

On the Starting and Stopping Problem: Application in reversible investments.

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Abstract

In this work we solve completely the *starting and stopping* problem when the dynamics of the system are a general adapted stochastic process. We use backward stochastic differential equations and Snell envelopes. Finally we give some numerical results.

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0. Introduction: First let us deal with an example in order to introduce the problem we consider in this paper.

Assume that a power station produces electricity whose selling price fluctuates and depends on many factors such as consumer demand, oil prices, weather and so on. It is well known that electricity cannot be stored and when produced it should be consumed. Now for obvious economic reasons we suppose that electricity is produced only when its profitability is satisfactory. Otherwise the power station is closed up to time when the profitability is coming back, *i.e.*, till the time when the market selling price of electricity reaches a level which makes the production profitable again.

So for the power station there are two modes: operating and closed. At the initial time, we assume it is in its operating mode. On the other hand, like every economic unit, there are expenditures when the station is in its operating mode as well as in the closed one. In addition, switching from a mode to another is not free and generates sunk costs.

The problem we are interested in is to find the sequence of stopping times where one should make decisions to stop the production and to resume it again successively in order to maximize the profitability of the station and then to determine the maximum profit.

More precisely suppose the electricity market selling price is given by a stochastic process $X = (X_t)_{t \leq T}$. As it is discussed previously, a management strategy of the power station is an increasing sequence of stopping times $\delta = (\tau_n)_{n \geq 1}$ where for $n \geq 1$, $\tau_n \leq \tau_{n+1}$ and τ_{2n-1} (resp. τ_{2n}) are the times where the station is switched from the operating to the closed mode (resp. conversely). Now let $J(\delta)$ be the profit of the power station provided by the implementation of the strategy δ . Naturally it depends on the given process X . Therefore we look for a strategy δ^* such that $J(\delta^*) \geq J(\delta)$ for any other δ .

The problem we consider in this paper is of *real options* type. It is usually called the *reversible investment problem*. In recent years, *real options* area has attracted considerable interest ([BO],[BS],[DP],[DZ],...). The motivations are mainly related to decision making in the economic sphere. For more details on this subject see *e.g.* the book by Dixit & Pindyck [DP] and the references therein.

In the literature, our problem is also called *starting and stopping* (or *switching*). In the previous example, we have considered electricity production. However there are many real cases where this problem intervenes. Among others, we can quote the management of oil tankers, oil fields,...

From the economic point of view, the problem of *starting and stopping* has been already considered by A.Dixit [D] in the case when T is infinite and X is a geometric Brownian motion. His approach is based on elliptic PDEs.

In this article we solve completely the *starting and stopping* problem when the dynamics of the system is a stochastic process $X = (X_t)_{t \leq T}$ adapted with respect to a Brownian filtration, whatever it may be and when T is finite. The main tools are the notions of reflected backward stochastic differential equation (BSDE in short) and Snell envelope. We show that our problem turns into the existence of a pair of adapted processes (Y^1, Y^2) which satisfies a system expressed by means of Snell envelopes. In a second step, we show that the existence of an optimal strategy and its expression is given. At the end we discuss a method to simulate the optimal strategy and we give some numerical results.

Another interest of our work is that we bring a new point of view to tackle the *starting and*

stopping problem when the dynamic of the system is affected by the control. With respect to the above example, it means that the process X depends on the running implemented strategy δ . Such problems are met in the management of raw material mines like copper, gold, steel,... In [BO] and [DZ], the approach is based on PDEs, and solution are provided only under fairly stringent conditions. We think that our approach based on BSDEs could bring new results. Though this is a task with which we deal with in a forthcoming paper.

This paper is organized as follows: Section 1 is devoted to the setting of the *starting and stopping problem*. Further we show that our problem reduces to the existence of a pair of processes (Y^1, Y^2) solution of a system expressed by means of Snell envelopes. Then we construct the optimal strategy. In Section 3 we show the existence of (Y^1, Y^2) and we present examples. Finally we give some numerical results.

1 Setting of the problem. Preliminary results

Throughout this paper (Ω, \mathcal{F}, P) is a fixed probability space on which is defined a standard d -dimensional Brownian motion $B = (B_t)_{t \leq T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{t \leq T}$. Let $\mathbf{F} = (\mathcal{F}_t)_{t \leq T}$ be the completed filtration of $(\mathcal{F}_t^0)_{t \leq T}$ with the P -null sets of \mathcal{F} , hence $(\mathcal{F}_t)_{t \leq T}$ satisfies the usual conditions, *i.e.*, it is right continuous and complete. We now introduce the following notations: let

- \mathcal{P} be the σ -algebra on $[0, T] \times \Omega$ of \mathbf{F} -progressively measurable sets

- $\mathcal{M}^{2,k}$ be the set of \mathcal{P} -measurable and \mathbb{R}^k -valued processes $w = (w_t)_{t \leq T}$ such that $E[\int_0^T |w_s|^2 ds] < \infty$

- \mathcal{S}^2 be the set of \mathcal{P} -measurable, continuous processes $w = (w_t)_{t \leq T}$ such that $E[\sup_{t \leq T} |w_t|^2] < \infty$

- \mathcal{S}_i^2 be the subset of \mathcal{S}^2 of processes $K := (K_t)_{t \leq T}$ which are non-decreasing and satisfy $K_0 = 0$.

In particular, if $K \in \mathcal{S}_i^2$, then $E(K_T^2) < \infty$.

For any stopping time $\tau \in [0, T]$, \mathcal{T}_τ denotes the set of all stopping times θ such that $\tau \leq \theta \leq T$.

A management strategy is an increasing sequence of \mathbf{F} -stopping times $\delta := (\tau_n)_{n \geq 1}$ where for any $n \geq 1$, τ_{2n} (resp. τ_{2n-1}) are the moments where the production is frozen (resp. on).

A strategy $\delta := (\tau_n)_{n \geq 1}$ is called admissible if P -*a.s.*, $\lim_{n \rightarrow \infty} \tau_n = T$. The set of admissible strategies is denoted \mathcal{D}_a .

Now in a short period of time dt , when the production is open, it provides a profit which is equal to $\psi_1(t, X_t)dt$. The quantity $\psi_1(t, X_t)$ can be negative. Such a situation happens when the electricity price is low enough at point that management expenses are not recovered. On the other hand, when the production is frozen there are sunk costs which are equal to $\psi_2(t, X_t)dt$. At least because one

should maintain the production equipment in a good state in order to operate in due time. Finally there are also costs linked to stop the production or to start it again. For example, one can think of the fees generated by laying of the workers or engaging them again.

So the outcome of those considerations is that when an admissible management strategy $\delta := (\tau_n)_{n \geq 1}$ is implemented, the average global profit is given by:

$$J(\delta) = E\left[\int_0^T \Phi(s, X_s, u_s) ds - \sum_{n \geq 1} \{D \mathbb{1}_{[\tau_{2n-1} < T]} + a \mathbb{1}_{[\tau_{2n} < T]}\}\right]$$

where :

[i] X_t is the electricity market price at t ; the process $(X_t)_{t \leq T}$ belongs to $\mathcal{M}^{2,1}$

[ii] for any $t \leq T$, $u_t = 1$ if the production is open and 0 otherwise. Actually the process $u = (u_t)_{t \leq T}$ is linked to the implemented strategy δ and for any $t \leq T$ we have $u_t = \mathbb{1}_{[0, \tau_1]}(t) + \sum_{n \geq 1} \mathbb{1}_{[\tau_{2n-1}, \tau_{2n}]}(t)$

[iii] $\Phi(t, x, 0) = \psi_2(t, x)$ and $\Phi(t, x, 1) = \psi_1(t, x)$

[iv] D (resp. a) stands for the sunk cost when the production is stopped (resp. starts)

[v] the functions $\psi_j(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^d$ and $j = 1, 2$, are sub-linearly growing, *i.e.*, there exists C such that $|\psi_j(t, x)| \leq C(1 + |x|)$.

Now the problem we are interested in is to find an optimal strategy for the manager, *i.e.*, a strategy $\delta^* = (\tau_n^*)_{n \geq 1} \in \mathcal{D}_a$ such that $J(\delta^*) \geq J(\delta)$, for any admissible strategy δ . In a second stage we deal with the numerical results of the optimal profit and strategy.

Note that here, the function Φ may depend on time in a general way, which is not the case in Dixit.

1.1 The Snell envelope notion

Let $U = (U_t)_{t \leq T}$ be an \mathbf{F} -adapted \mathbb{R} -valued càdlàg process without negative jumps and which belongs to the class [D], *i.e.*, the set of random variables $\{U_\tau, \tau \in \mathcal{T}_0\}$ is uniformly integrable. Then, it is well known that there exists an \mathbf{F} -adapted \mathbb{R} -valued continuous process $Z := (Z_t)_{t \leq T}$ (see e.g. [CK], [EK], [H]) such that:

(i) Z is the smallest super-martingale which dominates U , *i.e.*, if $(\bar{Z}_t)_{t \leq T}$ is another càdlàg super-martingale such that $\forall t \leq T, \bar{Z}_t \geq U_t$ then $\bar{Z}_t \geq Z_t$ for any $t \leq T$

(ii) For any \mathbf{F} -stopping time γ ,

$$Z_\gamma = \text{esssup}_{\tau \in \mathcal{T}_\gamma} E[U_\tau | \mathcal{F}_\gamma] \tag{1}$$

The process Z is called the *Snell envelope* of U . Moreover, the following properties hold true:

(iii) Let γ be an \mathbf{F} -stopping time and $\tau_\gamma^* = \inf\{s \geq \gamma, Z_s = U_s\} \wedge T$ then τ_γ^* is optimal after γ , *i.e.*,

$$Z_\gamma = E[Z_{\tau_\gamma^*} | \mathcal{F}_\gamma] = E[U_{\tau_\gamma^*} | \mathcal{F}_\gamma] = \text{esssup}_{\tau \geq \gamma} E[U_\tau | \mathcal{F}_\gamma]. \quad (2)$$

(iv) If U_n , $n \geq 0$, and U are càdlàg and uniformly square integrable processes such that the sequence $(U_n)_{n \geq 0}$ converges increasingly and pointwisely to U then $(Z^{U_n})_{n \geq 0}$ converges increasingly and pointwisely to Z^U ; Z^{U_n} and Z^U are the Snell envelopes of respectively U_n and U .

The proof of (iv) is given in the appendix of [CK]. For more details on Snell's envelope, one can refer to [EK] \diamond

Let $\delta = (\tau_n)_{n \geq 1}$ be an admissible strategy. The strategy δ is called *finite* if during the time interval $[0, T]$ it allows to the manager to make only a finite number of decisions, *i.e.*, $P(\omega, \tau_n(\omega) < T, \forall n \geq 0) = 0$. Hereafter the set of finite strategies will be denoted \mathcal{D} .

Proposition 1 : *The supremum over admissible strategies and finite strategies are the same: $\sup_{\delta \in \mathcal{D}_a} J(\delta) = \sup_{\delta \in \mathcal{D}} J(\delta)$.*

Proof: If $\delta = (\tau_n)_{n \geq 1}$ is an admissible strategy which does not belong to \mathcal{D} , then $J(\delta) = -\infty$. Indeed, let $A = \{\omega, \tau_n(\omega) < T, \forall n \geq 1\}$. Since $\delta \in \mathcal{D}_a \setminus \mathcal{D}$ then $P(A) > 0$, hence

$$\begin{aligned} J(\delta) &\leq E\left[\int_0^T (|\psi_1(s, X_s)| \vee |\psi_2(s, X_s)|) ds \right. \\ &\quad \left. - (\sum_{n \geq 1} D \mathbb{1}_{[\tau_{2n-1} < T]} + a \mathbb{1}_{[\tau_{2n} < T]}) \mathbb{1}_A - (\sum_{n \geq 1} D \mathbb{1}_{[\tau_{2n-1} < T]} + a \mathbb{1}_{[\tau_{2n} < T]}) \mathbb{1}_{\bar{A}}\right] = -\infty \end{aligned}$$

since the processes $(\psi_i(s, X_t))_{t \leq T}$ belong to $\mathcal{M}^{2,1}$ and $P(A) > 0$. Indeed, the process X belongs to $\mathcal{M}^{2,1}$ and ψ_i is sub-linearly growing. It implies that $J(\delta) = -\infty$ and then $\sup_{\delta \in \mathcal{D}_a} J(\delta) = \sup_{\delta \in \mathcal{D}} J(\delta)$.

\diamond

We now focus on the optimal profit.

Proposition 2 : *Assume there exist two \mathbb{R} -valued processes $Y^1 = (Y_t^1)_{t \leq T}$ and $Y^2 = (Y_t^2)_{t \leq T}$ of \mathcal{S}^2 such that $\forall t \leq T$,*

$$Y_t^1 = \text{esssup}_{\tau \in \mathcal{T}_t} E\left[\int_t^\tau \psi_1(s, X_s) ds + (-D + Y_\tau^2) \mathbb{1}_{[\tau < T]} | \mathcal{F}_t\right], \quad (3)$$

$$Y_t^2 = \text{esssup}_{\tau \in \mathcal{T}_t} E\left[\int_t^\tau \psi_2(s, X_s) ds + (-a + Y_\tau^1) \mathbb{1}_{[\tau < T]} | \mathcal{F}_t\right]. \quad (4)$$

Then $Y_0^1 = \sup_{\delta \in \mathcal{D}} J(\delta)$. Moreover the strategy $\delta^* = (\tau_n^*)_{n \geq 1}$ defined as follows:

$$\begin{aligned} \tau_0^* &= 0 \\ \forall n \geq 1, \tau_{2n-1}^* &= \inf\{s \geq \tau_{2n-2}^*, Y_s^1 = -D + Y_s^2\} \wedge T \\ \tau_{2n}^* &= \inf\{s \geq \tau_{2n-1}^*, Y_s^2 = -a + Y_s^1\} \wedge T \end{aligned}$$

is optimal.

Proof: Actually we have

$$Y_t^1 + \int_0^t \psi_1(s, X_s) ds = \text{esssup}_{\tau \geq t} E[\int_0^\tau \psi_1(s, X_s) ds + (-D + Y_\tau^2) \mathbb{1}_{[\tau < T]} | \mathcal{F}_t].$$

Now Y_0^1 is \mathcal{F}_0 -measurable then it is $P - a.s.$ a constant and then $Y_0^1 = E[Y_0^1]$. On the other hand, according to (2),

$$Y_0^1 = E[\int_0^{\tau_1^*} \psi_1(s, X_s) ds + (-D + Y_{\tau_1^*}^2) \mathbb{1}_{[\tau_1^* < T]}],$$

where τ_1^* is given as in the proposition. From the properties of Snell's envelope

$$\begin{aligned} Y_{\tau_1^*}^2 &= \text{esssup}_{\tau \in \mathcal{T}_{\tau_1^*}} E[\int_{\tau_1^*}^\tau \psi_2(s, X_s) ds + (-a + Y_\tau^1) \mathbb{1}_{[\tau < T]} | \mathcal{F}_{\tau_1^*}] \\ &= E[\int_{\tau_1^*}^{\tau_2^*} \psi_2(s, X_s) ds + (-a + Y_{\tau_2^*}^1) \mathbb{1}_{[\tau_2^* < T]} | \mathcal{F}_{\tau_1^*}]. \end{aligned}$$

It implies that

$$\begin{aligned} Y_0^1 &= E[\int_0^{\tau_1^*} \psi_1(s, X_s) ds + \int_{\tau_1^*}^{\tau_2^*} \psi_2(s, X_s) ds - D \mathbb{1}_{[\tau_1^* < T]} + E[(-a + Y_{\tau_2^*}^1) \mathbb{1}_{[\tau_2^* < T]} | \mathcal{F}_{\tau_1^*}] \mathbb{1}_{[\tau_1^* < T]}] \\ &= E[\int_0^{\tau_1^*} \psi_1(s, X_s) ds + \int_{\tau_1^*}^{\tau_2^*} \psi_2(s, X_s) ds - D \mathbb{1}_{[\tau_1^* < T]} - a \mathbb{1}_{[\tau_2^* < T]} + Y_{\tau_2^*}^1 \mathbb{1}_{[\tau_2^* < T]}] \end{aligned}$$

since $[\tau_1^* < T] \in \mathcal{F}_{\tau_1^*}$ and $[\tau_2^* < T] \subset [\tau_1^* < T]$. Therefore

$$Y_0^1 = E[\int_0^{\tau_2^*} \Phi(X_s, u_s) ds - D \mathbb{1}_{[\tau_1^* < T]} - a \mathbb{1}_{[\tau_2^* < T]} + Y_{\tau_2^*}^1 \mathbb{1}_{[\tau_2^* < T]}]$$

since between 0 and τ_1^* (resp. τ_1^* and τ_2^*) the production is open (resp. suspended) and then $u_t = 1$ (resp. $u_t = 0$) which implies that

$$\int_0^{\tau_2^*} \Phi(X_s, u_s) ds = \int_0^{\tau_1^*} \psi_1(s, X_s) ds + \int_{\tau_1^*}^{\tau_2^*} \psi_2(s, X_s) ds.$$

Now following this reasoning as many times as necessary we obtain

$$Y_0^1 = E[\int_0^{\tau_{2n}^*} \Phi(X_s, u_s) ds - \sum_{1 \leq k \leq n} (D \mathbb{1}_{[\tau_{2k-1}^* < T]} + a \mathbb{1}_{[\tau_{2k}^* < T]}) + Y_{\tau_{2n}^*}^1 \mathbb{1}_{[\tau_{2n}^* < T]}]. \quad (5)$$

But the strategy δ^* is finite. Indeed let $A = \{\omega, \tau_n^* < T, \forall n \geq 1\}$ and let us show that $P(A) = 0$.

Suppose that $P(A) > 0$ then for any $n \geq 1$,

$$\begin{aligned} Y_0^1 &\leq E[\int_0^T (|\psi_1(s, X_s)| \vee |\psi_2(s, X_s)|) ds - (\sum_{1 \leq k \leq n} (D \mathbb{1}_{[\tau_{2k-1}^* < T]} + a \mathbb{1}_{[\tau_{2k}^* < T]})) \mathbb{1}_A \\ &\quad - (\sum_{1 \leq k \leq n} (D \mathbb{1}_{[\tau_{2k-1}^* < T]} + a \mathbb{1}_{[\tau_{2k}^* < T]})) \mathbb{1}_{\bar{A}} + \sup_{s \leq T} |Y_s^1| \mathbb{1}_{[\tau_{2n}^* < T]}]. \end{aligned}$$

Now the right-hand side converges to $-\infty$ as $n \rightarrow \infty$ since the process Y^1 belongs to \mathcal{S}^2 and $\psi_i(\cdot, X) \in \mathcal{M}^{2,1}$, then $Y_0^1 = -\infty$. But this is contradictory because, once again, $Y^1 \in \mathcal{S}^2$. Henceforth the strategy δ^* is finite. Going back to (5) and taking the limit as $n \rightarrow \infty$ we obtain $Y_0^1 = J(\delta^*)$.

Now let us show that $Y_0^1 \geq J(\delta)$ for any $\delta \in \mathcal{D}$. Let $\delta = (\tau_n)_{n \geq 1}$ be a finite strategy. According to (2), τ_1^* is optimal and then

$$Y_0^1 \geq E\left[\int_0^{\tau_1} \psi_1(s, X_s) ds + (-D + Y_{\tau_1}^2) \mathbb{1}_{[\tau_1 < T]}\right].$$

On the other hand

$$Y_{\tau_1}^2 \geq E\left[\int_{\tau_1}^{\tau_2} \psi_2(s, X_s) ds + (-a + Y_{\tau_2}^1) \mathbb{1}_{[\tau_2 < T]} | F_{\tau_1}\right]$$

and then

$$\begin{aligned} Y_0^1 &\geq E\left[\int_0^{\tau_1} \psi_1(s, X_s) ds - D \mathbb{1}_{[\tau_1 < T]} + E[(-a + Y_{\tau_2}^1) \mathbb{1}_{[\tau_2 < T]} | F_{\tau_1}] \mathbb{1}_{[\tau_1 < T]}\right] \\ &\geq E\left[\int_0^{\tau_1} \psi_1(s, X_s) ds + \int_{\tau_1}^{\tau_2} \psi_2(s, X_s) ds - D \mathbb{1}_{[\tau_1 < T]} - a \mathbb{1}_{[\tau_2 < T]} + Y_{\tau_2}^1 \mathbb{1}_{[\tau_2 < T]}\right] \end{aligned}$$

since $[\tau_1 < T] \in F_{\tau_1}$ and $[\tau_2 < T] \subset [\tau_1 < T]$. Therefore we have,

$$Y_0^1 \geq E\left[\int_0^{\tau_2} \Phi(X_s, u_s) ds - D \mathbb{1}_{[\tau_1 < T]} - a \mathbb{1}_{[\tau_2 < T]} + Y_{\tau_2}^1 \mathbb{1}_{[\tau_2 < T]}\right].$$

Now making this reasoning as many times as necessary we obtain for any $n \geq 0$,

$$Y_0^1 \geq E\left[\int_0^{\tau_{2n}} \Phi(X_s, u_s) ds - \sum_{1 \leq k \leq n} (D \mathbb{1}_{[\tau_{2k-1} < T]} + a \mathbb{1}_{[\tau_{2k} < T]}) + Y_{\tau_{2n}}^1 \mathbb{1}_{[\tau_{2n} < T]}\right]. \quad (6)$$

As the strategy δ is finite then the right-hand side of (6) converges to $J(\delta)$ as $n \rightarrow \infty$. Therefore we have $Y_0^1 = J(\delta^*) \geq J(\delta)$ which implies that the strategy δ^* is optimal \diamond

2 Existence of the pair (Y^1, Y^2) .

We now focus on the existence of the pair (Y^1, Y^2) . Let us recall the following result related to reflected BSDEs with one lower barrier (El-Karoui et al. [EKal]).

Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}; dP)$ and

$$f : (t, \omega, y, z) \in \Omega \times [0, T] \times \mathbb{R}^{1+d} \longrightarrow f(t, \omega, y, z) \in \mathbb{R}$$

be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{1+d})$ -measurable mapping. In addition we assume that the process $(f(t, 0, 0))_{t \leq T}$ belongs to $H^{2,1}$, and that there exists a constant $k \geq 0$ such that for any $y, y', z, z' \in \mathbb{R}$:

$$|f(\omega, t, y, z) - f(\omega, t, y', z')| \leq k(|y - y'| + |z - z'|).$$

Finally let $S := (S_t)_{t \leq 1}$ be an \mathbb{R} -valued process of \mathcal{S}^2 such that $S_T \leq \xi$. Then we have :

Theorem 1 (EKal) : *There exists a triple $(Y, Z, K) := (Y_t, Z_t, K_t)_{t \leq T}$ of \mathcal{P} -measurable processes, with values in $\mathbb{R}^1 \times \mathbb{R}^d \times \mathbb{R}^1$ such that:*

$$\begin{cases} Y \in \mathcal{S}^2, Z \in H^{2,d} \text{ and } K \in \mathcal{S}_i^2 (K_0 = 0) \\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, t \leq T \\ \forall t \leq T, Y_t \geq S_t \text{ and } \int_0^T (Y_t - S_t) dK_t = 0 \diamond \end{cases} \quad (7)$$

In addition, Y can be interpreted as a Snell envelope in the following way: $\forall t \leq T$,

$$Y_t = \text{esssup}_{\tau \geq t} E \left[\int_t^\tau f(s, Y_s, Z_s) ds + S_\tau \mathbb{1}_{[\tau < T]} + \xi \mathbb{1}_{[\tau = T]} \middle| \mathcal{F}_t \right] \diamond \quad (8)$$

Let $(Y^{1,n}, Y^{2,n}) \in (\mathcal{S}^2)^2$ be the sequence of processes defined as follows:

$$Y_t^{1,0} = E \left[\int_t^T \psi_1(s, X_s) ds \middle| \mathcal{F}_t \right], t \leq T$$

$$\begin{cases} -dY_t^{2,n} = \psi_2(t, X_t) dt + dK_t^{2,n} - Z_t^{2,n} dB_t, Y_T^{2,n} = 0 \\ Y_t^{2,n} \geq -a + Y_t^{1,n-1} \text{ and } (Y_t^{2,n} + a - Y_t^{1,n-1}) dK_t^{2,n} = 0 \\ -dY_t^{1,n} = \psi_1(t, X_t) dt + dK_t^{1,n} - Z_t^{1,n} dB_t, Y_T^{1,n} = 0 \\ Y_t^{1,n} \geq -D + Y_t^{2,n} \text{ and } (Y_t^{1,n} + D - Y_t^{2,n}) dK_t^{1,n} = 0, \end{cases}$$

As in (8), the processes $Y^{1,n}$ and $Y^{2,n}$ ($n \geq 1$) have representations as Snell envelopes in the following way: for any $t \leq T$,

$$Y_t^{1,n} = \text{esssup}_{\tau \geq t} E \left[\int_t^\tau \psi_1(s, X_s) ds + (-D + Y_\tau^{2,n}) \mathbb{1}_{[\tau < T]} \middle| \mathcal{F}_t \right] \quad (9)$$

and

$$Y_t^{2,n} = \text{esssup}_{\tau \geq t} E \left[\int_t^\tau \psi_2(s, X_s) ds + (-D + Y_\tau^{1,n-1}) \mathbb{1}_{[\tau < T]} \middle| \mathcal{F}_t \right] \quad (10)$$

First we begin to focus on the properties of $Y^{1,n}$ and $Y^{2,n}$.

Proposition 3 *The following properties hold true:*

[i] for $n \geq 1$ the processes $Y^{1,n}$ and $Y^{2,n}$ are continuous

[ii] the sequences $(Y^{1,n})_{n \geq 1}$ and $(Y^{2,n})_{n \geq 0}$ are convergent in $H^{2,1}$ to càdlàg processes \tilde{Y}^1 and \tilde{Y}^2 respectively. In addition we have $E[\sup_{t \leq T} |\tilde{Y}_t^i|^2] < \infty$, $i = 1, 2$.

Proof: Since $-a < 0$, $-D < 0$ and, $Y^{1,0}$ is continuous and $Y_T^{1,0} = 0$ then it easily seen by induction, through Theorem 1, that the processes $Y^{1,n}$ and $Y^{2,n}$ are defined, continuous and satisfy $Y_T^{1,n} = 0$, $Y_T^{2,n} = 0$.

For $t \leq T$ let \mathcal{D}_n^t be the set of admissible strategies $\delta = (\tau_k)_{k \geq 0}$ such that $\tau_1 \geq t$ and $\tau_{n+1} = T$, P-*a.s.*. On the other hand for $\delta \in \mathcal{D}_n^t$ let us set $\bar{\Phi}(t, \delta) := \int_t^T \Phi(s, X_s, u_s) ds - \sum_{n \geq 0} (D \mathbb{1}_{[\tau_{2n+1} < T]} + a \mathbb{1}_{[\tau_{2n} < T]})$, P-*a.s.*, where $(u_t)_{t \leq T}$ is the underlying process associated with the strategy δ . It holds true that the process $Y^{1,n}$ satisfies the following property:

$$Y_t^{1,n} = \text{esssup}_{\delta \in \mathcal{D}_n^t} E[\bar{\Phi}(t, \delta) | \mathcal{F}_t]. \quad (11)$$

Actually let us consider the following sequence of stopping times:

$$\begin{aligned} \tau_t^1 &:= \{s \geq t, Y_s^{1,n} = -D + Y_s^{2,n}\} \wedge T, \quad \tau_t^2 := \{s \geq \tau_t^1, Y_s^{2,n} = -a + Y_s^{1,n-1}\} \wedge T \\ \tau_t^3 &:= \{s \geq \tau_t^2, Y_s^{1,n-1} = -D + Y_s^{2,n-1}\} \wedge T, \dots, \\ \tau_t^{2n} &:= \{s \geq \tau_t^{2n-1}, Y_s^{2,1} = -a + Y_s^{1,1}\} \wedge T \text{ and } \tau_t^{2n+1} = T. \end{aligned}$$

Let us set $\tilde{\delta} := (\tau_t^1, \dots, \tau_t^{2n+1})$, then $\tilde{\delta} \in \mathcal{D}_n^t$. On the other hand

$$Y_t^{1,n} = Y_{\tau_t^1}^{1,n} + \int_t^{\tau_t^1} \psi_1(s, X_s) ds - \int_t^{\tau_t^1} Z_s^{1,n} dB_s$$

since between t and τ_t^1 the process $Y^{1,n}$ does not reach the barrier $-D + Y^{2,n}$ and therefore the process $K^{1,n}$ does not increase. It follows that:

$$\begin{aligned} Y_t^{1,n} &= E\left[\int_t^{\tau_t^1} \psi_1(s, X_s) ds + Y_{\tau_t^1}^{1,n} | \mathcal{F}_t\right] \\ &= E\left[\int_t^{\tau_t^1} \psi_1(s, X_s) ds + (-D + Y_{\tau_t^1}^{2,n}) \mathbb{1}_{[\tau_t^1 < T]} | \mathcal{F}_t\right] \end{aligned} \quad (12)$$

since $Y_{\tau_t^1}^{1,n} = (-D + Y_{\tau_t^1}^{2,n}) \mathbb{1}_{[\tau_t^1 < T]}$. But

$$Y_{\tau_t^1}^{2,n} = Y_{\tau_t^1}^{2,n} + \int_{\tau_t^1}^{\tau_t^2} \psi_2(s, X_s) ds - \int_{\tau_t^1}^{\tau_t^2} Z_s^{1,n} dB_s$$

since, once again, $K_{\tau_t^2}^{2,n} = K_{\tau_t^1}^{2,n}$ due to the fact that $K^{2,n}$ increases only when $Y^{2,n}$ reaches the barrier $Y^{1,n} - a$. Now as $Y_{\tau_t^2}^{2,n} = (Y_{\tau_t^2}^{1,n-1} - a) \mathbb{1}_{[\tau_t^2 < T]}$ then

$$Y_{\tau_t^1}^{2,n} = E\left[\int_{\tau_t^1}^{\tau_t^2} \psi_2(s, X_s) ds - a \mathbb{1}_{[\tau_t^2 < T]} + Y_{\tau_t^2}^{1,n-1} \mathbb{1}_{[\tau_t^2 < T]} | \mathcal{F}_{\tau_t^1}\right].$$

Now plug that equality in (12) to obtain:

$$Y_t^{1,n} = E\left[\int_t^{\tau_t^1} \psi_1(s, X_s) ds + \mathbb{1}_{[\tau_t^1 < T]} \int_{\tau_t^1}^{\tau_t^2} \psi_2(s, X_s) ds - D \mathbb{1}_{[\tau_t^1 < T]} - a \mathbb{1}_{[\tau_t^2 < T]} + Y_{\tau_t^2}^{1,n-1} \mathbb{1}_{[\tau_t^2 < T]} | \mathcal{F}_t\right]$$

since $\mathbb{1}_{[\tau_t^1 < T]}$ is $\mathcal{F}_{\tau_t^1}$ -measurable and $[\tau_t^2 < T] \subset [\tau_t^1 < T]$. But

$$\int_t^{\tau_t^1} \psi_1(s, X_s) ds + \mathbb{1}_{[\tau_t^1 < T]} \int_{\tau_t^1}^{\tau_t^2} \psi_2(s, X_s) ds = \int_t^{\tau_t^1} \psi_1(s, X_s) ds + \int_{\tau_t^1}^{\tau_t^2} \psi_2(s, X_s) ds$$

since when $\tau_t^1 = T$ the stopping time τ_t^2 is also equal to T . Henceforth

$$Y_t^{1,n} = E\left[\int_t^{\tau_t^1} \psi_1(s, X_s)ds + \int_{\tau_t^1}^{\tau_t^2} \psi_2(s, X_s)ds - D\mathbb{1}_{[\tau_t^1 < T]} - a\mathbb{1}_{[\tau_t^2 < T]} + Y_{\tau_t^2}^{1,n-1}\mathbb{1}_{[\tau_t^2 < T]}|\mathcal{F}_t\right].$$

Now making this reasoning as many times as necessary we obtain :

$$\begin{aligned} Y_t^{1,n} &= E\left[\int_t^{\tau_t^1} \psi_1(s, X_s)ds + \int_{\tau_t^1}^{\tau_t^2} \psi_2(s, X_s)ds + \dots + \int_{\tau_t^{2n}}^T \psi_1(s, X_s)ds - \sum_{k=1,n} (D\mathbb{1}_{[\tau_t^{2k-1} < T]} + a\mathbb{1}_{[\tau_t^{2k} < T]})|\mathcal{F}_t\right] \\ &= E\left[\int_t^T \Phi(s, X_s, \tilde{u}_s)ds - \sum_{k=1,n} (D\mathbb{1}_{[\tau_t^{2k-1} < T]} + a\mathbb{1}_{[\tau_t^{2k} < T]})|\mathcal{F}_t\right] \\ &= E[\bar{\Phi}(t, \tilde{\delta})|\mathcal{F}_t]; \end{aligned}$$

\tilde{u} is the underlying process associated with the strategy $\tilde{\delta}$.

Now let us consider another strategy δ which belongs to \mathcal{D}_t^n , i.e, $\delta = (\tau^1, \tau^2, \dots, \tau^{2n}, T)$ with $\tau^1 \geq t$, P-a.s.. Since the process $K^{1,n}$ is non-decreasing then

$$Y_t^{1,n} \geq E\left[Y_{\tau^1}^{1,n} + \int_t^{\tau^1} \psi_1(s, X_s)ds|\mathcal{F}_t\right].$$

But $Y_{\tau^1}^{1,n} \geq (Y_{\tau^1}^{2,n} - D)\mathbb{1}_{[\tau^1 < T]}$ then

$$Y_t^{1,n} \geq E\left[\int_t^{\tau^1} \psi_1(s, X_s)ds - D\mathbb{1}_{[\tau^1 < T]} + Y_{\tau^1}^{2,n}\mathbb{1}_{[\tau^1 < T]}|\mathcal{F}_t\right].$$

As $Y_{\tau^1}^{2,n} \geq E\left[Y_{\tau^2}^{2,n} + \int_{\tau^1}^{\tau^2} \psi_2(s, X_s)ds|\mathcal{F}_{\tau^1}\right]$, through $dK^{2,n} \geq 0$, and since $Y_{\tau^2}^{2,n} \geq (Y_{\tau^2}^{1,n-1} - a)\mathbb{1}_{[\tau^2 < T]}$ then we have

$$Y_{\tau^1}^{2,n} \geq E\left[\int_{\tau^1}^{\tau^2} \psi_2(s, X_s)ds - a\mathbb{1}_{[\tau^2 < T]} + Y_{\tau^2}^{1,n-1}\mathbb{1}_{[\tau^2 < T]}|\mathcal{F}_{\tau^1}\right].$$

Once again plug that inequality in the previous one to obtain

$$\begin{aligned} Y_t^{1,n} &\geq E\left[\int_t^{\tau^1} \psi_1(s, X_s)ds + \int_{\tau^1}^{\tau^2} \psi_2(s, X_s)ds - D\mathbb{1}_{[\tau^1 < T]} - a\mathbb{1}_{[\tau^2 < T]} + Y_{\tau^2}^{1,n-1}\mathbb{1}_{[\tau^2 < T]}|\mathcal{F}_t\right] \\ &\geq E\left[\int_t^{\tau^2} \Phi(s, X_s, u_s)ds - D\mathbb{1}_{[\tau^1 < T]} - a\mathbb{1}_{[\tau^2 < T]} + Y_{\tau^2}^{1,n-1}\mathbb{1}_{[\tau^2 < T]}|\mathcal{F}_t\right]. \end{aligned}$$

Now making this reasoning as many times as necessary to obtain $Y_t^{1,n} \geq E[\bar{\Phi}(t, \delta)|\mathcal{F}_t]$. Therefore (11) follows.

We now focus on the convergence. Since $\mathcal{D}_t^n \subset \mathcal{D}_t^{n+1}$ then through (11) we have P-a.s., $Y_t^{1,n+1} \geq Y_t^{1,n}$ for any $t \leq T$. As the processes $Y^{1,n}$'s are continuous then we have also P-a.s., $Y^{1,n+1} \geq Y^{1,n}$.

On the other hand, for any $t \leq T$ and for any $\delta \in \mathcal{D}_t^n$, we have

$$|\bar{\Phi}(t, \delta)| \leq E\left[\int_0^T \{|\psi_1(s, X_s)| + |\psi_2(s, X_s)|\}ds|\mathcal{F}_t\right].$$

Henceforth $Y_t^{1,n} \leq E\left[\int_0^T \{|\psi_1(s, X_s)| + |\psi_2(s, X_s)|\}ds|\mathcal{F}_t\right]$. It follows that the sequence $(Y^{1,n})_{n \geq 0}$ converges in $H^{2,1}$ to a process $(\tilde{Y}_t^1)_{t \leq T}$ which, in addition, satisfies $E[\sup_{t \leq T} |\tilde{Y}_t^1|^2] < \infty$ through Doob's inequality and since the processes $(\psi_i(s, X_s))_{s \leq T}$, $i = 1, 2$, belong to $H^{2,1}$.

Now let us show that \tilde{Y}^1 is càdlàg . For any $n \geq 1$, from (9), the process $(Y_t^{1,n} + \int_0^t \psi_1(s, X_s) ds)_{t \leq T}$ is an \mathbf{F} -supermartingale which converges increasingly and pointwisely to $(\tilde{Y}_t^1 + \int_0^t \psi_1(s, X_s) ds)_{t \leq T}$, therefore the limit is also a càdlàg \mathbf{F} -supermartingale (see e.g. [KS], pp. xx), henceforth the process \tilde{Y}^1 is càdlàg .

Finally let us focus on the convergence of $(Y^{2,n})_{n \geq 1}$. But this a direct consequence of the increasing convergence of $(Y^{1,n})_{n \geq 1}$ to \tilde{Y}^1 , the relation (10) and Point 1.1-(iv) above. Henceforth there exists a \mathcal{P} -measurable càdlàg process \tilde{Y}^2 which is the $H^{2,1}$ -limit of $(Y^{2,n})_{n \geq 1}$ and which is uniformly bounded in $L^2(dP) \diamond$

Finally we are ready now to prove the existence of a pair of processes (Y^1, Y^2) which satisfies (3)-(4).

Theorem 2 : *There exists a pair of continuous processes (Y^1, Y^2) which satisfies (3)- (4)*

Proof: Once again according to Point 1.1-(iv) and the characterizations (9)-(10), the limits \tilde{Y}^1 and \tilde{Y}^2 satisfy to : for any $t \leq T$,

$$\tilde{Y}_t^1 = \text{esssup}_{\tau \geq t} E \left[\int_t^\tau \psi_1(s, X_s) ds + (-D + \tilde{Y}_\tau^2) \mathbb{1}_{[\tau < T]} \middle| \mathcal{F}_t \right] \quad (13)$$

and

$$\tilde{Y}_t^2 = \text{esssup}_{\tau \geq t} E \left[\int_t^\tau \psi_2(s, X_s) ds + (-D + \tilde{Y}_\tau^1) \mathbb{1}_{[\tau < T]} \middle| \mathcal{F}_t \right]. \quad (14)$$

It remains to prove that the processes \tilde{Y}^i are continuous.

A result by Hamadène [H] related to reflected BSDEs with càdlàg barriers implies the existence of processes $(Z_t^i)_{t \leq T} \in H^{2,1}$, $(K_t^i)_{t \leq T}$ (resp. $(A_t^i)_{t \leq T}$) non-decreasing, \mathcal{P} -measurable, continuous (resp. purely discontinuous) and $E[(K_T^i)^2] < \infty$ (resp. $E[(A_T^i)^2] < \infty$) ($K_0^i = 0$ (resp. $A_0^i = 0$)), $i = 1, 2$, such that: for any $t \leq T$,

$$\begin{cases} -d\tilde{Y}_t^1 = \psi_1(t, X_t) dt - dK_t^1 - dA_t^1 - Z_t^1 dB_t, \tilde{Y}_T^1 = 0 \\ \tilde{Y}_t^1 \geq -D + \tilde{Y}_t^2, (\tilde{Y}_t^1 + D - \tilde{Y}_t^2) dK_t^1 = 0 \text{ and } \Delta_t \tilde{Y}^1 = (-D + \tilde{Y}_{t-}^2 - \tilde{Y}_t^1)^+ \mathbb{1}_{[\Delta_t \tilde{Y}^1 < 0]} \end{cases}$$

and

$$\begin{cases} -d\tilde{Y}_t^2 = \psi_2(t, X_t) dt - dK_t^2 - dA_t^2 - Z_t^2 dB_t, \tilde{Y}_T^2 = 0 \\ \tilde{Y}_t^2 \geq -a + \tilde{Y}_t^1, (\tilde{Y}_t^2 + a - \tilde{Y}_t^1) dK_t^2 = 0 \text{ and } \Delta_t \tilde{Y}^2 = (-D + \tilde{Y}_{t-}^1 - \tilde{Y}_t^2)^+ \mathbb{1}_{[\Delta_t \tilde{Y}^2 < 0]} ; \end{cases}$$

here $\Delta_t \tilde{Y}^i$ ($i = 1, 2$) is the jump size of \tilde{Y}^i at t and \tilde{Y}_{t-}^i is the left limit of \tilde{Y}^i at t . Those backward equations imply in particular that if \tilde{Y}^1 (resp. \tilde{Y}^2) has a negative jump at t then \tilde{Y}^2 (resp. \tilde{Y}^1) has also a negative jump at t . In addition the jump size of \tilde{Y}^1 (resp. \tilde{Y}^2) is smaller than the one \tilde{Y}^2 (resp. \tilde{Y}^1). It follows that the processes A^1 and A^2 are undistinguishable.

Now let us set, for any $t \leq T$, $Y_t^1 = \tilde{Y}_t^1 + A_t^1 - E[A_T^1 | \mathcal{F}_t]$ and $Y_t^2 = \tilde{Y}_t^2 + A_t^2 - E[A_T^2 | \mathcal{F}_t]$. Therefore Y^1 and Y^2 belong to \mathcal{S}^2 (note that they are continuous processes). On the other hand, let η^i , $i = 1, 2$, be the processes of $H^{2,1}$ such that $E[A_T^1 | \mathcal{F}_t] = E[A_T^1] + \int_0^t \eta_s^1 dB_s$, $t \leq T$. Then the triples $(Y^i, Z^i + \eta^1, K^i)$, $i = 1, 2$ satisfy: for any $t \leq T$,

$$\begin{cases} -dY_t^1 = \psi_1(t, X_t)dt - dK_t^1 - (Z_t^1 + \eta_t^1)dB_t, & Y_T^1 = 0 \\ Y_t^1 \geq -D + Y_t^2, & (Y_t^1 + D - Y_t^2)dK_t^1 = 0 \end{cases}$$

and

$$\begin{cases} -dY_t^2 = \psi_2(t, X_t)dt - dK_t^2 - (Z_t^2 + \eta_t^1)dB_t, & Y_T^2 = 0 \\ Y_t^2 \geq -a + Y_t^1, & (Y_t^2 + a - Y_t^1)dK_t^2 = 0. \end{cases}$$

Finally the characterization (8) implies that the pair of processes (Y^1, Y^2) satisfies (3)-(4). The proof is now complete \diamond

3 Properties of the optimal strategy. Examples and numerical results.

Now let us set $Y := Y^1 - Y^2$ and $Z := Z^1 - Z^2$ then the quadruple of processes (Y, Z, K^1, K^2) is solution of the following BSDE with two reflecting barriers:

$$\begin{cases} -dY_t = (\psi_1(t, X_t) - \psi_2(t, X_t))dt - dK_t^1 + dK_t^2 - Z_t dB_t, & t \leq T ; Y_T = 0 \\ -D \leq Y_t \leq a, \forall t \leq T \\ \int_0^T (Y_t - a)dK_t^2 = \int_0^T (Y_t + D)dK_t^1 = 0 \end{cases} \quad (15)$$

In addition the solution is unique since $-a < D$ (see e.g. [CK], [HL]). On the other hand, it is easily seen that the stopping times τ_n^* are the ones where the process Y reaches successively the barriers a and $-D$, i.e.,

$$\forall n \geq 1, \tau_{2n-1}^* = \inf\{s \geq \tau_{2n-2}^*, Y_s = -D\} \wedge T \quad (\tau_0^* = 0) \quad \text{and} \quad \tau_{2n}^* = \inf\{s \geq \tau_{2n-1}^*, Y_s = a\} \wedge T.$$

Now suppose the process X is the unique solution of the following standard differential equation :

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \leq T; \quad X_0 = x \in \mathbb{R}^k$$

where the function b and σ , with appropriate dimensions, satisfy :

$$|b(t, x)| + |\sigma(t, x)| \leq k(1 + |x|) \quad \text{and} \quad |\sigma(t, x) - \sigma(t, x')| + |b(t, x) - b(t, x')| \leq k|x - x'|$$

for any $t \leq T$ and $x, x' \in \mathbb{R}^k$.

In [HH], it has been shown the existence of a function $u(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^k$, such that $Y_t = u(t, X_t)$, for any $t \leq T$. Moreover the function u is solution, in viscosity sense, of the following double obstacle partial differential inequality :

$$\begin{cases} \min\{u(t, x) + D, \max[-\frac{\partial u}{\partial t}(t, x) - L_t u(t, x) - (\psi(t, x) - c), u(t, x) - a]\} = 0, \\ u(T, x) = 0 \end{cases}$$

where the operator L is the one associated with X , *i.e.*,

$$L_t = \frac{1}{2} \sum_{i,j=1}^k (\sigma \sigma^*(t, x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^k b_i(t, x) \frac{\partial}{\partial x_i}.$$

Therefore the optimal strategy can be expressed via the beginnings of some deterministic sets. Actually we have: ($\tau_0^* = 0$),

$$\forall n \geq 1, \tau_{2n-1}^* = \inf\{s \geq \tau_{2n-2}^*, u(s, X_s) = -D\} \wedge T \quad \text{and} \quad \tau_{2n}^* = \inf\{s \geq \tau_{2n-1}^*, u(s, X_s) = a\} \wedge T \diamond$$

3.1. Examples.

Assume we have $\psi_1 \geq \psi_2$ then $Y \geq 0$. Indeed let $(\tilde{Y}, \tilde{Z}, \tilde{K}^+, \tilde{K}^-)$ be the solution of the following BSDE (which exists and is unique see e.g. [HL] or [CK]):

$$\begin{cases} -d\tilde{Y}_t = d\tilde{K}_t^+ - d\tilde{K}_t^- - \tilde{Z}_t dB_t, \quad t \leq T; \quad \tilde{Y}_T = 0 \\ -D \leq \tilde{Y}_t \leq a \\ (\tilde{Y}_t + D)d\tilde{K}_t^+ = (\tilde{Y}_t - a)d\tilde{K}_t^- = 0. \end{cases}$$

Now the comparison theorem of solutions of BSDEs with two reflecting barriers (see e.g. [HH]) implies that $Y \geq \tilde{Y}$. But uniqueness implies also that $(\tilde{Y}, \tilde{Z}, \tilde{K}^+, \tilde{K}^-) \equiv (0, 0, 0, 0)$ therefore $Y \geq 0$. It implies that Y never reaches D and the production should be kept open all time up to T . In that case the optimal profit is $E[\int_0^T \psi_1(s, X_s) ds]$.

In a symmetric way if $\psi_1 \leq \psi_2$, the process Y is negative and does not reach the barrier a . Therefore when Y reaches $-D$ at τ_1^* , the production should be suspended and kept in that state up to T . The optimal profit is given by $E[\int_0^{\tau_1^*} \psi_1(s, X_s) ds - D + \int_{\tau_1^*}^T \psi_2(s, X_s) ds] \diamond$

3.2. Numerical results.

For $n, m \geq 0$, let $(Y_t^{n,m}, Z_t^{n,m})_{t \leq T}$ be the process solution of the following standard BSDE:

$$-dY_t^{n,m} = \{(\psi_1(t, X_t) - \psi_2(t, X_t)) - n(Y_t^{n,m} - a)^+ + m(Y_t^{n,m} + D)^-\} dt - Z_t^{n,m} dB_t; \quad Y_T^{n,m} = 0.$$

The process $(Y_t^{n,m})_{n,m \geq 0}$ is decreasing (resp. increasing) in n (resp. m) for a fixed m (resp. n). Moreover the uniform limit of that sequence when $n \rightarrow \infty$ and then $m \rightarrow \infty$ is Y (see e.g. [HH])

for more details on this subject). On the other hand, there exists a continuous function $u^{n,m}(t, x)$, $(t, x) \in [0, T] \times R^k$, such that $Y_t^{n,m} = u^{n,m}(t, X_t)$, $t \leq T$ where $u^{n,m}$ is a viscosity solution of the following PDE:

$$\begin{cases} \frac{\partial u^{n,m}}{\partial t}(t, x) + L_t u^{n,m}(t, x) + (\psi_1(t, x) - \psi_2(t, x)) - n(u^{n,m}(t, x) - a)^+ + m(u^{n,m}(t, x) + D)^- = 0 \\ u(T, x) = 0. \end{cases} \quad (16)$$

Now when the functions b , σ and ψ are smooth enough then $u^{n,m}$ is also smooth and becomes a classical solution for the PDE (16) (see e.g. [PP]). Therefore we can simulate $u^{n,m}$. Now if k, p are large enough then $Y_t \sim u^{p,k}(t, X_t)$. Therefore we can expect that, since the convergence is uniform, for $n \geq 1$, τ_n^* is not far from $\tau_n^{*k,p}$ where ($\tau_0^{*k,p} = 0$)

$$\tau_{2n-1}^{*k,p} = \inf\{s \geq \tau_{2n-2}^{*k,p}, u^{k,p}(s, X_s) = -D\} \wedge T \quad \text{and} \quad \tau_{2n}^{*k,p} = \inf\{s \geq \tau_{2n-1}^{*k,p}, u^{k,p}(s, X_s) = a\} \wedge T.$$

On the ground of those considerations, in Appendix are some simulations obtained with X a geometric Brownian motion, *i.e.*, $dX_t = \alpha X_t dt + \sigma X_t dB_t$, $t \leq 1$; $X_0 = 1$.

3.3. Final remark: Let us point out that, in the resolution of the problem, there is no particular difficulty to replace D (resp. a) by a cost which depends on the state of the system, *i.e.*, $g(X_{\tau_{2n-1}})$ (resp. $\bar{g}(X_{\tau_{2n}})$) where g (resp. \bar{g}) is a positive function bonded by below by a positive constant $\gamma \diamond$

Appendix

As it is pointed out previously *Fig.1* shows that when $\psi_1 \geq \psi_2$ then one should keep the production in its working mode. As for *Fig.2*, it shows that production must be stopped when Y reaches $-D$. Finally, *Fig.3* and *Fig.4* are related to the case where the sign of $\psi_1 - \psi_2$ is whatever.

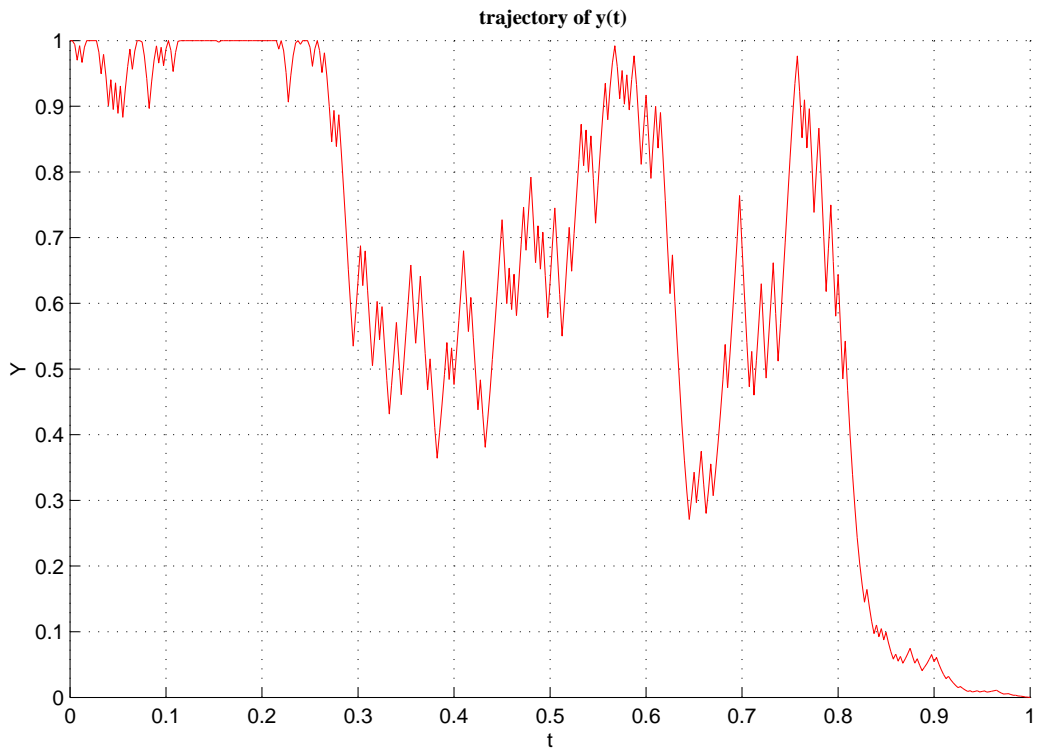


Fig.1: $X_0 = 1, \alpha = 1, \sigma = -3, a = 1, D = 0.5, (\psi_1 - \psi_2)(x) = 0.1x$.

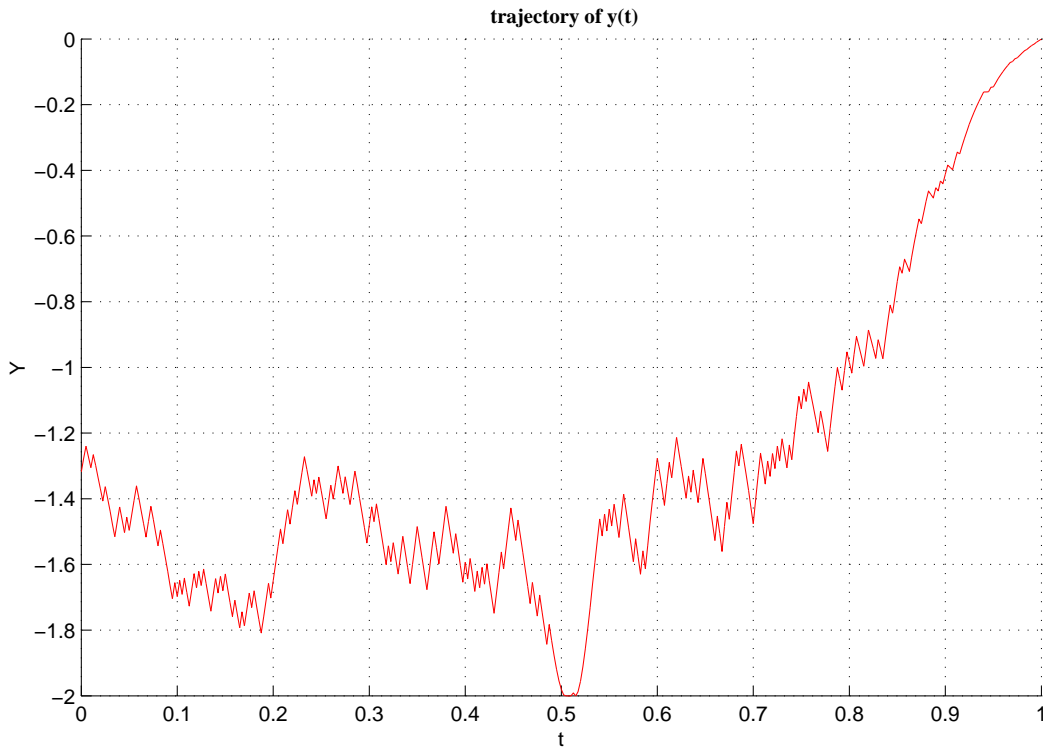


Fig.2: $X_0 = 1, \alpha = 1, \sigma = 1, a = 1, D = 2, (\psi_1 - \psi_2)(x) = -0.01x - 0.5$.

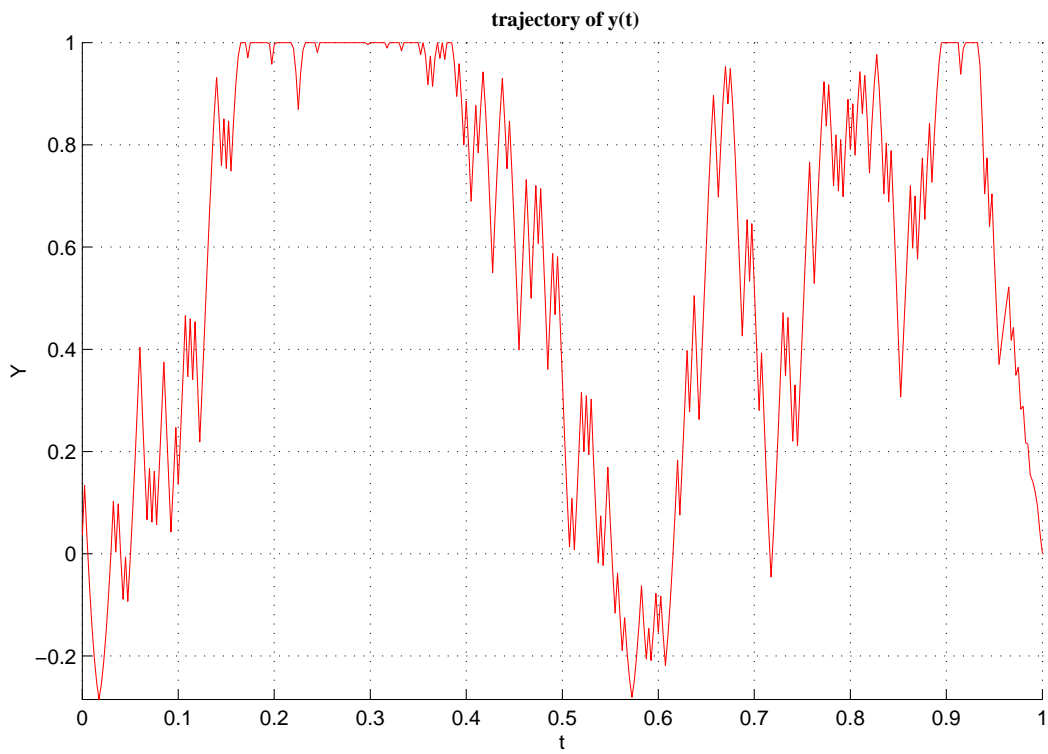


Fig.3: $X_0 = 1, \alpha=1, \sigma=2, a=1, D=0.3, (\psi_1 - \psi_2)(x) = 0.1x - 6$.

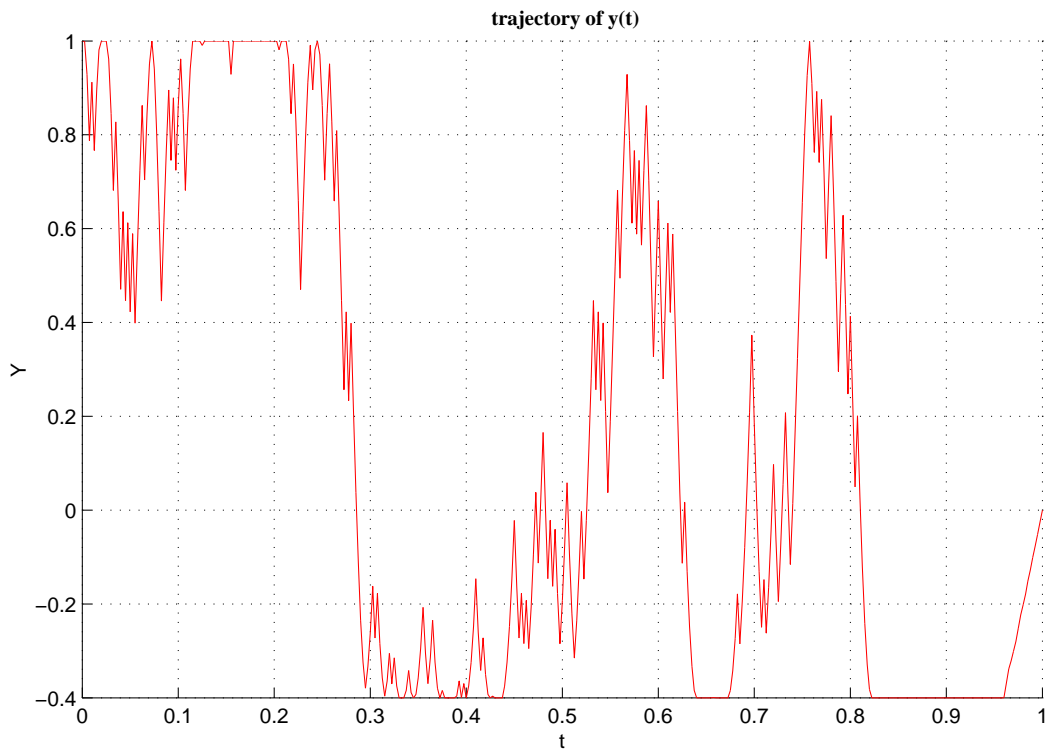


Fig.4: $X_0 = 1, \alpha=0.5, \sigma=-3, a=1, D=0.4, (\psi_1 - \psi_2)(x) = 0.7x - 11$.

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