Some remarks on the Navier-Stokes equations in $\mathbb{R}^3$

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Abstract : We study various existence and uniqueness results for solutions of the Navier-Stokes equations in connection with function spaces related to real harmonic analysis.

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In this paper, we are mainly interested in reviewing recent results concerning weak solutions of the Navier-Stokes equations which belong to $\mathcal{C}([0,+\infty), L^3(\mathbb{R}^3))$. Such solutions were first studied by T. Kato [KAT] in 1984. L^3 estimates have been used as well in the study of Leray's weak solutions, leading to the uniqueness theorem of H. Sohr and W. Von Wahl [WAH].

A new impetus has been given to the study of such solutions by the recent book of M. Cannone [CAN]. Following the idea of P. Federbush [FED] that a good time-frequency analysis of the fluctuation term (see below) could give new existence theorems, M. Cannone uses the Littlewood-Paley decomposition instead of divergence-free wavelet bases (used in [FED]) to provide a precise analysis of the fluctuation. In [FLT], we show how this *frequential* analysis of the fluctuation, in terms of a *Besov norm*, could solve the uniqueness problem for solutions in $\mathcal{C}([0,+\infty), L^3(\mathbb{R}^3))$. This frequential approach is highlighted by a result of Y. Le Jan and A. S. Sznitman [LJS], which shows how to deal with the Fourier transform of the Navier-Stokes equations. A short time after our result, Y. Meyer proposed a new proof using only *spatial* estimates, in terms of a *Lorentz norm* [MEY 2]. We find very striking the fact that there is yet no actual *time-frequency* analysis of the Navier-Stokes equations.

We will show as well the relation between our result and the Sohr and Von Wahl theorem. This implies introducing energy estimates for L^3 solutions, a technique which is useful for proving existence of weak solutions for initial value with infinite energy [LEM].

Besides, we will pay a few words on the self-similar solutions described by M. Cannone [CAN] and some related asymptotic results of F. Planchon [PLA].

1. The Navier-Stokes equations in $\mathbb{R}^3$.

The classical *Navier-Stokes equations* describe the motion of a Newtonian fluid; we consider only the case when the fluid is viscous, homogeneous, incompressible and fills the entire space and is not submitted to external forces; then, the equations describing the evolution of the motion $\vec{u}(t,x)$ of the fluid element at time t and position x are given by:

$$\rho\ \partial_t \vec{u} = \mu\ \Delta\vec{u} - \rho\ (\vec{u}.\vec{\nabla})\ \vec{u} - \vec{\nabla}p \tag{1}$$

$$\vec{\nabla}.\vec{u} = 0 \tag{2}$$

ρ is the (constant) *density* of the fluid, and we may assume with no loss of generality that $\rho = 1$. μ is the *viscosity* coefficient, and we may assume as well that $\mu = 1$. p is the (unknown) *pressure*, whose action is to maintain the divergence of $\vec{u}$ to be 0 (this *divergence free* condition expresses the incompressibility of the fluid). p can be expressed as a function of $\vec{u}$, provided that $\vec{u}$ vanishes at infinity; indeed, taking the divergence of (1), we obtain:

$$\Delta p = -\vec{\nabla}.(\vec{u}.\vec{\nabla})\ \vec{u} \tag{3}$$

and this formula allows one to eliminate p in many cases: indeed, formula (3) determines p up to an harmonic function and the requirement that $\vec{u}$ remains equal to 0 at infinity allows one to get rid of the contribution this harmonic term could have given to $\partial_t\vec{u}$.

Since $\vec{\nabla}.\vec{u} = 0$, the equation (1) can be rewritten as well as:

$$\partial_t\vec{u} = \Delta\vec{u} - \vec{\nabla}.(\vec{u}\otimes\vec{u}) - \vec{\nabla}p \tag{4}$$

and we shall thus study solutions of (4) and (2) which vanish in a suitable sense at $x = \infty$. Moreover, we complement (4) and (2) (and the boundary condition $\vec{u} = 0$ at infinity) by an initial value $\vec{u} = \vec{u}_0$ at $t = 0$, and we study solutions for $t > 0$.

There is a huge literature on the mathematical theory of the Navier-Stokes equations; classical references are the books by R. Temam [TEM], O.A. Ladyzhenskaya [LAD] or P. Constantin and C. Foias [COF]; a more recent reference is the book by P.L. Lions [LIO].

A *classical solution* of (2) and (4) is a time-dependent vector field which has one bounded and continuous time derivative and two bounded and continuous space derivatives. Existence of such solutions was obtained first by C.W. Oseen in the beginning of the century [OSE]. Following the Picard iteration scheme, $\vec{u}$ is obtained as a fixed point of the associated integral transform: one considers first the linear problems (P1) and (P2):

$$(P1) \qquad \vec{\nabla}.\vec{u_1} = 0,\ \partial_t \vec{u_1} = \Delta \vec{u_1} - \vec{\nabla} p_1,\ \vec{u_1}(0,.) = \vec{u}_0$$

$$(P2) \qquad \vec{\nabla}.\vec{u_2} = 0,\ \partial_t \vec{u_2} = \Delta \vec{u_2} + \vec{f} - \vec{\nabla} p_2,\ \vec{u_2}(0,.) = 0$$

The first one is easily solved by $\vec{u_1} = e^{t\Delta}\ \vec{u}_0 = W_t * \vec{u}_0$ where $W_t(x) = (4\pi t)^{-3/2} e^{\frac{-|x|^2}{4t}}$ is the heat kernel on $\mathbb{R}^3$. The second one is solved by the integral formula: $\vec{u_2} = \int_0^t \ \mathbb{P} e^{(t-s)\Delta} \vec{f}(s)\ ds = \int_0^t O_{t-s} * \vec{f}(s)\ ds$ where $\mathbb{P}$, the *Leray-Hopf projection operator* on divergence free vector fields, is formally given as the operator matrix $Id_3 - \frac{1}{\Delta}\vec{\nabla} \otimes \vec{\nabla}$ and where the *Oseen kernel* $O_t(x) = t^{-3/2}\ O(\frac{x}{\sqrt{t}})$ is a matrix function given by $O_{k,l(x)} = (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{|\xi|^2 \delta_{k,l} - \xi_k \xi_l}{|\xi|^2} e^{-|\xi|^2}\ e^{ix.\xi}\ d\xi$. The functions $O_{k,l}$ are analytical on $\mathbb{R}^3$ and satisfy for all $\alpha \in \mathbb{N}^3$ the estimate $|\frac{\partial^{|\alpha|}}{\partial x^\alpha} O_{k,l}(x)| \leq C_\alpha (1+|x|)^{-(3+|\alpha|)}$. Then, we seek a solution $\vec{u}$ of the Navier-Stokes equations as a solution of the integral equation:

$$(5) \qquad \vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t (\vec{\nabla} \otimes O_{t-s}) * (\vec{u} \otimes \vec{u})\ ds$$

where the j-th component of $(\vec{\nabla} \otimes O_{t-s}) * (\vec{u} \otimes \vec{u})$ is given by $\sum_k \sum_l \frac{1}{(t-s)^2} \partial_k O_{j,l}(\frac{x}{\sqrt{t-s}}) * (u_k(s,x) u_l(s,x))$. Following P. Federbush, we call $e^{t\Delta}\vec{u}_0$ the *tendency* and $\int_0^t (\vec{\nabla} \otimes O_{t-s}) * (\vec{u} \otimes \vec{u})\ ds$ the *fluctuation*.

Existence of a classical solution, whatever regular the initial value may be, is guaranteed only for a finite interval of time. If we look for *global solutions*, i.e. solutions defined for every $t > 0$, we have to consider *weak solutions*, which were introduced in 1934 by J. Leray [LER] for $\vec{u}_0 \in (L^2)^3$, or *mild solutions*, which were introduced by T. Kato in the early 60's for the equations on a bounded domain and an initial value in a Sobolev space [FUK] and in 1984 [KAT 1] for the equations on the whole space and $\vec{u}_0 \in (L^3)^3$. Weak solutions satisfy equations (2) and (4) in the sense of distributions and mild solutions satisfy the integral equation (5). As a matter of fact, the notion of mild solutions refers to an abstract approach of evolution equations in a Banach space, but we shall stick to a more simple definition, because in the whole space the Oseen kernel is quite explicit and so the integral equation can be handled in a more direct way.

2. Weak and mild solutions

We begin by defining weak and mild solutions of the Navier-Stokes equations on the whole space.

Definition 1: [Weak solutions]

A *weak solution* of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^3$ is a distribution vector field $\vec{u}(t,x)$ in $(\mathcal{D}'((O,T) \times \mathbb{R}^3)^3$ such that:

a) $\vec{u}$ is locally square integrable on $(0,T) \times \mathbb{R}^3$

b) $\vec{\nabla}.\vec{u} = 0$

c) $\exists p \in \mathcal{D}'((0,T)\times \mathbb{R}^3)\ \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.(\vec{u}\otimes \vec{u}) - \vec{\nabla} p$

For defining mild solutions, we consider only uniformly locally square integrable vector fields:

Definition 2: [Uniformly locally integrable functions]

For $1 \le p < \infty$, we define F_p of functions which are uniformly locally in L^p by: $f \in F_p$ if and only if: $\sup_{x\in\mathbb{R}^3} \int_{|y-x|\le 1} |f(y)|^p\, dy \ < +\infty$

It is normed by:

$$\|f\|_{F_p} = \sup_{x\in\mathbb{R}^3} (\int_{|y-x|\le 1} |f(y)|^p\, dy)^{\frac{1}{p}}$$

Moreover we define E_p as the closure in F_p of $\mathcal{D}(\mathbb{R}^3)$; this is the space of the elements in F_p which vanish at infinity and it is characterized by $\lim_{x\to\infty} \int_{|y-x|\le 1} |f(y)|^p\, dy = 0$.

Definition 3: [Mild solutions]

A *mild solution* of the Navier-Stokes equations on $(0,T^*)\times\mathbb{R}^3$ with initial value $\vec{u}_0 \in (\mathcal{S}'(\mathbb{R}^3))^3$, $\vec{\nabla}.\vec{u}_0 = 0$, is a vector field $\vec{u}(t,x)$ in $(\mathcal{D}'((0,T^*)\times\mathbb{R}^3)^3$ such that:
a) $\vec{u} \in \cap_{T<T^*} L^2((0,T),(F_2)^3)$
b) $\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t (\vec{\nabla}\otimes O_{t-s}) * (\vec{u}\otimes\vec{u})\, ds$.

The point is that when $v \in F_1$ then $\partial_j v$ belongs to a Banach space Y of tempered distributions whose norm is shift invariant (so that convolution with functions in L^1 operate boundedly on Y) and on which the Riesz transforms $\frac{\partial_j}{\sqrt{-\Delta}}$ operate boundedly (for instance, Y can be defined as the space of distributions which split in a sum of a low-frequency term in $\dot{B}_\infty^{-1,\infty}$ and a high-frequency term in $\dot{B}_\infty^{-4,\infty}$). Thus, if $\vec{u} \in \cap_{T<T^*} L^2((0,T),(F_2)^3)$ we find that $\vec{w} = \int_0^t (\vec{\nabla}\otimes O_{t-s}) * (\vec{u}\otimes\vec{u})\, ds$ is well defined and belongs to $\cap_{T<T^*} L^\infty((0,T),(Y)^3)$; moreover it satisfies $\partial_t \vec{w} = \Delta\vec{w} + \mathbb{P}(\vec{\nabla}.(\vec{u}\otimes\vec{u}))$ and $\lim_{t\to 0} \|\vec{w}\|_Y = 0$. See [FLT] for further details.

The two notions of weak and mild solutions are equivalent for solutions which vanish at infinity [FLT]:

Proposition1: Let $\vec{u} \in \cap_{T<T^*} L^2((0,T),(F_2)^3)$. Then the following assertions are equivalent:

(A) $\vec{u}$ satisfies the following requirements:
a) $\vec{\nabla}.\vec{u} = 0$
b) $\partial_t \vec{u} = \Delta\vec{u} - \mathbb{P}(\vec{\nabla}.(\vec{u}\otimes\vec{u}))$
where $\mathbb{P}$ is the Leray-Hopf projection operator on divergence free vector fields.

(B) $\vec{u}$ is a mild solution of the Navier-Stokes equations: $\exists \vec{u}_0 \in (\mathcal{S}'(\mathbb{R}^3))^3$ such that
a) $\vec{\nabla}.\vec{u}_0 = 0$
b) $\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}(\vec{\nabla}.(\vec{u}\otimes\vec{u}))\, ds$

If we assume that $\vec{u}$ vanishes at infinity (i.e. we assume more precisely that $\vec{u} \in \cap_{T<T^*} L^2((0,T),(E_2)^3)$) then (A) and (B) are equivalent to the following assertion:

(C) $\vec{u}$ is a weak solution of the Navier-Stokes equation:

a) $\vec{\nabla}.\vec{u} = 0$
b) $\exists p \in \mathcal{D}'((0,T^*) \times \mathbb{R}^3)$ $\partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p$

Sketch of the proof:
Equivalence between (A) and (B) is a simple exercise on distributions (derivatives and integrals are weakly defined: apply equations to a test function in $\mathcal{C}^\infty_{comp}(\ (0,T^*) \times \mathbb{R}^3))$. For equivalence between (A) and (C), we have to prove that we may apply $\mathbb{P}$ to (C) and that it gives (A); we may apply $\mathbb{P}$ to high frequency components without any problem: let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ be such that $|\xi| \leq \frac{1}{2} \Rightarrow \varphi(\xi) = 1$ and $|\xi| \geq 1 \Rightarrow \varphi(\xi) = 0$ and let S_j be defined as the Fourier multiplier $\mathcal{F}(S_j f) = \varphi(\xi/2^j)\mathcal{F}f$; then $\mathbb{P}(Id - S_j)$ operates boundedly on $\mathcal{S}^3$ and $(\mathcal{S}')^3$; thus (C) gives $\partial_t(Id - S_j)\vec{u} = \Delta(Id - S_j)\vec{u} - (Id - S_j)\mathbb{P}(\vec{\nabla}.(\vec{u} \otimes \vec{u}))$ and we may conclude since for $u \in E_2$ we have $S_j u \to 0$ weakly in $\mathcal{S}'$ when $j \to -\infty$ while $\|S_j \mathbb{P}(\vec{\nabla}.(\vec{u} \otimes \vec{u}))\|_\infty \leq C 2^j \|\vec{u}\|^2_{F_2}$. •

Almost all the solutions we shall consider in this paper are weak/mild solutions in the sense of Proposition 1 : for instance, the Leray solutions (which are in $L^\infty((L^2)^3)$), the Kato solutions (which are in $L^\infty((L^p)^3)$ for some $p \in [3,\infty)$), the Cannone self-similar solutions (which satisfy for some $p \in (3,\infty)$ $t^{1/2-3/2p}\vec{u} \in L^\infty((L^p)^3)$), the Meyer solutions in $L^\infty((L^{3,\infty})^3)$, and so on .

Since a mild solution is defined as a fixed point of an integral transform, one may expect some uniqueness results. The following one is quite obvious:

Definition 4: [Regular mild solutions]
A mild solution $\vec{u}$ of the Navier-Stokes equations on $(0,T^*) \times \mathbb{R}^3$ is called *regular* if it satisfies the following requirements:
a) $\vec{u} \in \cap_{T<T^*} L^\infty((0,T), (F_2)^3)$
b) $\vec{u} \in \cap_{0<\epsilon<T<T^*} L^\infty((\epsilon,T), (L^\infty)^3)$
c) $\lim_{t\to 0} \sqrt{t}\ \|\vec{u}(t,.)\|_\infty = 0$

Proposition2: Let $\vec{u}_0 \in (F_2)^3, \vec{\nabla}.\vec{u}_0 = 0$. Then there exists at most one regular mild solution with $\vec{u}_0$ as initial value.

Proof:
Define $B(\vec{f},\vec{g}) = \int_0^t (\vec{\nabla} \otimes O_{t-s}) * (\vec{f} \otimes \vec{g})\, ds$. Let $\vec{u}$ and $\vec{v}$ be two regular mild solutions: $\vec{u} = e^{t\Delta}\ \vec{u}_0 - B(\vec{u},\vec{u})$ and $\vec{v} = e^{t\Delta}\ \vec{u}_0 - B(\vec{v},\vec{v})$ so that $\vec{w} = \vec{u} - \vec{v}$ is solution of the equation $\vec{w} = -B(\vec{w},\vec{v}) - B(\vec{u},\vec{w})$. Now, we use the fact that the norm of F_2 is shift invariant and the fact that $\vec{\nabla} \otimes O_{t-s}$ is a convolution matrix with functions whose L^1 norm is $O(1/\sqrt{t-s})$ to get:

$$\|\vec{w}(t,.)\|_{F_2} \leq C \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}}\, ds \sup_{0<s<t} \|\vec{w}(s,.)\|_{F_2} \sup_{0<s<t} \sqrt{s}(\|\vec{u}(s,.)\|_\infty + \|\vec{v}(s,.)\|_\infty)$$

Since $\int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}}\, ds = \pi$, we get local uniqueness (as long as $\pi\ C\ \sup_{0<s<t} \sqrt{s}(\|\vec{u}(s,.)\|_\infty + \|\vec{v}(s,.)\|_\infty) < 1$): indeed, we then have $\sup_{0<s<t} \|\vec{w}(s,.)\|_{F_2} \leq \pi\ C\ \sup_{0<s<t} \sqrt{s}(\|\vec{u}(s,.)\|_\infty + \|\vec{v}(s,.)\|_\infty) \sup_{0<s<t} \|\vec{w}(s,.)\|_{F_2}$, which gives $\sup_{0<s<t} \|\vec{w}(s,.)\|_{F_2} = 0$. Then, global uniqueness follows by looking at the largest t such that $\vec{u} = \vec{v}$ on $[0,t)$ and applying local uniqueness again on $[t, t+\delta t]$: since F_2 is a dual space and since $\vec{u}$ and $\vec{v}$ are weakly continuous into $(\mathcal{S}'(\mathbb{R}^3))^3$, we find $\vec{u}(t,.) = \vec{v}(t,.) \in (F_2)^3$ while $\vec{u}$ and $\vec{v}$ are mild solutions on (t,T^*) with $\vec{u}(t,.)$ as initial value; moreover, since $\vec{u}$ and $\vec{v}$ are bounded on (t,T) for all $T < T^*$, they are regular mild solutions. •

3. Existence of regular mild solutions

Existence of regular mild solutions has been proved for an initial value $\vec{u}_0$ in many spaces: Sobolev spaces H^s for $s \geq 1/2$ [FUK], Lebesgue spaces L^p for $p \in [3, \infty)$ [KAT 1], Besov spaces, Morrey-Campanato spaces and others. We will focus on the Morrey-Campanato space $M_{2,3}$:

Definition 5: [Morrey-Campanato space]

For $1 \leq p \leq q \leq \infty$, the Morrey-Campanato space $M_{p,q}$ is defined as the spaces of functions f which are locally in L^p and such that:

$$\sup_{x\in\mathbb{R}^3,\ 0<r\leq 1} \frac{1}{r^{3(\frac{1}{p}-\frac{1}{q})}} \Big(\int_{|x-y|\leq r} |f(y)|^p\, dy\Big)^{\frac{1}{p}} < \infty \tag{6}$$

where the left hand side of inequality (6) is the norm of f in $M_{p,q}$.

The homogeneous Morrey-Campanato space $\dot{M}_{p,q}$ is defined in the same way, by taking the supremum on all $r \in (0, \infty)$ instead of $r \in (0, 1]$.

It is easy to check that $M_{p,p} = F_p$, $F_q \subset M_{p,q}$ and $M_{p,\infty} = L^\infty$. Existence of solutions in Morrey-Campanato spaces has been discussed by several authors (see for instance [CAN], [FED], [FLT], [KAT 2], [KOY], [TAY]). We shall specialize our attention to two subspaces of $M_{2,3}$:

Theorem 1: [Mild solutions in $M_{2,3}$]

Let $\vec{u}_0 \in (M_{2,3})^3, \vec{\nabla}.\vec{u}_0 = 0$. Then:

a) $\vec{U} = e^{t\Delta}\vec{u}_0$ satisfies $\sup_{t>0} \|\vec{U}\|_{M_{2,3}} + \sup_{0<t\leq 1} \sqrt{t}\|\vec{U}\|_\infty < C\ \|\vec{u}_0\|_{M_{2,3}}$.

b) If $\lim_{t\to 0} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$, then there exists $T^* = T^*(\vec{u}_0) > 0$ and a regular mild solution $\vec{u}$ on $(0, T^*) \times \mathbb{R}^3$ with initial value $\vec{u}_0$. Moreover, $\vec{u} \in \cap_{T<T^*} L^\infty((0,T), (M_{2,3})^3)$.

c) There exists two constants $\epsilon_0, \epsilon_1 > 0$ such that if $\vec{u}_0 \in (\dot{M}_{2,3})^3$ and $\|\vec{u}_0\|_{\dot{M}_{2,3}} < \epsilon_0$ then there exists a mild solution $\vec{u}$ on $(0, \infty) \times \mathbb{R}^3$ with initial value $\vec{u}_0$ such that $\sup_{t>0} \|\vec{u}\|_{\dot{M}_{2,3}} < \infty$ and $\sup_{t>0} \sqrt{t}\|\vec{u}\|_\infty < \epsilon_1$. Moreover such a solution is unique.

d) Under the hypotheses of c), if $\|\vec{u}_0\|_{\dot{M}_{2,3}} < \epsilon_0$ and $\vec{u}_0$ is homogeneous ($\forall\lambda > 0\ \vec{u}_0(\lambda x) = \frac{1}{\lambda}\vec{u}_0(x)$) then $\vec{u}$ is selfsimilar: $\forall\lambda > 0\ \vec{u}(\lambda^2 t, \lambda x) = \frac{1}{\lambda}\vec{u}(t, x)$.

Sketch of the proof:

The proof is by now quite classical. It is enough to check that the operator B (defined as $B(\vec{f}, \vec{g}) = \int_0^t (\vec{\nabla} \otimes O_{t-s}) * (\vec{f} \otimes \vec{g})\, ds$) is bilinear continuous on E and on $\dot{E}$ where

$$E = \{\vec{f} \in L^\infty((0,T), (M_{2,3})^3) / \sup_{0<t<T} \sqrt{t}\ \|\vec{f}\|_\infty < \infty,\ \lim_{t\to 0} \sqrt{t}\ \|\vec{f}\|_\infty = 0\}$$

$$\dot{E} = \{\vec{f} \in L^\infty((0,\infty), (\dot{M}_{2,3})^3) / \sup_{0<t} \sqrt{t}\ \|\vec{f}\|_\infty < \infty\}$$

Indeed, we have the inequalities $\|(\vec{\nabla} \otimes O_{t-s}) * (\vec{f} \otimes \vec{g})\|_{M_{2,3}} \leq C\frac{1}{\sqrt{t-s}}\|(\vec{f} \otimes \vec{g}\|_{M_{2,3}} \leq C\frac{1}{\sqrt{t-s}}\|\vec{f}\|_{M_{2,3}}\|\vec{g}\|_\infty$ and $\|(\vec{\nabla} \otimes O_{t-s}) * (\vec{f} \otimes \vec{g})\|_\infty \leq C \inf(\frac{\|\vec{f}\otimes\vec{g}\|_\infty}{\sqrt{t-s}}, \frac{\|\vec{f}\otimes\vec{g}\|_{M_{2,3}}}{t-s}) \leq C \inf(\frac{\|\vec{f}\|_\infty}{\sqrt{t-s}}, \frac{\|\vec{f}\|_{M_{2,3}}}{t-s})\|g\|_\infty$, from which we get $\|B(\vec{f}, \vec{g})\|_E \leq C_0\|f\|_E \sup_{0<s<T} \sqrt{s}\|g(s, .)\|_\infty$ where C_0 does not depend on T.

Then a contraction argument shows existence and uniqueness of the solutions. We show how to prove point b) (point c) is shown in a very similar way and point d) is obvious since the norm of $\dot{M}_{2,3}$ is invariant through the transformation $f(x) \to \lambda\ f(\lambda x)$). We first notice that we have $\|B(\vec{f},\vec{f}) - B(\vec{g},\vec{g})(t,x)\|_E \leq C_0\ \sup_{0<s<t} \|(\vec{f}-\vec{g})(s,x)\|_E\ \sup_{0<s<t}(\sqrt{s}\ \|\vec{f}(s,x)\|_\infty + \sqrt{s}\ \|\vec{g}(s,x)\|_\infty)$. Moreover we have $\|B(\vec{f},\vec{f})\|_\infty \leq C_1 \sup_{0<s<t} \|\vec{f}\|_{M_{2,3}}^{1/2}\ \sup_{0<s<t} \|\vec{f}\|_\infty^{3/2}$. Thus if $R_0 = \|\vec{u}_0\|_{M_{2,3}}$, if $\epsilon_0 > 0$ is chosen such that $C_1\sqrt{2R_0\epsilon_0} < 1$ and $2C_0\epsilon_0 < 1$, and if T^* is chosen such that $\sup_{0<s<T^*} \|e^{s\Delta}\vec{u}_0\|_\infty < \epsilon_0$, then the map $\vec{f} \to e^{t\Delta}\vec{u}_0 - B(\vec{f},\vec{f})$ maps the closed subset of E defined by $\{\vec{u}\ /\ \sup_{0<s<T^*} \|\vec{u}\|_{M_{2,3}} < 2R_0$ and $\sup_{0<s<T^*} \|\vec{u}\|_\infty < \epsilon_0\}$ into itself and is a contraction on it; thus we have a fixed point in this set, which can be obtained by iterating the map. •

Remark : The smoothness of regular solutions in Morrey-Campanato spaces is discussed in details by T. Kato in [KAT 2] and by M. E. Taylor in [TAY]. There is some difficulty in the fact that these spaces are not separable, hence $e^{t\Delta}$ is not a $\mathcal{C}_0$-semigroup on these spaces.

4. The Littlewood-Paley decomposition

In the next section, we shall discuss the theory of M. Cannone about adapted spaces for the Navier-Stokes equations. It relies on the Littlewood-Paley decomposition of tempered distributions. In this section, we recall briefly some points of the Littlewood-Paley theory.

Definition 6: [Littlewood-Paley decomposition]
Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ be such that $|\xi| \leq \frac{1}{2} \Rightarrow \varphi(\xi) = 1$ and $|\xi| \geq 1 \Rightarrow \varphi(\xi) = 0$. Let ψ be defined as $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$. Let S_j and Δ_j be defined as the Fourier multipliers $\mathcal{F}(S_j f) = \varphi(\xi/2^j)\mathcal{F}f$ and $\mathcal{F}(\Delta_j f) = \psi(\xi/2^j)\mathcal{F}f$. Then for all $N \in \mathbb{Z}$ and all $f \in \mathcal{S}'(\mathbb{R}^3)$ we have $f = S_N f + \sum_{j\geq N} \Delta_j f$ in $\mathcal{S}'(\mathbb{R}^3)$. This equality is called the *Littlewood-Paley decomposition* of the distribution f. If moreover $\lim_{N\to-\infty} S_N f = 0$ in $\mathcal{S}'$, then the equality $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ is called the homogeneous Littlewood-Paley decomposition of f.

$\Delta_j f$ can be viewed as a filtering of f through a bandpass filter which selects the frequencies around $|\xi| \approx 2^j$.

A very basic tool in using the Littlewood-Paley decomposition is the so-called Bernstein inequalities which express how to control the L^p norm of the derivatives of the dyadic blocks:

$$\forall \alpha \in \mathbb{N}^3\ \exists C_\alpha\ \forall p \in [1,\infty]\ \forall f \in L^p \quad \|\partial^\alpha S_j f\|_p \leq C_\alpha\ 2^{j|\alpha|}\ \|S_j f\|_p$$

$$\forall \alpha \in \mathbb{N}^3\ \exists C_\alpha\ \forall p \in [1,\infty]\ \forall f \in L^p \quad \|\partial^\alpha \Delta_j f\|_p \leq C_\alpha\ 2^{j|\alpha|}\ \|\Delta_j f\|_p$$

$$\forall \alpha \in \mathbb{R}\ \exists C_\alpha\ \forall p \in [1,\infty]\ \forall f \in L^p \quad \|(\sqrt{-\Delta})^\alpha \Delta_j f\|_p \leq C_\alpha\ 2^{j|\alpha|}\ \|\Delta_j f\|_p$$

whereas we easily control the L^q norm of a dyadic block by its L^p norm if $q > p$:

$$\exists C\ \forall p \in [1,\infty]\ \forall q \in [p,\infty]\ \forall f \in L^p \quad \|S_j f\|_q \leq C\ 2^{j(3/p-3/q)}\ \|S_j f\|_p$$

Besov space $B_p^{s,q}$ may be easily defined through the Littlewood-Paley decomposition as the space of tempered distributions f such that for all $j \in \mathbb{N}$ we have $S_j f \in L^p$ and such that moreover $2^{js}\|\Delta_j f\|_p \in$

$l^q(\mathbb{N})$. The space $B_\infty^{-1,\infty}$ plays a key role in finding regular mild solutions since $f \in B_\infty^{-1,\infty}$ if and only if $\forall T > 0 \sup_{O<t<T} \sqrt{t}\|e^{t\Delta} f\|_\infty < \infty$.

For further informations on the Littlewood-Paley decomposition, the reader may consult the classical book of Bergh and Löfström on interpolation spaces [BEL], the nice booklet of Frazier, Jawerth and Weiss on Littlewood-Paley decomposition and wavelets [FJW], the book of Y. Meyer on Wavelets and Operators ([MEY 1], vol. 3) and the various treatises on paradifferential calculus.

5. Adapted spaces for the Navier-Stokes equations

Definition 7 [CAN] : [adapted spaces]

A Banach space X is *adapted* to the Navier-Stokes equations if the following assertions are satisfied:

a) $\mathcal{S}(\mathbb{R}^3) \subset X \subset \mathcal{S}'(\mathbb{R}^3)$ [continuous imbeddings]

b) The norm of X is shift invariant:

$$\forall f \in X \ \forall x_0 \in \mathbb{R}^3 \ \|f(x)\|_X = \|f(x-x_0)\|_X$$

c) the pointwise product between two elements of X is still well defined as a tempered distribution

d) there is a sequence of real numbers $\eta_j > 0,\ j \in \mathbb{Z}$, such that

$$\sum_{j\in\mathbb{Z}} 2^{-|j|}\ \eta_j < \infty$$

and such that

$$\forall j \in \mathbb{Z}, \forall f \in X, \forall g \in X\ \|\Delta_j(fg)\|_X \leq \eta_j\ \|f\|_X\ \|g\|_X$$

Theorem 2 [CAN]: Let X be a Banach space adapted to the Navier-Stokes equations. Then, for all $\vec{u_0} \in X^3$ such that $\vec{\nabla}.\vec{u_0} = 0$, there is a $T^* = T^*(\|\vec{u_0}\|_X)$ and a unique solution $\vec{u}$ of the Navier-Stokes equations on $(0,T^*)$ with initial value $\vec{u_0}$ such that $\vec{u} - e^{t\Delta}\vec{u_0} \in \mathcal{C}([0,T^*), X^3)$. Moreover we have the inequality:

$$\forall t \in (0,T^*)\ \|\vec{u} - e^{t\Delta}\vec{u_0}\|_X \leq C\ (t \sum_{4^j t\leq 1} 2^j \eta_j + \sum_{4^j t>1} 2^{-j}\eta_j)\ \|\vec{u_0}\|_X^2$$

Sketch of the proof: The proof lies again on the Picard iteration scheme. The main point is to prove that the bilinear transform $\vec{u},\vec{v} \to \vec{w} = \int_0^t e^{(t-s)\Delta}\mathbb{P}(\vec{\nabla}.(\vec{u}\otimes\vec{v}))\ ds$ is continuous on $L^\infty(X^3)$. It is enough to notice the following properties of this operator: α) $e^{(t-s)\Delta}\mathbb{P}(\vec{\nabla}.\ )$ is a matrix of convolution operators; β) since the norm of X is shift invariant, we have for $f \in X$ and $g \in L^1$ $\|f * g\|_X \leq \|f\|_X\ \|g\|_1$; γ) for all j we have $\Delta_j = (\sum_{k=j-2}^{k=j+2} \Delta_k)\ \Delta_j$; δ) we thus may conclude that

$$\|e^{(t-s)\Delta}\mathbb{P}(\vec{\nabla}.\Delta_j(\vec{u}\otimes\vec{v}))\|_X \leq C\min(2^j, 4^{-j}(t-s)^{-3/2})\ \|\Delta_j(\vec{u}\otimes\vec{v}))\|_X$$

The continuity of the bilinear transform is then easily established, writing

$$\|\vec{w}\|_X \leq \sum_{j\in\mathbb{Z}} \|\Delta_j \vec{w}\|_X \leq \int_0^t \sum_{j\in\mathbb{Z}} \|e^{(t-s)\Delta}\mathbb{P}(\vec{\nabla}.\Delta_j(\vec{u}\otimes\vec{v}))\|_X \; ds$$

and estimating the integral $\int_0^t \sum_{j\in\mathbb{Z}} \min(2^j, 4^{-j}(t-s)^{-3/2}) \; \eta_j \; ds.$ •

If we look precisely at the definition of an adapted space (and at the examples developed by M. Cannone), we may see that the theorem is not really a result about *existence* of solutions but a result about *regularity* at $t = 0$ of mild solutions. Indeed, let us assume that assertion c) in the definition is ensured by the embedding of X in L^2_{loc}, the space of locally square integrable functions, and that X is invariant under dilations as well as translations: $\forall\lambda > 0 \quad f \to f(\lambda x)$ is a continuous mapping in X. Then we have the following result:

Proposition3: Let X be an adapted space for the Navier-Stokes equations. If X is continuously embedded in L^2_{loc} anf if $\forall\lambda > 0 \quad f \to f(\lambda x)$ is a continuous mapping in X, then X is continuously embedded in $M_{2,3}$ and for all $f \in X$ we have $\lim_{t\to 0} \sqrt{t}\|e^{t\Delta} f\|_\infty = 0$.

Proof: Since the norm of X is shift invariant, the embedding $X \subset L^2$ is equivalent to $X \subset F_2$. Thus, we may define the following numbers:

$$\alpha(r) = \sup_{\|f\|_X\leq 1,\; x\in\mathbb{R}^3} \int_{|x-y|\leq r} |f(y)|^2 dy$$

We are going to estimate $\alpha(r)$ for $r \leq 1$. We choose a smooth non-negative function θ which is equal to 1 on $|x| \leq 1$ ant to 0 for $|x| \geq 2$, and we write $f^2 = S_0(f^2) + (\sum_{j\geq 0} \Delta_j(f^2)) = g + h$. It is easy to see that g is bounded (since $f^2 \in F_1$ and that S_0 maps F_1 into L^∞) so that $|\int g(y) \; \theta(\frac{y-x}{r}) \; dy \; | \leq Cr^3 \|f\|_X^2$. We may write h as $\sum_{1\leq j\leq 3} \partial_j h_j$ where $h_j \in X$ and $\|h_j\|_X \leq C\|f\|_X^2$; we then have: $|\int h(y) \; \theta(\frac{y-x}{r}) \; dy \; | \leq \sum_{1\leq j\leq 3} 1/r \; |\int h_j(y) \; \partial_j\theta(\frac{y-x}{r}) \; dy \; | \leq C \; \sqrt{r} \; \sqrt{\alpha(2r)}\|f\|_X^2$; since $\alpha(2r) \leq C\alpha(r)$ and $\int_{|x-y|\leq r} |f(y)|^2 dy \leq \int (g(y) + h(y)) \; \theta(\frac{y-x}{r}) \; dy$, we obtain that for $r \leq 1$, $\alpha(r) \leq C(r^3 + \sqrt{\alpha(r)}\sqrt{r})$ which gives $\alpha(r) \leq Cr$: thus X is embedded in the Morrey-Campanato space $M_{2,3}$.

This implies that $X \subset B_\infty^{-1,\infty}$ (and in particular that $\sup_{0\leq t\leq 1} \sqrt{t} \; \|e^{t\Delta} f\|_\infty \leq C\|f\|_X$). We are now going to show that we have also $\lim_{t\to 0} \|e^{t\Delta} f\|_\infty = 0$. Indeed let us define

$$C_j = \sup_{\|f\|_X\leq 1} \|\Delta_j f\|_\infty$$

. Since X is invariant through dyadic dilations, we see that C_j is slowly varying: $\exists\gamma > 1 \; \forall j \in \mathbb{Z} \; \frac{1}{\gamma} \leq \frac{C_j}{C_{j+1}} \leq \gamma$. If f_j satisfies $\|f_j\|_X = 1$ and $|\Delta_j f_j(0)| \geq 1/2 \; C_j$, then $f_j \; f_{j+3} = \sum_{j+1\leq k\leq j+5} \Delta_k(f_j \; f_{j+3})$, hence $1/4 \; C_j \; C_{j+3} \leq (\sum_{j+1\leq k\leq j+5} C_k) \; (\sum_{j\leq l\leq j+6} \eta_l)$. Thus, we get $\lim_{j\to+\infty} 2^{-j} C_j = 0$. This is just what we needed to show: write for $f \in X$ $f = S_N f + \sum_{j\geq N} \Delta_j f = f_N + g_N$; then for $t \in (0,1]$ and $N \geq 1$ we have: $\sqrt{t} \; \|e^{t\Delta} f\|_\infty \leq \sqrt{t} \; \|e^{t\Delta} f_N\|_\infty + \sqrt{t} \; \|e^{t\Delta} g_N\|_\infty \leq \sqrt{t} \; \|f_N\|_\infty + C \; \|g_N\|_{B_\infty^{-1,\infty}} \leq C(2^N\sqrt{t} + sup_{j\geq N} 2^{-j} C_j)\|f\|_X$
•

Thus, we have shown that the spaces studied by M. Cannone are subspaces of one space, the space $M_{2,3}$. Moreover, for all the examples quoted in [CAN] [Sobolev spaces $H_p^s(p < 3, s > \frac{3}{p} - 1)$, Morrey-Campanato spaces $M^{2,p}(p > 3)$, Lebesgue spaces $L^p, (p > 3)$], we have the property that $\forall f, g \in X \cap L^\infty \; \|fg\|_X \leq C \; (\|f\|_X\|g\|_\infty + \|g\|_X\|f\|_\infty)$ so that the regular mild solutions described in Theorem 1 are as well in $\cap_{T<T^*} L^\infty((0,T), X^3)$ (and we may notice that conversely a mild solution in $\cap_{T<T^*} L^\infty((0,T), X^3)$ where X is one of these spaces is in fact a regular mild solution). It means that we do not need a very precise

analysis in frequency for proving existence of a solution. Moreover, the tools used for exhibiting solutions in $M_{2,3}$ can be used as well for obtaining regularity results in the limit cases of Cannone's theorem, when $\eta_j = 2^j$ (think of L^3: this is the same recipee that we use in limit cases [FLT]).

6 Selfsimilar solutions

In view of theorem 1, we see clearly the strategy for exhibiting selfsimilar solutions to the Navier-Stokes equations. We take a space of tempered distributions X which contains non-trivial homogeneous distributions and try to get a theorem of global existence and uniqueness for solutions of the Navier-Stokes equations. Again, we assume that the norm of X is shift-invariant, and we assume, for sake of homogeneity, that $\|f(\lambda x)\|_X = \frac{1}{\lambda}\|f\|_X$. This implies that $X \subset B_\infty^{-1,\infty}$, or equivalently that for all $f \in X$ we have $\sup_{0<t} \sqrt{t}\ \|e^{t\Delta} f\|_\infty \leq C\|f\|_X$.

We write again $B(\vec{f}, \vec{g}) = \int_0^t (\vec{\nabla} \otimes O_{t-s}) * (\vec{f} \otimes \vec{g})\ ds$ and we have to check on which Banach space based on X and containing the tendencies $e^{t\Delta}\vec{u}_0$ for $\vec{u}_0 \in X^3$ the bilinear transform B is continuous. The first space we can try is $E = L^\infty((0,\infty), X^3)$. Of course, the bilinear product uv should then be defined for elements of X, hence we should assume that X is embedded in L^2_{loc}; we then find that $X \subset \dot{M}_{2,3}$, so that this case has already been discussed. There are many instances of spaces X which can be treated this way and provide selfsimilar solutions: homogeneous Besov spaces $\dot{B}_p^{s,\infty}$ where $p < 3$ and $s = 3/p - 1$ [CAN], [CHE], [FLT], the Lorentz space $L^{3,\infty}$ [MEY 2] and the space $\dot{B}_{PM}^{2,\infty} = \{f / |\xi|^2\ \mathcal{F}f \in L^\infty\}$ used by Y. Le Jan and A. S. Sznitman [LJS].

We may use the smoothing effect of $e^{t\Delta}$ and try to start from more singular initial value. M. Cannone [CAN] and F. Planchon [PLA] have shown that one could take $\vec{u}_0 \in \dot{B}_p^{s,\infty}$ where $p \in (3,\infty)$ and $s = 3/p - 1$ Then $E = \{\vec{f} / \sup_{0<t} t^{1/2-3/2p}\|\vec{f}(t,.)\|_p < \infty\}$ is a good choice.

This latter example can even be generalized by replacing L^p by F_p. We thus may consider a $\vec{u}_0 \in (\dot{B}_\infty^{-1,\infty})^3$ such that $\sup_{0<t} t^{1/2-3/2p}\|e^{t\Delta}\vec{u}_0\|_{F_p} < \infty$; then if this quantity is small enough, we shall have a mild solution in $X_p = \{\vec{f} / \sup_{0<t} t^{1/2-3/2p}\|\vec{f}(t,.)\|_{F_p} < \infty\}$. As a matter of fact, the first instance of selfsimilar solutions was constructed with help of Morrey-Campanato spaces [GIM]. X_p is a kind of Besov space above a Morrey-Campanato space, as in [KOY].

Self-similar solutions are connected to the asymptotic behaviour of solutions as t goes to ∞. F. Planchon [PLA] has studied the problem of solutions $\vec{u}(t,x)$ such that $\sup_{t>0} t^{1/2-3/2p}\|\vec{u}\|_p < \infty$ and $\lim_{t\to\infty} \sqrt{t}\,\vec{u}(t,\sqrt{t}x)$ exists in L^p where p is fixed in $(3,\infty)$.Then the initial value $\vec{u}_0$ (well defined since $\vec{u} \in \cap_{T>0} L^2((0,T),(F_2)^3))$ belongs to $\dot{B}_p^{3/p-1,\infty}$, the limit $\vec{V} = \lim_{t\to\infty} \sqrt{t}\ \vec{u}(t,\sqrt{t}x)$ satisfies that $\frac{1}{\sqrt{t}}\vec{V}(\frac{x}{\sqrt{t}})$ is a self-similar solution of the Navier-Stokes equations and the initial value $\vec{v}_0 \in \dot{B}_p^{3/p-1,\infty}$ of $\frac{1}{\sqrt{t}}\vec{V}(\frac{x}{\sqrt{t}})$ satisfies $e^\Delta\vec{v}_0 = \lim_{t\to\infty} \sqrt{t}\ \{e^{t\Delta}\vec{u}_0\}(t,\sqrt{t}x)$ in L^p. Notice that this proves that the asymptotic behaviour is governed only by the lowest frequencies of the initial value $\vec{u}_0$, since for all $N \in \mathbb{Z}$ we have $\sqrt{t}\ \|\{e^{t\Delta}(Id - S_N)\vec{u}_0\}(t,\sqrt{t}x)\|_p \leq C\ \frac{1}{t}\|\frac{Id-S_N}{\Delta}\vec{u}_o\|_{\dot{B}_p^{3/p-1,\infty}} \leq C\ \frac{1}{2^N t}\|(Id - S_N)\vec{u}_o\|_{\dot{B}_p^{3/p-1,\infty}}$ and thus $e^\Delta\vec{v}_0 = \lim_{t\to\infty} \sqrt{t}\ \{e^{t\Delta}S_N\vec{u}_0\}(t,\sqrt{t}x)$.

7. Uniqueness in limit spaces

In this section, we pay a few words to the uniqueness problems for mild solutions in $\mathcal{C}([0,T^*),(L^3)^3)$. As a matter of fact,we may not construct directly a solution in $\mathcal{C}([0,T^*),(L^3)^3)$, given $\vec{u}_0 \in (L^3)^3$, because L^3 is a limit space and this is not clear whether B is continuous or not on $\mathcal{C}([0,T^*),(L^3)^3)$. The reader may find various constructions in [KAT 1], [CAN] or [PLA], and uniqueness was granted only in the subspaces of $\mathcal{C}([0,T^*),(L^3)^3)$ where the iteration algorithm converge (for instance, for regular mild solutions). The constructions aimed to prove that the fluctuation was in $\mathcal{C}([0,T^*),(L^3)^3)$, and even in a smaller subspace (such as $\mathcal{C}([0,T^*),(Y)^3)$ with $Y = \dot{B}_3^{0,1}$ [CAN], [FLT] or $Y = \dot{B}_3^{0,2}$ [PLA] or, as one may easily check for a regular mild solution, $Y = L^{3,1}$). The idea we had in [FLT] was that the problem of uniqueness could be treated in a *greater* space, since one did not bother to get an estimate especially in the L^3 norm, the solutions being already given.

The example of Le Jan and Sznitman [LJS] is particularly interesting. They consider the limit space $X = \dot{B}^{2,\infty}_{PM}$ and show in a very simple way that the operator B is well behaved on this space. As a matter of fact, since the Riesz transform are obviously bounded on this space, we may just look at the scalar bilinear operator $A(u,v) = \int_0^t e^{(t-s)\Delta}\sqrt{-\Delta}(uv)ds$ operating on $L^\infty(X)$. Now, they do not try to prove that $\int_0^t \|e^{(t-s)\Delta}\sqrt{-\Delta}(uv)\|_X ds$ can be controlled; this is a too gross estimate. But the estimate they use is even more direct: $\frac{1}{\sqrt{-\Delta}}(uv) \in X$ since (taking the Fourier transform) it is enough to notice that $\frac{1}{|\xi|^2} * \frac{1}{|\xi|^2} \le \frac{C}{|\xi|}$ (an obvious inequality due to radial invariance and homogeneity), and then one writes:

$$|\mathcal{F}\{A(uv)(t,.)\}(\xi) \le \frac{C}{|\xi|^2} \sup_{0<s<t} \|u\|_X \sup_{0<s<t} \|v\|_X \int_0^t e^{-(t-s)|\xi|^2}|\xi|^2 ds \le \frac{C}{|\xi|^2} \sup_{0<s<t} \|u\|_X \sup_{0<s<t} \|v\|_X \int_0^\infty e^{-\tau} d\tau$$

These simple estimates suggest that one can deal with certain limit spaces, as far as their norm can be computed *frequency by frequency* or at least by frequency packets. This is the case of the Besov l^∞ spaces: if X is a Banach space of tempered distributions with a shift invariant norm and if $\dot{B}^{s,\infty}_X$ for $s > 0$ is the space of distributions $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ such that $2^{-js}\|\Delta_j f\|_X \in l^\infty(\mathbb{Z})$, then $\forall s \in (0,2)\ \forall f \in L^\infty((0,\infty), \dot{B}^{s,\infty}_X)\ \ \int_0^\infty \tau\Delta e^{\tau\Delta} f(\tau,.)\frac{d\tau}{\tau} \in \dot{B}^{s,\infty}_X$.

Thus we see for instance that B operates boundedly on $\dot{B}^{1/2,\infty}_2$ and this implies readily uniqueness of mild solutions in $\mathcal{C}((0,T^*), \dot{H}^{1/2})$. This argument does not work directly for the limit space L^3 since we then should work with the Besov space $\dot{B}^{0,\infty}_3$ which is not embedded in L^2_{loc} so that we would be bound again to use the smoothing effect of the heat kernel.

The next idea is to split the solutions in tendency and fluctuation. We consider two mild solutions $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u},\vec{u}) = e^{t\Delta}\vec{u}_0 - \vec{w}_1$ and $\vec{v} = e^{t\Delta}\vec{u}_0 - B(\vec{v},\vec{v}) = e^{t\Delta}\vec{u}_0 - \vec{w}_2$ in $\mathcal{C}((0,T^*),(L^3)^3)$ and write $\vec{w} = \vec{u} - \vec{v} = \vec{w}_2 - \vec{w}_1 = -B(\vec{w},\vec{v}) - B(\vec{u},\vec{w})$. Thus we see clearly the role of the fluctuations: they control the behaviour of $\vec{w}$. Indeed, we now turn to the scalar operator A (since the Riesz transforms operate on all the spaces involved in our computations) and write first:

$$A(u,v) = -(-\Delta)^{-1/4} \int_0^t (t-s)\Delta e^{(t-s)\Delta}(-\Delta)^{-1/4}(uv)\frac{ds}{t-s}$$

and find that A maps $\mathcal{C}([0,T],L^3) \times \mathcal{C}([0,T],L^3)$ into $L^\infty([0,T],(\dot{B}_2^{1/2,\infty})$ due to Sobolev inequalities. Then we use the continuity of u to split it in $u_1 + u_2$ (and the same for v) where $\forall t < T^*\ \|u_1(t,.)\|_3 < \epsilon$ and $\sup_{0<s<t}\|u_2(s,.)\|_4 < \infty$. Now we notice that, due to Sobolev and Hölder inequalities, $\forall f \in L^p (-\Delta)^{-3/2p} M_f$ (where M_f is the operator of pointwise multiplication by f) is continuous on L^q for all $q \in (\frac{p}{p-1},\infty)$ and on $\dot{L}^{3/p}_q$ for all $q \in (1,p)$, hence by interpolation on $\dot{B}^{s,\infty}_q$ for all $q \in (\frac{p}{p-1},p)$ and all $s \in (0,3/p)$. Taking $p = 3,4, q = 2, s = 1/2$, we easily get that:

$$\forall T < T^* \forall t < T\ \ \|A(uw)(t,.)\|_{\dot{B}_2^{1/2,\infty}} \le C\ (\epsilon + C(T,u)\ t^{1/4}) \sup_{0<s<t} \|w(s,.)\|_{\dot{B}_2^{1/2,\infty}}$$

which implies $w = 0$ as far as $C\ (\epsilon + C(T,u)\ t^{1/4}) < 1$; hence we find that we have local uniqueness and this uniqueness propagates to the whole interval $[0,T^*)$ because of the continuity of $\vec{u}$ and $\vec{v}$.

Thus, a precise analysis of the frequencies involved in the fluctuation allows one to prove uniqueness in $\mathcal{C}([0,T^*),(L^3)^3)$ This proof works in a more general pattern:

Theorem 3 [FLT]: Let $\dot{m}_{p,3}$ is the closure of the smooth compactly supported functions in $\dot{M}_{p,3}$. If $p > 2$ and $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier-Stokes equations on $(0,T^*) \times \mathbb{R}^3$ such that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0,T^*),(\dot{m}_{p,3})^3)$ and have the same initial value, then $\vec{u} = \vec{v}$.

Hence we have uniqueness in $\mathcal{C}([0,T^*),(Y)^3)$ for all limit space Y in which the test functions are dense and which is embedded into L^p_{loc} for some $p>2$.

But for L^3, there is now a much simpler way to get uniqueness: the space $\dot{B}^{0,\infty}_3$ was not good for proving uniqueness but it can be replaced by another l^∞ space: $L^{3,\infty}$ [MEY 2]. Moreover this result is not really *new*: the fact that $f\to\int_0^t(t-s)\Delta e^{(t-s)\Delta}\frac{1}{\sqrt{-\Delta}}f(s,.)\frac{ds}{(t-s)}$ maps $L^\infty((0,T),L^{3/2,\infty})$ into $L^\infty((0,T),L^{3,\infty})$ can be considered as a dual result to the fact that $f\to\int_s^T(t-s)\Delta e^{(t-s)\Delta}\frac{1}{\sqrt{-\Delta}}f(t,.)\frac{dt}{(t-s)}$ maps $L^1((0,T),L^{3/2,1})$ into $L^1((0,T),L^{3,1})$; this latter result is proved in exactly the same way than the fact that $f\to\int_0^t(t-s)\Delta e^{(t-s)\Delta}\frac{1}{\sqrt{-\Delta}}f(s,.)\frac{ds}{(t-s)}$ maps $L^1((0,T),L^{3/2,1})$ into $L^1((0,T),L^{3,1})$ or the fact that $f\to\int_0^t e^{(t-s)\Delta}f(s,.)ds$ maps $L^1((0,T),L^{3/2,1})$ into $L^1((0,T),\mathcal{C}_0)$ [remember that $\frac{1}{\sqrt{-\Delta}}$ maps $L^{3,1}$ into $\mathcal{C}_0$], a fact which is used by P.L.Lions in [LIO] to get the Foias-Guillopé-Temam theorem that the Leray weak solutions belong to $\cap_{T>0}L^1((0,T),\mathcal{C}_0)$!

8. Weak solutions with infinite energy

There is still another way to prove uniqueness in L^3. We split $\vec{u}_0$ in $\vec{v}_0+\vec{w}_0$ where $\vec{v}_0\in(L^2)^3$ and $\|\vec{w}_0\|_3<\epsilon$. Then Kato's theorem allows one to find a regular mild solution $\vec{w}$ on $(0,\infty)\times\mathbb{R}^3$ with initial value $\vec{w}_0$ and Leray's theory allows one to find a weak solution $\vec{v}\in L^\infty((0,\infty),(L^2)^3)\cap L^2((0,\infty),(\dot{H}^1)^3)$ of the problem

$$\vec{v}(0,.)=\vec{v}_0,\vec{\nabla}.\vec{v}=0,\ \partial_t\vec{v}=\Delta\vec{v}-\mathbb{P}(\vec{\nabla}.[\vec{v}\otimes\vec{v}+\vec{v}\otimes\vec{w}+\vec{w}\otimes\vec{v}])$$

with an energy estimate:

$$\forall t>0\ \|\vec{v}(t,.)\|_2^2+2\int_0^t\|\vec{\nabla}\otimes\vec{v}(s,.)\|_2^2ds\le\ \|\vec{v}_0\|_2^2+2\int_0^t\vec{\nabla}\otimes\vec{v}.\vec{v}\otimes\vec{w}\ ds$$

Now, if $\vec{u}$ is a mild solution of Navier-Stokes in $\mathcal{C}([0,T^*),(L^3)^3)$ with initial value $\vec{u}_0$, then $\vec{z}=\vec{u}-\vec{w}$ is a solution for the same problem as $\vec{v}$ and belongs to $\mathcal{C}([0,T^*),(L^3)^3)$. If we can prove that $\vec{z}$ belongs to $\cap_{T<T^*}L^\infty((0,T),(L^2)^3)\cap L^2((0,T),(\dot{H}^1)^3)$, then we may apply the Sohr and Von Wahl uniqueness theorem [WAH] to get $\vec{u}=\vec{v}+\vec{w}$ on $(0,T^*)$. Since $\vec{v}+\vec{w}$ does not depend on $\vec{u}$, this proves uniqueness of $\vec{u}$! This is exactly what is done in [LEM], and moreover this method provides a way to prove existence of global suitable weak solutions for the Navier-Stokes equations with initial value with infinite energy.

Definition 8 [CKN]: [suitable weak solutions]

A *suitable weak solution* of the Navier-Stokes equations on $(0,T)$ is a distribution vector field $\vec{u}(t,x)$ in $(\mathcal{D}'((0,T)\times\mathbb{R}^3)^3$ such that:

a) $\vec{u}$ and $\vec{\nabla}\otimes\vec{u}$ are locally square integrable on $(0,T)\times\mathbb{R}^3$

b) $\vec{\nabla}.\vec{u}=0$

c) $\exists p\in L^{4/3}_{loc}((0,T)\times\mathbb{R}^3)\ \partial_t\vec{u}=\Delta\vec{u}-(\vec{u}.\vec{\nabla})\ \vec{u}-\vec{\nabla}p$

d) for all compact subset K of $(0,T)\times\mathbb{R}^3$ we have $\sup_{t\in(0,T)}\int_{(t,x)\in K}|\vec{u}|^2\ dx<+\infty$

e) $\forall\varphi\in\mathcal{D}((0,+\infty)\times\mathbb{R}^3)$ such that $\varphi\ge0$ we have:

$$2\int\int|\vec{\nabla}\otimes\vec{u}|^2\ \varphi\ dx\ dt\le\int\int|\vec{u}|^2\ (\partial_t\varphi+\Delta\varphi)\ dx\ dt+\int\int(|\vec{u}|^2+2p)\ (\vec{u}.\vec{\nabla})\varphi\ dx\ dt$$

We recall that suitable solutions were proved in [CKN] to be regular outside from a subset of $(0,T)\times\mathbb{R}^3$ whose 1-dimensional Hausdorff measure is equal to 0. The theorem proved in [LEM] states the existence of global suitable weak solutions for an initial value $\vec{u_0}\in L^q(\mathbb{R}^3)$ where $2\leq q<\infty$:

Theorem 4: [suitable weak solutions with unbounded energy]
Let $q\in[2,+\infty)$ and let $\vec{u_0}\in L^q(\mathbb{R}^3)$ such that $\vec{\nabla}.\vec{u_0}=0$. Then there exists a suitable weak solution $\vec{u}$ of the Navier-Stokes equations on $(0,+\infty)$ such that $\vec{u}$ belongs to $\cap_{T>0}L^2((0,T),(E_2)^3)\cap L^r((0,T),(L^q)^3)$ (with $\frac{1}{r}=\frac{3}{2}(\frac{1}{2}-\frac{1}{q})$ if $q\leq 6$ and $\frac{1}{r}=1-\frac{3}{q}$ if $q\geq 6$) and such that $\vec{u}(0,.)=\vec{u_0}$.

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