UNIQUENESS FOR THE NAVIER–STOKES EQUATIONS AND MULTIPLIERS BETWEEN SOBOLEV SPACES

Pierre Gilles LEMARIÉ-RIEUSSET & Ramzi MAY

ABSTRACT

We reprove various uniqueness theorems for the Navier–Stokes equations, stating the assumptions in terms of multipliers between Sobolev spaces instead of Lebesgue or Lorentz spaces.

1. LEBESGUE SPACES AND UNIQUENESS FOR THE NAVIER–STOKES EQUATIONS

We consider the Navier-Stokes equations on the whole space $\mathbb{R}^d$:

$$\begin{cases} \exists p \in \mathcal{D}'((0,T)\times\mathbb{R}^d) & \partial_t \vec{u} = \Delta\vec{u} - \vec{\nabla}.(\vec{u}\otimes\vec{u}) - \vec{\nabla} p \\ & \vec{\nabla}.\vec{u} = 0 \end{cases} \tag{1}$$

We shall speak of weak solutions when the derivatives in (1) are taken in the sense of distributions theory.

Leray [8] studied the Cauchy initial value problem for equations (1) with a square-integrable initial value. He proved the existence of weak solutions, which satisfy moreover an energy inequality :

Definition 1 : (Leray solutions)

A Leray solution on $(0,T)$ for the Navier-Stokes equations with initial value $\vec{u}_0 \in (L^2)^d$ is a solution $\vec{u}$ such that

i) $t \mapsto \vec{u}(t,.)$ is weakly continuous from $(0,T)$ to $(L^2)^d$

ii) $\vec{u}(t,.)$ converges weakly to $\vec{u}_0$ as $t \to 0^+$,

iii) $\vec{u} \in L^\infty((0,T),(L^2)^d) \cap L^2((0,T),(\dot{H}^1)^d)$,

iv) $\vec{u}$ satisfies the Leray energy inequality

$$\text{for all } t \in (0,T),\ \|\vec{u}(t,.)\|_2^2 + 2\int_0^t \int_{\mathbb{R}^d} |\vec{\nabla}\otimes\vec{u}|^2 \, dx \, ds \le \|\vec{u}_0\|_2^2. \tag{2}$$

Weak continuity of (a representant of) $\vec{u}$ is a consequence of the Navier–Stokes equations and of the hypothesis iii). An easy consequence of inequality (2) is then the strong continuity at $t = 0$:

$$\lim_{t\to 0^+} \|\vec{u} - \vec{u}_0\|_2 = 0.$$

But, when $d \geq 3$, it is still not known whether we have continuity for all time t and whether we have uniqueness in the class of Leray solutions. Serrin's theorem [17] gives a criterion for uniqueness :

Theorem 1 : (Serrin's uniqueness theorem)
Let $\vec{u}_0 \in (L^2(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^d$ (for some $T \in (0,+\infty]$) with initial value $\vec{u}_0$ such that :
i) $\vec{u} \in L^\infty((0,T),(L^2(\mathbb{R}^d)^d)$;
ii) $\vec{u} \in L^2((0,T),(\dot{H}^1(\mathbb{R}^d)^d)$;
iii) For some $r \in [0,1)$, $\vec{u}$ belongs to $(L^\sigma((0,T),L^{d/r}))^d$ with $2/\sigma = 1-r$.
Then, $\vec{u}$ satisfies the Leray energy inequality and it is the unique Leray solution associated to $\vec{u}_0$ on $(0,T)$.

The limit case $r = 1$ is dealt with Von Wahl's theorem [WAH] :

Theorem 2 : (Von Wahl's uniqueness theorem)
Let $\vec{u}_0 \in (L^2(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^d$ (for some $T \in (0,+\infty]$) with initial value $\vec{u}_0$ such that :
i) $\vec{u} \in L^\infty((0,T),(L^2(\mathbb{R}^d)^d)$;
ii) $\vec{u} \in L^2((0,T),(\dot{H}^1(\mathbb{R}^d)^d)$;
iii) $\vec{u}$ belongs to $(\mathcal{C}([0,T],L^d))^d$.
Then, $\vec{u}$ satisfies the Leray energy inequality and it is the unique Leray solution associated to $\vec{u}_0$ on $(0,T)$.

Another class of solutions has been classically studied in the following of Kato [4] : mild solutions. In the case of mild solutions, the initial value may have infinite energy ($\|\vec{u}\|_2 = +\infty$) and the method of proof does not use energy inequalities nor compactness criteria, but an iterative method for finding a fixed-point in a Banach space. Kato proved that, when $\vec{u}_0 \in (L^p)^d$ with $d < p < \infty$, then the Navier–Stokes equations (1) have a unique solution in $\mathcal{C}((0,T),(L^p)^d)$ where T depends on the size of $\|\vec{u}_0\|_p$. In the limit case, $p = d$, Kato proved existence but not uniqueness. This has been proved by Furioli, Lemarié–Rieusset and Terraneo [3] :

Theorem 3 : (Furioli, Lemarié–Rieusset and Terraneo's uniqueness theorem.)
If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier-Stokes equations on $(0,T^*) \times \mathbb{R}^d$ such that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0,T^*),(L^d(\mathbb{R}^d))^d)$ and have the same initial value, then $\vec{u} = \vec{v}$.

This theorem has been reproved by many authors and various methods (Meyer [15], Monniaux [16], Lions and Masmoudi [9]). Further, it was extended to $(L^\infty([0,T],L^d(\mathbb{R}^d))^d$ in dimension $d \geq 4$ by Lions and Masmoudi [9]:

Theorem 4 : (Lions and Masmoudi's uniqueness theorem.)

Let $d \geq 4$. If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier-Stokes equations on $(0,T^*) \times \mathbb{R}^d$ such that $\vec{u}$ and $\vec{v}$ belong to $L^\infty([0,T^*),(L^d(\mathbb{R}^d))^d)$ and have the same initial value, then $\vec{u} = \vec{v}$.

Many authors noticed that L^d in theorems 2, 3 and 4 could be replaced by the Lorentz space $L^{d,\infty}$ (at least, by the closure of the test functions in $L^{d,\infty}$) [18] [3] [9] or by Morrey–Campanato spaces [3]. The aim of this paper is to show how all those results are enclosed in more general results, where we use the space of multipliers between H^1 and L^2.

2. THE MULTIPLIER SPACES X_r AND $\tilde{X}_r$

The pointwise multipliers between different spaces of differentiable functions have been studied by Maz'ya and co-workers [11] [12] [13] [14]. They are a useful tool to state minimal regularity requirements on the coefficients of partial differential operators for proving regularity or uniqueness of solutions.

More precisely, we define the space $X_r(\mathbb{R}^d)$ of pointwise multipliers which map H^r into L^2 in the following way:

Definition 2 : (Pointwise multipliers of negative order.)

For $0 \leq r < d/2$, we define the space $X_r(\mathbb{R}^d)$ as the space of functions which are locally square integrable on $\mathbb{R}^d$ and such that pointwise multiplication with these functions maps boundedly the Sobolev space $H^r(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. The norm in X_r is given by the operator norm of pointwise multiplication:

$$\|f\|_{X_r} = \sup\{ \ \|fg\|_2 \ / \ \|g\|_{H^r} \leq 1\}. \tag{3}$$

The closure of the space $\mathcal{D}$ of smooth test functions in X_r will be denoted by $\tilde{X}_r$.

The spaces X_r have been characterized by Maz'ya [11] in terms of Sobolev capacities. A weaker result establishes a comparison between the spaces X_r and the Morrey–Campanato spaces $M^{2,p}$ [2] [6] [7].

We first recall the definition of Morrey–Campanato spaces:

Definition 3 : (Morrey–Campanato spaces)

For $1 < p < \infty$, the Morrey space L^p_{uloc} of uniformly locally L^p functions is defined by: $f \in L^p_{uloc}$ if and only if f is locally L^p on $\mathbb{R}^d$ and $\|f\|_{L^p_{uloc}} < \infty$ where

$$\|f\|_{L^p_{uloc}} = \sup_{x\in\mathbb{R}^d} \Big(\int_{|x-y|\leq 1} |f(y)|^p \, dy\Big)^{1/p}. \tag{3}$$

Moreover, for $q \in [p,\infty]$, the Morrey–Campanato space $M^{p,q}$ is defined by: $f \in M^{p,q}$ if and only if f is locally L^p on $\mathbb{R}^d$ and $\|f\|_{M^{p,q}} < \infty$ where

$$\|f\|_{M^{p,q}} = \sup_{x\in\mathbb{R}^d} \sup_{0<R<1} R^{d/q-d/p} \Big(\int_{|x-y|\leq R} |f(y)|^p \, dy\Big)^{1/p}. \tag{4}$$

It is easy to check that $M^{p,p} = L^p_{uloc}$ and $M^{p,\infty} = L^\infty$.

Proposition 1: (Comparison theorem)
For $2 < p \leq d/r$ and $0 < r$ we have $M^{p,d/r} \subset X_r \subset M^{2,d/r}$.

Another easy result is the embedding $L^{d/r,\infty} \subset X_r$ for $r < d/2$.

3. SERRIN'S UNIQUENESS THEOREM

In this section, we generalize the uniqueness theorem of Serrin [17]. This generalization has been announced in the preprint [6] and proved in full details in the book [7], hence we shall only sketch the proof below.

Theorem 5 : (Generalization of Serrin's uniqueness theorem)
Let $\vec{u}_0 \in (L^2(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^d$ (for some $T \in (0,+\infty]$) with initial value $\vec{u}_0$ such that :
i) $\vec{u} \in L^\infty((0,T),(L^2(\mathbb{R}^d)^d)$;
ii) $\vec{u} \in L^2((0,T),(\dot{H}^1(\mathbb{R}^d)^d)$;
iii) For some $r \in (0,1)$, $\vec{u}$ belongs to $L^\sigma((0,T),(X_r)^d)$ with $2/\sigma = 1-r$.
Then, $\vec{u}$ is the unique Leray solution associated to $\vec{u}_0$ on $(0,T)$.

Proof: Let $\vec{v}$ be another solution associated to $\vec{u}_0$ on $(0,T)$ (with associated pressure q) such that $\vec{v} \in L^\infty((0,T),(L^2(\mathbb{R}^d)^d) \cap L^2((0,T),(H^1(\mathbb{R}^d)^d)$. The main point is to check the validity of the formula

$$\partial_t \int \vec{u}.\vec{v}\, dx = -2 \int \vec{\nabla} \otimes \vec{u}.\vec{\nabla} \otimes \vec{v}\, dx + \int \vec{u}.(\vec{u}.\vec{\nabla})\vec{v}\, dx - \int \vec{u}.(\vec{v}.\vec{\nabla})\vec{v}\, dx. \tag{5}$$

This is checked by regularizing $\vec{u}$ and $\vec{v}$: we use a smoothing function $\theta(t,x) = \alpha(t)\beta(x) \in \mathcal{D}(\mathbb{R}^{d+1})$, where α is supported in $[-1,1]$, with $\int\int \theta\, dx\, dt = 1$, and define, for $\epsilon > 0$, $\theta_\epsilon(t,x) = \frac{1}{\epsilon^{d+1}}\theta(\frac{t}{\epsilon},\frac{x}{\epsilon})$. Then, $\theta_\epsilon * \vec{u}$ and $\theta_\epsilon * \vec{v}$ are smooth functions on $(\epsilon, T-\epsilon) \times \mathbb{R}^d$ and we may write $\partial_t\big((\theta_\epsilon * \vec{u}).(\theta_\epsilon * \vec{v})\big) = (\theta_\epsilon * \partial_t\vec{u}).(\theta_\epsilon * \vec{v}) + (\theta_\epsilon * \vec{u}).(\theta_\epsilon * \partial_t\vec{v})$. We then get by an integration with respect to x :

$$\begin{aligned} \partial_t \textstyle\int (\theta_\epsilon * \vec{u}).(\theta_\epsilon * \vec{v})\, dx = \quad & -2 \textstyle\int (\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}]).(\theta_\epsilon * [\vec{\nabla} \otimes \vec{v}])\, dx \\ & + \textstyle\int (\theta_\epsilon * [\vec{u} \otimes \vec{u}]).(\theta_\epsilon * [\vec{\nabla} \otimes \vec{v}])\, dx \\ & + \textstyle\int (\theta_\epsilon * [\vec{v} \otimes \vec{v}]).(\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}])\, dx \end{aligned} \tag{6}$$

We may rewrite the last summand in

$$\begin{aligned} \textstyle\int (\theta_\epsilon * [\vec{v} \otimes \vec{v}]).(\theta_\epsilon * [\vec{\nabla} \otimes \vec{u}])\, dx \quad & = - \textstyle\int (\theta_\epsilon * [\vec{\nabla}.(\vec{v} \otimes \vec{v})]).(\theta_\epsilon * \vec{u})\, dx \\ & = - \textstyle\int (\theta_\epsilon * [\{\vec{v}.\vec{\nabla}\}\vec{v})]).(\theta_\epsilon * \vec{u})\, dx \end{aligned} \tag{7}$$

To deal with $\theta_\epsilon * [\vec{u}\otimes\vec{u}]$, we write that the pointwise product maps $L^{2/r}H^r \times L^{2/(1-r)}X_r$ to L^2L^2, hence $\theta_\epsilon * [\vec{u}\otimes\vec{u}]$ converges strongly to $\vec{u}\otimes\vec{u}$ in $(L^2((0,T)\times \mathbb{R}^d))^{d\times d}$. To deal with $\theta_\epsilon * [\{\vec{v}.\vec{\nabla}\}\vec{v})]$, we write that the pointwise product maps $H^r \times L^2$ to the pre-dual Y_r of X_r and that smooth functions are dense in Y_r [7]; thus, $\theta_\epsilon * [(\vec{v}.\vec{\nabla})\vec{v}]$ converges strongly to $(\vec{v}.\vec{\nabla})\vec{v}$ in $(L^{\frac{2}{1+r}}Y_r)^d$ while $\theta_\epsilon * \vec{u}$ converges weakly to $\vec{u}$ in $(L^{\frac{2}{1-r}}X_r)^d$. This proves (5).

Since $\vec{u}\otimes\vec{u} \in (L^2((0,T)\times\mathbb{R}^d))^{d\times d}$ and $\vec{u} = e^{t\Delta}\vec{u}(0) - \mathbb{P}\int_0^t e^{(t-s)\Delta}\vec{\nabla}.(\vec{u}\otimes\vec{u})\ ds$, $t\mapsto\vec{u}$ is continuous from $[0,T]$ to $(L^2(dx))^d$ and (since $t\mapsto\vec{v}$ is weakly continuous from $[0,T]$ to $(L^2(dx))^d$) $t\mapsto\int \vec{u}.\vec{v}\ dx$ is continuous. Thus, we may integrate equality (5) and obtain

$$\begin{array}{c}\int \vec{u}(t,x).\vec{v}(t,x)\ dx + 2\int_0^t\int_{\mathbb{R}^d} \vec{\nabla}\otimes\vec{u}.\vec{\nabla}\otimes\vec{v}\ dx\ ds = \\ \|\vec{u}_0\|_2^2 + \int_0^t\int_{\mathbb{R}^d}\vec{u}.(\vec{u}.\vec{\nabla})\vec{v}\ dx\ ds - \int_0^t\int_{\mathbb{R}^d}\vec{u}.(\vec{v}.\vec{\nabla})\vec{v}\ dx\ ds\end{array} \tag{8}$$

Of course, this equality holds as well for $\vec{v}=\vec{u}$.

Now, if we assume moreover that $\vec{v}$ satisfies the Leray inequality

$$\|\vec{v}(t)\|_2^2 + 2\int_0^t \|\vec{\nabla}\otimes\vec{v}\|_2^2\ ds \leq \|\vec{u}_0\|_2^2, \tag{9}$$

we get the following inequality for $\vec{u}-\vec{v}$:

$$\|\vec{u}(t,.)-\vec{v}(t,.)\|_2^2 \leq -2\int_0^t\int_{\mathbb{R}^d} |\vec{\nabla}\otimes(\vec{u}-\vec{v})|^2\ dx\ ds - 2\int_0^t\int_{\mathbb{R}^d}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)\vec{v}\ dx\ ds \tag{10}$$

Moreover, we have the antisymmetry property $\int_0^t\int_{\mathbb{R}^d}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)\vec{u}\ dx\ ds = 0$.

We then write

$$\begin{array}{l}|\int_\tau^t\int_{\mathbb{R}^d}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)(\vec{v}-\vec{u})\ dx\ ds| \\ \leq C_r(\int_\tau^t \|\vec{u}\|_{X_r}^{\frac{2}{1-r}}\ ds)^{(1-r)/2}(\int_0^t\|\vec{v}-\vec{u}\|_{H^1}^2\ ds)^{1/2}(\int_\tau^t\|\vec{v}-\vec{u}\|_{H^r}^{\frac{2}{r}}\ ds)^{\frac{r}{2}} \\ \leq C_r'(\int_\tau^t\|\vec{u}\|_{X_r}^{\frac{2}{1-r}}\ ds)^{(1-r)/2}(\int_0^t\|\vec{v}-\vec{u}\|_{H^1}^2\ ds)^{(1+r)/2}(\sup_{\tau<s<t}\|\vec{v}-\vec{u}\|_2^2)^{\frac{(1-r)}{2}}\end{array} \tag{11}$$

We now write with help of the Young inequality

$$C_r' a^{(1-r)/2}b^{(1+r)/2} \leq \frac{1-r}{2}C_r'^{\,2/(1-r)}a + \frac{1+r}{2}b \tag{12}$$

Thus, if $\vec{u}=\vec{v}$ on $[0,\tau]$, we find from (10) and (11) that

$$\sup_{0<s\leq t}\|\vec{u}-\vec{v}\|_2^2 \leq C_r''(\int_\tau^t \|\vec{u}\|_{X_r}^{\frac{2}{1-r}}\ ds)^{1/2}\sup_{0<s\leq t}\|\vec{u}-\vec{v}\|_2^2 \tag{13}$$

and uniqueness is valid on a bigger interval. By weak continuity of $t\mapsto\vec{v}$, we find $\vec{u}=\vec{v}$.◇

4. VON WAHL'S UNIQUENESS THEOREM

Similarly, we have the following generalization of Von Wahl's theorem :

Theorem 6 : (Generalization of Von Wahl's uniqueness theorem)

Let $\vec{u}_0 \in (L^2(\mathbb{R}^d))^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^d$ (for some $T \in (0,+\infty]$) with initial value $\vec{u}_0$ such that :
i) $\vec{u} \in L^\infty((0,T),(L^2(\mathbb{R}^d)^d)$;
ii) $\vec{u} \in L^2((0,T),(\dot{H}^1(\mathbb{R}^d)^d)$;
iii) $\vec{u}$ belongs to $(\mathcal{C}([0,T],\tilde{X}_1))^d$.

Then, $\vec{u}$ is the unique Leray solution associated to $\vec{u}_0$ on $(0,T)$.

Proof: We start again from the inequality

$$\|\vec{u}(t,.)-\vec{v}(t,.)\|_2^2 \leq -2\int_0^t\int_{\mathbb{R}^d} |\vec{\nabla}\otimes(\vec{u}-\vec{v})|^2\,dx\,ds - 2\int_0^t\int_{\mathbb{R}^d} \vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)(\vec{v}-\vec{u})\,dx\,ds \quad (14)$$

$\vec{u}$ belongs to $(\mathcal{C}([0,T],\tilde{X}_1))^d$ where $\tilde{X}_1$ is the closure of $\mathcal{D}$ in X_1; thus, if $T_0 < T$, then for each $\epsilon > 0$ we may split $\vec{u}$ on $[0,T_0]$ in $\vec{u} = \vec{\alpha} + \vec{\beta}$ with $\|\vec{\alpha}\|_{L^\infty X_1} < \epsilon$ and $\vec{\beta} \in (L^\infty((0,T_0)\times\mathbb{R}^d)^d$. Then we write

$$\begin{array}{c} |\int_0^t\int_{\mathbb{R}^d} \vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)(\vec{v}-\vec{u})\,dx\,ds| \\ \leq C\|\vec{\alpha}\|_{L^\infty X_1}\int_0^t \|\vec{v}-\vec{u}\|_{H^1}^2\,ds \\ +\|\vec{\beta}\|_\infty(\int_0^t\int_{\mathbb{R}^d}|\vec{\nabla}\otimes(\vec{v}-\vec{u})|^2\,dx\,ds)^{1/2}(\int_0^t\int_{\mathbb{R}^d}|\vec{v}-\vec{u}|^2\,dx\,ds)^{1/2} \\ \leq 2C\epsilon\int_0^t\int_{\mathbb{R}^d}|\vec{\nabla}\otimes(\vec{v}-\vec{u})|^2\,dx\,ds + (\frac{4}{C\epsilon}\|\vec{\beta}\|_\infty^2 + C\epsilon)\int_0^t\int_{\mathbb{R}^d}|\vec{v}-\vec{u}|^2\,dx\,ds. \end{array} \quad (15)$$

Choosing ϵ such that $2C\epsilon < 1$, we get that

$$\|\vec{v}(t,.)-\vec{u}(t,.)\|_2^2 \leq (\frac{4}{C\epsilon}\|\vec{\beta}\|_\infty^2 + C\epsilon)\int_0^t \|\vec{v}(s,.)-\vec{u}(s,.)\|_2^2\,ds. \quad (16)$$

The Gronwall lemma gives then that $\vec{u} = \vec{v}$. ◇

5. FURIOLI, LEMARIÉ–RIEUSSET AND TERRANEO'S UNIQUENESS THEOREM

In 1997, Furioli, Lemarié-Rieusset and Terraneo [3] proved uniqueness of mild solutions in $\mathcal{C}([0,T^*),(L^d)^d)$. They extended their proof to the case of Morrey-Campanato spaces by using the Besov spaces over Morrey-Campanato spaces described by Kozono and Yamazaki [5] and found that uniqueness holds as well in the class $\mathcal{C}([0,T^*),(\tilde{M}^{p,d})^d)$ for $p > 2$, where $\tilde{M}^{p,d}$ is the closure of the smooth compactly supported functions in the Morrey-Campanato space $M^{p,d}$. In his thesis dissertation, May [10] proved a slightly more general result by extending the approach of Monniaux (i.e. by using the maximal L^pL^q property of the heat kernel) :

Theorem 7 : (Generalization of Furioli, Lemarié–Rieusset and Terraneo's uniqueness theorem.)

If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier-Stokes equations on $(0,T^*)\times\mathbb{R}^d$ such that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0,T^*),(\tilde{X}_1)^d)$ and have the same initial value, then $\vec{u} = \vec{v}$.

In order to prove the theorem, we need to get rid of the pressure. This is done by using the Leray projection operator. Indeed, we have the following result of Furioli, Lemarié–Rieusset and Terraneo [3] :

Proposition 2:

Let $\vec{u}$ be a solution of the Navier–Stokes equations on $(0,T)\times\mathbb{R}^d$:

$$\begin{cases} \exists p \in \mathcal{D}'((0,T)\times\mathbb{R}^d) & \partial_t\vec{u} = \Delta\vec{u} - \vec{\nabla}.(\vec{u}\otimes\vec{u}) - \vec{\nabla}p \\ & \vec{\nabla}.\vec{u} = 0 \end{cases} \tag{17}$$

and assume that $\vec{u}$ belongs to $(L^2((0,T),\tilde{L}^2_{uloc}))^d$, where $\tilde{L}^2_{uloc}$ is the closure of the space $\mathcal{D}$ of smooth test functions in L^2_{uloc}. Then:
i) $\vec{u} = (Id-\Delta)^{\frac{d+4}{2}}\vec{U}$ where $\vec{U}$ is a continuous bounded vector field on $[0,T]\times\mathbb{R}^d$. Hence, the mapping $t\mapsto\vec{u}(t,.)$ is continuous from $[0,T]$ to $(B^{-d-4,\infty}_{\infty})^d$ and the initial value $\vec{u}_0=\vec{u}(0,.)$ is well-defined.
ii) $\vec{\nabla}.(\vec{u}\otimes\vec{u})$ may be decomposed into $\vec{\nabla}.(\vec{u}\otimes\vec{u}) = \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) - \vec{\nabla}p$ where $\mathbb{P}$ is the Leray projection operator (orthogonal projection onto solenoidal vector fields)
iii) $\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\,ds$

Proof of Theorem 7 :

Let us define the bilinear operator B as

$$B(\vec{\alpha},\vec{\beta}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{\alpha}\otimes\vec{\beta})\,ds. \tag{18}$$

Let us define $\vec{u}_0=\vec{u}(0,.)=\vec{v}(0,.)$, $\vec{w}_1 = B(\vec{u},\vec{u}) = e^{t\Delta}\vec{u}_0-\vec{u}$, $\vec{w}_2 = B(\vec{v},\vec{v}) = e^{t\Delta}\vec{u}_0-\vec{v}$ and $\vec{w}=\vec{u}-\vec{v}=B(\vec{v},\vec{v})-B(\vec{u},\vec{u})$. Due to the bilinearity of B, we may write as well

$$\vec{w} = B(\vec{w}_1,\vec{w}) + B(\vec{w},\vec{w}_2) - B(e^{t\Delta}\vec{u}_0,\vec{w}) - B(\vec{w},e^{t\Delta}\vec{u}_0). \tag{19}$$

We consider for $T<T^*$ the space E_T defined by

$$E_T = \{f\in L^2_{loc}((0,T)\times\mathbb{R}^d))\ /\ \sup_{x_0\in\mathbb{R}^d}\Big(\int_{|x-x_0|<1}\int_0^T |f(t,x)|^2\,dx\,dt\Big)^{1/2}<\infty\} \tag{20}$$

and normed by

$$\|f\|_{E_T} = \sup_{x_0\in\mathbb{R}^d}\Big(\int_{|x-x_0|<1}\int_0^T |f(t,x)|^2\,dx\,dt\Big)^{1/2}. \tag{21}$$

Since $\vec{w}$ belongs to $\mathcal{C}([0,T^*),(\tilde{X}_1)^d)$, we have clearly that $\vec{w}$ belongs to $(E_T)^d$ for all $T<T^*$. Moreover, if we split $\vec{u}_0$ into $\vec{u}_0=\vec{\alpha}+\vec{\beta}$ with $\vec{\alpha}$ in $(\tilde{X}_1)^d$ with small norm and $\vec{\beta}\in(\mathcal{C}_0)^d$, we have the following estimates on $\vec{w}_1$, $\vec{w}_2$, $e^{t\Delta}\vec{\alpha}$ and $e^{t\Delta}\vec{\beta}$:

- for $i=1,2$, $\vec{w}_i\in\mathcal{C}([0,T^*),(\tilde{X}_1)^d)$ with $\lim_{t\to 0}\|\vec{w}_i(t,.)\|_{X_1}=0$
- $\sup_{0<t<T^*}\|e^{t\Delta}\vec{\alpha}\|_{X_1}\le\|\vec{\alpha}\|_{X_1}$

• $\sup_{0<t<T^*} \|e^{t\Delta}\vec\beta\|_\infty \leq \|\vec\beta\|_\infty$

Thus, we shall study $B(\vec f,\vec g)$ and $B(\vec g,\vec f)$ for $\vec f \in (E_T)^d$ and $\vec g \in \mathcal{C}([0,T],(X_1)^d)$ or $\vec g \in \mathcal{C}([0,T],(L^\infty)^d)$. Since the operator $e^{(t-s)\Delta}\mathbb{P}\vec\nabla.$ is a matrix of convolution operators and since the norms of L^∞, X_1 and E_T are invariant through a shift of the space variable, it is enough to estimate $\big(\int_{|x|<1}\int_0^T |B(\vec f,\vec g)|^2\,dx\,dt\big)^{1/2}$ and $\big(\int_{|x|<1}\int_0^T |B(\vec g,\vec f)|^2\,dx\,dt\big)^{1/2}$ to estimate the norms of $B(\vec f,\vec g)$ and $B(\vec g,\vec f)$ in $(E_T)^d$.

We split $\vec f$ in $\vec f_0$ supported in $[0,T]\times\{x\in\mathbb{R}^d \ / \ |x|\leq 10\}$ and $\vec f_1$ supported in $[0,T]\times\{x\in\mathbb{R}^d \ / \ |x|\geq 10\}$. It is well known that the kernel of the operators defining $e^{(t-s)\Delta}\mathbb{P}\vec\nabla.$ are bounded by

$$C\frac{1}{(t-s)^{(d+1)/2}+|x-y|^{d+1}}, \tag{22}$$

so that

$$\begin{aligned}\Big(\int_{|x|<1}\int_0^T |B(\vec f_1,\vec g)|^2\,dx\,dt\Big)^{1/2} &\leq C\sqrt{T}\int_{|y|\geq 10}\int_0^T \tfrac{|\vec f(s,y))|\,|\vec g(s,y)|}{|y|^{d+1}}\,dy\,ds\\ &\leq C'\sqrt{T}\|\vec f\|_{E_T}\|\vec g\|_{E_T}\end{aligned} \tag{23}$$

and we easily conclude since $L^\infty\subset L^2_{uloc}$ and $X_1\subset L^2_{uloc}$ (so that we have $\|\vec g\|_{E_T}\leq C\sqrt{T}\sup_{t\in[0,T]}\|\vec g(t,.)\|_{L^\infty}$ and $\|\vec g\|_{E_T}\leq C\sqrt{T}\sup_{t\in[0,T]}\|\vec g(t,.)\|_{X_1}$).

Moreover, we have

$$\|e^{(t-s)\Delta}\mathbb{P}\vec\nabla.\vec f_0\otimes\vec g\|_{L^2(dx)}\leq C\frac{1}{\sqrt{t-s}}\|\vec f_0(s,.)\|_2\|\vec g(s,.)\|_\infty \tag{24}$$

and thus

$$\Big(\int_{|x|<1}\int_0^T |B(\vec f_0,\vec g)|^2 dx\,dt\Big)^{1/2}\leq C\sup_{t\in[0,T]}\|\vec g(t,.)\|_\infty\Big(\int_0^T\Big(\int_0^t\|\vec f_0(s,.)\|_2\frac{ds}{\sqrt{t-s}}\Big)^2dt\Big)^{1/2} \tag{25}$$

and thus

$$\begin{aligned}\Big(\int_{|x|<1}\int_0^T|B(\vec f_0,\vec g)|^2\,dx\,dt\Big)^{1/2} &\leq C\sqrt{T}\sup_{t\in[0,T]}\|\vec g(t,.)\|_\infty\Big(\int_0^T\|\vec f_0(s,.)\|_2^2\,ds\Big)^{1/2}\\ &\leq C'\sqrt{T}\|\vec f\|_{E_T}\sup_{t\in[0,T]}\|\vec g(t,.)\|_\infty\end{aligned} \tag{26}$$

On the other hand, the pointwise multipliers from H^1 to L^2 map as well L^2 to H^{-1} (by duality), so that we have

$$\|(Id-\Delta)^{-1}\mathbb{P}\vec\nabla.\vec f_0\otimes\vec g\|_{L^2(dx)}\leq C\|\vec f_0(s,.)\|_{L^2}\|\vec g(s,.)\|_{X_1}, \tag{27}$$

while we have the following well-known energy estimates

$$\|\int_0^t e^{(t-s)\Delta}h(s,.)\,ds\|_{L^2((0,T)\times\mathbb{R}^d)}\leq\sqrt{T}\int_0^T\|h(s,.)\|_2\,ds\leq T\|h\|_{L^2((0,T)\times\mathbb{R}^d)} \tag{28}$$

and

$$\| \int_0^t e^{(t-s)\Delta} \Delta h(s,.)\; ds\|_{L^2((0,T)\times \mathbb{R}^d)} \leq C \|h\|_{L^2((0,T)\times\mathbb{R}^d)}. \tag{29}$$

Thus far, we have proved that there exists a constant C_0 such that for all $T > 0$ we have

$$\|B(\vec f,\vec g)\|_{E_T} + \|B(\vec g,\vec f)\|_{E_T} \leq C_0 \max(\sqrt{T}, T) \|\vec f\|_{E_T} \sup_{t\in[0,T]} \|\vec g(t,.)\|_\infty \tag{30}$$

and

$$\|B(\vec f,\vec g)\|_{E_T} + \|B(\vec g,\vec f)\|_{E_T} \leq C_0 \max(1, T) \|\vec f\|_{E_T} \sup_{t\in[0,T]} \|\vec g(t,.)\|_{X_1}. \tag{31}$$

Hence, we have, for every $T \in (0, \min(T^*, 1))$ and every decomposition $\vec u_0 = \vec\alpha + \vec\beta$, that

$$\|\vec w\|_{E_T} \leq C_0 \|\vec w\|_{E_T} (2\|\vec\alpha\|_{X_1} + 2\sqrt{T}\|\vec\beta\|_\infty + \sup_{0<t<T} \|\vec w_1(t,.)\|_{X_1} + \sup_{0<t<T} \|\vec w_2(t,.)\|_{X_1}). \tag{32}$$

Thus, if we choose a decomposition such that $\|\vec\alpha\|_{X_1} < \frac{1}{4C_0}$ and then choose a time T small enough to grant that

$$2\sqrt{T}\|\vec\beta\|_\infty \leq \frac{1}{4C_0} \text{ and } \sup_{0<t<T} \|\vec w_1(t,.)\|_{X_1} + \sup_{0<t<T} \|\vec w_2(t,.)\|_{X_1} \leq \frac{1}{4C_0}, \tag{33}$$

we get that $\|\vec w\|_{E_T} \leq \delta \|\vec w\|_{E_T}$ with $\delta < 1$, hence $\vec w = 0$ on $(0,T)$. Thus, $\vec u = \vec v$ on $[0,T)$, hence at $t = T$ as well by continuity. Now, if T_{MAX} is the greatest time for which $\vec u = \vec v$ on $[0, T_{MAX})$ and if $T_{MAX} < T^*$, we may reiterate our proof with initial value $\vec u(T_{MAX}, .)$ (since the Navier–Stokes equations are invariant through time-translations) to get a little time T such that $\vec u = \vec v$ on $[T_{MAX}, T_{MAX} + T]$, and thus get a contradiction. Thus, we have proved $\vec u = \vec v$ on the whole interval $[0, T^*)$. ◇

6. LIONS AND MASMOUDI'S UNIQUENESS THEOREM

We shall now slightly generalize Theorem 4 :

Theorem 8 : (Generalization of Lions and Masmoudi's uniqueness theorem.)
Let $d \geq 3$. If $\vec u$ and $\vec v$ are two weak solutions of the Navier-Stokes equations on $(0,T^*) \times \mathbb{R}^d$ such that
i) $\sup_{0<t<T^*} \|\vec u\|_{X_1} < \infty$ *and* $\sup_{0<t<T^*} \|\vec v\|_{X_1} < \infty$
ii) for every $t \in [0, T^*)$, $\vec u(t,.)$ and $\vec v(t,.)$ belong more precisely to $(\tilde X_1)^d$
iii) $\sup_{x_0\in\mathbb{R}^d} \left(\int_{|x-x_0|<1} \int_0^{T^*} |\vec u(t,x)|^4 \, dx\, dt\right)^{1/4} < \infty$
iv) $\sup_{x_0\in\mathbb{R}^d} \left(\int_{|x-x_0|<1} \int_0^{T^*} |\vec v(t,x)|^4 \, dx\, dt\right)^{1/4} < \infty$
v) $\vec u$ and $\vec v$ have the same initial value,
then $\vec u = \vec v$.

Remark: This is a generalization of Theorem 4, since we have $L^{d,\infty} \subset X_1$ for $d \geq 3$ and $L^{d,\infty} \subset L^4_{uloc}$ for $d \geq 5$.

The key ingredients in the proof of Theorem 4 by Lions and Masmoudi [9] were the following: first, they checked that, when $\vec{u}$ and $\vec{v}$ are weak solutions of the Navier–Stokes equations that are bounded in the L^d norm, the difference $\vec{w} = \vec{u} - \vec{v}$ belongs to $L^2((0,T^*),(L^2)^d) \cap L^2((0,T^*),(\dot{H}^1)^d)$, then they establish a Leray inequality on $\|\vec{w}\|_2$

$$\partial_t \|\vec{w}\|_2^2 + 2\int_{\mathbb{R}^d} |\vec{\nabla} \otimes \vec{w}|^2 \, dx = 2\int_{\mathbb{R}^d} \vec{u}.[\vec{w}.\vec{\nabla}]\vec{w} - \vec{w}.[\vec{v}.\vec{\nabla}]\vec{w} \, dx \tag{34}$$

and then use the antisymmetry property $\int_0^t \int_{\mathbb{R}^d} \vec{w}.(\vec{v}.\vec{\nabla})\vec{w} \, dx \, ds = 0$ to get rid of $\vec{v}$ while they use Kato's theory of mild solutions to assume that $\vec{u}$ is smooth.

Similarly, in order to prove Theorem 8, we first prove another generalization of Lions and Masmoudi's theorem :

Theorem 9: (Second generalization of Lions and Masmoudi's uniqueness theorem.)

Let $d \geq 3$. Let $\vec{u}_0 \in (\tilde{X}_1)^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. Let $\vec{u} \in (\mathcal{C}([0,T_0]),\tilde{X}_1)^d$ be the Kato solution of the Navier-Stokes equations on $[0,T_0]$ with initial value $\vec{u}_0$. Let $\vec{v}$ be another solution of the Navier-Stokes equations on $(0,T_0) \times \mathbb{R}^d$ such that

i) $\sup_{0<t<T_0} \|\vec{v}\|_{X_1} < \infty$

ii) for every $t \in [0,T_0)$, $\vec{v}(t,.)$ belongs more precisely to $(\tilde{X}_1)^d$

iii) $\sup_{x_0 \in \mathbb{R}^d} \left(\int_{|x-x_0|<1} \int_0^{T_0} |\vec{\nabla} \otimes \vec{v}|^2 \, dx \, dt \right)^{1/2} < \infty$

iv) $\vec{u}$ and $\vec{v}$ have the same initial value $\vec{u}_0$,

then $\vec{u} = \vec{v}$ on some interval $[0,\tau)$.

Of course, we must first show some useful regularity results on mild solutions for initial values in $(\tilde{X}_1)^d$. This will be done in the next section, then in the last section we shall prove Theorems 8 and 9.

7. KATO'S SOLUTIONS IN $(\tilde{X}_1)^d$.

In this section, we shall prove the following result on mild solutions in $(\tilde{X}_1)^d$:

Proposition 3:

Let $\vec{u}_0 \in (\tilde{X}_1)^d$ with $\vec{\nabla}.\vec{u}_0 = 0$. Then there exists a positive T such that the initial value problem for the Navier-Stokes equations with initial data $\vec{u}_0$ has a solution $\vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) \, ds$ with the following properties:

i) $\vec{u} \in \mathcal{C}([0,T),(\tilde{X}_1)^d)$

ii) $\sup_{x_0 \in \mathbb{R}^d} \left(\int_{|x-x_0|<1} \int_0^T |\vec{\nabla} \otimes \vec{u}|^2 \, dx \, dt \right)^{1/2} < \infty$.

Before proving Proposition 3, we recall a general tool for estimating L^2_{uloc} norms [6] [7]: in this section and in the following, we shall use a given smooth function $\varphi_0 \in \mathcal{D}(\mathbb{R}^d)$

such that $\varphi_0 \geq 0$ and $\sum_{k\in\mathbb{Z}^d} \varphi_0(x-k) = 1$. We define $\mathcal{B} = \{\varphi_0(x - x_0) \ / \ x_0 \in \mathbb{R}^d\}$; the norm $\|f\|_{L^2_{uloc}}$ is then equivalent to $\sup_{\varphi\in\mathcal{B}} \|f\varphi\|_2$. We fix as well ω_0 and $\psi_0 \in \mathcal{D}(\mathbb{R}^d)$ such that ω_0 is identically equal to 1 in the neighbourhood of the support of φ_0 and similarly ψ_0 is identically equal to 1 in the neighbourhood of the support of ω_0. Finally, for $\varphi \in \mathcal{B}$, $\varphi = \varphi_0(x - x_\varphi)$, we define $\omega = \omega_0(x - x_\varphi)$ and $\psi = \psi_0(x - x_\varphi)$.

Proof:

Let $\mathcal{E}_T$ be the space of functions f on $[0, T] \times \mathbb{R}^d$ such that the mapping $t \mapsto f(t, .)$ belongs to $\mathcal{C}([0, T], \tilde{X}_1)$ and the mapping $t \mapsto \sqrt{t} f(t, .)$ belongs to $\mathcal{C}([0, T], \mathcal{C}_0)$ with value 0 at $t = 0$, with norm

$$\|f\|_{\mathcal{E}_T} = \sup_{0<t<T} \|f\|_{X_1} + \sqrt{t}\|f\|_\infty. \tag{35}$$

Let $\mathcal{F}_T$ be the subspace of functions f in $\mathcal{E}_T$ such that the norm

$$\|f\|_{\mathcal{F}_T} = \|f\|_{\mathcal{E}_T} + \sup_{x_0\in\mathbb{R}^d} \Big(\int_{|x-x_0|<1} \int_0^T |\vec{\nabla} f|^2 \ dx \ dt\Big)^{1/2} \tag{36}$$

is finite.

We first check that, whenever $f_0 \in \tilde{X}_1$, $e^{t\Delta} f_0 \in \mathcal{F}_T$. First, we notice that we have $\sup_{0<t<T} \sqrt{t}\|f_0\|_\infty \leq C_T \|f_0\|_{M^{2,3}}$. It is easy to check that $\tilde{X}_1$ is stable under convolution with an integrable kernel, and more precisely that

$$\|f * g\|_{X_1} \leq \|f\|_{X_1} \|g\|_1 \tag{37}$$

since the norm of $\tilde{X}_1$ is invariant under translations of the argument $f(.) \mapsto f(. - x_0)$. Thus, it is obvious that $e^{t\Delta} f_0 \in \mathcal{E}_T$. Hence, we just have to check that we have

$$\sup_{\varphi\in\mathcal{B}} \Big(\int\int_0^T \varphi(x)^2 |\vec{\nabla} e^{t\Delta} f_0|^2 \ dx \ dt\Big)^{1/2} < \infty. \tag{38}$$

Let $f = e^{t\Delta} f_0$; we have

$$\begin{aligned} \partial_t \|\varphi \ f\|_2^2 = 2\int \varphi^2 f \Delta f \ dx \ &= -2\int \varphi^2 |\vec{\nabla} f|^2 \ dx - 4\int \varphi \ f \ \vec{\nabla}\varphi.\vec{\nabla} f \ dx \\ &\leq -\int \varphi^2 |\vec{\nabla} f|^2 \ dx + 4\int f^2 \ |\vec{\nabla}\varphi|^2 \ dx. \end{aligned} \tag{39}$$

This gives

$$\begin{aligned} \int\int_0^T \varphi^2 |\vec{\nabla} f|^2 \ dx \ dt \ &\leq \int (f_0(x)^2 - f(T,x)^2)\varphi(x)^2 \ dx \quad +4\int\int_0^T f^2 \ |\vec{\nabla}\varphi|^2 \ dx \ dt \\ &\leq C(1+T)\|f_0\|^2_{L^2_{uloc}}. \end{aligned} \tag{40}$$

Let us consider again the bilinear operator B defined by

$$B(\vec{\alpha}, \vec{\beta}) = \int_0^t e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{\alpha} \otimes \vec{\beta}) \ ds. \tag{41}$$

We consider $T \leq 1$ and we define the semi-norms N_0^T and N_∞^T as

$$N_0^T(\vec{\alpha}) = \sup_{0<t<T} \|\vec{\alpha}\|_{X_1} \text{ and } N_\infty^T(\vec{\alpha}) = \sup_{0<t<T} \sqrt{t}\ \|\vec{\alpha}\|_\infty. \tag{42}$$

We have the following easy estimates for $0 < s < t < T$:

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{\alpha}\otimes\vec{\beta}\|_{X_1} \leq C\frac{1}{\sqrt{t-s}}\|\vec{\alpha}\otimes\vec{\beta}\|_{X_1} \leq C'\frac{1}{\sqrt{s}\sqrt{t-s}}N_\infty^T(\vec{\alpha})N_0^T(\vec{\beta}) \tag{43}$$

and

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{\alpha}\otimes\vec{\beta}\|_\infty \leq C\min((1+\frac{1}{\sqrt{t-s}})\frac{1}{\sqrt{t-s}}\|\vec{\alpha}\otimes\vec{\beta}\|_{X_1}, \frac{1}{\sqrt{t-s}}\|\vec{\alpha}\otimes\vec{\beta}\|_\infty), \tag{44}$$

hence

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{\alpha}\otimes\vec{\beta}\|_\infty \leq C'\frac{N_\infty^T(\vec{\alpha})}{\sqrt{s}\sqrt{t-s}}\min((1+\frac{1}{\sqrt{t-s}})N_0^T(\vec{\beta}), \frac{1}{\sqrt{s}}N_\infty^T(\vec{\beta})). \tag{45}$$

If we assume moreover $T < 1$, we get (writing, for positive x and y, $\min(x,y) \leq \sqrt{xy}$) that

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{\alpha}\otimes\vec{\beta}\|_\infty \leq C\frac{1}{s^{3/4}(t-s)^{3/4}}N_\infty^T(\vec{\alpha})\sqrt{N_\infty^T(\vec{\beta})N_0^T(\vec{\beta})} \tag{46}$$

Thus, we get, for $T < 1$,

$$\|B(\vec{\alpha},\vec{\beta})\|_{\mathcal{E}_T} \leq C_0 N_\infty^T(\vec{\alpha})\|\vec{\beta}\|_{\mathcal{E}_T} \text{ and, similarly, } \|B(\vec{\alpha},\vec{\beta})\|_{\mathcal{E}_T} \leq C_0 N_\infty^T(\vec{\beta})\|\vec{\alpha}\|_{\mathcal{E}_T}, \tag{47}$$

and

$$N_\infty^T(B(\vec{\alpha},\vec{\beta})) \leq C_0 N_\infty^T(\vec{\alpha})\sqrt{N_\infty^T(\vec{\beta})N_0^T(\vec{\beta})}, \tag{48}$$

where the constant C_0 does not depend on T. If δ is chosen small enough to ensure the inequalities

$$\delta \leq \|\vec{u}_0\|_{X_1}, \quad 4\delta C_0 < 1 \quad \text{and} \quad 32\delta C_0^2\|u_0\|_{X_1}^2 \leq 1 \tag{49}$$

and if T is chosen small enough to ensure that

$$N_\infty^T(e^{t\Delta}\vec{u}_0) \leq \delta, \tag{50}$$

we easily check by induction that the sequence $(\vec{v}_n)_{n\in\mathbb{N}}$ recursively defined by

$$\vec{v}_0 = e^{t\Delta}\vec{u}_0 \text{ and } \vec{v}_{n+1} = e^{t\Delta}\vec{u}_0 - B(\vec{v}_n,\vec{v}_n) \tag{51}$$

satisfies the inequalities

$$\|\vec{v}_n\|_{\mathcal{E}_T} \leq 4\|\vec{u}_0\|_{X_1}, \quad N_\infty^T(\vec{v}_n) \leq 2\delta \quad \text{and } \|\vec{v}_{n+1}-\vec{v}_n\|_{\mathcal{E}_T} \leq (4C_0\delta)^n 4\|\vec{u}_0\|_{X_1}. \tag{52}$$

Thus, the sequence $(v_n)_{n\in\mathbb{N}}$ converges to a solution $\vec{u}$ of

$$\vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\ ds \tag{53}$$

with $\vec{u}\in\mathcal{C}([0,T),(\tilde{X}_1)^d)$.

We now check, for $0<T<1$, the inequalities

$$\|B(\vec{\alpha},\vec{\beta})\|_{\mathcal{F}_T} \le C_1\|\vec{\alpha}\|_{\mathcal{E}_T}\|\vec{\beta}\|_{\mathcal{F}_T} \text{ and } \|B(\vec{\alpha},\vec{\beta})\|_{\mathcal{F}_T} \le C_1\|\vec{\beta}\|_{\mathcal{E}_T}\|\vec{\alpha}\|_{\mathcal{F}_T}. \tag{54}$$

We must estimate, for $\varphi\in\mathcal{B}$, the norm $\|\varphi\vec{\nabla}\otimes B(\vec{\alpha},\vec{\beta})\|_{L^2((0,T)\times\mathbb{R}^d)}$. We have

$$\begin{aligned}\|\varphi\vec{\nabla}\otimes B(\psi\vec{\alpha},\vec{\beta})\|_{L^2((0,T)\times\mathbb{R}^d)} &\le \|\vec{\nabla}\otimes B(\psi\vec{\alpha},\vec{\beta})\|_{L^2((0,T)\times\mathbb{R}^d)}\\ &\le \|\textstyle\int_0^t e^{(t-s)\Delta}\Delta(\psi\vec{\alpha}\otimes\vec{\beta})\ ds\|_{L^2((0,T)\times\mathbb{R}^d)}\end{aligned} \tag{55}$$

hence,

$$\begin{aligned}\|\varphi\vec{\nabla}\otimes B(\psi\vec{\alpha},\vec{\beta})\|_{L^2((0,T)\times\mathbb{R}^d)} &\le C\|\psi\vec{\alpha}\otimes\vec{\beta}\|_{L^2((0,T)\times\mathbb{R}^d)}\\ &\le C'\sup_{0<t<T}\|\vec{\alpha}\|_{X_1}\|\psi\vec{\beta}\|_{L^2((0,T),H^1)} \le C''\|\vec{\alpha}\|_{\mathcal{E}_T}\|\vec{\beta}\|_{\mathcal{F}_T}\end{aligned} \tag{56}$$

On the other hand, if Supp $\varphi_0\subset B(0,r_0)$ and if ψ_0 is chosen such that $\psi=1$ on $B(0,r_1)$ with $r_1>r_0$, we have

$$\begin{aligned}|\varphi\vec{\nabla}\otimes B((1-\psi)\vec{\alpha},\vec{\beta})| &\le C\textstyle\int_0^t\int_{|y-x_\varphi|\ge r_1}\frac{1}{|y-x_\varphi|^{d+2}}|\vec{\alpha}(s,y)\otimes\vec{\beta}(s,y)|\ ds\ dy\\ &\le C'\frac{\sqrt{T}}{r_1}\big(\textstyle\int_0^T\int_{|y-x_\varphi|\ge r_1}\frac{1}{|y-x_\varphi|^{d+2}}|\vec{\alpha}(s,y)\otimes\vec{\beta}(s,y)|^2\ ds\ dy\big)^{1/2}\\ &\le C''\|\big(\textstyle\int_0^T|\vec{\alpha}(s,y)\otimes\vec{\beta}(s,y)|^2\ ds\ dy\big)^{1/2}\|_{L^2_{uloc}}\end{aligned} \tag{57}$$

hence,

$$\begin{aligned}\|\varphi\vec{\nabla}\otimes B((1-\psi)\vec{\alpha},\vec{\beta})\|_{L^2((0,T)\times\mathbb{R}^d)} &\le C'''\|\big(\textstyle\int_0^T|\vec{\alpha}(s,y)\otimes\vec{\beta}(s,y)|^2\ ds\ dy\big)^{1/2}\|_{L^2_{uloc}}\\ &\le C''''\sup_{0<t<T}\|\vec{\alpha}\|_{X_1}\sup_{\theta\in\mathcal{B}}\|\theta\vec{\beta}\|_{L^2((0,T),H^1)}\\ &\le C'''''\|\vec{\alpha}\|_{\mathcal{E}_T}\|\vec{\beta}\|_{\mathcal{F}_T}.\end{aligned} \tag{58}$$

Thus, we get inequalities (54).

From (54), we get by induction that $\vec{v}_n$ belongs more precisely to $\mathcal{F}_T$. Moreover, for $n\ge1$,

$$\|\vec{v}_{n+1}-\vec{v}_n\|_{\mathcal{F}_T} \le 2C_1\|\vec{v}_n\|_{\mathcal{F}_T}\|\vec{v}_{n-1}-\vec{v}_n\|_{\mathcal{E}_T} \le 8C_1(4C_0\delta)^{n-1}\|\vec{u}_0\|_{X_1}\|\vec{v}_n\|_{\mathcal{F}_T}. \tag{59}$$

Now, we take η such that $4C_0\delta<\eta<1$ and n_0 such that for every $n\ge n_0$ we have

$$8C_1(4C_0\delta)^{n-1}\|\vec{u}_0\|_{X_1} \le \eta^{n+1}(1-\eta). \tag{60}$$

We define Γ as

$$\Gamma = \max(\|\vec{v}_0\|_{\mathcal{F}_T}, \sup_{0\le n\le n_0}\eta^{-n-1}\|\vec{v}_{n+1}-\vec{v}_n\|_{\mathcal{F}_T}) \tag{61}$$

and we find by induction on n that $\|\vec{v}_{n+1} - \vec{v}_n\|_{\mathcal{F}_T} \leq \Gamma\eta^{n+1}$, so that $\vec{u} = \lim_{n\to\infty} \vec{v}_n$ belongs to $\mathcal{F}_T$. $\diamond$

8. PROOFS OF THEOREMS 8 AND 9.

Proof of Theorem 9:

We proved in the previous section that the Kato solution $\vec{u}$ is smooth on $(0, T_0) \times \mathbb{R}^d$ and belongs to $\mathcal{C}([0, T_0), (\tilde{X}_1)^d)$, while

$$\sup_{x_0\in\mathbb{R}^d} \Big(\int_{|x-x_0|<1} \int_0^{T_0} |\vec{\nabla} \otimes \vec{u}|^2 \, dx \, dt\Big)^{1/2} < \infty. \tag{62}$$

Let us define again $\vec{u}_0 = \vec{u}(0, .) = \vec{v}(0, .)$, $\vec{w}_1 = B(\vec{u}, \vec{u}) = e^{t\Delta}\vec{u}_0 - \vec{u}$, $\vec{w}_2 = B(\vec{v}, \vec{v}) = e^{t\Delta}\vec{u}_0 - \vec{v}$ and $\vec{w} = \vec{u} - \vec{v} = B(\vec{v}, \vec{v}) - B(\vec{u}, \vec{u})$. We are going to estimate the norm of $\vec{w}(t, .)$ in $(L^2_{uloc}(\mathbb{R}^d))^d$, following the lines of the study of uniformly locally square integrable solutions in [6] [7]. To estimate L^2_{uloc} norms, we use the family $\mathcal{B}$ of smooth functions we introduced in the proof of Proposition 3 and we define the functions

$$\alpha(t) = \sup_{\varphi\in\mathcal{B}} \|\vec{w}(t, .)\varphi(x)\|_2^2 \tag{63}$$

and

$$\beta(t) = \sup_{\varphi\in\mathcal{B}} \int_0^t \|(\vec{\nabla} \otimes \vec{w}(s, .))\varphi(x)\|_2^2 ds. \tag{64}$$

1st step : Computing $\partial_t|\vec{w}|^2$.

We begin by proving that we have in the sense of distributions that

$$\partial_t|\vec{w}|^2 = 2(\partial_t\vec{w}).\vec{w}. \tag{65}$$

We know that $\vec{w}$ is (locally in x) in $(L^2_t H^1_x)^d$. Thus, it will be enough to prove that $\partial_t\vec{w}$ belongs (locally in x) in $(L^2_t H^{-1}_x)^d$. We write

$$\partial_t\vec{w} = \Delta\vec{w} - \vec{\nabla}.(\vec{u} \otimes \vec{u} - \vec{v} \otimes \vec{v}) - \vec{\nabla}(p - q) \tag{66}$$

where p and q are the pressures associated with $\vec{u}$ and $\vec{v}$. We have that $\Delta\vec{w}$ belongs locally to $(L^2_t H^{-1}_x)^d$; moreover, since $\vec{u}\otimes\vec{u} - \vec{v}\otimes\vec{v} = \vec{w}\otimes\vec{u} + \vec{v}\otimes\vec{w}$ and thus $\vec{u}\otimes\vec{u} - \vec{v}\otimes\vec{v}$ belongs locally to $(L^2_t L^2_x)^d$, we have that $\vec{\nabla}.(\vec{u} \otimes \vec{u} - \vec{v} \otimes \vec{v})$ belongs locally to $(L^2_t H^{-1}_x)^d$. In order to estimate the remaining term $\vec{\nabla}(p-q)$, we notice that, since $\vec{u}$ belongs to $\tilde{X}_1$, it vanishes at infinity in the sense of [3] and thus $\vec{\nabla}p$ is computed as

$$\vec{\nabla}p = (Id - \mathbb{P})\vec{\nabla}.\vec{u} \otimes \vec{u} = -\vec{\nabla}\frac{1}{\Delta}(\vec{\nabla} \otimes \vec{\nabla}.\vec{u} \otimes \vec{u}), \tag{67}$$

and similarly $\vec{\nabla} q = (Id - \mathbb{P})\vec{\nabla}.\vec{v}\otimes\vec{v}$. We are going to show that, for $\varphi\in\mathcal{B}$, we have $\varphi(x)\vec{\nabla}(p-q)\in(L^2_t H^{-1}_x)^d$: indeed, we write again $\vec{u}\otimes\vec{u}-\vec{v}\otimes\vec{v}=\vec{w}\otimes\vec{u}+\vec{v}\otimes\vec{w}$ and we further decompose $\vec{w}$ into $\psi\vec{w}+(1-\psi)\vec{w}$; for the contribution of $\psi\vec{w}$, we notice that $\psi\vec{w}\otimes\vec{u}=\vec{w}\otimes(\psi\vec{u})$ and that $\vec{w}$ and $\vec{v}$ are divergence free, so that

$$(Id-\mathbb{P})\vec{\nabla}.(\psi\vec{w}\otimes\vec{u}+\vec{v}\otimes\psi\vec{w}) = -\vec{\nabla}\frac{1}{\Delta}\vec{\nabla}.((\vec{w}.\vec{\nabla})(\psi\vec{u})+(\vec{v}.\vec{\nabla})(\psi\vec{w})) \tag{68}$$

and we conclude since $\psi\vec{w}$ and $\psi\vec{u}$ belong to $(L^2_t H^1_x)^d$ while $\vec{w}(t,.)$ and $\vec{v}(t,.)$ are uniformly bounded in $(X_1)^d$ and since pointwise multiplication maps $X_1\times L^2$ to H^{-1}; for the contribution of $(1-\psi)\vec{w}$, we notice that the kernel of the convolution operator $(Id-\mathbb{P})\vec{\nabla}.$ is $O(\frac{1}{|x-y|^{d+1}})$ off the diagonal, so that, defining $\vec{f}$ for typographic convenience as

$$\vec{f}=\varphi(x)(Id-\mathbb{P})\vec{\nabla}.((1-\psi)\vec{w}\otimes\vec{u}+\vec{v}\otimes(1-\psi)\vec{w}), \tag{69}$$

we get

$$|\vec{f}(x)|\leq C\int_{|y-x_\varphi|\geq\gamma}\frac{1}{|y-x_\varphi|^{d+1}}(|\vec{u}(t,y)|^2+|\vec{v}(t,y)|^2)\ dy \tag{70}$$

which gives that $\vec{f}$ belongs to $(L^2_t L^2_x)^d$ with a norm which is controlled by

$$\|\vec{f}\|_{L^2_t L^2_x}\leq C(\sup_{x_0\in\mathbb{R}^d}(\int_{|x-x_0|<1}\int_0^t|\vec{u}(s,x)|^2\ dx\ ds)+\sup_{x_0\in\mathbb{R}^d}(\int_{|x-x_0|<1}\int_0^t|\vec{v}(s,x)|^2\ dx\ ds)). \tag{71}$$

Thus, (65) is proved.

2nd step : the energy inequality.

We have

$$2\Delta\vec{w}.\vec{w}=\Delta|\vec{w}|^2-2|\vec{\nabla}\otimes\vec{w}|^2, \tag{72}$$

$$2\vec{\nabla}(p-q).\vec{w}=2\vec{\nabla}((p-q)\vec{w}) \tag{73}$$

since $\vec{w}$ is divergence free,

$$2(\vec{\nabla}.\vec{w}\otimes\vec{u}).\vec{w}=2\vec{\nabla}((\vec{u}.\vec{w})\vec{w})-2\vec{u}.(\vec{w}.\vec{\nabla})\vec{w} \tag{74}$$

and

$$2(\vec{\nabla}.\vec{v}\otimes\vec{w}).\vec{w}=\vec{\nabla}(|\vec{w}|^2\vec{v}). \tag{75}$$

Finally, we get the equality

$$\partial_t|\vec{w}|^2=\Delta|\vec{w}|^2-2|\vec{\nabla}\otimes\vec{w}|^2-2\vec{\nabla}((p-q)\vec{w})-2\vec{\nabla}((\vec{u}.\vec{w})\vec{w})+2\vec{u}.(\vec{w}.\vec{\nabla})\vec{w}-\vec{\nabla}(|\vec{w}|^2\vec{v}) \tag{76}$$

In particular, we have that for $\varphi\in\mathcal{B}$ the mapping $t\mapsto\|\varphi\ \vec{w}(t,.)\|_2^2$ is continuous and fullfills the equality

$$\begin{array}{c}\|\vec{w}(t,.)\varphi(x)\|_2^2+2\int_0^t\|(\vec{\nabla}\otimes\vec{w}(s,.))\varphi(x)\|_2^2\ ds=\\ \iint_0^t|\vec{w}|^2\Delta(\varphi^2(x))\ dx\ ds\\ +2\iint_0^t(\vec{w}.\vec{u})\ \vec{w}.\vec{\nabla}\varphi^2(x)\ dx\ ds+2\iint_0^t\varphi^2(x)\vec{u}.(\vec{w}.\vec{\nabla})\vec{w}\ dx\ ds\\ +\iint_0^t|\vec{w}|^2\vec{v}.\vec{\nabla}\varphi^2(x)\ dx\ ds+2\iint_0^t(p-q)(\vec{w}.\vec{\nabla})\varphi^2(x)\ dx\ ds\end{array} \tag{77}$$

Let us notice that we may modify $p-q$ by adding a function $\varpi_\varphi(s)$ which does not depend on x: since $\vec{w}$ is divergence free, we have $\int \varpi_\varphi(s)\vec{w}(s,x).\vec{\nabla}\varphi^2(x)\ dx = 0$.

Now, we split again $\vec{u}_0$ into $\vec{u}_0 = \vec{\alpha} + \vec{\beta}$ and define $\vec{A} = \vec{u} - e^{t\Delta}\vec{\beta}$ and $\vec{B} = e^{t\Delta}\vec{\beta}$. We then check the following estimates for all $t \in (0, T^*)$ and all $\varphi \in \mathcal{B}$:

$$\int\int_0^t |\vec{w}|^2\ \Delta(\varphi^2(x))\ dx\ ds \le C_1 \int_0^t \alpha(s)\ ds \tag{78}$$

$$\begin{aligned} 2\iint_0^t (\vec{w}.\vec{u})\vec{w}.\vec{\nabla}\varphi^2(x)\ dx\ ds &\le C\int_0^t \|\vec{u}\|_{X_1}\|\psi\vec{w}\|_2\|\psi\vec{w}\|_{H^1}\ ds \\ &\le C_2 \sup_{0<s<t}\|\vec{u}\|_{X_1}\sqrt{\int_0^t \alpha\ ds}\sqrt{\beta(t)+\int_0^t\alpha\ ds} \end{aligned} \tag{79}$$

$$\int\int_0^t |\vec{w}|^2\vec{v}.\vec{\nabla}\varphi^2(x)\ dx\ ds \le C_3 \sup_{0<s<t}\|\vec{v}\|_{X_1}\sqrt{\int_0^t \alpha\ ds}\sqrt{\beta(t)+\int_0^t \alpha\ ds} \tag{80}$$

$$\begin{aligned} 2\iint_0^t (p-q)(\vec{w}.\vec{\nabla})\varphi^2(x)\ dx\ ds &\le C\int_0^t \|(p-q+\varpi_\varphi(s))\omega\|_2\|\psi\vec{w}\|_2 ds \\ &\le C_4\sqrt{\int_0^t \alpha\ ds}\sqrt{\int_0^t \|(p-q+\varpi_\varphi)\omega\|_2^2\ ds} \end{aligned} \tag{81}$$

$$\begin{aligned} 2\iint_0^t \varphi^2(x)\vec{A}.(\vec{w}.\vec{\nabla})\vec{w}\ dx\ ds &\le C\int_0^t \|\vec{A}\|_{X_1}\|\psi\vec{w}\|_{H_1}\|\psi\vec{\nabla}\otimes\vec{w}\|_2\ ds \\ &\le C_5 \sup_{0<s<t}\|\vec{A}\|_{X_1}(\beta(t)+\int_0^t\alpha(s)\ ds) \end{aligned} \tag{82}$$

$$\begin{aligned} 2\iint_0^t \varphi^2(x)\vec{B}.(\vec{w}.\vec{\nabla})\vec{w}\ dx\ ds &\le C\int_0^t \|\vec{B}\|_\infty\|\psi\vec{w}\|_2\|\psi\vec{\nabla}\otimes\vec{w}\|_2\ ds \\ &\le C_6 \sup_{0<s<t}\|\vec{B}\|_\infty\sqrt{\int_0^t\alpha\ ds}\sqrt{\beta(t)+\int_0^t\alpha\ ds} \end{aligned} \tag{83}$$

We still have to estimate $\int_0^t \|(p-q+\varpi_\varphi(s))\omega\|_2^2\ ds$ for a good choice of the function ϖ_φ. We postpone the (technical) proof to the next step and we claim that we shall be able to prove that

$$\begin{aligned} &\int_0^t \|(p-q+\varpi_\varphi(s))\omega\|_2^2\ ds \\ &\le C_7^2(\sup_{0<s<t}\|\vec{u}\|_{X_1} + \sup_{0<s<t}\|\vec{v}\|_{X_1})^2\ (\beta(t)+\int_0^t\alpha(s)\ ds) \end{aligned} \tag{84}$$

The right-hand terms in inequalities (78) to (80) and (82) to (84) do not depend on φ, thus taking the supremum on $\varphi \in \mathcal{B}$ we get that

$$\max(\alpha(t), 2\beta(t)) \le \left\{ \begin{array}{c} C_1\int_0^t \alpha(s)\ ds \\ +C_2\sup_{0<s<t}\|\vec{u}\|_{X_1}\sqrt{\int_0^t\alpha\ ds}\sqrt{\beta(t)+\int_0^t\alpha\ ds} \\ +C_3\sup_{0<s<t}\|\vec{v}\|_{X_1}\sqrt{\int_0^t\alpha\ ds}\sqrt{\beta(t)+\int_0^t\alpha\ ds} \\ +C_4C_7(\sup_{0<s<t}\|\vec{u}\|_{X_1}+\sup_{0<s<t}\|\vec{v}\|_{X_1})\sqrt{\int_0^t\alpha\ ds}\sqrt{\beta(t)+\int_0^t\alpha\ ds} \\ +C_5\sup_{0<s<t}\|\vec{A}\|_{X_1}(\beta(t)+\int_0^t\alpha(s)\ ds) \\ +C_6\sup_{0<s<t}\|\vec{B}\|_\infty\sqrt{\int_0^t\alpha\ ds}\sqrt{\beta(t)+\int_0^t\alpha\ ds} \end{array}\right. \tag{85}$$

3rd step : estimating the pressure.

We first recall that p is computed, at least formally, as

$$p = \sum_{1\leq j\leq d} \sum_{1\leq k\leq d} R_j R_k \big(u_j u_k\big), \tag{86}$$

where $R_j = \frac{\partial_j}{\sqrt{-\Delta}}$ is the j-th Riesz transform and u_j is the j-th coordinate of the vector $\vec u$ [3] [7]. We write more compactly

$$p = \mathcal{G}.\vec u \otimes \vec u = \int G(x,y).\vec u(t,y) \otimes \vec u(t,y)\ dy. \tag{87}$$

Similarly, we have $q = \mathcal{G}.\vec v \otimes \vec v$. We thus compute $p - q + \varpi_\varphi(t)$ on the support of $\omega = \omega_0(. - x_\varphi)$ through the formula:

$$p - q + \varpi_\varphi(t) = p_1(t,x) + p_2(t,x) \tag{88}$$

with

$$p_1(t,x) = \mathcal{G}\big(\psi_0^2(y - x_\varphi)(\vec u(t,y)\otimes \vec u(t,y) - \vec v(t,y)\otimes \vec v(t,y))\big)(x) \tag{89}$$

and

$$p_2(t,x) = \int (G(x,y) - G(x_\varphi,y)).(1-\psi_0^2(y-x_\varphi))(\vec u(t,y)\otimes\vec u(t,y) - \vec v(t,y)\otimes \vec v(t,y))\ dy. \tag{90}$$

We write one more time $\vec u\otimes\vec u - \vec v\otimes\vec v = \vec w\otimes\vec u + \vec v\otimes\vec w$. Since $\mathcal{G}$ is a matrix of Calderón–Zygmund operators, we have

$$\|p_1(t,x)\|_{L^2(dx)} \leq C\|\psi\vec w\|_{H^1}(\|\vec u\|_{X_1} + \|\vec v\|_{X_1}). \tag{91}$$

Moreover, we have

$$|p_2(t,x)\omega(x)| \leq C \sum_{k\in\mathbb{Z}^d, k\neq 0} \frac{1}{|k|^{d+1}} \int (|\vec u(t,y)\otimes\vec w(t,y)| + |\vec v(t,y)\otimes \vec w(t,y)|)\varphi_0(y - x_\varphi - k)\ dy \tag{92}$$

and thus

$$\|p_2(t,x)\omega(x)\|_{L^2(dx)} \leq C(\|\vec u\|_{X_1} + \|\vec v\|_{X_1}) \sum_{k\in\mathbb{Z}^d, k\neq 0} \frac{1}{|k|^{d+1}} \|\vec w(t,x)\psi_0(x - x_\varphi - k)\|_{H^1}. \tag{93}$$

Writing

$$A_t(\vec u, \vec v) = (\sup_{0<s<t} \|\vec u\|_{X_1} + \sup_{0<s<t}\|\vec v\|_{X_1})^2, \tag{94}$$

we get from (91) and (93) the following inequalities

$$\begin{gathered} \int_0^t \|(p - q + \varpi_\varphi(s))\omega\|_2^2\ ds \\ \leq CA_t(\vec u, \vec v)\ \int_0^t (\sum_{k\in\mathbb{Z}^d} \frac{1}{1+|k|^{d+1}} \|\vec w(s,x)\psi(x - x_\varphi - k)\|_{H^1})^2\ ds \\ \leq CA_t(\vec u, \vec v) (\sum_{k\in\mathbb{Z}^d} \frac{1}{1+|k|^{d+1}}) \sum_{k\in\mathbb{Z}^d} \frac{\int_0^t \|\vec w(s,x)\psi(x - x_\varphi - k)\|_{H^1}^2\, ds}{1+|k|^{d+1}} \\ \leq C_7^2 A_t(\vec u, \vec v)\ (\beta(t) + \int_0^t \alpha(s)\ ds) \end{gathered} \tag{95}$$

Thus, (84) is proved.

4th step : using a Gronwall lemma.

Finally, we use the obvious inequality $XY \leq \epsilon X^2 + \frac{1}{4\epsilon} Y^2$ to get from (85) the inequalities

$$\begin{array}{rc} \alpha(t) \leq & C_1 \int_0^t \alpha(s)\ ds \\ & +C_2 \sup_{0<s<t} \|\vec{u}\|_{X_1} \sqrt{\int_0^t \alpha\ ds} \sqrt{\beta(t) + \int_0^t \alpha\ ds} \\ & +C_3 \sup_{0<s<t} \|\vec{v}\|_{X_1} \sqrt{\int_0^t \alpha\ ds} \sqrt{\beta(t) + \int_0^t \alpha\ ds} \\ & +C_4 C_7 (\sup_{0<s<t} \|\vec{u}\|_{X_1} + \sup_{0<s<t} \|\vec{v}\|_{X_1}) \sqrt{\int_0^t \alpha\ ds} \sqrt{\beta(t) + \int_0^t \alpha(s)\ ds} \\ & +C_5 \sup_{0<s<t} \|\vec{A}\|_{X_1} (\beta(t) + \int_0^t \alpha(s)\ ds) \\ & +C_6 \sup_{0<s<t} \|\vec{B}\|_\infty \sqrt{\int_0^t \alpha\ ds} \sqrt{\beta(t) + \int_0^t \alpha\ ds} \end{array} \tag{96}$$

and

$$\begin{array}{rc} 2\beta(t) \leq & C_1 \int_0^t \alpha(s)\ ds \\ & +C_2 \sup_{0<s<t} \|\vec{u}\|_{X_1} \big((\epsilon + \frac{1}{4\epsilon}) \int_0^t \alpha\ ds + \epsilon\beta(t)\big) \\ & +C_3 \sup_{0<s<t} \|\vec{v}\|_{X_1} \big((\epsilon + \frac{1}{4\epsilon}) \int_0^t \alpha\ ds + \epsilon\beta(t)\big) \\ & +C_4 C_7 (\sup_{0<s<t} \|\vec{u}\|_{X_1} + \sup_{0<s<t} \|\vec{v}\|_{X_1}) \big((\epsilon + \frac{1}{4\epsilon}) \int_0^t \alpha\ ds + \epsilon\beta(t)\big) \\ & +C_5 \sup_{0<s<t} \|\vec{A}\|_{X_1} (\beta(t) + \int_0^t \alpha(s)\ ds) \\ & +C_6 \sup_{0<s<t} \|\vec{B}\|_\infty \big((\epsilon + \frac{1}{4\epsilon}) \int_0^t \alpha\ ds + \epsilon\beta(t)\big) \end{array} \tag{97}$$

which are fullfilled for every decomposition $\vec{u}_0 = \vec{\alpha} + \vec{\beta}$ and every $t \in (0, T^*)$.

We now use the fact that $\vec{u}_0 \in (\tilde{X}_1)^d$ and that $\vec{u} \in \mathcal{C}([0, T^*), (X_1)^d)$. We choose a decomposition such that $\|\vec{\alpha}\|_{X_1} < \frac{1}{4C_5}$ and then choose a time T small enough to grant that

$$\sup_{0<t<T} \|\vec{w}_1(t, .)\|_{X_1} \leq \frac{1}{4C_5}, \tag{98}$$

so that

$$\sup_{0<t<T} \|\vec{A}(t, .)\|_{X_1} \leq \frac{1}{2C_5}. \tag{99}$$

We then choose $\epsilon > 0$ small enough to grant that

$$\big((C_2 + C_4 C_7) \sup_{0<s<T} \|\vec{u}\|_{X_1} + (C_3 + C_4 C_7) \sup_{0<s<T} \|\vec{v}\|_{X_1} + C_6 \|\vec{\beta}\|_\infty\big)\epsilon < \frac{1}{2}. \tag{100}$$

This gives obviously

$$\beta(t) \leq D \int_0^t \alpha(s)\ ds \tag{101}$$

for $t \in (0, T)$, with

$$D = \begin{array}{c} C_1 + \big(\epsilon + \frac{1}{4\epsilon}\big)\big((C_2 + C_4 C_7) \sup_{0<s<T} \|\vec{u}\|_{X_1} \\ +(C_3 + C_4 C_7) \sup_{0<s<T} \|\vec{v}\|_{X_1} + C_6 \|\vec{\beta}\|_\infty\big) + C_5 \sup_{0<s<T} \|\vec{A}\|_{X_1} \end{array}. \tag{102}$$

We get finally

$$\alpha(t) \leq 2D \int_0^t \alpha(s)\ ds \tag{103}$$

and finally $\alpha = 0$ on $[0, T)$, which gives $\vec{u} = \vec{v}$ on $[0, T)$. ⋄

Proof of Theorem 8 : As in the proof of Theorem 7, it is enough to prove that $\vec{u} = \vec{v}$ on some small interval $[0, T]$ (with $T > 0$). Moreover, as in the proof of Lions and Masmoudi, we shall prove that they coincide on a small interval with the Kato solution associated to the common initial value $\vec{u}_0$. Thus, it is enough to prove that $\vec{u}$ and $\vec{v}$ satisfy the assumptions of Theorem 9.

More precisely, writing $\vec{u} = e^{t\Delta}\vec{u}_0 - \vec{w}$, we shall prove that

$$\sup_{x_0 \in \mathbb{R}^d} \Big(\int_{|x-x_0|<1} \int_0^{T^*} |\vec{\nabla} \otimes \vec{w}|^2\ dx\ dt \Big)^{1/2} < \infty, \tag{104}$$

or equivalently that

$$\sup_{\varphi \in \mathcal{B}} \int_0^{T^*} \|(\vec{\nabla} \otimes \vec{w}(t,.))\varphi(x)\|_2^2 dt < \infty \tag{105}$$

(where we use the same notation $\mathcal{B}$ as in the proof of Theorem 9). We write that

$$(\vec{\nabla} \otimes \vec{w})\ \varphi = \vec{X} + \vec{Y} \text{ with } \begin{cases} \vec{X} = \varphi \int_0^t e^{(t-s)\Delta} \vec{\nabla} \otimes (\mathbb{P}\vec{\nabla}.[\omega \vec{u} \otimes \vec{u}])\ ds \\ \vec{Y} = \varphi \int_0^t e^{(t-s)\Delta} \vec{\nabla} \otimes (\mathbb{P}\vec{\nabla}.[(1-\omega)\vec{u} \otimes \vec{u}])\ ds \end{cases}. \tag{106}$$

We have

$$\|\frac{1}{\Delta} \vec{\nabla} \otimes (\mathbb{P}\vec{\nabla}.[\omega \vec{u} \otimes \vec{u}])\|_{L^2(dx)} \leq C \|\psi \vec{u}\|^2_{L^4(dx)} \tag{107}$$

and thus

$$\int_0^{T^*} \|\vec{X}\|_2^2\ dt \leq C \sup_{x_0 \in \mathbb{R}^d} \int_{|x-x_0|<1} \int_0^{T^*} |\vec{u}(t,x)|^4\ dx\ dt. \tag{108}$$

On the other hand, the kernel of the operators $e^{(t-s)\Delta}\vec{\nabla} \otimes (\mathbb{P}\vec{\nabla}.)$ are controlled by

$$C \frac{1}{(t-s)^{(d+2)/2} + |x-y|^{d+2}},$$

thus we have for two positive constants C and γ

$$\begin{aligned} |\vec{Y}(t,x)| &\leq C\varphi(x) \int_0^t \int_{|y-x_\varphi| \geq \gamma} \frac{1}{|y-x_\varphi|^{d+2}} |\vec{u}(s,y)|^2\ dy\ ds \\ &\leq C'\varphi(x) \sup_{x_0 \in \mathbb{R}^d} \Big(\int_{|x-x_0|<1} \int_0^t |\vec{u}(s,x)|^4\ dx\ ds \Big)^{1/2} \end{aligned} \tag{109}$$

and thus

$$\int_0^{T^*} \|\vec{Y}\|_2^2\ dt \leq CT^* \sup_{x_0 \in \mathbb{R}^d} \int_{|x-x_0|<1} \int_0^{T^*} |\vec{u}(t,x)|^4\ dx\ dt. \tag{110}$$

Thus, we may conclude by applying Theorem 9. ◇

REFERENCES

[1] Cannone, M. *Ondelettes, paraproduits et Navier–Stokes* ; Diderot Editeur: Paris, 1995.

[2] Fefferman, C. The uncertainty principle. Bull. Amer. Math. Soc **1983**, *9*, 129–206.

[3] Furioli, G.; Lemarié–Rieusset, P.G.; Terraneo, E. Unicité dans $L^3(\mathbb{R}^3)$ et d'autres espaces limites pour Navier–Stokes. Revista Mat. Iberoamer **2000**, *16*, 605–667.

[4] Kato, T. Strong L^p solutions of the Navier–Stokes equations in $\mathbb{R}^m$ with applications to weak solutions. Math. Zeit **1984**, *187*, 471–480.

[5] Kozono, H.; Yamazaki, Y. Semilinear heat equations and the Navier–Stokes equations with distributions in new function spaces as initial data. Comm. P.D. E **1994**, *19*, 959–1014.

[6] Lemarié–Rieusset, P.G. Weak infinite-energy solutions for the Navier–Stokes equations in $\mathbb{R}^3$. **1998**, *Preprint.*

[7] Lemarié–Rieusset, P.G. *Recent developments in the Navier–Stokes problem*; Chapman & Hall/CRC: 2002.

[8] Leray, J. Essai sur le mouvement d'un fluide visqueux emplissant l'espace. Acta Math. **1934**, *63*, 193–248.

[9] Lions, P.L.; Masmoudi, N. Unicité des solutions faibles de Navier–Stokes dans $L^N(\Omega)$. C. R. Acad. Sci. Paris (Série I) **1998**, *327*, 491–496.

[10] May, R. *Thesis*; Université d'Evry: 2002.

[11] Maz'ya, V.G. On the theory of the n-dimensional Schrödinger operator. Izv. Akad. Nauk SSSR (Ser. Mat.) **1964**, *28*, 1145–1172.

[12] Maz'ya, V.G.; Shaposhnikova, T.O. *Theory of multipliers in spaces of differentiable functions*; Pitman: 1985.

[13] Maz'ya, V.G.; Verbitsky, I.E. Capacitary estimates for fractional integrals, with applications to partial differential equations and Sobolev multipliers. Ark. Mat **1995**, *33*, 81–115.

[14] Maz'ya, V.G.; Verbitsky, I.E. The Schrödinger operator on the energy space: boundedness and compactness criteria. Acta Math. **2002**, *188*, 263-302.

[15] Meyer, Y. *Wavelets, paraproducts and Navier–Stokes equations*, Current developments in mathematics 1996; International Press: PO Box 38-2872, Cambridge, MA 02238-2872, 1999.

[16] Monniaux, S. Uniqueness of mild solutions of the Navier–Stokes equation and maximal L^p-regularity. C. R. Acad. Sci. Paris (Série I) **1999**, *328*, 663–668.

[17] Serrin, J. On the interior regularity of weak solutions of the Navier–Stokes equations. Arch. Rat. Mech. Anal **1962**, *9*, 187–195.

[18] Sohr, H. A regularity class for the Navier–Stokes equations in Lorentz spaces. J. Evol. Eq **2001**, *1*, 441–467.

[19] Von Wahl, W. *The equations of Navier–Stokes and abstract parabolic equations*; Vieweg & Sohn: Wiesbaden, 1985.

Pierre-Gilles Lemarié-Rieusset
Pierre-Gilles.Lemarie@univ-evry.fr
Département de Mathématiques
Université d'Evry
Bd F. Mitterand
91025-Evry cedex, France

Ramzi May
Département de Mathématiques
Faculté des Sciences
7021 Jarzouna-Bizerte, Tunisie