but it is better avoiding the intensity-based approach to valuation since working with the conditional hazard processes is direct and much simpler.

While some of the previous studies obtain valuation formulas for defaultable bonds by numerical methods, we provide analytical solutions for the prices of bonds with zero recovery in the two subfiltrations. For the first type of incomplete information, we also provide a price of a bond with non-zero recovery, paid at the time when asset value hits the default barrier. While bond prices with full information are continuous, in the two asset subfiltrations, discrete information arrivals induce surprise jumps in them. The information reductions result in higher credit spreads of defaultable bonds than in the case of continuous observation and the difference is most pronounced for short maturities. For the same sequence of observation times and asset value processes, a firm with the first type of incomplete asset information is more transparent to its bondholders and would has a lower credit spread on its debt than a firm with the second type of incomplete information. Assuming the same default barrier for equityholders, we examine in detail the case when the two asset subfiltrations of bondholders are enlarged with observations of whether equityholders have liquidated the firm. In these expanded information sets, credit spreads are the same. However, this will not be true in the general case, in which the default barriers of the holders of the different contingent claims are different.

The rest of the paper is organized as follows. In Section 2, we specify a standard first passage model for the default event assuming fractional recovery of par for defaultable bonds. Section 3 introduces a first type of incomplete information. At selected discrete times, bondholders obtain information about the contemporaneous and past asset values. Section 4 investigates a second type of incomplete information, namely, the case when the investors only observe the contemporaneous asset value at these discrete dates. Section 5 provides analysis of credit spreads. In Section 6, we compare the valuation of defaultable bonds by the hazard process and intensity approaches. Section 7 contains concluding remarks.

2 Full information: continuous observations of asset value

In the first-passage models, it is usually assumed that the unlevered asset value V of a firm follows the geometric Brownian motion process

$$dV_t = V_t((\mu - \delta)dt + \sigma dW_t^*), \tag{1}$$

where μ and σ, (with $\sigma > 0$) are the drift rate and the percentage volatility of asset value, δ is the rate of payments to the holders of claims on the firm's assets and W^* is a Brownian motion under the historical probability P^*.

In the classical models such as Black and Cox [3], the information flows of the bondholders are modeled by filtrations generated by continuous observations of asset value. In particular, we denote by $\mathbf{F}^V$ the complete filtration generated by V, i.e., $\mathbf{F}^V = (\mathcal{F}_t^V, t \geq 0)$, where $\mathcal{F}_t^V = \sigma(V_s, s \leq t)$. Such information sets are plausible for the firm's managers, but not so for the secondary bond market. In the subsequent analysis, we assume that the firm managers are also the equityholders of the firm.

The first passage approach accounts for the possibility of liquidation by equityholders as well as for such provisions of corporate indentures as safety covenants and debt subordination. Safety covenants provide bondholders with the rights to enforce default, reorganization and liquidation of a firm prior to default when its financial performance is poor.

In particular, suppose that default occurs when the firm's asset value falls below some threshold level α, which is lower than the initial firm value $V_0 = v$. Then, the time of default τ is given by

$$\tau = \inf\{t \, : \, V_t \leq \alpha\} = \inf\{t \, : \, X_t \leq a\} \tag{2}$$

with $X_t = \frac{1}{\sigma} \ln(V_t/v)$ and $a = \dfrac{1}{\sigma} \ln(\alpha/v)$. Default can be forced by either the bondholders or the equityholders.

Suppose that it exists a savings account paying a constant short term interest rate r. Hence, if one assumes that the firm's asset value is a tradable asset, then the market is complete and arbitrage-free. In what follows, the computations are carried under the risk neutral probability measure P. We recall that under P

$$dV_t = V_t((r - \delta)dt + \sigma dW_t), \tag{3}$$

where $(W_t, t \geq 0)$ is a $(P, \mathcal{F}_t^V)$-Brownian motion. The solution of (3) is $V_t = ve^{\sigma(W_t + \nu t)}$, where $\nu = \dfrac{1}{\sigma}(r - \delta - \dfrac{\sigma^2}{2})$. We rewrite the solution in the more convenient form

$$V_t = ve^{\sigma X_t},$$

where $X_t = \nu t + W_t$ is a $(P, \mathcal{F}_t^V)$-Brownian motion with drift. It is important to notice that $\mathbf{F}^V$ is equal to the filtrations $\mathbf{F}^X$ and $\mathbf{F}^W$ generated by X and W, respectively.

Figure 1 displays $(X_t, 0 \leq t \leq T)$ and the default boundary with the base values of the parameters that we shall use in this study

$$\sigma = 0.30; \; r = 0.04; \; \delta = 0.03; \; v = 100; \; \alpha = 80; \; T = 2. \tag{4}$$

We choose a very high default boundary for illustrative purposes, since, in the simulations, we want to obtain paths close to default.

We denote the probability that the process $(X_s = W_s + \nu s, s \geq 0)$ remains above the barrier z till time t by

$$\Phi(\nu, t, z) = P\left(\inf_{s \leq t} X_s > z\right). \tag{5}$$

It is important to notice that this probability depends on the drift rate ν. The reflection principle and elementary considerations lead to

$$\begin{aligned} \Phi(\nu, t, z) &= \mathcal{N}\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z}\mathcal{N}\left(\frac{z + \nu t}{\sqrt{t}}\right), \qquad \text{for } z < 0,\ t > 0, \\ &= 0, \qquad \text{for } z \geq 0,\ t \geq 0, \\ \Phi(\nu, 0, z) &= 1, \qquad \text{for } z < 0. \end{aligned}$$

The event that the firm does not default up to the maturity T of the bond can be expressed as

$$\begin{aligned} \{\tau > T\} &= \left\{\inf_{s \leq T} V_s > \alpha\right\} = \left\{\inf_{s \leq t} V_s > \alpha\right\} \cap \left\{\inf_{t < s \leq T} V_s > \alpha\right\} \\ &= \left\{\inf_{s \leq t} X_s > a\right\} \cap \left\{\inf_{t < s \leq T} X_s - X_t > a - X_t\right\} \\ &= \left\{\inf_{s \leq t} X_s > a\right\} \cap \left\{\inf_{s \leq T-t} \widehat{X}_{s-t} > a - X_t\right\}, \end{aligned}$$

where $\widehat{X} = (\widehat{X}_u = X_{t+u} - X_t, u \geq 0)$ is independent of $\mathcal{F}_t^V$. Because of the stationarity and the independence of the increments of the Brownian motion, the process $(\widehat{X}_u, u \geq 0)$ is a $(P, \mathcal{F}_{t+u}^V)$-Brownian motion with drift.

Suppose that the firm has issued defaultable bonds with unit face values that pay their face values at maturity if there is no default or the recovery rate otherwise. We assume that there are no bankruptcy costs, so that bondholders receive all asset value at default, i.e. α. Let the total face value of the firm's outstanding bonds be D. Then, in the case of default, the residual firm value is distributed to all the bondholders and each bonds recovers $\frac{\alpha}{D}$.

To facilitate the subsequent analysis, we introduce two approximations to the Black and Cox [3] valuation problem. First, we assume that the default-triggering barrier for asset value is constant and equals α. Black and Cox provide a valuation formula for a barrier of the form $\alpha e^{-\eta(T-t)}$. In fact, the case of an exponential barrier $f(t) = \alpha e^{\eta t}$, $\tau = \inf\{t : V_t \leq f(t)\}$ reduces to the constant case up to a change of parameters

$$\tau = \inf\{t : xe^{\sigma X_t} \leq \alpha e^{\eta t}\} = \inf\{t : ye^{\sigma Y_t} \leq \alpha\}$$

with $Y_t = X_t - \frac{\eta}{\sigma}t, \quad y = x.$

Second, in the formulation of Black and Cox, the default barrier would be α up to T and D at T. Following this approach, one would define the default time (see also Bielecki and Rutkowski

[1]) as $\tau^* = \tau \wedge \tau^m$, where τ^m is the Merton's default time, i.e., $\tau^m = T$ if $V_T < D$ and $\tau^m = \infty$ otherwise. We work with the default time τ rather than τ^*.

With the above assumptions, when the information available to the market at time t is the σ-algebra $\mathcal{F}_t^V$, the default-risky bond price $B_d(t,T)$ is given by

$$\begin{aligned} B_d(t,T) &= E\left(e^{-r(T-t)}\mathbb{1}_{\tau>T} + \frac{\alpha}{D}e^{-r(\tau-t)}\mathbb{1}_{\tau\le T}\Big|\mathcal{F}_t^V\right) \\ &= e^{-r(T-t)}P(\tau>T|\mathcal{F}_t^V) + \frac{\alpha}{D}E\left(e^{-r(\tau-t)}\mathbb{1}_{\tau\le T}\Big|\mathcal{F}_t^V\right). \end{aligned}$$

The price of the defaultable bond is a sum of two components. We write $Z(t,T) := e^{-r(T-t)}P\left(\tau > T\big|\mathcal{F}_t^V\right)$ for the first term, which is the price of a defaultable bond with zero recovery. It pays the face value of the bond at maturity if there is no default and zero otherwise. The second term is the value of the recovery, $R(t,T) := \frac{\alpha}{D}E\left(e^{-r(\tau-t)}\mathbb{1}_{\tau\le T}\big|\mathcal{F}_t^V\right)$, in the case of default.[1]

The conditional survival probability $P(\tau > T|\mathcal{F}_t^V)$ plays a key role in the term $Z(t,T)$. It can be expressed in terms of the probability of hitting the default barrier

$$\begin{aligned} P(\tau > T|\mathcal{F}_t^V) &= P\left(\left\{\inf_{s\le t} X_s > a\right\} \cap \left\{\inf_{t<s\le T} V_s > \alpha\right\}\Big|\mathcal{F}_t^V\right) \\ &= \mathbb{1}_{t<\tau}P\left(\inf_{t\le s\le T} V_s > \alpha\Big|\mathcal{F}_t^V\right) \\ &= \mathbb{1}_{t<\tau}P\left(\inf_{s\le T-t} \widehat{X}_s > a - X_t\Big|\mathcal{F}_t^V\right) \\ &= \mathbb{1}_{t<\tau}P\left(\inf_{s\le T-t} \widehat{X}_s > a - x\right)\Big|_{x=X_t} \\ &= \mathbb{1}_{t<\tau}\Phi(\nu, T-t, a-X_t), \end{aligned} \tag{6}$$

hence

$$Z(t,T) = e^{-r(T-t)}\mathbb{1}_{t<\tau}\Phi(\nu, T-t, a-X_t).$$

As for the recovery value, using absolute continuity relationship, we can write

$$\begin{aligned} R(t,T) &= \frac{\alpha}{D}e^{rt}E\left(e^{-r\tau}\mathbb{1}_{\tau\le T}|\mathcal{F}_t^V\right) \\ &= \frac{\alpha}{D}e^{rt}\frac{E^{(\gamma)}\left(L_\tau e^{-r\tau}\mathbb{1}_{\tau\le T}|\mathcal{F}_t^V\right)}{L_t} \\ &= \frac{\alpha}{D}e^{rt}E^{(\gamma)}\left(e^{-r\tau}e^{(\nu-\gamma)(X_\tau - X_t) - \frac{\nu^2-\gamma^2}{2}(\tau-t)}\mathbb{1}_{\tau\le T}\Big|\mathcal{F}_t^V\right), \end{aligned}$$

where $E^{(\gamma)}$ denotes expectation under the probability measure $P^{(\gamma)}$, defined by

$$L_t = \frac{dP}{dP^{(\gamma)}}\bigg|_{\mathcal{F}_t^V} = \exp\left((\nu-\gamma)X_t - \frac{\nu^2-\gamma^2}{2}t\right).$$

[1] If the recovery is paid at the maturity date, the computation is simpler.

Under $P^{(\gamma)}$, the process X is a Brownian motion with drift γ. If we choose γ such that $\gamma = \sqrt{2r+\nu^2}$, we obtain

$$\begin{aligned} R(t,T) &= \frac{\alpha}{D} e^{(\nu-\gamma)(a-X_t)} E^{(\gamma)}\left(\mathbb{1}_{\tau \le T} | \mathcal{F}_t^V\right) \\ &= \frac{\alpha}{D} e^{(\nu-\gamma)(a-X_t)}\left(1 - E^{(\gamma)}\left(\mathbb{1}_{T<\tau} | \mathcal{F}_t^V\right)\right) \\ &= \frac{\alpha}{D} e^{(\nu-\gamma)(a-X_t)}\Big(1 - \mathbb{1}_{t<\tau}\Phi(\gamma, T-t, a-X_t)\Big). \end{aligned} \tag{7}$$

Finally, the price of the defaultable bond is

$$\begin{aligned} B_d(t,T) &= e^{-r(T-t)}\mathbb{1}_{t<\tau}\Phi(\nu, T-t, a-X_t) \\ &+ \frac{\alpha}{D} e^{(\nu-\gamma)(a-X_t)}\Big(1 - \mathbb{1}_{t<\tau}\Phi(\gamma, T-t, a-X_t)\Big). \end{aligned} \tag{8}$$

Equation (8) gives the price of a defaultable bond with fractional recovery (α/D) of par as opposed to the recovery payoff in the original Black and Cox [3] model. In the numerical examples in [3], fractional recovery of the net present value of par is assumed, that is, $(\alpha/D)e^{-r(T-t)}$. Moreover, we assumed that the total value of the bonds at the maturity date if the firm is not reorganized before is $\min(\alpha, D)$, while in [3], it is $\min(V_T, D)$. So our valuation formula for the defaultable bond is an approximation to the Black and Cox formula.

As a check, for $r = 0$, the term on the right-hand-side reduces to $\mathbb{1}_{t<\tau}\Phi(\nu, T-t, a-X_t)(1 - \frac{\alpha}{D}) + \frac{\alpha}{D}$ as expected from the definition. The problem with the structural approach is that the time of default τ is a predictable stopping time in the filtration $\mathbf{F}^V$ and the default probability in this filtration is simply given by $F_t = P(\tau \le t|\mathcal{F}_t^V) = \mathbb{1}_{\tau\le t}$. It is obvious that, in this case, the price of the defaultable bond does not have surprise jumps. In the case of default, bond price converges continuously to its recovery value of $\frac{\alpha}{D}$ as the Brownian motion X approaches the barrier a. This is inconsistent with the empirical regularities, since the prices of defaultable bonds have multiple jumps prior to the time of default. Also, credit spreads generated by the model are too low when compared to the actual market spreads. In fact, the short term credit spreads are close to zero and, for short maturities, the term structure of credit spreads is upwardly sloping, which is in contrast to the market credit spreads. In the following section we examine bond prices with two types of partial information.

3 First type of incomplete information: full observation at discrete times

The previous section shows that the asset value process is a necessary input in the valuation formula. In this and the following section, we retain the stochastic process for the firm's asset value, but change the information sets available to the secondary bond market. We address the important empirical regularity that bondholders are not completely informed about the financial status of the firm to the extent that asset value is not traded and is not perfectly and continuously observable. In such cases it has to be estimated and the most natural way to do this is by looking at the prices of the contingent claims against its assets.

In this section, we examine the case of a publicly traded firm with observation lags for the prices of some of its contingent claims. Such a situation may arise if some of the contingent claims of the firm are less liquid and are traded over-the-counter. Then prices of these claims may not be available to the bond market continuous, but at some discrete time interval, such prices cold possibly become available by averaging the different brokerage quotations. Then, at the observation time of the prices of these claims, bondholders can compute asset value during the interval.

To address such a situation, we assume that the bondholders have access only to a subfiltration of the full-information filtration $\mathbf{F}^V$. More specifically, we assume that the bondholders obtain all the information about contemporaneous and past values of the firm's assets at the dates

$$\mathbb{T} = \{t_1, t_2, t_3, \dots\},$$

where $(t_i, 1 \leq i)$ is an increasing sequence of observation times.

We denote by $\mathbf{H} = (\mathcal{H}_t, t \geq 0)$ the filtration generated by the observations of the past V at times $t_1, \dots, t_n$ with $t_n \leq t < t_{n+1}$, that is,

$$\begin{aligned}
\mathcal{H}_t &= \{\emptyset, \Omega\} \qquad \text{for } t < t_1, \\
\mathcal{H}_t &= \mathcal{F}^V_{t_1} = \sigma(V_s, s \leq t_1) \quad \text{for } t_1 \leq t < t_2, \\
\mathcal{H}_t &= \mathcal{F}^V_{t_n} = \sigma(V_s, s \leq t_n) \quad \text{for } t_n \leq t < t_{n+1},
\end{aligned}$$

where Ω is the sample space. Our aim is to characterize the distributional properties of τ when the available information is $\mathbf{H}$.

The filtration $\mathbf{H}$ is a subfiltration of $\mathbf{F}^V$ (i.e., $\mathbf{H} \subset \mathbf{F}^V$). It is constant between the observation dates. At the observation date t_i, the σ-algebra $\mathcal{H}_t$ is enlarged by $\sigma(V_s, t_{i-1} \leq s < t_i)$, that is, $\mathcal{H}_{t_i} = \mathcal{H}_{t_{i-1}} \vee \sigma(V_s, t_{i-1} \leq s < t_i)$. We write $F^1_t := P(\tau \leq t|\mathcal{H}_t)$ for the $\mathbf{H}$-conditional default probability, where the superscript 1 stands for the first type of information. We denote by $\Gamma^1_t := -\ln(1 - F^1_t)$ the $\mathbf{H}$-hazard process of τ. In this and the following section, we shall write $\Phi(t, a)$ for $\Phi(\nu, t, a)$.

3.1 First interval

In the case $t < t_1$, the conditional default probability F_t^1 is deterministic and is equal to the distribution function of τ. Using the fact that $a < 0$, we obtain

$$F_t^1 = P(\tau \leq t) = P\left(\inf_{s \leq t} X_s \leq a\right) = 1 - \Phi(t, a)$$

and F^1 is continuous on $[0, t_1[$.

3.2 Second interval

In the case $t_1 \leq t < t_2$, we have

$$\begin{aligned} F_t^1 &= P(\tau \leq t | \mathcal{H}_{t_1}) = 1 - P(\tau > t | \mathcal{H}_{t_1}) \\ &= 1 - P\left(\inf_{s<t} X_s > a \middle| \mathcal{F}_{t_1}^V\right) = 1 - \mathbb{1}_{\{\inf_{s<t_1} X_s > a\}} P\left(\inf_{t_1 \leq s < t} X_s > a \middle| \mathcal{F}_{t_1}^V\right). \end{aligned} \tag{9}$$

By using (6) and (9), we obtain

$$F_t^1 = 1 - \mathbb{1}_{\tau > t_1} \Phi(t - t_1, a - X_{t_1}).$$

We denote by ΔF_s^1 the jump of F^1 at s, i.e., $\Delta F_s^1 = F_s^1 - F_{s-}^1$. The process F^1 is continuous, increasing and deterministic on $[t_1, t_2[$; the jump of F^1 at time t_1 on the set $\{\tau \leq t_1\}$ is a positive jump with size

$$\Delta F_{t_1}^1 = 1 - (1 - \Phi(t_1, a)) = \Phi(t_1, a),$$

while on the set $\{\tau > t_1\}$ it is a negative jump equal to

$$\begin{aligned} \Delta F_{t_1}^1 &= 1 - \Phi(0, a - X_{t_1}) - (1 - \Phi(t_1, a)) \\ &= \Phi(t_1, a) - 1. \end{aligned}$$

In the above computation, we have used that, on $\{\tau > t_1\}$, the inequality $a - X_{t_1} < 0$ holds, hence $\Phi(0, a - X_{t_1}) = 1$. Note that if $F_t^1 = 1$, then $F_{t+s}^1 = 1 \quad \forall s > 0$.

3.3 General case

We easily extend the previous result to general observation times. For $t_i \leq t < t_{i+1}$, we can write

$$\begin{aligned} F_t^1 &= P(\tau \leq t | \mathcal{H}_t) = 1 - P(\tau > t | \mathcal{F}_{t_i}^V) \\ &= 1 - P\left(\inf_{s<t} X_s > a \middle| \mathcal{F}_{t_i}^V\right) = 1 - \mathbb{1}_{\tau > t_i} P\left(\inf_{t_i \leq s < t} X_s > a \middle| \mathcal{F}_{t_i}^V\right) \\ &= 1 - \mathbb{1}_{\tau > t_i} \Phi(t - t_i, a - X_{t_i}). \end{aligned}$$

If there is default between two observations, the process F^1 jumps to one on the next observation date. The **H**-hazard process is given by

$$\Gamma_t^1 = -\ln(1\!\!1_{\tau \geq t_i} \Phi(t - t_i, a - X_{t_i})).$$

Prior to default, there are jumps in F^1 and Γ^1 to zero at the times of observation of the firm's assets.

The process F^1 is continuous and increasing on $[t_i, t_{i+1}[$. Its jump at time t_i on the set $\{\tau \leq t_i\}$ is a positive jump

$$\begin{aligned} \Delta F_{t_i}^1 &= 1 - (1 - 1\!\!1_{\tau > t_{i-1}} \Phi(t_i - t_{i-1}, a - X_{t_{i-1}})) \\ &= 1\!\!1_{\tau > t_{i-1}} \Phi(t_i - t_{i-1}, a - X_{t_{i-1}}), \end{aligned}$$

while on the set $\{\tau > t_i\}$ it is a negative jump equal to

$$\begin{aligned} \Delta F_{t_i}^1 &= 1 - \Phi(0, a - X_{t_i}) - (1 - \Phi(t_i - t_{i-1}, a - X_{t_{i-1}})) \\ &= \Phi(t_i - t_{i-1}, a - X_{t_{i-1}}) - 1. \end{aligned}$$

Apparently, the process F^1 jumps to either one or zero at each observation date and this is illustrated in the following two figures. Figure 2 displays the conditional default probability $(F_t^1, 0 \leq t \leq T)$ and the hazard process $(\Gamma_t^1, 0 \leq t \leq T)$ for the following sequence of observation dates: $t_1 = 0.5$, $t_2 = 1$ and $t_3 = 1.5$. We have simulated the probability with the typical values of the parameters and the simulations resulted in a path with no default. These jumps are with stochastic sizes depending on V_{t_1}, V_{t_2} and V_{t_3} for t_2, t_3 and t_4, respectively.

Figure 3 plots other simulation results for the same processes. In this example, default occurs at time $\tau = 0.624$. At the next observation date, $t_2 = 1$, the conditional default probability jumps to one and the hazard process jumps to $+\infty$.

Lemma 1 *The pure jump process ζ^1 defined by $\zeta_t^1 = \sum_{i, t_i \leq t} \Delta F_{t_i}^1$ is an **H**-martingale.*

Proof: Consider first the times $t_i \leq s < t \leq t_{i+1}$. In this case, it is obvious that $E(\zeta_t^1 | \mathcal{H}_s) = \zeta_s^1$ since $\zeta_t^1 = \zeta_s^1 = \zeta_{t_i}^1$, which is $\mathcal{H}_s$-measurable.

It suffices to show that $E(\zeta_t^1 | \mathcal{H}_s) = \zeta_s^1$ for $t_i \leq s < t_{i+1} \leq t < t_{i+2}$. In this case $\zeta_t^1 = \zeta_{t_i}^1 + \Delta F_{t_{i+1}}^1$

and $\zeta^1_s = \zeta^1_{t_i}$. We have

$$\begin{aligned} E(\Delta F^1_{t_{i+1}}|\mathcal{H}_s) &= E\Big(1\!\!1_{\tau>t_i}1\!\!1_{\tau\le t_{i+1}}\Phi(t_{i+1}-t_i, a-X_{t_i}) + 1\!\!1_{\tau>t_{i+1}}(\Phi(t_{i+1}-t_i, a-X_{t_i})-1)|\mathcal{H}_s\Big) \\ &= 1\!\!1_{\tau>t_i}\Phi(t_{i+1}-t_i, a-X_{t_i}) - E(1\!\!1_{\tau>t_{i+1}}|\mathcal{H}_s) \\ &= 1\!\!1_{\tau>t_i}\Phi(t_{i+1}-t_i, a-X_{t_i}) - E(1\!\!1_{\tau>t_{i+1}}|\mathcal{F}^V_{t_i}) \\ &= 1\!\!1_{\tau>t_i}\Phi(t_{i+1}-t_i, a-X_{t_i}) - 1\!\!1_{\tau>t_i}\Phi(t_{i+1}-t_i, a-X_{t_i}) \\ &= 0. \end{aligned}$$

△

The probability of the bond surviving without default is given by

$$\begin{aligned} P(\tau > T|\mathcal{H}_t) &= P\left(\inf_{s<T} X_s > a\Big|\mathcal{F}^V_{t_i}\right) \\ &= 1\!\!1_{\tau>t_i}P\left(\inf_{t_i<s<T} X_s > a\Big|\mathcal{F}^V_{t_i}\right) \\ &= 1\!\!1_{\tau>t_i}\Phi(T-t_i, a-X_{t_i}) \qquad \text{for } t_i \le t < T. \end{aligned} \tag{10}$$

3.4 Default process, its compensator and hazard process

Let $\mathbf{F}^i = (\mathcal{F}^i_t, t \ge 0)$ be some reference filtration. We denote the probability of default in this filtration by $F^i_t = P(\tau \le t|\mathcal{F}^i_t)$. We write $\mathcal{G}^i_t = \mathcal{F}^i_t \vee \sigma(\tau \wedge t)$ for the σ-algebra enlarged with observations of the default event and $\mathbf{G}^i = (\mathcal{G}^i_t, t \ge 0)$ for the enlarged filtration. We also write $\Gamma^i_t = -\ln(1-F^i_t)$ for the $\mathbf{F}^i$-hazard process, where we set $\Gamma^i_t = +\infty$ for $F^i_t = 1$.

The hazard process approach (Elliott et al. [11]; Jeanblanc and Rutkowski [17]) states that if X is some integrable, $\mathcal{G}^i$-measurable random variable, then

$$E[X1\!\!1_{T<\tau}|\mathcal{G}^i_t] = 1\!\!1_{t<\tau}E\left[Xe^{\Gamma^i_t-\Gamma^i_T}\Big|\mathcal{F}^i_t\right]. \tag{11}$$

Thus, the hazard process approach provides a method of evaluation of conditional expectations with respect to $\mathbf{G}^i$. We emphasize that this formula holds true even if $(\Gamma^i_t, t \ge 0)$ does not enjoy the increasing property. In particular, for $X = 1$, that is, the case of a defaultable bond, we are back to

$$P(T<\tau|\mathcal{G}^i_t) = 1\!\!1_{t<\tau}\frac{P(T<\tau|\mathcal{F}^i_t)}{P(t<\tau|\mathcal{F}^i_t)}.$$

The hypothesis (H) (Jeanblanc and Rutkowski [17]) is that every square integrable martingale in the filtration $\mathbf{F}^i$ is invariant under the enlargement of filtration with observations of default time, that is, it is also a $\mathbf{G}^i$-square integrable martingale.

In what follows, we will write $D_t := 1\!\!1_{\tau\le t}$ for the default process and $\mathbf{D}$ for the filtration generated by it. The properties of the compensator of D in its $\mathbf{F}^i$-Doob-Meyer decomposition depend on the default probability F^i. In the case where F^i is increasing and continuous, the process

$$D_t - \Gamma^i_{t\wedge\tau} = D_t - \int_0^{t\wedge\tau} \frac{dF^i_s}{1-F^i_s}$$

is a martingale.

If F^i is increasing (this is called condition (G) in Jeanblanc and Rutkowski [17]), the process

$$D_t - \int_0^{t\wedge\tau} \frac{dF^i_s}{1-F^i_{s-}}$$

is a martingale and $\Lambda^i_{t\wedge\tau} = \int_0^{t\wedge\tau} \frac{dF^i_s}{1-F^i_{s-}}$ is an increasing process.

In the general case, the submartingale F^i admits the decomposition $F^i = Z^i + A^i$ where A^i is a predictable increasing process and Z^i is an $\mathbf{F}^i$-martingale, we have that

$$D_t - \int_0^{t\wedge\tau} \frac{dA^i_s}{1-F^i_{s-}} \tag{12}$$

is a martingale.

Accordingly, the Doob-Meyer decomposition of the submartingale $(F^1_t, t \geq 0)$ is

$$F^1_t = \zeta^1_t + (F^1_t - \zeta^1_t),$$

where ζ^1 is an $\mathbf{H}$-martingale; $F^{1,c} := F^1 - \zeta^1$ is a continuous, hence predictable increasing process. Since F^1 is not increasing, the hypothesis (H) does not hold.

3.5 Implications for bond prices

3.5.1 Bond prices with observation of the time of default

In the literature, it is usually assumed that default is announced as it occurs and the default time is observed by the market. The question that arises naturally is how could bondholders with discrete information about asset value observe whether the default threshold has been reached. According to the US Bankruptcy Code, default and liquidation of the firm's assets can be done by the equityholders. In the subsequent analysis, we assume that the reorganization boundary for the equityholders is the same as for the bondholders and that the equityholders announce publicly their decision to liquidate the firm.

The following theorem provides an analytical valuation formula for a defaultable bond with zero recovery when the flow of information is $\mathbf{H}$ and observations of the default time.

Theorem 1 *When the information available to bondholders is $\mathcal{H}_t \vee \sigma(\tau \wedge t)$, the price of the zero-recovery defaultable bond is, for $t_i < t < t_{i+1}$; $t < T$,*

$$Z^1(t,T) = e^{-r(T-t)} \mathbb{1}_{t<\tau} \frac{\Phi(T-t_i, a - X_{t_i})}{\Phi(t-t_i, a-X_{t_i})}. \tag{13}$$

Figure 1: Path of the drifted Brownian motion $(X_t, 0 \leq t \leq T)$ and the default boundary a

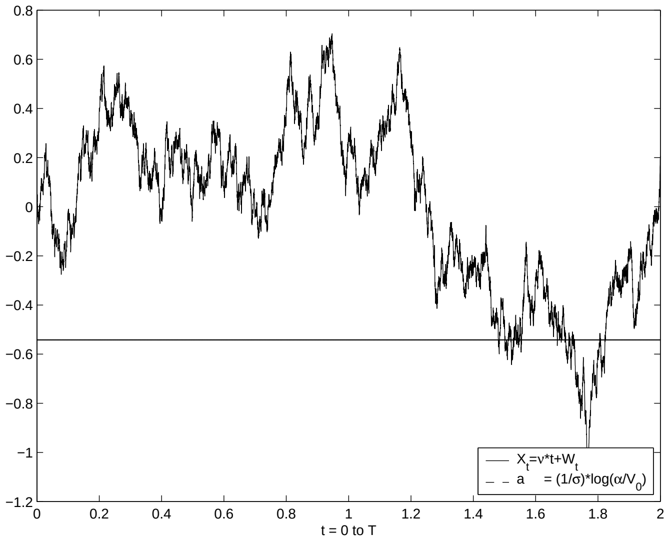


Figure 2: Default probability $F_t^1 = P(\tau < t|\mathcal{H}_t)$ and hazard process $\Gamma_t^1 = -\ln(1 - F_t^1)$ with information of the first type (path without default)

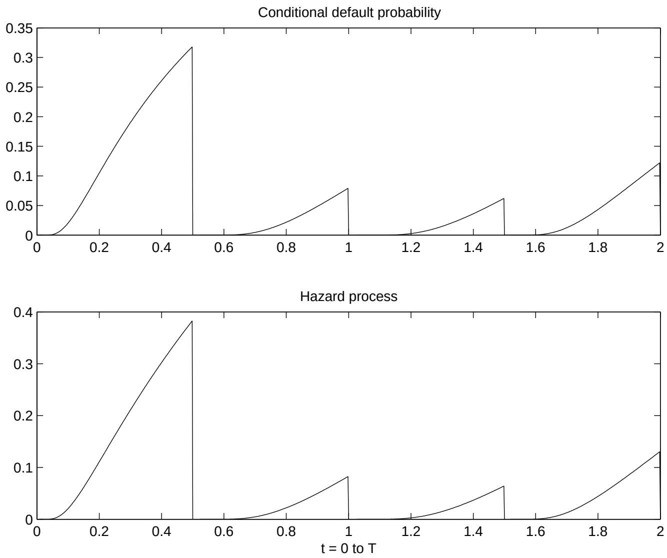


Figure 3: Default probability $F_t^1 = P(\tau < t|\mathcal{H}_t)$ and hazard process $\Gamma_t^1 = -\ln(1-F_t^1)$ with information of the first type (path with default)

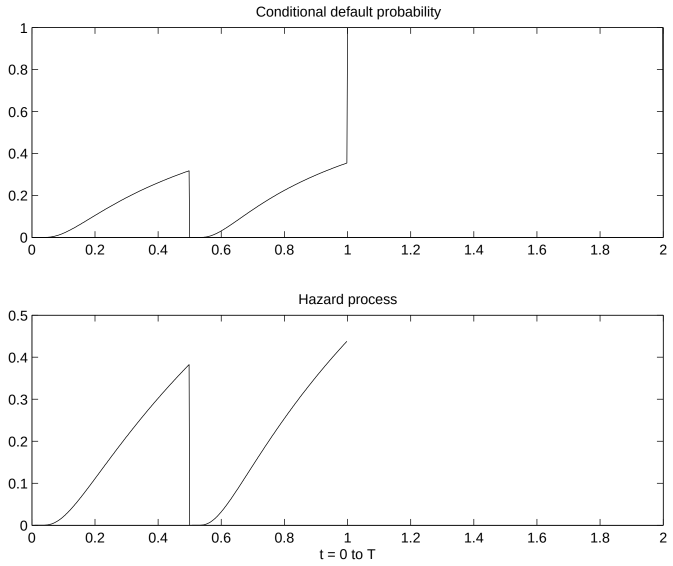
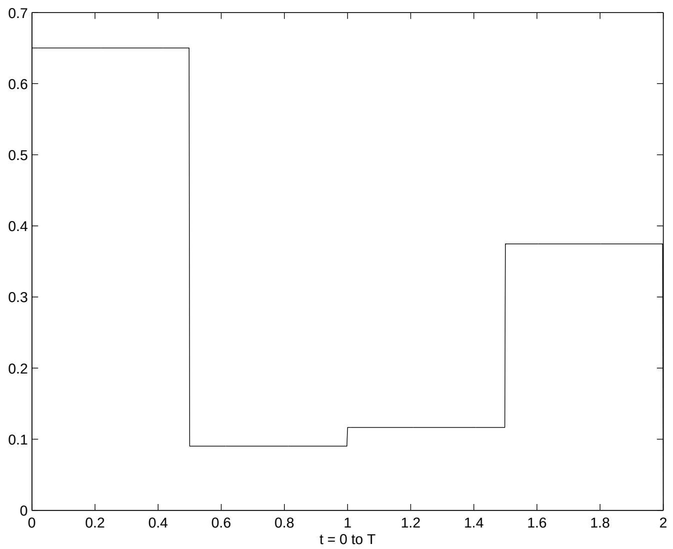


Figure 4: Probability of default prior to maturity with information of the first type $P(\tau < T|\mathcal{H}_t)$



Proof: The price of a zero-recovery defaultable bond is given by the discounted value of the conditional survival probability, i.e.,

$$Z^1(t,T) = e^{-r(T-t)} P\left(T < \tau | \mathcal{H}_t \vee \sigma(\tau \wedge t)\right) = e^{-r(T-t)} \mathbb{1}_{t<\tau} \frac{P(T < \tau|\mathcal{H}_t)}{P(t < \tau|\mathcal{H}_t)}, \tag{14}$$

where the second equality follows from the hazard process approach (11). To compute the fraction on the right-hand-side, we use the fact that

$$P(\tau > s|\mathcal{H}_t) = \mathbb{1}_{\tau > t_i} \Phi(s - t_i, a - X_{t_i}) \qquad \text{for } s \geq t,$$

hence,

$$P\left(T < \tau | \mathcal{H}_t \vee \sigma(\tau \wedge t)\right) = \mathbb{1}_{t<\tau} \frac{\Phi(T - t_i, a - X_{t_i})}{\Phi(t - t_i, a - X_{t_i})}.$$

A substitution of this expression in (14) yields the result.

△

The next theorem provides a formula for a defaultable bond with fractional recovery of par.

Theorem 2 *For $t_i < t < t_{i+1}$ and $t < T$, the price of the defaultable bond when the information available on the market is $\mathcal{H}_t \vee \sigma(\tau \wedge t)$ is given by*

$$\begin{aligned} B_d^1(t,T) &= e^{-r(T-t)} \mathbb{1}_{t<\tau} \frac{\Phi(T - t_i, a - X_{t_i})}{\Phi(t - t_i, a - X_{t_i})} \\ &+ \frac{\alpha}{D} \frac{\mathbb{1}_{t<\tau} e^{r(t-t_i)} e^{(\nu-\gamma)(a - X_{t_i})}}{\Phi(t - t_i, a - X_{t_i})} \left\{ \Phi(\gamma, t - t_i, a - X_{t_i}) - \Phi(\gamma, T - t_i, a - X_{t_i}) \right\}. \end{aligned} \tag{15}$$

Proof: With information $\mathcal{H}_t \vee \sigma(\tau \wedge t)$, the value of a defaultable bond is

$$B_d^1(t,T) = Z^1(t,T) + R^1(t,T),$$

where $R^1(t,T)$ is the value of the rebate part. It is given by

$$R^1(t,T) = \frac{\alpha}{D} E\left(e^{-r(\tau-t)} \mathbb{1}_{t<\tau<T} \middle| \mathcal{H}_t \vee \sigma(\tau \wedge t)\right) = \frac{\alpha}{D} \frac{\mathbb{1}_{t<\tau}}{P(t < \tau|\mathcal{H}_t)} E\left(e^{-r(\tau-t)} \mathbb{1}_{t<\tau<T} \middle| \mathcal{H}_t\right).$$

Assume that $t_i < t < t_{i+1}$. From the Markov property,

$$E\left(e^{-r(\tau-t)} \mathbb{1}_{t<\tau<T} | \mathcal{H}_t\right) = E\left(e^{-r(\tau-t)} \mathbb{1}_{t<\tau<T} | \mathcal{F}_{t_i}^V\right). \tag{16}$$

One can write

$$E\left(e^{-r(\tau-t_i)} \mathbb{1}_{t_i<\tau<T} | \mathcal{H}_t\right) = \Psi(a - X_{t_i}, T - t_i),$$

where $\Psi(x,u) = E(e^{-r\tau}\mathbb{1}_{\tau<u})$ with $\tau = \inf\{s : \nu s + W_s \le x\}$. It follows that

$$\begin{aligned}
\mathbb{1}_{t<\tau}E\left(e^{-r(\tau-t)}\mathbb{1}_{t<\tau<T}\middle|\mathcal{H}_t\right) &= \mathbb{1}_{t<\tau}E\left(e^{-r(\tau-t)}\mathbb{1}_{t<\tau<T}\middle|\mathcal{F}^V_{t_i}\right) \\
&= \mathbb{1}_{t<\tau}\Big[e^{r(t-t_i)}E\left(e^{-r(\tau-t_i)}\mathbb{1}_{t_i<\tau<T}\middle|\mathcal{F}^V_{t_i}\right) \\
&\qquad -e^{r(t-t_i)}E\left(e^{-r(\tau-t_i)}\mathbb{1}_{t_i<\tau<t}\middle|\mathcal{F}^V_{t_i}\right)\Big] \\
&= \mathbb{1}_{t<\tau}e^{r(t-t_i)}\Big[\Psi(a-X_{t_i},T-t_i)-\Psi(a-X_{t_i},t-t_i)\Big].
\end{aligned}$$

The computation of $\Psi(\cdot,\cdot)$ can be found in Jeanblanc et al. [18]. For $x = a - X_{t_i} < 0$,

$$\begin{aligned}
\Psi(x,t) &= e^{(\nu-\gamma)x}\mathcal{N}\left(\frac{-\gamma t+x}{\sqrt{t}}\right)+e^{(\nu+\gamma)x}\mathcal{N}\left(\frac{\gamma t+x}{\sqrt{t}}\right) \\
&= e^{(\nu-\gamma)x}\left[1-\mathcal{N}\left(\frac{\gamma t-x}{\sqrt{t}}\right)+e^{2\gamma x}\mathcal{N}\left(\frac{\gamma t+x}{\sqrt{t}}\right)\right] \\
&= e^{(\nu-\gamma)x}\left[1-\Phi(\gamma,t,x)\right] \qquad (17)
\end{aligned}$$

with $\gamma = \sqrt{2r+\nu^2}$. Therefore,

$$\Psi(a-X_{t_i},T-t_i)-\Psi(a-X_{t_i},t-t_i) = e^{(\nu-\gamma)(a-X_{t_i})}\left\{\Phi(\gamma,t-t_i,a-X_{t_i})-\Phi(\gamma,T-t_i,a-X_{t_i})\right\}.$$

Consequently, on the st $t<\tau$, the value of the rebate is given by

$$R^1(t,T) = \frac{\alpha}{D}\frac{\mathbb{1}_{t<\tau}e^{r(t-t_i)}e^{(\nu-\gamma)(a-X_{t_i})}}{\Phi(t-t_i,a-X_{t_i})}\left\{\Phi(\gamma,t-t_i,a-X_{t_i})-\Phi(\gamma,T-t_i,a-X_{t_i})\right\}.$$

A substitution in the price of defaultable bond leads to the result. △

A comparison between $B_d(t,T)$ and $B^1_d(t,T)$ shows that the former depends on contemporaneous asset value (as indicated by the presence of X_t in (8)), while the latter depends on asset value at the last observation date (through X_{t_i} in (15)). Of course, both bond prices depend on whether the default-triggering barrier has been reached. However, it is clear that when the bondholders have access to $\mathbf{F}^V$, they use more up-to-date information to value default-risky bonds. Also, $B^1_d(t,T)$ has multiple jumps to the $B_d(t,T)$ values at the observation dates prior to default. This is consistent with the empirically observed jumps in the prices of defaultable bonds.

3.5.2 Bond prices when H is the only information

Theorem 3 *For $t_i < t < t_{i+1}$ and $t < T$, the price of the defaultable bond with a maturity date T when the information available on the market is $\mathcal{H}_t$ is given by*

$$\begin{aligned}
\widetilde{B}^1_d(t,T) &= \mathbb{1}_{\tau>t_i}e^{-r(T-t)}\Phi(T-t_i,a-X_{t_i}) \\
&+ \frac{\alpha}{D}\mathbb{1}_{t<\tau}e^{r(t-t_i)}e^{(\nu-\gamma)(a-X_{t_i})}\left\{\Phi(\gamma,t-t_i,a-X_{t_i})-\Phi(\gamma,T-t_i,a-X_{t_i})\right\}. \qquad (18)
\end{aligned}$$

Proof: When bondholders have access only to the filtration **H**, the price of the defaultable bond is

$$\begin{aligned}\widetilde{B}_d^1(t,T) &= E\left(e^{-r(T-t)}\mathbb{1}_{\tau>T} + \frac{\alpha}{D}e^{-r(\tau-t)}\mathbb{1}_{\tau\leq T}\big|\mathcal{H}_t\right)\\ &= e^{-r(T-t)}P(\tau>T|\mathcal{H}_t) + \frac{\alpha}{D}E\left(e^{-r(\tau-t)}\mathbb{1}_{\tau\leq T}\big|\mathcal{H}_t\right)\\ &= \widetilde{Z}^1(t,T) + \widetilde{R}^1(t,T).\end{aligned}$$

The first term on the right-hand side is the value of a zero-recovery defaultable bond in the filtration **H**, while the second term is the value of the rebate in the same filtration. The price of the zero recovery bond is the discounted value of the survival probability in (10)

$$\widetilde{Z}^1(t,T) = \mathbb{1}_{\tau>t_i}e^{-r(T-t)}\Phi(T-t_i, a-X_{t_i}). \tag{19}$$

Using the previous results, on the set $\{t<\tau\}$, the value of the rebate is given by

$$\widetilde{R}^1(t,T) = \frac{\alpha}{D}\mathbb{1}_{t<\tau}e^{r(t-t_i)}e^{(\nu-\gamma)(a-X_{t_i})}\left\{\Phi(\gamma, t-t_i, a-X_{t_i}) - \Phi(\gamma, T-t_i, a-X_{t_i})\right\}$$

and the result follows. △

Both the value of the zero recovery bond and the value of the rebate depend on the probability of survival to the maturity date, given in (10), which is equal to one minus the probability of default prior to the maturity. Figure 4 displays the probability of default prior to the maturity. It is piecewise constant between two observations and jumps at the observation dates with stochastic sizes depending on asset values. These jumps induce jumps in the price of the defaultable bond.

4 Second type of partial information

In this section, we examine default probability and bond prices with a second type of incomplete information, which is a further restriction of that of the first type. As in Duffie and Lando [10], suppose that the bondholders observe asset value at selected discrete times. As pointed out by these authors, such a situation is relevant for a privately held firm, whose stockholders are not allowed to trade on the bond markets and whose bondholders obtain accounting information about its assets. The observation times may correspond to the dates of release of periodic accounting reports, such as balance sheets, profit and loss statements and cash flow statements by the firm. We write

$$\mathbb{T} = \{t_1, t_2, \ldots, t_n\}$$

for the sequence of discrete observation times. In this case, the secondary bond markets receives information only about the contemporaneous asset value rather than the contemporaneous and past asset values as was the case examined in the previous section. It is important to note that we consider the case of perfect observations, while [10] examine the case where the log of asset value is observed in Gaussian noise.

We denote by $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$ the filtration generated by V at dates $t_1, \ldots, t_n$. It follows that $\mathcal{F}_t$ is trivial for $t < t_1$, i.e.,

$$\begin{aligned}
\mathcal{F}_t &= \{\emptyset, \Omega\} \qquad \text{for } t < t_1, \\
\mathcal{F}_t &= \mathcal{F}_{t_1} = \sigma(V_{t_1}) = \sigma(X_{t_1}) \quad \text{for } t_1 \leq t < t_2, \\
\mathcal{F}_t &= \mathcal{F}_{t_2} = \sigma(V_{t_1}, V_{t_2}) = \sigma(X_{t_1}, X_{t_2}) \quad \text{for } t_2 \leq t < t_3,
\end{aligned}$$

and so on. At each observation date t_i, the filtration $\mathbf{F}$ is enlarged with the observation of asset value at that date, that is, $\mathcal{F}_{t_i} = \mathcal{F}_{t_{i-1}} \vee \sigma(V_{t_i})$. Therefore, bondholders have access to a subfiltration of the filtration $\mathbf{H}$ (i.e., $\mathbf{F} \subset \mathbf{H} \subset \mathbf{F}^V$). We denote the conditional default probability with respect to $\mathbf{F}$ by $F_t^2 := P(\tau \leq t | \mathcal{F}_t)$, where the superscript 2 stands for the second type of information.

4.1 On $t < t_1$

As with the first type of information, before t_1 bondholders do not observe anything, and we have

$$F_t^2 = 1 - \Phi(t, a). \tag{20}$$

The default probability is deterministic.

4.2 On $t_1 < t < t_2$

In this interval, we have

$$F_t^2 = P(\tau \leq t | X_{t_1}) = 1 - P(\tau > t | X_{t_1}), \tag{21}$$

where

$$\begin{aligned}
P(\tau > t | X_{t_1}) &= P\left(\inf_{s<t} X_s > a \middle| X_{t_1} \right) \\
&= P\left(\inf_{s<t_1} X_s > a, \inf_{t_1 \leq u < t} X_u > a \middle| X_{t_1} \right) \\
&= E\left(\mathbb{1}_{\inf_{s<t_1} X_s > a} \, P\left(\inf_{t_1 \leq s < t} X_s > a \middle| \mathcal{F}_{t_1}^V \right) \middle| X_{t_1} \right)
\end{aligned}$$

Using (6), we can evaluate the probability inside the expectation

$$P\left(\inf_{t_1 \leq s < t} X_s > a \middle| \mathcal{F}_{t_1}^V \right) = \Phi(t - t_1, a - X_{t_1}).$$

Substituting above, we obtain

$$F_t^2 = 1 - \Phi(t - t_1, a - X_{t_1}) P\left(\inf_{s<t_1} X_s > a \big| X_{t_1}\right). \tag{22}$$

The term $P(\inf_{s<t_1} X_s > a|X_{t_1})$ corresponds to a drifted Brownian bridge. In the following lemma, this term is computed by using the joint law of the infimum and the current position of a Brownian motion with drift.

Lemma 2 *Let $X_t = W_t + \nu t$ and $m_t^X = \inf_{s\le t} X_s$. Then, for $y < 0$, $y < x$*

$$P(m_t^X \le y | X_t = x) = \exp\left(-\frac{2}{t} y(y - x)\right). \tag{23}$$

PROOF: This is a classic result, we provide a demonstration to facilitate understanding. We know that, for $y \le 0$, $y \le x$,

$$P(X_t \ge x, m_t^X \le y) = e^{2\nu y} \mathcal{N}\left(\frac{-x + 2y + \nu t}{\sqrt{t}}\right). \tag{24}$$

Taking a derivative with respect to x, leads to

$$P(X_t \in dx, m_t^X \le y) = \frac{1}{\sqrt{2\pi t}} e^{2\nu y} \exp\left(-\frac{1}{2}\left(\frac{-x + 2y + \nu t}{\sqrt{t}}\right)^2\right) dx\,.$$

By the definition of X, we have

$$P(X_t \in dx) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\left(\frac{x - \nu t}{\sqrt{t}}\right)^2\right) dx.$$

Since

$$P(m_t^X \le y | X_t = x) = \frac{P(X_t \in dx, m_t^X \le y)}{P(X_t \in dx)},$$

from

$$\begin{aligned} 2\nu y - \frac{1}{2}\left(\frac{-x + 2y + \nu t}{\sqrt{t}}\right)^2 + \frac{1}{2}\left(\frac{x - \nu t}{\sqrt{t}}\right)^2 &= 2\nu y - \frac{1}{2t}\left(4y^2 + 4y(-x + \nu t)\right) \\ &= -\frac{2}{t} y(y - x), \end{aligned}$$

the equality follows.

△

Note that the right-hand side of (23) does not depend on ν. This is, in fact, a consequence of Girsanov's theorem. Since $P(\inf_{s<t_1} X_s > a|X_{t_1}) = 1 - P(\inf_{s<t_1} X_s \le a|X_{t_1})$, we can use (23) to evaluate (22).

For $X_{t_1} > a$, we obtain

$$F_t^2 = 1 - \Phi(t - t_1, a - X_{t_1}) \left[1 - \exp\left(-\frac{2}{t_1} a(a - X_{t_1})\right)\right]. \tag{25}$$

The case $X_{t_1} \leq a$ corresponds to default and, therefore, for $X_{t_1} \leq a$, $F_t^2 = 1$.

The process F^2 is continuous and increasing in $[t_1, t_2[$. When t approaches t_1 from above, for $X_{t_1} > a$, $F^2_{t_1^+} = \exp\left[-\frac{2}{t_1} a(a - X_{t_1})\right]$, because $\lim_{t \to t_1^+} \Phi(t - t_1, a - X_{t_1}) = 1$.

For $X_{t_1} > a$, the jump of F^2 at t_1 is

$$\Delta F^2_{t_1} = \exp\left[-\frac{2}{t_1} a(a - X_{t_1})\right] - 1 + \Phi(t_1, a).$$

For $X_{t_1} \leq a$, $\Phi(t - t_1, a - X_{t_1}) = 0$ by the definition of $\Phi(\cdot)$ and

$$\Delta F^2_{t_1} = \Phi(t_1, a).$$

4.3 General observation times $t_i < t < t_{i+1} < T$, $i \geq 2$

For $t_i < t < t_{i+1}$,

$$\begin{aligned} P(\tau > t | X_{t_1}, \ldots, X_{t_i}) &= P\left(\inf_{s \leq t_i} X_s > a \; P(\inf_{t_i \leq s < t} X_s > a | \mathcal{F}_{t_i}) \Big| X_{t_1}, \ldots, X_{t_i}\right) \\ &= \Phi(t - t_i, a - X_{t_i}) P\left(\inf_{s \leq t_i} X_s > a \Big| X_{t_1}, \ldots, X_{t_i}\right). \end{aligned}$$

Write K_i for the second term on the right-hand-side. It can be rewritten as

$$\begin{aligned} K_i &= P\left(\inf_{s \leq t_i} X_s > a \Big| X_{t_1}, \ldots, X_{t_i}\right) \\ &= P\left(\inf_{s \leq t_{i-1}} X_s > a \; P\left(\inf_{t_{i-1} \leq s < t_i} X_s > a \big| X_s : s \leq t_{i-1}, X_{t_i}\right) \Big| X_{t_1}, \ldots, X_{t_i}\right). \end{aligned}$$

An analytical expression for the inside member can be obtained as follows:

$$\begin{aligned} P\left(\inf_{t_{i-1} \leq s < t_i} X_s > a \big| X_s : s \leq t_{i-1}, X_{t_i}\right) &= P\left(\inf_{t_{i-1} \leq s < t_i} X_s - X_{t_{i-1}} > a - X_{t_{i-1}} | X_{t_i}, X_{t_{i-1}}\right) \\ &= \exp\left(-\frac{2}{t_i - t_{i-1}} (a - X_{t_{i-1}})(a - X_{t_i})\right). \end{aligned}$$

Therefore,

$$K_i = K_{i-1} \exp\left(-\frac{2}{t_i - t_{i-1}} (a - X_{t_{i-1}})(a - X_{t_i})\right), \tag{26}$$

which can be solved recursively by using the initial condition K_1 computed in Subsection 4.1.

Hence, the general formula for the conditional default probability F_t^2 reads

$$\begin{aligned} P(\tau \leq t | \mathcal{F}_t) &= 1 \qquad \text{if } X_{t_j} < a \text{ for at least one } t_j,\ t_j < t \\ &= 1 - \Phi(t - t_i, a - X_{t_i}) K_i, \end{aligned}$$

where

$$\begin{aligned} K_i &= \left(1-\exp\left(-\frac{2}{t_1}a(a-X_{t_1})\right)\right) \\ &\quad \left(1-\exp\left(-\frac{2}{t_2-t_1}(a-X_{t_1})(a-X_{t_2})\right)\right) \\ &\quad \dots \\ &\quad \left(1-\exp\left(-\frac{2}{t_i-t_{i-1}}(a-X_{t_{i-1}})(a-X_{t_i})\right)\right). \end{aligned}$$

Accordingly, the **F**-hazard process is given by

$$\begin{aligned} \Gamma^2_t &= \infty \qquad \text{if } X_{t_j} < a \text{ for at least one } t_j,\ t_j < t \\ &= -\ln(\Phi(t-t_i, a-X_{t_i})K_i). \end{aligned}$$

We know that the Doob-Meyer decomposition of F^2 is

$$F^2_t = \zeta^2_t + (F^2_t - \zeta^2_t),$$

where ζ^2 is an **F**-martingale and $F^{2,c} := F^2 - \zeta^2$ is a continuous, hence predictable, increasing process.

Let ζ^2 be the pure jump process defined by

$$\zeta^2_t = \sum_{i, t_i \le t} \Delta F^2_{t_i}.$$

Lemma 3 *The process ζ^2 is an* **F***-martingale.*

Proof: Consider first the times $t_i \le s < t \le t_{i+1}$. In this case, it is obvious that $E(\zeta^2_t|\mathcal{H}_s) = \zeta^2_s$ since $\zeta^2_t = \zeta^2_s = \zeta^2_{t_i}$, which is $\mathcal{H}_s$-measurable.

It suffices to show that $E(\zeta^2_t|\mathcal{F}_s) = \zeta^2_s$ for $t_i \le s < t_{i+1} \le t < t_{i+2}$. In this case, $\zeta^2_s = \zeta^2_{t_i}$ and $\zeta^2_t = \zeta^2_{t_i} + \Delta F^2_{t_{i+1}}$. Therefore,

$$\begin{aligned} E(\zeta^2_t|\mathcal{F}_s) &= E(\zeta^2_{t_i} + \Delta F^2_{t_{i+1}}|\mathcal{F}_s) \\ &= \zeta^2_{t_i} + E(\Delta F^2_{t_{i+1}}|\mathcal{F}_s), \end{aligned}$$

which shows that it is necessary to prove that $E(\Delta F^2_{t_{i+1}}|\mathcal{F}_s) = 0$.

Let $s < u < t_{i+1} < v < t$. Then,

$$E(F^2_v - F^2_u|\mathcal{F}_s) = E(1\!\!1_{u<\tau\le v}|\mathcal{F}_s).$$

When

$$\begin{aligned} &v \to t_{i+1}, \quad v > t_{i+1} \quad \text{and} \\ &u \to t_{i+1}, \quad u < t_{i+1}, \qquad F^2_v - F^2_u \to \Delta F^2_{t_{i+1}}. \end{aligned}$$

It follows that

$$\begin{aligned} E(\Delta F^2_{t_{i+1}}|\mathcal{F}_s) &\to \lim_{\substack{u\to t_{i+1}\\ v\to t_{i+1}}} E(1\!\!1_{u<\tau\leq v}|\mathcal{F}_s) \\ &= E(1\!\!1_{\tau=t_{i+1}}|\mathcal{F}_s) = 0. \end{aligned}$$

△

Duffie and Lando [10] established that when the information on the market is $\mathbf{F}\vee\sigma(t\wedge\tau)$, the time of default is totally inaccessible in this filtration and admits intensity. From the results of Elliott et al. [11], the generalized $\mathbf{F}$-intensity of τ is the process Λ^2 defined as

$$d\Lambda^2_t = \frac{d(F^2_t - \zeta^2_t)}{1-F^2_{t-}}.$$

It can be checked that the process $(F^2_t - \zeta^2_t, t\geq 0)$ is absolutely continuous with respect to Lebesgue measure.

4.4 Bond prices

When the information available on the market is $\mathcal{F}_t \vee \sigma(\tau\wedge t)$, and the reorganization boundary for the equityholders is also α, the price of a defaultable bond with zero recovery is given by

$$\begin{aligned} Z^2(t,T) &= e^{-r(T-t)}P(T<\tau|\mathcal{F}_t\vee\sigma(\tau\wedge t)) \\ &= 1\!\!1_{\tau>t}e^{-r(T-t)}\frac{P(T<\tau|\mathcal{F}_t)}{P(t<\tau|\mathcal{F}_t)} = 1\!\!1_{\tau>t}e^{-r(T-t)}\frac{\Phi(T-t_i,a-X_{t_i})K_i}{\Phi(t-t_i,a-X_{t_i})K_i} \\ &= 1\!\!1_{\tau>t}e^{-r(T-t)}\frac{\Phi(T-t_i,a-X_{t_i})}{\Phi(t-t_i,a-X_{t_i})}, \qquad \text{for } t_i<t<t_{i+1}<T. \end{aligned}$$

If the sequences of observation dates for the two types of information are the same, this is the same as when the information is $\mathcal{H}_t\vee\sigma(\tau\wedge t)$, that is, $Z^1(t,T)=Z^2(t,T)$.

With $\mathcal{F}_t$ information only, the secondary bond market values the zero recovery bond as the discounted value of a survival without default to the maturity date:

$$\widetilde{Z}^2(t,T) = 1\!\!1_{\tau>t_i}e^{-r(T-t)}\Phi(T-t_i,a-X_{t_i})K_i, \qquad \text{for } t_i<t<t_{i+1}<T. \tag{27}$$

It is important to note, that while F_t has a single jump from zero to one at τ, F^1_t and F^2_t have multiple jumps at the observation dates prior to default and these jumps induce jumps in the price of the defaultable bond.

5 Analysis of credit spreads

One of the failures of the structural approach is that it produces counterfactually low spreads for the short-maturity default-risky bonds (see, e.g., Jones et al. [19]). This is a consequence of the fact that τ is a stopping time in the filtration generated by the asset prices and, in a diffusion setting, if asset value is substantially larger than the default barrier and the time to the maturity is short, the probability that it falls to the barrier is small. As a result, the short-maturity credit spreads are quite low, which is in contrast to the empirically observed spreads.

To facilitate the subsequent analysis, we examine only the spreads of a zero-recovery defaultable bond with the different types of information. We write $S^2(t,T)$ for the yield spread of the defaultable bond when the information is $\mathcal{F}_t \vee \sigma(\tau \wedge t)$. It can be computed by taking a log of the ratio of a defaultable to risk-free bond with the same maturity, that is, on the set $\{\tau > t\}$,

$$\begin{aligned} S^2(t,T) &= -\frac{1}{(T-t)} \ln \frac{Z^2(t,T)}{e^{-r(T-t)}} \\ &= -\frac{1}{(T-t)} \ln \left(\frac{\Phi(T-t_i, a-X_{t_i})}{\Phi(t-t_i, a-X_{t_i})} \right). \end{aligned}$$

If the two sequences of observation dates for the two types of information are the same, $S^2(t,T) = S^1(t,T)$ holds, where $S^1(t,T)$ is the yield spread of the defaultable bond when the information is $\mathcal{H}_t \vee \sigma(\tau \wedge t)$. However, we again emphasize that this is only true if the equityholders have the same default threshold as the bondholders.

Since $\Phi(t-t_i, z)$ is a decreasing function of $t-t_i$, the credit spreads are increasing in the length of observation lag $t-t_i$. It is obvious that the spreads are decreasing in the last observation of the log of asset value, that is, a high observed X_{t_i} lead to lower credit spreads. For $X_t = X_{t_i}$, with the two types of information, the short-term credit spreads are higher than in the case of complete information $S(t,T) = -\frac{1}{(T-t)} \ln\left(\mathbb{1}_{\tau>t}\Phi(T-t, a-X_t)\right)$.

At each observation date t_{i+1}, prior to default, the jump in the credit spread with the first type of information is a negative jump, given by

$$\Delta S^1(t_{i+1},T) = -\frac{1}{(T-t_{i+1})} \left[\ln\left(\mathbb{1}_{\tau>t}\Phi(T-t_{i+1}, a-X_{t_{i+1}})\right) - \ln\left(\mathbb{1}_{\tau>t_{i+1}} \frac{\Phi(T-t_i, a-X_{t_i})}{\Phi(t_{i+1}-t_i, a-X_{t_i})}\right)\right].$$

The same is the jump with the second type of information.

It is important to note that only the last observed asset value enters into the formulas for credit spreads. This is not the case when the market has access to **H** and **F** only. Then, assuming the same sequence of observation dates, and using (19),

$$\widetilde{S}^1(t,T) = -\frac{1}{(T-t)} \ln(\mathbb{1}_{\tau \geq t_i}\Phi(T-t_i, a-X_{t_i})).$$

If the current time is close to the last observation time, that is, $t \to t_i$, $\widetilde{S}^1(t,T) \to S^1(t,T)$.

With the second type of information, using (27),

$$\widetilde{S}^2(t,T) = \widetilde{S}^1(t,T) - \frac{1}{(T-t)} \ln K_i.$$

Also, as $t \to t_i$, $\widetilde{S}^2(t,T) \to S^2(t,T)$. However, the spread converges slower, since there is the additional term $-\frac{1}{(T-t)} \ln K_i > 0$. This shows that credit spreads are increasing with decrease of the information about asset value, because $\mathbf{F} \subset \mathbf{H}$. We can conclude that, when the default time is not immediately announced, a firm with $\mathbf{F}$-type of information is less transparent to the the bondholders than a firm with an $\mathbf{H}$-type of information and has higher short-term credit spreads.

6 Comparison of hazard process and intensity methods

In this section, we study the properties of the compensator of the default process. The $\mathbf{F}^i$-martingale hazard process of a random time τ is the predictable increasing process Λ^i, such that $1\!\!1_{\tau \le t} - \Lambda^i_{t\wedge\tau} = M^i_t$ is a martingale in the reference filtration. Since with full information, τ is a $\mathbf{F}^V$-stopping time, $\Lambda_t = D_t = 1\!\!1_{\tau\le t}$ and the compensated martingale $M_t = 0$.

The most interesting cases are when τ is a totally inaccessible stopping time in the reference filtration and the compensated martingale M^i_t, associated with the default process, is different from zero. In order to characterize the default process $(D_t, t \ge 0)$ with the two types of partial information, it is necessary to project it on the incomplete information filtrations $\mathbf{H}$ and $\mathbf{F}$ of the secondary bond market. Such projections would lead to different martingale hazard processes. If we consider the first type of partial information, using the results of Elliott et al. [11], or, more generally, some results from enlargement of filtrations, the $\mathbf{H}$-martingale hazard process of τ is the $\mathbf{H}$-adapted process Λ^1 given by

$$\Lambda^1_t = \int_{]0,t]} \frac{dF^{1,c}_s}{1 - F^1_{s-}}.$$

Moreover, for $t_n < t < t_{n+1}$, from the continuity of $F^{1,c}$

$$\Lambda^1_t = \int_{]0,t]} \frac{dF^{1,c}_s}{1 - F^1_{s-}} = \sum_{i=0}^{n-1} \int_{]t_i,t_{i+1}]} \frac{dF^{1,c}_s}{1 - F^1_{s-}} + \int_{]t_n,t]} \frac{dF^{1,c}_s}{1 - F^1_{s-}}$$

and on $]t_i, t_{i+1}]$, $F^{1,c}$ is differentiable. Therefore, we can write

$$\begin{aligned}
\Lambda^1_t &= \sum_{i=0}^{n-1} \int_{]t_i,t_{i+1}]} \frac{dF^{1,c}_s}{1 - F^{1,c}_{s-} - \zeta^1_{t_i}} + \int_{]t_n,t]} \frac{dF^{1,c}_s}{1 - F^{1,c}_{s-} - \zeta^1_{t_n}} \\
&= \sum_{i=0}^{n-1} - \ln(1 - F^{1,c}_{s-} - \zeta^1_{t_i})\Big|_{t_i}^{t_{i+1}} - \ln(1 - F^{1,c}_{s-} - \zeta^1_{t_n})\Big|_{t_n}^{t} \\
&= -\sum_{i=0}^{n-1} \ln \frac{(1 - F^{1,c}_{t_{i+1}} - \zeta^1_{t_i})}{(1 - F^{1,c}_{t_i} - \zeta^1_{t_i})} - \ln \frac{(1 - F^{1,c}_{t} - \zeta^1_{t_n})}{(1 - F^{1,c}_{t_n} - \zeta^1_{t_n})}.
\end{aligned} \tag{28}$$

The process $F^{1,c}_s$ is differentiable and $dF^{1,c}_s = f^{1,c}_s ds$. Hence, the $\mathbf{H}$-martingale hazard process Λ^1

admits the following integral representation:

$$\Lambda_t^1 = \int_0^t \lambda_s^1 ds,$$

where the $\mathbf{H}$-intensity of τ is $\lambda_s^1 = \frac{f_s^{1,c}}{1-F_{s-}^{1,c}-\zeta_{t_i}^1}$ on the interval $]t_i, t_{i+1}[$.

One can use this result for the valuation of defaultable bonds. However, when the information set of bondholders includes also observations of τ, the process

$$\begin{aligned} C_t^1 &= e^{-r(T-t)} E\left[e^{\Lambda_t^1 - \Lambda_T^1} \middle| \mathcal{H}_t \vee \sigma(\tau \wedge t) \right] \\ &= \left[e^{-\int_t^T (r+\lambda_s^1) ds} \middle| \mathcal{H}_t \vee \sigma(\tau \wedge t) \right] \end{aligned}$$

admits a jump at time τ. It is necessary to compute the jump ΔC_τ^1 in order to obtain the price of a contingent claim. Indeed, using the results of [6] and [17],

$$Z^1(t,T) = 1\!\!1_{\tau \geq t} \left(C_t^1 - E\left(e^{-r(\tau-t)} \Delta C_\tau^1 \middle| \mathcal{H}_t \vee \sigma(\tau \wedge t) \right) \right). \tag{29}$$

However, these computations in the intensity framework are long and non-trivial.

In contrast, by the hazard process approach, we compute this jump in a direct way and obtaining the price $Z^1(t,T)$ in the filtration $(\mathcal{H}_t \vee \sigma(\tau \wedge t))_{t\geq 0}$is much simpler, as indicated by Theorem 1. Working with the hazard process Γ^1 in the filtration $\mathbf{H}$ only leads to a price of a defaultable bond with zero recovery

$$\begin{aligned} \widetilde{Z}^1(t,T) &= 1\!\!1_{t<\tau} e^{-r(T-t)} E\left\{ \exp(\Gamma_t^1 - \Gamma_T^1) \middle| \mathcal{H}_t \right\} \\ &= 1\!\!1_{t<\tau} e^{-r(T-t)} \frac{1}{1-F_t^1} E\{1 - F_T^1 | \mathcal{H}_t\} \\ &= 1\!\!1_{t<\tau} e^{-r(T-t)} \frac{1}{1-F_t^1} \{E(1 - F_T^{1,c} - \zeta_T^1 | \mathcal{H}_t)\} \\ &= 1\!\!1_{t<\tau} e^{-r(T-t)} \frac{1}{1-F_t^{1,c} - \zeta_t^1} \{1 - E(F_T^{1,c} | \mathcal{H}_t) - \zeta_t^1\}, \end{aligned} \tag{30}$$

where the fourth equality follows because the process ζ^1 is an $\mathbf{H}$-martingale.

7 Conclusion

As our results in Section 2 show, bond prices in the classical structural models with continuous and perfect observation of asset value are continuous. Yet, market prices of defaultable bonds display frequent jumps at times of release of important information about the credit quality of the issuers. In our view, the two partial information models display realistic information flows to the secondary bond market. These models also yield more realistic bond prices. The discrepancies among the partial information and full information prices are increasing with the observation lags. We also find that credit spreads are increasing with the reduction of the information filtrations and with the increases of the observation lags. The models described here could serve as a basis for a study of the two valuation approaches with more complicated information structures (e.g., combining discrete with continuous but noisy observations).

As discussed in the previous sections, we have the following containment relation between the different filtrations: $\mathbf{F}^V \supset \mathbf{H} \supset \mathbf{F}$. The default time changes from a predictable stopping time in the filtration $\mathbf{F}^V$ to a totally inaccessible time when information of the secondary bond market is reduced to $\mathbf{H}$ or $\mathbf{F}$. Consequently, the default process compensated by the martingale hazard process is the identically zero constant martingale with full information. With partial information, the compensated martingale is not equal to zero. Since in these cases the martingale hazard process is absolutely continuous, τ admits risk-neutral default arrival intensities and the well developed reduced form models could be used to model the default event. However, with each of the two information-constrained models, the hypothesis (H) and condition (G) are not satisfied. In fact, default probability in the two partial information filtrations could have negative jumps. In such settings, the hazard approach allows for a direct evaluation of conditional expectations while the intensity approach requires an evaluation of the jump at τ of the conditional expectation of an exponential of the martingale hazard process. With the two types of information, intensities depend on asset value at the observation times. Another observation scheme (e.g. cash-flow induced default) can make them dependent on other firm-specific information. However, the hazard process approach offers similar flexibility, because default hazard process also depends on these firm-specific characteristics while, at the same time, computations based on it are much simpler and direct.

If the information to bondholders is augmented with announcement of default (forced by equityholders) as it occurs, we have the following containment relation: $\mathbf{F}^V \supset (\mathbf{H} \vee \mathbf{D}) \supset (\mathbf{F} \vee \mathbf{D})$. Bond prices and credit spreads in the two expanded asset subfiltrations, $\mathbf{H} \vee \mathbf{D}$ and $\mathbf{F} \vee \mathbf{D}$, are the same. However, this is true only if the equityholders have the same default threshold as the bondholders and the sequences of observation dates for the two types of information are the same. In such cases, the last observed asset value is a sufficient statistics for the prices of defaultable bonds and credit spreads. As pointed out in Section 6, prior to default, there is a downward jump in the credit spreads at the next observation time to the spread with full information. We can conclude that firms in good financial conditions should have strong incentives to release more information to their bondholders since this would lower the costs of their new debt financing. Even for firms with moderate financial results, it may be beneficial to release more information for the values of their assets to the secondary bond market.

In addition, discontinuities associated with asset value process are also important to the extend that, in practice, stock and defaultable bond prices have common jumps at times of release of important information about the issuer. However, because of the mathematical complexity of the first passage problem in a Levy setting, analytical solutions for default probability and bond prices are still not available, even in the base case of full information. Finding solutions with partial information remains a major challenge for future research.

It is important to note that the information sets of investors in the secondary markets for credit-risky bonds include additional non firm-specific information. Such information could be economy-wide information such as the short-term interest rate, industrial production, and the exchange rate. Another example of relevant non firm-specific information is the case of extraneous risk, in which market perception of asset value dynamics changes with default of another firm within the same industry (market). In practice, defaultable bonds are held in large portfolios by institutional investors. Default of a single name in their portfolio may cause them to revise their beliefs about the likelihood of default of the other issuers. For example, in the case of a single default, followed by a debt crisis, the other firms may not be able to roll-over the existing bonds and may experience financial difficulties. Then, the distributional properties of τ depend on default times of the other firms. Therefore, information about the the assets and liabilities of other firms also plays a key role in determining default probability of the reference entity. It is a major challenge for future research to provide valuations of defaultable bonds with such expanded information sets, consistent with the total information sets of the market. In such models, the structure of the different asset value processes and the links between would be very important.

References

[1] Bielecki, T., and M. Rutkowski, 2002, *Credit Risk: Modeling, Valuation and Hedging*, Springer-Verlag, Berlin-Heidelberg-New York.

[2] Black, F., and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637-654.

[3] Black, F., and J. Cox, 1976, "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions," *Journal of Finance*, 31, 351-367.

[4] Blanchet-Scalliet, C., and M. Jeanblanc, 2004, "Hedging Defaultable Contingent Claims," *Finance and Stochastics*, 8, 145 - 159.

[5] Cetin, U., R. Jarrow, P. Protter and Y. Yildirim, 2004, "Modeling Default Risk with Partial Information," *Annals of Applied Probability*, 14, 1167-1178.

[6] Duffie, D., M. Scroder and C. Skiadas, 1996, "Recursive Valuation of Defaultable Securities and the Timing of Resolution of Uncertainty," *Annals of Applied Probability*, 6, 1075-1090.

[7] Duffie, D., 1998a, "First-to-Default Valuation," Working Paper, Institut de Finance, University of Paris, Dauphine and Graduate School of Business, Stanford University.

[8] Duffie, D., 1998b, "Defaultable Term Structure Models with Fractional Recovery of Par," Working Paper, Graduate School of Business, Stanford University.

[9] Duffie, D., and K. Singleton, 1999, "Modeling Term Structures of Defaultable Bonds," *Review of Financial Studies*, 12, 687-720.

[10] Duffie D., and D. Lando, 2001, "Term Structure of Credit Spreads with Incomplete Accounting Information," *Econometrica*, 69, 633-664.

[11] Elliott, R., M. Jeanblanc, and M. Yor, 2000, "On Models of Default Risk," *Mathematical Finance*, 10, 179-195.

[12] Giesecke, K., 2004, "Correlated Default with Incomplete Information," *Journal of Banking and Finance*, 28, 1521-1545.

[13] Gupton, G., Finger, C., and M. Bhatia, 1997, CreditMetrics – Technical Document, RiskMetrics Group, New York.

[14] Harrison, J. M., and S. Pliska, 1981, "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Their Applications*, 11, 215-260.

[15] Jarrow, R., and S. Turnbull, 1995, "Pricing Options on Financial Securities Subject to Default Risk," *Journal of Finance*, 50, 53-86.

[16] Jeanblanc, M., and M. Rutkowski, 2000a, "Modelling of Default Risk: An Overview," in *Mathematical Finance: Theory and Practice*, Higher Education Press, Beijing, 171-269.

[17] Jeanblanc, M., and M. Rutkowski, 2000b, "Modelling of Default Risk: Mathematical Tools," working paper, Université d'Evry.

[18] Jeanblanc, M., M. Yor and M. Chesney, 2003, *Mathematical Methods for Financial Markets*, forthcoming, Springer-Verlag, Berlin-Heidelberg-New York.

[19] Jones, E., S. Mason, and E. Rosenfeld, 1984, "Contingent Claims Analysis of Corporate Capital Structures: An Empirical Investigation," *Journal of Finance*, 39, 611-625.

[20] Kusuoka, S., 1999, "A Remark on Default Risk Models," *Advances in Mathematical Economics*, 1, 69-82.

[21] Lando, D., 1998, "On Cox Processes and Credit Risky Securities," *Review of Derivatives Research*, 2, 99-120.

[22] Leland, H., 1994, "Debt Value, Bond Covenants, and Optimal Capital Structure," *Journal of Finance*, 49, 1213-1252.

[23] Leland, H., and K. Toft, 1996, "Optimal Capital Structure, Endogenous Bankruptsy and the Term Structure of Credit Spreads," *Journal of Finance*, 51, 987-1019.

[24] Leland, H., 1998, "Agency Costs, Risk Management, and Capital Structure," *Journal of Finance*, 53, 1213-1243.

[25] Longstaff, F., and E. Schwartz, 1995, "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt," *Journal of Finance*, 50, 789-819.

[26] Madan, D., and H. Unal, 1998, "Pricing the Risks of Default," *Review of Derivatives Research*, 2, 121-160.

[27] Merton, R., 1974, "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance*, 29, 449-470.

[28] Nakagawa, 2001, "A Filtering Model of Default Risk," *Journal of Mathematical Sciences University of Tokyo*, 8, 107-142.

[29] Rutkowski, M., 2002, "First Passage Time Structural Models with Interest Rate Risk," in Proceedings of the Conference on Stochastic Processes, H.J. Engelbert ed., North Holland.

[30] Zhou, C., 2001, "The Term Structure of Credit Spreads with Jump," *Journal of Banking and Finance*, 25, 2015-2040.