UNIFORM LAW OF LARGE NUMBERS AND CONSISTENCY OF ESTIMATORS FOR HARRIS DIFFUSIONS

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ABSTRACT. Consider a family of local martingales depending on a parameter θ running through some compact in $\mathbb{R}^d$. We show that if their quadratic variations are Hölder in θ, then the family satisfies a uniform law of large numbers. We apply it to deduce the almost sure consistency of maximum likelihood estimators for drift parameters of a multidimensional Harris recurrent diffusion, thereby extending a recent result of J. H. van Zanten for one-dimensional ergodic diffusions.

1. INTRODUCTION

The important role of uniform laws of large numbers (ULLN) in parameter estimation is well-known. Though the general problem has been extensively studied (see [1, 2, 8, 12]), many recent articles contribute to a better understanding of sufficient conditions for ULLN in particular settings (see [4, 5, 7, 11, 13]). So, J. H. van Zanten has recently proved a version of uniform ergodic theorem that allows him to loosen the assumptions for the consistency of maximum likelihood estimators (MLE) in ergodic diffusion models (see [13, 14]). Namely, consider a family of one-dimensional diffusions given by

$$dX_t^\theta = b_\theta(X_t^\theta)\,dt + \sigma(X_t^\theta)\,dW_t,$$

where the unknown parameter θ belongs to a compact $\Theta \subset \mathbb{R}^d$. The main result of [13] states (roughly speaking) that, if for all $\theta \in \Theta$ the diffusion X_t^θ is ergodic, and if b_θ is Hölder w.r.t. θ, then a MLE of θ exists and is consistent in probability.

In the present article we prove a ULLN that allows to extend this result to the much less studied general case of multi-dimensional recurrent Harris diffusions. Recall that such a diffusion $X_t \in \mathbb{R}^n$ admits an invariant measure μ such that $\mu(A) > 0$ implies $\int_0^\infty \mathbf{1}_A(X_s)\,ds = \infty$ a.s. for every measurable $A \subseteq \mathbb{R}^n$. According to the case $\mu(\mathbb{R}^n) < \infty$ or $\mu(\mathbb{R}^n) = \infty$ the diffusion X_t is said ergodic or null-recurrent. Hence the class of Harris recurrent diffusions includes the ergodic ones, but also the Brownian motion in dimensions 1 and 2, the Bessel processes of low dimensions, and so on.

The few known results about MLE for null-recurrent diffusions concern only some special models. For example, Höpfner and Kutoyants have established in [4] the consistency and the limit distribution of MLE for a process

$$dX_t = -\theta \frac{X_t}{1 + X_t^2}\,dt + \sigma\,dW_t.$$

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Note that in the case $b_\theta(x) = \theta \cdot b(x)$ of linear dependence w.r.t. θ the consistency follows immediately from the explicit expression of $\hat{\theta}_t$ (see Lemma 17.4 of Liptser, Shiryaev [9]), so our work mainly concerns the nonlinear models.

Our proofs are simple and self-consistent, since we use only classical results from the theory of stochastic processes. To sketch the main lines, denote by θ_0 the true value of the parameter. Girsanov theorem implies that the logarithm of the likelihood ratio can be written as

$$l_t(\theta) = M_t(\theta) - \frac{1}{2}A_t(\theta),$$

where $M_t(\theta)$ is a continuous local martingale and $A_t(\theta)$ is its quadratic variation, with $l_t(\theta_0) = 0$. Hence to establish the consistency of MLE it suffices to prove that for t large enough $\sup_\theta l_t(\theta) < 0$ on any compact not containing θ_0.

In the ergodic case one disposes of the following LLN (Birkhoff theorems) for martingales and additive functionals associated with such diffusions:

$$\frac{1}{t}M_t \to 0 \quad \text{and} \quad \frac{1}{t}\int_0^t f(X_s)\,ds \to \mu(f).$$

Their uniform versions (see [13]) allow to prove that $l_t(\theta)/t$ converges uniformly to some strictly negative limit $l(\theta)$, whence $\lim_{t\to\infty} \sup_\theta l_t(\theta) = -\infty$, as desired.

But for null-recurrent diffusions there is no universal deterministic normalization like $1/t$ above (see [3]). Nevertheless, Chacon-Ornstein theorem provides that

$$\frac{\int_0^t f(X_s)\,ds}{\int_0^t g(X_s)\,ds} \to \frac{\mu(f)}{\mu(g)}$$

for all integrable f and g with $\mu(g) \neq 0$. Hence a natural idea consists to replace t by some additive functional V_t. Following this idea, we firstly obtain in Section 2 a ULLN (theorem 2) for randomly normalized martingale families. Then we show that this ULLN holds for martingales associated with Harris recurrent diffusions under Hölder assumption on the drift coefficient. Finally, in Section 3 we apply it to deduce the consistency of MLE.

2. Regularity of martingale families

In this section we prove a ULLN for continuous local martingales. Let $d \in \mathbb{N}$ and $\Theta = [0,1]^d$. Consider a probability space $\mathbf{\Omega} = (\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ and a family of real processes $\{M_t(\theta);\ t \geq 0,\ \theta \in \Theta\}$ on $\mathbf{\Omega}$ such that for all θ the process $\{M_t(\theta)\}$ is a continuous local martingale starting at zero. Denote by $A_t(\theta)$ the quadratic variation of $M_t(\theta)$ and by $A_t(\theta,\theta')$ that of $M_t(\theta) - M_t(\theta')$.

Our first theorem concerns the Hölder property of normalized martingale families. The Kolmogorov-type proof is quite classical, but the result we obtain does not depend on the dimension d of Θ.

Theorem 1. *Suppose there exist a constant $\delta \in\ ]0,1]$ and a continuous increasing process $V_t > 0$ such that $V_\infty = \infty$ a.s. and*

$$\forall (\theta,\theta') \in \Theta^2,\ \forall t, \quad A_t(\theta,\theta') \leq |\theta - \theta'|^{2\delta} V_t \quad a.s. \tag{1}$$

Let $\gamma < \delta$ and $\alpha > 1/2$.

Then there exists a modification $\tilde{M}_t(\theta)$ *of* $M_t(\theta)$ *such that for almost all* $\omega \in \Omega$ *and some* $t(\omega)$ *the family* $\left\{\frac{\tilde{M}_t(\theta)}{V_t^\alpha}\right\}_{t>t(\omega)}$ *of functions of* θ *is uniformly Hölder of order* γ *on* Θ*. In particular,* $\tilde{M}_t(\theta)$ *is continuous in* θ *for all* $t > t(\omega)$*.*

Proof. Denote by D_m the set of the m-th order dyadic points of Θ, and by D the union of D_m, i.e.

$$D_m = \left\{\frac{\mathbf{k}}{2^m} : \mathbf{k} \in \mathbb{Z}^d\right\} \cap \Theta \quad \text{and} \quad D = \bigcup_{m=0}^{\infty} D_m.$$

We start by showing the "local" regularity of M_t/V_t^α on D. Denote by $\tilde{D}_m^2$ the set of all couples $(\theta, \theta') \in D_m^2$ such that $|\theta - \theta'| = 2^{-m}$, with (θ, θ') assimilated to (θ', θ). Let $\tau_n = \inf\{t : \ V_t = n\}$. By assumption $V_t \to \infty$, so $\tau_n < \infty$ a.s. For $n \geq 1$ denote

$$B_n = \left\{\exists m \geq 0, \ \exists (\theta, \theta') \in \tilde{D}_m^2, \quad \sup_{[\tau_{n-1}, \tau_n[} \frac{|M_t(\theta) - M_t(\theta')|}{V_t^\alpha} > |\theta - \theta'|^\gamma\right\}.$$

We have for n great enough and any $p > 1$

$$\begin{aligned}
\mathbf{P}(B_n) &\leq \sum_{m \geq 0} \sum_{\tilde{D}_m^2} \mathbf{P}\left(\sup_{[\tau_{n-1}, \tau_n[} |M_t(\theta) - M_t(\theta')|^p > |\theta - \theta'|^{\gamma p} (n-1)^{\alpha p}\right) \\
&\leq \sum_{m \geq 0} \sum_{\tilde{D}_m^2} \frac{\mathbb{E}\left(\sup_{t < \tau_n} |M_t(\theta) - M_t(\theta')|^p\right)}{|\theta - \theta'|^{\gamma p} (n-1)^{\alpha p}} \\
&\leq \sum_{m \geq 0} \sum_{\tilde{D}_m^2} C(p) \frac{\mathbb{E} A_{\tau_n}(\theta, \theta')^{p/2}}{(n-1)^{\alpha p} |\theta - \theta'|^{\gamma p}} \leq \sum_{m \geq 0} \sum_{\tilde{D}_m^2} C(p) \frac{n^{p/2} |\theta - \theta'|^{\delta p}}{(n-1)^{\alpha p} |\theta - \theta'|^{\gamma p}} \\
&\leq C(p) \frac{n^{p/2}}{(n-1)^{\alpha p}} \sum_{m \geq 0} d 2^{m(d - p(\delta - \gamma))},
\end{aligned}$$

where we have used the Burkholder-Davis-Gundy inequality and the fact that $\operatorname{Card} \tilde{D}_m^2 \leq d 2^{md}$.

Now choose $p > \max\left(\frac{d}{\delta - \gamma}, \frac{2}{2\alpha - 1}\right)$. We see that the m-series converges and $\mathbf{P}(B_n) \leq C n^{-p(\alpha - 1/2)}$. So $\sum_n \mathbf{P}(B_n)$ converges too, and by the Borel-Cantelli lemma, $\mathbf{P}(\limsup B_n) = 0$.

Since $\tau_n \to +\infty$, for almost all $\omega \in \Omega$ and t greater than some $t(\omega)$ we have:

$$\forall m \geq 0, \ \forall (\theta, \theta') \in \tilde{D}_m^2, \quad \frac{|M_t(\theta) - M_t(\theta')|}{V_t^\alpha} \leq |\theta - \theta'|^\gamma.$$

A well-known Kolmogorov argument (see [10]) implies that the family $\left\{\frac{M_t(\theta)}{V_t^\alpha}\right\}_{t>t(\omega)}$ is then Hölder of order γ on D with some constant $C(\gamma, d)$ not depending on t.

Now the existence of a version claimed in the theorem follows by classical technique, since D is dense in Θ, taking

$$\tilde{M}_t(\theta) = \lim_{\substack{\theta' \to \theta \\ \theta' \in D}} M_t(\theta')$$

for t large enough. □

Remark. It is not difficult to adapt the proof for any compact $\Theta \subset \mathbb{R}^d$, and even for infinite-dimensional compacts under some entropy condition, but we omit it for the sake of brevity.

The following easy proposition shows that the assumption (1) implies the uniform Hölder continuity of $A_t(\theta)/V_t$ on Θ, provided its boundedness.

Proposition 1. *If $\{M_t(\theta)\}$ satisfies the assumptions of theorem 1 and*

$$\exists \theta' \in \Theta, \quad \limsup_{t\to\infty} \frac{A_t(\theta')}{V_t} < \infty \quad a.s. \tag{2}$$

then almost surely the family $\{A_t(\theta)/V_t\}_{t>0}$ is Hölder of order δ on Θ, with a constant not depending on t.

Proof. Take $C = C(\omega)$ such that $A_t(\theta')/V_t < C$ for all $t > 0$. From the inequality

$$\begin{aligned} |A_t(\theta) - A_t(\theta')| &= |A_t(\theta,\theta') + 2\left\langle M(\theta'), M(\theta) - M(\theta')\right\rangle_t| \\ &\le V_t|\theta-\theta'|^{2\delta} + 2\sqrt{A_t(\theta')}\sqrt{V_t|\theta-\theta'|^{2\delta}} \end{aligned} \tag{3}$$

we deduce that almost surely the family $\left\{\frac{A_t(\theta)}{V_t}\right\}$ is bounded by $C + d^\delta + 2\sqrt{Cd^\delta}$ on Θ. Rewriting (3) for any (θ_1, θ_2) in place of (θ, θ') gives the claimed result. □

Now we are ready to prove the following ULLN.

Theorem 2. *Suppose that the family $\{M_t(\theta)\}$ satisfies the assumptions of theorem 1 with some V_t and δ. Suppose also that (2) holds for some $\theta' \in \Theta$.*

Then there exists a continuous in θ modification $\tilde{M}_t(\theta)$ of $M_t(\theta)$ such that

$$\lim_{t\to\infty} \sup_{\theta\in K} \frac{|\tilde{M}_t(\theta)|}{A_t(\theta)} = 0 \quad a.s.$$

for any compact $K \subset \Theta$ satisfying

$$\forall \theta \in K,\ \liminf_{t\to\infty} \frac{A_t(\theta)}{V_t} > 0 \quad a.s. \tag{4}$$

Proof. Take the modification provided by theorem 1 with $\alpha = 1$. We have

$$\sup_{\theta\in K} \frac{|\tilde{M}_t(\theta)|}{A_t(\theta)} \le \sup_{\theta\in K} \frac{|\tilde{M}_t(\theta)|}{V_t} \cdot \sup_{\theta\in K} \frac{V_t}{A_t(\theta)}.$$

Proposition 1 ensures that the family $\left\{\frac{A_t(\theta)}{V_t}\right\}$ is equicontinuous on K, so it's an easy exercise to show that (4) implies that $\sup_{\theta\in K} \frac{V_t}{A_t(\theta)} \le c(\omega) < \infty$ for some $c(\omega)$ and all $t > t(\omega)$ large enough.

Since $V_t \to \infty$, for all $\theta \in K$ we have $A_t(\theta) \to \infty$. By the law of large numbers for continuous martingales,

$$\forall \theta \in K, \quad \lim_{t\to\infty} \frac{|\tilde{M}_t(\theta)|}{A_t(\theta)} = 0 \quad \text{a.s.}$$

Hence, by comparison between $A_t(\theta)$ and V_t we can choose a countable set K' dense in K such that almost surely

$$\forall \theta \in K', \quad \lim_{t\to\infty} \frac{|\tilde{M}_t(\theta)|}{V_t} = 0.$$

By theorem 1 the family of function $\left\{\frac{\tilde{M}_t(\theta)}{V_t}\right\}_{t>t(\omega)}$ is equicontinuous on K, and it is convergent to zero on a dense subset of K. According to Ascoli lemma the convergence is uniform on K, whence our theorem. □

We end this section with an example of martingale families $\{M_t(\theta)\}$ for which the theorems above hold.

Let

$$M_t(\theta) = \int_0^t f(\theta, X_s)dB_s, \quad \theta \in \Theta, \tag{5}$$

where B_t is a k-dimensional brownian motion and X_t is a n-dimensional Harris recurrent diffusion. It means that X_t possesses an invariant measure μ such that for any measurable function $g : \mathbb{R}^n \to \mathbb{R}^+$ with $\mu(g) > 0$, one has $\int_0^\infty g(X_s)\, ds = \infty$.

Suppose that $f : \Theta \times \mathbb{R}^n \to \mathbb{R}^k$ is Hölder of some order δ w.r.t. θ:

$$\forall(\theta, \theta', x) \in \Theta^2 \times \mathbb{R}^n, \quad |f(\theta, x) - f(\theta', x)| \leq C(x)|\theta - \theta'|^\delta, \tag{6}$$

with some borelian $C(x) > 0$. Then $A_t(\theta) = \int_0^t |f(\theta, X_s)|^2 ds$ and

$$A_t(\theta, \theta') = \int_0^t |f(\theta, X_s) - f(\theta', X_s)|^2 ds \leq |\theta - \theta'|^{2\delta} \int_0^t C^2(X_s)ds,$$

so the inequality (1) holds with

$$V_t = \int_0^t C^2(X_s)ds. \tag{7}$$

The following proposition is a uniform version of the mentioned above Lemma 17.4 in [9].

Proposition 2. *Let $M_t(\theta)$ be given by* (5) *with f verifying* (6) *for some δ. Suppose that $0 < \int C^2(x)\mu(dx) < \infty$ and $\int |f(\theta', x)|^2 \mu(dx) < \infty$ for some θ'. Then there exists a version $\tilde{M}_t(\theta)$ of $M_t(\theta)$ such that*

- *almost surely, for t great enough $\tilde{M}_t(\theta)$ is uniformly continuous on Θ;*
- *if $K \subset \Theta$ is a compact such that*

$$\forall\theta \in K, \quad \int |f(\theta, x)|^2 \mu(dx) > 0 \tag{8}$$

then

$$\lim_{t\to\infty} \sup_{\theta \in K} \frac{|\tilde{M}_t(\theta)|}{A_t(\theta)} = 0 \quad a.s.$$

Proof. Take V_t given by (7) and $\alpha = 1$. Since V_t is an additive functional of a Harris recurrent diffusion X_t and since $C(x) > 0$, we get $V_\infty = \infty$ a.s. The first point now follows by theorem 1.

To prove the second assertion we will show that (2) and (4) of theorem 2 are satisfied. Let $\|\cdot\|_2 = \|\cdot\|_{L^2(\mu)}$. Note that for all $(\theta_1, \theta_2) \in \Theta^2$

$$\Big|\, \|f(\theta_1, \cdot)\|_2 - \|f(\theta_2, \cdot)\|_2 \,\Big| \leq \|f(\theta_1, \cdot) - f(\theta_2, \cdot)\|_2 \leq \|C(\cdot)\|_2 |\theta_1 - \theta_2|^\delta,$$

and since $\|f(\theta', \cdot)\|_2 < \infty$, the function $\|f(\theta, \cdot)\|_2$ is continuous and bounded on Θ. Now Chacon-Ornstein ratio-limit theorem (see [10]) implies

$$\lim_{t\to\infty} \frac{A_t(\theta)}{V_t} = \frac{\int |f(\theta, x)|^2 \mu(dx)}{\int C^2(x)\mu(dx)} = l(\theta) < \infty \quad \text{a.s.}$$

In particular, (2) holds for θ', and the assumption (8) implies (4). Hence the proof follows by theorem 2. □

3. Consistency of MLE for Harris diffusions

We show in this section the existence and the strong consistency of maximum likelihood estimators for drift parameters of Harris recurrent diffusions.

Let Θ be as above and $x \in \mathbb{R}^n$. Consider a family of n-dimensional diffusions given by

$$dX_t^\theta = \sigma(X_t^\theta)dB_t + b(X_t^\theta, \theta)dt, \quad X_0^\theta = x, \quad \theta \in \Theta, \tag{9}$$

where B_t is a k-dimensional brownian motion. We suppose that $a = \sigma\sigma^T$ is positively definite and that for each $\theta \in \Theta$ there exists a solution on $[0; +\infty[$.

Let θ_0 be the (unknown) true value of the parameter and denote by $\mathbf{P}_\theta$ the law of X^θ. The measures $\mathbf{P}_\theta$ are locally absolutely continuous with respect to each other and also with respect to the law $\mathbf{P}$ of the solution of the equation (9) without drift: $dX_t = \sigma(X_t)dB_t$.

Let $L_t(\theta)$ be the local density $d\mathbf{P}_\theta/d\mathbf{P}$ (the likelihood ratio):

$$L_t(\theta) = \exp\left[\int_0^t b_\theta^T a^{-1}\sigma(X_s)\, dB_s - \frac{1}{2}\int_0^t b_\theta^T a^{-1} b_\theta(X_s)\, ds\right],$$

where $a = a(\,\cdot\,)$ and $b_\theta = b(\theta, \cdot\,)$. The maximum likelihood estimator $\hat{\theta}_t$ of θ_0 is defined as a maximizer of the random map $\theta \to L_t(\theta)$ provided it exists.

Theorem 3. *Consider a family of diffusions given by* (9)*. Suppose that $X_t^{\theta_0}$ is Harris recurrent and let μ_0 be its invariant measure. Suppose that*

- *b is Hölder in θ : $|b(\theta, x) - b(\theta', x)| \le |\theta - \theta'|^\delta C(x)$, $C(x) > 0$;*
- *the function $C^2(x)||a^{-1}||(x)$ is μ_0-integrable;*
- *for any $\theta \neq \theta_0$*

$$\int_{R^n} (b_\theta - b_{\theta_0})^T a^{-1}(b_\theta - b_{\theta_0})(x)\, \mu_0(dx) > 0. \tag{10}$$

Then $\mathbf{P}_{\theta_0}$-a.s. an estimator $\hat{\theta}_t$ of θ exists for all t great enough and any MLE converges to θ_0 as $t \to \infty$.

Proof. Observe that $\hat{\theta}_t$ also maximizes $\mathcal{L}_t(\theta) = L_t(\theta)/L_t(\theta_0)$. Under $\mathbf{P}_{\theta_0}$

$$\mathcal{L}_t(\theta) = \exp\left[\int_0^t (b_\theta - b_{\theta_0})^T a^{-1}\sigma(X_s^{\theta_0})\, dW_s \right.$$
$$\left. - \frac{1}{2}\int_0^t (b_\theta - b_{\theta_0})^T a^{-1}(b_\theta - b_{\theta_0})(X_s^{\theta_0})\, ds\right],$$

where W is a Brownian motion under $\mathbf{P}_{\theta_0}$. We can write

$$\mathcal{L}_t(\theta) = \exp\left[M_t^\theta - \frac{1}{2}A_t^\theta\right], \tag{11}$$

where

$$M_t^\theta = \int_0^t (b_\theta - b_{\theta_0})^T a^{-1}\sigma(X_s^{\theta_0})\, dW_s$$

is a martingale under $\mathbf{P}_{\theta_0}$ and $A_t^\theta = \langle M^\theta \rangle_t$.

The martingale M_t^θ is of the form $M_t^\theta = \int_0^t f(\theta, X_s^{\theta_0}) dW_s$ with $f : \Theta \times \mathbb{R}^n \to \mathbb{R}^k$ given by

$$f(\theta, x) = (b_\theta - b_{\theta_0})^T a^{-1} \sigma(x).$$

Note that $\forall x \in \mathbb{R}^n,\ f(\theta_0, x) = 0$ and

$$|f(\theta, x) - f(\theta', x)|^2 = (b_\theta - b_{\theta'})^T a^{-1} (b_\theta - b_{\theta'})(x) \leq |\theta - \theta'|^{2\delta} C^2(x) ||a^{-1}(x)||$$

So f is Hölder in θ of order δ with the constant $C(x)\sqrt{||a^{-1}(x)||}$ which by our assumptions belongs to $L^2(\mu_0)$. We see also that f satisfies (8) on any compact K not containing θ_0. Recall that $X_t^{\theta_0}$ is Harris recurrent, so $\{M_t^\theta\}$ is a martingale family satisfying all conditions of the proposition 2. It follows, in particular, that M_t^θ and A_t^θ are (up to a modification) continuous in θ for $t > t(\omega)$. Hence $\mathcal{L}_t(\theta)$ is continuous on Θ and since Θ is compact, a MLE $\hat{\theta}_t$ exists for t great enough.

Now, according to the Wald's method, in order to prove that $\hat{\theta}_t \to \theta_0$ it suffices to show that for any compact $K \subset \Theta$ not containing θ_0

$$\lim_{t\to\infty} \sup_{\theta \in K} \mathcal{L}_t(\theta) = 0 \quad P_{\theta_0} \text{ - almost surely.}$$

Indeed, if we choose K_n as the exterior of an open ball centered in θ_0 with the radius $1/n$, the preceding limit means that, almost surely $L_t(\theta) < L_t(\theta_0)$ on K_n for t great enough. Hence $\hat{\theta}_t$ that maximizes $L_t(\theta)$ on Θ belongs to this ball. Making $n \to \infty$ will then give the convergence as claimed.

Rewrite (11) as

$$\mathcal{L}_t(\theta) = \exp\left[A_t^\theta \left(\frac{M_t^\theta}{A_t^\theta} - \frac{1}{2}\right)\right]$$

Since $X_t^{\theta_0}$ is Harris, we have $\lim_{t\to\infty} A_t^\theta = \infty$ a.s. Being monotone, this convergence is uniform by Dini lemma. Again by proposition 2

$$\lim_{t\to\infty} \sup_{\theta \in K} \frac{M_t^\theta}{A_t^\theta} = 0 \quad \text{a.s.}$$

Hence

$$\lim_{t\to\infty} \sup_{\theta \in K} A_t^\theta \left(\frac{M_t^\theta}{A_t^\theta} - \frac{1}{2}\right) = -\infty,$$

whence the theorem follows. □

Let us end the paper by a brief discussion of the assumptions of theorem 3. The first and the third points are quite standard. The second one (the integrability of $C^2(x)$) can seem restrictive for null-recurrent diffusions (since $\mu_0(\mathbb{R}^n)$ is infinite), but in fact it is not easy to find a model where it is not satisfied. Take, for example, the following SDE

$$dX_t = \frac{\theta X_t}{1 + \theta^2 + X_t^2}\, dt + dW_t,$$

that we borrow (with a slight modification, to avoid linearity) from Kutoyants [8]. It is easy to see that its solution is recurrent if $\theta \leq 1/2$ and null-recurrent for $|\theta| \leq 1/2$, and that $\mu_0(x) \sim (1 + |x|)^{2\theta_0}\, dx$. A simple calculation shows that $\delta = 1$ and $C(x) \sim (1+|x|)^{-1}$ for $|x|$ great enough, so the integrability condition is fulfilled and theorem 3 applies for $\theta_0 < 1/2$.

Actually, the Hölder assumption about $b(\theta, x)$ is too strong, since the inequality $A(\theta, \theta') \leq V_t |\theta - \theta'|^{2\delta}$ can be satisfied even for some discontinuous trends $b(\theta, x)$ (see p. 269 in [8]), and this will be the subject of a future work.

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