Wavelet bases on the L-shaped domain.

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Abstract : We present in this paper two elementary constructions of multiresolution analyses on the L-shaped domain. In the first one, we use Meyer's method to define an orthonormal multiresolution analysis. In the second one, we use Cieselski and Figiel's decomposition method for constructing a biorthogonal multiresolution analysis.

Keywords : Multiresolution analyses; Wavelet bases; Sobolev spaces; Extension operators.

Introduction

The search for wavelet bases on a domain of $\mathbb{R}^2$ has been an active field for many years, since the beginning of the 1990's. Most constructions are based on the decomposition method, introduced by Z.Ciesielski and T. Figiel in 1983 [CF 83] to construct spline bases of generalized Sobolev spaces $W_p^k(M)$ ($k \in \mathbb{Z}$ and $1 < p < \infty$) on a Riemannian manifold M. This method has been adapted to the wavelet setting to construct generalized multiresolution analyses on bounded domains. In 1992, we constructed biorthogonal wavelet bases on a two-dimensional manifold Ω [J 93] [JL 92]; these bases were adapted to the study of Sobolev spaces $H^1(\Omega)$ or to $H_0^1(\Omega)$.

More recently, in 1997, the decomposition method has been used by A. Cohen, W. Dahmen and R. Schneider [DS 99a] [DS 99b] to construct biorthogonal wavelet bases $(\psi_\lambda, \widetilde{\psi}_\lambda)_{\lambda\in\nabla}$ of $L^2(\Omega)$ where Ω is a bounded domain of $\mathbb{R}^d$ ($d \in \mathbb{N}$); these bases were shown to be bases of Sobolev spaces $H^s(\Omega)$ for $|s| < \frac{3}{2}$. There are related constructions as well by A. Canuto and coworkers [CTU 99] and by R. Masson [Mas 99]. All these constructions are based on the decomposition method; there is a slight difficulty in their presentation, due to notational burden; moreover, it is often unclear how to get higher regularity Sobolev estimates.

In this paper, we aim to construct in an elementary way two multiresolution analyses on the L-shaped domain which are adapted to higher regularity analysis. The first one is a direct method, which turns out to be well adapted to the wavelet setting due to the simple geometry of the domain. The second one is an illustration of the decomposition method in this simple case; the specific geometry of the domain allows to get higher regularity estimates in a straightforward manner.

1. The space $V_j(I)$.

Throughout this paper, we consider a univariate orthonormal scaling function φ, with support $[0, 2N-1]$ and Sobolev regularity H^σ for some $\sigma > 0$. We call ψ the associated orthonormal wavelet with support $[0, 2N-1]$.

Let us recall that I. Daubechies has constructed compactly supported scaling functions with arbitrarily high regularity [D 88]. (More precisely, she constructed a family $(\varphi_N)_{N\geq 1}$ of scaling functions, such that φ_N is supported in $[0, 2N-1]$ and has Sobolev regularity H^{s_N} with $s_N = (1 - \frac{\ln 3}{\ln 4})N + o(N))$.

We define $(V_j(\mathbb{R}))_{j\in\mathbb{Z}}$ the multiresolution analysis associated to φ: $V_j(\mathbb{R})$ is the closed linear span in L^2 of the functions $\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$, $k \in \mathbb{Z}$.

Definition 1:

Let I be a bounded interval of $\mathbb{R}$. The space $V_j(I)$ is defined as the space of restrictions to I of elements of $V_j(\mathbb{R})$. This is the linear span of the functions $\varphi_{(j,k)} = (\varphi_{j,k})_{|I}$, $k \in \mathbb{Z}$. More precisely, we may keep only the indexes k such that $(2^{-j}k, 2^{-j}(k + 2N - 1)) \cap I \neq \emptyset$.

Recall the following important result of Meyer [Mey 91] and Malgouyres [Mal 93] [KL 95]:

Proposition 1:

Let $I=[\alpha,\beta]$. For $j\in\mathbb{Z}$, let α_j the smallest integer which is greater than $2^j\alpha-2N+1$ and let β_j the greatest integer which is smaller than $2^j\beta$. The functions $\varphi_{(j,k)}$, $\alpha_j\le k\le\beta_j$, are linearly independent, and thus they are a basis for $V_j(I)$.

In particular, there exists a constant $c(j,I)$ such that for all sequences $(\lambda_k)_{\alpha_j\le k\le\beta_j}$ we have the inequality

$$c(j,I)\sum_{\alpha_j\le k\le\beta_j}|\lambda_k|^2\le\int_\alpha^\beta|\sum_{\alpha_j\le k\le\beta_j}\lambda_k\varphi_{j,k}(x)|^2\,dx\le\sum_{\alpha_j\le k\le\beta_j}|\lambda_k|^2.$$

If α or β is not a dyadic number, we may have $\liminf_{j\to+\infty}c(j,I)=0$: we have obviously $c(j,I)\le\min(\int_\alpha^{2^{-j}\alpha_j}|\varphi|^2\,dx,\int_{2^{-j}\beta_j}^\beta|\varphi|^2\,dx)$. On the other hand, when α and β are dyadic numbers, $c(j,I)$ does not depend on j when j is big enough:

Definition 2: (Meyer's border functions)

Let φ be a compactly supported orthonormal scaling function with support $[0,2N-1]$. The associated Meyer border functions are defined in the following way:
i) [left border functions] for $1\le p\le 2N-2$, the functions $\varphi_p^{[l]}$ belong to the linear span of the functions $\varphi(x-k)_{|(0,+\infty)}$ with $-2N+2\le k\le -1$ and satisfy $\int_0^\infty\varphi(x-k)\varphi_p^{[l]}(x)\,dx=\delta_{k,-p}$.
ii) [right border functions] for $1\le p\le 2N-2$, the functions $\varphi_p^{[r]}$ belong to the linear span of the functions $\varphi(x-k)_{|(-\infty,0)}$ with $-2N+2\le k\le -1$ and satisfy $\int_{-\infty}^0\varphi(x-k)\varphi_p^{[r]}(x)\,dx=\delta_{k,-p}$.

Proposition 2:

Let $(\varphi^*_{(j,k)})_{\alpha_j\le k\le\beta_j}$ be the dual system of the basis $(\varphi_{(j,k)})_{\alpha_j\le k\le\beta_j}$. If α and β are dyadic numbers and if moreover j_0 is the smallest integer j such that $2^j\alpha$ and $2^j\beta$ belong to $\mathbb{Z}$ and $2^j(\beta-\alpha)\ge 2N-1$, then for $j\ge j_0$ we have $\alpha_j=2^j\alpha-2N+2$ and $\beta_j=2^j\beta-1$, and
i) [interior functions] for $2^j\alpha\le k\le 2^j\beta-2N+1$, we have $\varphi^*_{(j,k)}=\varphi_{(j,k)}=\varphi_{j,k}$
ii) [left border functions] for $2^j\alpha-2N+2\le k\le 2^j\alpha-1$, $k=2^j\alpha-p$, we have $\varphi^*_{(j,k)}(x)=2^{j/2}\varphi_p^{[l]}(2^j(x-\alpha))$
iii) [right border functions] for $2^j\beta-2N+2\le k\le 2^j\beta-1$, $k=2^j\beta-p$, we have $\varphi^*_{(j,k)}(x)=2^{j/2}\varphi_p^{[r]}(2^j(x-\beta))$.
In particular, $c(j,I)=c(j_0,I)$.

2. The space $V_j(\Omega)$.

As usually in wavelet theory, we define $V_j(\mathbb{R}^2)$ the multiresolution analysis associated to the separable scaling function $\varphi\otimes\varphi$: $V_j(\mathbb{R}^2)$ is the tensor product $V_j(\mathbb{R}^2)=V_j(\mathbb{R})\widehat{\otimes}V_j(\mathbb{R})$. For Ω a bounded open domain in $\mathbb{R}^2$, we define $V_j(\Omega)$ as the space of restrictions to Ω of elements of $V_j(\mathbb{R}^2)$. For a generic Ω, we cannot expect a simple description of the space $V_j(\Omega)$: even in the univariate case and in the case of an elementary interval, we had difficulties in estimating the dual basis. On the other hand, for very simple cases, we get an easy description. The first elementary case is the case of the unit cube:

Proposition 3:

Let $\Omega=(0,1)^2$ and $I=(0,1)$. Then, $V_j(\Omega)=V_j(I)\otimes V_j(I)$. Thus, for j such that $2^j\ge 2N-1$, a basis for $V_j(\Omega)$ is given by the family $\phi_{(j,k_1,k_2)}=(\varphi_{j,k_1}\otimes\varphi_{j,k_2})_{|\Omega}=\varphi_{(j,k_1)}\otimes\varphi_{(j,k_2)}$, $-2N+2\le k_1\le 2^j-1$ and $-2N+2\le k_2\le 2^j-1$.

Let $(\phi^*_{(j,k_1,k_2)})_{-2N+2\le k_1,k_2\le 2^j-1}$ be the dual system of the basis $(\phi_{(j,k_1,k_2)})_{-2N+2\le k_1,k_2\le 2^j-1}$. Then:
i) [interior functions] for $0\le k_1,k_2\le 2^j-2N+1$, we have $\phi^*_{(j,k_1,k_2)}=\phi_{(j,k_1,k_2)}=\varphi_{j,k_1}\otimes\varphi_{j,k_2}$
ii) [edge functions]
- *for $-2N+2\le k_1\le -1$, $k_1=-p$, and $0\le k_2\le 2^j-2N+1$, $\phi^*_{(j,k_1,k_2)}=2^j\varphi_p^{[l]}(2^jx_1)\varphi(2^jx_2-k_2)$*
- *for $0\le k_1\le 2^j-2N+1$ and $-2N+2\le k_2\le -1$, $k_2=-p$, $\phi^*_{(j,k_1,k_2)}=2^j\varphi(2^jx_1-k_1)\varphi_p^{[l]}(2^jx_2)$*

• *for* $2^j - 2N + 2 \leq k_1 \leq 2^j - 1$, $k_1 = 2^j - p$, *and* $0 \leq k_2 \leq 2^j - 2N + 1$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi^{[r]}_p(2^j(x_1 - 1))\varphi(2^j x_2 - k_2)$
• *for* $0 \leq k_1 \leq 2^j - 2N + 1$ *and* $2^j - 2N + 2 \leq k_2 \leq 2^j - 1$, $k_2 = 2^j - p$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi(2^j x_1 - k_1)\varphi^{[r]}_p(2^j(x_2 - 1))$
iii) [corner functions]
• *for* $-2N + 2 \leq k_1 \leq -1$, $k_1 = -p$, *and* $-2N + 2 \leq k_2 \leq -1$, $k_2 = -q$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi^{[l]}_p(2^j x_1)\varphi^{[l]}_q(2^j x_2)$
• *for* $2^j - 2N + 2 \leq k_1 \leq 2^j - 1$, $k_1 = 2^j - p$, *and* $-2N + 2 \leq k_2 \leq -1$, $k_2 = -q$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi^{[r]}_p(2^j(x_1 - 1))\varphi^{[l]}_q(2^j x_2)$
• *for* $-2N + 2 \leq k_1 \leq -1$, $k_1 = -p$, *and* $2^j - 2N + 2 \leq k_2 \leq 2^j - 1$, $k_2 = 2^j - q$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi^{[l]}_p(2^j x_1)\varphi^{[r]}_q(2^j(x_2 - 1))$
• *for* $2^j - 2N + 2 \leq k_1 \leq 2^j - 1$, $k_1 = 2^j - p$, *and* $2^j - 2N + 2 \leq k_2 \leq 2^j - 1$, $k_2 = 2^j - q$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi^{[r]}_p(2^j(x_1 - 1))\varphi^{[r]}_q(2^j(x_2 - 1))$.

The next easy domain we shall consider is the L-shaped domain $\Omega = \{(x, y) \in (-1, 1)^2 \ / \ x < 0 \text{ or } y < 0\}$. We intoduce some new functions associated to the scaling function φ:

Definition 3: (Interior border functions)
Let $\mathcal{O} = \{(x, y) \in \mathbb{R}^2 \ / \ x < 0 \text{ or } y < 0\}$. *Let* φ *be a compactly supported orthonormal scaling function with support* $[0, 2N - 1]$. *The associated interior border functions are defined in the following way: for* $p, q \in \{1, \ldots, 2N - 2\}$, *the functions* $\phi^{[b]}_{p,q}$ *belong to the linear span of the functions* $\varphi(x_1 - k_1)\varphi(x_2 - k_2)_{|\mathcal{O}}$ *with* $-2N + 2 \leq k_1, k_2 \leq -1$ *and satisfy* $\int_{\mathcal{O}} \varphi(x_1 - k_1, x_2 - k_2)\phi^{[b]}_{p,q}(x) \ dx = \delta_{k_1,-p} \ \delta_{k_2,-q}$.

Proposition 3 then becomes:

Proposition 4:
Let $\Omega = \{(x, y) \in (-1, 1)^2 \ / \ x < 0 \text{ or } y < 0\}$. *Then, for* j *such that* $2^j \geq 2N - 1$, *a basis for* $V_j(\Omega)$ *is given by the family* $\phi_{(j,k_1,k_2)} = (\varphi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $-2^j - 2N + 2 \leq k_1 \leq 2^j - 1$, $-2^j - 2N + 2 \leq k_2 \leq 2^j - 1$ *and* $k_1 < 0$ *or* $k_2 < 0$.
Let $\Delta_j = \{(k_1, k_2) \in \mathbb{Z}^2 \ / \ -2^j - 2N + 2 \leq k_1 \leq 2^j - 1, \ -2^j - 2N + 2 \leq k_2 \leq 2^j - 1; \ k_1 < 0 \text{ or } k_2 < 0\}$. *Let* $(\phi^*_{(j,k_1,k_2)})_{(k_1,k_2)\in\Delta_j}$ *be the dual system of the basis* $(\phi_{(j,k_1,k_2)})_{(k_1,k_2)\in\Delta_j}$. *Then:*
i) [interior functions] for $-2^j \leq k_1, k_2 \leq 2^j - 2N + 1$, $k_1 \leq -2N + 1$ *or* $k_2 \leq -2N + 1$, *we have* $\phi^*_{(j,k_1,k_2)} = \phi_{(j,k_1,k_2)} = \varphi_{j,k_1} \otimes \varphi_{j,k_2}$
ii) [edge functions]
• *for* $-2^j - 2N + 2 \leq k_1 \leq -2^j - 1$, $k_1 = -2^j - p$, *and* $-2^j \leq k_2 \leq 2^j - 2N + 1$,
$\phi^*_{(j,k_1,k_2)} = 2^j \varphi^{[l]}_p(2^j(x_1 + 1))\varphi(2^j x_2 - k_2)$
• *for* $-2^j \leq k_1 \leq 2^j - 2N + 1$ *and* $-2^j - 2N + 2 \leq k_2 \leq -2^j - 1$, $k_2 = -2^j - p$,
$\phi^*_{(j,k_1,k_2)} = 2^j \varphi(2^j x_1 - k_1)\varphi^{[l]}_p(2^j(x_2 + 1))$
• *for* $2^j - 2N + 2 \leq k_1 \leq 2^j - 1$, $k_1 = 2^j - p$, *and* $-2^j \leq k_2 \leq -2N + 1$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi^{[r]}_p(2^j(x_1 - 1))\varphi(2^j x_2 - k_2)$
• *for* $-2N + 2 \leq k_1 \leq -1$, $k_1 = -p$, *and* $0 \leq k_2 \leq 2^j - 2N + 1$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi^{[r]}_p(2^j x_1)\varphi(2^j x_2 - k_2)$
• *for* $-2^j \leq k_1 \leq -2N + 1$ *and* $2^j - 2N + 2 \leq k_2 \leq 2^j - 1$, $k_2 = 2^j - p$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi(2^j x_1 - k_1)\varphi^{[r]}_p(2^j(x_2 - 1))$
• *for* $0 \leq k_1 \leq 2^j - 2N + 1$ *and* $-2N + 2 \leq k_2 \leq -1$, $k_2 = -p$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi(2^j x_1 - k_1)\varphi^{[r]}_p(2^j x_2)$
iii) [exterior corner functions]
• *for* $-2^j - 2N + 2 \leq k_1 \leq -2^j - 1$, $k_1 = -2^j - p$, *and* $-2^j - 2N + 2 \leq k_2 \leq -2^j - 1$, $k_2 = -2^j - q$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi^{[l]}_p(2^j(x_1 + 1))\varphi^{[l]}_q(2^j(x_2 + 1))$
• *for* $2^j - 2N + 2 \leq k_1 \leq 2^j - 1$, $k_1 = 2^j - p$, *and* $-2^j - 2N + 2 \leq k_2 \leq -2^j - 1$, $k_2 = -2^j - q$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi^{[r]}_p(2^j(x_1 - 1))\varphi^{[l]}_q(2^j(x_2 + 1))$
• *for* $-2^j - 2N + 2 \leq k_1 \leq -2^j - 1$, $k_1 = -2^j - p$, *and* $2^j - 2N + 2 \leq k_2 \leq 2^j - 1$, $k_2 = 2^j - q$, $\phi^*_{(j,k_1,k_2)} = 2^j \varphi^{[l]}_p(2^j(x_1 + 1))\varphi^{[r]}_q(2^j(x_2 - 1))$
• *for* $2^j - 2N + 2 \leq k_1 \leq 2^j - 1$, $k_1 = 2^j - p$, *and* $-2N + 2 \leq k_2 \leq -1$, $k_2 = -q$,
$\phi^*_{(j,k_1,k_2)} = 2^j \varphi^{[r]}_p(2^j(x_1 - 1))\varphi^{[r]}_q(2^j x_2)$

- *for* $-2N+2 \leq k_1 \leq -1$, $k_1 = -p$, *and* $2^j - 2N + 2 \leq k_2 \leq 2^j - 1$, $k_2 = 2^j - q$,
  $\phi^*_{(j,k_1,k_2)} = 2^j \varphi_p^{[r]}(2^j x_1)\varphi_q^{[r]}(2^j(x_2-1))$

iv) [interior corner functions] for $-2N+2 \leq k_1, k_2 \leq -1$, $k_1 = -p$, $k_2 = -q$, *we have* $\phi^*_{(j,k_1,k_2)} = 2^j \phi^{[b]}_{p,q}(2^j x)$.

3. A regularity lemma.

We now proceed to prove some elementary lemmas which will be useful in regularity analysis for functions defined on Ω.

Lemma 1:
Let ω_1, ω_2 *be two square integrable compactly supported functions on* $\mathbb{R}^2$. *Then the operator* $f \mapsto \sum_{k\in\mathbb{Z}^2}\langle f|\omega_1(.-k)\rangle\omega_2(.-k)$ *is bounded on* L^2.

Proof: Let $M \in \mathbb{N}$ be such that the supports of ω_1 and ω_2 are contained in $(-M,M)^2$. Then we have

$$\int |\sum_{k\in\mathbb{Z}^2} \lambda_k \omega_2(x-k)|^2 \, dx \leq 4M^2 \int \sum_{k\in\mathbb{Z}^2} |\lambda_k \omega_2(x-k)|^2 \, dx = 4M^2 \|\omega_2\|_2^2 \sum_{k\in\mathbb{Z}^2} |\lambda_k|^2$$

and

$$\sum_{k\in\mathbb{Z}^2} |\int f(y)\omega_1(y-k)\, dy|^2 \leq \|\omega_1\|_2^2 \sum_{k\in\mathbb{Z}^2} \int |f(y)|^2 1_{(-M,M)^2}(y-k)\, dy \leq 4M^2\|\omega_1\|_2^2\|f\|_2^2.$$

Thus, the lemma is obvious. ◇

Definition 4: (Extension operators)
Let $\Omega = \{(x,y) \in (-1,1)^2 \;/\; x < 0 \text{ or } y < 0\}$. *Let us consider, for* j *such that* $2^j \geq 2N-1$, *the bases for* $V_j(\Omega)$ *given by the families* $(\phi_{(j,k_1,k_2)})_{(k_1,k_2)\in\Delta_j}$ *and* $(\phi^*_{(j,k_1,k_2)})_{(k_1,k_2)\in\Delta_j}$ *described in Proposition 4. Then we define the extension operator* E_j *from* $V_j(\Omega)$ *to* $V_j(\mathbb{R}^2)$ *by the formula*

$$E_j(f) = \sum_{(k_1,k_2)\in\Delta_j} \langle f|\phi^*_{(j,k_1,k_2)}\rangle_\Omega \; \varphi_{j,k_1}\otimes\varphi_{j,k_2},$$

where $\langle f|g\rangle_\Omega = \int_\Omega f\bar{g}\, dx$.

A direct consequence of Lemma 1 and Proposition 4 is the following lemma:

Lemma 2: (Uniform estimates for extension operators)
Let $\Omega = \{(x,y) \in (-1,1)^2 \;/\; x < 0 \text{ or } y < 0\}$. *There exists a positive constant* C_0 *such that, for al* j *such that* $2^j \geq 2N-1$ *and all* $f \in V_j(\Omega)$, $\|E_j f\|_{L^2(\mathbb{R}^2)} \leq C_0\|f\|_{L^2(\Omega)}$.

We now recall the following result of wavelet theory [Mey 90] [B 95] [KL 95]:

Lemma 3:
Let $\sigma > 0$ *and assume that* φ *belongs to the Sobolev space* $H^\sigma(\mathbb{R})$. *Then, for all* $s \in (0,\sigma)$, *there exists a constant* D_s *such that for all sequences* $(f_j)_{j\geq 0} \in (L^2(\mathbb{R}^2))^{\mathbb{N}}$ *such that* $f_j \in V_j(\mathbb{R}^2)$ *for all* $j \in \mathbb{N}$, *we have the inequality*

$$\|\sum_{j\in\mathbb{N}} f_j\|_{H^s} \leq D_s\sqrt{\sum_{j\in\mathbb{N}} 4^{js}\|f_j\|_2^2}.$$

Moreover, if A_j *is the orthogonal projection operator from* $L^2(\mathbb{R}^2)$ *onto* $V_j(\mathbb{R}^2)$, *there exists a constant* d_s *such that for all* $f \in H^s$ *we have*

$$\|A_0 f\|_2 + \sqrt{\sum_{j\in\mathbb{N}} 4^{js}\|(A_{j+1}-A_j)f\|_2^2} \leq d_s\|f\|_{H^s}.$$

Lemmas 2 and 3 then gives the following useful regularity criterion:

Theorem 1:

Let $\Omega=\{(x,y)\in(-1,1)^2\ /\ x<0 \text{ or } y<0\}$ *and let* $j_0\in\mathbb{N}$ *such that* $2^{j_0}\geq 2N-1$. *Let* $\sigma>0$ *and assume that* φ *belongs to the Sobolev space* $H^\sigma(\mathbb{R})$. *Moreover, let* $(P_j)_{j\geq j_0}$ *be a sequence of operators such that*

- *for every* $j\geq j_0$, P_j *is a projection operator from* $L^2(\Omega)$ *onto* $V_j(\Omega)$
- *there exists a constant* β *such that, for all* $j\geq j_0$ *and all* $f\in L^2(\Omega)$, $\|P_jf\|_{L^2(\Omega)}\leq\beta\|f\|_{L^2(\Omega)}$.

Then, for all $s\in(0,\sigma)$, *there exists two positive constants* A_s *and* B_s *such that for all* $f\in H^s(\Omega)$ *we have*

$$A_s\|f\|_{H^s(\Omega)}\leq\|P_{j_0}f\|_{L^2(\Omega)}+\sqrt{\sum_{j\geq j_0}4^{js}\|(P_{j+1}-P_j)f\|^2_{L^2(\Omega)}}\leq B_s\|f\|_{H^s(\Omega)}.$$

Proof: Let $f_{j_0}=P_{j_0}f$ and $f_j=(P_j-P_{j-1})f$ for $j\geq j_0+1$. Since Ω is a Lipschitz domain, $H^s(\Omega)$ is the space of functions on Ω which are restrictions to Ω of functions in $H^s(\mathbb{R}^2)$ and the norm $\|f\|_{H^s(\Omega)}$ is equivalent to $\min\{\|F\|_{H^s(\mathbb{R}^2)}\ /\ F_{|\Omega}=f\}$. Thus, we have

$$\|\sum_{j\geq j_0}f_j\|_{H^s(\Omega)}\leq C(s)\|\sum_{j\geq j_0}E_jf_j\|_{H^s(\mathbb{R}^2)}\leq C(s)D_s\sqrt{\sum_{j\geq j_0}4^{js}\|E_jf_j\|_2^2}\leq C(s)D_sC_0\sqrt{\sum_{j\geq j_0}4^{js}\|f_j\|^2_{L^2(\Omega)}}.$$

Conversely, let us write $f=F_{|\Omega}$ and $F=(F-A_jF)+A_jF$. We have

$$\|P_{j+1}f-P_jf\|_{L^2(\Omega)}=\|(P_{j+1}-P_j)\big((F-A_jF)_{|\Omega}\big)\|_{L^2(\Omega)}\leq\beta\|F-A_jF\|_{L^2(\Omega)};$$

hence,

$$\sum_{j\geq j_0}2^{2js}\|P_{j+1}f-P_jf\|^2_{L^2(\Omega)}\leq\beta^2\sum_{j\geq j_0}2^{2js}\|F-A_jF\|^2_{L^2(\mathbb{R}^2)}$$

$$=\beta^2\sum_{j\geq j_0}2^{2js}\sum_{p\geq j}\|(A_{p+1}-A_p)F\|^2_{L^2(\mathbb{R}^2)}\leq\frac{4^s\beta^2}{4^s-1}\sum_{p\geq j_0}2^{2ps}\|(A_{p+1}-A_p)F\|^2_{L^2(\mathbb{R}^2)}\leq\frac{4^s\beta^2d_s^2}{4^s-1}\|F\|^2_{H^s}.$$

Thus, Theorem 1 is proved. ◇

Theorem 1 is the basis for our strategy : in order to get a good tool for regularity analysis of functions defined on the L-shaped domain, we shall try to define nice equicontinuous families of projection operators on the spaces $V_j(\Omega)$. Moreover, following Proposition 4, the spaces $V_j(\Omega)$ have dual bases that are generated through dilations and translations from a finite set of basic functions with small supports (the scaling function $\varphi\otimes\varphi$, the edge functions $\varphi_p^{[l]}\otimes\varphi$, $\varphi\otimes\varphi_q^{[l]}$, $\varphi_p^{[r]}\otimes\varphi$ and $\varphi\otimes\varphi_q^{[r]}$, and the corner functions $\varphi_p^{[l]}\otimes\varphi_q^{[l]}$, $\varphi_p^{[r]}\otimes\varphi_q^{[l]}$, $\varphi_p^{[l]}\otimes\varphi_q^{[r]}$, $\varphi_p^{[r]}\otimes\varphi_q^{[r]}$, $\phi_{p,q}^{[b]}$) and we shall try to keep this feature in all our constructions.

4. Meyer's analysis.

In his fundamental paper on wavelets on the interval [Mey 91], Meyer intoduced the projection operators P_j (orthogonal projection from $L^2((0,1))$ onto $V_j((0,1))$) and $Q_j=P_{j+1}-P_j$, and he showed that the ranges $V_j=\mathrm{Im}\ P_j$ and $W_j=\mathrm{Im}\ Q_j$ have elementary Hilbertian bases. This is based on the remark that, while $V_{j+1}((0,1))$ is clearly generated by the restrictions of the scaling functions $\varphi_{j,k}$ and the wavelet functions $\psi_{j,k}$, $-2N+2\leq k\leq 2^j-1$, the restrictions of the extreme wavelets $\psi_{j,k}$ $(-2N+2\leq k\leq -N$ and $2^j-N+1\leq k\leq 2^j-1$ belong to $V_j((0,1))$, so that their elimination gives a generating families of $2^{j+1}+2N-2$ vectors of $V_{j+1}((0,1))$, hence a Riesz basis of $V_{j+1}((0,1))$.

Definition 5: (Meyer's border wavelets)

Let φ be a compactly supported orthonormal scaling function with support $[0, 2N-1]$. The associated Meyer border wavelets are defined in the following way:
i) [left border scaling functions] the family $(\varphi_p^{\{l\}})_{1\leq p\leq 2N-2}$ is the Gram-Schmidt orthonormalization of the family $(\varphi_p^{[l]})_{1\leq p\leq 2N-2}$.
ii) [right border scaling functions] the family $(\varphi_p^{\{r\}})_{1\leq p\leq 2N-2}$ is the Gram-Schmidt orthonormalization of the family $(\varphi_p^{[r]})_{1\leq p\leq 2N-2}$.
iii) [left border wavelets] the family $(\varphi_p^{\{r\}})_{1\leq p\leq 2N-2}\cup(\psi_q^{\{l\}})_{1\leq q\leq N-1}$ is the Gram-Schmidt orthonormalization of the family $(\varphi_p^{[l]})_{1\leq p\leq 2N-2}\cup(\psi(x+q)_{|(0,+\infty)})_{1\leq q\leq N-1}$.
iv) [right border wavelets] the family $(\varphi_p^{\{r\}})_{1\leq p\leq 2N-2}\cup(\psi_q^{\{r\}})_{1\leq q\leq N-1}$ is the Gram-Schmidt orthonormalization of the family $(\varphi_p^{[r]})_{1\leq p\leq 2N-2}\cup(\psi(x-2+N+q)_{|(-\infty,0)})_{1\leq q\leq N-1}$.

Then, Meyer's theorem reads as :

Proposition 5: (Meyer's theorem)

Let j such that $2^j\geq 2N-1$. Then
i) A Hilbertian basis for $V_j((0,1))$ is given by the family $(\varphi_{j,k}^{\perp})_{-2N+2\leq k\leq 2^j-1}$, with

- *[interior functions] for $0\leq k\leq 2^j-2N+1$, $\varphi_{j,k}^{\perp}=\varphi_{j,k}$*
- *[left border functions] for $-2N+2\leq k\leq -1$, $k=-p$, $\varphi_{j,k}^{\perp}=2^{j/2}\varphi_p^{\{l\}}(2^jx)$*
- *[right border functions] for $2^j-2N+2\leq k\leq 2^j-1$, $k=2^j-p$, $\varphi_{j,k}^{\perp}=2^{j/2}\varphi_p^{\{r\}}(2^j(x-1))$*

ii) A Hilbertian basis for $W_j((0,1))$ is given by the family $(\psi_{j,k}^{\perp})_{-N+1\leq k\leq 2^j-N}$, with

- *[interior wavelets] for $0\leq k\leq 2^j-2N+1$, $\psi_{j,k}^{\perp}=\psi_{j,k}$*
- *[left border wavelets] for $-N+1\leq k\leq -1$, $k=-q$, $\psi_{j,k}^{\perp}=2^{j/2}\psi_q^{\{l\}}(2^jx)$*
- *[right border wavelets] for $2^j-2N+2\leq k\leq 2^j-N$, $k=2^j-N+1-q$, $\psi_{j,k}^{\perp}=2^{j/2}\psi_p^{\{r\}}(2^j(x-1))$*

Our purpose is to show similar results on the L-shaped domain.

5. Orthogonal multi-resolution analysis on the L-shaped domain.

The most direct way to construct an equicontinuous family of projection operators is obviously to use orthogonal projections; the problem is then just to check that this leads to well-supported basic functions.

Let $\Omega=\{(x,y)\in(-1,1)^2\ /\ x<0 \text{ or } y<0\}$. Then, for j such that $2^j\geq 2N-1$, Proposition 4 describes the Riesz basis $(\phi_{(j,k_1,k_2)})_{(k_1,k_2)\in\Delta_j}$ of $V_j(\Omega)$ as split into four families (and thirtenn subfamilies) of functions :

i) [interior functions] $-2^j\leq k_1,k_2\leq 2^j-2N+1$, $k_1\leq -2N+1$ or $k_2\leq -2N+1$

ii) [edge functions]

- $-2^j-2N+2\leq k_1\leq -2^j-1$ and $-2^j\leq k_2\leq 2^j-2N+1$
- $-2^j\leq k_1\leq 2^j-2N+1$ and $-2^j-2N+2\leq k_2\leq -2^j-1$
- $2^j-2N+2\leq k_1\leq 2^j-1$ and $-2^j\leq k_2\leq -2N+1$
- $-2N+2\leq k_1\leq -1$ and $0\leq k_2\leq 2^j-2N+1$
- $-2^j\leq k_1\leq -2N+1$ and $2^j-2N+2\leq k_2\leq 2^j-1$
- $0\leq k_1\leq 2^j-2N+1$ and $-2N+2\leq k_2\leq -1$

iii) [exterior corner functions]

- $-2^j-2N+2\leq k_1\leq -2^j-1$ and $-2^j-2N+2\leq k_2\leq -2^j-1$
- $2^j-2N+2\leq k_1\leq 2^j-1$ and $-2^j-2N+2\leq k_2\leq -2^j-1$
- $-2^j-2N+2\leq k_1\leq -2^j-1$ and $2^j-2N+2\leq k_2\leq 2^j-1$
- $2^j-2N+2\leq k_1\leq 2^j-1$ and $-2N+2\leq k_2\leq -1$
- $-2N+2\leq k_1\leq -1$ and $2^j-2N+2\leq k_2\leq 2^j-1$

iv) [interior corner functions] $-2N+2\leq k_1,k_2\leq -1$.

The main point is that these thirteen subfamilies are orthogonal one to each other and that orthonormalization can be processed independently for each subfamily. Moreover, the twelve first families can be reduced to Meyer's analysis, and only the last one gives a new family of basic functions.

Definition 6: (Interior border orthonormal scaling functions)

Let $\mathcal{O} = \{(x,y) \in \mathbb{R}^2 \ / \ x < 0 \text{ or } y < 0\}$. Let φ be a compactly supported orthonormal scaling function with support $[0, 2N-1]$. The associated interior border orthonormal scaling functions are defined in the following way: $(\phi^{\{b\}}_{p,q})_{1 \le p,q \le 2N-2}$ is a Gram–Schmidt orthonormalization of $(\varphi(x_1+p)\varphi(x_2+q)_{|\mathcal{O}})_{1 \le p,q \le 2N-2}$.

We then get :

Theorem 2:

Let Let $\Omega = \{(x,y) \in (-1,1)^2 \ / \ x < 0 \text{ or } y < 0\}$. Then, for j such that $2^j \ge 2N-1$, a Hilbertian basis for $V_j(\Omega)$ is given by the family $(\phi^{\perp}_{(j,k_1,k_2)})_{(k_1,k_2) \in \Delta_j}$, where

i) [interior functions] for $-2^j \le k_1, k_2 \le 2^j - 2N + 1$, $k_1 \le -2N+1$ or $k_2 \le -2N+1$, we have $\phi^{\perp}_{(j,k_1,k_2)} = \varphi_{j,k_1} \otimes \varphi_{j,k_2}$

ii) [edge functions]

- *for $-2^j - 2N + 2 \le k_1 \le -2^j - 1$, $k_1 = -2^j - p$, and $-2^j \le k_2 \le 2^j - 2N + 1$,*
  $\phi^{\perp}_{(j,k_1,k_2)} = 2^j \varphi^{\{l\}}_p(2^j(x_1+1))\varphi(2^j x_2 - k_2)$
- *for $-2^j \le k_1 \le 2^j - 2N + 1$ and $-2^j - 2N + 2 \le k_2 \le -2^j - 1$, $k_2 = -2^j - p$,*
  $\phi^{\perp}_{(j,k_1,k_2)} = 2^j \varphi(2^j x_1 - k_1)\varphi^{\{l\}}_p(2^j(x_2+1))$
- *for $2^j - 2N + 2 \le k_1 \le 2^j - 1$, $k_1 = 2^j - p$, and $-2^j \le k_2 \le -2N + 1$,*
  $\phi^{\perp}_{(j,k_1,k_2)} = 2^j \varphi^{\{r\}}_p(2^j(x_1-1))\varphi(2^j x_2 - k_2)$
- *for $-2N + 2 \le k_1 \le -1$, $k_1 = -p$, and $0 \le k_2 \le 2^j - 2N + 1$, $\phi^{\perp}_{(j,k_1,k_2)} = 2^j \varphi^{\{r\}}_p(2^j x_1)\varphi(2^j x_2 - k_2)$*
- *for $-2^j \le k_1 \le -2N+1$ and $2^j - 2N + 2 \le k_2 \le 2^j - 1$, $k_2 = 2^j - p$, $\phi^{\perp}_{(j,k_1,k_2)} = 2^j \varphi(2^j x_1 - k_1)\varphi^{\{r\}}_p(2^j(x_2-1))$*
- *for $0 \le k_1 \le 2^j - 2N + 1$ and $-2N + 2 \le k_2 \le -1$, $k_2 = -p$, $\phi^{\perp}_{(j,k_1,k_2)} = 2^j \varphi(2^j x_1 - k_1)\varphi^{\{r\}}_p(2^j x_2)$*

iii) [exterior corner functions]

- *for $-2^j - 2N + 2 \le k_1 \le -2^j - 1$, $k_1 = -2^j - p$, and $-2^j - 2N + 2 \le k_2 \le -2^j - 1$, $k_2 = -2^j - q$, $\phi^{\perp}_{(j,k_1,k_2)} = 2^j \varphi^{\{l\}}_p(2^j(x_1+1))\varphi^{\{l\}}_q(2^j(x_2+1))$*
- *for $2^j - 2N + 2 \le k_1 \le 2^j - 1$, $k_1 = 2^j - p$, and $-2^j - 2N + 2 \le k_2 \le -2^j - 1$, $k_2 = -2^j - q$, $\phi^{\perp}_{(j,k_1,k_2)} = 2^j \varphi^{\{r\}}_p(2^j(x_1-1))\varphi^{\{l\}}_q(2^j(x_2+1))$*
- *for $-2^j - 2N + 2 \le k_1 \le -2^j - 1$, $k_1 = -2^j - p$, and $2^j - 2N + 2 \le k_2 \le 2^j - 1$, $k_2 = 2^j - q$, $\phi^{\perp}_{(j,k_1,k_2)} = 2^j \varphi^{\{l\}}_p(2^j(x_1+1))\varphi^{\{r\}}_q(2^j(x_2-1))$*
- *for $2^j - 2N + 2 \le k_1 \le 2^j - 1$, $k_1 = 2^j - p$, and $-2N + 2 \le k_2 \le -1$, $k_2 = -q$,*
  $\phi^{\perp}_{(j,k_1,k_2)} = 2^j \varphi^{\{r\}}_p(2^j(x_1-1))\varphi^{\{r\}}_q(2^j x_2)$
- *for $-2N + 2 \le k_1 \le -1$, $k_1 = -p$, and $2^j - 2N + 2 \le k_2 \le 2^j - 1$, $k_2 = 2^j - q$,*
  $\phi^{\perp}_{(j,k_1,k_2)} = 2^j \varphi^{\{r\}}_p(2^j x_1)\varphi^{\{r\}}_q(2^j(x_2-1))$

iv) [interior corner functions] for $-2N+2 \le k_1, k_2 \le -1$, $k_1 = -p$, $k_2 = -q$, we have $\phi^{\perp}_{(j,k_1,k_2)} = 2^j \phi^{\{b\}}_{p,q}(2^j x)$.

We now describe a supplementary space X_j of $V_j(\Omega)$ in $V_{j+1}(\Omega)$. We have an obvious generating family of $V_{j+1}(\Omega)$ by taking the functions $(\varphi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $(\psi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$ and $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2^j - 2N + 2 \le k_1, k_2 \le 2^j - 1$, $k_1 \le -1$ or $k_2 \le -1$. We have thus $4((2^{j+1} + 2N - 2)^2 - 2^{2j})$ functions, while the dimension of $V_{j+1}(\Omega)$ is $(2^{j+2} + 2N - 2)^2 - 2^{2j+2}$. Thus, we must eliminate $8(2N-2)2^j + 3(2N-2)^2$ functions.

Lemma 4:

The following family is a Riesz basis of $V_{j+1}(\Omega)$:

- $(\varphi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $-2^j - 2N + 2 \le k_1 \le 2^j - 1$, $-2^j - 2N + 2 \le k_2 \le 2^j - 1$, $k_1 \le -1$ *or* $k_2 \le -1$*;*
- $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2^j - 2N + 2 \le k_1 \le 2^j - 1$, $-2^j - N + 1 \le k_2 \le 2^j - N$, $k_1 \le -1$ *or* $k_2 \le -N$*;*

- $(\psi_{j,k_1}\otimes\varphi_{j,k_2})_{|\Omega}$, $-2^j-N+1\le k_1\le 2^j-N$, $-2^j-2N+2\le k_2\le 2^j-1$, $k_1\le -N$ *or* $k_2\le -1$;
- $(\psi_{j,k_1}\otimes\psi_{j,k_2})_{|\Omega}$, $-2^j-N+1\le k_1\le 2^j-N$, $-2^j-N+1\le k_2\le 2^j-N$, $k_1\le -N$ *or* $k_2\le -N$.

Proof: Since we have the right number of functions, it is enough to prove that they are linearly independent. Let us consider sequences (a_{j,k_1,k_2}), (b_{j,k_1,k_2}), (c_{j,k_1,k_2}), and (d_{j,k_1,k_2}), such that

$$\begin{aligned}&\sum\sum_{(k_1,k_2)\in A_j} a_{j,k_1,k_2}\varphi_{j,k_1}\otimes\varphi_{j,k_2}+\sum\sum_{(k_1,k_2)\in B_j} b_{j,k_1,k_2}\varphi_{j,k_1}\otimes\psi_{j,k_2}\\ &+\sum\sum_{(k_1,k_2)\in C_j} c_{j,k_1,k_2}\psi_{j,k_1}\otimes\varphi_{j,k_2}+\sum\sum_{(k_1,k_2)\in D_j} d_{j,k_1,k_2}\psi_{j,k_1}\otimes\psi_{j,k_2}\quad =0 \text{ on } \Omega\end{aligned}$$

Thus, defining $Q_1=(-1,0)\times(-1,0)$, $Q_2=(-1,0)\times(0,1)$ and $Q_3=(0,1)\times(-1,0)$, we have that

$$\begin{aligned}&\sum_{k_1=-2^j-2N+2}^{-1}\sum_{k_2=-2^j-2N+2}^{-1} a'_{j,k_1,k_2}\varphi_{j,k_1}\otimes\varphi_{j,k_2}+\sum_{k_1=-2^j-2N+2}^{-1}\sum_{k_2=-2^j-N+1}^{-N} b'_{j,k_1,k_2}\varphi_{j,k_1}\otimes\psi_{j,k_2}\\ &+\sum_{k_1=-2^j-N+1}^{-N}\sum_{k_2=-2^j-2N+2}^{-1} c'_{j,k_1,k_2}\psi_{j,k_1}\otimes\varphi_{j,k_2}+\sum_{k_1=-2^j-N+1}^{-N}\sum_{k_2=-2^j-N+1}^{-N} d_{j,k_1,k_2}\psi_{j,k_1}\otimes\psi_{j,k_2}\\ &=0 \text{ on } Q_1,\end{aligned}$$

$$\begin{aligned}&\sum_{k_1=-2^j-2N+2}^{-1}\sum_{k_2=-2N+2}^{2^j-1} a''_{j,k_1,k_2}\varphi_{j,k_1}\otimes\varphi_{j,k_2}+\sum_{k_1=-2^j-2N+2}^{-1}\sum_{k_2=-N+1}^{2^j-N} b''_{j,k_1,k_2}\varphi_{j,k_1}\otimes\psi_{j,k_2}\\ &+\sum_{k_1=-2^j-N+1}^{-N}\sum_{k_2=-2N+2}^{2^j-1} c''_{j,k_1,k_2}\psi_{j,k_1}\otimes\varphi_{j,k_2}+\sum_{k_1=-2^j-N+1}^{-N}\sum_{k_2=-N+1}^{2^j-N} d_{j,k_1,k_2}\psi_{j,k_1}\otimes\psi_{j,k_2}\\ &=0 \text{ on } Q_2\end{aligned}$$

and

$$\begin{aligned}&\sum_{k_1=-2N+2}^{2^j-1}\sum_{k_2=-2^j-2N+2}^{-1} a'''_{j,k_1,k_2}\varphi_{j,k_1}\otimes\varphi_{j,k_2}+\sum_{k_1=-2N+2}^{2^j-1}\sum_{k_2=-2^j-N+1}^{-N} b'''_{j,k_1,k_2}\varphi_{j,k_1}\otimes\psi_{j,k_2}\\ &+\sum_{k_1=-N+1}^{2^j-N}\sum_{k_2=-2^j-2N+2}^{-1} c'''_{j,k_1,k_2}\psi_{j,k_1}\otimes\varphi_{j,k_2}+\sum_{k_1=-N+1}^{2^j-N}\sum_{k_2=-2^j-N+1}^{-N} d_{j,k_1,k_2}\psi_{j,k_1}\otimes\psi_{j,k_2}\\ &=0 \text{ on } Q_3.\end{aligned}$$

Then the tensorization of Meyer's analysis gives us that the coefficients d_{j,k_1,k_2} are all equal to 0. This gives in turn that

$$\begin{aligned}&\sum_{k_1=-2^j-2N+2}^{-1}\sum_{k_2=-2^j-2N+2}^{-1} a'_{j,k_1,k_2}\varphi_{j,k_1}\otimes\varphi_{j,k_2}+\sum_{k_1=-2^j-2N+2}^{-1}\sum_{k_2=-2^j-N+1}^{-N} b_{j,k_1,k_2}\varphi_{j,k_1}\otimes\psi_{j,k_2}\\ &+\sum_{k_1=-2^j-N+1}^{-N}\sum_{k_2=-2^j-2N+2}^{-1} c_{j,k_1,k_2}\psi_{j,k_1}\otimes\varphi_{j,k_2}\\ &=0 \text{ on } Q_1,\end{aligned}$$

$$\begin{aligned}&\sum_{k_1=-2^j-2N+2}^{-1}\sum_{k_2=-2N+2}^{2^j-1} a''_{j,k_1,k_2}\varphi_{j,k_1}\otimes\varphi_{j,k_2}+\sum_{k_1=-2^j-2N+2}^{-1}\sum_{k_2=-N+1}^{2^j-N} b_{j,k_1,k_2}\varphi_{j,k_1}\otimes\psi_{j,k_2}\\ &+\sum_{k_1=-2^j-N+1}^{-N}\sum_{k_2=-2N+2}^{2^j-1} c_{j,k_1,k_2}\psi_{j,k_1}\otimes\varphi_{j,k_2}\\ &=0 \text{ on } Q_2\end{aligned}$$

and

$$\begin{aligned}&\sum_{k_1=-2N+2}^{2^j-1}\sum_{k_2=-2^j-2N+2}^{-1} a'''_{j,k_1,k_2}\varphi_{j,k_1}\otimes\varphi_{j,k_2}+\sum_{k_1=-2N+2}^{2^j-1}\sum_{k_2=-2^j-N+1}^{-N} b_{j,k_1,k_2}\varphi_{j,k_1}\otimes\psi_{j,k_2}\\ &+\sum_{k_1=-N+1}^{2^j-N}\sum_{k_2=-2^j-2N+2}^{-1} c_{j,k_1,k_2}\psi_{j,k_1}\otimes\varphi_{j,k_2}\\ &=0 \text{ on } Q_3.\end{aligned}$$

Tensorization of Meyer's analysis gives then us that the coefficients b_{j,k_1,k_2} and c_{j,k_1,k_2} are all equal to 0. Finally, we have that the coefficents b_{j,k_1,k_2} are all equal to 0 as well. ⋄

One more time, we may group the remaining functions in families according to the geometrical position of their support : the basis of X_j is given by the families

i) [interior functions]

- $(\varphi_{j,k_1}\otimes\psi_{j,k_2})_{|\Omega}$, $-2^j\le k_1,k_2\le 2^j-2N+1$, $k_1\le -2N+1$ or $k_2\le -2N+1$
- $(\psi_{j,k_1}\otimes\varphi_{j,k_2})_{|\Omega}$, $-2^j\le k_1,k_2\le 2^j-2N+1$, $k_1\le -2N+1$ or $k_2\le -2N+1$
- $(\psi_{j,k_1}\otimes\psi_{j,k_2})_{|\Omega}$, $-2^j\le k_1,k_2\le 2^j-2N+1$, $k_1\le -2N+1$ or $k_2\le -2N+1$

ii) [edge functions]

- $(\varphi_{j,k_1}\otimes\psi_{j,k_2})_{|\Omega}$, $-2^j-2N+2\le k_1\le -2^j-1$ and $-2^j\le k_2\le 2^j-2N+1$
- $(\psi_{j,k_1}\otimes\varphi_{j,k_2})_{|\Omega}$, $-2^j-N+1\le k_1\le -2^j-1$ and $-2^j\le k_2\le 2^j-2N+1$

- $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2^j - N + 1 \leq k_1 \leq -2^j - 1$ and $-2^j \leq k_2 \leq 2^j - 2N + 1$
- $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2^j \leq k_1 \leq 2^j - 2N + 1$ and $-2^j - N + 1 \leq k_2 \leq -2^j - 1$
- $(\psi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $-2^j \leq k_1 \leq 2^j - 2N + 1$ and $-2^j - 2N + 2 \leq k_2 \leq -2^j - 1$
- $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2^j \leq k_1 \leq 2^j - 2N + 1$ and $-2^j - N + 1 \leq k_2 \leq -2^j - 1$
- $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $2^j - 2N + 2 \leq k_1 \leq 2^j - 1$ and $-2^j \leq k_2 \leq -2N + 1$
- $(\psi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $2^j - 2N + 2 \leq k_1 \leq 2^j - N$ and $-2^j \leq k_2 \leq -2N + 1$
- $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $2^j - 2N + 2 \leq k_1 \leq 2^j - N$ and $-2^j \leq k_2 \leq -2N + 1$
- $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2N + 2 \leq k_1 \leq -1$ and $0 \leq k_2 \leq 2^j - 2N + 1$
- $(\psi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $-2N + 2 \leq k_1 \leq -N$ and $0 \leq k_2 \leq 2^j - 2N + 1$
- $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2N + 2 \leq k_1 \leq -N$ and $0 \leq k_2 \leq 2^j - 2N + 1$
- $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2^j \leq k_1 \leq -2N + 1$ and $2^j - 2N + 2 \leq k_2 \leq 2^j - N$
- $(\psi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $-2^j \leq k_1 \leq -2N + 1$ and $2^j - 2N + 2 \leq k_2 \leq 2^j - 1$
- $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2^j \leq k_1 \leq -2N + 1$ and $2^j - 2N + 2 \leq k_2 \leq 2^j - N$
- $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $0 \leq k_1 \leq 2^j - 2N + 1$ and $-2N + 2 \leq k_2 \leq -N$
- $(\psi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $0 \leq k_1 \leq 2^j - 2N + 1$ and $-2N + 2 \leq k_2 \leq -1$
- $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $0 \leq k_1 \leq 2^j - 2N + 1$ and $-2N + 2 \leq k_2 \leq -N$

iii) [exterior corner functions]

- $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2^j - 2N + 2 \leq k_1 \leq -2^j - 1$ and $-2^j - N + 1 \leq k_2 \leq -2^j - 1$
- $(\psi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $-2^j - N + 1 \leq k_1 \leq -2^j - 1$ and $-2^j - 2N + 2 \leq k_2 \leq -2^j - 1$
- $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2^j - N + 1 \leq k_1 \leq -2^j - 1$ and $-2^j - N + 1 \leq k_2 \leq -2^j - 1$
- $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $2^j - 2N + 2 \leq k_1 \leq 2^j - 1$ and $-2^j - N + 1 \leq k_2 \leq -2^j - 1$
- $(\psi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $2^j - 2N + 2 \leq k_1 \leq 2^j - N$ and $-2^j - 2N + 2 \leq k_2 \leq -2^j - 1$
- $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $2^j - 2N + 2 \leq k_1 \leq 2^j - N$ and $-2^j - N + 1 \leq k_2 \leq -2^j - 1$
- $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2^j - 2N + 2 \leq k_1 \leq -2^j - 1$ and $2^j - 2N + 2 \leq k_2 \leq 2^j - N$
- $(\psi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $-2^j - N + 1 \leq k_1 \leq -2^j - 1$ and $2^j - 2N + 2 \leq k_2 \leq 2^j - 1$
- $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2^j - N + 1 \leq k_1 \leq -2^j - 1$ and $2^j - 2N + 2 \leq k_2 \leq 2^j - N$
- $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $2^j - 2N + 2 \leq k_1 \leq 2^j - 1$ and $-2N + 2 \leq k_2 \leq -N$
- $(\psi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $2^j - 2N + 2 \leq k_1 \leq 2^j - N$ and $-2N + 2 \leq k_2 \leq -1$
- $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $2^j - 2N + 2 \leq k_1 \leq 2^j - N$ and $-2N + 2 \leq k_2 \leq -N$
- $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2N + 2 \leq k_1 \leq -1$ and $2^j - 2N + 2 \leq k_2 \leq 2^j - N$
- $(\psi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $-2N + 2 \leq k_1 \leq -N$ and $2^j - 2N + 2 \leq k_2 \leq 2^j - 1$
- $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2N + 2 \leq k_1 \leq -N$ and $2^j - 2N + 2 \leq k_2 \leq 2^j - N$

iv) [interior corner functions]

- $(\varphi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2N + 2 \leq k_1 \leq -1$ and $-2N + 2 \leq k_2 \leq -1$
- $(\psi_{j,k_1} \otimes \varphi_{j,k_2})_{|\Omega}$, $-2N + 2 \leq k_1 \leq -1$ and $-2N + 2 \leq k_2 \leq -1$
- $(\psi_{j,k_1} \otimes \psi_{j,k_2})_{|\Omega}$, $-2N + 2 \leq k_1 \leq -1$ and $-2N + 2 \leq k_2 \leq -1$, $k_1 \leq -N$ or $k_2 \leq -N$

Gram-Schmidt orthonormalization allows then to get a nice basis for $W_j^{\perp}(\Omega)$, the orthogonal complement of $V_j(\Omega)$ in $V_{j+1}(\Omega)$:

Theorem 3:

Let Let $\Omega = \{(x, y) \in (-1, 1)^2 \ / \ x < 0 \ or \ y < 0\}$. Let $W_j^{\perp}(\Omega)$ be the the orthogonal complement of $V_j(\Omega)$ in $V_{j+1}(\Omega)$. Then, there exists $11(N-1)^2$ functions Ψ_p, $1 \leq p \leq 11(N-1)^2$, compactly supported in $\{(x, y) \in \mathbb{R}^2 \ / \ x \leq 0 \ or \ y \leq 0\}$ such that, for j such that $2^j \geq 4N - 2$, a Hilbertian basis for $W_j^{\perp}(\Omega)$ is given by the following family

i) [interior functions]

- $2^j \varphi(2^j x_1 - k_1) \psi(2^j x_2 - k_2)$, $-2^j \leq k_1, k_2 \leq 2^j - 2N + 1$, $k_1 \leq -2N + 1$ *or* $k_2 \leq -2N + 1$
- $2^j \psi(2^j x_1 - k_1) \varphi(2^j x_2 - k_2)$, $-2^j \leq k_1, k_2 \leq 2^j - 2N + 1$, $k_1 \leq -2N + 1$ *or* $k_2 \leq -2N + 1$
- $2^j \psi(2^j x_1 - k_1) \psi(2^j x_2 - k_2)$, $-2^j \leq k_1, k_2 \leq 2^j - 2N + 1$, $k_1 \leq -2N + 1$ *or* $k_2 \leq -2N + 1$

ii) [edge functions]

- $2^j \varphi_p^{\{l\}}(2^j(x_1 + 1)) \psi(2^j x_2 - k_2)$, $1 \leq p \leq 2N - 2$ *and* $-2^j \leq k_2 \leq 2^j - 2N + 1$
- $2^j \psi_p^{\{l\}}(2^j(x_1 + 1)) \varphi(2^j x_2 - k_2)$, $1 \leq p \leq N - 1$ *and* $-2^j \leq k_2 \leq 2^j - 2N + 1$
- $2^j \psi_p^{\{l\}}(2^j(x_1 + 1)) \psi(2^j x_2 - k_2)$, $1 \leq p \leq N - 1$ *and* $-2^j \leq k_2 \leq 2^j - 2N + 1$

- $2^j\varphi(2^jx_1-k_1)\psi_p^{\{l\}}(2^j(x_2+1))$, $-2^j \le k_1 \le 2^j-2N+1$ *and* $1\le p\le N-1$
- $2^j\psi(2^jx_1-k_1)\varphi_p^{\{l\}}(2^j(x_2+1))$, $-2^j \le k_1 \le 2^j-2N+1$ *and* $1\le p\le 2N-2$
- $2^j\psi(2^jx_1-k_1)\psi_p^{\{l\}}(2^j(x_2+1))$, $-2^j \le k_1 \le 2^j-2N+1$ *and* $1\le p\le N-1$
- $2^j\varphi_p^{\{r\}}(2^j(x_1-1))\psi(2^jx_2-k_2)$, $1\le p\le 2N-2$ *and* $-2^j\le k_2\le -2N+1$
- $2^j\psi_p^{\{r\}}(2^j(x_1-1))\varphi(2^jx_2-k_2)$, $1\le p\le N-1$ *and* $-2^j\le k_2\le -2N+1$
- $2^j\psi_p^{\{r\}}(2^j(x_1-1))\psi(2^jx_2-k_2)$, $1\le p\le N-1$ *and* $-2^j\le k_2\le -2N+1$
- $2^j\varphi_p^{\{r\}}(2^jx_1)\psi(2^jx_2-k_2)$, $1\le p\le 2N-2$ *and* $0\le k_2\le 2^j-2N+1$
- $2^j\psi_p^{\{r\}}(2^jx_1)\varphi(2^jx_2-k_2)$, $1\le p\le N-1$ *and* $0\le k_2\le 2^j-2N+1$
- $2^j\psi_p^{\{r\}}(2^jx_1)\psi(2^jx_2-k_2)$, $1\le p\le N-1$ *and* $0\le k_2\le 2^j-2N+1$
- $2^j\varphi(2^jx_1-k_1)\psi_p^{\{r\}}(2^j(x_2-1))$, $-2^j\le k_1\le -2N+1$ *and* $1\le p\le N-1$
- $2^j\psi(2^jx_1-k_1)\varphi_p^{\{r\}}(2^j(x_2-1))$, $-2^j\le k_1\le -2N+1$ *and* $1\le p\le 2N-2$
- $2^j\psi(2^jx_1-k_1)\psi_p^{\{r\}}(2^j(x_2-1))$, $-2^j\le k_1\le -2N+1$ *and* $1\le p\le N-1$
- $2^j\varphi(2^jx_1-k_1)\psi_p^{\{r\}}(2^jx_2)$, $0\le k_1\le 2^j-2N+1$ *and* $1\le p\le N-1$
- $2^j\psi(2^jx_1-k_1)\varphi_p^{\{r\}}(2^jx_2)$, $0\le k_1\le 2^j-2N+1$ *and* $1\le p\le 2N-2$
- $2^j\psi(2^jx_1-k_1)\psi_p^{\{r\}}(2^jx_2)$, $0\le k_1\le 2^j-2N+1$ *and* $1\le p\le N-1$

iii) [*exterior corner functions*]

- $2^j\varphi_p^{\{l\}}(2^j(x_1+1))\psi_q^{\{l\}}(2^j(x_2+1))$, $1\le p\le 2N-2$ *and* $1\le q\le N-1$
- $2^j\psi_p^{\{l\}}(2^j(x_1+1))\varphi_q^{\{l\}}(2^j(x_2+1))$, $1\le p\le N-1$ *and* $1\le q\le 2N-2$
- $2^j\psi_p^{\{l\}}(2^j(x_1+1))\psi_q^{\{l\}}(2^j(x_2+1))$, $1\le p\le N-1$ *and* $1\le q\le N-1$
- $2^j\varphi_p^{\{r\}}(2^j(x_1-1))\psi_q^{\{l\}}(2^j(x_2+1))$, $1\le p\le 2N-2$ *and* $1\le q\le N-1$
- $2^j\psi_p^{\{r\}}(2^j(x_1-1))\varphi_q^{\{l\}}(2^j(x_2+1))$, $1\le p\le N-1$ *and* $1\le q\le 2N-2$
- $2^j\psi_p^{\{r\}}(2^j(x_1-1))\psi_q^{\{l\}}(2^j(x_2+1))$, $1\le p\le N-1$ *and* $1\le q\le N-1$
- $2^j\varphi_p^{\{l\}}(2^j(x_1+1))\psi_q^{\{r\}}(2^j(x_2-1))$, $1\le p\le 2N-2$ *and* $1\le q\le N-1$
- $2^j\psi_p^{\{l\}}(2^j(x_1+1))\varphi_q^{\{r\}}(2^j(x_2-1))$, $1\le p\le N-1$ *and* $1\le q\le 2N-2$
- $2^j\psi_p^{\{l\}}(2^j(x_1+1))\psi_q^{\{r\}}(2^j(x_2-1))$, $1\le p\le N-1$ *and* $1\le q\le N-1$
- $2^j\varphi_p^{\{r\}}(2^j(x_1-1))\psi_q^{\{r\}}(2^jx_2)$, $1\le p\le 2N-2$ *and* $1\le q\le N-1$
- $2^j\psi_p^{\{r\}}(2^j(x_1-1))\varphi_q^{\{r\}}(2^jx_2)$, $1\le p\le N-1$ *and* $1\le q\le 2N-2$
- $2^j\psi_p^{\{r\}}(2^j(x_1-1))\psi_q^{\{r\}}(2^jx_2)$, $1\le p\le N-1$ *and* $1\le q\le N-1$
- $2^j\varphi_p^{\{r\}}(2^jx_1)\psi_q^{\{r\}}(2^j(x_2-1))$, $1\le p\le 2N-2$ *and* $1\le q\le N-1$
- $2^j\psi_p^{\{r\}}(2^jx_1)\varphi_q^{\{r\}}(2^j(x_2-1))$, $1\le p\le N-1$ *and* $1\le q\le 2N-2$
- $2^j\psi_p^{\{r\}}(2^jx_1)\psi_q^{\{r\}}(2^j(x_2-1))$, $1\le p\le N-1$ *and* $1\le q\le N-1$

iv) [*interior corner functions*]

- $2^j\Psi_p(2^jx_1,2^jx_2)$, $1\le p\le 11(N-1)^2$.

Proof: We proceed the orthonormalization by following the order in the description of the basis of X_j. The interior functions are already orthogonal to $V_j(\Omega)$ and orthonormal : they give the interior wavelets. Then, we orthonormalize the edge functions. This can be reduced to the analysis on the interval and gives the edge wavelets. Then, we proceed to the orthonormalization of the exterior corner functions. Due to the control of the supports of the wavelets involved in those computations, we do not see in those computations the global geometry of the open set and for each corner, the computations are the same as if we were in the case of a cube $(0,1)\times(0,1)$, and we find wavelets provided by tensor products of the analysis on the interval (more precisely, we find the corner elements of the tensor basis). Finally, we must orthonormalize the interior corner functions, which may be written as $2^jf_p(2^jx_1,2^jx_2)$, $1\le p\le 11(N-1)^2$. Indeed, if we define $\alpha_{k_1,k_2}=(\varphi_{0,k_1}\otimes\psi_{0,k_2})_{|\mathcal{O}}$, $\beta_{k_1,k_2}=(\psi_{0,k_1}\otimes\varphi_{0,k_2})_{|\mathcal{O}}$ and $\gamma_{k_1,k_2}=(\psi_{0,k_1}\otimes\psi_{0,k_2})_{|\mathcal{O}}$, we must proceed with the functions

- $(\varphi_{j,k_1}\otimes\psi_{j,k_2})_{|\Omega}=2^j\alpha_{k_1,k_2}(2^jx_1,2^jx_2)$, $-2N+2\le k_1\le -1$ and $-2N+2\le k_2\le -1$
- $(\psi_{j,k_1}\otimes\varphi_{j,k_2})_{|\Omega}=2^j\beta_{k_1,k_2}(2^jx_1,2^jx_2)$, $-2N+2\le k_1\le -1$ and $-2N+2\le k_2\le -1$
- $(\psi_{j,k_1}\otimes\psi_{j,k_2})_{|\Omega}=2^j\gamma_{k_1,k_2}(2^jx_1,2^jx_2)$, $-2N+2\le k_1\le -1$ and $-2N+2\le k_2\le -1$, $k_1\le -N$ or $k_2\le -N$.

We must first substract from those functions their components on the other elements og the Hilbertian basis. The only elements which may interfere have a support which intersect $(-\frac{2N-2}{2^j}, \frac{2N-2}{2^j})^2$. Those are the functions :

- $2^j\varphi(2^jx_1-k_1)\varphi(2^jx_2-k_2)$, for $-4N+2\leq k_1,k_2\leq 2N-3$, $k_1\leq -2N+1$ or $k_2\leq -2N+1$
- $2^j\varphi_p^{\{r\}}(2^jx_1)\varphi(2^jx_2-k_2)$, for $1\leq p\leq 2N-2$ and $0\leq k_2\leq 2N-3$
- $2^j\varphi(2^jx_1-k_1)\varphi_p^{\{r\}}(2^jx_2)$, for $0\leq k_1\leq 2N-3$ and $1\leq p\leq 2N-2$
- $2^j\phi_{p,q}^{\{b\}}(2^jx)$, for $1\leq p,q\leq 2N-2$
- $2^j\varphi(2^jx_1-k_1)\psi(2^jx_2-k_2)$, for $-4N+2\leq k_1,k_2\leq 2N-3$, $k_1\leq -2N+1$ or $k_2\leq -2N+1$
- $2^j\psi(2^jx_1-k_1)\varphi(2^jx_2-k_2)$, for $-4N+2\leq k_1,k_2\leq 2N-3$, $k_1\leq -2N+1$ or $k_2\leq -2N+1$
- $2^j\psi(2^jx_1-k_1)\psi(2^jx_2-k_2)$, for $-4N+2\leq k_1,k_2\leq 2N-3$, $k_1\leq -2N+1$ or $k_2\leq -2N+1$
- $2^j\varphi_p^{\{r\}}(2^jx_1)\psi(2^jx_2-k_2)$, for $1\leq p\leq 2N-2$ and $0\leq k_2\leq 2N-3$
- $2^j\psi_p^{\{r\}}(2^jx_1)\varphi(2^jx_2-k_2)$, for $1\leq p\leq N-1$ and $0\leq k_2\leq 2N-3$
- $2^j\psi_p^{\{r\}}(2^jx_1)\psi(2^jx_2-k_2)$, for $1\leq p\leq N-1$ and $0\leq k_2\leq 2N-3$
- $2^j\varphi(2^jx_1-k_1)\psi_p^{\{r\}}(2^jx_2)$, for $0\leq k_1\leq 2N-3$ and $1\leq p\leq N-1$
- $2^j\psi(2^jx_1-k_1)\varphi_p^{\{r\}}(2^jx_2)$, for $0\leq k_1\leq 2N-3$ and $1\leq p\leq 2N-2$
- $2^j\psi(2^jx_1-k_1)\psi_p^{\{r\}}(2^jx_2)$, for $0\leq k_1\leq 2N-3$ and $1\leq p\leq N-1$

This is a finite set of functions $2^j\delta_n(2^jx_1,2^jx_2)$, $1\leq n\leq n_{\max}$, so that we finally have to orthonormalize the set of functions $g_p=f_p-\sum_{n=1}^{n_{\max}}\langle f_p|\delta_n\rangle\delta_n$, $1\leq p\leq 11(N-1)^2$, to find the functions Ψ_p. ◇

6. Cieselski and Figiel's decomposition method.

We shall now describe an altenative method for getting a good multi-resolution analysis on the L-shaped domain. This method is based on Cieselski and Figiel's construction of (generalized) spline bases on a manifold [CF 83]. Their construction is based on a quadrangulation of the manifold. For the L-shaped domain, Cieselski and Figiel's quadrangulation is very easy to perform:

Definition 7: (Cieselski and Figiel's quadrangulation)

Let Ω be the domain $\Omega=\{(x,y)\in(-1,1)^2\ /\ x<0\ or\ y<0\}$. Cieselski and Figiel's quadrangulation of Ω is the following decomposition of Ω into a union of cubes $\Omega=Q_1\cup Q_2\cup Q_3$ where $Q_1=(-1,0)\times(-1,0)$, $Q_2=(-1,0)\times[0,1)$ and $Q_3=[0,1)\times(-1,0)$.

For $s\geq 0$, we consider the following Sobolev spaces on Ω and the cubes Q_i:

- $H^s(\Omega)$ is the usual Sobolev space on Ω;
- $H^s(Q_i)$ is the usual Sobolev space on Q_i; it may be defined as well as the space of restrictions to Q_i of elements of $H^s(\Omega)$;
- $H^{s,i}(\Omega)$ is the subspace of $H^s(\Omega)$ of functions in $H^s(\Omega)$ which are identically equal to 0 outside from Q_i;
- $H^{s,i}(Q_i)$ is the space of restrictions to Q_i of elements of $H^{s,i}(\Omega)$.

We shall need the following lemma on those Sobolev spaces :

Lemma 5:

If $f\in H^s(\Omega)$ is identically equal to 0 inside Q_1, then $f_{|Q_i}$ belongs to $H^{s,i}(Q_i)$ for $i=2,\ 3$.

Proof: f may be extended to $f_0\in H^s(\mathbb{R}^2)$ (since Ω is Lipschitz); thus, we may consider the translates $f_0(x,y-\epsilon)_{|Q_2}$ as approximations of f in $H^s(Q_2)$; but $f_0(x,y-\epsilon)_{|Q_2}$ is equal to 0 for $0<y<\epsilon$, so that it may be extended by 0 on $\Omega-Q_2$ to define an element of $H^{s,2}(\Omega)$ and we find that $f_0(x,y-\epsilon)_{|Q_2}$ belongs to $H^{s,2}(Q_2)$; letting ϵ go to 0^+, we find that $f_{|Q_2}$ belongs to $H^{s,2}(Q_2)$. ◇

Moreover, we define the operators $E_i^{(0)}$ of extension by 0 as the operators defined from $L^2(Q_i)$ to $L^2(\Omega)$ by $(E_i^{(0)}f)_{|Q_i}=f$ and $(E_i^{(0)}f)_{|\Omega-Q_i}=0$. Obviously, $E_i^{(0)}$ maps as well $H^{s,i}(Q_i)$ onto $H^{s,i}(\Omega)$. We then have the following decomposition lemma for $H^s(\Omega)$:

Proposition 6: (Cieselski and Figiel's lemma)

Let E_1 be a continuous extension operator from $H^s(Q_1)$ to $H^s(\Omega)$: $f \in H^s(Q_1) \mapsto E_1 f \in H^s(\Omega)$ with $E_1 f_{|Q_1} = f$. Then, the mapping E defined by $(f_1, f_2, f_3) \mapsto E(f) = E_1 f_1 + E_2^{(0)} f_2 + E_3^{(0)} f_3$ is an isomorphism between $H^s(Q_1) \times H^{s,2}(Q_2) \times H^{s,3}(Q_3)$ and $H^s(\Omega)$.

Proof: This is a direct consequence of Lemma 5. ◇

Proposition 6 suggests that, in order to have a multi-resolution analysis on Ω that is well-adapted to the analysis of H^s functions, we may try to find multi-resolution analyses on the cubes Q_i that are well-adapted to the analysis of the Sobolev spaces $H^s(Q_1)$, $H^{s,2}(Q_2)$ and $H^{s,3}(Q_3)$. This will be done in section 8 by tensorizing multi-resolution analyses on the interval that are well adapted to boundary conditions (section 7).

7. Wavelets on the interval.

There are many construction of wavelet bases on the interval, besides Meyer's analysis. The main concern is to get better condition numbers in the orthonormalization process or to deal with boundary conditions [CDV 93]. A most flexible method was introduced by Sweldens [S 96]. While most constructions are based on approximating L^2 as the increasing union $\cup_{j\geq j_0} V_j$ of spaces $V_j \subset V_j((0,1))$ [JL 93], there are even some constructions which introduce at scale j some boundary functions which does not belong to $V_j((0,1))$ [Mel 01].

Here, we shall only consider Meyer's analysis ($V_j^- = V_j((-1,0))$, $W_j^- = V_{j+1}^- \cap (V_j^-)^\perp$) and the analysis $V_j^+ = V_{j,0}((0,1)) =$ Span $\left((\varphi_{j,k})_{|(0,1)}, 0 \leq k \leq 2^j - 1\right)$ (the restrictions to $(0,1)$ of the elements of $V_j(\mathbb{R})$ which are supported in $[0, +\infty)$) and $W_j^+ = V_{j+1}^+ \cap (V_j^+)^\perp$. This analysis was described in [JL 93].

Let us call P_j^- the orthogonal projection operator on V_j^- and $Q_j^- = P_{j+1}^- - P_j^-$ the orthogonal projection operator on W_j^-. According to Proposition 2, we have, for $2^j \geq 2N-1$,

$$\begin{aligned} P_j^- f = & \textstyle\sum_{p=1}^{2N-2} \langle f | 2^{j/2} \varphi_p^{[l]}(2^j(x+1)) \rangle \, 2^{j/2} \varphi(2^j(x+1)+p)_{|(-1,0)} \\ & + \textstyle\sum_{k=-2^j}^{-2N+1} \langle f | 2^{j/2} \varphi(2^j x - k) \rangle \, 2^{j/2} \varphi(2^j x - k) \\ & + \textstyle\sum_{q=1}^{2N-2} \langle f | 2^{j/2} \varphi_q^{[r]}(2^j x) \rangle \, 2^{j/2} \varphi(2^j x + q)_{|(-1,0)} \end{aligned}$$

and, according to Proposition 5, we have

$$\begin{aligned} P_j^- f = & \textstyle\sum_{p=1}^{2N-2} \langle f | 2^{j/2} \varphi_p^{\{l\}}(2^j(x+1)) \rangle \, 2^{j/2} \varphi_p^{\{l\}}(2^j(x+1)) \\ & + \textstyle\sum_{k=-2^j}^{-2N+1} \langle f | 2^{j/2} \varphi(2^j x - k) \rangle \, 2^{j/2} \varphi(2^j x - k) \\ & + \textstyle\sum_{q=1}^{2N-2} \langle f | 2^{j/2} \varphi_q^{\{r\}}(2^j x) \rangle \, 2^{j/2} \varphi_q^{\{r\}}(2^j x) \end{aligned}$$

and

$$\begin{aligned} Q_j^- f = & \textstyle\sum_{p=1}^{N-1} \langle f | 2^{j/2} \psi_p^{\{l\}}(2^j(x+1)) \rangle \, 2^{j/2} \psi_p^{\{l\}}(2^j(x+1)) \\ & + \textstyle\sum_{k=-2^j}^{-2N+1} \langle f | 2^{j/2} \psi(2^j x - k) \rangle \, 2^{j/2} \psi(2^j x - k) \\ & + \textstyle\sum_{q=1}^{N-1} \langle f | 2^{j/2} \psi_q^{\{r\}}(2^j x) \rangle \, 2^{j/2} \psi_q^{\{r\}}(2^j x) \end{aligned}$$

Similarly, let us call P_j^+ the orthogonal projection operator on V_j^+ and $Q_j^+ = P_{j+1}^+ - P_j^+$ the orthogonal projection operator on W_j^+. According to Proposition 2, we have, for $2^j \geq 2N-1$,

$$\begin{aligned} P_j^+ f = & \textstyle\sum_{k=0}^{2^j-2N+1} \langle f | 2^{j/2} \varphi(2^j x - k) \rangle \, 2^{j/2} \varphi(2^j x - k) \\ & + \textstyle\sum_{q=1}^{2N-2} \langle f | 2^{j/2} \varphi_q^{[r]}(2^j(x-1)) \rangle \, 2^{j/2} \varphi(2^j(x-1)+q)_{|(0,1)} \end{aligned}$$

and, according to Proposition 5, we have

$$P_j^+ f = \begin{array}{l} \sum_{k=0}^{2^j-2N+1} \langle f | 2^{j/2}\varphi(2^j x - k)\rangle\ 2^{j/2}\varphi(2^j x - k) \\ + \sum_{q=1}^{2N-2} \langle f | 2^{j/2}\varphi_q^{\{r\}}(2^j(x-1))\rangle\ 2^{j/2}\varphi_q^{\{r\}}(2^j(x-1)) \end{array}$$

Now, according to [JL 93], a Riesz basis for V_{j+1}^+ is given by the family :

- $(\varphi_{j,k})_{|(0,1)}$, $0 \le k \le 2^j - 1$
- $(\psi_{j,k})_{|(0,1)}$, $0 \le k \le 2^j - N$
- $\varphi_{j+1,2p+1}$, $1 \le p \le N-1$

Orthonormalization then gives us $N-1$ new functions $\Psi_p^{\{l\}}$ supported in $[0, 4N-2]$ such that, for $2^j \ge 4N-4$, we have

$$Q_j^+ f = \begin{array}{l} \sum_{p=1}^{N-1} \langle f | 2^{j/2}\Psi_p^{\{l\}}(2^j x)\rangle\ 2^{j/2}\Psi_p^{\{l\}}(2^j x) \\ + \sum_{k=-2^j}^{-2N+1} \langle f | 2^{j/2}\psi(2^j x - k)\rangle\ 2^{j/2}\psi(2^j x - k) \\ + \sum_{q=1}^{N-1} \langle f | 2^{j/2}\psi_q^{\{r\}}(2^j(x-1))\rangle\ 2^{j/2}\psi_q^{\{r\}}(2^j(x-1)) \end{array}$$

8. Bi-orthogonal multi-resolution analysis on the L-shaped domain: scaling functions.

We consider again the L-shaped domain Ω and the space $V_j(\Omega)$.

Theorem 4:

Let $\Omega = \{(x, y) \in (-1,1)^2 \ / \ x < 0 \text{ or } y < 0\}$. Let $Q_1 = (-1,0)\times(-1,0)$, $Q_2 = (-1,0)\times[0,1)$ and $Q_3 = [0,1)\times(-1,0)$. Let R_i be the restriction operator to Q_i : $f \in L^2(\Omega) \mapsto R_i f = f_{|Q_i} \in L^2(Q_i)$. Let $E_{j,1}$, $E_2^{(0)}$ and $E_3^{(0)}$ be the extension operators

$E_{j,1} : V_j^- \otimes V_j^- \mapsto L^2(\Omega)$, $(\varphi_{j,k_1}\otimes\varphi_{j,k_2})_{|(-1,0)\times(-1,0)} \mapsto (\varphi_{j,k_1}\otimes\varphi_{j,k_2})_{|\Omega}$

$E_2^{(0)} : L^2(Q_2) \mapsto L^2(\Omega)$, $(E_2^{(0)} f)_{|Q_2} = f$ and $(E_2^{(0)} f)_{|\Omega - Q_2} = 0$.

$E_3^{(0)} : L^2(Q_3) \mapsto L^2(\Omega)$, $(E_3^{(0)} f)_{|Q_3} = f$ and $(E_3^{(0)} f)_{|\Omega - Q_3} = 0$.

Then, we may decompose $V_j(\Omega)$ in a direct sum

$$V_j(\Omega) = E_{j,1}(V_j^- \otimes V_j^-)\ \oplus\ E_2^{(0)}(V_j^- \otimes V_j^+)\ \oplus\ E_3^{(0)}(V_j^+ \otimes V_j^-).$$

. Moreover, the following operators are projection operators :

- *$P_{j,1} = E_{j,1}\circ(P_j^-\otimes P_j^-)\circ R_1$ is a projection operator from $L^2(\Omega)$ onto $E_{j,1}(V_j^-\otimes V_j^-)$*
- *$P_{j,2} = E_2^{(0)}\circ(P_j^-\otimes P_j^+)\circ R_2$ is a projection operator from $L^2(\Omega)$ onto $E_2^{(0)}(V_j^-\otimes V_j^+)$*
- *$P_{j,3} = E_3^{(0)}\circ(P_j^+\otimes P_j^-)\circ R_3$ is a projection operator from $L^2(\Omega)$ onto $E_3^{(0)}(V_j^+\otimes V_j^-)$*
- *$P_j = P_{j,1} + P_{j,2} + P_{j,3}$ is a projection operator from $L^2(\Omega)$ onto $V_j(\Omega)$.*

Moreover, the sequence of operators $(P_j)_{j\ge j_0}$ (with $2^{j_0} \ge 4N-2$) is equicontinuous on L^2.

Proof: According to Proposition 4, we have a Riesz basis of $V_j(\Omega)$ by taking the family $\phi_{(j,k_1,k_2)} = (\varphi_{j,k_1}\otimes\varphi_{j,k_2})_{|\Omega}$, $-2^j-2N+2 \le k_1 \le 2^j-1$, $-2^j-2N+2 \le k_2 \le 2^j-1$ and $k_1 < 0$ or $k_2 < 0$. By distinguishing three subfamilies by considering the cases $k_1 < 0$ and $k_2 < 0$, $k_2 \ge 0$, $k_1 \ge 0$, we decompose $V_j(\Omega)$ into the direct sum $V_j(\Omega) = E_{j,1}(V_j^-\otimes V_j^-)\ \oplus\ E_2^{(0)}(V_j^-\otimes V_j^+)\ \oplus\ E_3^{(0)}(V_j^+\otimes V_j^-)$. $P_{j,i}$ are obviously projection operators, since $R_1\circ E_{j,1} = Id_{Q_1}$, $R_2\circ E_2^{(0)} = Id_{Q_2}$ and $R_3\circ E_3^{(0)} = Id_{Q_3}$. P_j is easily checked to be a projection operator since Im $P_{j,1}\subset$ Ker $P_{j,2}\cap$ Ker $P_{j,3}$, Im $P_{j,2}\subset$ Ker $P_{j,1}\cap$ Ker $P_{j,3}$ and Im $P_{j,3}\subset$ Ker $P_{j,1}\cap$ Ker $P_{j,2}$.

Moreover, the equicontinuity of the family $(P_j)_{j\ge j_0}$ is a direct consequence of the equicontinuity of the family $(E_{j,1})_{j\ge j_0}$ ⋄

We may give a more precise description of the projection operators P_j. We define $E_1^{(0)}$ as $E_1^{(0)}$: $L^2(Q_1)\mapsto L^2(\Omega)$, $(E_1^{(0)} f)_{|Q_1} = f$ and $(E_2^{(0)} f)_{|\Omega - Q_1} = 0$. Then:

Theorem 5:

Let P_j, $P_{j,1}$, $P_{j,2}$ and $P_{j,3}$ be the projection operators introduced in Theorem 4. Then :

- *Ran* $P_{j,1} = E_{j,1}(V_j^- \otimes V_j^-)$ *and* *Ran* $P_{j,1}^* = E_1^{(0)}(V_j^- \otimes V_j^-)$
- *Ran* $P_{j,2} =$ *Ran* $P_{j,2}^* = E_2^{(0)}(V_j^- \otimes V_j^+)$
- *Ran* $P_{j,3} =$ *Ran* $P_{j,3}^* = E_3^{(0)}(V_j^+ \otimes V_j^-)$
- *Ran* $P_j = V_j(\Omega)$ *and* *Ran* $P_j^* = E_1^{(0)}(V_j^- \otimes V_j^-) + E_2^{(0)}(V_j^- \otimes V_j^+) + E_3^{(0)}(V_j^+ \otimes V_j^-)$.

In particular, $P_{j+1} - P_j$ is a projection operator.

Moreover, using Meyer's analysis, we find that for j such that $2^j \geq 2N-1$ we have

i) $P_{j,1}f = \sum_{-2^j-2N+2\leq k_1\leq -1} \sum_{-2^j-2N+2\leq k_2\leq -1} \langle f|\phi^*_{j,k_1,k_2}\rangle \phi_{j,k_1,k_2}$ *with*

- $\phi_{j,k_1,k_2} = 2^j\varphi(2^jx_1-k_1)\varphi(2^jx_2-k_2)_{|\Omega}$
- $\phi^*_{j,k_1,k_2} = 2^j\varphi(2^jx_1-k_1)\varphi(2^jx_2-k_2)$, $-2^j \leq k_1 \leq -2N+1$, $-2^j \leq k_2 \leq -2N+1$
- $\phi^*_{j,k_1,k_2} = E_1^{(0)}\big(2^j\varphi_p^{[l]}(2^j(x_1+1))\varphi(2^jx_2-k_2)\big)$, $1\leq p\leq 2N-2$, $-2^j \leq k_2 \leq -2N+1$, $k_1 = -2^j-p$
- $\phi^*_{j,k_1,k_2} = E_1^{(0)}\big(2^j\varphi_p^{[r]}(2^jx_1)\varphi(2^jx_2-k_2)\big)$, $1\leq p\leq 2N-2$, $-2^j \leq k_2 \leq -2N+1$, $k_1 = -p$
- $\phi^*_{j,k_1,k_2} = E_1^{(0)}\big(2^j\varphi(2^jx_1-k_1)\varphi_p^{[l]}(2^j(x_2+1))\big)$, $-2^j \leq k_1 \leq -2N+1$, $1\leq p\leq 2N-2$, $k_2 = -2^j-p$
- $\phi^*_{j,k_1,k_2} = E_1^{(0)}\big(2^j\varphi(2^jx_1-k_1)\varphi_p^{[r]}(2^jx_2)\big)$, $-2^j \leq k_1 \leq -2N+1$, $1\leq p\leq 2N-2$, $k_2 = -p$
- $\phi^*_{j,k_1,k_2} = E_1^{(0)}\big(2^j\varphi_p^{[l]}(2^j(x_1+1))\varphi_q^{[l]}(2^j(x_2+1))\big)$, $1\leq p\leq 2N-2$, $1\leq q\leq 2N-2$, $k_1 = -2^j-p$, $k_2 = -2^j-q$
- $\phi^*_{j,k_1,k_2} = E_1^{(0)}\big(2^j\varphi_p^{[l]}(2^j(x_1+1))\varphi_q^{[r]}(2^jx_2)\big)$, $1\leq p\leq 2N-2$, $1\leq q\leq 2N-2$, $k_1 = -2^j-p$, $k_2 = -q$
- $\phi^*_{j,k_1,k_2} = E_1^{(0)}\big(2^j\varphi_p^{[r]}(2^jx_1)\varphi_q^{[l]}(2^j(x_2+1))\big)$, $1\leq p\leq 2N-2$, $1\leq q\leq 2N-2$, $k_1 = -p$, $k_2 = -2^j-q$
- $\phi^*_{j,k_1,k_2} = E_1^{(0)}\big(2^j\varphi_p^{[r]}(2^jx_1)\varphi_q^{[r]}(2^jx_2)\big)$, $1\leq p\leq 2N-2$, $1\leq q\leq 2N-2$, $k_1 = -p$, $k_2 = -q$

ii) $P_{j,2}f = \sum_{-2^j-2N+2\leq k_1\leq -1} \sum_{0\leq k_2\leq 2^j-1} \langle f|\phi^*_{j,k_1,k_2}\rangle \phi_{j,k_1,k_2}$ *with*

- $\phi_{j,k_1,k_2} = 2^j\varphi(2^jx_1-k_1)\varphi(2^jx_2-k_2)_{|\Omega}$
- $\phi^*_{j,k_1,k_2} = 2^j\varphi(2^jx_1-k_1)\varphi(2^jx_2-k_2)$, $-2^j \leq k_1 \leq -2N+1$, $0 \leq k_2 \leq 2^j-2N+1$
- $\phi^*_{j,k_1,k_2} = E_2^{(0)}\big(2^j\varphi_p^{[l]}(2^j(x_1+1))\varphi(2^jx_2-k_2)\big)$, $1\leq p\leq 2N-2$, $0 \leq k_2 \leq 2^j-2N+1$, $k_1 = -2^j-p$
- $\phi^*_{j,k_1,k_2} = E_2^{(0)}\big(2^j\varphi_p^{[r]}(2^jx_1)\varphi(2^jx_2-k_2)\big)$, $1\leq p\leq 2N-2$, $0 \leq k_2 \leq 2^j-2N+1$, $k_1 = -p$
- $\phi^*_{j,k_1,k_2} = E_2^{(0)}\big(2^j\varphi(2^jx_1-k_1)\varphi_p^{[r]}(2^j(x_2-1))\big)$, $-2^j \leq k_1 \leq -2N+1$, $1\leq p\leq 2N-2$, $k_2 = 2^j-p$
- $\phi^*_{j,k_1,k_2} = E_2^{(0)}\big(2^j\varphi_p^{[l]}(2^j(x_1+1))\varphi_q^{[r]}(2^j(x_2-1))\big)$, $1\leq p\leq 2N-2$, $1\leq q\leq 2N-2$, $k_1 = -2^j-p$, $k_2 = 2^j-q$
- $\phi^*_{j,k_1,k_2} = E_2^{(0)}\big(2^j\varphi_p^{[r]}(2^jx_1)\varphi_q^{[r]}(2^j(x_2-1))\big)$, $1\leq p\leq 2N-2$, $1\leq q\leq 2N-2$, $k_1 = -p$, $k_2 = 2^j-q$

iii) $P_{j,3}f = \sum_{0\leq k_1\leq 2^j-1} \sum_{-2^j-2N+2\leq k_2\leq -1} \langle f|\phi^*_{j,k_1,k_2}\rangle \phi_{j,k_1,k_2}$ *with*

- $\phi_{j,k_1,k_2} = 2^j\varphi(2^jx_1-k_1)\varphi(2^jx_2-k_2)_{|\Omega}$
- $\phi^*_{j,k_1,k_2} = 2^j\varphi(2^jx_1-k_1)\varphi(2^jx_2-k_2)$, $0 \leq k_1 \leq 2^j-2N+1$, $-2^j \leq k_2 \leq -2N+1$
- $\phi^*_{j,k_1,k_2} = E_3^{(0)}\big(2^j\varphi_p^{[r]}(2^j(x_1-1))\varphi(2^jx_2-k_2)\big)$, $1\leq p\leq 2N-2$, $-2^j \leq k_2 \leq -2N+1$, $k_1 = 2^j-p$
- $\phi^*_{j,k_1,k_2} = E_3^{(0)}\big(2^j\varphi(2^jx_1-k_1)\varphi_p^{[l]}(2^j(x_2+1))\big)$, $0 \leq k_1 \leq 2^j-2N+1$, $1\leq p\leq 2N-2$, $k_2 = -2^j-p$
- $\phi^*_{j,k_1,k_2} = E_3^{(0)}\big(2^j\varphi(2^jx_1-k_1)\varphi_p^{[r]}(2^jx_2)\big)$, $0 \leq k_1 \leq 2^j-2N+1$, $1\leq p\leq 2N-2$, $k_2 = -p$
- $\phi^*_{j,k_1,k_2} = E_3^{(0)}\big(2^j\varphi_p^{[r]}(2^j(x_1-1))\varphi_q^{[l]}(2^j(x_2+1))\big)$, $1\leq p\leq 2N-2$, $1\leq q\leq 2N-2$, $k_1 = 2^j-p$, $k_2 = -2^j-q$
- $\phi^*_{j,k_1,k_2} = E_3^{(0)}\big(2^j\varphi_p^{[r]}(2^j(x_1-1))\varphi_q^{[r]}(2^jx_2)\big)$, $1\leq p\leq 2N-2$, $1\leq q\leq 2N-2$, $k_1 = 2^j-p$, $k_2 = -q$

iv) Let $\Delta_j = \{(k_1,k_2)\in \mathbb{Z}^2 \ / \ -2^j-2N+2 \leq k_1 \leq 2^j-1,\ -2^j-2N+2 \leq k_2 \leq 2^j-1, k_1<0$ *or* $k_2<0\}$. *Then*

$$P_jf = \sum\sum_{(k_1,k_2)\in\Delta_j} \langle f|\phi^*_{j,k_1,k_2}\rangle\ \phi_{j,k_1,k_2}$$

Proof: This is a direct consequence of Meyer's analysis. $P_{j+1}-P_j$ is a projection operator, since we find that Ran $P_j \subset$ Ran P_{j+1} and Ran $P_j^* \subset$ Ran P_{j+1}^*. $\diamond$

9. Bi-orthogonal multi-resolution analysis on the L-shaped domain: wavelets.

Theorem 1 and Theorem 4 give that the family of projection operators $(P_j)_{j\geq j_0}$ is well-adapted to the study of Sobolev regularity on Ω. Moreover, from Theorem 5, we may prove that this analysis is linked to some unconditional basis for $H^s(\Omega)$, $0<s<\sigma$. We shall state it now in a precise way, prove that the basis is as well a Riesz basis for $L^2(\Omega)$ and explain the connexion with Cieselski and Figiel's method.

Theorem 6:

Let $\Omega=\{(x,y)\in(-1,1)^2\ /\ x<0$ or $y<0\}$. Let $\varphi\in H^\sigma$. Let j_0 be such that $2^{j_0}\geq 4N-2$. Then :

A) The following family $(\psi_{1,\lambda})_{\lambda\in\Lambda_1}$ is a Hilbertian basis of $L^2(Q_1)$ and a Riesz basis of $H^s(Q_1)$ for $s\in(0,\sigma)$:

- $\psi_{1,\lambda}=2^{j_0}\varphi(2^{j_0}x_1-k_1)\varphi(2^{j_0}x_2-k_2)$, $-2^{j_0}\leq k_1\leq -2N+1$, $-2^{j_0}\leq k_2\leq -2N+1$, $\lambda=(j_0,0,0,k_1,k_2)$
- $\psi_{1,\lambda}=2^{j_0}\varphi_p^{\{l\}}(2^{j_0}(x_1+1))\varphi(2^{j_0}x_2-k_2)$, $1\leq p\leq 2N-2$, $-2^{j_0}\leq k_2\leq -2N+1$, $\lambda=(j_0,0,0,-2^{j_0}-p,k_2)$
- $\psi_{1,\lambda}=2^{j_0}\varphi_p^{\{r\}}(2^{j_0}x_1)\varphi(2^{j_0}x_2-k_2)$, $1\leq p\leq 2N-2$, $-2^{j_0}\leq k_2\leq -2N+1$, $\lambda=(j_0,0,0,-p,k_2)$
- $\psi_{1,\lambda}=2^{j_0}\varphi(2^{j_0}x_1-k_1)\varphi_p^{\{l\}}(2^{j_0}(x_2+1))$, $-2^{j_0}\leq k_1\leq -2N+1$, $1\leq p\leq 2N-2$, $\lambda=(j_0,0,0,k_1,-2^{j_0}-p)$
- $\psi_{1,\lambda}=2^{j_0}\varphi(2^{j_0}x_1-k_1)\varphi_p^{\{r\}}(2^{j_0}x_2)$, $-2^{j_0}\leq k_1\leq -2N+1$, $1\leq p\leq 2N-2$, $\lambda=(j_0,0,0,k_1,-p)$
- $\psi_{1,\lambda}=2^{j_0}\varphi_p^{\{l\}}(2^{j_0}(x_1+1))\varphi_q^{\{l\}}(2^{j_0}(x_2+1))$, $1\leq p\leq 2N-2$, $1\leq q\leq 2N-2$, $\lambda=(j_0,0,0,-2^{j_0}-p,-2^{j_0}-q)$
- $\psi_{1,\lambda}=2^{j_0}\varphi_p^{\{l\}}(2^{j_0}(x_1+1))\varphi_q^{\{r\}}(2^{j_0}x_2)$, $1\leq p\leq 2N-2$, $1\leq q\leq 2N-2$, $\lambda=(j_0,0,0,-2^{j_0}-p,-q)$
- $\psi_{1,\lambda}=2^{j_0}\varphi_p^{\{r\}}(2^{j_0}x_1)\varphi_q^{\{l\}}(2^{j_0}(x_2+1))$, $1\leq p\leq 2N-2$, $1\leq q\leq 2N-2$, $\lambda=(j_0,0,0,-p,-2^{j_0}-q)$
- $\psi_{1,\lambda}=2^{j_0}\varphi_p^{\{r\}}(2^{j_0}x_1)\varphi_q^{\{r\}}(2^{j_0}x_2)$, $1\leq p\leq 2N-2$, $1\leq q\leq 2N-2$, $\lambda=(j_0,0,0,-p,-q)$
- $\psi_{1,\lambda}=2^{j}\varphi(2^{j}x_1-k_1)\psi(2^{j}x_2-k_2)$, $j\geq j_0$, $-2^{j}\leq k_1\leq -2N+1$, $-2^{j}\leq k_2\leq -2N+1$, $\lambda=(j+1,0,1,k_1,k_2)$
- $\psi_{1,\lambda}=2^{j}\varphi_p^{\{l\}}(2^{j}(x_1+1))\psi(2^{j}x_2-k_2)$, $j\geq j_0$, $1\leq p\leq 2N-2$, $-2^{j}\leq k_2\leq -2N+1$, $\lambda=(j+1,0,1,-2^{j}-p,k_2)$
- $\psi_{1,\lambda}=2^{j}\varphi_p^{\{r\}}(2^{j}x_1)\psi(2^{j}x_2-k_2)$, $j\geq j_0$, $1\leq p\leq 2N-2$, $-2^{j}\leq k_2\leq -2N+1$, $\lambda=(j+1,0,1,-p,k_2)$
- $\psi_{1,\lambda}=2^{j}\varphi(2^{j}x_1-k_1)\psi_p^{\{l\}}(2^{j}(x_2+1))$, $j\geq j_0$, $-2^{j}\leq k_1\leq -2N+1$, $1\leq p\leq N-1$, $\lambda=(j+1,0,1,k_1,-2^{j}-p)$
- $\psi_{1,\lambda}=2^{j}\varphi(2^{j}x_1-k_1)\psi_p^{\{r\}}(2^{j}x_2)$, $j\geq j_0$, $-2^{j}\leq k_1\leq -2N+1$, $1\leq p\leq N-1$, $\lambda=(j+1,0,1,k_1,-p-N+1)$
- $\psi_{1,\lambda}=2^{j}\varphi_p^{\{l\}}(2^{j}(x_1+1))\psi_q^{\{l\}}(2^{j}(x_2+1))$, $j\geq j_0$, $1\leq p\leq 2N-2$, $1\leq q\leq N-1$, $\lambda=(j+1,0,1,-2^{j}-p,-2^{j}-q)$
- $\psi_{1,\lambda}=2^{j}\varphi_p^{\{l\}}(2^{j}(x_1+1))\psi_q^{\{r\}}(2^{j}x_2)$, $j\geq j_0$, $1\leq p\leq 2N-2$, $1\leq q\leq N-1$, $\lambda=(j+1,0,1,-2^{j}-p,-q-N+1)$
- $\psi_{1,\lambda}=2^{j}\varphi_p^{\{r\}}(2^{j}x_1)\psi_q^{\{l\}}(2^{j}(x_2+1))$, $j\geq j_0$, $1\leq p\leq 2N-2$, $1\leq q\leq N-1$, $\lambda=(j+1,0,1,-p,-2^{j}-q)$
- $\psi_{1,\lambda}=2^{j}\varphi_p^{\{r\}}(2^{j}x_1)\psi_q^{\{r\}}(2^{j}x_2)$, $j\geq j_0$, $1\leq p\leq 2N-2$, $1\leq q\leq N-1$, $\lambda=(j+1,0,1,-p,-q-N+1)$
- $\psi_{1,\lambda}=2^{j}\psi(2^{j}x_1-k_1)\varphi(2^{j}x_2-k_2)$, $j\geq j_0$, $-2^{j}\leq k_1\leq -2N+1$, $-2^{j}\leq k_2\leq -2N+1$, $\lambda=(j+1,1,0,k_1,k_2)$
- $\psi_{1,\lambda}=2^{j}\psi_p^{\{l\}}(2^{j}(x_1+1))\varphi(2^{j}x_2-k_2)$, $j\geq j_0$, $1\leq p\leq N-1$, $-2^{j}\leq k_2\leq -2N+1$, $\lambda=(j+1,1,0,-2^{j}-p,k_2)$
- $\psi_{1,\lambda}=2^{j}\psi_p^{\{r\}}(2^{j}x_1)\varphi(2^{j}x_2-k_2)$, $j\geq j_0$, $1\leq p\leq N-1$, $-2^{j}\leq k_2\leq -2N+1$, $\lambda=(j+1,1,0,-p-N+1,k_2)$
- $\psi_{1,\lambda}=2^{j}\psi(2^{j}x_1-k_1)\varphi_p^{\{l\}}(2^{j}(x_2+1))$, $j\geq j_0$, $-2^{j}\leq k_1\leq -2N+1$, $1\leq p\leq 2N-2$, $\lambda=(j+1,1,0,k_1,-2^{j}-p)$
- $\psi_{1,\lambda}=2^{j}\psi(2^{j}x_1-k_1)\varphi_p^{\{r\}}(2^{j}x_2)$, $j\geq j_0$, $-2^{j}\leq k_1\leq -2N+1$, $1\leq p\leq 2N-2$, $\lambda=(j+1,1,0,k_1,-p)$
- $\psi_{1,\lambda}=2^{j}\psi_p^{\{l\}}(2^{j}(x_1+1))\varphi_q^{\{l\}}(2^{j}(x_2+1))$, $j\geq j_0$, $1\leq p\leq N-1$, $1\leq q\leq 2N-2$, $\lambda=(j+1,1,0,-2^{j}-p,-2^{j}-q)$
- $\psi_{1,\lambda}=2^{j}\psi_p^{\{l\}}(2^{j}(x_1+1))\varphi_q^{\{r\}}(2^{j}x_2)$, $j\geq j_0$, $1\leq p\leq N-1$, $1\leq q\leq 2N-2$, $\lambda=(j+1,1,0,-2^{j}-p,-q)$
- $\psi_{1,\lambda}=2^{j}\psi_p^{\{r\}}(2^{j}x_1)\varphi_q^{\{l\}}(2^{j}(x_2+1))$, $j\geq j_0$, $1\leq p\leq N-1$, $1\leq q\leq 2N-2$, $\lambda=(j+1,1,0,-p-N+1,-2^{j}-q)$
- $\psi_{1,\lambda}=2^{j}\psi_p^{\{r\}}(2^{j}x_1)\varphi_q^{\{r\}}(2^{j}x_2)$, $j\geq j_0$, $1\leq p\leq N-1$, $1\leq q\leq 2N-2$, $\lambda=(j+1,1,0,-p-N+1,-q)$
- $\psi_{1,\lambda}=2^{j}\psi(2^{j}x_1-k_1)\psi(2^{j}x_2-k_2)$, $j\geq j_0$, $-2^{j}\leq k_1\leq -2N+1$, $-2^{j}\leq k_2\leq -2N+1$, $\lambda=(j+1,1,1,k_1,k_2)$
- $\psi_{1,\lambda}=2^{j}\psi_p^{\{l\}}(2^{j}(x_1+1))\psi(2^{j}x_2-k_2)$, $j\geq j_0$, $1\leq p\leq N-1$, $-2^{j}\leq k_2\leq -2N+1$, $\lambda=(j+1,1,1,-2^{j}-p,k_2)$

• $\psi_{1,\lambda} = 2^j \psi_p^{\{r\}}(2^j x_1)\psi(2^j x_2 - k_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $-2^j \leq k_2 \leq -2N+1$, $\lambda = (j+1, 1, 1, -p-N+1, k_2)$
• $\psi_{1,\lambda} = 2^j \psi(2^j x_1 - k_1)\psi_p^{\{l\}}(2^j(x_2+1))$, $j \geq j_0$, $-2^j \leq k_1 \leq -2N+1$, $1 \leq p \leq N-1$, $\lambda = (j+1, 1, 1, k_1, -2^j - p)$
• $\psi_{1,\lambda} = 2^j \psi(2^j x_1 - k_1)\psi_p^{\{r\}}(2^j x_2)$, $j \geq j_0$, $-2^j \leq k_1 \leq -2N+1$, $1 \leq p \leq N-1$, $\lambda = (j+1, 1, 1, k_1, -p-N+1)$
• $\psi_{1,\lambda} = 2^j \psi_p^{\{l\}}(2^j(x_1+1))\psi_q^{\{l\}}(2^j(x_2+1))$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq N-1$, $\lambda = (j+1, 1, 1, -2^j - p, -2^j - q)$
• $\psi_{1,\lambda} = 2^j \psi_p^{\{l\}}(2^j(x_1+1))\psi_q^{\{r\}}(2^j x_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq N-1$, $\lambda = (j+1, 1, 1, -2^j - p, -q - N+1)$
• $\psi_{1,\lambda} = 2^j \psi_p^{\{r\}}(2^j x_1)\psi_q^{\{l\}}(2^j(x_2+1))$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq N-1$, $\lambda = (j+1, 1, 1, -p-N+1, -2^j - q)$
• $\psi_{1,\lambda} = 2^j \psi_p^{\{r\}}(2^j x_1)\psi_q^{\{r\}}(2^j x_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq N-1$, $\lambda = (j+1, 1, 1, -p-N+1, -q-N+1)$.

More precisely, we have for $f \in L^2(Q_1)$,

$$f = \sum_{\lambda \in \Lambda_1} \langle f | \psi_{1,\lambda} \rangle \ \psi_{1,\lambda}$$

and, for $s \in [0, \sigma)$, there exists two positive constants A_s and B_s such that

$$\text{for } f \in H^s(Q_1), \quad A_s \|f\|_{H^s}^2 \leq \sum_{\lambda \in \Lambda_1} 4^{j(\lambda)} |\langle f | \psi_{1,\lambda} \rangle|^2 \leq B_S \|f\|_{H^s}^2$$

B) The following family $(\psi_{2,\lambda})_{\lambda \in \Lambda_2}$ is a Hilbertian basis of $L^2(Q_2)$ and a Riesz basis of $H^{s,2}(Q_2)$ for $s \in (0, \sigma)$:
• $\psi_{2,\lambda} = 2^{j_0} \varphi(2^{j_0} x_1 - k_1)\varphi(2^{j_0} x_2 - k_2)$, $-2^{j_0} \leq k_1 \leq -2N+1$, $0 \leq k_2 \leq 2^{j_0} - 2N+1$, $\lambda = (j_0, 0, 0, k_1, k_2)$
• $\psi_{2,\lambda} = 2^{j_0} \varphi_p^{\{l\}}(2^{j_0}(x_1+1))\varphi(2^{j_0} x_2 - k_2)$, $1 \leq p \leq 2N-2$, $0 \leq k_2 \leq 2^{j_0} - 2N+1$, $\lambda = (j_0, 0, 0, -2^{j_0} - p, k_2)$
• $\psi_{2,\lambda} = 2^{j_0} \varphi_p^{\{r\}}(2^{j_0} x_1)\varphi(2^{j_0} x_2 - k_2)$, $1 \leq p \leq 2N-2$, $0 \leq k_2 \leq 2^{j_0} - 2N+1$, $\lambda = (j_0, 0, 0, -p, k_2)$
• $\psi_{2,\lambda} = 2^{j_0} \varphi(2^{j_0} x_1 - k_1)\varphi_p^{\{r\}}(2^{j_0}(x_2-1))$, $-2^{j_0} \leq k_1 \leq -2N+1$, $1 \leq p \leq 2N-2$, $\lambda = (j_0, 0, 0, k_1, 2^{j_0} - p)$
• $\psi_{2,\lambda} = 2^{j_0} \varphi_p^{\{l\}}(2^{j_0}(x_1+1))\varphi_q^{\{r\}}(2^{j_0}(x_2-1))$, $1 \leq p \leq 2N-2$, $1 \leq q \leq 2N-2$, $\lambda = (j_0, 0, 0, -2^{j_0} - p, 2^{j_0} - q)$
• $\psi_{2,\lambda} = 2^{j_0} \varphi_p^{\{r\}}(2^{j_0} x_1)\varphi_q^{\{r\}}(2^{j_0}(x_2-1))$, $1 \leq p \leq 2N-2$, $1 \leq q \leq 2N-2$, $\lambda = (j_0, 0, 0, -p, 2^{j_0} - q)$
• $\psi_{2,\lambda} = 2^j \varphi(2^j x_1 - k_1)\psi(2^j x_2 - k_2)$, $j \geq j_0$, $-2^j \leq k_1 \leq -2N+1$, $0 \leq k_2 \leq 2^j - 2N+1$, $\lambda = (j+1, 0, 1, k_1, k_2)$
• $\psi_{2,\lambda} = 2^j \varphi_p^{\{l\}}(2^j(x_1+1))\psi(2^j x_2 - k_2)$, $j \geq j_0$, $1 \leq p \leq 2N-2$, $0 \leq k_2 \leq 2^j - 2N+1$, $\lambda = (j+1, 0, 1, -2^j - p, k_2)$
• $\psi_{2,\lambda} = 2^j \varphi_p^{\{r\}}(2^j x_1)\psi(2^j x_2 - k_2)$, $j \geq j_0$, $1 \leq p \leq 2N-2$, $0 \leq k_2 \leq 2^j - 2N+1$, $\lambda = (j+1, 0, 1, -p, k_2)$
• $\psi_{2,\lambda} = 2^j \varphi(2^j x_1 - k_1)\Psi_p^{\{l\}}(2^j x_2)$, $j \geq j_0$, $-2^j \leq k_1 \leq -2N+1$, $1 \leq p \leq N-1$, $\lambda = (j+1, 0, 1, k_1, -p)$
• $\psi_{2,\lambda} = 2^j \varphi(2^j x_1 - k_1)\psi_p^{\{r\}}(2^j(x_2-1))$, $j \geq j_0$, $-2^j \leq k_1 \leq -2N+1$, $1 \leq p \leq N-1$, $\lambda = (j+1, 0, 1, k_1, 2^j - p - N+1)$
• $\psi_{2,\lambda} = 2^j \varphi_p^{\{l\}}(2^j(x_1+1))\Psi_q^{\{l\}}(2^j x_2)$, $j \geq j_0$, $1 \leq p \leq 2N-2$, $1 \leq q \leq N-1$, $\lambda = (j+1, 0, 1, -2^j - p, -q)$
• $\psi_{2,\lambda} = 2^j \varphi_p^{\{l\}}(2^j(x_1+1))\psi_q^{\{r\}}(2^j(x_2-1))$, $j \geq j_0$, $1 \leq p \leq 2N-2$, $1 \leq q \leq N-1$, $\lambda = (j+1, 0, 1, -2^j - p, 2^j - q - N+1)$
• $\psi_{2,\lambda} = 2^j \varphi_p^{\{r\}}(2^j x_1)\Psi_q^{\{l\}}(2^j x_2)$, $j \geq j_0$, $1 \leq p \leq 2N-2$, $1 \leq q \leq N-1$, $\lambda = (j+1, 0, 1, -p, -q)$
• $\psi_{2,\lambda} = 2^j \varphi_p^{\{r\}}(2^j x_1)\psi_q^{\{r\}}(2^j(x_2-1))$, $j \geq j_0$, $1 \leq p \leq 2N-2$, $1 \leq q \leq N-1$, $\lambda = (j+1, 0, 1, -p, 2^j - q - N+1)$
• $\psi_{2,\lambda} = 2^j \psi(2^j x_1 - k_1)\varphi(2^j x_2 - k_2)$, $j \geq j_0$, $-2^j \leq k_1 \leq -2N+1$, $0 \leq k_2 \leq 2^j - 2N+1$, $\lambda = (j+1, 1, 0, k_1, k_2)$
• $\psi_{2,\lambda} = 2^j \psi_p^{\{l\}}(2^j(x_1+1))\varphi(2^j x_2 - k_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $0 \leq k_2 \leq 2^j - 2N+1$, $\lambda = (j+1, 1, 0, -2^j - p, k_2)$
• $\psi_{2,\lambda} = 2^j \psi_p^{\{r\}}(2^j x_1)\varphi(2^j x_2 - k_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $0 \leq k_2 \leq 2^j - 2N+1$, $\lambda = (j+1, 1, 0, -p-N+1, k_2)$
• $\psi_{2,\lambda} = 2^j \psi(2^j x_1 - k_1)\varphi_p^{\{r\}}(2^j(x_2-1))$, $j \geq j_0$, $-2^j \leq k_1 \leq -2N+1$, $1 \leq p \leq 2N-2$, $\lambda = (j+1, 1, 0, k_1, 2^j - p)$

• $\psi_{2,\lambda} = 2^j \psi_p^{\{l\}}(2^j(x_1+1))\varphi_q^{\{r\}}(2^j(x_2-1))$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq 2N-2$, $\lambda = (j+1, 1, 0, -2^j - p, 2^j - q)$
• $\psi_{2,\lambda} = 2^j \psi_p^{\{r\}}(2^j x_1)\varphi_q^{\{r\}}(2^j(x_2-1))$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq 2N-2$, $\lambda = (j+1, 1, 0, -p-N+1, 2^j - q)$
• $\psi_{2,\lambda} = 2^j \psi(2^j x_1 - k_1)\psi(2^j x_2 - k_2)$, $j \geq j_0$, $-2^j \leq k_1 \leq -2N+1$, $0 \leq k_2 \leq 2^j - 2N+1$, $\lambda = (j+1, 1, 1, k_1, k_2)$
• $\psi_{2,\lambda} = 2^j \psi_p^{\{l\}}(2^j(x_1+1))\psi(2^j x_2 - k_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $0 \leq k_2 \leq 2^j - 2N+1$, $\lambda = (j+1, 1, 1, -2^j - p, k_2)$
• $\psi_{2,\lambda} = 2^j \psi_p^{\{r\}}(2^j x_1)\psi(2^j x_2 - k_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $0 \leq k_2 \leq 2^j - 2N+1$, $\lambda = (j+1, 1, 1, -p-N+1, k_2)$
• $\psi_{2,\lambda} = 2^j \psi(2^j x_1 - k_1)\Psi_p^{\{l\}}(2^j x_2)$, $j \geq j_0$, $-2^j \leq k_1 \leq -2N+1$, $1 \leq p \leq N-1$, $\lambda = (j+1, 1, 1, k_1, -p)$
• $\psi_{2,\lambda} = 2^j \psi(2^j x_1 - k_1)\psi_p^{\{r\}}(2^j(x_2-1))$, $j \geq j_0$, $-2^j \leq k_1 \leq -2N+1$, $1 \leq p \leq N-1$, $\lambda = (j+1, 1, 1, k_1, 2^j - p - N + 1)$
• $\psi_{2,\lambda} = 2^j \psi_p^{\{l\}}(2^j(x_1+1))\Psi_q^{\{l\}}(2^j x_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq N-1$, $\lambda = (j+1, 1, 1, -2^j - p, -q)$
• $\psi_{2,\lambda} = 2^j \psi_p^{\{l\}}(2^j(x_1+1))\psi_q^{\{r\}}(2^j(x_2-1))$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq N-1$, $\lambda = (j+1, 1, 1, -2^j - p, 2^j - q - N + 1)$
• $\psi_{2,\lambda} = 2^j \psi_p^{\{r\}}(2^j x_1)\Psi_q^{\{l\}}(2^j x_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq N-1$, $\lambda = (j+1, 1, 1, -p-N+1, -q)$
• $\psi_{2,\lambda} = 2^j \psi_p^{\{r\}}(2^j x_1)\psi_q^{\{r\}}(2^j(x_2-1))$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq N-1$, $\lambda = (j+1, 1, 1, -p-N+1, 2^j - q - N + 1)$.

More precisely, we have for $f \in L^2(Q_2)$,

$$f = \sum_{\lambda \in \Lambda_2} \langle f | \psi_{2,\lambda} \rangle \; \psi_{2,\lambda}$$

and, for $s \in [0, \sigma)$, *there exists two positive constants* A_s *and* B_s *such that*

$$\text{for } f \in H^{s,2}(Q_2), \quad A_s \|f\|_{H^s}^2 \leq \sum_{\lambda \in \Lambda_2} 4^{j(\lambda)} |\langle f | \psi_{2,\lambda} \rangle|^2 \leq B_S \|f\|_{H^s}^2$$

C) The following family $(\psi_{3,\lambda})_{\lambda \in \Lambda_3}$ *is a Hilbertian basis of* $L^2(Q_3)$ *and a Riesz basis of* $H^{s,3}(Q_3)$ *for* $s \in (0, \sigma)$ *:*
• $\psi_{3,\lambda} = 2^{j_0} \varphi(2^{j_0} x_1 - k_1)\varphi(2^{j_0} x_2 - k_2)$, $0 \leq k_1 \leq 2^{j_0} - 2N+1$, $-2^{j_0} \leq k_2 \leq -2N+1$, $\lambda = (j_0, 0, 0, k_1, k_2)$
• $\psi_{3,\lambda} = 2^{j_0} \varphi_p^{\{r\}}(2^{j_0}(x_1-1))\varphi(2^{j_0} x_2 - k_2)$, $1 \leq p \leq 2N-2$, $-2^{j_0} \leq k_2 \leq -2N+1$, $\lambda = (j_0, 0, 0, 2^{j_0} - p, k_2)$
• $\psi_{3,\lambda} = 2^{j_0} \varphi(2^{j_0} x_1 - k_1)\varphi_p^{\{l\}}(2^{j_0}(x_2+1))$, $0 \leq k_1 \leq 2^{j_0} - 2N+1$, $1 \leq p \leq 2N-2$, $\lambda = (j_0, 0, 0, k_1, -2^{j_0} - p)$
• $\psi_{3,\lambda} = 2^{j_0} \varphi(2^{j_0} x_1 - k_1)\varphi_p^{\{r\}}(2^{j_0} x_2)$, $0 \leq k_1 \leq 2^{j_0} - 2N+1$, $1 \leq p \leq 2N-2$, $\lambda = (j_0, 0, 0, k_1, -p)$
• $\psi_{3,\lambda} = 2^{j_0} \varphi_p^{\{r\}}(2^{j_0}(x_1-1))\varphi_q^{\{l\}}(2^{j_0}(x_2+1))$, $1 \leq p \leq 2N-2$, $1 \leq q \leq 2N-2$, $\lambda = (j_0, 0, 0, 2^{j_0} - p, -2^{j_0} - q)$
• $\psi_{3,\lambda} = 2^{j_0} \varphi_p^{\{r\}}(2^{j_0}(x_1-1))\varphi_q^{\{r\}}(2^{j_0} x_2)$, $1 \leq p \leq 2N-2$, $1 \leq q \leq 2N-2$, $\lambda = (j_0, 0, 0, 2^{j_0} - p, -q)$
• $\psi_{3,\lambda} = 2^j \varphi(2^j x_1 - k_1)\psi(2^j x_2 - k_2)$, $j \geq j_0$, $0 \leq k_1 \leq 2^j - 2N+1$, $-2^j \leq k_2 \leq -2N+1$, $\lambda = (j+1, 0, 1, k_1, k_2)$
• $\psi_{3,\lambda} = 2^j \varphi_p^{\{r\}}(2^j(x_1-1))\psi(2^j x_2 - k_2)$, $j \geq j_0$, $1 \leq p \leq 2N-2$, $-2^j \leq k_2 \leq -2N+1$, $\lambda = (j+1, 0, 1, 2^j - p, k_2)$
• $\psi_{3,\lambda} = 2^j \varphi(2^j x_1 - k_1)\psi_p^{\{l\}}(2^j(x_2+1))$, $j \geq j_0$, $0 \leq k_1 \leq 2^j - 2N+1$, $1 \leq p \leq N-1$, $\lambda = (j+1, 0, 1, k_1, -2^j - p)$
• $\psi_{3,\lambda} = 2^j \varphi(2^j x_1 - k_1)\psi_p^{\{r\}}(2^j x_2)$, $j \geq j_0$, $0 \leq k_1 \leq 2^j - 2N+1$, $1 \leq p \leq N-1$, $\lambda = (j+1, 0, 1, k_1, -p-N+1)$
• $\psi_{3,\lambda} = 2^j \varphi_p^{\{r\}}(2^j(x_1-1))\psi_q^{\{l\}}(2^j(x_2+1))$, $j \geq j_0$, $1 \leq p \leq 2N-2$, $1 \leq q \leq N-1$, $\lambda = (j+1, 0, 1, 2^j - p, -2^j - q)$
• $\psi_{3,\lambda} = 2^j \varphi_p^{\{r\}}(2^j(x_1-1))\psi_q^{\{r\}}(2^j x_2)$, $j \geq j_0$, $1 \leq p \leq 2N-2$, $1 \leq q \leq N-1$, $\lambda = (j+1, 0, 1, 2^j - p, -q - N + 1)$
• $\psi_{3,\lambda} = 2^j \psi(2^j x_1 - k_1)\varphi(2^j x_2 - k_2)$, $j \geq j_0$, $0 \leq k_1 \leq 2^j - 2N+1$, $-2^j \leq k_2 \leq -2N+1$, $\lambda = (j+1, 1, 0, k_1, k_2)$
• $\psi_{3,\lambda} = 2^j \Psi_p^{\{l\}}(2^j x_1)\varphi(2^j x_2 - k_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $-2^j \leq k_2 \leq -2N+1$, $\lambda = (j+1, 1, 0, -p, k_2)$
• $\psi_{3,\lambda} = 2^j \psi_p^{\{r\}}(2^j(x_1-1))\varphi(2^j x_2 - k_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $-2^j \leq k_2 \leq -2N+1$, $\lambda = (j+1, 1, 0, 2^j - p - N + 1, k_2)$

• $\psi_{3,\lambda} = 2^j\psi(2^jx_1 - k_1)\varphi_p^{\{l\}}(2^j(x_2+1))$, $j \geq j_0$, $0 \leq k_1 \leq 2^j - 2N + 1$, $1 \leq p \leq 2N-2$, $\lambda = (j+1,1,0,k_1,-2^j-p)$
• $\psi_{3,\lambda} = 2^j\psi(2^jx_1 - k_1)\varphi_p^{\{r\}}(2^jx_2)$, $j \geq j_0$, $0 \leq k_1 \leq 2^j - 2N + 1$, $1 \leq p \leq 2N-2$, $\lambda = (j+1,1,0,k_1,-p)$
• $\psi_{3,\lambda} = 2^j\Psi_p^{\{l\}}(2^jx_1)\varphi_q^{\{l\}}(2^j(x_2+1))$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq 2N-2$, $\lambda = (j+1,1,0,-p,-2^j-q)$
• $\psi_{3,\lambda} = 2^j\Psi_p^{\{l\}}(2^jx_1)\varphi_q^{\{r\}}(2^jx_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq 2N-2$, $\lambda = (j+1,1,0,-p,-q)$
• $\psi_{3,\lambda} = 2^j\psi_p^{\{r\}}(2^j(x_1-1))\varphi_q^{\{l\}}(2^j(x_2+1))$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq 2N-2$, $\lambda = (j+1,1,0,2^j-p-N+1,-2^j-q)$
• $\psi_{3,\lambda} = 2^j\psi_p^{\{r\}}(2^j(x_1-1))\varphi_q^{\{r\}}(2^jx_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq 2N-2$, $\lambda = (j+1,1,0,2^j-p-N+1,-q)$
• $\psi_{3,\lambda} = 2^j\psi(2^jx_1-k_1)\psi(2^jx_2-k_2)$, $j \geq j_0$, $0 \leq k_1 \leq 2^j-2N+1$, $-2^j \leq k_2 \leq -2N+1$, $\lambda = (j+1,1,1,k_1,k_2)$
• $\psi_{3,\lambda} = 2^j\Psi_p^{\{l\}}(2^jx_1)\psi(2^jx_2-k_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $-2^j \leq k_2 \leq -2N+1$, $\lambda = (j+1,1,1,-p,k_2)$
• $\psi_{3,\lambda} = 2^j\psi_p^{\{r\}}(2^j(x_1-1))\psi(2^jx_2-k_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $-2^j \leq k_2 \leq -2N+1$, $\lambda = (j+1,1,1,2^j-p-N+1,k_2)$
• $\psi_{3,\lambda} = 2^j\psi(2^jx_1 - k_1)\psi_p^{\{l\}}(2^j(x_2+1))$, $j \geq j_0$, $0 \leq k_1 \leq 2^j - 2N + 1$, $1 \leq p \leq N-1$, $\lambda = (j+1,1,1,k_1,-2^j-p)$
• $\psi_{3,\lambda} = 2^j\psi(2^jx_1-k_1)\psi_p^{\{r\}}(2^jx_2)$, $j \geq j_0$, $0 \leq k_1 \leq 2^j-2N+1$, $1 \leq p \leq N-1$, $\lambda = (j+1,1,1,k_1,-p-N+1)$
• $\psi_{3,\lambda} = 2^j\Psi_p^{\{l\}}(2^jx_1)\psi_q^{\{l\}}(2^j(x_2+1))$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq N-1$, $\lambda = (j+1,1,1,-p,-2^j-q)$
• $\psi_{3,\lambda} = 2^j\Psi_p^{\{l\}}(2^jx_1)\psi_q^{\{r\}}(2^jx_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq N-1$, $\lambda = (j+1,1,1,-p,-q-N+1)$
• $\psi_{3,\lambda} = 2^j\psi_p^{\{r\}}(2^j(x_1-1))\psi_q^{\{l\}}(2^j(x_2+1))$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq N-1$, $\lambda = (j+1,1,1,2^j-p-N+1,-2^j-q)$
• $\psi_{3,\lambda} = 2^j\psi_p^{\{r\}}(2^j(x_1-1))\psi_q^{\{r\}}(2^jx_2)$, $j \geq j_0$, $1 \leq p \leq N-1$, $1 \leq q \leq N-1$, $\lambda = (j+1,1,1,2^j-p-N+1,-q-N+1)$.

More precisely, we have for $f \in L^2(Q_3)$,

$$f = \sum_{\lambda\in\Lambda_3} \langle f|\psi_{3,\lambda}\rangle\ \psi_{3,\lambda}$$

and, for $s \in [0,\sigma)$, there exists two positive constants A_s and B_s such that

$$\text{for } f \in H^{s,3}(Q_3), \quad A_s\|f\|^2_{H^s} \leq \sum_{\lambda\in\Lambda_3} 4^{j(\lambda)}|\langle f|\psi_{3,\lambda}\rangle|^2 \leq B_S\|f\|^2_{H^s}$$

D) The following family $(\tilde\psi_{i,\lambda})_{(i,\lambda)\in\Lambda}$ is a Riesz basis of $L^2(\Omega)$ and of $H^s(\Omega)$ for $s \in (0,\sigma)$:
• $\tilde\psi_{1,\lambda} = E_{j(\lambda),1}\psi_{1,\lambda}$, $\lambda \in \Lambda_1$
• $\tilde\psi_{2,\lambda} = E_2^{(0)}\psi_{2,\lambda}$, $\lambda \in \Lambda_2$
• $\tilde\psi_{3,\lambda} = E_3^{(0)}\psi_{3,\lambda}$, $\lambda \in \Lambda_3$

The dual basis is given by $(\tilde\psi^*_{i,\lambda})_{(i,\lambda)\in\Lambda}$ with :
• $\tilde\psi^*_{1,\lambda} = E_1^{(0)}\psi_{1,\lambda}$, $\lambda \in \Lambda_1$
• $\tilde\psi^*_{2,\lambda} = E_2^{(0)}\psi_{2,\lambda}$, $\lambda \in \Lambda_2$, $j(\lambda) = j_0$
• $\tilde\psi^*_{2,\lambda} = (Id - P^*_{j(\lambda)-1,1})E_2^{(0)}\psi_{2,\lambda}$, $\lambda \in \Lambda_2$, $j(\lambda) > j_0$
• $\tilde\psi^*_{3,\lambda} = E_3^{(0)}\psi_{3,\lambda}$, $\lambda \in \Lambda_3$, $j(\lambda) = j_0$
• $\tilde\psi^*_{3,\lambda} = (Id - P^*_{j(\lambda)-1,1})E_3^{(0)}\psi_{3,\lambda}$, $\lambda \in \Lambda_3$, $j(\lambda) > j_0$

More precisely, we have for $f \in L^2(\Omega)$,

$$f = \sum\sum_{(i,\lambda)\in\Lambda} \langle f|\tilde\psi^*_{i,\lambda}\rangle\ \tilde\psi_{i,\lambda}$$

and, for $s \in [0,\sigma)$, there exists two positive constants A_s and B_s such that

$$\text{for } f \in H^s(\Omega), \quad A_s\|f\|^2_{H^s} \leq \sum\sum_{(i,\lambda)\in\Lambda} 4^{j(\lambda)}|\langle f|\tilde\psi^*_{i,\lambda}\rangle|^2 \leq B_S\|f\|^2_{H^s}$$

Proof: Using the tensorization of the multiresolution analyses on intervals described in section 7, we find that the families $(\psi_{i,\lambda})_{\lambda\in\Lambda_i}$ are Hilbertian bases of $L^2(Q_i)$ for $i=1,\dots,3$. Moreover, we have

- on Q_1 :
  - $P_{j_0}^- \otimes P_{j_0}^- f = \sum_{j(\lambda)=j_0} \langle f|\psi_{1,\lambda}\rangle_{L^2(Q_1)}\psi_{1,\lambda}$
  - $(P_{j+1}^- \otimes P_{j+1}^- - P_j^- \otimes P_j^-)f = \sum_{j(\lambda)=j+1}\langle f|\psi_{1,\lambda}\rangle_{L^2(Q_1)}\psi_{1,\lambda}$ for $j\geq j_0$
- on Q_2 :
  - $P_{j_0}^- \otimes P_{j_0}^+ f = \sum_{j(\lambda)=j_0} \langle f|\psi_{2,\lambda}\rangle_{L^2(Q_2)}\psi_{2,\lambda}$
  - $(P_{j+1}^- \otimes P_{j+1}^+ - P_j^- \otimes P_j^+)f = \sum_{j(\lambda)=j+1}\langle f|\psi_{2,\lambda}\rangle_{L^2(Q_2)}\psi_{2,\lambda}$ for $j\geq j_0$
- on Q_3 :
  - $P_{j_0}^+ \otimes P_{j_0}^- f = \sum_{j(\lambda)=j_0} \langle f|\psi_{3,\lambda}\rangle_{L^2(Q_3)}\psi_{3,\lambda}$
  - $(P_{j+1}^+ \otimes P_{j+1}^- - P_j^+ \otimes P_j^-)f = \sum_{j(\lambda)=j+1}\langle f|\psi_{3,\lambda}\rangle_{L^2(Q_3)}\psi_{3,\lambda}$ for $j\geq j_0$

Recall that $P_j = P_{j,1}+P_{j,2}+P_{j,3} = E_{j,1}\circ(P_j^-\otimes P_j^-)\circ R_1 + E_2^{(0)}\circ(P_j^-\otimes P_j^+)\circ R_2 + E_3^{(0)}\circ(P_j^+\otimes P_j^-)\circ R_3$. Since we have $R_i^* = E_i^{(0)}$ for $i=1,\dots,3$, we find that $P_{j_0}f = \sum_{i=1,\dots,3}\sum_{j(\lambda)=j_0}\langle f|\tilde\psi_{i,\lambda}^*\rangle\tilde\psi_{i,\lambda}$, while we easily check that, for $j(\lambda)=j(\mu)=j_0$ and $i,k\in\{1,2,3\}$, we have $\langle\tilde\psi_{i,\lambda}|\tilde\psi_{k,\mu}^*\rangle = \delta_{\lambda,\mu}$.

We now write $P_{j+1}-P_j = E_{j+1,1}\circ(P_{j+1}^-\otimes P_{j+1}^- - P_j^-\otimes P_j^-)\circ R_1 + (E_{j+1,1}-E_{j,1})\circ(P_j^-\otimes P_j^-)\circ R_1 + E_2^{(0)}\circ(P_{j+1}^-\otimes P_{j+1}^+ - P_j^-\otimes P_j^+)\circ R_2 + E_3^{(0)}\circ(P_{j+1}^+\otimes P_{j+1}^- - P_j^+\otimes P_j^-)\circ R_3$. Moreover, we see that the range of $E_{j+1,1}-E_{j,1}$ is contained in $E_2^0(V_{j+1}^-\otimes V_{j+1}^+) + E_3^0(V_{j+1}^+\otimes V_{j+1}^-)$ ans is orthogonal to $E_2^0(V_j^-\otimes V_j^+) + E_3^0(V_j^+\otimes V_j^-)$. This gives that $E_{j+1,1}-E_{j,1} = E_2^{(0)}\circ(P_{j+1}^-\otimes P_{j+1}^+ - P_j^-\otimes P_j^+)\circ R_2\circ(E_{j+1,1}-E_{j,1}) + E_3^{(0)}\circ(P_{j+1}^+\otimes P_{j+1}^- - P_j^+\otimes P_j^-)\circ R_3\circ(E_{j+1,1}-E_{j,1})$. Moreover, the range of E_{j+1} is orthogonal to $E_2^0(V_{j+1}^-\otimes V_{j+1}^+) + E_3^0(V_{j+1}^+\otimes V_{j+1}^-)$ and we find finally that

$$E_{j+1,1}-E_{j,1} = -E_2^{(0)}\circ(P_{j+1}^-\otimes P_{j+1}^+ - P_j^-\otimes P_j^+)\circ R_2\circ E_{j,1} - E_3^{(0)}\circ(P_{j+1}^+\otimes P_{j+1}^- - P_j^+\otimes P_j^-)\circ R_3\circ E_{j,1}$$

and thus $(E_{j+1,1}-E_{j,1})\circ(P_j^-\otimes P_j^-)\circ R_1 = E_2^{(0)}\circ(P_{j+1}^-\otimes P_{j+1}^+ - P_j^-\otimes P_j^+)\circ R_2\circ P_{j,1} - E_3^{(0)}\circ(P_{j+1}^+\otimes P_{j+1}^- - P_j^+\otimes P_j^-)\circ R_3\circ P_{j,1}$. Thus, we have

$$\begin{aligned} P_{j+1}-P_j = \quad & E_{j+1,1}\circ(P_{j+1}^-\otimes P_{j+1}^- - P_j^-\otimes P_j^-)\circ R_1 \\ & +E_2^{(0)}\circ(P_{j+1}^-\otimes P_{j+1}^+ - P_j^-\otimes P_j^+)\circ R_2\circ(Id-P_{j,1}) \\ & +E_3^{(0)}\circ(P_{j+1}^+\otimes P_{j+1}^- - P_j^+\otimes P_j^-)\circ R_3\circ(Id-P_{j,1}) \end{aligned}$$

and thus

$$\begin{aligned} (P_{j+1}-P_j)f \quad = \quad & \textstyle\sum_{j(\lambda)=j+1}\langle R_1 f|\psi_{1,\lambda}\rangle\ E_{j+1,1}\psi_{1,\lambda} \\ & +\textstyle\sum_{j(\lambda)=j+1}\langle R_2\circ(Id-P_{j,1})f|\psi_{2,\lambda}\rangle\ E_2^{(0)}\psi_{2,\lambda} \\ & +\textstyle\sum_{j(\lambda)=j+1}\langle R_3\circ(Id-P_{j,1})f|\psi_{3,\lambda}\rangle\ E_3^{(0)}\psi_{3,\lambda} \\ = \quad & \textstyle\sum_{i=1,\dots,3}\sum_{j(\lambda)=j+1}\langle f|\tilde\psi_{i,\lambda}^*\rangle\ \tilde\psi_{i,\lambda} \end{aligned}$$

Besides, we easily check that, for $j(\lambda)=j(\mu)$ and $i,k\in\{1,2,3\}$, we have $\langle\tilde\psi_{i,\lambda}|\tilde\psi_{k,\mu}^*\rangle = \delta_{i,k}\delta_{\lambda,\mu}$. On the other hand, if $j(\lambda)\neq j(\mu)$ and $i,k\in\{1,2,3\}$, we have $\langle\tilde\psi_{i,\lambda}|\tilde\psi_{k,\mu}^*\rangle = 0$: if $j(\lambda)>j(\mu)$, we have obviously that $\tilde\psi_{i,\lambda}\in\left(E_1^{(0)}V_{j(\lambda)-1}^-\otimes V_{j(\lambda)-1}^- + E_2^{(0)}V_{j(\lambda)-1}^-\otimes V_{j(\lambda)-1}^+ + E_3^{(0)}V_{j(\lambda)-1}^+\otimes V_{j(\lambda)-1}^-\right)^\perp$, while we have $\tilde\psi_{k,\mu}^*\in E_1^{(0)}V_{j(\mu)}^-\otimes V_{j(\mu)}^- + E_2^{(0)}V_{j(\mu)}^-\otimes V_{j(\mu)}^+ + E_3^{(0)}V_{j(\mu)}^+\otimes V_{(\mu)}^-$; if $j(\lambda)<j(\mu)$, we have proved that $\tilde\psi_{i,\lambda}\in E_{j(\mu)-1,1}V_{j(\mu)-1}^-\otimes V_{j(\mu)-1}^- + E_2^{(0)}V_{j(\mu)-1}^-\otimes V_{j(\mu)-1}^+ + E_3^{(0)}V_{j(\mu)-1}^+\otimes V_{j(\mu)-1}^-$, while we have by construction that $\tilde\psi_{k,\mu}^*\in\left(E_{j(\mu)-1,1}V_{j(\mu)-1}^-\otimes V_{j(\mu)-1}^- + E_2^{(0)}V_{j(\mu)-1}^-\otimes V_{j(\mu)-1}^+ + E_3^{(0)}V_{j(\mu)-1}^+\otimes V_{j(\mu)-1}^-\right)^\perp$.

We now use the homogeneity of the Lebesgue measure and Lemma 1 to get that there exists a positive constant C_0 such that, for every $j\geq j_0$ and every $f\in L^2(\Omega)$, we have

$$\frac{1}{C_0}\|(P_{+1}-P_j)f\|_2^2 \le \sum_{i=1,\dots,3}\sum_{j(\lambda)=j} |\ \langle f|\psi^*_{i,\lambda}\rangle\ |^2 \le C_0\|(P_{+1}-P_j)f\|_2^2$$

Thus, Theorem 1 gives us that the family $(\tilde{\psi}_{i,\lambda})_{(i,\lambda)\in\Lambda}$ is a Riesz basis of H^s for $0 < s < \sigma$ and more precisely that every $f \in H^s(\Omega)$ can be written as $f = \sum\sum_{(i,\lambda)\in\Lambda}\langle f|\tilde{\psi}^*_{i,\lambda}\rangle\ \tilde{\psi}_{i,\lambda}$ and that we have for two positive constants A_s and B_s the inequalities $A_s\|f\|^2_{H^s} \le \sum\sum_{(i,\lambda)\in\Lambda} 4^{j(\lambda)s}|\ \langle f|\tilde{\psi}^*_{i,\lambda}\rangle\ |^2 \le A_s\|f\|^2_{H^s}$.

In particular, the partial sums $Z_i f = \sum_{\lambda\in\Lambda_i}\langle f|\tilde{\psi}^*_{i,\lambda}\rangle\ \tilde{\psi}_{i,\lambda}$ for $i = 1,\dots,3$ are bounded projection operators on $H^s(\Omega)$. We have $Z_1(H^{s,2}(\Omega)) = Z_3(H^{s,2}(\Omega)) = \{0\}$, and thus Z_2 is a projection operator onto $H^{s,2}(\Omega)$; similarly, Z_3 is a projection operator onto $H^{s,3}(\Omega)$. Moreover, on $H^{s,2}(Q_2)$, we have $E_2^{(0)}f \in H^{s,2}(\Omega)$ and $\langle E_2^{(0)}f|\tilde{\psi}^*_{2,\lambda}\rangle_{L^2(\Omega)} = \langle f|\psi_{2,\lambda}\rangle_{L^2(Q_2)}$; this gives that $(\psi_{2,\lambda})_{\lambda\in\Lambda_2}$ is a Riesz basis of $H^{s,2}(Q_2)$. Similarly, $(\psi_{3,\lambda})_{\lambda\in\Lambda_3}$ is a Riesz basis of $H^{s,3}(Q_3)$.

Since $Z_1 f$ depends only on $R_1 f$ (we have $\langle f|\tilde{\psi}^*_{1,\lambda}\rangle_{L^2(\Omega)} = \langle R_1 f|\psi_{1,\lambda}\rangle_{L^2(Q_1)}$), the operator E_1 defined by $E_1 f = \sum_{\lambda\in\Lambda_1}\langle f|\psi_{1,\lambda}\rangle\ E_{j(\lambda),1}\psi_{1,\lambda}$ is a bounded extension operator from $H^s(Q_1)$ to $H^s(\Omega)$. This gives that $(\psi_{1,\lambda})_{\lambda\in\Lambda_1}$ is a Riesz basis of $H^s(Q_1)$. Moreover, following Cieselski and Figiel's method, we found n extension operator E_1 from $H^s(Q_1)$ to $H^s(\Omega)$; the isomorphism $I : (f_1,f_2,f_3) \in H^s(Q_1)\times H^{s,2}(Q_2)\times H^{s,3}(Q_3) \mapsto I(f_1,f_2,f_3) = E_1 f_1 + E_2^{(0)}f_2 + E_3^{(0)}f_3 \in H^s(\Omega)$ is then invertible with $L^{-1}f = (R_1 Z_1 f, R_2 Z_2 f, R_3 Z_3 f)$.

The only point that we still have to check is that we have a Riesz basis for $L^2(\Omega)$. It is equivalent to prove that for some constant C_0 we have the inequalities

$$\|\sum_{(i,\lambda)\in\Lambda}\alpha_{i,\lambda}\tilde{\psi}_{i,\lambda}\|_2^2 \le C_0\sum_{(i,\lambda)\in\Lambda}|\alpha_{i,\lambda}|^2 \text{ and } \|\sum_{(i,\lambda)\in\Lambda}\alpha_{i,\lambda}\tilde{\psi}^*_{i,\lambda}\|_2^2 \le C_0\sum_{(i,\lambda)\in\Lambda}|\alpha_{i,\lambda}|^2.$$

This will be done at the end of the next section. ◇

10. The vaguelettes lemma.

We begin this section with two lemmas on Sobolev spaces. For a subset E of $\mathbb{R}^2$, we define χ_E by $\chi_E(x) = 1$ when $x \in E$ and $\chi_E(x) = 0$ when $x \notin E$

Lemma 6:

Let $\epsilon \in (0,1/2)$. Then for any cube Q, the pointwise multiplication by χ_Q is a bounded operator on $H^\epsilon(\mathbb{R}^2)$.

Proof: Since a cube is the intersectoin of four half-spaces, it is enough to check that the pointwise multiplication by χ is a bounded operator on $H^\epsilon(\mathbb{R}^2)$, where $\chi = \chi_{\{(x_1,x_2)\ /\ x_2>0\}}$. We shall write $\chi(x_2)$ instead of $\chi(x_1,x_2)$ since χ does not depend on the first variable. Moreover, since $H^\epsilon = [L^2, B_2^{\frac{1/2+\epsilon}{2},\infty}]_{\frac{2\epsilon}{1+2\epsilon},2}$, it is enough to prove that the pointwise multiplication by χ is a bounded operator on $B_2^{\eta,\infty}(\mathbb{R}^2)$ for $\eta \in (0,1/2)$. Recall that f belongs to the Besov space $B_2^{\eta,\infty}(\mathbb{R}^2)$ if and only if $f \in L^2$, $\sup_{h>0} h^{-\eta}\|f(x_1,x_2) - f(x_1+h,x_2)\|_2 < \infty$ and $\sup_{h>0} h^{-\eta}\|f(x_1,x_2) - f(x_1,x_2+h)\|_2 < \infty$. Since we have the inequalities

$$\begin{cases} \|\chi\, f\|_2 \le \|f\|_2 \\ \|\chi(x_2)f(x_1,x_2) - \chi(x_2)f(x_1+h,x_2)\|_2 \le \|f(x_1,x_2) - f(x_1+h,x_2)\|_2 \\ \|\chi(x_2)f(x_1,x_2) - \chi(x_2+h)f(x_1,x_2+h)\|_2 \le \|f(x_1,x_2) - f(x_1,x_2+h)\|_2 + \|f(x_1,x_2)(\chi(x_2) - \chi(x_2+h))\|_2 \end{cases}$$

we have only to estimate $\|f(x_1,x_2)(\chi(x_2) - \chi(x_2+h))\|_2$. Let $g(x_2) = (\int_{\mathbb{R}}|f(x_1,x_2)|^2\ dx_1)^{1/2}$. We want to prove that $\int_{-h}^0 g(x_2)^2\ dx_2 \le Ch^{2\eta}$. We have

$$\|g(x_2) - g(x_2+h)\|_{L^2(\mathbb{R})} \le \|f(x_1,x_2) - f(x_1,x_2+h)\|_{L^2(\mathbb{R}^2)} \le Ch^\eta$$

Thus, g belongs to the Besov space $B_2^{\eta,\infty}(\mathbb{R})$, hence to the Lorentz space $L^{r,\infty}$ with $\frac{1}{r}=\frac{1}{2}-\eta$. Then , g^2 belongs to $L^{r/2,\infty}$, hence

$$\int_{-h}^{0} g(x_2)^2\ dx_2 \leq \|g\|_{L^{r,\infty}}^2 \|\chi_{[-h,0]}\|_{L^{\frac{1}{2\eta},1}} \leq C h^{2\eta}.$$

Thus, the lemma is proved. ⋄

Lemma 7:

Let $\epsilon \in (0,1/2)$. If $\omega \in H^\epsilon(\mathbb{R}^2)$ is compactly supported and satisfies $\int_{\mathbb{R}^2}\omega\ dx = 0$, then there exists a positive constant C_0 such that for every $(\alpha_k)_{k\in\mathbb{Z}^2} \in l^2(\mathbb{Z}^2)$ we have

$$\|(-\Delta)^{\epsilon/2}\sum_{k\in\mathbb{Z}^2}\alpha_k\ \omega(x-k)\|_2^2 + \|(-\Delta)^{-\epsilon/2}\sum_{k\in\mathbb{Z}^2}\alpha_k\ \omega(x-k)\|_2^2 \leq C_0 \sum_{k\in\mathbb{Z}^2}|\alpha_k|^2.$$

Proof: Let $\theta \in \mathcal{D}(\mathbb{R}^2)$ such that $\sum_{k\in\mathbb{Z}^2}\theta(x-k)=1$. For any $s\in\mathbb{R}$, the norm in $H^s(\mathbb{R}^2)$ is equivalent to $\big(\sum_{k\in\mathbb{Z}^2}\|f\theta(x-k)\|_{H^s}^2\big)^{1/2}$. This gives

$$\|(-\Delta)^{\epsilon/2}\sum_{k\in\mathbb{Z}^2}\alpha_k\ \omega(x-k)\|_2^2 \leq C\sum_{p\in\mathbb{Z}^2}\|\theta(x-p)\sum_{k\in\mathbb{Z}^2}\alpha_k\ \omega(x-k)\|_{H^\epsilon}^2 \leq C'\sum_{k\in\mathbb{Z}^2}|\alpha_k|^2.$$

Moreover, since $\int\omega\ dx=0$, the functions $\omega_1=\omega(x_1,x_2)-\theta(x_1)\int\omega(y_1,x_2)\ dy_1$ and $\omega_2=\omega-\omega_1$ belong to H^ϵ, are compactly supported and satisfy $\int\omega_i(x_1,x_2)\ dx_i=0$; thus, $\omega_1=\partial_i\Omega_1$, with $\Omega_1=\int_{-\infty}^{x_1}\omega(y_1,x_2)\ dy_1$ and similarly $\omega_2=\partial_2\Omega_2$; the functions Ω_i are compactly supported and still belong to H^ϵ. We thus have

$$\|(-\Delta)^{\frac{\epsilon-1}{2}}\sum_{k\in\mathbb{Z}^2}\alpha_k\ \omega(x-k)\|_2^2 \leq C(\|(-\Delta)^{\frac{\epsilon}{2}}\sum_{k\in\mathbb{Z}^2}\alpha_k\ \Omega_1(x-k)\|_2^2+\|(-\Delta)^{\frac{\epsilon}{2}}\sum_{k\in\mathbb{Z}^2}\alpha_k\ \Omega_2(x-k)\|_2^2) \leq C'\sum_{k\in\mathbb{Z}^2}|\alpha_k|^2$$

Thus, we control the norm of $\sum_{k\in\mathbb{Z}^2}\alpha_k\ \omega(x-k)$ in the homogeneous Sobolev spaces $\dot{H}^\epsilon$ and $\dot{H}^{\epsilon-1}$, hence by interpolation in the homogeneous Sobolev space $\dot{H}^{-\epsilon}$. ⋄

A direct consequence of Lemma 7 is then the vaguelettes lemma of Meyer [KL 95] [Mey 90]

Proposition 7: (The vaguelettes lemma)

Let $\epsilon\in(0,1/2)$. If $\omega\in H^\epsilon(\mathbb{R}^2)$ is compactly supported and satisfies $\int_{\mathbb{R}^2}\omega\ dx=0$, then there exists a positive constant C_0 such that for every $(\alpha_{j,k})_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^2}\in l^2(\mathbb{Z}\times\mathbb{Z}^2)$ we have

$$\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^2}\alpha_{j,k}\ 2^j\omega(2^jx-k)\|_2^2 \leq C_0\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^2}|\alpha_{j,k}|^2.$$

Proof: We just write

$$\langle 2^j f(2^jx)|2^p g(2^px)\rangle_{L^2(\mathbb{R}^2)} = 2^{\epsilon(j-p)}\langle 2^j[(-\Delta)^{\epsilon/2}f](2^jx)|2^p[(-\Delta)^{-\epsilon/2}g](2^px)\rangle_{L^2(\mathbb{R}^2)}$$

to get

$$|\ \langle\sum_{k\in\mathbb{Z}^2}\alpha_{j,k}\ 2^j\omega(2^jx-k)|\sum_{k\in\mathbb{Z}^2}\alpha_{p,k}\ 2^p\omega(2^px-k)\rangle\ | \leq 2^{-\epsilon|j-p|}\big(\sum_{k\in\mathbb{Z}^2}|\alpha_{j,k}|^2\big)^{1/2}\big(\sum_{k\in\mathbb{Z}^2}|\alpha_{p,k}|^2\big)^{1/2}$$

which is enough to prove the vaguelettes lemma. ⋄

End of the proof of Theorem 6:

We want to estimate $I_i = \|\sum_{\lambda\in\Lambda_i}\alpha_{i,\lambda}\tilde{\psi}_{i,\lambda}\|_2^2$ and $I_i^* = \|\sum_{\lambda\in\Lambda_i}\alpha_{i,\lambda}\tilde{\psi}^*_{i,\lambda}\|_2^2$. Since $(\psi_{i,\lambda})_{\lambda\in\Lambda_i}$ is a Hilbertian basis of $L^2(Q_i)$, we find that

$$\begin{cases} I_1 = & \sum_{\lambda\in\Lambda_1}|\alpha_{1,\lambda}|^2 & +\int_{Q_2}|\sum_{\lambda\in\Lambda_1}\alpha_{1,\lambda}E_{j(\lambda),1}\psi_{1,\lambda}|^2\,dx & +\int_{Q_3}|\sum_{\lambda\in\Lambda_1}\alpha_{1,\lambda}E_{j(\lambda),1}\psi_{1,\lambda}|^2\,dx \\ I_2 = & \sum_{\lambda\in\Lambda_2}|\alpha_{2,\lambda}|^2 \\ I_3 = & \sum_{\lambda\in\Lambda_3}|\alpha_{3,\lambda}|^2 \\ I_1^* = & \sum_{\lambda\in\Lambda_1}|\alpha_{1,\lambda}|^2 \\ I_2^* = & \sum_{\lambda\in\Lambda_2}|\alpha_{2,\lambda}|^2 & +\int_{Q_1}|\sum_{\lambda\in\Lambda_2,j(\lambda)>j_0}\alpha_{2,\lambda}P^*_{j(\lambda)-1,1}E_2^{(0)}\psi_{2,\lambda}|^2\,dx \\ I_3^* & = \sum_{\lambda\in\Lambda_3}|\alpha_{3,\lambda}|^2 & +\int_{Q_1}|\sum_{\lambda\in\Lambda_3,j(\lambda)>j_0}\alpha_{3,\lambda}P^*_{j(\lambda)-1,1}E_3^{(0)}\psi_{3,\lambda}|^2\,dx \end{cases}$$

Now, due to the invariance properties under (local) dyadic translations and dilations, we find that there exists a positive numbers K and a finite number of functions $\theta_u^{(l)}$, $\zeta_v^{(l)}$ $\theta_w^{(r)}$ and $\zeta_x^{(r)}$, $1\le u,v,w,x\le R$ such that

- $\theta_u^{(l)}$, $\zeta_v^{(l)}$ $\theta_w^{(r)}$ and $\zeta_x^{(r)}$ belong to $H^\epsilon(\mathbb{R}^2)$ for some $\epsilon\in(0,1/2)$ (since they are constructed with restrictions of $\varphi\otimes\varphi$ on cubes)
- the functions $\theta_u^{(l)}$ are supported in the half space $x_1\ge 0$, $\zeta_v^{(l)}$ in the half space $x_2\ge 0$, $\theta_w^{(r)}$ in the half space $x_1\le 0$, and $\zeta_x^{(r)}$ in the half space $x_2\le 0$

and

$$\begin{cases} \chi_{Q_2}\sum_{\lambda\in\Lambda_1}\alpha_{1,\lambda}E_{j(\lambda),1}\psi_{1,\lambda} = & \sum_{\lambda\in\Lambda_1,-K\le k_2(\lambda)\le -1}\alpha_{1,\lambda}2^{j(\lambda)}\zeta^{(l)}_{v(\lambda)}(2^{j(\lambda)}x_1-k_1(\lambda),2^{j(\lambda)}x_2) \\ \chi_{Q_3}\sum_{\lambda\in\Lambda_1}\alpha_{1,\lambda}E_{j(\lambda),1}\psi_{1,\lambda} = & \sum_{\lambda\in\Lambda_1,-K\le k_1(\lambda)\le -1}\alpha_{1,\lambda}2^{j(\lambda)}\theta^{(l)}_{u(\lambda)}(2^{j(\lambda)}x_1,2^{j(\lambda)}x_2-k_2(\lambda)) \\ \chi_{Q_1}\sum_{\lambda\in\Lambda_2,j(\lambda)>j_0}\alpha_{2,\lambda}P^*_{j(\lambda)-1,1}E_2^{(0)}\psi_{2,\lambda} = & \sum_{\lambda\in\Lambda_2,-K\le k_2(\lambda)\le -1}\alpha_{2,\lambda}2^{j(\lambda)}\zeta^{(r)}_{x(\lambda)}(2^{j(\lambda)}x_1-k_1(\lambda),2^{j(\lambda)}x_2) \\ \chi_{Q_1}\sum_{\lambda\in\Lambda_3,j(\lambda)>j_0}\alpha_{3,\lambda}P^*_{j(\lambda)-1,1}E_3^{(0)}\psi_{3,\lambda} = & \sum_{\lambda\in\Lambda_3,-K\le k_1(\lambda)\le -1}\alpha_{3,\lambda}2^{j(\lambda)}\theta^{(r)}_{w(\lambda)}(2^{j(\lambda)}x_1,2^{j(\lambda)}x_2-k_2(\lambda)) \end{cases}$$

We now define $\theta_u^{[l]} = \theta_u^{(l)}(x_1,x_2)-\theta_u^{(l)}(-x_1,x_2)$, $\zeta_v^{[l]} = \zeta_v^{(l)}(x_1,x_2)-\zeta_v^{(l)}(x_1,-x_2)$, $\theta_w^{[r]} = \theta_w^{(r)}(x_1,x_2)-\theta_w^{(r)}(-x_1,x_2)$ and $\zeta_x^{[l]} = \theta_x^{(r)}(x_1,x_2)-\zeta_x^{(r)}(-x_1,x_2)$ and we use the vaguelettes lemma to get

$$\|\sum_{j\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}2^j\theta_p^l(2^jx_1,2^jx_2-k_2)\|_2^2 = \frac{1}{2}\|\sum_{j\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\alpha_{j,k_1}2^j\theta_p^{[l]}(2^jx_1,2^jx_2-k_2)\|_2^2 \le C\sum_{j\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}|\alpha_{j,k_1}|^2$$

and thus

$$\|\chi_{Q_2}\sum_{\lambda\in\Lambda_1}\alpha_{1,\lambda}E_{j(\lambda),1}\psi_{1,\lambda}\|_2^2 \le 3KC\sum_{\lambda\in\Lambda_1,-K\le k_2(\lambda)\le -1}|\alpha_{1,\lambda}|^2.$$

We obtain similar estimates for $\|\chi_{Q_3}\sum_{\lambda\in\Lambda_1}\alpha_{1,\lambda}E_{j(\lambda),1}\psi_{1,\lambda}\|_2^2$, $\|\chi_{Q_1}\sum_{\lambda\in\Lambda_2,j(\lambda)>j_0}\alpha_{2,\lambda}P^*_{j(\lambda)-1,1}E_2^{(0)}\psi_{2,\lambda}\|_2^2$ and $\|\chi_{Q_1}\sum_{\lambda\in\Lambda_3,j(\lambda)>j_0}\alpha_{3,\lambda}P^*_{j(\lambda)-1,1}E_3^{(0)}\psi_{3,\lambda}\|_2^2$. Thus, Theorem 6 is proved. ⋄

Conclusion.

In the special setting of the L-shaped domain Ω, we have presented two multiresolution analyses associated to the scale of the spaces $V_j(\Omega)$, i.e. the spaces of restrictions to Ω of the elements of a Daubechies multi-resolution analysis $V_j(\mathbb{R}^2)$. In both constructions, we were able to preserve two essential features of wavelet analysis :

- the basis is obtained through translations and dilations from a finite number of basic compactly supported functions
- the basis is a Riesz basis of $L^2(\Omega)$ and an unconditional basis of $H^s(\Omega)$ for $0<s<\sigma$, where σ is the Sobolev regularity exponent of the Daubechies wavelet we are dealing with.

Though we have thus no limitations on the regularity of the Sobolev spaces we may want to analyze through wavelets on our domain and though those constructions may easily be extended to a slightly more general setting (any quadrangulation by N cubes $Q_i = (k_1(i), k_2(i)) + [0,1]^2$, $(k_1(i), k_2(i)) \in \mathbb{Z}^2$, $1 \leq i \leq N$, satisfying Cieselski and Figiel's requirement that $Q_i \cap \cup_{j<i} Q_j$ is a union of faces of Q_i), we should notice however that those constructions are far from being optimal: they rely on the construction of Meyer's border wavelets, which can be achieved numerically in a satisfactory way only for the first Daubechies scaling functions [CDV 93].

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