Uniqueness for the Navier–Stokes problem : remarks on a theorem of Jean–Yves Chemin.

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Abstract :
We prove uniqueness for the tridimensional Navier–Stokes problem in the class $L^2H^1 \cap \mathcal{C}([0,T], B_\infty^{-1,\infty})$.

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We consider the following Navier–Stokes equations for a vector field $\vec{u}(t,x)$ defined on $(0,T) \times \mathbb{R}^3$:

$$(1) \qquad \begin{cases} \partial_t \vec{u} = \Delta \vec{u} - (\vec{u}.\vec{\nabla})\,\vec{u} - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

In [CHE 99], Chemin proved the following uniqueness theorem for the Navier–Stokes equations :

Theorem 1 :
Let $\vec{u}$ and $\vec{v}$ be two solutions of the Navier–Stokes equations (1) such that
i) $\vec{u}$ and $\vec{v}$ belong to $L^2([0,T], H^1(\mathbb{R}^3))$
ii) $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0,T], B_\infty^{-1,\infty})$ and $\vec{u}(0,.) = \vec{v}(0,.)$
iii) For some $p \in (1,\infty)$, $\vec{u}(0,.)$ belongs to the closure of the test functions in $B_\infty^{3/p-1,\infty}$.
Then $\vec{u} = \vec{v}$ on $[0,T]$.

We are going to get rid of the hypothesis iii) in Theorem 1. Our result is the following theorem :

Theorem 2 :
Let $\vec{u}$ and $\vec{v}$ be two solutions of the Navier–Stokes equations (1) such that
i) $\vec{u}$ and $\vec{v}$ belong to $L^2([0,T], H^1(\mathbb{R}^3))$
ii) $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0,T], B_\infty^{-1,\infty})$ and $\vec{u}(0,.) = \vec{v}(0,.)$.
Then $\vec{u} = \vec{v}$ on $[0,T]$.

We shall even prove a more general result. We shall see (Lemma 3) that we have, for $f \in B_\infty^{-1,\infty} \cap H^1$, the estimate

$$\|f\|_4 \le C\sqrt{\|f\|_{B_\infty^{-1,\infty}} \|f\|_{H^1}}. \tag{2}$$

Thus, when $\vec{u}$ belongs to $L^2([0,T], H^1(\mathbb{R}^3)) \cap \mathcal{C}([0,T], B_\infty^{-1,\infty})$, then $\vec{u} \in L^4([0,T], L^4(\mathbb{R}^3))$. Moreover, when $\vec{u} \in L^2([0,T], H^1)$ and $\vec{u}$ is divergence free ($\vec{\nabla}.\vec{u} = 0$), we have

$$(\vec{u}.\vec{\nabla})\vec{u} = \vec{\nabla}.(\vec{u} \otimes \vec{u}) - (\vec{\nabla}.\vec{u})\vec{u} = \vec{\nabla}.(\vec{u} \otimes \vec{u}). \tag{3}$$

Thus, Theorem 2 will be a straightforward corollary of our main result :

Theorem 3 :
Let $\vec{u}$ and $\vec{v}$ be two solutions of the Navier–Stokes equations

$$\left\{ \begin{array}{ll} \exists p \in \mathcal{D}'((0,T) \times \mathbb{R}^3) & \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.\,(\vec{u} \otimes \vec{u}) - \vec{\nabla} p \\ & \vec{\nabla}.\vec{u} = 0 \end{array} \right. \tag{4}$$

such that
i) $\vec{u}$ and $\vec{v}$ belong to $L^p([0,T], L^q(\mathbb{R}^3))$ for some $p \in (2,\infty)$ and $q \in (3,\infty)$.
ii) $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0,T], B_\infty^{-1,\infty})$ and $\vec{u}(0,.) = \vec{v}(0,.)$.
Then $\vec{u} = \vec{v}$ on $[0,T]$.

Remark : If $p \in (2,\infty)$ and $q \in (3,\infty)$ satisfy the Serrin condition $2/p + 3/q \le 1$, then we can prove directly uniqueness in the class L^pL^q (so that hypothesis ii) is not useful) and that, if $\vec{u}$ is a solution in $L^p([0,T], L^q)$, then it is easy to check that $\vec{u}$ belongs to $\mathcal{C}([0,T], B_\infty^{-1,\infty})$ (so that hypothesis ii) is redundant). Thus, theorem 3 is actually new only in the range $1 < 2/p + 3/q < 2$.

1. The bilinerar operator B_a.

We shall systematically get rid of the pressure p in equations (4) by using the Leray projection operator, which is the orthogonal projection onto solenoidal vector fields. We shall use the following lemma of Furioli, Lemarié–Rieusset and Terraneo [FUR 00] [LEM 02] :

Lemma 1 :
Let E_2 be the closure of the test functions in the Morrey space L^2_{uloc} :

$$f \in E_2 \Leftrightarrow \sup_{x_0 \in \mathbb{R}^3} \int_{|x-x_0|<1} |f(x)|^2 \, dx < \infty \text{ and } \lim_{x_0 \to \infty} \int_{|x-x_0|<1} |f(x)|^2 \, dx = 0.$$

If $\vec{u} \in L^2([a,b], E_2)$, then the following assertion are equivalent :

(A) $\vec u$ is solution of the Navier–Stokes equations

$$(5)\qquad \exists p\in\mathcal{D}'((a,b)\times\mathbb{R}^3)\left\{\begin{aligned}\partial_t\vec u = &\ \Delta\vec u - \vec\nabla.\,(\vec u\otimes\vec u)-\vec\nabla p\\ &\vec\nabla.\vec u=0\end{aligned}\right.$$

(B) $\vec u$ is solution of the Navier–Stokes equations :

$$(6)\qquad \left\{\begin{aligned}\partial_t\vec u = &\ \Delta\vec u - \mathbb{P}\vec\nabla.\,(\vec u\otimes\vec u)\\ &\vec\nabla.\vec u=0\end{aligned}\right.$$

where $\mathbb{P}$ is the Leray projection operator

$$(7)\qquad \mathbb{P}\vec f=\vec f-\vec\nabla\frac{1}{\Delta}(\vec\nabla.\vec f)$$

(C) $\vec u$ is solution of the integral Navier–Stokes equations :

$$(8)\qquad \exists\vec u_a\in\mathcal{S}'(\mathbb{R}^3)\left\{\begin{aligned}&\vec u=e^{(t-a)\Delta}\vec u_a-\textstyle\int_a^t e^{(t-s)\Delta}\mathbb{P}\vec\nabla.\,(\vec u\otimes\vec u)\,ds\\ &\vec\nabla.\vec u_a=0\end{aligned}\right.$$

We shall apply this Lemma to solutions in L^pL^q, since we assume that $p>2$ and that $3<q<\infty$ (so that $L^q\subset E_2$). We shall rewrite equations (8) as

$$(9)\qquad \vec u=e^{(t-a)\Delta}\vec u_a-B_a(\vec u,\vec u)$$

where the bilinear operator B_a is defined in the following way :

Definition 1 :
For $\vec u$ and $\vec v\in L^2([a,b],E_2)$, we define $B_a(\vec u,\vec v)$ as the distribution on $(a,b)\times\mathbb{R}^3$ computed as

$$(10)\qquad B_a(\vec u,\vec v)=\int_a^t e^{(t-s)\Delta}\mathbb{P}\vec\nabla.(\vec u\otimes\vec v)\,ds$$

In order to analyze B_a, we shall use well-known size estimates on the kernel of the operator $e^{(t-s)\Delta}\mathbb{P}\vec\nabla.$ or use the maximal regularity of the heat kernel :

Lemma 2 :
B_a may be written in the following ways

(A) For $\alpha \in \mathbb{R}$ with $\alpha > -1$

$$B_a(\vec{u}, \vec{f}) = \int_a^t e^{(t-s)\Delta} (-\Delta)^{\alpha/2} \mathbb{P}\vec{\nabla}.(-\Delta)^{-\alpha/2} (\vec{u} \otimes \vec{v}) \, ds \tag{11}$$

where $(-\Delta)^{\alpha/2} e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.$ is a matrix of convolution operators with integrable kernels

$$K_{i,j,\alpha,t-s}(x) = \frac{1}{(t-s)^{\frac{3}{2}+\frac{\alpha+1}{2}}} K_{i,j}(\frac{x}{\sqrt{t-s}}) \tag{12}$$

with $K_{i,j,\alpha} \in L^1 \cap L^\infty$
(B) Defining W_a as

$$W_a f = \int_a^t e^{(t-s)\Delta} \Delta f \, ds \tag{13}$$

and $\mathcal{R}$ as

$$\mathcal{R}(\vec{u}, \vec{v}) = -\frac{1}{\sqrt{-\Delta}} \mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{v}) \tag{14}$$

(so that $\mathcal{R}$ may be defined as a sum of products of Riesz transforms), we have

$$B_a(\vec{u}, \vec{v}) = W_a(\frac{1}{\sqrt{-\Delta}} \mathcal{R}(\vec{u}, \vec{v})) \tag{15}$$

Sometimes, we shall use the paraproduct formalism in order to deal with the product $\vec{u} \otimes \vec{v}$. More precisely, we use the Littlewood–Paley decomposition

$$\vec{u} = S_0 \vec{u} + \sum_{j=0}^{\infty} \Delta_j \vec{u} \tag{16}$$

(where $\Delta_j \vec{u} = S_{j+1}\vec{u} - S_j \vec{u}$ has its spectrum contained in a corona $2^{j-1} \leq |\xi| \leq 2^{j+1}$ [see [LEM 02] for instance]) and we use the paraproduct operators of Bony and write

$$\vec{u} \otimes \vec{v} = \pi(\vec{u}, \vec{v}) + \tilde{\pi}(\vec{u}, \vec{v}) + \rho(\vec{u}, \vec{v}) + \tilde{\rho}(\vec{u}, \vec{v}) \tag{17}$$

with

$$\begin{cases} \pi(\vec{u}, \vec{v}) = & \sum_{j=2}^{\infty} \Delta_j \vec{u} \otimes S_{j-2} \vec{v} \\ \tilde{\pi}(\vec{u}, \vec{v}) = & \sum_{j=2}^{\infty} S_{j-2} \vec{u} \otimes \Delta_j \vec{v} \\ \rho(\vec{u}, \vec{v}) = & \sum_{j=2}^{\infty} \Delta_j \vec{u} \otimes \sum_{k=j-2}^{j+2} \Delta_k \vec{v} \\ \tilde{\rho}(\vec{u}, \vec{v}) = & S_0\vec{u} \otimes S_0 \vec{v} + S_0 \vec{u} \otimes \Delta_0 \vec{v} + S_0 \vec{u} \otimes \Delta_1 \vec{v} + \Delta_0 \vec{u} \otimes S_0 \vec{v} + \Delta_1 \vec{u} \otimes S_0 \vec{v} \\ & + \Delta_0 \vec{u} \otimes \Delta_0 \vec{v} + \Delta_0 \vec{u} \otimes \Delta_1 \vec{v} + \Delta_0 \vec{u} \otimes \Delta_2 \vec{v} \\ & + \Delta_1 \vec{u} \otimes \Delta_0 \vec{v} + \Delta_1 \vec{u} \otimes \Delta_1 \vec{v} + \Delta_1 \vec{u} \otimes \Delta_2 \vec{v} + \Delta_1 \vec{u} \otimes \Delta_3 \vec{v} \end{cases} \tag{18}$$

Then, we decompose B_a into

$$B_a(\vec{u},\vec{v}) = B_{a,\pi}(\vec{u},\vec{v}) + B_{a,\tilde{\pi}}(\vec{u},\vec{v}) + B_{a,\rho}(\vec{u},\vec{v}) + B_{a,\tilde{\rho}}(\vec{u},\vec{v}) \tag{19}$$

where, for $\tau \in \{\pi, \tilde{\pi}, \rho, \tilde{\rho}\}$, we have

$$B_{a,\tau}(\vec{u},\vec{v}) = \int_a^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\tau(\vec{u},\vec{v})\ ds. \tag{20}$$

2. Refined Sobolev inequalities.

We recall the refined Sobolev inequalities of [GER 96] involving the norms of Sobolev spaces H_q^α (the Sobolev space H_q^α is the space $H_q^\alpha = (Id-\Delta)^{-\alpha/2}L^q$ for $1 < q < \infty$) and Besov spaces $B_\infty^{\sigma,\infty}$. We give here a very simple proof of those inequalities (following Hedberg's proof of classical Sobolev inequalities [HED 72] [ADA 96]) :

Lemma 3 :
Let $\alpha \in (0,2)$ and $1 < q < \infty$. Assume that $f \in H_q^\alpha \cap B_\infty^{-1,\infty}$. Then $f \in L^{q(1+\alpha)}$ and we have

$$\|f\|_{q(1+\alpha)} \le C_{q,\alpha}\|(Id-\Delta)^{\alpha/2}f\|_q^{\frac{1}{1+\alpha}}\|f\|_{B_\infty^{-1,\infty}}^{\frac{\alpha}{1+\alpha}} \tag{21}$$

Proof : We cut f in low and high frequency components : $f = f_0 + f_1$, where $f_0 = S_0 f$ is the block of low frequencies in the Littlewood-Paley decomposition of f. We have $f_0 \in L^q \cap L^\infty$ and

$$\|f_0\|_{q(1+\alpha)} \le \|f_0\|_q^{\frac{1}{1+\alpha}}\|f_0\|_\infty^{\frac{\alpha}{1+\alpha}} \le C_{q,\alpha}\|(Id-\Delta)^{\alpha/2}f\|_q^{\frac{1}{1+\alpha}}\|f\|_{B_\infty^{-1,\infty}}^{\frac{\alpha}{1+\alpha}}$$

Since f_1 has no low frequencies, we have $f_1 \in \dot{B}_\infty^{-1,\infty}$ and we may write

$$f_1 = \int_0^\infty e^{t\Delta}\Delta f_1\ dt.$$

We have

$$\|e^{t\Delta}\Delta f_1\|_\infty \le C\|f_1\|_{\dot{B}_\infty^{-1,\infty}}t^{-3/2}.$$

We now use the fact that, for all $g \in L^1 + L^\infty$ and for $0 < \alpha < 2$ we have

$$\sup_{t>0}|e^{t\Delta}(-t\Delta)^{1-\alpha/2}g(x)| \le C_\alpha M_g(x)$$

where M_g is the Hardy–Littlewood maximal function of g. We use this for $g = (-\Delta)^{\alpha/2} f_1$, so that, for every $A > 0$, we have

$$|f_1(x)| \leq C_\alpha (\int_0^A t^{\alpha/2-1} \, dt \; M_g(x) + \int_A^\infty t^{-3/2} \, dt \|f_1\|_{\dot B^{-1,\infty}_\infty})$$

hence, choosing $A = (\|f_1\|_{\dot B^{-1,\infty}_\infty}/M_g(x))^{2/(1+\alpha)}$, we get

$$|f_1(x)| \leq C_\alpha (A^{\alpha/2} M_g(x) + A^{-1/2} \|f_1\|_{\dot B^{-1,\infty}_\infty}) = 2C_\alpha M_g(x)^{1/(1+\alpha)} \|f_1\|_{\dot B^{-1,\infty}_\infty}^{\alpha/(1+\alpha)}.$$

Thus, Lemma 3 is proved.

3. Mild solutions in L^q.

In this section, we recall some classical results on Kato's mild solutions in L^q for $3 < q < \infty$ [KAT 84]. We start from the following lemma :

Lemma 4 :
For $a < b$, $q \in (3, \infty)$ and $\alpha \in (1, 2 - 3/q)$, let

$$E_{a,b,q,\alpha} = \{\vec u \in \mathcal{C}([a,b], L^q) / \sup_{a<t<b} (t-a)^{\frac{3}{2q}} \|\vec u(t,.)\|_\infty < \infty \text{ and } \sup_{a<t<b} (t-a)^{\frac{\alpha}{2}} \|\vec u(t,.)\|_{H^\alpha_q} < \infty\}$$

normed with

$$\|\vec u\|_{E_{a,b,q,\alpha}} = \sup_{a<t<b} \|\vec u(t,.)\|_q + \sup_{a<t<b} (t-a)^{\frac{3}{2q}} \|\vec u(t,.)\|_\infty + \sup_{a<t<b} (t-a)^{\frac{3}{2q}} \|(-\Delta)^{\alpha/2} \vec u(t,.)\|_q. \tag{22}$$

Then B_a is bounded on $E_{a,b,q,\alpha}$:

$$\|B_a(\vec u, \vec v)\|_{E_{a,b,q,\alpha}} \leq C_{q,\alpha} (b-a)^{1/2-3/2q} \|\vec u\|_{E_{a,b,q,\alpha}} \|\vec v\|_{E_{a,b,q,\alpha}} \tag{23}$$

Proof : In order to estimate the L^q and L^∞ norms, we just write

$$\|e^{(t-s)\Delta} \mathbb{P} \vec\nabla . \vec u \otimes \vec v\|_q \leq C \frac{1}{\sqrt{t-s}} (t-s)^{-3/2q} \|\vec u\|_q \|\vec v\|_q \tag{24}$$

and

$$\|e^{(t-s)\Delta} \mathbb{P} \vec\nabla . \vec u \otimes \vec v\|_\infty \leq C \frac{1}{\sqrt{t-s}} (s-a)^{-3/q} (s-a)^{3/2q} \|\vec u\|_\infty (s-a)^{3/2q} \|\vec v\|_\infty. \tag{25}$$

We now consider the homogeneous Sobolev norm

$$\|f\|_{\dot H^\alpha_q} = \|(-\Delta)^{\alpha/2} f\|_q.$$

We shall use the following well-known inequality (for positive α)

$$\|uv\|_{\dot H^\alpha_q} \le C_{\alpha,q}(\|u\|_{\dot H^\alpha_q}\|v\|_\infty + \|v\|_{\dot H^\alpha_q}\|u\|_\infty). \tag{26}$$

This gives

$$\|e^{(t-s)\Delta}\mathbb{P}\vec\nabla.\vec u\otimes\vec v\|_{\dot H^\alpha_q} \le C_{\alpha,q}\frac{(s-a)^{\frac{\alpha}{2}}\|\vec u\|_{\dot H^\alpha_q}(s-a)^{\frac{3}{2q}}\|\vec v\|_\infty + (s-a)^{\frac{3}{2q}}\|\vec u\|_\infty(s-a)^{\frac{\alpha}{2}}\|\vec v\|_{\dot H^\alpha_q}}{\sqrt{t-s}\ (s-a)^{\frac{3}{2q}+\frac{\alpha}{2}}} \tag{27}$$

Thus, Lemma 4 is proved.

A direct consequence of Lemma 4 and of the fixed-point theorem is the existence of mild solutions in L^q :

Lemma 5 :
Let $q\in(3,\infty)$ and $\alpha\in(1,2-3/q)$. There exists a positive constant $C(\alpha,q)$ such that, for all $a\in\mathbb{R}$, all $\vec u(a)\in L^q$ with $\vec\nabla.\vec u(a)=0$, there exists a solution $\vec u$ of the Navier-Stokes equations (5) on $[a,b]$ (with $b=a+C(\alpha,q)\|\vec u(a)\|_q^{\frac{1}{1/2-3/2q}}$) such that $\vec u\in\mathcal{C}([a,b],L^q)$, $\sup_{a<t<b}(t-a)^{3/2q}\|\vec u(t,.)\|_\infty<\infty$ and $\sup_{a<t<b}(t-a)^{\alpha/2}\|\vec u(t,.)\|_{H^\alpha_q}<\infty$

We finish this section by describing maximal solutions which are continuous in L^q norm :

Lemma 6 :
Let $q\in(3,\infty)$ and $\alpha\in(1,2-3/q)$. Let $a<b^*$ and let $\vec u$ be a solution of the Navier-Stokes equations (5) on (a,b^*) such that $\vec u\in\mathcal{C}([a,b^*),L^q)$. Then :
(A) If $\vec v$ is a solution of the Navier-Stokes equations (5) on (a,b^*) such that $\vec v\in\mathcal{C}([a,b^*),L^q)$ and $\vec v(a,.)=\vec u(a,.)$, then $\vec v=\vec u$ on $[a,b^*)$.
(B) For all $b\in(a,b^*)$,we have

$$\sup_{a<t<b}(t-a)^{3/2q}\|\vec u(t,.)\|_\infty<\infty \text{ and } \sup_{a<t<b}(t-a)^{\alpha/2}\|\vec u(t,.)\|_{H^\alpha_q}<\infty.$$

(C) b^* is maximal (i.e. $\vec u$ can not be extended as a solution of (5) on a larger interval $[a,b')$ with $b'>b^*$ and $\vec u\in\mathcal{C}([a,b'),L^q)$) if and only if $\vec u$ can not be extended at b^* as a function in $\mathcal{C}([a,b^*],B^{-1,\infty}_\infty)$.

Proof :
(A) is easy : we write $\vec w=\vec u-\vec v$; then we have

$$\vec w=-B_a(\vec w,\vec u)-B_a(\vec v,\vec w) \tag{28}$$

Then, we use (24) and get, for $a<b<b_1<b^*$,

$$\sup_{a<t<b}\|\vec w(t,.)\|_q\le C(b-a)^{\frac12-\frac{3}{2q}}\sup_{a<t<b}\|\vec w(t,.)\|_q(\sup_{a<t<b_1}\|\vec u(t,.)\|_q+\sup_{a<t<b_1}\|\vec v(t,.)\|_q). \tag{29}$$

Thus, for b close enough to a, we get $\vec{w} = 0$, so that we have local uniqueness. This propoagates to global uniqueness by continuity.

(B) is a straightforward consequence of Lemma 5 and of uniqueness.

We now consider (C). This result is by now classical, due to the works of Kozono [KOZ 97] [KOZ 04], and their generalization by May [MAY 03]. Because of uniqueness and of Lemma 5, it is enough to prove that, if the solution $\vec{u}$ of the Navier–Stokes equations satisfies $\vec{u} \in \mathcal{C}([a, b^*), L^q) \cap \mathcal{C}([a, b^*], B_\infty^{-1,\infty})$, then the norm L^q of $\vec{u}$ remains bounded. We consider $\alpha \in (1, 2 - 3/q)$ and we shall prove more precisely that the norm of $\vec{u}$ in $B_q^{\alpha,\infty}$ can not blow up.

Assume that $\vec{u} \in \mathcal{C}([a, b^*), L^q) \cap \mathcal{C}([a, b^*], B_\infty^{-1,\infty})$. Let $\tilde{B}_\infty^{1,\infty}$ be the closure of the space $\mathcal{D}$ of test functions in $B_\infty^{1,\infty}$. Since $L^q \subset \tilde{B}_\infty^{-1,\infty}$ (recall that $q \in (3, \infty)$), we have more precisely that $\vec{u} \in \mathcal{C}([a, b^*], \tilde{B}_\infty^{-1,\infty})$. Thus, following Sohr and Von Wahl's idea [WAH 85], we see that, for every $\epsilon > 0$, we may decompose $\vec{u}$ into

$$\vec{u} = \vec{U}_\epsilon + \vec{V}_\epsilon \tag{30}$$

with

$$\sup_{a \le t \le b^*} \|\vec{U}_\epsilon\|_{B_\infty^{-1,\infty}} < \epsilon \text{ and } \sup_{a \le t \le b^*} \|\vec{V}_\epsilon\|_\infty < \infty.$$

We shall write

$$M_\epsilon = \sup_{a \le t \le b^*} \|\vec{V}_\epsilon\|_\infty < \infty. \tag{31}$$

From (B), we know that, if $a < a^* < b < b^*$, then $\vec{u} \in L^\infty([a^*, b], H_q^\alpha)$. We replace the norm in H_q^α by the weaker norm $B_q^{\alpha,\infty}$. We now estimate $\sup_{a^* < t < b} \|\vec{u}\|_{B_q^{\alpha,\infty}}$ by writing, for $a^* < t < b$, $\vec{u}(t, .) = e^{(t-a^*)\Delta}\vec{u}(a^*, .) - B_{a^*}(\vec{u}, \vec{u})$. In order to estimate the norm of $B_{a^*}(\vec{u}, \vec{u})$, we use the following estimates on homogeneous Besov norms [LEM 02]

$$\sup_{a^* < t < b} \|W_{a^*} f\|_{\dot{B}_q^{\alpha,\infty}} \le C_{\alpha,q} \sup_{a^* < t < b} \|f\|_{\dot{B}_q^{\alpha,\infty}} \tag{32}$$

and

$$\sup_{a^* < t < b} \|W_{a^*} f\|_{\dot{B}_q^{\alpha,\infty}} \le C_{\alpha,q} \sqrt{b - a^*} \sup_{a^* < t < b} \|f\|_{B_q^{\alpha+1,\infty}} \tag{33}$$

where W_{a^*} is defined by (13) (replacing a with a^*). We write

$$\begin{aligned} \vec{u}(t, .) = \quad & e^{(t-a^*)\Delta}\vec{u}(a^*, .) - B_{a^*,\pi}(\vec{U}_\epsilon, \vec{u}) - B_{a^*,\pi}(\vec{V}_\epsilon, \vec{u}) - B_{a^*,\tilde{\pi}}(\vec{u}, \vec{U}_\epsilon) - B_{a^*,\tilde{\pi}}(\vec{u}, \vec{V}_\epsilon) \\ & -B_{a^*,\rho}(\vec{U}_\epsilon, \vec{u}) - B_{a^*,\rho}(\vec{V}_\epsilon, \vec{u}) - B_{a^*,\tilde{\rho}}(\vec{U}_\epsilon, \vec{u}) - B_{a^*,\tilde{\rho}}(\vec{V}_\epsilon, \vec{u}) \end{aligned}$$

We then use (32) to get that

$$A_1 = \sup_{a^* < t < b} \|(Id - S_0)\big(B_{a^*,\pi}(\vec{U}_\epsilon, \vec{u}) + B_{a^*,\tilde{\pi}}(\vec{u}, \vec{U}_\epsilon) + B_{a^*,\rho}(\vec{U}_\epsilon, \vec{u}) + B_{a^*,\tilde{\rho}}(\vec{U}_\epsilon, \vec{u})\big)\|_{B_q^{\alpha,\infty}}$$

is controlled by

$$A_1 \le C_{q,\alpha} \sup_{a^*<t<b} \|\vec u\|_{B^{\alpha,\infty}_q} \sup_{a^*<t<b} \|\vec U_\epsilon\|_{B^{-1,\infty}_\infty} \le \epsilon\, C_{q,\alpha} \sup_{a^*<t<b} \|\vec u\|_{B^{\alpha,\infty}_q}. \tag{34}$$

Similarly, we use (33) to get that

$$A_2 = \sup_{a^*<t<b} \|(Id - S_0)\big(B_{a^*,\pi}(\vec V_\epsilon, \vec u) + B_{a^*,\tilde\pi}(\vec u, \vec V_\epsilon) + B_{a^*,\rho}(\vec V_\epsilon, \vec u) + B_{a^*,\tilde\rho}(\vec V_\epsilon, \vec u)\big)\|_{B^{\alpha,\infty}_q}$$

is controlled by

$$A_2 \le C_{q,\alpha}\sqrt{b-a^*} \sup_{a^*<t<b} \|\vec u\|_{B^{\alpha,\infty}_q} \sup_{a^*<t<b} \|\vec V_\epsilon\|_\infty \le M_\epsilon\, C_{q,\alpha}\sqrt{b-a^*} \sup_{a^*<t<b} \|\vec u\|_{B^{\alpha,\infty}_q}. \tag{35}$$

For the low frequencies, we just write that

$$\|e^{(t-s)\Delta}\mathbb{P}\vec\nabla.\vec f \otimes \vec g\|_q \le C(t-s)^{-1/2}\|\vec f\otimes \vec g\|_q$$

and get that

$$A_3 = \sup_{a^*<t<b} \|S_0\big(B_{a^*,\pi}(\vec U_\epsilon, \vec u) + B_{a^*,\tilde\pi}(\vec u, \vec U_\epsilon) + B_{a^*,\rho}(\vec U_\epsilon, \vec u) + B_{a^*,\tilde\rho}(\vec U_\epsilon, \vec u)\big)\|_{B^{\alpha,\infty}_q}$$

is controlled by

$$A_3 \le C_{q,\alpha}\sqrt{b-a^*} \sup_{a^*<t<b} \|\vec u\|_{B^{\alpha,\infty}_q} \sup_{a^*<t<b} \|\vec U_\epsilon\|_{B^{-1,\infty}_\infty} \le \epsilon\, C_{q,\alpha}\sqrt{b-a^*} \sup_{a^*<t<b} \|\vec u\|_{B^{\alpha,\infty}_q}. \tag{36}$$

and that

$$A_4 = \sup_{a^*<t<b} \|S_0\big(B_{a^*,\pi}(\vec V_\epsilon, \vec u) + B_{a^*,\tilde\pi}(\vec u, \vec V_\epsilon) + B_{a^*,\rho}(\vec V_\epsilon, \vec u) + B_{a^*,\tilde\rho}(\vec V_\epsilon, \vec u)\big)\|_{B^{\alpha,\infty}_q}$$

is controlled by

$$A_4 \le C_{q,\alpha}\sqrt{b-a^*} \sup_{a^*<t<b} \|\vec u\|_{B^{\alpha,\infty}_q} \sup_{a^*<t<b} \|\vec V_\epsilon\|_\infty \le M_\epsilon\, C_{q,\alpha}\sqrt{b-a^*} \sup_{a^*<t<b} \|\vec u\|_{B^{\alpha,\infty}_q}. \tag{37}$$

Thus, we find that, for a constant $D_{\alpha,q}$ which depend neither on a^* nor on b nor on ϵ, we have :

$$\sup_{a^*<t<b} \|\vec u\|_{B^{\alpha,\infty}_q} \le \|\vec u(a^*,.)\|_{B^{\alpha,\infty}_q} + D_{\alpha,q}(\epsilon + (M_\epsilon+\epsilon)\sqrt{b-a^*}) \sup_{a^*<t<b} \|\vec u\|_{B^{\alpha,\infty}_q} \tag{38}$$

Thus, if we choose ϵ small enough to grant that $D_{\alpha,q}\epsilon < 1/4$ and then we choose a^* close enough to b^* to grant that $D_{\alpha,q}(M_\epsilon+\epsilon)\sqrt{b^*-a^*} < 1/4$, we find

$$\sup_{a^*<t<b^*} \|\vec u\|_{B^{\alpha,\infty}_q} \le 2\|\vec u(a^*,.)\|_{B^{\alpha,\infty}_q}. \tag{39}$$

Thus, Lemma 6 is proved.

4. A weak-strong uniqueness lemma.

The main tool in the proof of Theorem 3 will be the following result of weak-strong uniqueness :

Lemma 7 :
Let $\vec{u}$ be a solution of the Navier–Stokes equations (2) such that

$$\vec{u} \in L^p([0,T], L^q(\mathbb{R}^3)) \cap \mathcal{C}([0,T], B_\infty^{-1,\infty})$$

with $2 < p < \infty$ and $3 < q < \infty$.
If $[a,b] \subset [0,T]$ and $\vec{v}$ is a solution of the Navier–Stokes equations (2) on $(a,b) \times \mathbb{R}^3$ such that $\vec{v} \in \mathcal{C}([a,b], L^q)$ and $\vec{v}(a) = \vec{u}(a)$, then $\vec{u} = \vec{v}$ on $[a,b]$.

Proof :
Once again, we write $\vec{w} = \vec{u} - \vec{v}$; then we have

$$\vec{w} = -B_a(\vec{w}, \vec{v}) - B_a(\vec{v}, \vec{w}) - B_a(\vec{w}, \vec{w}) \tag{40}$$

We begin by estimating the norm of $B_a(\vec{w}, \vec{v}) + B_a(\vec{v}, \vec{w})$ in $B_\infty^{-1,\infty}$. Following (17), we write

$$\vec{w} \otimes \vec{v} = U + V$$

with

(41). $$U = \pi(\vec{w}, \vec{v}) + \tilde{\pi}(\vec{w}, \vec{v}) + \tilde{\rho}(\vec{w}, \vec{v}) \text{ and } V = \rho(\vec{w}, \vec{v})$$

Let $\gamma = 1 - 3/q$. We have $L^q \subset B_\infty^{-1+\gamma,\infty}$. We find easily that

$$\|U\|_{B_\infty^{-2+\gamma,\infty}} \leq C_q \|\vec{v}\|_{B_\infty^{-1+\gamma,\infty}} \|\vec{w}\|_{B_\infty^{-1,\infty}} \leq C_q' \|\vec{v}\|_q \|\vec{w}\|_{B_\infty^{-1,\infty}} \tag{42}$$

and

$$\|e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}.U\|_{B_\infty^{-1,\infty}} \leq C_q \big(\frac{1}{(t-s)^{1/2}} + \frac{1}{(t-s)^{1-\gamma/2}} \big) \|U\|_{B_\infty^{-2+\gamma,\infty}}. \tag{43}$$

Thus, we get, for $a < c < b$, :

$$\sup_{a<t<c} \| \int_a^t e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}.U \, ds \|_{B_\infty^{-1,\infty}} \leq C_q \big((c-a)^{\frac{1}{2}} + (c-a)^{\frac{\gamma}{2}} \big) \sup_{a<t<b} \|\vec{v}\|_q \sup_{a<t<b} \|\vec{w}\|_{B_\infty^{-1,\infty}}. \tag{44}$$

Those computations do not apply to V. We now use the regularity of V and take $\alpha \in (1, 1+\gamma)$ such that $\alpha < 2 - \gamma$. We write

$$\|V\|_{B_\infty^{\alpha+\gamma-2,\infty}} \leq C_{\alpha,q} \|V\|_{B_q^{\alpha-1,\infty}} \leq C_{\alpha,q}' \|\vec{v}\|_{H_q^\alpha} \|\vec{w}\|_{B_\infty^{-1,\infty}} \tag{45}$$

and

$$e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.V = e^{(t-s)\Delta}\mathbb{P}(-\Delta)^{\frac{2-\alpha-\gamma}{2}}\frac{\vec{\nabla}}{\sqrt{-\Delta}}.(-\Delta)^{\frac{\alpha+\gamma-1}{2}}V, \tag{46}$$

hence

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.V\|_{B_\infty^{-1,\infty}} \leq C_{\alpha,q}\frac{1}{(t-s)^{\frac{2-\alpha-\gamma}{2}}}\|V\|_{B_\infty^{\alpha+\gamma-2,\infty}}. \tag{47}$$

Thus, we get, for $a < c < b$, :

$$\sup_{a<t<c}\|\int_a^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.V\, ds\|_{B_\infty^{-1,\infty}} \leq C_{\alpha,q}(c-a)^{\frac{\gamma}{2}}\sup_{a<t<b}(t-a)^{\alpha/2}\|\vec{v}\|_{H_q^\alpha}\sup_{a<t<b}\|\vec{w}\|_{B_\infty^{-1,\infty}}. \tag{48}$$

This gives

$$\lim_{c\to a^+}\|B(\vec{w},\vec{v})\|_{B_\infty^{-1,\infty}} = 0. \tag{49}$$

Similar computations give

$$\lim_{c\to a^+}\|B(\vec{v},\vec{w})\|_{B_\infty^{-1,\infty}} = 0. \tag{50}$$

Moreover, we have

$$\lim_{c\to a^+}\|\vec{w}\|_{B_\infty^{-1,\infty}} = \|\vec{u}(a)-\vec{v}(a)\|_{B_\infty^{-1,\infty}} = 0, \tag{51}$$

hence we get the following estimate :

$$\lim_{c\to a^+}\|B(\vec{w},\vec{w})\|_{B_\infty^{-1,\infty}} = 0. \tag{52}$$

We now estimate the norm of $\vec{w}$ in L^pL^q. We have obvioulsly

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{w}\otimes\vec{v}\|_q \leq C_q\frac{1}{(t-s)^{\frac{2-\gamma}{2}}}\|\vec{w}\|_q\|\vec{v}\|_q. \tag{53}$$

We have a similar estimate for $e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{w}\otimes\vec{v}$. This gives

$$\|B(\vec{w},\vec{v})\|_{L^p([a,c],L^q)} + \|B(\vec{v},\vec{w})\|_{L^p([a,c],L^q)} \leq C_q(c-a)^{\gamma/2}\|\vec{w}\|_{L^p([a,c],L^q)}\sup_{a<t<b}\|\vec{v}\|_q. \tag{54}$$

The difficult term is $B(\vec{w},\vec{w})$. We have obviously

$$\|B(\vec{w},\vec{w})\|_{L^{p/2}([a,c],L^{q/2})} \leq C(c-a)^{1/2}\|\vec{w}\|^2_{L^p([a,c],L^q)}. \tag{55}$$

Moreover, using the maximal $L^{p/2}L^{q/2}$ regularity of the heat kernel, we have

$$\|\sqrt{-\Delta}B(\vec{w},\vec{w})\|_{L^{p/2}([a,c],L^{q/2})} \leq C_{p,q}\|\vec{w}\|^2_{L^p([a,c],L^q)}. \tag{56}$$

Using the refined Sobolev inequality in $H^1_{q/2} \cap B^{-1,\infty}_\infty$ given by Lemma 3, we get

$$\|B(\vec{w},\vec{w})\|_{L^p([a,c],L^q)} \leq C_q\sqrt{\|B(\vec{w},\vec{w})\|_{L^{p/2}([a,c],H^1_{q/2})}\|B(\vec{w},\vec{w})\|_{L^\infty([a,c],B^{-1,\infty}_\infty)}} \tag{57}$$

Thus, we have

$$\|\vec{w}\|_{L^p([a,c],L^q)} \leq \eta(c)\|\vec{w}\|_{L^p([a,c],L^q)} \tag{58}$$

with

$$\eta(c) \leq C_{p,q,\alpha}\big((c-a)^{\gamma/2}\|\sup_{a<t<b}\|\vec{v}\|_q + (c-a)^{1/4} + \sqrt{\sup_{a<t<c}\|B(\vec{w},\vec{w})\|_{B^{-1,\infty}_\infty}}\big) \tag{59}$$

From (52), we get that, for c close enough to a, $\eta(c) < 1$, hence $\vec{w} = 0$ on $[a,c]$. Thus, we have local uniqueness, and by continuity of $\vec{u}$ and $\vec{v}$ in the $B^{-1,\infty}_\infty$ norm, this uniqueness holds on the whole $[a,b]$.

5. Regularity of the weak solutions.

A direct consequence of Lemma 7 is that the class of solutions we deal with is a class of smooth solutions :

Lemma 8 :
Let $\vec{u}$ be a weak solution of the Navier–Stokes equations

$$\begin{cases} \exists p \in \mathcal{D}'((0,T)\times \mathbb{R}^3) \quad \partial_t\vec{u} = \Delta\vec{u} - \vec{\nabla}.\,(\vec{u}\otimes\vec{u}) - \vec{\nabla}p \\ \vec{\nabla}.\vec{u} = 0 \end{cases} \tag{60}$$

such that
i) $\vec{u}$ belongs to $L^p([0,T],L^q(\mathbb{R}^3))$ for some $p \in (2,\infty)$ and $q \in (3,\infty)$.
ii) $\vec{u}$ belongs to $\mathcal{C}([0,T],B^{-1,\infty}_\infty)$.
Then $\vec{u}$ belongs to $\mathcal{C}((0,T],L^q)$. Hence, $\vec{u}$ is smooth on $(0,T)\times\mathbb{R}^3$: $\vec{u} \in \mathcal{C}^\infty((0,T]\times \mathbb{R}^3)$ and moreover, for all $\sigma > 0$, $\vec{u}$ belongs to $\mathcal{C}((0,T],B^{\sigma,\infty}_\infty)$.

Proof :
If we consider $t_0 \in (0,T)$, there exists $a \in [0,t_0)$ such that $\vec{u}(a,.) \in L^q$. Lemma 5 gives us a small interval $[a,b]$ and a solution $\vec{v} \in \mathcal{C}([a,b],L^q)$ with $\vec{v}(a,.) = \vec{u}(a,.)$. Let b^* be the supremum of the b such that we have a solution in $\mathcal{C}([a,b],L^q)$. By Lemma 7, we have $\vec{u} = \vec{v}$ on $[a,\min(T,b^*))$. But then $\vec{v} \in \mathcal{C}([a,\min(T,b^*)],B^{\sigma,\infty}_\infty)$ and Lemma 6 gives that $\vec{v}$

can be extended beyond $\min(T, b^*)$, so that $b^* > T$. Thus, $\vec u$ belongs to $\mathcal{C}([t_0,T], L^q)$ and satisfies $\sup_{t_0<t<T}(t-t_0)^{3/2q}\|\vec u\|_\infty < \infty$.

From this, it is classical to deduce that $\vec u$ is smooth on $(0,T)$. We consider the Besov space $B^{\sigma,\infty}_\infty$. Using the Littlewood-Paley decomposition, we define the homogeneous norm

$$\|f\|_{\dot B^{\sigma,\infty}_\infty} = \sup_{j\in\mathbb{Z}} 2^{j\sigma}\|\Delta_j f\|_\infty$$

on $B^{\sigma,\infty}_\infty$ for $\sigma>0$. If $\sigma>0$, and if f and g belong to $B^{\sigma,\infty}_\infty$, then we have the inequality

$$\|uv\|_{\dot B^{\sigma,\infty}_\infty} \le C_\sigma(\|u\|_{\dot B^{\sigma,\infty}_\infty}\|v\|_\infty + \|v\|_{\dot B^{\sigma,\infty}_\infty}\|u\|_\infty).$$

This gives

$$(61) \qquad \|e^{(t-s)\Delta}\mathbb{P}\vec\nabla.(\vec u\otimes\vec u)\|_{\dot B^{\sigma+1/2,\infty}_\infty} \le C_q\frac{1}{(s-t)^{3/4}}\|\vec u\|_{\dot B^{\sigma,\infty}_\infty}\|\vec u\|_\infty.$$

For $a\in(0,T)$ and $a<t<T$, we write

$$\vec u(t) = e^{(t-a)\Delta}\vec u(a) - B_a(\vec u,\vec u).$$

We have, for all $\sigma\ge 0$,

$$\sup_{a<t<T}(t-a)^{\sigma/2}\|e^{(t-a)\Delta}\vec u(a)\|_{\dot B^{\sigma,\infty}_\infty} \le C_\sigma\|\vec u(a)\|_\infty.$$

Using (61), we get

$$\text{for } 0\le\sigma\le 1/2, \quad \sup_{a<t<T}(t-a)^{\sigma/2-1/2}\|B_a(\vec u,\vec u)\|_{\dot B^{\sigma,\infty}_\infty} \le \sup_{a<t<T}\|\vec u\|^2_\infty.$$

In the same way, we get, for $\sigma>0$ and $a<t<T$,

$$(62) \quad \|\vec u(t)\|_{\dot B^{\sigma+1/2,\infty}_\infty} \le C_\sigma(t-a)^{-\frac12}\left(\|\vec u(a)\|_{\dot B^{\sigma,\infty}_\infty} + (T-a)^{3/4}\sup_{a<t<T}\|\vec u\|_{\dot B^{\sigma,\infty}_\infty}\sup_{a<t<T}\|\vec u\|_\infty\right)$$

Thus, Lemma 8 is proved.

6. Size of the weak solution.

In this section, we prove that we can easily control the size of the weak solution in the neighbourhood of $t=0$:

Lemma 9 :

Let $\vec u$ be a weak solution of the Navier–Stokes equations

$$(63) \qquad \begin{cases} \exists p\in\mathcal{D}'((0,T)\times\mathbb{R}^3) \quad \partial_t\vec u = \Delta\vec u - \vec\nabla.\,(\vec u\otimes\vec u) - \vec\nabla p \\ \vec\nabla.\vec u = 0 \end{cases}$$

such that
i) $\vec u$ *belongs to* $L^p([0,T],L^q(\mathbb{R}^3))$ *for some* $p\in(2,\infty)$ *and* $q\in(3,\infty)$.
ii) $\vec u$ *belongs to* $\mathcal{C}([0,T],B_\infty^{-1,\infty})$.
Then we have

$$\sup_{0<t<T}\sqrt{t}\|\vec u(t,.)\|_\infty<\infty \quad \textit{and} \quad \lim_{t\to 0}\sqrt{t}\|\vec u(t,.)\|_\infty=0.$$

Proof : One more time, for every $\epsilon>0$, we may decompose $\vec u$ into

$$\vec u=\vec U_\epsilon+\vec V_\epsilon \tag{64}$$

with

$$\sup_{0\le t\le T}\|\vec U_\epsilon\|_{B_\infty^{-1,\infty}}<\epsilon \text{ and } \sup_{0\le t\le T}\|\vec V_\epsilon\|_\infty<\infty. \tag{65}$$

We write

$$M_\epsilon=\sup_{0\le t\le T}\|\vec V_\epsilon\|_\infty<\infty. \tag{66}$$

We choose $\sigma\in(1,2)$ and, for $0<a<c\le T$, we define

$$\omega_\sigma(a,c)=\sup_{a<t<c}(t-a)^{(1+\sigma)/2}\|\vec u(t,.)\|_{B_\infty^{\sigma,\infty}}. \tag{67}$$

In order to estimate ω_σ, we write, for $a<t<c$,

$$\vec u(t,.)=e^{\frac{t-a}{2}\Delta}\vec u(\frac{t+a}{2},.)-B_{(t+a)/2}(\vec u,\vec u) \tag{68}$$

where

$$B_{(t+a)/2}(\vec u,\vec u)=\int_{(t+a)/2}^t e^{(t-s)\Delta}\mathbb{P}\vec\nabla.\vec u\otimes\vec u\, ds. \tag{69}$$

In order to estimate the norm of $B_{(t+a)/2}(\vec u,\vec u)$, we use the following estimates on homogeneous Besov norms

$$\|W_{(t+a)/2}f(t,.)\|_{\dot B_\infty^{\sigma,\infty}}\le C_\sigma\sup_{(a+t)/2<s<t}\|f\|_{\dot B_\infty^{\sigma,\infty}} \tag{70}$$

and

$$\|W_{(t+a)/2}f(t,.)\|_{\dot B_\infty^{\sigma,\infty}}\le C_\sigma\sqrt{(t-a)/2}\sup_{(a+t)/2<s<t}\|f\|_{B_\infty^{\sigma+1,\infty}}. \tag{71}$$

We write

$$\vec u(t,.) = e^{\frac{t-a}{2}\Delta}\vec u(\tfrac{t+a}{2},.) - B_{\frac{t+a}{2},\pi}(\vec U_\epsilon,\vec u) - B_{\frac{t+a}{2},\pi}(\vec V_\epsilon,\vec u) - B_{\frac{t+a}{2},\tilde\pi}(\vec u,\vec U_\epsilon)$$
$$-B_{\frac{t+a}{2},\tilde\pi}(\vec u,\vec V_\epsilon) - B_{\frac{t+a}{2},\rho}(\vec U_\epsilon,\vec u) - B_{\frac{t+a}{2},\rho}(\vec V_\epsilon,\vec u) - B_{\frac{t+a}{2},\tilde\rho}(\vec U_\epsilon,\vec u) - B_{\frac{t+a}{2},\tilde\rho}(\vec V_\epsilon,\vec u)$$

We then use (70) to get that

$$A_1 = \|(Id - S_0)\big(B_{\frac{t+a}{2},\pi}(\vec U_\epsilon,\vec u) + B_{\frac{t+a}{2},\tilde\pi}(\vec u,\vec U_\epsilon) + B_{\frac{t+a}{2},\rho}(\vec U_\epsilon,\vec u) + B_{\frac{t+a}{2},\tilde\rho}(\vec U_\epsilon,\vec u)\big)\|_{B^{\sigma,\infty}_\infty}$$

is controlled by

$$(72)\qquad A_1 \le C_\sigma \sup_{(a+t)/2<s<t}\|\vec u\|_{B^{\sigma,\infty}_\infty} \sup_{(a+t)/2<s<t}\|\vec U_\epsilon\|_{B^{-1,\infty}_\infty} \le \epsilon\, C_\sigma \sup_{(a+t)/2<s<t}\|\vec u\|_{B^{\sigma,\infty}_\infty}$$

Similarly, we use (71) to get that

$$A_2 = \|(Id - S_0)\big(B_{\frac{t+a}{2},\pi}(\vec V_\epsilon,\vec u) + B_{\frac{t+a}{2},\tilde\pi}(\vec u,\vec V_\epsilon) + B_{\frac{t+a}{2},\rho}(\vec V_\epsilon,\vec u) + B_{\frac{t+a}{2},\tilde\rho}(\vec V_\epsilon,\vec u)\big)\|_{B^{\sigma,\infty}_\infty}$$

is controlled by

$$(73)\quad A_2 \le C_\sigma\sqrt{\frac{t-a}{2}} \sup_{(a+t)/2<s<t}\|\vec u\|_{B^{\sigma,\infty}_\infty} \sup_{(a+t)/2<s<t}\|\vec V_\epsilon\|_\infty \le M_\epsilon C_\sigma\sqrt{\frac{t-a}{2}} \sup_{(a+t)/2<s<t}\|\vec u\|_{B^{\sigma,\infty}_\infty}.$$

For the low frequencies, we just write that

$$\|e^{(t-s)\Delta}\mathbb{P}\vec\nabla.\vec f\otimes\vec g\|_\infty \le C(t-s)^{-1/2}\|\vec f\otimes\vec g\|_\infty$$

and get that

$$A_3 = \|S_0\big(B_{\frac{t+a}{2},\pi}(\vec U_\epsilon,\vec u) + B_{\frac{t+a}{2},\tilde\pi}(\vec u,\vec U_\epsilon) + B_{\frac{t+a}{2},\rho}(\vec U_\epsilon,\vec u) + B_{\frac{t+a}{2},\tilde\rho}(\vec U_\epsilon,\vec u)\big)\|_{B^{\sigma,\infty}_\infty}$$

is controlled by

$$(74)\quad A_3 \le C_\sigma\sqrt{\frac{t-a}{2}} \sup_{\frac{a+t}{2}<s<t}\|\vec u\|_{B^{\sigma,\infty}_\infty} \sup_{\frac{a+t}{2}<s<t}\|\vec U_\epsilon\|_{B^{-1,\infty}_\infty} \le \epsilon\, C_\sigma\sqrt{\frac{t-a}{2}} \sup_{\frac{a+t}{2}<s<t}\|\vec u\|_{B^{\sigma,\infty}_\infty}$$

and that

$$A_4 = \|S_0\big(B_{\frac{t+a}{2},\pi}(\vec V_\epsilon,\vec u) + B_{\frac{t+a}{2}\tilde\pi}(\vec u,\vec V_\epsilon) + B_{\frac{t+a}{2},\rho}(\vec V_\epsilon,\vec u) + B_{\frac{t+a}{2},\tilde\rho}(\vec V_\epsilon,\vec u)\big)\|_{B^{\sigma,\infty}_\infty}$$

is controlled by

$$(75)\quad A_4 \le C_\sigma\sqrt{\frac{t-a}{2}} \sup_{\frac{a+t}{2}<s<t}\|\vec u\|_{B^{\sigma,\infty}_\infty} \sup_{\frac{a+t}{2}<s<t}\|\vec V_\epsilon\|_\infty \le M_\epsilon\, C_\sigma\sqrt{\frac{t-a}{2}} \sup_{\frac{a+t}{2}<s<t}\|\vec u\|_{B^{\sigma,\infty}_\infty}.$$

Moreover, we have

$$\|e^{\frac{t-a}{2}\Delta}\vec{u}(\frac{t+a}{2},.)\|_{B^{\sigma,\infty}_\infty} \leq \|e^{\frac{t-a}{2}\Delta}\vec{U}_\epsilon(\frac{t+a}{2},.)\|_{B^{\sigma,\infty}_\infty} + \|e^{\frac{t-a}{2}\Delta}\vec{V}_\epsilon(\frac{t+a}{2},.)\|_{B^{\sigma,\infty}_\infty},$$

hence

$$(76)\quad (t-a)^{\frac{1+\sigma}{2}}\|e^{\frac{t-a}{2}\Delta}\vec{u}(\frac{t+a}{2},.)\|_{B^{\sigma,\infty}_\infty} \leq C_\sigma\big(\epsilon(1+(t-a)^{\frac{1+\sigma}{2}})+M_\epsilon(\sqrt{t-a}+(t-a)^{\frac{1+\sigma}{2}})\big)$$

Thus, we find that, for a constant D_σ which depend neither on a nor on c nor on ϵ, we have :

$$(77)\qquad \omega_\sigma(a,c) \leq D_\sigma\Big(\Omega_{\sigma,\epsilon}(a,c) + \big(\epsilon + (M_\epsilon+\epsilon)\sqrt{c-a}\big)\omega_\sigma(a,c)\Big)$$

with

$$(78)\qquad \Omega_{\sigma,\epsilon}(a,c) = \epsilon(1+(c-a)^{\frac{1+\sigma}{2}}) + M_\epsilon(\sqrt{c-a}+(c-a)^{\frac{1+\sigma}{2}}.$$

Thus, if we choose ϵ small enough to grant that $D_\sigma\epsilon < 1/4$ and then we choose c^*_ϵ small enough to grant that $(M_\epsilon+\epsilon)\sqrt{c^*_\epsilon} < \epsilon$ and $\epsilon {c^*_\epsilon}^{\frac{1+\sigma}{2}} + M_\epsilon(\sqrt{c^*_\epsilon} + {c^*_\epsilon}^{\frac{1+\sigma}{2}}) < \epsilon$, we find for $0 < a < c^*_\epsilon$

$$(79)\qquad \sup_{a<t<c^*_\epsilon}(t-a)^{\frac{1+\sigma}{2}}\|\vec{u}\|_{B^{\sigma,\infty}_\infty} \leq 4D_\sigma\ \epsilon.$$

Letting a go to 0, we get

$$(80)\qquad \sup_{0<t<c^*_\epsilon} t^{\frac{1+\sigma}{2}}\|\vec{u}\|_{B^{\sigma,\infty}_\infty} \leq 4D_\sigma\ \epsilon.$$

and by interpolation (since $\|f\|_\infty \leq C_\sigma\|f\|^{\frac{1}{1+\sigma}}_{B^{\sigma,\infty}_\infty}\|f\|^{\frac{\sigma}{1+\sigma}}_{B^{-1,\infty}_\infty}$)

$$(81)\qquad \sup_{0<t<c^*_\epsilon} t^{1/2}\|\vec{u}\|_\infty \leq C_\sigma\ \epsilon^{\frac{1}{1+\sigma}} \sup_{0<t<c^*_\epsilon}\|\vec{u}\|^{\frac{\sigma}{1+\sigma}}_{B^{-1,\infty}_\infty}.$$

Thus, Lemma 9 is proved.

7. Uniqueness of the weak solution.

We may now finish the proof of Theorem 3 (and of Theorem 2, due to Lemma 3) with this easy lemma :

Lemma 10 :

Let $\vec{u}$ and $\vec{v}$ be two solutions of the Navier–Stokes equations (63) such that
i) $\vec{u}$ and $\vec{v}$ belong to $L^p([0,T], L^q(\mathbb{R}^3))$ with $2 < p \leq \infty$, $1 \leq q \leq \infty$ and

ii) $\sup_{0<t<T}\sqrt{t}\|\vec u(t,.)\|_\infty<\infty$ *and* $\lim_{t\to 0}\sqrt{t}\|\vec u(t,.)\|_\infty=0$.
iii) $\sup_{0<t<T}\sqrt{t}\|\vec v(t,.)\|_\infty<\infty$ *and* $\lim_{t\to 0}\sqrt{t}\|\vec v(t,.)\|_\infty=0$.
iv) $\vec u(0,.)=\vec v(0,.)$.

Then $\vec u=\vec v$ *on* $[0,T]$.

Proof : Once again, we write $\vec w=\vec u-\vec v$; then we have

$$\vec w=-B_0(\vec w,\vec u)-B_0(\vec v,\vec w) \tag{82}$$

We write

$$\|e^{(t-s)\Delta}\mathbb{P}\vec\nabla.\vec w\otimes\vec u\|_q\le C\frac{1}{\sqrt{t-s}\sqrt{s}}\|\vec w\|_q\sqrt{s}\|\vec u\|_\infty. \tag{83}$$

Since

$$f\to\int_0^t\frac{1}{\sqrt{t-s}\sqrt{s}}f(s)\,ds$$

is bounded on L^p for $p\in(2,\infty]$, we get, for $c\in(0,T]$

$$\|\vec w\|_{L^p([0,c],L^q}\le C_p\|\vec w\|_{L^p([0,c],L^q}(\sup_{0<s<c}\sqrt{s}\|\vec u(s,.)\|_\infty+\sup_{0<s<c}\sqrt{s}\|\vec v(s,.)\|_\infty). \tag{84}$$

From (84), we get that, for c close enough to 0, $\vec w=0$ on $[0,c]$. Thus, we have local uniqueness. This uniqueness can be propagated ($\vec u$ and $\vec v$ being weakly continuous as time-dependent distributions on $\mathbb{R}^3$) to the whole $[0,T]$.

8. Uniformly vanishing high frequencies.

We now explain a criterion to check continuity in the Besov norm. In most cases, this can be checked by establishing some uniform smallness in high frequencies.

Definition 2 :

A distribution $u\in\mathcal{D}'((0,T)\times\mathbb{R}^3)$ *such that* $t\mapsto u(t,.)$ *is weakly continuous from* $[0,T]$ *to* $\mathcal{D}'(\mathbb{R}^3)$ *and satisfies*

$$\sup_{0<t<T}\|u(t,.)\|_{B_\infty^{-1,\infty}}<\infty$$

has uniformly vanishing high frequencies if it satisfies

$$\lim_{j\to\infty}\sup_{0<t<T}2^{-j}\|\Delta_j u(t,.)\|_\infty=0$$

This uniform vanishing condition may be viewed equivalently in the following ways :

Lemma 11 :

Let u be a distribution in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ such that $t\mapsto u(t,.)$ is weakly continuous from $[0,T]$ to $\mathcal{D}'(\mathbb{R}^3)$ and satisfies

(85)
$$\sup_{0<t<T}\|u(t,.)\|_{B_\infty^{-1,\infty}}<\infty.$$

Then the following assertions are equivalent :
(A) u has uniformly vanishing high frequencies :

(86)
$$\lim_{j\to\infty}\sup_{0<t<T}2^{-j}\|\Delta_j u(t,.)\|_\infty=0$$

(B) $e^{\theta\Delta}u$ is uniformly small for small θ's :

(87)
$$\lim_{\theta\to 0}\sup_{0<t<T}\sqrt{\theta}\|e^{\theta\Delta}u(t,.)\|_\infty=0$$

(C) For every $\epsilon>0$, u may be decomposed as a sum of a uniformly bounded function and a distribution whose Besov norm is less than ϵ :

(88)
$$u=U_\epsilon+V_\epsilon \text{ with } \sup_{0<t<T}\|U_\epsilon(t,.)\|_{B_\infty^{-1,\infty}}<\epsilon \text{ and } \sup_{0<t<T}\|V_\epsilon(t,.)\|_\infty<\infty$$

Proof : $(A)\Rightarrow(C)$ is easy : we use the Littlewood–Paley decomposition and we write $u=U_j+V_j$ with $V_j=S_ju$ and $U_j=\sum_{k=j}^{+\infty}\Delta_k u$. Then we have

$$\|V_j(t,.)\|_\infty\le C2^j\|u(t,.)\|_{B_\infty^{-1,\infty}}$$

and

$$\|U_j(t,.)\|_{B_\infty^{-1,\infty}}\le C\sup_{k\ge j}2^k\|\Delta_k u(t,.)\|_\infty.$$

$(C)\Rightarrow(B)$ is obvious : if $u=U_\epsilon+V_\epsilon$, we have

$$\sqrt{\theta}\|e^{\theta\Delta}u(t,.)\|_\infty\le\sqrt{\theta}\|V_\epsilon(t,.)\|_\infty+C\|U_\epsilon(t,.)\|_{B_\infty^{-1,\infty}}.$$

$(B)\Rightarrow(A)$ is classical : we write $\Delta_j u=e^{-4^{-j}\Delta}\Delta_j e^{4^{-j}\Delta}u$ and we find that

$$\|\Delta_j u(t,.)\|_\infty\le C\|e^{4^{-j}\Delta}u(t,.)\|_\infty.$$

Thus, Lemma 11 is proved.

We may now state our criterion :

Theorem 4 :

Let $\vec{u}$ be a solution of the Navier–Stokes equations

$$(89)\qquad \begin{cases} \exists p\in\mathcal{D}'((0,T)\times\mathbb{R}^3) & \partial_t\vec{u} = \Delta\vec{u} - \vec{\nabla}.\,(\vec{u}\otimes\vec{u}) - \vec{\nabla}p \\ & \vec{\nabla}.\vec{u} = 0 \end{cases}$$

such that $\vec{u}$ belongs to $L^p([0,T],L^q(\mathbb{R}^3))$ for some $p\in[2,\infty)$ and $q\in[3,\infty)$. Then the following assertions are equivalent :
(A) $\vec{u}$ belongs to $\mathcal{C}([0,T],B_\infty^{-1,\infty})$.
(B) u is bounded in the Besov norm and has uniformly vanishing high frequencies :

$$\sup_{0<t<T}\|\vec{u}(t,.)\|_{B_\infty^{-1,\infty}}<\infty \text{ and } \lim_{j\to\infty}\sup_{0<t<T}2^{-j}\|\Delta_j\vec{u}(t,.)\|_\infty = 0.$$

Proof : We have already seen that $(A)\Rightarrow(B)$ (since $L^q\subset\tilde{B}_\infty^{-1,\infty}$). Conversely, let us assume that (B) is satisfied. Then $\vec{u}(0,.)$ belongs to $B_\infty^{-1,\infty}$. If $\vec{u}_j = S_j\vec{u}$, we have $\vec{u}_j(0,.)\in B_\infty^{-1,\infty}$ and $\partial_t\vec{u}_j\in L^1([0,T],B_\infty^{-1,\infty})$:

$$\|\partial_t\vec{u}_j(t,.)\|_{B_\infty^{-1,\infty}} = \|S_j\partial_t\vec{u}(t,.)\|_{B_\infty^{-1,\infty}} \le C_q(2^{2j}\|\vec{u}(t,.)\|_q + 2^j\max(1,2^{\frac{j(6-q)}{q}})\|\vec{u}(t,.)\|_q^2).$$

Hence, we find that $\vec{u}_j$ belongs to $\mathcal{C}([0,T],B_\infty^{-1,\infty})$. Since $\vec{u}_j$ converges uniformly in t to $\vec{u}$ in the Besov norm, we find that $\vec{u}$ belongs to $\mathcal{C}([0,T],B_\infty^{-1,\infty})$. Theorem 4 is proved.

9. The case of $L^\infty L^3$ solutions.

Following similar lines, we can deal with $L^\infty L^3$ solutions :

Theorem 5 :
Let $\vec{u}$ be a solution of the Navier–Stokes equations

$$(90)\qquad \begin{cases} \exists p\in\mathcal{D}'((0,T)\times\mathbb{R}^3) & \partial_t\vec{u} = \Delta\vec{u} - \vec{\nabla}.\,(\vec{u}\otimes\vec{u}) - \vec{\nabla}p \\ & \vec{\nabla}.\vec{u} = 0 \end{cases}$$

such that $\vec{u}$ belongs to $L^\infty([0,T],L^3(\mathbb{R}^3))\cap\mathcal{C}([0,T],B_\infty^{-1,\infty})$.
Then $\vec{u}$ belongs to $\mathcal{C}([0,T],L^3)$.

Proof : We write

$$(91)\qquad \vec{u} = e^{t\Delta}\vec{u}_0 + \vec{w} \text{ with } \vec{w} = -B_0(\vec{u},\vec{u}).$$

We write

$$(92)\qquad \|\frac{1}{\sqrt{-\Delta}}(\vec{u}\otimes\vec{u})\|_{\dot{B}^{1,\infty}_{3/2}} \le C\|\vec{u}\|_3^2$$

hence

$$\sup_{0<t<T} \|\vec{w}\|_{\dot{B}^{1,\infty}_{3/2}} \leq C \sup_{0<t<T} \|\vec{u}\|_3^2. \tag{93}$$

On the other hand, we have

$$\sup_{0<t<T} \|\vec{w}\|_{3/2} \leq C\sqrt{T} \sup_{0<t<T} \|\vec{u}\|_3^2. \tag{94}$$

Moreover, by weak continuity of $\vec{u}$ in L^3, we find that $\vec{u}_0 \in L^3$, hence $e^{t\Delta}\vec{u}_0 \in \mathcal{C}([0,T], L^3)$.

Thus, $\vec{w} \in L^\infty([0,T], B^{1,\infty}_{3/2}) \cap \mathcal{C}([0,T], B^{-1,\infty}_\infty)$. This implies that $\vec{w} \in \mathcal{C}([0,T], L^{3,\infty})$. Indeed, if $f \in B^{1,\infty}_{3/2} \cap B^{-1,\infty}_\infty$, we have $S_0 f \in L^3$ with

$$\|S_0 f\|_3 \leq \sqrt{\|S_0 f\|_{3/2}\|S_0 f\|_\infty} \leq C\sqrt{\|f\|_{B^{1,\infty}_{3/2}}\|f\|_{B^{-1,\infty}_\infty}};$$

for $j \geq 0$, we have

$$\|\Delta_j f\|_{3/2} \leq C2^{-j}\|f\|_{B^{1,\infty}_{3/2}} \text{ and } \|\Delta_j f\|_\infty \leq C2^j\|f\|_{B^{-1,\infty}_\infty}$$

which gives (since $L^{3,\infty} = [L^{3/2}, L^\infty]_{1/2,\infty}$)

$$\|(Id - S_0)f\|_3 \leq C\sqrt{\|f\|_{B^{1,\infty}_{3/2}}\|f\|_{B^{-1,\infty}_\infty}}.$$

Thus far, we got that $\vec{u} \in \mathcal{C}([0,T], L^{3,\infty}) \cap L^\infty([0,T], L^3)$. By weak continuity, $\vec{u}(t,.)$ belongs to L^3 for all $t \in [0,T]$, hence $\vec{u}(t,.) \in \tilde{L}^{3,\infty}$ (the closure of the test functions in $L^{3,\infty}$. But we have uniqueness in the class $\mathcal{C}([0,T], \tilde{L}^{3,\infty})$, as it was proved by Meyer [MEY 99]. It is then easy to conclude that the Kato solution coincides with $\vec{u}$.

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