Weak-strong uniqueness criterions for the critical quasi-geostrophic equation

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Abstract : We give two weak-strong uniqueness results for the weak solutions to the critical dissipative quasi-geostrophic equation when the initial data belongs to $H^{-1/2}$. The first one shows that we can construct a unique $H^{-1/2}$-solution when the initial data belongs moreover to $L^\infty$ with a small $L^\infty$ norm. The other one gives the uniqueness of a $H^{-1/2}$-solution which belongs to $C([0,T), CMO)$.

1 Introduction

We consider the surface 2D quasi-geostrophic equation for a scalar $\theta(t,x)$ defined on $[0,T) \times \mathbb{R}^2$:

$$(QG)_\alpha \left\{ \begin{array}{l} \partial_t \theta + u \cdot \nabla \theta + k \Lambda^{2\alpha} \theta = 0 \\ u = -R^\perp \theta = -(-R_2 \theta R_1 \theta) \\ \theta(0,\cdot) = \theta_0 \end{array} \right.$$  \hspace{1cm} (1)

where $k \geq 0$ and $\Lambda$ is the operator $(-\Delta)^{1/2}$. Thus, $\widehat{\Lambda^{2\alpha}f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi)$ where the Fourier transform $\widehat{f} = \mathcal{F}(f)$ is defined by $\widehat{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-ix \cdot \xi} dx$. The Riesz transforms $R_1$ and $R_2$ are defined by $\widehat{R_k f}(\xi) = -\frac{i\xi_k}{|\xi|^2} \hat{f}(\xi)$. The inviscid model ($k = 0$) has been introduced by Constantin, Majda and Tabak [7] in order to study the formation of sharp fronts between masses of hot and cold air. The equation $(QG)_\alpha$ with $k > 0$ has been introduced by Constantin and Wu [6], and the case where $\alpha = 1/2$ arise in the study of rapidly rotating geophysical fluids.

The analysis of the subcritical case $\alpha > 1/2$ is now quite well understood while in the critical ($\alpha = 1/2$) and supercritical ($\alpha < 1/2$) cases there are still unanswered questions. Recently, Kiselev, Nazarov and Volberg [9] proved, in the critical case, that any $C^\infty$ periodic initial data give rise to a unique $C^\infty$ solution; the proof relies on the propagation of an appropriate moduli of continuity. Using the same argument, Abidi and Hmidi [1] proved the global well-posed of $(QG)_{1/2}$ for an initial data in the Besov space $\dot{B}_\infty^{0,1}(\mathbb{R}^2)$. Another new recent result concerning $(QG)_{1/2}$ is the global Hölder regularity of the weak Leray-Hopf $L^2$-solutions (global weak solutions with an initial data in $L^2$), this result is due to Caffarelli and Vasseur [3]. We point out that such result does not give the uniqueness of these solutions.

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In this paper we address the question of uniqueness of weak $\dot{H}^{-1/2}$-solutions (see definition 1.1 below) to the critical quasi-geostrophic equation $(QG)_{1/2}$.

In the following, a weak solution to $(QG)\alpha$ will be a solution to $(QG)\alpha$ in $\mathcal{D}'((0,T) \times \mathbb{R}^2)$.

We define the class of $\dot{H}^{-1/2}$-solutions that we will consider for uniqueness by:

**Definition 1.1.** A $\dot{H}^{-1/2}$-solution on $(0,T)$ to the quasi-geostrophic equation $(QG)\alpha$ with initial value $\theta_0 \in \dot{H}^{-1/2}$ is a weak solution $\theta$ to $(QG)\alpha$ which satisfies the energy inequality

$$\|\theta(t)\|_{\dot{H}^{-1/2}} + 2k \int_0^t \int_{\mathbb{R}^2} |\Lambda^{\alpha-1/2}\theta|^2 dxds \leq \|\theta_0\|_{\dot{H}^{-1/2}}$$

for all $t \in (0,T)$.

**Remark 1.1.** When the spatial domain is the 2D-torus, we may restrict to functions with mean value zero in space; in that case a weak Leray-hopf $L^2$-solution is in particular a $\dot{H}^{-1/2}$-solution. This is not true when the spatial domain is $\mathbb{R}^2$ since $L^2(\mathbb{R}^2)$ is not embedded in $\dot{H}^{-1/2}(\mathbb{R}^2)$.

We recall that the class of $\dot{H}^{-1/2}$-solutions is not empty since the author have proved in [12] the following result:

**Theorem 1.1.** Let $0 < \alpha \leq 1$ and $\theta_0 \in \dot{H}^{-1/2}(\mathbb{R}^2)$. There exists a weak solution $\theta$ on $(0,T) \times \mathbb{R}^2$ to $(QG)\alpha$ such that $\theta \in L^\infty((0,T),\dot{H}^{-1/2}) \cap L^2((0,T),H^{\alpha-1/2})$ and satisfies the global inequality:

$$\|\theta(t)\|_{\dot{H}^{-1/2}} + 2k \int_0^t \int_{\mathbb{R}^2} |\Lambda^{\alpha-1/2}\theta|^2 dxds \leq \|\theta_0\|_{\dot{H}^{-1/2}}$$

for all $t \in (0,T)$.

**Remark 1.2.** - For a definition of the the non-linear term in the case where $0 < \alpha < 1/2$ we refer the reader to [12].
- In particular, an initial data in $L^{4/3}$ give rise to a $\dot{H}^{-1/2}$-solution to $(QG)\alpha$.

In [6], Constantin and Wu give a weak-strong uniqueness result in the sub-critical case $1/2 < \alpha \leq 1$:

**Theorem 1.2.** Let $\theta_0 \in \dot{H}^{-1/2}$, assume that there exists a weak solution to $(QG)\alpha$ on $(0,T) \times \mathbb{R}^2$ with initial value $\theta_0$ such that:

$$\theta \in L^\infty((0,T),\dot{H}^{-1/2}) \cap L^2((0,T),H^{\alpha-1/2})$$

and

$$\theta \in L^p((0,T),L^q)$$

with $(p,q) \in [1,\infty)$ and $\frac{1}{p} + \frac{\alpha}{q} = \alpha - 1/2$. Then $\theta$ satisfies the inequality (2) and is the unique $\dot{H}^{-1/2}$-solution on $(0,T) \times \mathbb{R}^2$ with initial data $\theta_0$.

**Remark 1.3.** In the case where $q = \infty$ and $p = \frac{2}{2\alpha-1}$ we still have the same conclusion, this is an analogue to the result of Kozono and Sohr [10] for the Navier-Stokes equations.
If we combine the existence theorem 1 to the maximum principle property satisfied by \((QG)_\alpha\) (see [5]), we get the existence of a weak solution which is in \(L^\infty((0, T), H^{-1/2}) \cap L^2((0, T), H^{\alpha-1/2}) \cap L^\infty((0, T), L^p)\) for an initial data in \(\dot{H}^{-1/2} \cap L^p\). When \(p \geq \frac{2}{\alpha-1}\), theorem 1.2 gives the uniqueness of this solution in the class of the \(\dot{H}^{-1/2}\)-solutions. Then, a natural question is whether we still have such a result in the critical case.

We now state and comment our two main results. In section 3, we prove the following result of uniqueness for a solution to \((QG)_{1/2}\) which belongs to \(L^\infty((0, T), \dot{H}^{-1/2}) \cap L^2((0, T), L^2) \cap L^\infty((0, T), BMO)\) with a small norm in \(L^\infty((0, T), BMO)\):

**Theorem 1.3.** Let \(\theta_0 \in \dot{H}^{-1/2}\) and \(T \in (0, +\infty]\), assume there exists a solution \(\theta\) to the quasigeostrophic equation, \((QG)_{1/2}\), on \((0, T)\) with initial value \(\theta_0\) such that:

\[
\theta \in L^\infty((0, T), \dot{H}^{-1/2}) \cap L^2((0, T), L^2) \cap L^\infty((0, T), BMO).
\]

Then \(\theta\) satisfies the equality in (2). Moreover, there exists a positive constant \(D_\infty\) such that, if \(\|\theta\|_{L^\infty((0,T),BMO)} < D_\infty k\), then \(\theta\) is the unique \(\dot{H}^{-1/2}\)-solution to \((QG)_{1/2}\) on \((0, T)\) with initial value \(\theta_0\).

**Remark 1.4.** This result is an improvement of a result given by Wu in [18]; the improvement relies on the observation that for a function \(f\) in \(L^2\), \(fR_t f\) belongs to the Hardy space \(\mathcal{H}^1\) (see section 2). This observation has first been used, in [8] by Córdoba A. and Córdoba D. to deal with \((QG)_{1/2}\) (see also [13]).

Since the solutions constructed in theorem 1.1 satisfy the maximum principle property we get the immediate corollary:

**Corollary 1.4.** Let \(T > 0\) and \(\theta_0 \in \dot{H}^{-1/2} \cap L^\infty\) with \(\|\theta_0\|_\infty < D_\infty\) (the constant \(D_\infty\) is given by theorem 1.3). The solution given by theorem 1.1 belongs to \(L^\infty((0, T), \dot{H}^{-1/2}) \cap L^2((0, T), L^2) \cap L^\infty((0, T), L^\infty)\) with \(\|\theta(t)\|_\infty \leq \|\theta_0\|_\infty\); this is the unique \(\dot{H}^{-1/2}\)-solution to \((QG)_{1/2}\) on \((0, T)\) with initial value \(\theta_0\).

In section 4 we show that we can get rid of the smallness assumption if we replace the \(L^\infty((0, T), BMO)\) condition by a \(C([0, T), CMO)\) condition; this result is a kind of analogue to the Won Wahl theorem for the Navier-Stokes equations:

**Theorem 1.5.** Let \(\theta_0 \in \dot{H}^{-1/2}\) and \(T \in (0, +\infty]\), assume there exists a solution \(\theta\) to the quasigeostrophic equation, \((QG)_{1/2}\), on \((0, T)\) with initial value \(\theta_0\) such that:

\[
\theta \in L^\infty((0, T), \dot{H}^{-1/2}) \cap L^2((0, T), L^2) \cap C([0, T), CMO).
\]

Then \(\theta\) satisfies the equality in (2) and is the unique \(\dot{H}^{-1/2}\)-solution to \((QG)_{1/2}\) on \((0, T)\) with initial value \(\theta_0\).
This result raises the interesting question of the existence of a weak solution which belongs to \( \theta \in L^\infty((0, T), H^{-1/2}) \cap L^2((0, T), L^2) \cap C([0, T), \text{CMO}) \) for an initial data in \( H^{-1/2} \cap \text{CMO} \).

2 Some definitions and technical tools

We start to recall the definition of homogeneous Sobolev spaces:

Definition 2.1. Let \( s \) be a real number and \( f \) a tempered distribution such that \( \hat{f} \in L^1_{\text{loc}} \). We say that \( f \) belongs to the homogeneous Sobolev space \( \dot{H}^s \) if

\[
\|f\|_{\dot{H}^s} = \left( \int |\xi|^{2s} |\hat{f}|^2 \, d\xi \right)^{1/2} < \infty.
\]

We now turn to the definition of the \( \text{BMO} \) space and other related spaces:

Definition 2.2. A distribution \( f \in \mathcal{D}'(\mathbb{R}^n) \) is said to belong to the space \( \text{BMO}(\mathbb{R}^n) \) if \( f \) is locally integrable and

\[
\sup_{B \in \mathcal{B}} \frac{1}{|B|} \int_B |f - m_B f| \, dx < \infty
\]

where \( \mathcal{B} \) is the collection of all open balls in \( \mathbb{R}^n \) and \( m_B f = \int_B f(x) \, dx \).

When seen as a distribution space modulo the constants, \( \text{BMO}(\mathbb{R}^n) \) is a Banach space for the norm \( \|f\|_{\text{BMO}} = \sup_{B \in \mathcal{B}} \frac{1}{|B|} \int_B |f - m_B f| \, dx \). The space \( L^\infty \) is obviously embedded in \( \text{BMO} \). For other characterizations and further details on \( \text{BMO} \), the reader can consult [11] and [2].

Remark 2.1. A Calderón-Zygmund operator \( T \) which satisfies \( T(1) = 0 \) is bounded from \( \text{BMO} \) to \( \text{BMO} \) (see [11]); in particular the Riesz transforms satisfies this boundedness property.

We will also make use of the space \( \text{CMO} \):

Definition 2.3. The space \( \text{CMO}(\mathbb{R}^n) \) is the completion of \( \mathcal{D}(\mathbb{R}^n) \) in \( \text{BMO}(\mathbb{R}^n) \) and \( \text{C}_0 \) the completion of \( \mathcal{D}(\mathbb{R}^n) \) in \( L^\infty(\mathbb{R}^n) \) (i.e. the space of continuous functions which goes to zero at infinity).

In [2] and [17] we can find the following interesting characterizations of \( \text{CMO} \):

Theorem 2.1. Let \( f \in \text{BMO}(\mathbb{R}^n) \), the following assertions are equivalent:

(i) \( f \in \text{CMO}(\mathbb{R}^n) \)

(ii) there exists \( (f_0, ..., f_n) \in \text{C}_0(\mathbb{R}^n)^{n+1} \) such that \( f = f_0 + \sum_{k=1}^n R_k(f_k) \)

(iii) \( \lim_{r \to 0} \sup_{|B| \leq t} M(f, B) = 0 \), \( \lim_{r \to +\infty} \sup_{|B| \geq t} M(f, B) = 0 \) and \( \lim_{a \to -\infty} M(f, B + a) = 0 \) with

\[
M(f, B) = \frac{1}{|B|} \int_B |f - m_B f| \, dx
\]

and \( B \) any ball in \( \mathbb{R}^n \).

We will also make use of the Hardy space \( \mathcal{H}^1 \):

Definition 2.4. A function \( f \) belongs to the Hardy space \( \mathcal{H}^1(\mathbb{R}^n) \) if and only if \( f \) can be written as

\[
f = \sum_{j \in \mathbb{N}} \lambda_j a_j
\]

where the support of \( a_j \) is contained in a ball \( B(x_j, r_j), \|a_j\|_\infty \leq r_j^{-n}, \int a_j \, dx = 0 \)

and \( \sum_{j \in \mathbb{N}} |\lambda_j| < \infty \).
Remark 2.2. - The norm is then defined by \( \| f \|_{\mathcal{H}^1} = \inf \{ \sum_{j \in \mathbb{N}} |\lambda_j|/f = \sum_{j \in \mathbb{N}} \lambda_j a_j \} \) where the infimum is taken over all the atomic decompositions of \( f \). The space \( \mathcal{H}^1 \) is then a Banach space which is continuously embedded in \( L^1 \).

- The dual of the Hardy space \( \mathcal{H}^1 \) is the space \( BMO \).

We will use the following property which can be seen as a direct consequence of the commutator theorem of Coifman, Rochberg and Weiss [4].

**Proposition 2.2.** There exists a constant \( C \) such that
\[
\| f R_i f \|_{\mathcal{H}^1} \leq C \| f \|_{L^2}^2,
\]
for all \( f \in L^2(\mathbb{R}^2) \) and \( i \in \{1, 2\} \).

When dealing with fractional derivatives one may use of the very useful formula that can be found in [15] or more recently in [8]:

**Proposition 2.3.** Let \( 0 < \alpha < 2 \), there exists a constant \( c_\alpha > 0 \) such that for all function \( f \) in the nonhomogeneous Hölder space \( C^{\alpha+\epsilon} \), \( \epsilon > 0 \). We have the formula
\[
\Lambda^\alpha f(x) = c_\alpha \text{ p.v.} \int_{\mathbb{R}^n} \frac{\delta_y f(x)}{|y|^{n+\alpha}} dy
\]
for all \( x \in \mathbb{R}^n \) with \( \delta_y f(x) = f(x - y) - f(x) \).

**Remark 2.3.** When \( 0 < \alpha < 1 \) we don’t need the use of the principal value since the integral is then absolutely convergent.

The formula allows us to compute the fractional derivative of a product and get a substitute for the Leibniz formula:

**Corollary 2.4.** Let \( 0 < \alpha < 2 \) and \( f, g \) be two functions in \( C^{\alpha+\epsilon} \), \( \epsilon > 0 \), such that the product \( fg \) belongs to \( C^{\alpha+\epsilon} \). We have
\[
\Lambda^\alpha (fg) = f \Lambda^\alpha g + g \Lambda^\alpha f - C_s(f, g) \tag{4}
\]
with
\[
C_s(f, g) = c_\alpha \text{ p.v.} \int_{\mathbb{R}^n} \frac{(\delta_y f)(\delta_y g)}{|y|^{n+\alpha}} dy.
\]
3 Proof of theorem 1.3

We assume, without loss of generality, that \( k = 1 \). We begin to check that if 
\( \vartheta \in L^\infty((0, T), \dot{H}^{-1/2}) \cap L^2((0, T), L^2) \) is another solution on \((0, T)\) to \((QG)_{1/2}\) with initial data \( \theta_0 \), then we have the equality in \( \mathcal{D}'((0, T)) \):

\[
\partial_t \int \Lambda^{-1/2} \vartheta \vartheta \, dx + 2 \int \vartheta \partial \vartheta \, dx = \int \vartheta \mathcal{R}^\perp \vartheta \cdot \mathcal{R} \vartheta \, dx + \frac{\partial \mathcal{R}^\perp \vartheta}{\partial t} \cdot \mathcal{R} \vartheta \, dx. \tag{5}
\]

In order to prove this equality, we need to regularize \( \Lambda^{-1/2} \vartheta \) and \( \Lambda^{-1/2} \vartheta \); let \( \rho(t, x) \in \mathcal{D}(\mathbb{R}^3) \) be such that \( \int_{\mathbb{R}^3} \rho = 1 \) and \( \text{supp } \rho \subset [-1, 1] \times B(0, R) \) for a \( R > 0 \). We note \( \rho_\epsilon(t, x) = \frac{1}{\epsilon^3} \rho(\frac{t}{\epsilon^2}, \frac{x}{\epsilon}) \). For all \( t \in (\epsilon, T - \epsilon) \) we have,

\[
\partial_t \int (\rho_\epsilon \ast \Lambda^{-1/2} \vartheta)(\rho_\epsilon \ast \Lambda^{-1/2} \vartheta) \, dx = -2 \int \rho_\epsilon \ast \theta \rho_\epsilon \ast \vartheta \, dx + \int \rho_\epsilon \ast (\theta \mathcal{R}^\perp \vartheta) \cdot \rho_\epsilon \ast \mathcal{R} \vartheta \, dx + \int \rho_\epsilon \ast (\partial \mathcal{R}^\perp \vartheta) \cdot \rho_\epsilon \ast \mathcal{R} \vartheta \, dx.
\]

We now deal with the convergence of these integrals as \( \epsilon \) goes to 0; we clearly have

\[
\partial_t \int (\rho_\epsilon \ast \Lambda^{-1/2} \vartheta)(\rho_\epsilon \ast \Lambda^{-1/2} \vartheta) \, dx \to \partial_t \int \Lambda^{-1/2} \vartheta \Lambda^{-1/2} \vartheta \, dx
\]

and

\[
\int \rho_\epsilon \ast \theta \rho_\epsilon \ast \vartheta \, dx \to \int \theta \vartheta \, dx
\]

in \( \mathcal{D}'((0, T)) \).

For the the other integrals, we first write that the pointwise product maps \( BMO \cap L^2 \times BMO \cap L^2 \) to \( L^2 \) thus \( \rho_\epsilon \ast (\theta \mathcal{R}^\perp \vartheta) \) converges strongly to \( \theta \mathcal{R}^\perp \vartheta \) in \( L^2((0, T), L^2) \), this gives the convergence of one of the last two integrals.

For the other one, we write that \( \partial \mathcal{R}^\perp \vartheta \in L^1((0, T), \mathcal{H}^1) \) (see 2.2) while \( \mathcal{R} \vartheta \in L^\infty((0, T), BMO) \); thus \( \rho_\epsilon \ast (\partial \mathcal{R}^\perp \vartheta) \) converges strongly to \( \partial \mathcal{R}^\perp \vartheta \) in \( L^1((0, T), \mathcal{H}^1) \) and \( \mathcal{R} \vartheta \) converge weakly in \( L^\infty((0, T), BMO) \), this is enough to get the convergence of the last integral and (5) is proved.

Now, since \( \theta \mathcal{R}^\perp \vartheta \in L^2((0, T), L^2) \) lemma 4.3.2 gives that \( \vartheta \in \mathcal{C}_b([0, T], \dot{H}^{-1/2}) \), and

\[
t \to \int \Lambda^{-1/2} \vartheta \Lambda^{-1/2} \vartheta \, dx
\]

is therefore continuous.

Integrating (5) on \((0, t)\), we get

\[
\int \Lambda^{-1/2} \vartheta \Lambda^{-1/2} \vartheta \, dx + 2 \int_0^t \int \vartheta \partial \vartheta \, dx = \|\theta_0\|_{\dot{H}^{-1/2}}^2 + 2 \int_0^t \int \theta \mathcal{R}^\perp \vartheta \cdot \mathcal{R} \vartheta \, dx + 2 \int_0^t \int \partial \mathcal{R}^\perp \vartheta \cdot \mathcal{R} \vartheta \, dx
\]

\[= 6\]
which implies in particular that $\theta$ satisfies (2) with an equality.

We assume now that $\vartheta$ also satisfies (2). We then obtain,

$$
\|\theta - \vartheta\|_{H^{-1/2}}^2 + 2 \int_0^t |\theta - \vartheta|^2 dxds \leq -2 \int_0^t \left[ \theta \mathcal{R}^\perp \theta \cdot \mathcal{R} \vartheta + \vartheta \mathcal{R}^\perp \vartheta \cdot \mathcal{R} \theta \right] dxds.
$$

Then, using that $\mathcal{R}^\perp \theta \cdot \mathcal{R} \vartheta = -\mathcal{R} \theta \cdot \mathcal{R}^\perp \vartheta$ and $\mathcal{R}^\perp \theta \cdot \mathcal{R} \theta = 0$ we get,

$$
\|\theta - \vartheta\|_{H^{-1/2}}^2 + 2 \int_0^t |\theta - \vartheta|^2 dxds \leq 2 \int_0^t \left( (\theta - \vartheta) \mathcal{R}^\perp \theta \mathcal{R}(\theta - \vartheta) \right) dx.
$$

As before we have the inequality

$$
\left| \int_0^t (\theta - \vartheta) \mathcal{R}^\perp \theta \mathcal{R}(\theta - \vartheta) \right| dxds \leq \|\theta - \vartheta\|_{H^{-1/2}}^2 \|\mathcal{R}^\perp \theta\|_{L^\infty((0,T),BMO)} \|\mathcal{R}^\perp \theta\|_{L^\infty((0,T),L^\infty)}
$$

Then, if $\|\theta\|_{L^\infty((0,T),BMO)}$ is small enough, we obtain

$$
\|\theta - \vartheta\|_{H^{-1/2}}^2 \leq 0
$$

for all $t \in [0,T)$ which gives $\theta = \vartheta$ on $[0,T)$.

4 Proof of theorem 1.5

The fact that $\theta$ satisfies the energy inequality (2) is an immediate consequence of theorem 1.3. For all $T_0 < T$ and $\eta > 0$, by uniform continuity (on $[0,T_0]$) we may find $N$ such that

$$
\|\theta - \sum_{k=1}^N \chi_{[k/N,(k+1)/N]}(t)\theta(k/N,\cdot)\|_{L^\infty((0,T_0),BMO)} < \eta/2
$$

where $\chi_A$ is the characteristic function of the set $A$. Then, by density of test functions in CMO, we may approximate each $\theta(k/N,\cdot)$ by a function $\gamma_k \in H^3$ with $\|\theta(k/N,\cdot) - \gamma_k\|_{BMO} < \eta/2$. We define the function $\gamma$ by $\gamma = \sum_{k=1}^N \chi_{[k/N,(k+1)/N]}(t)\gamma_k$ and we get the decomposition $\theta = \gamma + \delta$ with $\gamma \in L^\infty((0,T_0),H^3)$ and $\delta \in L^\infty((0,T_0),BMO)$ with $\|\delta\|_{L^\infty((0,T_0),BMO)} < \eta$.

If $\vartheta$ is another $\dot{H}^{-1/2}$-solution to $(QG)_{1/2}$, we can write

$$
\|\theta - \vartheta\|_{H^{-1/2}}^2 + 2 \int_0^t |\theta - \vartheta|^2 dxds \leq 2 \int_0^t \left( \mathcal{R}^\perp \gamma \right)(\theta - \vartheta) \mathcal{R}(\theta - \vartheta) dx + 2 \int_0^t \left( \mathcal{R}^\perp \delta \right)(\theta - \vartheta) \mathcal{R}(\theta - \vartheta) dx.
$$

For all $t \leq T_0$, the integral involving $\delta$ is easy to deal with,

$$
\left| \int_0^t \mathcal{R}^\perp \delta(\theta - \vartheta) \mathcal{R}(\theta - \vartheta) dxds \right| \leq \|\mathcal{R}^\perp \delta\|_{L^\infty((0,T),BMO)} \|\theta - \vartheta\|_{H^1} \mathcal{R}(\theta - \vartheta) \|_{H^1}
$$

$$
\leq C \|\delta\|_{L^\infty((0,T),BMO)} \|\theta - \vartheta\|_{L^2((0,t),L^2)}^2.
$$
In order to estimate the other integral we use an approximation of the identity, \( \rho_t(x) = e^{-2x} \) with \( \rho \in D \) and \( \int \rho \, dx = 1 \). An integration by parts and the use of formula 2.4 allow us to write,

\[
- \int (R^\perp \gamma) \rho_t * (\theta - \vartheta) \rho_t * R(\theta - \vartheta) \, dx = \int (R^\perp \gamma) \rho_t * \nabla(\theta - \vartheta) \rho_t * \Lambda^{-1}(\theta - \vartheta) \, dx \\
= \int \rho_t * \Lambda^{1/2} R(\theta - \vartheta) \Lambda^{1/2} (R^\perp \gamma) \rho_t * \Lambda^{-1}(\theta - \vartheta) \, dx \\
= \int \rho_t * \Lambda^{1/2} R(\theta - \vartheta) (R^\perp \Lambda^{1/2} \gamma) \rho_t * \Lambda^{-1}(\theta - \vartheta) \, dx \\
\int \rho_t * \Lambda^{1/2} R(\theta - \vartheta) C_{1/2} (R^\perp \gamma, \rho_t * \Lambda^{-1}(\theta - \vartheta)) \, dx \\
= (I) + (II).
\]

For the first integral we have,

\[
|I| \leq \int_0^1 \| \rho_t * \Lambda^{1/2} R(\theta - \vartheta) \|_{\dot{H}^{-1/2}} \| (R^\perp \Lambda^{1/2} \gamma) \rho_t * \Lambda^{-1}(\theta - \vartheta) \|_{\dot{H}^{1/2}} \, ds \\
\leq C \| R^\perp \Lambda^{1/2} \gamma \|_{L^\infty((0,T),H^2)} (\int_0^1 \| \theta - \vartheta \|_{L^2}^2 \, ds)^{1/2} (\int_0^t \| \theta - \vartheta \|_{\dot{H}^{-1/2}}^2 \, ds)^{1/2}.
\]

where we used that \( H^2 \times \dot{H}^{1/2} \hookrightarrow \dot{H}^{1/2} \) (this can be easily seen with the use of paradifferential calculus for example). This gives the following estimate,

\[
|I| \leq \eta \int_0^t \| \theta - \vartheta \|_{L^2}^2 \, ds + \frac{C}{\eta} \| \gamma \|_{L^\infty((0,T),H^2)}^2 \int_0^t \| \theta - \vartheta \|_{\dot{H}^{-1/2}}^2 \, ds
\]

for all \( \eta > 0 \).

We deal with the integral \( (II) \) in the same way as for integral \( (I) \); therefore, we need to control \( \| C_{1/2} (R^\perp \gamma, \rho_t * \Lambda^{-1}(\theta - \vartheta)) \|_{\dot{H}^{1/2}} \) and write

\[
\| C_{1/2} (R^\perp \gamma, \rho_t * \Lambda^{-1}(\theta - \vartheta)) \|_{\dot{H}^{1/2}} \leq C \| \theta - \vartheta \|_{\dot{H}^{-1/2}} \int \| \delta y \gamma \|_{H^2} \, dy.
\]

For a real non-negative number \( A \), we split the integral in \( |y| \leq A \) and \( |y| \geq A \); using the fact that \( \| \delta y \gamma \|_{H^2} \leq |y| \| \gamma \|_{H^3} \) we get that

\[
\int \| \delta y \gamma \|_{H^2} \, dy \leq A^{1/2} \| \gamma \|_{H^3} + A^{-1/2} \| \gamma \|_{H^2}.
\]

Then, taking \( A = \frac{\| \gamma \|_{H^2}^2}{\| \gamma \|_{H^3}^2} \) we have

\[
\| C_{1/2} (R^\perp \gamma, \rho_t * \Lambda^{-1}(\theta - \vartheta)) \|_{\dot{H}^{1/2}} \leq C \| \gamma \|_{H^2}^{1/2} \| \gamma \|_{H^3}^{1/2} \| \theta - \vartheta \|_{\dot{H}^{-1/2}}.
\]
We obtain an estimate for integral (II),

\[
| \int_0^t (II) ds | \leq \eta \int_0^t \| \theta - \vartheta \|_{L^2}^2 ds + \frac{C}{\eta} \| \gamma \|_{L^\infty((0,T_0),H^3)}^2 \int_0^t \| \theta - \vartheta \|_{H^{-1/2}}^2 ds
\]

for all \( \eta > 0 \). Then, taking \( \eta \) small enough we get

\[
\| \theta - \vartheta \|_{H^{-1/2}} \leq \frac{C}{\eta} \| \gamma \|_{L^\infty((0,T_0),H^3)}^2 \int_0^t \| \theta - \vartheta \|_{H^{-1/2}}^2 ds
\]

for all \( t \in (0,T) \); the Gronwall lemma allows us to conclude that \( \theta = \vartheta \) on \([0,T_0)\).

Références


[12] Marchand, F., Existence and regularity of weak solutions to the Quasi-Geostrophic equations in the spaces \( L^p \) or \( \dot{H}^{-1/2} \), sumitted.


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