Measurable metrics, intrinsic metrics and Lipschitz functions

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In the paper [29], N. Weaver introduced the notion of measurable metric and that of Lipschitz function with respect to such a metric. The study of these notions was pursued by the same author in subsequent papers ([30, 32, · · ·]) and in the book [31].

In our paper [16], we treated various important examples and, in particular, we studied the intrinsic measurable metric associated with a local Dirichlet form and notably the case of Wiener spaces.

On the other hand, M. Hino and J.A. Ramirez ([15]) showed that the intrinsic measurable metric associated with a symmetric diffusion semigroup was strongly involved in the description of the Gaussian behavior in small time of such a semigroup. Moreover, we introduced in [17] the concentration function related to a measurable metric and studied the corresponding Gaussian concentration property.

In this paper, we shall give a survey of this set of results.

We first fix the framework and some notation.
All functions that we consider in this paper are supposed to be $\mathbb{R}$-valued.
In what follows, we consider a $\sigma$-finite measure space $(X, \mu)$. Spaces $L^\infty$ or $L^2$ are related to this measure space.
If $S$ is a set of classes (with respect to the $\mu$-a.e. equality) of measurable functions, $S$ has a least upper bound or essential supremum which is a class denoted by $\bigvee S$, and a greatest lower bound or essential infimum denoted by $\bigwedge S$.
We denote by $\Omega$ the collection of all positive measure subsets of $X$. If $A$ and $B$ belong to $\Omega$, we denote by $A \sim B$ the fact that $A$ and $B$ only differ from a null set ($\mu(A \setminus B) = \mu(B \setminus A) = 0$).

If $h$ is any function on $X^2$ and if $A$ and $B$ belong to $\Omega$, we denote by

$$\text{ess inf}\{h(x, y) : x \in A, x \in B\}$$

the greatest $C \in \mathbb{R}$ such that $C \leq h(x, y)$ for $\mu$-almost every $x \in A$ and for $\mu$-almost every $y \in B$. 

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1 Measurable metrics

In [29], N. Weaver introduced the following definition.

**Definition 1.1** A measurable pseudometric is a map \( \rho : \Omega^2 \rightarrow [0, \infty] \) such that

1. \( A' \sim A \implies \rho(A', B) = \rho(A, B) \)
2. \( \rho(A, A) = 0 \)
3. \( \rho(A, B) = \rho(B, A) \)
4. \( \rho(\bigcup_{n=1}^{\infty} A_n, B) = \inf_{n \geq 1} \rho(A_n, B) \)
5. for any \( A, B, C \), \( \rho(A, C) \leq \sup_{B' \subset B} (\rho(A, B') + \rho(B', C)) \)

where all subsets which appear above belong to \( \Omega \).

We now give two classes of examples which will be of special interest.

1.1 Example

Let \( \rho_0 \) be a map from \( X^2 \) into \([0, \infty]\) satisfying, for any \( x, y, z \in X \),

\[
\rho_0(x, x) = 0, \quad \rho_0(x, y) = \rho_0(y, x) \quad \text{and} \quad \rho_0(x, z) \leq \rho_0(x, y) + \rho_0(y, z).
\]

Such a map \( \rho_0 \) will be called a pointwise semimetric.

For \( A, B \in \Omega \), we set

\[
\rho_0(A, B) = \inf\{\rho_0(x, y) : x \in A, y \in B\},
\]

and

\[
\rho(A, B) = \sup\{\rho_0(A', B') : A' \sim A, B' \sim B\}.
\]

We also have

\[
\rho(A, B) = \text{ess inf}\{\rho_0(x, y) : x \in A, y \in B\}.
\]

For \( A \subset X \), we denote by \( \rho_0^A \) the function

\[
\rho_0^A(x) = \inf\{\rho_0(x, y) : y \in A\}.
\]

Generally, \( \rho \) satisfies conditions 1 to 4 of Definition 1.1, but, without additional assumption, condition 5 is not satisfied.

**Proposition 1.2 ([16, Proposition 2.3.])** Suppose that \( \rho_0 \) satisfies the following additional assumption:

For all \( A \in \Omega \), there exists \( A' \in \Omega \) with \( A' \subset A \), \( A' \sim A \) and \( \rho_0^{A'} \) measurable.

Then \( \rho \) is a measurable pseudometric which will be called associated with \( \rho_0 \).
The hypothesis in the previous statement is in particular satisfied under the following assumptions: $X$ is a topological space, $\rho_0$ is a lower semicontinuous function on $X \times X$, and $\mu$ is a Borel measure which is inner regular, which means that, for any Borel set $A$,

$$\mu(A) = \sup\{\mu(K) : K \text{ compact and } K \subseteq A\}.$$ 

In particular, if $(X, \mu)$ satisfies these conditions and if $\Phi$ denotes a set of real continuous functions on $X$, we can take

$$\rho_\Phi^0(x, y) = \sup\{|u(x) - u(y)| : u \in \Phi\}$$

and the associated measurable pseudometric will be denoted by $\rho^\Phi$.

1.2 Example

We first introduce a notation which will be used in all what follows.

Let $u$ be a $\mu$-class of measurable real functions. If $A \in \Omega$, we denote by $F_A(u)$ the support of the image measure $(u_A)_*(\mu_A)$, where the index $A$ indicates the restriction to $A$. This support $F_A(u)$ also is the essential image of the restriction $u_A$ of $u$ to $A$. We set, for $A, B \in \Omega$,

$$\rho_u(A, B) = d(F_A(u), F_B(u))$$

where we denote by $d$ the usual distance in $\mathbb{R}$:

$$\rho_u(A, B) = \inf\{|x - y| : x \in F_A(u), y \in F_B(u)\}.$$

If we set, for a particular representative $\tilde{u}$ of $u$,

$$\rho_\tilde{u}^0(x, y) = |\tilde{u}(x) - \tilde{u}(y)|,$$

then $\rho_u$ is nothing but the map $\rho$ associated with $\rho_\tilde{u}^0$ by the method of the previous example 1.1:

$$\rho_u(A, B) = \text{ess inf}\{|\tilde{u}(x) - \tilde{u}(y)| : x \in A, y \in B\}.$$ 

It is easy to see that the map $\rho_u$ is a measurable pseudometric.

Consider now a collection $\Psi$ of classes and set

$$\rho_\Psi(A, B) = \sup\{\rho_u(A, B) : u \in \Psi\}.$$ 

Then, the map $\rho_\Psi$ satisfies properties 1, 2, 3 and 5 of Definition 1.1. But generally, condition 4 is not satisfied.

The following proposition is essentially [31, Example 6.2.5] (see [16]).

**Proposition 1.3** Let $F$ be a subspace of $L^\infty$ containing the constant function $1$, and let $\| \cdot \|_0$ be a seminorm on $F$ such that $\|1\|_0 = 0$. We assume that, for any $M, N \geq 0$, for any subset $S$ of the set

$$F_{M,N} = \{f \in F : \|f\|_\infty \leq M \text{ and } \|f\|_0 \leq N\},$$

the essential supremum $\bigvee S$ belongs to $F_{M,N}$. Then, denoting by $\Psi$ the set $\{f \in F : \|f\|_0 \leq 1\}$, $\rho_\Psi$ is a measurable pseudometric.

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In the proof of this result, the following important lemma, which is a reformulation of [31, Lemma 6.2.4]) is useful.

**Lemma 1.4** Let $F$ be as above. Then, for any $f \in F$, for any $A, B \in \Omega$, there exists $g \in F$ such that

$$
\|g\|_0 \leq \|f\|_0, \quad 0 \leq g \leq \rho_f(A, B), \quad \rho_g(A, B) = \rho_f(A, B), \quad g = 0 \text{ on } A, \quad g = \rho_f(A, B) \text{ on } B.
$$

### 2 Lipschitz functions

We henceforth consider a measurable pseudometric $\rho$, on the $\sigma$-finite measure space $(X, \mu)$.

**Definition 2.1** Let $u$ be a $\mu$-class of measurable real functions. Then $u$ is said to be a $\rho$-Lipschitz function if there exists a constant $C \geq 0$ such that

$$
\forall A, B \in \Omega \quad \rho_u(A, B) \leq C \rho(A, B).
$$

In this case,

$$
L_\rho(u) = \sup\{\rho_u(A, B)/\rho(A, B) : A, B \in \Omega \text{ and } \rho(A, B) > 0\}
$$

is finite and called the Lipschitz constant of $u$.

We shall denote by $\text{Lip}(\rho)$ the set of all $\rho$-Lipschitz functions, and by $\text{Lip}^\infty(\rho)$ (resp. $\text{Lip}^2(\rho)$) the set $\text{Lip}(\rho) \cap L^\infty$ (resp. $\text{Lip}(\rho) \cap L^2$).

**Definition 2.2** The measurable pseudometric $\rho$ is called a measurable metric if $\text{Lip}^\infty(\rho)$ is weak*-dense in $L^\infty$.

For an equivalent definition, we refer to [32, Proposition 14].

The following statement comes from [31]. Notice that the proof is much less obvious than that of the similar results for classical Lipschitz functions. Actually, the proof in [31] has to be modified ([33]).

**Proposition 2.3 ([31, Theorem 6.2.7])** Let $F = \text{Lip}^\infty(\rho)$ and, for $f \in F$, set $\|f\|_0 = L_\rho(f)$. Then

1. $F$ is a subalgebra of $L^\infty$ containing the constant function $1$, and $\|\cdot\|_0$ is a seminorm on $F$ such that $\|1\|_0 = 0$. Moreover, for any $f, g \in F$

$$
\|fg\|_0 \leq \|f\|_L^\infty \|g\|_0 + \|g\|_L^\infty \|f\|_0.
$$

2. $F$ equipped with the norm

$$
\|f\|_\rho = \|f\|_L^\infty + \|f\|_0
$$

is a Banach space.

3. For any $M, N \geq 0$, for any subset $S$ of the set

$$
F_{M,N} = \{f \in F : \|f\|_L^\infty \leq M \text{ and } \|f\|_0 \leq N\},
$$

the essential supremum $\bigvee S$ belongs to $F_{M,N}$.
Consequently, the space \( F = \text{Lip}^\infty(\rho) \) equipped with the seminorm \( \| \cdot \|_0 = L_\rho \) satisfies the hypotheses of Proposition 1.3. Actually, we also have the following converse.

**Proposition 2.4 ([16, Proposition 2.10.])** Suppose that a subspace \( F \) of \( L^\infty \) equipped with a seminorm \( \| \cdot \|_0 \) satisfies the hypotheses of Proposition 1.3. Then

\[
F = \text{Lip}^\infty(\rho_\Psi) \quad \text{and} \quad \forall f \in F \quad \|f\|_0 = L_\rho(\Psi(f)),
\]

where the measurable pseudometric \( \rho_\Psi \) is that which is defined in the statement of Proposition 1.3.

In what follows, if \( B \subset X \), we denote by \( 1_B \) the indicator function of \( B \).

The following result also comes from [31].

**Proposition 2.5 ([31, Lemma 6.2.8])** Let \( A \in \Omega \) and \( a > 0 \). Set

\[
\rho_a^A = \bigvee \left\{ \min(\rho(A, B), a) 1_B : B \in \Omega \right\}.
\]

Then \( \rho_a^A \in \text{Lip}^\infty(\rho) \) and \( L_\rho(\rho_a^A) \leq 1 \). Moreover, \( \rho_a^A(x) = 0 \) almost everywhere on \( A \).

We then get easily the following Corollary ([17]).

**Corollary 2.6** For any \( A, B \in \Omega \),

\[
\rho(A, B) = \sup \{ f(A, B) : f \in \text{Lip}^\infty(\rho) \text{ and } L_\rho(f) \leq 1 \}.
\]

By previous Proposition 2.3 and Corollary 2.6, we therefore see that any measurable pseudometric may be defined by the method of Proposition 1.3.

We now introduce as in [17] the distance function to a subset.

**Definition 2.7** Set, for \( A \in \Omega \),

\[
\rho_A = \bigvee \{ \rho(A, B) 1_B : B \in \Omega \}.
\]

The class \( \rho_A \) is called the *distance function to \( A \).*

It is easy to verify that, for any \( a > 0 \), \( \rho_a^A = \rho_A \wedge a \).

The characterization below shows in particular that this distance function coincides with that introduced in [15] in the framework of local Dirichlet forms.

**Proposition 2.8 ([17, Theorem 2.7])** Let \( A \in \Omega \). Then

\[
\rho_A = \bigvee \{ f : f \in \text{Lip}^\infty(\rho), L_\rho(f) \leq 1, f \geq 0, f(x) = 0 \text{ a.e. on } A \}.
\]

Moreover, \( \rho_A \) is the unique class \( \varphi \) satisfying the following properties:

1. \( \varphi \geq 0 \) and \( \varphi(x) = 0 \) a.e. on \( A \)
2. \( \forall B \in \Omega, \text{ess inf} \{ \varphi(x) : x \in B \} = \rho(A, B) \)
3. \( \forall a > 0, (\varphi \land a) \in \text{Lip}(\rho) \) and \( L_{\rho}(\varphi \land a) \leq 1 \).

The following Proposition gives a necessary and sufficient condition in order to \( \rho_A \) itself be a Lipschitz function.

**Proposition 2.9 ([17, Proposition 2.8])** Let \( A \in \Omega \). Then \( \rho_A(x) < +\infty \) a.e. if and only if, for any \( B \in \Omega \), \( \rho(A, B) < +\infty \). If this is satisfied, \( \rho_A \in \text{Lip}(\rho) \) and \( L_{\rho}(\rho_A) \leq 1 \).

In fact, any \( \rho \)-Lipschitz function may be described in terms of the distance function as it is shown in the following proposition (see [31, Theorem 6.2.9]).

**Proposition 2.10** Let \( f \in \text{Lip}(\rho) \) and \( C \geq L_{\rho}(f) \). Then

\[
f = \bigvee \{ f_A - C \rho_A : A \in \Omega \},
\]

where \( f_A \) denotes the essential infimum of \( f(x) \) for \( x \in A \).

We now consider the situation of Example 1.1. We assume that \( \rho_0 \) is a pointwise semi-metric satisfying the hypothesis of Proposition 1.2 and that \( \rho \) is the associated measurable pseudometric. We then have:

**Proposition 2.11 ([17, Proposition 2.9])** For \( A \in \Omega \),

\[
\rho_A = \bigvee \{ \rho_0^{A'} : A' \subset A, A' \sim A, \rho_0^{A'} \text{ measurable} \}.
\]

As a consequence, we have the following corollary.

**Corollary 2.12** For any \( A \in \Omega \), there exists \( A' \subset A \) such that \( A' \sim A \), \( \rho_0^{A'} \) is measurable and \( \rho_0^{A'} \) is a representative of \( \rho_A \).

We now recall an elementary definition.

**Definition 2.13** A real valued function \( h \) is said to be \( \rho_0 \)-Lipschitz continuous if \( h \) is measurable and if there exists \( C \geq 0 \) such that

\[
\forall x, y \in X \quad |h(x) - h(y)| \leq C \rho_0(x, y).
\]

For such a function \( h \), the \( \rho_0 \)-Lipschitz constant \( L_{\rho_0}(h) \) is the smallest constant \( C \) satisfying the above inequalities.

Obviously, if for some \( A \in \Omega \) \( \rho_0^A \) is measurable, then for any \( a \geq 0 \), \( \rho_0^A \land a \) is \( \rho_0 \)-Lipschitz continuous and its \( \rho_0 \)-Lipschitz constant is \( \leq 1 \). Therefore, we can obtain the following consequence of Proposition 2.10 and Corollary 2.12.

**Theorem 2.14** If \( f \in \text{Lip}^\infty(\rho) \), there exists a bounded \( \rho_0 \)-Lipschitz continuous representative \( \tilde{f} \) of \( f \) such that \( L_{\rho}(f) = L_{\rho_0}(\tilde{f}) \). As a consequence, if \( f \in L^\infty \), \( f \in \text{Lip}^\infty(\rho) \) if and only if \( f \) has a \( \rho_0 \)-Lipschitz continuous representative, and then

\[
L_{\rho}(f) = \min \{ L_{\rho_0}(h) : h \ \rho_0 \text{-Lipschitz continuous representative of } f \}.
\]
3 Dirichlet forms

We consider in this section a Dirichlet structure \((X, \mathcal{B}, \mu, \mathcal{D}, \mathcal{E})\) in the sense of [3] to which we refer for the main definitions, properties and examples. In particular, \(\mu\) is as before a \(\sigma\)-finite measure on \(X\) equipped with the \(\sigma\)-algebra \(\mathcal{B}\), and \(\mathcal{E}\) is a Dirichlet form on \(L^2\) whose domain is \(\mathcal{D}\). We assume that the Dirichlet form \(\mathcal{E}\) is local (which means e.g.
\[
\forall f, g \in \mathcal{D} \forall a \in \mathbb{R} \quad (f + a)g = 0 \implies \mathcal{E}(f, g) = 0
\]
and there exists a carré du champ operator \(\Gamma\) (which means that there exists a continuous map \(\Gamma : \mathcal{D} \times \mathcal{D} \to L^1\) such that
\[
\forall f, g, h \in \mathcal{D} \cap L^\infty \quad \mathcal{E}(fh, g) + \mathcal{E}(gh, f) - \mathcal{E}(fg, h) = \int h \Gamma(f, g) d\mu.
\]
In what follows, we write \(\Gamma(f)\) instead of \(\Gamma(f, f)\). We set
\[
\mathcal{D}^\infty = \{ f \in \mathcal{D} \cap L^\infty : \Gamma(f) \in L^\infty \}.
\]
All the results of this section come from [16].

3.1 Case 1 \(\in \mathcal{D}\)

We first assume that the constant function \(1\) belongs to \(\mathcal{D}\). This implies of course that the measure \(\mu\) is finite.

**Proposition 3.1** The space \(F = \mathcal{D}^\infty\), equipped with the seminorm \(\|f\|_0 = \|\Gamma(f)^{1/2}\|_{L^\infty}\) satisfies the assumptions of Proposition 1.3.

We now define the intrinsic measurable pseudometric \(\rho\) by
\[
\forall A, B \in \Omega \quad \rho(A, B) = \sup\{\rho_f(A, B) : f \in \mathcal{D}^\infty \text{ and } \Gamma(f) \leq 1\}.
\]
Then, by Proposition 1.3 and Proposition 2.4, we have:

**Corollary 3.2** \(\rho\) is a measurable pseudometric, \(\text{Lip}^\infty(\rho) = \mathcal{D}^\infty\) and, for any \(f \in \mathcal{D}^\infty\),
\[
L_\rho(f) = \|\Gamma(f)^{1/2}\|_{L^\infty}.
\]
We also have:

**Corollary 3.3**
\[
\text{Lip}^2(\rho) = \{ f \in \mathcal{D} : \Gamma(f) \in L^\infty \}
\]
and, for any \(f \in \text{Lip}^2(\rho)\), \(L_\rho(f) = \|\Gamma(f)^{1/2}\|_{L^\infty}\). Moreover, for any \(A, B \in \Omega\),
\[
\rho(A, B) = \sup\{\rho_f(A, B) : f \in \mathcal{D} \text{ and } \Gamma(f) \leq 1\}.
\]
3.2 General case

We no longer assume $1 \in D$. However, we may use the following fact: There exists $\phi \in D$ such that $\phi(x) > 0$ $\mu$-a.e. This is easy to see, using the $\sigma$-finiteness of $\mu$ and the density of $D$ in $L^2$. Let now $\varphi_p = (p \phi) \wedge 1$ and $A_p = \{\varphi_p = 1\}$. Then $\varphi_p \in D$, $0 \leq \varphi_p \leq 1$ and $\cup_p A_p = X$ a.e. We set

$$\tilde{D} = \{f \in L^\infty : \forall \varphi \in D \varphi f \in D\}.$$ 

Then, $\tilde{D} \subset D_{loc}$, where the space $D_{loc}$ of functions locally in $D$ is defined in [3]: Actually, if $f \in \tilde{D}$, then $f = \varphi_p f$ on $A_p$, $\varphi_p f \in D$ and $\cup_p A_p = X$ a.e. By the hypothesis of locality, the carré du champ operator can be extended to $D_{loc} \times D_{loc}$ and therefore to $\tilde{D} \times \tilde{D}$ by setting

$$\forall f, g \in \tilde{D} \quad \Gamma(f, g) = \Gamma(\varphi f, \psi g)$$

on $\{\varphi = 1\} \cap \{\psi = 1\}$ for any $\varphi, \psi \in D$. We then set

$$\tilde{D}^\infty = \{f \in \tilde{D} : \Gamma(f) \in L^\infty\}$$

where, as previously, $\Gamma(f)$ is put for $\Gamma(f, f)$.

**Proposition 3.4** $1 \in \tilde{D}^\infty$, $D^\infty \subset \tilde{D}^\infty$ and, if $1 \in D$, $D^\infty = \tilde{D}^\infty$.

**Proposition 3.5** The space $F = \tilde{D}^\infty$, equipped with the seminorm $\|f\|_0 = \|\Gamma(f)^{1/2}\|_{L^\infty}$ satisfies the assumptions of Proposition 1.3.

We now define again the intrinsic measurable pseudometric $\rho$ by

$$\forall A, B \in \Omega \quad \rho(A, B) = \sup\{\rho_f(A, B) : f \in \tilde{D}^\infty \text{ and } \Gamma(f) \leq 1\}.$$ 

Using again Proposition 1.3 and Proposition 2.4, we obtain:

**Corollary 3.6** $\rho$ is a measurable pseudometric, $\text{Lip}^\infty(\rho) = \tilde{D}^\infty$ and, for any $f \in \tilde{D}^\infty$,

$$L_\rho(f) = \|\Gamma(f)^{1/2}\|_{L^\infty}.$$ 

3.3 Classical intrinsic metric

Consider again the situation of the previous subsection and suppose moreover that $X$ is a topological space and that $\mu$ is an inner regular Borel $\sigma$-finite measure. We set

$$\Phi = \{u \in D \cap C : \Gamma(u) \leq 1\}$$

where $C$ denotes the set of real continuous functions on $X$. As in Subsection 1.1, we define the pointwise semimetric

$$\rho^\Phi_0(x, y) = \sup\{|u(x) - u(y)| : u \in \Phi\}.$$ 

This is the usual intrinsic semimetric (see for instance [4, 5]). We also may consider the measurable pseudometric $\rho^\Phi$ associated to $\rho^\Phi_0$ in the sense of Subsection 1.1. It is an open problem in general, to compare $\rho^\Phi$ with the intrinsic measurable pseudometric $\rho$ defined in the previous subsection 3.2. K.-Th. Sturm gives in [25, 26] conditions for the equality in the case of classical regular Dirichlet forms (on a locally compact space). In the next section, we also study this problem in the case of Wiener spaces.
4 Abstract Wiener spaces

We consider in this section the particular setting of an abstract Wiener space \((X, H, \mu)\) in the sense of L. Gross ([13]). We recall that, in particular, \(X\) is a separable Banach space, \(\mu\) is a centered Gaussian probability measure on \(X\) with support \(X\), and \((H, \|\cdot\|_H)\) is a Hilbert space compactly embedded in \(X\), which is called the Cameron-Martin space. We refer to [3] for a precise definition and for the notation we adopt here. Actually, we could take the more general framework considered in [8].

As shown in [3], there is a canonical Dirichlet structure \((X, \mathcal{B}, \mu, D, E)\), where \(\mathcal{B}\) denotes the Borel \(\sigma\)-algebra of \(X\), which is local and admits a carré du champ \(\Gamma\). Moreover, \(1 \in D\) and if we set \(D_0 = \{f(l_0, \ldots, l_n) : n \in \mathbb{N}, l_0, \ldots, l_n \in X'\text{ and } f \in C_b^\infty(\mathbb{R}^{n+1})\}\), where \(C_b^\infty(\mathbb{R}^{n+1})\) denotes the set of \(C^\infty\)-functions on \(\mathbb{R}^{n+1}\) which are bounded as well as all their derivatives, then \(D_0 \subset D_\infty\) and \(D_0\) is dense in \(L^2\). As a consequence, \(D_\infty\) is weak*-dense in \(L^\infty\).

The results of Subsection 3.1 are available and we can define the intrinsic measurable metric:

\[
\rho(A, B) = \sup\{\rho_f(A, B) : f \in D_\infty\text{ and } \Gamma(f) \leq 1\} = \sup\{\rho_f(A, B) : f \in D\text{ and } \Gamma(f) \leq 1\}.
\]

According to Corollary 3.2 and Corollary 3.3,

\[
\text{Lip}_\infty(\rho) = D_\infty, \text{ Lip}_2(\rho) = \{f \in D : \Gamma(f) \in L^\infty\},
\]

and for any \(f \in D\) such that \(\Gamma(f) \in L^\infty\), \(L_\rho(f) = \|\Gamma(f)^{1/2}\|_{L^\infty}\).

4.1 \(\mu\)-a.e. \(H\)-Lipschitz continuous functions

In [6], O. Enchev and D.W. Stroock introduced another notion of Lipschitz function (see also [19, 20]):

**Definition 4.1** Let \(f\) be a \(\mu\)-class of real Borel functions. Then \(f\) is said \(\mu\)-a.e. \(H\)-Lipschitz continuous if there is a representative \(\tilde{f}\) of \(f\) and a real constant \(C\) such that

\[
\forall x \in X \forall h \in H \quad |\tilde{f}(x + h) - \tilde{f}(x)| \leq C \|h\|_H.
\]

In this case, the \(H\)-Lipschitz constant of \(f\) is defined by

\[
L_H(f) = \text{ess sup} \{ \sup_{h \in H \setminus \{0\}} \frac{1}{\|h\|_H} |\tilde{f}(x + h) - \tilde{f}(x)| : x \in X\}.
\]

The main result of [6] is the following:

**Theorem 4.2** Let \(f \in L^2\). Then \(f\) is \(\mu\)-a.e. \(H\)-Lipschitz continuous if and only if \(f \in D\) and \(\Gamma(f) \in L^\infty\).

In this case, \(L_H(f) = \|\Gamma(f)^{1/2}\|_{L^\infty}\).

A direct consequence of the above theorem and Corollary 3.3 is the following:

**Corollary 4.3**

\[
\text{Lip}_2(\rho) = \{f \in L^2 : f \text{ is } \mu\text{-a.e. } H\text{-Lipschitz continuous}\}
\]

and, if \(f \in \text{Lip}_2(\rho)\), then \(L_\rho(f) = L_H(f)\).
4.2 $H$-metric

We shall now give an equivalent definition of the intrinsic metric in terms of the Cameron-Martin space. We set, for $x, y \in X$,

$$
\rho_H(x, y) = \begin{cases} 
|y - x|_H & \text{if } x - y \in H \\
+\infty & \text{otherwise}
\end{cases}
$$

and, for $B \subset X$, $\rho_H^B(x) = \inf\{\rho_H(x, y) : y \in B\}$.

This metric was first considered by S. Kusuoka ([20]). In [7], S. Fang proved (see also [9]) the following result.

**Proposition 4.4** Let $B$ be a $K_\sigma$ (i.e. a countable union of compact sets) of $X$, with positive measure. Then $\rho_H^B \in L^2$.

Moreover, we have the following easy result:

**Proposition 4.5** Let $B$ be as in the previous proposition. Then $\rho_H^B$ is $\mu$-a.e. $H$-Lipschitz continuous and $L_H(\rho_H^B) \leq 1$.

We now define, like in Subsection 1.1, for $A, B \in \Omega$,

$$
\rho_H(A, B) = \inf\{\rho_H(x, y) : x \in A, y \in B\},
$$

and

$$
\rho_H^L(A, B) = \sup\{\rho_H(A', B') : A' \sim A, B' \sim B\}.
$$

By the fact that, for any $B$ a $K_\sigma$ of positive measure, $\rho_H^B$ is a Borel function, and by Proposition 1.2, $\rho_H$ is a measurable pseudometric. Actually, $\mu$ is a Borel probability measure on the Polish space $X$ and $\rho_H$ is lower semicontinuous on $X \times X$. Therefore, the remark following Proposition 1.2 applies. Using Theorem 4.2 and Propositions 4.4 and 4.5 above, one can show the following result.

**Theorem 4.6 ([16, Theorem 4.6.])** For any $A, B \in \Omega$,

$$
\rho_H(A, B) = \rho(A, B).
$$

Notice that in view of this theorem, Corollary 4.3 appears as a particular case of Theorem 2.14.

We deduce directly from Corollary 4.3 and Theorem 4.6 the following Corollary.

**Corollary 4.7** Let $f \in L^2$. Then $f$ is $\mu$-a.e. $H$-Lipschitz continuous if and only if there exists a constant $C \geq 0$ such that

$$
\forall A, B \in \Omega \quad \rho_f(A, B) \leq C \rho_H(A, B).
$$

Then,

$$
L_H(f) = \sup\{\rho_f(A, B)/\rho_H(A, B) : A, B \in \Omega \text{ and } \rho_H(A, B) > 0\}.
$$
Finally, the next proposition states that the $H$-metric $\rho_H$ is also the pointwise intrinsic semimetric, in the sense of Subsection 3.3, associated with the Dirichlet space $\mathcal{D}$. Therefore in this case, the intrinsic measurable metric $\rho$ also is the measurable pseudometric associated with the pointwise intrinsic semimetric.

**Proposition 4.8 ([16, Proposition 4.8.])** For every $x, y \in X$,

$$\rho_H(x, y) = \sup\{|u(x) - u(y)| : u \in \mathcal{D} \cap \mathcal{C} \text{ and } \Gamma(u) \leq 1\},$$

where $\mathcal{C}$ denotes the space of continuous functions on $X$.

## 5 Gaussian estimates

We consider in this section the general framework and the notation of Subsection 3.2. In particular, $\rho$ denotes the intrinsic measurable pseudometric associated to a local Dirichlet form $\mathcal{E}$ with domain $\mathcal{D}$ and admitting a carré du champ operator $\Gamma$. There exists an $L^2$-strongly continuous semigroup $(P_t)_{t \geq 0}$ consisting of sub-Markovian symmetric operators such that

$$\forall f \in \mathcal{D} \quad \mathcal{E}(f) = \lim_{t \to 0} \frac{1}{t} ((I - P_t)f, f)_{L^2}.$$

We denote by $\lambda$ the bottom of the spectrum of $-A$, where $A$ is the infinitesimal generator of $(P_t)_{t \geq 0}$:

$$\lambda = \inf \left\{ \frac{\mathcal{E}(u)}{\|u\|^2_{L^2}} : u \in \mathcal{D}, u \neq 0 \right\}.$$

We first give integrated upper Gaussian estimates. The method to obtain such estimates is wellknown and is probably due to M.P. Gaffney ([12]). In our general situation, we refer to [26, Theorem 1.2] (see also [15]). Actually, in [26] as well as in [15], the existence of a carré du champ operator is not assumed. The proof of Theorem 5.2 below is based on the following proposition.

**Proposition 5.1** Let $f \in L^2$ and $\psi \in \tilde{D}\infty$. Assume $\Gamma(\psi) \leq 2\gamma$. Then, for every $t \geq 0$,

$$\|e^{\psi} P_t f\|_{L^2} \leq e^{(\gamma - \lambda) t} \|e^{\psi} f\|_{L^2}.$$  

**Theorem 5.2** Suppose that $A$ and $B$ belong to $\Omega$ and $\mu(A) < \infty$, $\mu(B) < \infty$. Then, for all $t > 0$,

$$(P_t 1_A, 1_B)_{L^2} \leq (\mu(A) \mu(B))^{1/2} \exp \left(-\frac{\rho(A, B)^2}{2t} - \lambda t\right).$$

**Corollary 5.3** Suppose that $A$ and $B$ belong to $\Omega$ and $\mu(A) < \infty$, $\mu(B) < \infty$. Then,

$$\rho(A, B) = +\infty \implies \forall t > 0 \quad (P_t 1_A, 1_B)_{L^2} = 0.$$
If in particular, $\mu$ is a probability measure and $(P_t)_{t \geq 0}$ is ergodic (this is actually the case in the situation of Section 4), then

$$\lim_{t \to +\infty} (P_t 1_A, 1_B)_{L^2} = \mu(A) \mu(B),$$

and hence, for any $A, B \in \Omega$, $\rho(A, B) < +\infty$. Proposition 2.9 may then apply.

We now assume that $\mu$ is a probability measure and $1 \in D$. The Gaussian behavior in small time of the semigroup $(P_t)_{t \geq 0}$ may be precisely described in terms of the intrinsic measurable pseudometric $\rho$ and of the distance functions $\rho_A$ associated with $\rho$:

**Theorem 5.4 ([15, Theorem 1.1])** For any $A$ and $B$ in $\Omega$,

$$\lim_{t \to 0} \{2t \log((P_t 1_A, 1_B)_{L^2})\} = -\rho(A, B)^2.$$

**Theorem 5.5 ([15, Theorem 1.3])** Let $A \in \Omega$. For any $t > 0$,

$$\{P_t 1_A = 0\} \sim \{\rho_A = +\infty\}.$$

Moreover, $2t \log(P_t 1_A)$ converges in probability to $-\rho_A^2$ on the set $\{\rho_A < +\infty\}$.

### 6 Gaussian Concentration

We assume, in this section, that $\mu$ is a probability measure on $X$. As before, $\rho$ denotes a measurable pseudometric on $(X, \mu)$. The results presented in this section are essentially taken from [17].

#### 6.1 Concentration function

We begin with a definition.

**Definition 6.1** For $A \in \Omega$ and $r > 0$, we denote by $A_r$ the class (with respect to the relation $\sim$) $\{\rho_A < r\}$. We then define the concentration function $\alpha$ (related to the triplet $(X, \mu, \rho)$) by

$$\forall r > 0 \quad \alpha(r) = 1 - \inf \{\mu(A_r) : A \in \Omega, \mu(A) \geq 1/2\}.$$

Consider first the particular case described in Subsection 1.1: The measurable pseudometric $\rho$ is supposed to be associated with a pointwise semimetric $\rho_0$ satisfying the hypothesis of Proposition 1.2. For $A \subset X$ and $r > 0$, we set $A_r^0 = \{\rho_0^A < r\}$. We then define the “classical” concentration function $\alpha_0$ (related to $(X, \mu, \rho_0)$) by

$$\forall r > 0 \quad \alpha_0(r) = 1 - \inf \{\mu_*(A_r^0) : A \in \Omega, \mu(A) \geq 1/2\},$$

where $\mu_*$ denotes, for example, the interior measure. The next proposition shows that this classical concentration function coincides with the concentration function in the sense of Definition 6.1.
Proposition 6.2 If \( \alpha \) is the concentration function related to \( (X, \mu, \rho) \), where \( \rho \) is the measurable pseudometric associated with \( \rho_0 \), then \( \alpha_0 = \alpha \).

We now come back to the general situation. The following proposition is the analogue, in the general setting, of a classical result (see for example [1, Proposition 7.2.5]).

Proposition 6.3 We have

\[
\alpha(r) = \sup \{ \mu(F \geq m_F + r) : F \in \text{Lip}(\rho), L_{\rho}(F) \leq 1, m_F \text{ median of } F \}
\]

and we may replace above \( \text{Lip}(\rho) \) by \( \text{Lip}^\infty(\rho) \).

We finish this subsection with the definition of the Gaussian concentration property.

Definition 6.4 Let \( c \) be a positive constant. We say that the triplet \( (X, \mu, \rho) \) satisfies the Gaussian concentration property denoted by \( G(c) \) if the associated concentration function \( \alpha \) satisfies

\[
\exists C \geq 0 \quad \forall r > 0 \quad \alpha(r) \leq C \exp\left(-\frac{r^2}{c}\right).
\]

We see by Proposition 6.2 that, in the particular case where the measurable pseudometric \( \rho \) is associated with a pointwise semimetric \( \rho_0 \) satisfying the hypothesis of Proposition 1.2, the above Gaussian concentration property coincides with the classical one related to \( (X, \mu, \rho_0) \) (see for example [1, Définition 7.2.3]).

6.2 Kantorovich metric and Gaussian concentration

We denote by \( \Pi(\mu) \) the set of all probability measures on \( X \) which are absolutely continuous with respect to \( \mu \). We first define, in this setting, the Kantorovich metric (see, for example, [24, p.88]).

Definition 6.5 We define the Kantorovich metric \( \kappa \) on \( \Pi(\mu) \) by

\[
\forall \xi, \eta \in \Pi(\mu) \quad \kappa(\xi, \eta) = \sup\{ \int f \, d\xi - \int f \, d\eta : f \in \text{Lip}^\infty(\rho), L_{\rho}(f) \leq 1 \} \leq +\infty.
\]

Generally, \( \kappa \) is not a true metric on \( \Pi(\mu) \). We have

\[
\forall \xi, \eta \in \Pi(\mu) \quad \kappa(\xi, \eta) = 0 \quad \Rightarrow \quad \xi = \eta
\]

if and only if \( \rho \) is a measurable metric (Definition 2.2).

We now define transportation type inequalities (this terminology being taken from [1]). We recall that, if \( \xi = \varphi \, d\mu \) belongs to \( \Pi(\mu) \), the entropy of \( \xi \) is defined by

\[
\text{Ent}_\mu(\xi) = E_{\mu}(\varphi \log \varphi)
\]

where \( E_{\mu} \) denotes the expectation with respect to \( \mu \).
Definition 6.6 Let $c$ be a positive constant. We say that the triplet $(X, \mu, \rho)$ satisfies the transportation inequality denoted by $T(c)$ if

$$\forall \xi \in \Pi(\mu) \quad \kappa(\xi, \mu) \leq \sqrt{c \operatorname{Ent}_\mu(\xi)}.$$  

We can now state in this framework a result of S.G. Bobkov and F. Götze ([2]). In fact, by our definition of the transportation inequality from the Kantorovich metric, the proof of [2, Theorem 3.1] (see also [1, Théorème 8.3.2]) works without modification.

Proposition 6.7 The transportation inequality $T(c)$ is equivalent to the following property: For all $\psi \in \operatorname{Lip}_\infty(\rho)$ such that $L_\rho(\psi) \leq 1$, one has

$$\forall t \in \mathbb{R} \quad E_\mu(\exp(t \psi)) \leq \exp \left( \frac{c}{4} t^2 + t E_\mu(\psi) \right).$$

The following theorem then shows the equivalence between a transportation inequality and the Gaussian concentration property. For the proof of the first part, Marton’s argument ([22, 2, 21, 1, ...]) can be used.

Theorem 6.8 Let $c$ be a positive real number. If $T(c)$ holds, then $G(c')$ holds for any $c' > c$. Conversely, if $G(c)$ holds, then there exists $c' > 0$ such that $T(c')$ holds.

6.3 Logarithmic Sobolev inequality and Gaussian concentration

In this subsection, we particularize the previous framework. We consider the situation of Subsection 3.1, with the same notation.

We recall that, if $f \in L^2$, the entropy of $f^2$ is defined by

$$\operatorname{Ent}_\mu(f^2) = E_\mu(f^2 \log f^2) - E_\mu(f^2) \log E_\mu(f^2).$$

Definition 6.9 Let $c$ be a positive constant. We say that the Dirichlet form $(X, \mu, \mathcal{D}, \mathcal{E})$ satisfies the logarithmic Sobolev inequality denoted by $LS(c)$ if

$$\forall f \in \mathcal{D} \quad \operatorname{Ent}_\mu(f^2) \leq c \mathcal{E}(f).$$

An important example is that of the canonical Dirichlet form on a Wiener space (Section 4). In this case, according to [14] the logarithmic Sobolev inequality $LS(4)$ holds.

In the rest of this subsection, $c$ denotes a positive real number.

Using, as in classical cases, the so-called argument of Herbst (see [21, 1]) together with Proposition 6.7, we can prove that, in this general situation too, a logarithmic Sobolev inequality implies a transportation inequality.

Theorem 6.10 If the Dirichlet form $(X, \mu, \mathcal{D}, \mathcal{E})$ satisfies the logarithmic Sobolev inequality $LS(c)$, then the triplet $(X, \mu, \rho)$ satisfies the transportation inequality $T(c/2)$.

As a consequence of the above theorem and Theorem 6.8, we get:
Corollary 6.11 If the Dirichlet form \((X, \mu, \mathcal{D}, \mathcal{E})\) satisfies the logarithmic Sobolev inequality \(LS(c)\), then the triplet \((X, \mu, \rho)\) satisfies the Gaussian concentration inequality \(G(c')\) for any \(c' > (c/2)\). More precisely, one has
\[
r \geq \sqrt{\frac{c}{2} \log 2} \implies \alpha(r) \leq \exp \left[ -\frac{2}{c} \left( r - \sqrt{\frac{c}{2} \log 2} \right)^2 \right],
\]
where \(\alpha\) denotes the concentration function related to \((X, \mu, \rho)\).

We also obtain, as in [21, 1], the following result.

Theorem 6.12 If the Dirichlet form \((X, \mu, \mathcal{D}, \mathcal{E})\) satisfies the logarithmic Sobolev inequality \(LS(c)\), then, for any \(F \in \mathbb{D}\) such that \(\Gamma(F) \leq 1\), one has
\[
\forall r > 0 \quad \mu(F \geq E_{\mu}(F) + r) \leq \exp \left( -\frac{2}{c} r^2 \right).
\]

6.4 Wasserstein metrics

We suppose in this subsection that \(X\) is a Polish space and that \(\mu\) is a Borel probability. We consider a pointwise semimetric \(\rho_0\) which is assumed to be lower semicontinuous on \(X \times X\). Then, we suppose that \(\rho\) is the associated measurable pseudometric, in the sense of Subsection 1.1. This framework contains that of Section 4 (Wiener spaces) with \(\rho_0 = \rho_H\). The results of this subsection are given in [17] in the case of Wiener spaces, but their extension to the more general case considered here is easy, using Theorem 2.14 instead of Theorem 4.2.

If \(\xi\) and \(\eta\) are probability measures on \(X\), we denote by \(\Sigma(\xi, \eta)\) the set of all probability measures on \(X \times X\) with marginal distributions \(\xi\) and \(\eta\).

Definition 6.13 For \(p \in [1, \infty[\) we define the \(L^p\)-Wasserstein metric by
\[
W_p(\xi, \eta) = \left[ \inf \left\{ \int [\rho_0(x, y)]^p \, d\pi(x, y) : \pi \in \Sigma(\xi, \eta) \right\} \right]^{1/p}.
\]

We shall first extend to this situation the classical Kantorovich representation theorem (see for example [24, Theorem 2.5.6], [27, Theorem 1.14]).

Theorem 6.14 For every \(\xi, \eta \in \Pi(\mu)\),
\[
W_1(\xi, \eta) = \kappa(\xi, \eta),
\]
where \(\kappa\) denotes the Kantorovich metric defined in Definition 6.5.

As usual, the proof of this theorem uses the duality theorem [18, Theorem 2.6] (see also for example [27, Theorem 1.3]).

For \(p > 1\), we have the following extension of [24, Corollary 2.5.2] (see also [23, Theorem 5.2.1]). We also mention the deep study [10, 11] of the \(L^2\)-Wasserstein metric on abstract Wiener spaces related to transportation problems. In the following statement, if \(h \in \text{Lip}^\infty(\rho)\), \(\tilde{h}\) denotes a bounded \(\rho_0\)-Lipschitz continuous representative of \(h\) (see Theorem 2.14).
**Theorem 6.15** Let $p > 1$ and let $U$ be a $K_\sigma$ of $X$ with $\mu(U) = 1$. For every $\xi, \eta \in \Pi(\mu)$,

$$[W_p(\xi, \eta)]^p = \sup \{ \int f \, d\xi - \int g \, d\eta : f \in \operatorname{Lip}_\infty(\rho), g \in \operatorname{Lip}_\infty(\rho) \text{ and } \forall x \in X, \forall y \in U, \quad \tilde{f}(x) - \tilde{g}(y) \leq [\rho_0(x, y)]^p \}.$$

Notice that, in the above statement, we may assume moreover that $\tilde{f}$ (resp. $\tilde{g}$) is an u.s.c. (resp. l.s.c.) function. This follows from the proof of [17, Theorem 6.3].

In the case of Wiener spaces we may assume that $U = U + H$ (notation of Section 4). Then by modifying $\tilde{g}$ on the complement of $U$, we may replace $U$ by $X$ in the above statement.

**References**


[33] N. WEAVER Personal communication.