# REPLICATION OF DEFAULTABLE CLAIMS WITHIN THE REDUCED-FORM FRAMEWORK

Paris-Princeton Lectures on Mathematical Finance 2004

Tomasz R. Bielecki
Department of Applied Mathematics
Illinois Institute of Technology
Chicago, IL 60616, USA

Monique Jeanblanc
Département de Mathématiques
Université d'Évry Val d'Essonne
91025 Évry Cedex, France

Marek Rutkowski
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-661 Warszawa, Poland
and
Institute of Mathematics
Polish Academy of Sciences
00-956 Warszawa, Poland

April 13, 2004

# Contents

# 1 Introduction

The goal of this note is the present some methods and results related to the replication of defaultable claims within the *reduced-form approach* (also known as the *intensity-based approach*). In contrast to some other related works, in which this issue was addressed by invoking a suitable version of the martingale representation theorem (see, for instance, Bélanger et al. (2001) or Blanchet-Scalliet and Jeanblanc (2004)), we analyze here the possibility of an exact replication of a given defaultable claim through a trading strategy based on default-free and defaultable securities. Therefore, the important issue of the choice of primary assets that are used to replicate a defaultable claim (e.g., a vulnerable option or a credit derivative) is emphasized. Let us stress that replication of defaultable claims within the structural approach is rather standard since in this approach the default time is, typically, a predictable stopping time with respect to the filtration generated by prices of primary assets. By contrast, in the intensity-based approach the default time is not a stopping time with respect to the filtration generated by default-free primary assets, and it is a totally inaccessible stopping time with respect to the enlarged filtration, that is, the filtration generated by primary assets and the jump process associated with default time. Our research was motivated, in particular, by the paper by Vaillant (2001).

The note is organized as follows. Section 2 is devoted to a brief description of the basic concepts that are used in what follows. In Section 3, we formally introduce the definition of a generic defaultable claim $(X, Z, C, \tau)$ and we examine the basic features of its ex-dividend price and pre-default value. The well known valuation results for defaultable claims are also provided. In the next section, we analyze (following, in particular, Vaillant (2001)) various classes of self-financing trading strategies based on default-free and defaultable primary assets. Section 5 deals with applications of results obtained in the preceding section to financial models with default-free and defaultable primary assets. We develop a systematic approach to replication of a generic defaultable claim, and we provide closed-form expressions for prices and replicating strategies for several typical defaultable claims. Finally, in the last section a few examples of replicating strategies for particular credit derivatives are presented.

# 2 Preliminaries

In this section, we introduce the basic notions that will be used in what follows. First, we introduce a default-free market model; for the sake of concreteness we focus on default-free zero-coupon bonds. Subsequently, we shall examine the concept of a random time associated with a prespecified hazard process.

## 2.1 Default-Free Market

Consider an economy in continuous time, with the time parameter $t \in \mathbb{R}_+$. We are given a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P}^*)$ endowed with a $d$-dimensional standard Brownian motion $W^*$. It is convenient to assume that $\mathbb{F}$ is the $\mathbb{P}^*$-augmented and right-continuous version of the natural filtration generated by $W^*$. As we shall see in what follows, the filtration $\mathbb{F}$ will also play an important role of a *reference filtration* for the intensity of default event. Let us recall that any (local) martingale with respect to a Brownian filtration $\mathbb{F}$ is continuous; this well-known property will be of frequent use in what follows. In the first step, we introduce an arbitrage-free default-free market. In this market, we have the following primary assets:

- A *money market account* $B$ satisfying

$$dB_t = r_t B_t \, dt, \quad B_0 = 1,$$

  or, equivalently,

$$B_t = \exp\left(\int_0^t r_u \, du\right)$$

where $r$ is an $\mathbb{F}$-progressively measurable stochastic process. Thus, $B$ is an $\mathbb{F}$-adapted, continuous, and strictly positive process of finite variation.

- *Default-free zero-coupon bonds* with prices

$$B(t,T) = B_t \, \mathbb{E}_{\mathbb{P}^*}(B_T^{-1} \,|\, \mathcal{F}_t), \quad \forall\, t \le T,$$

where $T$ is the bond's maturity date. Since the filtration $\mathbb{F}$ is generated by a Brownian motion, for any maturity date $T > 0$ we have

$$dB(t,T) = B(t,T)\big(r_t\, dt + b(t,T)\, dW_t^*\big)$$

for some $\mathbb{F}$-predictable, $\mathbb{R}^d$-valued process $b(t,T)$, referred to as the *bond's volatility*.

For the sake of expositional simplicity, we shall postulate throughout that the default-free term structure model is complete. The probability $\mathbb{P}^*$ is thus the unique martingale measure for the default-free market model. This assumption is not essential, however. Notice that all price processes introduced above are continuous $\mathbb{F}$-semimartingales.

**Remarks.** The bond was chosen as a convenient and practically important example of a tradeable financial asset. We shall be illustrating our theoretical derivations with examples in which bond market will play a prominent role. Most results can be easily applied to other classes of financial models, such as: models of equity markets, futures markets, or currency markets, as well as to models of LIBORs and swap rates.

## 2.2 Random Time

Let $\tau$ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, termed a *random time* (it will be later referred to as a *default time*). We introduce the jump process $H_t = \mathbb{1}_{\{\tau \le t\}}$ and we denote by $\mathbb{H}$ the filtration generated by this process.

### 2.2.1 Hazard Process

We now assume that some *reference filtration* $\mathbb{F}$ such that $\mathcal{F}_t \subseteq \mathcal{G}$ is given. We set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t)$ for every $t \in \mathbb{R}_+$. The filtration $\mathbb{G}$ is referred to as to the *full filtration*: it includes the observations of default event. It is clear $\tau$ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time (but not necessarily an $\mathbb{F}$-stopping time). The concept of the hazard process of a random time $\tau$ is closely related to the process $F_t$ which is defined as follows:

$$F_t = \mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\}, \quad \forall\, t \in \mathbb{R}_+.$$

Let us denote $G_t = 1 - F_t = \mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\}$ and let us assume that $G_t > 0$ for every $t \in \mathbb{R}_+$ (hence, we exclude the case where $\tau$ is an $\mathbb{F}$-stopping time). Then the process $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$, given by the formula

$$\Gamma_t = -\ln(1 - F_t) = -\ln G_t, \quad \forall\, t \in \mathbb{R}_+,$$

is termed the *hazard process* of a random time $\tau$ with respect to the reference filtration $\mathbb{F}$, or briefly the $\mathbb{F}$-*hazard process* of $\tau$. Notice that $\Gamma_\infty = \infty$ and $\Gamma$ is an $\mathbb{F}$-submartingale, in general. We shall only consider the case when $\Gamma$ is an increasing process (for a construction of a random time associated with a given hazard process $\Gamma$, see Section 2.2.2). This indeed is not a serious compromise to generality. We refer to the Blanchet-Scalliet and Jeanblanc (2004) for a discussion regarding completeness of the underlying financial market and properties of process $\Gamma$. It is demonstrated there that if the underlying financial market is complete then the so-called (H) hypothesis is satisfied and, as a consequence, the process $\Gamma$ is indeed increasing.

**Remarks.** The simplifying assumption that $\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\} > 0$ for every $t \in \mathbb{R}_+$ can be relaxed. First, if we fix a maturity date $T$, it suffices to postulate that $\mathbb{Q}^*\{\tau > T \,|\, \mathcal{F}_T\} > 0$. Second, if

$\mathbb{Q}^*\{\tau \le T\} = 1$, so that the default time is bounded by some $U = \text{ess sup}\, \tau \le T$ it suffices to postulate that $\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\} > 0$ for every $t \in [0, U[$ and to consider an event $\{\tau = U\}$. For a general approach to hazard processes, the interested reader is referred to Bélanger et al. (2001).

**Deterministic intensity.** The study of a simple case when the reference filtration $\mathbb{F}$ trivial (or when a random time $\tau$ is independent of the filtration $\mathbb{F}$, and thus the hazard process is deterministic) may be instructive. Assume that $\tau$ is such that the cumulative distribution function $F(t) = \mathbb{Q}^*\{\tau \le t\}$ is an absolutely continuous function, that is,

$$F(t) = \int_0^t f(u)\, du$$

for some density function $f : \mathbb{R}_+ \to \mathbb{R}_+$. Then we have

$$F(t) = 1 - e^{-\Gamma(t)} = 1 - e^{-\int_0^t \gamma(u)\, du}, \quad \forall\, t \in \mathbb{R}_+,$$

where (recall that we postulated that $G(t) = 1 - F(t) > 0$)

$$\gamma(t) = \frac{f(t)}{1 - F(t)}, \quad \forall\, t \in \mathbb{R}_+.$$

It is clear that the function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ is non-negative and satisfies $\int_0^\infty \gamma(u)\, du = \infty$. The function $\gamma$ is called the *intensity function* of $\tau$ (or the *hazard rate*). It can be checked by direct calculations that the process $H_t - \int_0^{t\wedge\tau} \gamma(u)\, du$ is an $\mathbb{H}$-martingale.

**Stochastic intensity.** Assume that the hazard process $\Gamma$ is absolutely continuous with respect to the Lebesgue measure (and therefore an increasing process), so that there exists a process $\gamma$ such that $\Gamma_t = \int_0^t \gamma_u\, du$ for every $t \in \mathbb{R}_+$. Then the $\mathbb{F}$-predictable version of $\gamma$ is called the *stochastic intensity* of $\tau$ with respect to $\mathbb{F}$, or simply the $\mathbb{F}$-intensity of $\tau$. In terms of the stochastic intensity, the conditional probability of the default event $\{t < \tau \le T\}$, given the full information $\mathcal{G}_t$ available at time $t$, equals

$$\mathbb{Q}^*\{t < \tau \le T \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}}\, \mathbb{E}_{\mathbb{Q}^*}\Big(1 - e^{-\int_t^T \gamma_u\, du} \,\Big|\, \mathcal{F}_t\Big).$$

Thus

$$\mathbb{Q}^*\{\tau > T \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}}\, \mathbb{E}_{\mathbb{Q}^*}\Big(e^{-\int_t^T \gamma_u\, du} \,\Big|\, \mathcal{F}_t\Big).$$

It can be shown that the process

$$H_t - \Gamma_{\tau\wedge t} = H_t - \int_0^{\tau\wedge t} \gamma_u\, du = \int_0^t (1 - H_u)\gamma_u\, du, \quad \forall\, t \in \mathbb{R}_+,$$

is a (purely discontinuous) $\mathbb{G}$-martingale

### 2.2.2 Construction of a Random Time

We shall now briefly describe the most commonly used construction of a random time associated with a given hazard process $\Gamma$. It should be stressed that the random time obtained through this particular method – which will be called the *canonical construction* in what follows – has certain specific features that are not necessarily shared by all random times with a given $\mathbb{F}$-hazard process $\Gamma$. We start by assuming that we are given an $\mathbb{F}$-adapted, right-continuous, increasing process $\Gamma$ defined on a filtered probability space $(\widetilde{\Omega}, \mathbb{F}, \mathbb{P}^*)$. As usual, we assume that $\Gamma_0 = 0$ and $\Gamma_\infty = +\infty$. In many instances, the hazard process $\Gamma$ is given by the equality

$$\Gamma_t = \int_0^t \gamma_u\, du, \quad \forall\, t \in \mathbb{R}_+,$$

for some non-negative, $\mathbb{F}$-predictable, stochastic intensity $\gamma$. To construct a random time $\tau$ such that $\Gamma$ is the $\mathbb{F}$-hazard process of $\tau$, we need to enlarge the underlying probability space $\widetilde{\Omega}$. This also means that $\Gamma$ is not the $\mathbb{F}$-hazard process of $\tau$ under $\mathbb{P}^*$, but rather the $\mathbb{F}$-hazard process of $\tau$ under a suitable extension $\mathbb{Q}^*$ of the probability measure $\mathbb{P}^*$. Let $\xi$ be a random variable defined on some probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{Q}})$, uniformly distributed on the interval $[0,1]$ under $\widehat{\mathbb{Q}}$. We consider the product space $\Omega = \widetilde{\Omega} \times \widehat{\Omega}$, endowed with the product $\sigma$-field $\mathcal{G} = \mathcal{F}_\infty \otimes \widehat{\mathcal{F}}$ and the product probability measure $\mathbb{Q}^* = \mathbb{P}^* \otimes \widehat{\mathbb{Q}}$. The latter equality means that for arbitrary events $A \in \mathcal{F}_\infty$ and $B \in \widehat{\mathcal{F}}$ we have $\mathbb{Q}^*\{A \times B\} = \mathbb{P}^*\{A\}\widehat{\mathbb{Q}}\{B\}$. We define the random time $\tau : \Omega \to \mathbb{R}_+$ by setting

$$\tau = \inf\{\, t \in \mathbb{R}_+ : e^{-\Gamma_t} \le \xi \,\} = \inf\{\, t \in \mathbb{R}_+ : \Gamma_t \ge \eta \,\},$$

where the random variable $\eta = -\ln \xi$ has a unit exponential law under $\mathbb{Q}^*$. It is not difficult to find the process $F_t = \mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\}$. Indeed, since clearly $\{\tau > t\} = \{\xi < e^{-\Gamma_t}\}$ and the random variable $\Gamma_t$ is $\mathcal{F}_\infty$-measurable, we obtain

$$\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_\infty\} = \mathbb{Q}^*\{\xi < e^{-\Gamma_t} \,|\, \mathcal{F}_\infty\} = \widehat{\mathbb{Q}}\{\xi < e^{-x}\}_{x=\Gamma_t} = e^{-\Gamma_t}.$$

Consequently, we have

$$1 - F_t = \mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\} = \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_\infty\} \,|\, \mathcal{F}_t\big) = e^{-\Gamma_t},$$

and so $F$ is an $\mathbb{F}$-adapted, right-continuous, increasing process. It is also clear that $\Gamma$ is the $\mathbb{F}$-hazard process of $\tau$ under $\mathbb{Q}^*$. Finally, it can be checked that any $\mathbb{P}^*$-Brownian motion $W^*$ with respect to $\mathbb{F}$ remains a Brownian motion under $\mathbb{Q}^*$ with respect to the enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.

# 3 Defaultable Claims

A generic defaultable claim $(X, C, Z, \tau)$ with maturity date $T$ consists of:

- The *default time* $\tau$ specifying the random time of default and thus also the default events $\{\tau \le t\}$ for every $t \in [0,T]$. It is always assumed that $\tau$ is strictly positive with probability 1.
- The *promised payoff* $X$, which represents the random payoff received by the owner of the claim at time $T$, if there was no default prior to or at time $T$. The actual payoff at time $T$ associated with $X$ thus equals $X\mathbb{1}_{\{\tau > T\}}$.
- The finite variation process $C$ representing the *promised dividends* – that is, the stream of (continuous or discrete) random cash flows received by the owner of the claim prior to default or up to time $T$, whichever comes first. We assume that $C_T - C_{T-} = 0$.
- The *recovery process* $Z$, which specifies the recovery payoff $Z_\tau$ received by the owner of a claim at time of default, provided that the default occurs prior to or at maturity date $T$.

It is convenient to introduce the *dividend process* $D$, which represents all cash flows associated with a defaultable claim $(X, C, Z, \tau)$. Formally, the dividend process $D$ is defined through the formula

$$D_t = X\mathbb{1}_{\{\tau > T\}}\mathbb{1}_{[T,\infty[}(t) + \int_{]0,t]} (1 - H_u)\, dC_u + \int_{]0,t]} Z_u \, dH_u,$$

where both integrals are (stochastic) Stieltjes integrals.

**Definition 3.1** The *ex-dividend price process* $U$ of a defaultable claim of the form $(X, C, Z, \tau)$ which settles at time $T$ is given as

$$U_t = B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big( \int_{]t,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t \Big), \quad \forall\, t \in [0,T[.$$

where $\mathbb{Q}^*$ is the *spot martingale measure* and $B$ is the savings account. In addition, at maturity date we set $U_T = U_T(X) + U_T(Z) = X\mathbb{1}_{\{\tau > T\}} + Z_T\mathbb{1}_{\{\tau = T\}}$ .

Observe that $U_t = U_t(X) + U_t(Z) + U_t(C)$, where the meaning of $U_t(X), U_t(Z)$ and $U_t(C)$ is clear. Recall also that the filtration $\mathbb{G}$ models the full information, that is, the observations of the default-free market and of the default event.

## 3.1 Default Time

We assume from now on that we are given an $\mathbb{F}$-adapted, right-continuous, increasing process $\Gamma$ on $(\Omega, \mathbb{F}, \mathbb{P}^*)$ with $\Gamma_\infty = \infty$. The default time $\tau$ and the probability measure $\mathbb{Q}^*$ are constructed as in Section 2.2.2. The probability $\mathbb{Q}^*$ will play the role of the *martingale probability* for the defaultable market. It is essential to observe that:

- The Wiener process $W^*$ is also a Wiener process with respect to $\mathbb{G}$ under the probability measure $\mathbb{Q}^*$.
- We have $\mathbb{Q}^*{}_{|\mathcal{F}_t} = \mathbb{P}^*{}_{|\mathcal{F}_t}$ for every $t \in [0, T]$.

If the hazard process $\Gamma$ admits the integral representation $\Gamma_t = \int_0^t \gamma_u \, du$ then the process $\gamma$ is called the (stochastic) *intensity of default* with respect to the reference filtration $\mathbb{F}$.

## 3.2 Risk-Neutral Valuation

We shall now present the well-known valuation formulae for defaultable claims within the reduced-form setup (see, for instance, Lando (1998), Schönbucher (1998), Bielecki and Rutkowski (2002) or Bielecki et al. (2004)).

**Terminal payoff.** The valuation of the terminal payoff is based on the following classic result.

**Lemma 3.1** *For any $\mathcal{G}$-measurable, integrable random variable $X$ and any $t \le T$ we have*

$$\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{\tau>T\}} X \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{\tau>T\}} X \,|\, \mathcal{F}_t)}{\mathbb{Q}^*(\tau > t \,|\, \mathcal{F}_t)}.$$

*If, in addition, $X$ is $\mathcal{F}_T$-measurable then*

$$\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{\tau>T\}} X \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \, \mathbb{E}_{\mathbb{Q}^*}(e^{\Gamma_t - \Gamma_T} X \,|\, \mathcal{F}_t).$$

Let $X$ be an $\mathcal{F}_T$-measurable random variable representing the promised payoff at maturity date $T$. We consider a defaultable claim of the form $\mathbb{1}_{\{\tau>T\}} X$ with zero recovery in case of default (i.e., we set $Z = C = 0$). Using the definition of the ex-dividend price of a defaultable claim, we get the following *risk-neutral valuation formula*

$$U_t(X) = B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} \mathbb{1}_{\{\tau>T\}} X \,|\, \mathcal{G}_t)$$

which holds for any $t < T$. The next result is a straightforward consequence of Lemma 3.1.

**Proposition 3.1** *The price of the promised payoff $X$ satisfies for $t \in [0, T]$*

$$U_t(X) = B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} X \mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \widetilde{U}_t(X), \tag{1}$$

*where*

$$\widetilde{U}_t(X) = B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} e^{\Gamma_t - \Gamma_T} X \,|\, \mathcal{F}_t) = \widehat{B}_t \, \mathbb{E}_{\mathbb{Q}^*}(\widehat{B}_T^{-1} X \,|\, \mathcal{F}_t)$$

*where the risk-adjusted savings account $\widehat{B}_t$ equals $\widehat{B}_t = B_t e^{\Gamma_t}$. If, in addition, the default time admits the intensity process $\gamma$ then*

$$\widehat{B}_t = \exp\left(\int_0^t (r_u + \gamma_u) \, du\right).$$

The process $\widetilde{U}_t(X)$ represents the *pre-default value* at time $t$ of the promised payoff $X$. Notice that $\widetilde{U}_T(X) = X$ and the process $\widetilde{U}_t(X)/\widehat{B}_t$, $t \in [0,T]$, is a continuous $\mathbb{F}$-martingale (thus, the process $\widetilde{U}(X)$ is a continuous $\mathbb{F}$-semimartingale).

**Remarks.** The valuation formula (1), as well as the concept of pre-default value, should be supported by replication arguments. To this end, we need first to construct a suitable model of a defaultable market. In fact, if we wish to use formula (1), we need to know the joint law of all random variables involved, and this appears to be a non-trivial issue, in general.

**Recovery payoff.** The following result appears to be useful in the valuation of the recovery payoff $Z_\tau$ which occurs at time $\tau$. The process $\widetilde{U}(Z)$ introduced below represents the pre-default value of the recovery payoff.

For the proof of Proposition 3.2 we refer, for instance, to Bielecki and Rutkowski (2002) (see Propositions 5.1.1 and 8.2.1 therein).

**Proposition 3.2** *Let $\Gamma$ be a continuous process and let $Z$ be an $\mathbb{F}$-predictable bounded process. Then for every $t \in [0,T]$ we have*

$$\begin{aligned} U_t(Z) &= B_t\, \mathbb{E}_{\mathbb{Q}^*}(B_\tau^{-1} Z_\tau 1\!\!1_{\{t<\tau\le T\}} \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big( \int_t^T Z_u B_u^{-1} e^{\Gamma_t - \Gamma_u}\, d\Gamma_u \,\Big|\, \mathcal{F}_t \Big) \\ &= 1\!\!1_{\{\tau>t\}} \widehat{B}_t\, \mathbb{E}_{\mathbb{Q}^*}\Big( \int_t^T Z_u \widehat{B}_u^{-1} d\Gamma_u \,\Big|\, \mathcal{F}_t \Big) = 1\!\!1_{\{\tau>t\}} \widetilde{U}_t(Z). \end{aligned}$$

*If the default intensity $\gamma$ with respect to $\mathbb{F}$ exists then we have*

$$\widetilde{U}_t(Z) = \mathbb{E}_{\mathbb{Q}^*}\Big( \int_t^T Z_u e^{-\int_t^u (r_v+\gamma_v)\, dv} \gamma_u \, du \,\Big|\, \mathcal{F}_t \Big).$$

**Remarks.** Notice that $\widetilde{U}_T(Z) = 0$ while, in general, $U_T(Z) = Z_T 1\!\!1_{\{\tau=T\}}$ is non-zero. Note, however, that if the hazard process $\Gamma$ is assumed to be continuous then we have $\mathbb{P}\{\tau = T\} = 0$, and thus $\widetilde{U}_T(Z) = 0 = U_T(Z)$.

**Promised dividends.** To value the promised dividends $C$ that are paid prior to default time $\tau$ we shall make use of the following result. Notice that at any date $t < T$ the process $\widetilde{U}(C)$ gives the pre-default value of future promised dividends.

**Proposition 3.3** *Let $\Gamma$ be a continuous process and let $C$ be an $\mathbb{F}$-predictable, bounded process of finite variation. Then for every $t \in [0,T]$*

$$\begin{aligned} U_t(C) &= B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big( \int_{]t,T]} B_u^{-1}(1-H_u)\, dC_u \,\Big|\, \mathcal{G}_t \Big) = 1\!\!1_{\{\tau>t\}} B_t\, \mathbb{E}_{\mathbb{Q}^*}\Big( \int_{]t,T]} B_u^{-1} e^{\Gamma_t-\Gamma_u}\, dC_u \,\Big|\, \mathcal{F}_t \Big) \\ &= 1\!\!1_{\{\tau>t\}} \widehat{B}_t\, \mathbb{E}_{\mathbb{Q}^*}\Big( \int_{]t,T]} \widehat{B}_u^{-1}\, dC_u \,\Big|\, \mathcal{F}_t \Big) = 1\!\!1_{\{\tau>t\}} \widetilde{U}_t(C). \end{aligned}$$

*If, in addition, the default time $\tau$ admits the intensity $\gamma$ with respect to $\mathbb{F}$ then*

$$\widetilde{U}_t(C) = \mathbb{E}_{\mathbb{Q}^*}\Big( \int_{]t,T]} e^{-\int_t^u (r_v+\gamma_v)\, dv}\, dC_u \,\Big|\, \mathcal{F}_t \Big).$$

## 3.3 Defaultable Term Structure

For a defaultable discount bond with zero recovery[1] it is natural to adopt the following definition of the price

$$D^0(t,T) = B_t\, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} 1\!\!1_{\{\tau>T\}} \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \widetilde{D}^0(t,T)$$

[1] Note that the superscript 0 refers to the postulated zero recovery scheme.

where $\widetilde{D}^0(t,T)$ stands for the *pre-default value* of the bond, which is given by the following equality:

$$\widetilde{D}^0(t,T) = \widehat{B}_t \, \mathbb{E}_{\mathbb{Q}^*}(\widehat{B}_T^{-1} \,|\, \mathcal{F}_t).$$

Since $\mathbb{F}$ is assumed to be the Brownian filtration, the process $\widetilde{D}^0(t,T)/\widehat{B}_t$ is a continuous, strictly positive, $\mathbb{F}$-martingale. Therefore, the pre-default bond price $\widetilde{D}^0(t,T)$ is a continuous, strictly positive, $\mathbb{F}$-semimartingale. In the special case, when $\Gamma$ is deterministic, we have $\widetilde{D}^0(t,T) = e^{\Gamma_t - \Gamma_T} B(t,T)$.

**Remarks.** The case zero recovery is, of course, only a particular example. For more general recovery schemes and the corresponding bond valuation results, we refer, for instance, to Section 2.2.4 in Bielecki et al. (2004).

Let $\mathbb{Q}_T$ stand for the *forward martingale measure*, given on $(\Omega, \mathcal{G}_T)$ (as well as on $(\Omega, \mathcal{F}_T)$) through the formula

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}^*} = \frac{1}{B_T B(0,T)}\,, \quad \mathbb{Q}^*\text{-a.s.},$$

so that the process $W_t^T = W_t^* - \int_0^t b(u,T)\,du$ is a Brownian motion under $\mathbb{Q}_T$. Denote by $F(t,U,T) = B(t,U)(B(t,T))^{-1}$ the forward price of the $U$-maturity bond, so that

$$dF(t,U,T) = F(t,U,T)\big(b(t,U) - b(t,T)\big) dW_t^T.$$

Since the processes $B_t$ and $B(t,T)$ are $\mathbb{F}$-adapted, it can be shown (see, e.g., Jamshidian (2002)) that $\Gamma$ is also the $\mathbb{F}$-hazard process of $\tau$ under $\mathbb{Q}_T$, and thus

$$\mathbb{Q}_T\{t < \tau \le T \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}_T}(e^{\Gamma_t - \Gamma_T} \,|\, \mathcal{F}_t).$$

Let us define an auxiliary process $\Gamma(t,T) = \widetilde{D}^0(t,T)(B(t,T))^{-1}$ (for a fixed $T > 0$). The next result examines the basic properties of the process $\Gamma(t,T)$.

**Lemma 3.2** *Assume that the $\mathbb{F}$-hazard process $\Gamma$ is continuous. The process $\Gamma(t,T)$, $t \in [0,T]$, is a continuous $\mathbb{F}$-submartingale and*

$$d\Gamma(t,T) = \Gamma(t,T)\big(d\Gamma_t + \beta(t,T)\, dW_t^T\big) \tag{2}$$

*for some $\mathbb{F}$-predictable process $\beta(t,T)$. The process $\Gamma(t,T)$ is of finite variation if and only if the hazard process $\Gamma$ is deterministic. In the latter case, we have $\Gamma(t,T) = e^{\Gamma_t - \Gamma_T}$.*

*Proof.* Recall that $\widehat{B}_t = B_t e^{\Gamma_t}$ and notice that

$$\Gamma(t,T) = \frac{\widetilde{D}^0(t,T)}{B(t,T)} = \frac{\widehat{B}_t \, \mathbb{E}_{\mathbb{Q}^*}(\widehat{B}_T^{-1} \,|\, \mathcal{F}_t)}{B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} \,|\, \mathcal{F}_t)} = \mathbb{E}_{\mathbb{Q}_T}(e^{\Gamma_t - \Gamma_T} \,|\, \mathcal{F}_t) = e^{\Gamma_t} M_t$$

where we set $M_t = \mathbb{E}_{\mathbb{Q}_T}(e^{-\Gamma_T} \,|\, \mathcal{F}_t)$. We conclude that $\Gamma(t,T)$ is the product of a strictly positive, increasing, right-continuous, $\mathbb{F}$-adapted process $e^{\Gamma_t}$, and a strictly positive, continuous,[2] $\mathbb{F}$-martingale $M$. Furthermore, there exists an $\mathbb{F}$-predictable process $\widehat{\beta}(t,T)$ such that $M$ satisfies

$$dM_t = M_t \widehat{\beta}(t,T)\, dW_t^T$$

with the initial condition $M_0 = \mathbb{E}_{\mathbb{Q}_T}(e^{-\Gamma_T})$. Formula (2) follows by an application of Itô's formula, by setting $\beta(t,T) = e^{-\Gamma_t} \widehat{\beta}(t,T)$. To complete the proof, it suffices to recall that a continuous martingale is never of finite variation, unless it is a constant process. □

Suppose that $\Gamma_t = \int_0^t \gamma_u \, du$. Then (2) yields

$$d\Gamma(t,T) = \Gamma(t,T)\big(\gamma_t \, dt + \beta(t,T)\, dW_t^T\big).$$

[2]Recall that the filtration $\mathbb{F}$ is generated by a process $W^*$, which is a Wiener process with respect to $\mathbb{P}^*$ and $\mathbb{Q}^*$. All martingales with respect to a Brownian filtration are continuous processes.

Consequently, the pre-default price $\widetilde{D}^0(t,T) = \Gamma(t,T)B(t,T)$ is governed by

$$d\widetilde{D}^0(t,T) = \widetilde{D}^0(t,T)\Big(\big(r_t + \gamma_t + \beta(t,T)b(t,T)\big)\,dt + \widetilde{b}(t,T)\,dW_t^*\Big) \tag{3}$$

where the volatility process equals $\widetilde{b}(t,T) = \beta(t,T) + b(t,T)$.

# 4 Properties of Trading Strategies

In this section, we shall examine the most basic properties of the wealth process of a self-financing trading strategy. First, we concentrate on trading in default-free assets. In the next step, we also include defaultable assets in our portfolio.

## 4.1 Default-Free Primary Assets

Our goal in this section is to present some auxiliary results related to the concept of a self-financing trading strategy for a market model involving default-free and defaultable securities. For the sake of the reader's convenience, we shall first discuss briefly the classic concepts of self-financing cash and futures strategies in the context of default-free market model. It appears that in case of defaultable securities only minor adjustments of definitions and results are needed (see, Vaillant (2001) or Blanchet-Scalliet and Jeanblanc (2004)).

### 4.1.1 Cash Strategies

Let $Y_t^1$ and $Y_t^2$ stand for the cash prices at time $t \in [0,T]$ of two tradeable assets. We postulate that $Y^1$ and $Y^2$ are continuous semimartingales. We assume, in addition, that the process $Y^1$ is strictly positive, so that it can be used as a numeraire.

**Remarks.** We have chosen that price processes for default-free securities to be continuous semimartingales. Results of this section can be extended to the case of general semimartingales (for instance, jump diffusions). Our choice was motivated by the desire of providing relatively simple closed-form expressions.

Let $\phi = (\phi^1, \phi^2)$ be a trading strategy for default-free market so that, in particular, processes $\phi^1$ and $\phi^2$ are adapted to the reference filtration $\mathbb{F}$ (the same measurability assumption will be valid for components $\phi^1, \dots, \phi^k$ of a $k$-dimensional trading strategy).

Let $V_t(\phi)$ denote the wealth of the cash strategy $\phi = (\phi^1, \phi^2)$ at time $t$, so that

$$V_t(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2, \quad \forall\, t \in [0,T].$$

We say that the cash strategy $\phi$ is *self-financing* if

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^1\, dY_u^1 + \int_0^t \phi_u^2\, dY_u^2, \quad \forall\, t \in [0,T],$$

that is,

$$dV_t(\phi) = \phi_t^1\, dY_t^1 + \phi_t^2\, dY_t^2.$$

This yields

$$dV_t(\phi) = (V_t(\phi) - \phi_t^2 Y_t^2)(Y_t^1)^{-1}\, dY_t^1 + \phi_t^2\, dY_t^2.$$

Let us introduce relative values: $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ and $Y_t^{2,1} = Y_t^2(Y_t^1)^{-1}$. A simple application of Itô's formula yields

$$V_t^1(\phi) = V_0^1(\phi) + \int_0^t \phi_u^2\, dY_u^{2,1}.$$

It is well known that a similar result holds for any finite number of cash assets. Let $Y^1_t, Y^2_t, \dots, Y^k_t$ represent that cash values at time $t$ of $k$ assets. We postulate that $Y^1, Y^2, \dots, Y^k$ are continuous semimartingales. Then the wealth $V_t(\phi)$ of a trading strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ equals

$$V_t(\phi) = \sum_{i=1}^{k} \phi^i_t Y^i_t, \quad \forall\, t \in [0,T], \tag{4}$$

and $\phi$ is said to be a *self-financing cash strategy* if

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{k} \int_0^t \phi^i_u \, dY^i_u, \quad \forall\, t \in [0,T]. \tag{5}$$

Suppose that the process $Y^1$ is strictly positive. Then by combining the last two formulae, we obtain

$$dV_t(\phi) = \Big(V_t(\phi) - \sum_{i=2}^{k} \phi^i_t Y^i_t\Big)(Y^1_t)^{-1}\, dY^1_t + \sum_{i=2}^{k} \phi^i_t \, dY^i_t.$$

The latter representation shows that the wealth process depends only on $k-1$ components of $\phi$. Choosing $Y^1$ as a numeraire asset, and denoting $V^1_t(\phi) = V_t(\phi)(Y^1_t)^{-1}$, $Y^{i,1}_t = Y^i_t(Y^1_t)^{-1}$, we get the following well-known result.

**Lemma 4.1** *Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a self-financing cash strategy. Then we have*

$$V^1_t(\phi) = V^1_0(\phi) + \sum_{i=2}^{k} \int_0^t \phi^i_u \, dY^{i,1}_u, \quad \forall\, t \in [0,T].$$

### 4.1.2 Cash-Futures Strategies

Let us first consider the special case of two assets. Assume that $Y^1_t$ and $Y^2_t$ represent the cash and futures prices at time $t \in [0,T]$ of some assets, respectively. As before, we postulate that $Y^1$ and $Y^2$ are continuous semimartingales. Moreover, $Y^1$ is assumed to be a strictly positive process. In view of specific features of a futures contract, it is natural to postulate that the wealth $V_t(\phi)$ satisfies

$$V_t(\phi) = \phi^1_t Y^1_t + \phi^2_t 0 = \phi^1_t Y^1_t, \quad \forall\, t \in [0,T].$$

The cash-futures strategy $\phi = (\phi^1, \phi^2)$ is self-financing if

$$dV_t(\phi) = \phi^1_t \, dY^1_t + \phi^2_t \, dY^2_t, \tag{6}$$

which yields, provided that $Y^1$ is strictly positive,

$$dV_t(\phi) = V_t(\phi)(Y^1_t)^{-1}\, dY^1_t + \phi^2_t \, dY^2_t.$$

**Remarks.** Let us recall that the futures price $Y^2_t$ (that is, the quotation of a futures contract at time $t$) has different features than the cash price of an asset. Specifically, we make the standard assumption that it is possible to enter a futures contract at no initial cost. The gains or losses from futures contracts are associated with *marking to market* (see, for instance, Duffie (2003) or Musiela and Rutkowski (1997)). Note that 0 in the formula defining $V_t(\phi)$ is aimed to represent the value of a futures contract at time $t$, as opposed to the futures price $Y^2_t$ at this date.

**Lemma 4.2** *Let $\phi = (\phi^1, \phi^2)$ be a self-financing cash-futures strategy. Suppose that the processes $Y^1$ and $Y^2$ are strictly positive. Then the relative wealth process $V^1_t(\phi) = V_t(\phi)(Y^1_t)^{-1}$ satisfies*

$$V^1_t(\phi) = V^1_0(\phi) + \int_0^t \widehat{\phi}^{2,1}_u \, d\widehat{Y}^{2,1}_u, \quad \forall\, t \in [0,T],$$

*where* $\widehat{\phi}^{2,1}_t = \phi^2_t (Y^1_t)^{-1} e^{\alpha^{2,1}_t}$, $\widehat{Y}^{2,1}_t = Y^2_t e^{-\alpha^{2,1}_t}$ *and*

$$\alpha^{2,1}_t = \langle \ln Y^2, \ln Y^1 \rangle_t = \int_0^t (Y^2_u)^{-1} (Y^1_u)^{-1} \, d\langle Y^2, Y^1 \rangle_u .$$

*Proof.* For brevity, we write $V_t = V_t(\phi)$ and $V^1_t = V^1_t(\phi)$. The Itô formula, combined with (6), yields

$$\begin{aligned} dV^1_t &= (Y^1_t)^{-1} dV_t + V_t \, d(Y^1_t)^{-1} + d\langle (Y^1)^{-1}, V \rangle_t \\ &= \phi^1_t (Y^1_t)^{-1} \, dY^1_t + \phi^2_t (Y^1_t)^{-1} \, dY^2_t + \phi^1_t Y^1_t \, d(Y^1_t)^{-1} \\ &\quad - \phi^1_t (Y^1_t)^{-2} \, d\langle Y^1, Y^1 \rangle_t - \phi^2_t (Y^1_t)^{-2} \, d\langle Y^1, Y^2 \rangle_t \\ &= \phi^2_t (Y^1_t)^{-1} dY^2_t - \phi^2_t (Y^1_t)^{-2} \, d\langle Y^1, Y^2 \rangle_t \\ &= \phi^2_t e^{\alpha^{2,1}_t} (Y^1_t)^{-1} \big( e^{-\alpha^{2,1}_t} dY^2_t - Y^2_t e^{-\alpha^{2,1}_t} \, d\alpha^{2,1}_t \big) = \widehat{\phi}^{2,1}_t \, d\widehat{Y}^{2,1}_t \end{aligned}$$

and the result follows. □

Let $Y^1, \ldots, Y^l$ be the cash prices of $l$ assets, and let $Y^{l+1}, \ldots, Y^k$ represent the futures prices of $k - l$ assets. Then the wealth process of a trading strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ is given by the formula

$$V_t(\phi) = \sum_{i=1}^{l} \phi^i_t Y^i_t, \quad \forall \, t \in [0, T], \tag{7}$$

and $\phi$ is a *self-financing cash-futures strategy* whenever

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{k} \int_0^t \phi^i_u \, dY^i_u, \quad \forall \, t \in [0, T].$$

The proof of the next result relies on the similar calculations as the proofs of Lemmas 4.1 and 4.2.

**Lemma 4.3** *Let $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ be a self-financing cash-futures strategy. Suppose that the processes $Y^1$ and $Y^{l+1}, \ldots, Y^k$ are strictly positive. Then the relative wealth process $V^1_t(\phi) = V_t(\phi)(Y^1_t)^{-1}$ satisfies*

$$V^1_t(\phi) = V^1_0(\phi) + \sum_{i=2}^{l} \int_0^t \phi^i_u \, dY^{i,1}_u + \sum_{i=l+1}^{k} \int_0^t \widehat{\phi}^{i,1}_u \, d\widehat{Y}^{i,1}_u, \quad \forall \, t \in [0, T],$$

*where we denote* $Y^{i,1}_t = Y^i_t (Y^1_t)^{-1}$, $\widehat{\phi}^{i,1}_t = \phi^i_t (Y^1_t)^{-1} e^{\alpha^{i1}_t}$, $\widehat{Y}^{i,1}_t = Y^i_t e^{-\alpha^{i1}_t}$, *and*

$$\alpha^{i1}_t = \langle \ln Y^i, \ln Y^1 \rangle_t = \int_0^t (Y^i_u)^{-1} (Y^1_u)^{-1} \, d\langle Y^i, Y^1 \rangle_u .$$

### 4.1.3 Constrained Cash Strategies

We return to the analysis of cash strategies for some $k \geq 3$. Price processes $Y^1, Y^2, \ldots, Y^k$ are assumed to be continuous semimartingales. We postulate, in addition, that $Y^1$ and $Y^{l+1}, \ldots, Y^k$ are strictly positive processes, where $1 < l + 1 < k$. Let $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ be a self-financing trading strategy, so that the wealth process $V(\phi)$ satisfies (4)-(5). We shall consider three particular cases of increasing generality.

**Strategies with zero net investment in $Y^{l+1}, \ldots, Y^k$.** Assume first that at any time $t$ there is zero net investment in assets $Y^{l+1}, \ldots, Y^k$. Specifically, we postulate that the strategy is subject to the following constraint:

$$\sum_{i=l+1}^{k} \phi^i_t Y^i_t = 0, \quad \forall \, t \in [0, T], \tag{8}$$

so that $V_t(\phi)$ is given by (7). Equivalently, we have $\phi_t^k = -\sum_{i=l+1}^{k-1} \phi_t^i Y_t^i (Y_t^k)^{-1}$, provided that $Y^k$ is a strictly positive process. Combining the last equality with (5), we obtain

$$dV_t(\phi) = \Big(V_t(\phi) - \sum_{i=2}^{l} \phi_t^i Y_t^i\Big)(Y_t^1)^{-1}\, dY_t^1 + \sum_{i=2}^{l} \phi_t^i \, dY_t^i + \sum_{i=l+1}^{k-1} \phi_t^i \big(dY_t^i - Y_t^i (Y_t^k)^{-1}\, dY_t^k\big).$$

It is thus clear that the wealth process $V(\phi)$ depends only on $k-2$ components $\phi^2, \ldots, \phi^{k-1}$ of the $k$-dimensional trading strategy $\phi$. The following result, which can be seen as an extension of Lemma 4.2, provides a more convenient representation for the (relative) wealth process.

**Lemma 4.4** *Let $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ be a self-financing cash strategy such that* (8) *holds. Assume that the processes $Y^1, Y^{l+1}, \ldots, Y^k$ are strictly positive. Then the relative wealth process $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ satisfies*

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \widehat{\phi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1}, \quad \forall\, t \in [0,T],$$

*where we denote*

$$\widehat{\phi}_t^{i,k,1} = \phi_t^i (Y_t^{1,k})^{-1} e^{\alpha_t^{i,k,1}}, \quad \widehat{Y}_t^{i,k,1} = Y_t^{i,k} e^{-\alpha_t^{i,k,1}}, \tag{9}$$

*and*

$$\alpha_t^{i,k,1} = \langle \ln Y^{i,k}, \ln Y^{1,k} \rangle_t = \int_0^t (Y_u^{i,k})^{-1} (Y_u^{1,k})^{-1} \, d\langle Y^{i,k}, Y^{1,k} \rangle_u. \tag{10}$$

*Proof.* Let us consider the relative values of all processes, with the price $Y^k$ chosen as a numeraire, and let us consider the process $V_t^k(\phi) := V_t(\phi)(Y_t^k)^{-1} = \sum_{i=1}^k \phi_t^i Y_t^{i,k}$. In view of the constraint (8) we have that $V_t^k(\phi) = \sum_{i=1}^l \phi_t^i Y_t^{i,k}$. In addition, similarly as in Lemma 4.1 we get

$$dV_t^k(\phi) = \sum_{i=1}^{k-1} \phi_t^i \, dY_t^{i,k}$$

where $Y_t^{i,k} = Y_t^i (Y_t^k)^{-1}$. Since $Y_t^{i,k}(Y_t^{1,k})^{-1} = Y_t^{i,1}$ and $V_t^1(\phi) = V_t^k(\phi)(Y_t^{1,k})^{-1}$, using argument analogous as in proof of Lemma 4.2, we obtain

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \widehat{\phi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1}, \quad \forall\, t \in [0,T],$$

where the processes $\widehat{\phi}_t^{i,k,1}$, $\widehat{Y}_t^{i,k,1}$ and $\alpha_t^{i,k,1}$ are given by (9)-(10). □

**Strategies with a prespecified net investment $Z$ in $Y^{l+1}, \ldots, Y^k$.** We shall now postulate that the strategy $\phi$ is such that

$$\sum_{i=l+1}^{k} \phi_t^i Y_t^i = Z_t, \quad \forall\, t \in [0,T], \tag{11}$$

for a prespecified, $\mathbb{F}$-progressively measurable, process $Z$. The following result is a rather straightforward extension of Lemma 4.4.

**Lemma 4.5** *Let $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ be a self-financing cash strategy such that* (11) *holds. Assume that the processes $Y^1, Y^{l+1}, \ldots, Y^k$ are strictly positive. Then the relative wealth process $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ satisfies*

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \widehat{\phi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1} + \int_0^t Z_u (Y_u^k)^{-1} \, d(Y_u^{1,k})^{-1}$$

*where $\widehat{\phi}_t^{i,k,1}$, $\widehat{Y}_t^{i,k,1}$ and $\alpha_t^{i,k,1}$ are given by* (9)-(10).

*Proof.* Let us sketch the proof of the lemma for $k = 3$. Then $l = 2$ and $\phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 = Z_t$ for every $t \in [0, T]$. Consequently, for the process $V^3(\phi) = V(\phi)(Y^3)^{-1}$ we get

$$V_t^3(\phi) = \sum_{i=1}^{3} \phi_t^i Y_t^i (Y_t^3)^{-1} = \phi_t^1 Y_t^{1,3} + Z_t (Y_t^3)^{-1}, \quad \forall\, t \in [0, T].$$

Furthermore, the self-financing condition yields

$$dV_t^3(\phi) = \phi_t^1\, dY_t^{1,3} + \phi_t^2\, dY_t^{2,3}.$$

Proceeding as in the proof of Lemma 4.2, we obtain for $V^1(\phi) = V^3(\phi)(Y^{1,3})^{-1}$

$$\begin{aligned} dV_t^1(\phi) &= \phi_t^2 e^{\alpha_t^{2,3,1}} (Y_t^{1,3})^{-1} \big(e^{-\alpha_t^{2,3,1}} dY_t^{2,3} - Y_t^{2,3} e^{-\alpha_t^{2,3,1}} d\alpha_t^{2,3,1}\big) + Z_t (Y_t^3)^{-1} d(Y_t^{1,3})^{-1} \\ &= \widehat{\phi}_u^{2,3,1}\, d\widehat{Y}_u^{2,3,1} + Z_t (Y_t^3)^{-1} d(Y_t^{1,3})^{-1} \end{aligned}$$

where $\widehat{\phi}_t^{2,3,1} = \phi_t^2 (Y_t^{1,3})^{-1} e^{\alpha_t^{2,3,1}}$, $\widehat{Y}_t^{2,3,1} = Y_t^{2,3} e^{-\alpha_t^{2,3,1}}$ and

$$\alpha_t^{2,3,1} = \langle \ln Y^{2,3}, \ln Y^{1,3} \rangle_t = \int_0^t (Y_u^{2,3})^{-1} (Y_u^{1,3})^{-1}\, d\langle Y^{2,3}, Y^{1,3} \rangle_u.$$

The proof for the general case is based on similar calculations. □

**Strategies with consumption $A$ and a prespecified net investment $Z$ in $Y^{l+1}, \dots, Y^k$.** Let $A$ be an $\mathbb{F}$-adapted process of finite variation, with $A_0 = 0$. We consider a self-financing cash strategy $\phi$ with *consumption*[3] *process* $A$, so that the wealth process $V(\phi)$ satisfies:

$$V_t(\phi) = \sum_{i=1}^{k} \phi_t^i Y_t^i = \sum_{i=1}^{l} \phi_t^i Y_t^i + Z_t, \quad \forall\, t \in [0, T],$$

and

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{k} \int_0^t \phi_u^i\, dY_u^i + A_t, \quad \forall\, t \in [0, T].$$

Then it suffices to modify the formula established in Lemma 4.5 by adding a term associated with the consumption process $A$. Specifically, for the relative wealth process $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ we obtain the following integral representation, which is valid for every $t \in [0, T]$

$$\begin{aligned} V_t^1(\phi) &= V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i\, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \widehat{\phi}_u^{i,k,1}\, d\widehat{Y}_u^{i,k,1} \\ &\quad + \int_0^t Z_u (Y_u^k)^{-1}\, d(Y_u^{1,k})^{-1} + \int_0^t (Y_u^1)^{-1}\, dA_u. \end{aligned}$$

## 4.2 Defaultable and Default-Free Primary Assets

Let $Y^1, \dots, Y^m$ be price processes of $m$ defaultable assets, and let $Y^{m+1}, \dots, Y^k$ represent prices of $k - m$ default-free assets. Processes $Y^{m+1}, \dots, Y^k$ are assumed to be continuous semimartingales. We make here an essential assumption that $\tau$ is the default time for each defaultable asset $Y^i$, $i = 1, \dots, m$. Of course, in the case of defaultable assets with different default times (e.g., when dealing with the first-to-default claim), some definitions should be modified in a natural way. A special case of first-to-default claims is examined in Section 5.4.

[3]We use here a generic term 'consumption' to reflect the impact of $A$ on the wealth. The financial interpretation of $A$ depends on particular circumstances. For instance, an increasing process $A$ represents the inflows of cash, rather than the outflows of cash (the latter case is commonly referred to as consumption in the financial literature).

### 4.2.1 Self-Financing Trading Strategies

The following definition is a rather obvious extension of conditions (4)-(5). We postulate here that the processes $\phi^1, \dots, \phi^k$ are $\mathbb{G}$-adapted processes, in general.

**Definition 4.1** The wealth $V_t(\phi)$ of a trading strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ equals $V_t(\phi) = \sum_{i=1}^k \phi_t^i Y_t^i$ for every $t \in [0,T]$. A strategy $\phi$ is said to be *self-financing* if

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^m \int_0^t \phi_{u-}^i \, dY_u^i + \sum_{i=m+1}^k \int_0^t \phi_u^i \, dY_u^i, \quad \forall \, t \in [0,T].$$

Although Definition 4.1 is in a general setup, it can be simplified for our further purposes. Indeed, since we shall deal only with defaultable claims with default time $\tau$, we shall only examine a particular trading strategy $\phi$ prior to and at default time $\tau$ or, more precisely, on the stochastic interval $[\![0, \tau \wedge T]\!]$. In fact, we shall examine separately the following issues: (i) the behavior of the wealth process $V(\phi)$ on the random interval $[\![0, \tau \wedge T[\![$ and (ii) the size of its jump at the random time moment $\tau \wedge T$ or, equivalently, the value of $V_{\tau \wedge T}$. Such a study is, of course, sufficient in our setup, since we only consider the case where a recovery payment (if any) is made at the default time (and not after this date). Consequently, since we never deal with a trading strategy after the random time $\tau \wedge T$, we may and do assume from now on that all components $\phi^1, \phi^2, \dots, \phi^k$ of a portfolio $\phi$ are $\mathbb{F}$-adapted, rather than $\mathbb{G}$-predictable processes.

It is worthwhile to mention, that in the next two chapters we will examine the importance of the measurability property of an admissible trading strategy within the framework of optimization problems in incomplete market.

**Remarks.** It can be formally shown that for any $\mathbb{R}^k$-valued $\mathbb{G}$-predictable process $\phi$ there exists a unique $\mathbb{F}$-predictable process $\psi$ such that $\mathbb{1}_{\{\tau \ge t\}} \phi_t = \mathbb{1}_{\{\tau \ge t\}} \psi_t$ for every $t \in [0,T]$. In addition, we find it convenient to postulate, by convention, that the price processes $Y^{m+1}, \dots, Y^k$ are also stopped at the random time $\tau \wedge T$.

For the sake of simplicity of presentation, we find it convenient to postulate, in addition, the left-continuity of a trading strategy, and thus we have the following definition, which is sufficiently general to cover several cases of interest.

**Definition 4.2** By a *trading strategy* $\phi = (\phi^1, \phi^2, \dots \phi^k)$ we mean a family $\phi^1, \phi^2, \dots \phi^k$ of $\mathbb{F}$-adapted and left-continuous processes.

Let us stress that if a trading strategy considered in this section is self-financing on $[\![0, \tau \wedge T[\![$ then it it also self-financing on $[\![0, \tau \wedge T]\!]$. At the intuitive level, the portfolio is not rebalanced at time $\tau \wedge T$, but it is rather liquidated it in order to cover liabilities. Let $\widetilde{Y}_t^i$ stands for the pre-default value of the $i^{\text{th}}$ defaultable asset.

**Definition 4.3** The *pre-default wealth* $\widetilde{V}(\phi)$ of a trading strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ equals

$$\widetilde{V}_t(\phi) = \sum_{i=1}^m \phi_t^i \widetilde{Y}_t^i + \sum_{i=m+1}^k \phi_t^i Y_t^i, \quad \forall \, t \in [0,T].$$

A strategy $\phi$ is said to be *self-financing prior to default* if

$$\widetilde{V}_t(\phi) = \widetilde{V}_0(\phi) + \sum_{i=1}^m \int_0^t \phi_u^i \, d\widetilde{Y}_u^i + \sum_{i=m+1}^k \int_0^t \phi_u^i \, dY_u^i, \quad \forall \, t \in [0,T].$$

Note that $\widetilde{V}_0(\phi) = V_0(\phi)$, since $\mathbb{P}\{\tau > 0\} = 1$. Let us stress that if a trading strategy $\phi$ is self-financing prior to default then $\phi$ is also self-financing on $[0, T]$. Indeed, we always postulate that trading ceases at time of default, and the terminal wealth at time $\tau \wedge T$ equals

$$V_{\tau\wedge T}(\phi) = \sum_{i=1}^{k} \phi^i_{\tau\wedge T} Y^i_{\tau\wedge T} = \sum_{i=1}^{k} \phi^i_{(\tau\wedge T)-} Y^i_{\tau\wedge T}.$$

Of course, on the event $\{\tau > T\}$ we also have

$$V_{\tau\wedge T}(\phi) = V_T(\phi) = \widetilde{V}_T(\phi) = \sum_{i=1}^{m} \phi^i_T \widetilde{Y}^i_T + \sum_{i=m+1}^{k} \phi^i_T Y^i_T.$$

Hence, we shall not distinguish in what follows between the concept of a self-financing trading strategy and a trading strategy self-financing prior to default.

### 4.2.2 Zero Recovery for Defaultable Assets

The following assumption corresponds to the simplest situation of zero recovery for all defaultable primary assets that are used for replication. Manifestly, this assumption is not practical, and thus it will be later relaxed.

**Assumption (A)**. We assume that the defaultable primary assets $Y^1, \dots, Y^m$ are all subject to the zero recovery scheme, and they have a common default time $\tau$.

By virtue of Assumption (A), the prices $Y^1, \dots, Y^m$ vanish at default time $\tau$, and thus also after this date. Consequently, for every $i = 1, \dots, m$ we have $Y^i_t = \mathbb{1}_{\{\tau>t\}} \widetilde{Y}^i_t$ for every $t \in [0, T]$ for some $\mathbb{F}$-adapted processes $\widetilde{Y}^1, \dots, \widetilde{Y}^m$. In other words, for any $i = 1, \dots, m$ the price $Y^i$ jumps from $\widetilde{Y}^i_{\tau-}$ to $\widetilde{Y}^i_\tau = 0$ at the time of default. We make a technical assumption that the pre-default values $\widetilde{Y}^1, \dots, \widetilde{Y}^m$ are continuous $\mathbb{F}$-semimartingales.

In order to be able to use the price $Y^1$ as a numeraire prior to default, we assume that the pre-default price $\widetilde{Y}^1$ is a strictly positive continuous $\mathbb{F}$-semimartingale. Notice that $\widetilde{Y}^1_0 = Y^1_0$.

Assume first zero recovery for the defaultable contingent claim we wish to replicate. Thus, at time $\tau$ the wealth process of any strategy that is capable to replicate the claim $\mathbb{1}_{\{\tau>T\}} X$ should necessarily jump to zero, provided that $\tau \le T$. We can achieve this by considering only self-financing strategies $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ such that at any time the net investment in default-free assets $Y^{m+1}, \dots, Y^k$ equals zero, so that we have

$$\sum_{i=m+1}^{k} \phi^i_t Y^i_t = 0, \quad \forall\, t \in [0, T]. \tag{12}$$

In the general case, that is, when $Z$ is a prespecified non-zero recovery process for a defaultable claim under consideration, it suffices to consider self-financing strategies $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ such that

$$\sum_{i=m+1}^{k} \phi^i_t Y^i_t = Z_t, \quad \forall\, t \in [0, T]. \tag{13}$$

Notice that prior to default time (that is, on the event $\{\tau > t\}$) we have $V_t(\phi) = \sum_{i=1}^m \phi^i_t \widetilde{Y}^i_t + Z_t$, and the self-financing property of $\phi$ prior to default time $\tau$ takes the following form

$$dV_t(\phi) = \sum_{i=1}^{m} \phi^i_t \, d\widetilde{Y}^i_t + \sum_{i=m+1}^{k} \phi^i_t \, dY^i_t. \tag{14}$$

At default time $\tau$, we have $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \le T\}$.

The next goal is to examine the existence of $\phi$ with the properties described above. To this end, we denote $\widetilde{Y}^{i,1}_t = \widetilde{Y}^i_t(\widetilde{Y}^1_t)^{-1}$ for $i = 2,\dots,m$ and $\widetilde{Y}^{1,k}_t = \widetilde{Y}^1_t(Y^k_t)^{-1}$. As before, we write $Y^{i,k}_t = Y^i_t(Y^k_t)^{-1}$. Using Lemma 4.5, we obtain the following auxiliary result that will be later used to establish the existence of a replicating strategy for a defaultable claim.

**Proposition 4.1** (i) *Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a self-financing strategy such that* (13) *holds. Assume that the processes $\widetilde{Y}^1, Y^{m+1}, \dots, Y^k$ are strictly positive. Then the pre-default wealth process $\widetilde{V}(\phi)$ satisfies for every $t \in [0,T]$*

$$\widetilde{V}_t(\phi) = \widetilde{Y}^1_t\Big(\widetilde{V}^1_0(\phi) + \sum_{i=2}^m \int_0^t \phi^i_u\, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1} \int_0^t \widetilde{\phi}^{i,k,1}_u\, d\widehat{Y}^{i,k,1}_u + \int_0^t Z_u(Y^k_u)^{-1}\, d(\widetilde{Y}^{1,k}_u)^{-1}\Big)$$

*where we denote*

$$\widetilde{\phi}^{i,k,1}_t = \phi^i_t(\widetilde{Y}^{1,k}_t)^{-1} e^{\widetilde{\alpha}^{i,k,1}_t}, \quad \widehat{Y}^{i,k,1}_t = Y^{i,k}_t e^{-\widetilde{\alpha}^{i,k,1}_t},$$

*and*

$$\widetilde{\alpha}^{i,k,1}_t = \langle \ln Y^{i,k}, \ln \widetilde{Y}^{1,k}\rangle_t = \int_0^t (Y^{i,k}_u)^{-1}(\widetilde{Y}^{1,k}_u)^{-1}\, d\langle Y^{i,k}, \widetilde{Y}^{1,k}\rangle_u.$$

*In addition, at default time the wealth of $\phi$ equals $V_\tau(\phi) = Z_\tau$ on the event $\{\tau \le T\}$.*
(ii) *Conversely, suppose that the processes $\psi^i$, $i = 2,\dots,m$ and processes $\widetilde{\psi}^{i,k,1}$, $i = m+1,\dots,k-1$ are given. For an arbitrary constant $c \in \mathbb{R}$, we define the process $\widetilde{V}$ by setting, for $t \in [0,T]$,*

$$\widetilde{V}_t = c + \sum_{i=2}^m \int_0^t \psi^i_u\, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1} \int_0^t \widetilde{\psi}^{i,k,1}_u\, d\widehat{Y}^{i,k,1}_u + \int_0^t Z_u(Y^k_u)^{-1}\, d(\widetilde{Y}^{1,k}_u)^{-1}.$$

*Then there exists a self-financing trading strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ such that:*
(a) *$\phi^i_t = \psi^i_t$ for $i = 2,\dots,m$ and $\phi^i_t = \widetilde{\psi}^{i,k,1}_t \widetilde{Y}^{1,k}_t e^{-\widetilde{\alpha}^{i,k,1}_t}$ for $i = m+1,\dots,k-1$,*
(b) *$\phi$ satisfies* (13)*, so that $\sum_{i=m+1}^k \phi^i_t Y^i_t = Z_t$ for every $t \in [0,T]$,*
(c) *the pre-default wealth $\widetilde{V}(\phi)$ of $\phi$ equals $\widetilde{V}$,*
(d) *at default time the wealth of $\phi$ equals $V_\tau(\phi) = Z_\tau$ on the event $\{\tau \le T\}$.*

*Proof.* Part (i) is an almost immediate consequence of Lemma 4.5. Therefore, we shall focus on the second part. The idea of the proof of part (ii) is also rather clear. First, let $\phi^i$, $i = 2,\dots,m$ and $\phi^i$, $i = 2,\dots,m+1,\dots,k-1$ be defined from processes $\psi^i$ and $\widetilde{\psi}^{i,k,1}_t$ as in (a). Given the processes $\phi^i$ for $i = m+1,\dots,k-1$, we observe that the component $\phi^k$ is uniquely specified by condition (13). Thus, it remains to check that there exists a (unique) component $\phi^1$ such that the resulting $k$-dimensional trading strategy is self-financing prior to default, in the sense of Definition 4.3. Let us set

$$\phi^1_t = \Big(\widetilde{V}_t - \sum_{i=2}^m \phi^i_t Y^i_t - Z_t\Big)(\widetilde{Y}^1_t)^{-1} = \Big(\widetilde{V}_t - \sum_{i=2}^k \phi^i_t Y^i_t\Big)(\widetilde{Y}^1_t)^{-1}.$$

It is clear that $\widetilde{V}_t(\phi) = \widetilde{V}_t$ for every $t \in [0,T]$. To show that the strategy $(\phi^1, \phi^2, \dots, \phi^k)$ described above is self-financing prior to default, it suffices to show that for the discounted pre-default wealth

$$\widetilde{V}^1_t(\phi) = \sum_{i=1}^m \phi^i_t \widetilde{Y}^{i,1}_t + \sum_{i=m+1}^k \phi^i_t Y^{i,1}_t$$

we have for every $t \in [0,T]$

$$\widetilde{V}^1_t(\phi) = \widetilde{V}^1_0(\phi) + \sum_{i=2}^m \int_0^t \phi^i_u d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^k \int_0^t \phi^i_u dY^{i,1}_u.$$

Towards this end, it is enough observe that $\widetilde{V}_t^1(\phi) = (\widetilde{Y}_t^1)^{-1}\widetilde{V}_t = \widetilde{V}_t^1$, and then to verify that

$$\widetilde{V}_t^1 = \widetilde{V}_0^1 + \sum_{i=2}^{m} \int_0^t \phi_u^i d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k} \int_0^t \phi_u^i dY_u^{i,1}$$

for every $t \in [0,T]$. To establish that last equality, it suffices to use the definition of the process $\widetilde{V}^1$ and to observe that

$$\sum_{i=m+1}^{k-1} \widetilde{\psi}_t^{i,k,1}\, d\widehat{Y}_t^{i,k,1} = \sum_{i=m+1}^{k} \phi_t^i\, dY_t^{i,1},$$

which follows by direct calculations, using the definitions of $\phi^i$, $i = m+1,\dots,k$. It is easy to see that the strategy $\phi$ satisfies conditions (a)-(d). □

**Remarks.** Let us observe that the equality established in Proposition 4.1 is in fact valid on the random interval $[\![0,\tau[\![$ on the event $\{\tau \le T\}$ and on the interval $[0,T]$ on the event $\{\tau > T\}$. It is also important to notice that the assumption of zero recovery for $Y^1,\dots,Y^m$ is not essential for the validity for the statements of the last result, except for the last part, that is, the equality $V_\tau(\phi) = Z_\tau$. Indeed, the proof of Proposition 4.1 relies on conditions (13) and (14). Therefore, if defaultable primary assets $Y^1,\dots,Y^m$ are subject to non-zero recovery, it will be possible to modify Proposition 4.1 accordingly (see Section 4.2.3 below).

When dealing with defaultable claims with no recovery, that is, claims for which the recovery process $Z$ vanishes, it will be convenient to use directly the following corollary to Proposition 4.1.

**Corollary 4.1** *Let $\phi = (\phi^1,\phi^2,\dots,\phi^k)$ be a self-financing strategy such that condition* (12) *holds.*
(a) *Assume that the processes $\widetilde{Y}^1, Y^{m+1},\dots,Y^k$ are strictly positive. Then the wealth process $V(\phi)$ satisfies*

$$V_t(\phi) = Y_t^1\Big(V_0^1(\phi) + \sum_{i=2}^{m} \int_0^t \phi_u^i\, d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1} \int_0^t \widetilde{\phi}_u^{i,k,1}\, d\widehat{Y}_u^{i,k,1}\Big), \quad \forall\, t \in [0,T].$$

(b) *Assume that all primary assets are defaultable, that is, $m = k$, and the pre-default value $\widetilde{Y}^1$ is a strictly positive process. Then the wealth process $V(\phi)$ satisfies*

$$V_t(\phi) = Y_t^1\Big(V_0^1(\phi) + \sum_{i=2}^{m} \int_0^t \phi_u^i\, d\widetilde{Y}_u^{i,1}\Big), \quad \forall\, t \in [0,T].$$

**Remarks.** Of course, the counterparts of part (ii) in Proposition 4.1 are also valid and they will be used in what follows, although they are not explicitly formulated here.

**Remarks.** Consider the special case of two primary assets, defaultable and default-free, with prices $Y_t^1 = \mathbb{1}_{\{\tau>t\}}\widetilde{Y}_t^1$ and $Y_t^2$, respectively, where $\widetilde{Y}^1$ and $Y^2$ are strictly positive, continuous, $\mathbb{F}$-semimartingales. Suppose we wish to replicate a defaultable claim with zero recovery. We have

$$V_t(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = \phi_t^1 \mathbb{1}_{\{\tau>t\}}\widetilde{Y}_t^1 + \phi_t^2 Y_t^2$$

and

$$dV_t(\phi) = (V_{t-}(\phi) - \phi_t^2 Y_t^2)(\widetilde{Y}_t^1)^{-1} dY_t^1 + \phi_t^2\, dY_t^2.$$

It is rather clear that the equality $V_t(\phi) = 0$ on $\{\tau \le t\}$ implies that $\phi_t^2 = 0$ for every $t \in [0,T]$. Therefore,

$$dV_t(\phi) = V_{t-}(\phi)(\widetilde{Y}_t^1)^{-1} dY_t^1$$

and the existence of replicating strategy for a defaultable claim with zero-recovery is unlikely within the present setup (except for some trivial cases).

### 4.2.3 Non-Zero Recovery for Defaultable Assets

In this section, the assumption of zero recovery for defaultable primary assets $Y^1, \dots, Y^m$ is relaxed. To be more specific, Assumption (A) is replaced by the following weaker restriction.

**Assumption (B)**. We assume that the defaultable assets $Y^1, \dots, Y^m$ are subject to an arbitrary recovery scheme, and they have a common default time $\tau$.

Under Assumption (B), condition (13) no longer implies that $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \le T\}$. We can achieve this requirement by substituting (13) with the following constraint

$$\sum_{i=1}^{m} \phi_t^i \bar{Y}_t^i + \sum_{i=m+1}^{k} \phi_t^i Y_t^i = Z_t, \quad \forall\, t \in [0,T], \tag{15}$$

where $\bar{Y}^i$ represents the recovery payoff of the defaultable asset $Y^i$, so that $Y_\tau^i = \bar{Y}_\tau^i$ for $i = 1,2,\dots,m$. In this general setup, condition (15) does not seem to be sufficiently restrictive for more explicit calculations. It is plausible, however, that it can be used to derive a replicating strategy in several non-trivial and practically interesting cases.

It is not difficult to see that Proposition 4.1 can be extended to the case of non-zero recovery for defaultable assets, provided, of course, that we are in the position to find a priori the wealth invested in non-defaultable assets, that is, if the process $\beta_t := \sum_{i=m+1}^{k} \phi_t^i Y_t^i$ is known beforehand. By arguing as in Proposition 4.1, we then obtain for every $t \in [0,T]$

$$\widetilde{V}_t(\phi) = \widetilde{Y}_t^1 \Big( \widetilde{V}_0^1(\phi) + \sum_{i=2}^{m} \int_0^t \phi_u^i \, d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1} \int_0^t \widetilde{\phi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1} + \int_0^t \beta_u (Y_u^k)^{-1} \, d(\widetilde{Y}_u^{1,k})^{-1} \Big).$$

In view of (15), we also have that

$$\bar{\alpha}_t := \sum_{i=1}^{m} \phi_t^i \bar{Y}_t^i = Z_t - \beta_t, \quad \forall\, t \in [0,T], \tag{16}$$

thereby imposing an additional constraint on the wealth invested in defaultable assets. Condition (16) is not directly accounted for in the last formula for $\widetilde{V}(\phi)$, however, and thus the problem at hand is not solved. For further considerations related to non-zero recovery of defaultable primary assets, see Section 5.1.2 and 5.2.2.

**Fractional recovery of market value.** As an example of a non-zero recovery scheme, let us consider the so-called fractional recovery of (pre-default) market value (FRMV) scheme with constant recovery rates $\delta_i \neq 1$ (typically, $0 \le \delta_i < 1$). Then we have $\bar{Y}_t^i = \delta_i \widetilde{Y}_t^i$ for every $i = 1,2,\dots,m$, and thus (15) becomes

$$\sum_{i=1}^{m} \phi_t^i \delta_i \widetilde{Y}_t^i + \sum_{i=m+1}^{k} \phi_t^i Y_t^i = Z_t, \quad \forall\, t \in [0,T]. \tag{17}$$

Let us mention that in the case of a defaultable zero-coupon bond, the FRMV scheme results in the following expression for the pre-default value of a defaultable bond with unit face value (see, for instance, Section 2.2.4 in Bielecki et al. (2004))

$$\widetilde{D}_M^\delta(t,T) = \mathbb{E}_{\mathbb{Q}^*} \Big( e^{-\int_t^T (r_u + (1-\delta)\gamma_u) du} \,\Big|\, \mathcal{F}_t \Big),$$

where the recovery rate $\delta$ may depend on the bond's maturity $T$, in general. In particular, if the default intensity $\gamma$ is deterministic then we have

$$\widetilde{D}_M^\delta(t,T) = e^{-\int_t^T (1-\delta)\gamma(u)\, du} \, B(t,T).$$

Manifestly, we always have $D_M^\delta(\tau,T) = \delta D_M^\delta(\tau-,T)$ on the set $\{\tau \le T\}$ under the FRMV scheme.

# 5 Replication of Defaultable Claims

We are in the position to examine the issue of an exact replication of a generic defaultable claim. By a *replicating strategy* we mean here a self-financing trading strategy $\phi$ such that the wealth process $V(\phi)$ matches exactly the pre-default value of the claim at any time prior to default (and prior to the maturity date), as well as coincides with the claim's payoff at default time or at maturity date, whichever comes first. Using our notation introduced in Section 3, this can be formalized as follows.

**Definition 5.1** A self-financing trading strategy $\phi$ is a *replicating strategy* for a defaultable claim $(X, 0, Z, \tau)$ if and only if the following hold:
(i) $V_t(\phi) = \widetilde{U}_t$ on the random interval $[\![0, \tau \wedge T[\![$,
(ii) $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \leq T\}$,
(iii) $V_T(\phi) = X$ on the set $\{\tau > T\}$.
We say that a defaultable claim is *attainable* if it admits at least one replicating strategy.

The last definition is suitable only in the case of a defaultable claim with no promised dividends. Some comments regarding replication of promised dividends are given in Section 5.3.

## 5.1 Replication of a Promised Payoff

We shall first examine the possibility of an exact replication of a defaultable contingent claim of the form $(X, 0, 0, \tau)$, that is, a defaultable claim with zero recovery and with no promised dividends. Our approach will be based on Proposition 4.1. Thus, we assume that processes $Y^1, \ldots, Y^m$ represent prices of defaultable primary assets and $Y^{m+1}, \ldots, Y^k$ are prices of default-free primary assets. Processes $\widetilde{Y}^1, \ldots, \widetilde{Y}^m, Y^{m+1}, \ldots, Y^k$ are assumed to be continuous $\mathbb{F}$-semimartingales, and processes $\widetilde{Y}^1, Y^{m+1}, \ldots, Y^k$ are strictly positive.

### 5.1.1 Zero Recovery for Defaultable Primary Assets

Unless explicitly stated otherwise, we postulate that Assumption (A) is valid. Recall that $\widetilde{U}_t(X)$ stands for the pre-default value at time $t \in [0, T]$ of a defaultable claim $(X, 0, 0, \tau)$. In the statement of following result we preserve the notation of Proposition 4.1.

**Proposition 5.1** *Suppose that there exist a constant $\widetilde{V}_0^1$, processes $\psi^i$, $i = 2, \ldots, m$, and processes $\widetilde{\psi}^{i,k,1}$, $i = m+1, \ldots, k-1$ such that*

$$\widetilde{Y}_T^1 \Big( \widetilde{V}_0^1 + \sum_{i=2}^{m} \int_0^T \psi_u^i \, d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1} \int_0^T \widetilde{\psi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1} \Big) = X. \tag{18}$$

*Let $\widetilde{V}_t = \widetilde{Y}_t^1 \widetilde{V}_t^1$, where the process $\widetilde{V}_t^1$ is defined as*

$$\widetilde{V}_t^1 = \widetilde{V}_0^1 + \sum_{i=2}^{m} \int_0^t \psi_u^i \, d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1} \int_0^t \widetilde{\psi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1}, \quad \forall \, t \in [0, T].$$

*Then the trading strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ defined by*

$$\begin{aligned}
\phi_t^1 &= \Big( \widetilde{V}_t - \sum_{i=2}^{m} \psi_t^i Y_t^i \Big) (\widetilde{Y}_t^1)^{-1}, \\
\phi_t^i &= \psi_t^i, \quad i = 2, \ldots, m, \\
\phi_t^i &= \widetilde{\psi}_t^{i,k,1} \widetilde{Y}_t^{1,k} e^{-\widetilde{\alpha}_t^{i,k,1}}, \quad i = m+1, \ldots, k-1, \\
\phi_t^k &= - \sum_{i=m+1}^{k-1} \psi_t^i Y_t^i (Y_t^k)^{-1},
\end{aligned}$$

*is self-financing and it replicates* $(X,0,0,\tau)$*. In particular, we have* $\widetilde{V}_t(\phi)=\widetilde{V}_t=\widetilde{U}_t(X)$*, that is, the process* $\widetilde{V}$ *represents the pre-default value of* $(X,0,0,\tau)$*.*

*Proof.* The statement is an almost immediate consequence of part (ii) of Proposition 4.1 (see also Corollary 4.1). The strategy $(\phi^1,\phi^2,\dots,\phi^k)$ introduced in the statement of the proposition is self-financing, and at the default time $\tau$ the wealth $V(\phi)$ jumps to zero. Finally, $V_T(\phi)=\widetilde{V}_T(\phi)=X$ on the event $\{\tau>T\}$. We conclude that $\phi$ is self-financing and it replicates $(X,0,0,\tau)$. □

The following corollary to Proposition 5.1 provides the risk-neutral characterization of the process $\widetilde{U}_t(X)$, and thereby it furnishes a convenient method for the valuation of a promised payoff.

**Corollary 5.1** *Assume that a defaultable claim* $(X,0,0,\tau)$ *is attainable. Let* $\widetilde{\mathbb{Q}}$ *be a probability measure such that the processes* $\widetilde{Y}^{i,1}$, $i=2,\dots,m-1$ *and processes* $\widehat{Y}^{i,k,1}$, $i=m+1,\dots,k-1$ *are* $\mathbb{F}$*-martingales under* $\widetilde{\mathbb{Q}}$*. If all stochastic integrals in* (18) *are* $\widetilde{\mathbb{Q}}$*-martingales, rather than* $\widetilde{\mathbb{Q}}$*-local martingales, then the pre-default value of* $(X,0,0,\tau)$ *equals*

$$\widetilde{U}_t(X)=\widetilde{Y}^1_t\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(X(\widetilde{Y}^1_T)^{-1}\,\big|\,\mathcal{F}_t\big),\quad\forall\,t\in[0,T].$$

**Defaultable asset and two default-free assets.** In the case when $m=1$ and $k=2$, Proposition 5.1 reduces to the following result. Recall that we denote

$$\widetilde{\alpha}^{2,3,1}_t=\langle\ln Y^{2,3},\ln\widetilde{Y}^{1,3}\rangle_t=\int_0^t(Y^{2,3}_u)^{-1}(\widetilde{Y}^{1,3}_u)^{-1}\,d\langle Y^{2,3},\widetilde{Y}^{1,3}\rangle_u$$

where in turn $\widetilde{Y}^{1,3}_t=\widetilde{Y}^1_t(Y^3_t)^{-1}$ and $Y^{2,3}_t=Y^2_t(Y^3_t)^{-1}$. Moreover, $\widehat{Y}^{2,3,1}_t=Y^{2,3}_te^{-\widetilde{\alpha}^{2,3,1}_t}$. We postulate that the processes $\widetilde{Y}^1,Y^2$ and $Y^3$ are strictly positive.

**Corollary 5.2** *Suppose that there exists a constant* $\widetilde{V}^1_0$ *and a process* $\widetilde{\psi}^{2,3,1}$ *such that*

$$\widetilde{Y}^1_T\Big(\widetilde{V}^1_0+\int_0^T\widetilde{\psi}^{2,3,1}_u\,d\widehat{Y}^{2,3,1}_u\Big)=X. \tag{19}$$

*Let us set* $\widetilde{V}_t=\widetilde{Y}^1_t\widetilde{V}^1_t$*, where* $\widetilde{V}^1_t$ *is given by*

$$\widetilde{V}^1_t=\widetilde{V}^1_0+\int_0^t\widetilde{\psi}^{2,3,1}_u\,d\widehat{Y}^{2,3,1}_u,\quad\forall\,t\in[0,T]. \tag{20}$$

*Then the trading strategy* $\phi=(\phi^1,\phi^2,\phi^3)$*, given by the expressions*

$$\phi^1_t=\widetilde{V}_t(\widetilde{Y}^1_t)^{-1},\quad\phi^2_t=\widetilde{\psi}^{2,3,1}_t\widetilde{Y}^{1,3}_te^{-\widetilde{\alpha}^{2,3,1}_t},\quad\phi^3_t=-\phi^2_tY^2_t(Y^3_t)^{-1},$$

*is self-financing prior to default and it replicates a defaultable claim* $(X,0,0,\tau)$*.*

Assume that a claim $(X,0,0,\tau)$ is attainable, and let $\widetilde{\mathbb{Q}}$ be a probability measure such that $\widehat{Y}^{2,3,1}$ is an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$. Then the pre-default value of $(X,0,0,\tau)$ equals

$$U_t(X)=\widetilde{Y}^1_t\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(X(\widetilde{Y}^1_T)^{-1}\,\big|\,\mathcal{F}_t\big),\quad\forall\,t\in[0,T], \tag{21}$$

provided that the integral in (20) is also a $\widetilde{\mathbb{Q}}$-martingale.

**Example 5.1** Assume that $d\widetilde{Y}^1_t=Y^1_t(\mu_t\,dt+\sigma^1_t\,dW_t)$ and $dY^i_t=Y^i_t(r_t\,dt+\sigma^i_t\,dW^*_t)$ for $i=2,3$, where $W^*$ is a one-dimensional standard Brownian motion with respect to the filtration $\mathbb{F}=\mathbb{F}^{W^*}$

under the martingale measure $\mathbb{Q}^*$. Then for the processes $\widetilde{Y}^{1,2}_t = \widetilde{Y}^1_t (Y^3_t)^{-1}$ and $Y^{2,3}_t = Y^2_t (Y^3_t)^{-1}$ we get

$$\begin{aligned} d\widetilde{Y}^{1,3}_t &= \widetilde{Y}^{1,3}_t \Big( \big( \mu_t - r_t + \sigma^3_t(\sigma^3_t - \sigma^1_t) \big)\, dt + (\sigma^1_t - \sigma^3_t)\, dW^*_t \Big), \\ dY^{2,3}_t &= Y^{2,3}_t \Big( \sigma^3_t(\sigma^3_t - \sigma^2_t)\, dt + (\sigma^2_t - \sigma^3_t)\, dW^*_t \Big), \end{aligned}$$

and thus

$$\widetilde{\alpha}^{2,3,1}_t = \int_0^t (\sigma^3_u - \sigma^1_u)(\sigma^3_u - \sigma^2_u)\, du.$$

Hence, the process $\widehat{Y}^{2,3,1}_t = Y^{2,3}_t e^{-\widetilde{\alpha}^{2,3,1}_t}$ satisfies

$$d\widehat{Y}^{2,3,1}_t = \widehat{Y}^{2,3,1}_t \Big( \sigma^1_t(\sigma^3_t - \sigma^2_t)\, dt + (\sigma^2_t - \sigma^3_t) dW^*_t \Big).$$

If $\sigma^2 \neq \sigma^3$ then, under mild technical assumptions, there exists a probability measure $\widetilde{\mathbb{Q}}$ such that $\widehat{Y}^{2,3,1}$ is a martingale. To conclude, it suffices to use the fact that an $\mathcal{F}_T$-measurable random variable $X(\widetilde{Y}^1_T)^{-1}$ can be represented (by virtue of the predictable representation theorem) as follows

$$X(\widetilde{Y}^1_T)^{-1} = \widetilde{U}_0(X) + \int_0^T \widetilde{\phi}^{2,3,1}_u\, d\widehat{Y}^{2,3,1}_u$$

for some predictable process $\widetilde{\phi}^{2,3,1}$. It is natural to conjecture that within the present setup all defaultable claims with zero recovery and no promised dividends will be attainable, provided that the underlying default-free market is assumed to be complete, and provided we can use a defaultable asset in our hedging portfolio that is sensitive to the same default risk as the defaultable claim that we want to hedge.

**Two defaultable assets.** Let us now consider the case when $m = k = 2$. We thus consider two defaultable primary assets $Y^1$ and $Y^2$ with zero recovery at default.

**Corollary 5.3** *Suppose that there exists a constant $\widetilde{V}^1_0$ and a process $\psi^2$ such that*

$$\widetilde{Y}^1_T \Big( \widetilde{V}^1_0 + \int_0^T \psi^2_u\, d\widetilde{Y}^{2,1}_u \Big) = X \tag{22}$$

*where $\widetilde{Y}^{2,1}_t = \widetilde{Y}^2_t (\widetilde{Y}^1_t)^{-1}$. Let us set $\widetilde{V}_t = \widetilde{Y}^1_t \widetilde{V}^1_t$, where $\widetilde{V}^1_t$ is given by*

$$\widetilde{V}^1_t = \widetilde{V}^1_0 + \int_0^t \psi^2_u\, d\widetilde{Y}^{2,1}_u, \quad \forall\, t \in [0,T]. \tag{23}$$

*Then the trading strategy $\phi = (\phi^1, \phi^2)$ where*

$$\phi^1_t = (\widetilde{V}^1_t - \psi^2_t \widetilde{Y}^2_t)(\widetilde{Y}^1_t)^{-1}, \quad \phi^2_t = \psi^2_t, \quad \forall\, t \in [0,T],$$

*is self-financing and it replicates a defaultable claim $(X, 0, 0, \tau)$.*

Suppose that $(X, 0, 0, \tau)$ is an attainable claim. Let $\widetilde{\mathbb{Q}}$ be a probability measure such that $\widetilde{Y}^{2,1}$ is an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$. If the stochastic integral in (23) is a $\widetilde{\mathbb{Q}}$-martingale, then the pre-default value of $(X, 0, 0, \tau)$ satisfies

$$\widetilde{U}_t(X) = \widetilde{Y}^1_0\, \mathbb{E}_{\widetilde{\mathbb{Q}}} \big( X(\widetilde{Y}^1_T)^{-1} \,\big|\, \mathcal{F}_t \big), \quad \forall\, t \in [0,T]. \tag{24}$$

**Remarks.** Under the assumptions of Corollary 5.3, a defaultable claim $(X, 0, 0, \tau)$ is attainable since the associated promised payoff $X$ can be achieved by trading in the pre-default values $\widetilde{Y}^1$ and $\widetilde{Y}^2$. If we introduce, in addition, some default-free assets, a replicating strategy for an arbitrary defaultable claim $(X, 0, 0, \tau)$ will typically have a zero net investment in default-free assets. Therefore, default-free assets are not relevant if we restrict our attention to defaultable claims of the form $(X, 0, 0, \tau)$.

### 5.1.2 Non-Zero Recovery for Defaultable Primary Assets

We relax Assumption (A), and we postulate instead that Assumption (B) is valid. Specifically, let us consider $m$ defaultable primary assets with a common default time $\tau$ that are subject to a fractional recovery of market value (see Section 4.2.3) with $\delta_i = \delta \neq 1$ for $i = 1, 2, \ldots, m$. Let us denote

$$\widetilde{\alpha}_t = \sum_{i=1}^{m} \phi^i_t \widetilde{Y}^i_t, \quad \beta_t = \sum_{i=m+1}^{k} \phi^i_t Y^i_t.$$

so that $\widetilde{\alpha}_t + \beta_t$ represents the pre-default wealth of $\phi$. As usual, $\widetilde{U}_t(X)$ stands for the pre-default value at time $t$ of the promised payoff $X$. It is rather clear that the processes $\widetilde{\alpha}_t$ and $\beta_t$ should be chosen in such a way that $\widetilde{\alpha}_t + \beta_t = \widetilde{U}_t(X)$ and $\bar{\alpha}_t + \beta_t = \delta\widetilde{\alpha}_t + \beta_t = 0$ for every $t \in [0, T]$ (for the latter equality, see (16) and (17)). By solving these equations, we obtain

$$\widetilde{\alpha}_t = (1-\delta)^{-1}\widetilde{U}_t(X), \quad \beta_t = (\delta - 1)^{-1}\delta\widetilde{U}_t(X), \quad \forall\, t \in [0, T].$$

We end up with the following equation

$$\widetilde{Y}^1_T\Big(\widetilde{U}_0(X) + \sum_{i=2}^{m}\int_0^T \phi^i_u\, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1}\int_0^T \widetilde{\phi}^{i,k,1}_u\, d\widehat{Y}^{i,k,1}_u + \int_0^T \beta_u (Y^k_u)^{-1}\, d(\widetilde{Y}^{1,k}_u)^{-1}\Big) = X.$$

Using the latter equation, one may try to establish a suitable extension of Proposition 5.1. Notice that the process $\beta$ depends explicitly on the pre-default value $\widetilde{U}(X)$. In addition, we need to take care of the constraint $\widetilde{\alpha}_t = (1-\delta)^{-1}\widetilde{U}_t(X)$ for every $t \in [0, T]$. Thus, the problem of replication of a promised payoff under non-zero recovery for defaultable primary assets seems to be rather difficult to solve, in general.

## 5.2 Replication of a Recovery Payoff

Let us now focus on the recovery payoff $Z$ at time of default. As before, we write $\widetilde{U}_t(Z)$ to denote the pre-default value at time $t \in [0, T]$ of the claim $(0, 0, Z, \tau)$. Recall that $\widetilde{U}_T(Z) = 0$ (and $U_T(Z) = 0$ on the event $\{\tau > T\}$.

### 5.2.1 Zero Recovery for Defaultable Primary Assets

In order to examine the replicating strategy, we shall once again make use of Proposition 4.1. As already explained, in this case we need to assume that condition (11) is imposed on a strategy $\phi$ we are looking for, that is, we necessarily have $\sum_{i=l+1}^{k} \phi^i_t Y^i_t = Z_t$ for every $t \in [0, T]$.

**Proposition 5.2** *Suppose that there exist a constant $\widetilde{V}^1_0$, processes $\psi^i$, $i = 2, \ldots, m$, and processes $\widetilde{\psi}^{i,k,1}$, $i = m+1, \ldots, k-1$ such that*

$$\widetilde{Y}^1_T\Big(\widetilde{V}^1_0 + \sum_{i=2}^{m}\int_0^T \psi^i_u\, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1}\int_0^T \widetilde{\psi}^{i,k,1}_u\, d\widehat{Y}^{i,k,1}_u + \int_0^T Z_u (Y^k_u)^{-1}\, d(Y^{1,k}_u)^{-1}\Big) = 0. \tag{25}$$

*Let $\widetilde{V}_t = \widetilde{Y}^1_t \widetilde{V}^1_t$, where the process $\widetilde{V}^1_t$ is defined as*

$$\widetilde{V}^1_t = \widetilde{V}^1_0 + \sum_{i=2}^{m}\int_0^t \psi^i_u\, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1}\int_0^t \widetilde{\psi}^{i,k,1}_u\, d\widehat{Y}^{i,k,1}_u + \int_0^t Z_u (Y^k_u)^{-1}\, d(Y^{1,k}_u)^{-1}.$$

*Then the replicating strategy* $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ *for* $(0, 0, Z, \tau)$ *is given by*

$$\begin{aligned}
\phi^1_t &= \Big(\widetilde{V}_t - Z_t - \sum_{i=2}^{m} \phi^i_t Y^i_t\Big)(\widetilde{Y}^1_t)^{-1}, \\
\phi^i_t &= \psi^i_t, \quad \forall\, i = 2, \dots, m, \\
\phi^i_t &= \widetilde{\psi}^{i,k,1}_t \widetilde{Y}^{1,k}_t e^{-\widetilde{\alpha}^{i,k,1}_t}, \quad \forall\, i = m+1, \dots, k-1, \\
\phi^k_t &= \Big(Z_t - \sum_{i=m+1}^{k-1} \phi^i_t Y^i_t\Big)(Y^k_t)^{-1}.
\end{aligned}$$

*Proof.* The proof is based on an application of part (ii) of Proposition 4.1. First, notice that by virtue of the specification of the strategy $\phi$ we have $\widetilde{V}_t(\phi) = \widetilde{V}_t$ for every $t \in [0, T]$. Moreover, $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \le T\}$. Finally, $V_T(\phi) = \widetilde{V}_T(\phi) = 0$ on the event $\{\tau > T\}$. □

**Defaultable asset and two default-free assets.** For the ease of reference, we consider here a special case of Proposition 5.2. We take $m = 1$ and $k = 3$, and we postulate that the processes $\widetilde{Y}^1, Y^2$ and $Y^3$ are strictly positive. Recall that the recovery process $Z$, and thus also its pre-default value process $\widetilde{U}(Z)$, are prespecified.

**Corollary 5.4** *Suppose that there exists a constant* $\widetilde{V}^1_0$ *and a process* $\widetilde{\psi}^{2,3,1}$ *such that*

$$\widetilde{Y}^1_T\Big(\widetilde{V}^1_0 + \int_0^T \widetilde{\psi}^{2,3,1}_u \, d\widehat{Y}^{2,3,1}_u + \int_0^T Z_u (Y^3_u)^{-1} \, d(Y^{1,3}_u)^{-1}\Big) = 0. \tag{26}$$

*Let* $\widetilde{V}_t = \widetilde{Y}^1_t \widetilde{V}^1_t$*, where the process* $\widetilde{V}^1_t$ *is defined as*

$$\widetilde{V}_t = \widetilde{V}^1_0 + \int_0^t \widetilde{\psi}^{2,3,1}_u \, d\widehat{Y}^{2,3,1}_u + \int_0^t Z_u (Y^3_u)^{-1} \, d(Y^{1,3}_u)^{-1}.$$

*Then the replicating strategy for for the claim* $(0, 0, Z, \tau)$ *equals*

$$\phi^1_t = (\widetilde{V}_t - Z_t)(\widetilde{Y}^1_t)^{-1}, \quad \phi^2_t = \widetilde{\psi}^{2,3,1}_t \widetilde{Y}^{1,3}_t e^{-\widetilde{\alpha}^{2,3,1}_t}, \quad \phi^3_t = (Z_t - \phi^2_t Y^2_t)(Y^3_t)^{-1}.$$

The existence of $\widetilde{\psi}^{2,3,1}$, as well as the possibility of deriving a closed-form expression for $\phi$ are not obvious. One needs to impose more specific assumptions on the price processes of primary assets and the recovery process in order to proceed further.

If there exists a probability $\mathbb{Q}^*$ such that $\widehat{Y}^{2,3,1}$ is an $\mathbb{F}$-martingale, then the (ex-dividend) value of $Z^0$ equals

$$U_t(Z) = Y^1_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_t^T Z_u (Y^3_u)^{-1} \, d(Y^{1,3}_u)^{-1} \,\Big|\, \mathcal{F}_t\Big).$$

**Two defaultable assets.** Of course, if both defaultable primary assets are subject to the zero recovery scheme, and no other asset is available for trade, no replicating strategy exists in the case of a non-zero recovery process $Z$. Thus, we need to postulate a more general recovery scheme for defaultable assets if we wish to have a positive result.

### 5.2.2 Non-Zero Recovery for Defaultable Primary Assets

Suppose now that Assumption (B) is valid and $Y^1, \dots, Y^m$ are defaultable primary assets with a fractional recovery of market value. We assume that $\delta_i = \delta \neq 1$ for $i = 1, 2, \dots, m$, and we proceed along the similar lines as in Section 5.1.2. Recall that we denote

$$\widetilde{\alpha}_t = \sum_{i=1}^{m} \phi^i_t \widetilde{Y}^i_t, \quad \beta_t = \sum_{i=m+1}^{k} \phi^i_t Y^i_t.$$

We now postulate that $\widetilde{\alpha}_t + \beta_t = \widetilde{U}_t(Z)$ and $\bar{\alpha}_t + \beta_t = \delta\widetilde{\alpha}_t + \beta_t = Z_t$ for every $t \in [0,T]$, where $\widetilde{U}_t(Z)$ is the pre-default value of $(0,0,Z,\tau)$. Consequently, for every $t \in [0,T]$ we have

$$\widetilde{\alpha}_t = (\delta-1)^{-1}(Z_t - \widetilde{U}_t(Z)), \quad \beta_t = (\delta-1)^{-1}(\delta\widetilde{U}_t(Z) - Z_t).$$

To find a replicating strategy for a defaultable claim $(0,0,Z,\tau)$, we need, in particular, to find processes $\psi^i$ and $\widetilde{\psi}^{i,k,1}$ such that the equality

$$\widetilde{U}_t(Z) = \widetilde{Y}^1_t\Big(U_0(Z) + \sum_{i=2}^{m}\int_0^t \psi^i_u\, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1}\int_0^t \widetilde{\psi}^{i,k,1}_u\, d\widehat{Y}^{i,k,1}_u + \int_0^t \beta_u (Y^k_u)^{-1}\, d(\widetilde{Y}^{1,k}_u)^{-1}\Big)$$

is satisfied for every $t \in [0,T]$. Similarly as in Section 5.2.2, we conclude that the considered problem is non-trivial, in general.

## 5.3 Replication of Promised Dividends

We return to the case of zero recovery for defaultable primary assets, and we consider a defaultable claim $(0,C,0,\tau)$. In principle, replication of the stream of promised dividends can reduced to previously considered cases (that's why it was possible to postulate in Definition 5.1 that $C=0$). Specifically, it suffices to introduce the recovery process $Z^C$ generated by $C$ by setting

$$Z^C_t = \int_{]0,t[} B^{-1}(u,t)\, dC_u, \quad \forall\, t \in [0,T],$$

and to combine it with the terminal payoff $\mathbb{1}_{\{\tau>T\}}X^C$, where the promised payoff $X^C$ associated with $C$ equals

$$X^C = \int_{]0,T]} B^{-1}(u,T)\, dC_u.$$

It should be stressed, however, that the pre-default price of an "equivalent" defaultable claim $(X^C,0,Z^C,\tau)$ introduced above does not coincide with the pre-default price of the original claim $(0,C,0,\tau)$, that is, processes $\widetilde{U}(C)$ and $\widetilde{U}(Z^C)+\widetilde{U}(X^C)$ are not identical. But, clearly, the equality $U_0(C) = U_0(Z^C) + U_0(X^C)$ is satisfied, and thus at time 0 the replicating strategies for both claims coincide.

**Remarks.** It is apparent that the concept of the (ex-dividend) pre-default price $\widetilde{U}(C)$ is not well adapted to the study of replication of promised dividends if we consider non-dividend paying primary assets only. It would be much more convenient to use in the case of dividend-paying (default-free or defaultable) primary assets. For instance, it is sometimes legitimate to postulate the existence of a default-free version of the defaultable claim $(0,C,0,\tau)$, that is, a default-free asset with the dividend stream $C$.

If we insist on working directly with the process $\widetilde{U}(C)$, then we derive the following set of necessary conditions for a self-financing trading strategy $\phi$ with the consumption process $A=-C$

$$\sum_{i=m+1}^{k} \phi^i_t Y^i_t = 0, \quad V_t(\phi) = \sum_{i=1}^{m} \phi^i_t \widetilde{Y}^i_t = \widetilde{U}_t(C), \tag{27}$$

and

$$dV_t(\phi) = \sum_{i=1}^{m} \phi^i_t\, d\widetilde{Y}^i_t + \sum_{i=m+1}^{k} \phi^i_t\, dY^i_t - dC_t = d\widetilde{U}_t(C). \tag{28}$$

The existence of a strategy $\phi = (\phi^1,\phi^2,\ldots,\phi^k)$ with consumption process $A=-C$, which satisfies (27)-(28) is not evident, however.

**Example 5.2** Let us take, for instance, $m = 1$ and $k = 3$. Then conditions (27)-(28) become:

$$\phi_t^1 \widetilde{Y}_t^1 = \widetilde{U}_t(C), \quad \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 = 0,$$

and

$$\phi_t^1 \, d\widetilde{Y}_t^1 + \phi_t^2 \, dY_t^2 + \phi_t^3 dY_t^3 = d\widetilde{U}_t(C) + dC_t.$$

Assume that under $\mathbb{Q}^*$ we have

$$\begin{aligned} d\widetilde{Y}^1 &= \mu_t \, dt + \sigma_t^1 \, dW_t^*, \\ dY_t^i &= r_t \, dt + \sigma_t^i \, dW_t^*, \quad i = 2, 3, \\ d\widetilde{U}_t(C) &= a_t \, dt + b_t \, dW_t^*. \end{aligned}$$

If, in addition, $dC_t = c_t dt$ then we obtain the following system of equations for $\phi = (\phi^1, \phi^2, \phi^3)$

$$\begin{aligned} \phi_t^1 \widetilde{Y}_t^1 &= \widetilde{U}_t(C), \\ \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 &= 0, \\ \phi_t^1 \mu_t^1 + \phi_t^2 \mu_t^2 + \phi_t^3 \mu_t^3 &= a_t + c_t, \\ \phi_t^1 \sigma_t^1 + \phi_t^2 \sigma_t^2 + \phi_t^3 \sigma_t^3 &= b_t. \end{aligned}$$

## 5.4 Replication of a First-to-Default Claim

Until now, we have always postulated that a random time $\tau$ represents a common default time for all defaultable primary assets, as well as for a defaultable contingent claim under consideration. This simplifying assumptions manifestly fails to hold in the case of a credit derivative that explicitly depends on default times of several (possibly independent) reference entities. Consequently, the issue of replication of a so-called *first-to-default claim* is more challenging, and the approach presented in the preceding sections needs to be extended.

Let the random times $\tau_1, \ldots, \tau_m$ represent the default times of $m$ reference entities that underlie a given first-to-default claim. We assume that $\mathbb{Q}^*\{\tau_i = \tau_j\} = 0$ for every $i \neq j$, and we denote by $\tau_{(1)}$ the random moment of the first default, that is, we set $\tau_{(1)} = \min\{\tau_1, \tau_2, \ldots, \tau_n\} = \tau_1 \wedge \tau_2 \wedge \ldots \wedge \tau_n$. A *first-to-default claim* $(X, C, Z^1, \ldots, Z^m, \tau_1, \ldots, \tau_m)$ with maturity date $T$ can be described as follows. If $\tau_{(1)} = \tau_i \leq T$ for some $i = 1, \ldots, m$, then it pays at time $\tau_{(1)}$ the amount $Z^i_{\tau_{(1)}}$, where $Z^i$ is an $\mathbb{F}$-predictable recovery process. Otherwise, that is, if $\tau_{(1)} > T$, the claim pays at time $T$ an $\mathcal{F}_T$-measurable promised amount $X$. Finally, a claim pays promised dividends stream $C$ prior to the default time $\tau_{(1)}$, more precisely, on the random interval $\mathbb{1}_{\{\tau_{(1)} \leq T\}} [\![0, \tau_{(1)}[\![ \cup \mathbb{1}_{\{\tau_{(1)} > T\}} [0, T]$. It is clear the dividend process of a generic first-to-default claim equals, for every $t \in [0, T]$,

$$D_t = X \mathbb{1}_{\{\tau_{(1)} > T\}} \mathbb{1}_{[T, \infty[}(t) + \int_{]0,t]} (1 - H_u^{(i)}) \, dC_u + \int_{]0,t]} Z_u^i \mathbb{1}_{\{\tau_{(1)} = \tau_i\}} \, dH_u^{(i)},$$

where $H_t^{(i)} = 1 - \prod_{i=1}^m (1 - H_t^i)$ or, equivalently, $H_t^{(i)} = \mathbb{1}_{\{\tau_{(1)} \leq t\}}$. Let $\mathbb{H}^i$ be the filtration generated by the process $H_t^i = \mathbb{1}_{\{\tau_i \leq t\}}$ for $i = 1, 2, \ldots, m$, and let the filtration $\mathbb{G}$ be given as $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2 \vee \ldots \vee \mathbb{H}^m$. Then, by definition, the (ex-dividend) price of $(X, C, Z^1, \ldots, Z^m, \tau_1, \ldots, \tau_m)$ equals

$$U_t = B_t \, \mathbb{E}_{\mathbb{Q}^*} \Big( \int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big), \quad \forall \, t \in [0, T].$$

By a pre-default value of a claim we mean an $\mathbb{F}$-adapted process $\widetilde{U}$ such that $U_t = \widetilde{U}_t \mathbb{1}_{\{\tau_{(1)} > t\}}$ for every $t \in [0, T]$. The following definition is a direct extension of Definition 5.1 (thus, we maintain the assumption that $C = 0$). By a self-financing strategy we mean here a strategy which is self-financing prior to the first default (cf. Definition 4.3), and thus it is self-financing on $[0, T]$ as well.

**Definition 5.2** A self-financing strategy $\phi$ is a *replicating strategy* for a first-to-default contingent claim $(X, 0, Z^1, \dots, Z^m, \tau_1, \dots, \tau_m)$ if and only if the following hold:
(i) $V_t(\phi) = \widetilde{U}_t$ on the random interval $[\![0, \tau_{(1)} \wedge T[\![$,
(ii) $V_\tau(\phi) = Z^i_\tau$ on the event $\{\tau_{(1)} = \tau_i \le T\}$,
(iii) $V_T(\phi) = X$ on the event $\{\tau_{(1)} > T\}$.

In order to provide a replicating strategy for a first-to-default claim we postulate the existence of $m$ defaultable primary assets $Y^1, \dots, Y^m$ with the corresponding default times $\widetilde{\tau}_1, \dots, \widetilde{\tau}_m$. It is natural to postulate that the default times $\widetilde{\tau}_1, \dots, \widetilde{\tau}_m$ are also the default times of $m$ reference entities that underlie a first-to-default claim under consideration, so that, $\widetilde{\tau}_i = \tau_i$ for $i = 1, 2, \dots, m$. It should be stressed that, typically, the pre-default value $\widetilde{Y}^j$ will follow a discontinuous process (for instance, it may have jumps at default times of other entities). Finally, let us recall that $\bar{Y}^i_t$ represents the recovery payoff of the $i^{\text{th}}$ defaultable asset if its default occurs at time $t$.

**Case of zero promised dividends.** We shall assume from now on that $C = 0$. For arbitrary $i \neq j$, let $\widehat{Y}^{ij}_t$ represent the pre-default value of the $i^{\text{th}}$ asset conditioned on the event $\{\tau_{(1)} = \tau_j = t\}$. More explicitly, $\widehat{Y}^{ij}_t$ is equal to $\widetilde{Y}^i_t$ on the random interval $[\![\tau_{(1)} \mathbb{1}_D, \tau_{(2)} \mathbb{1}_D[\![$, where $D = \{\tau_{(1)} = \tau_j\}$ and $\tau_{(2)}$ is the time of the second default ($\widehat{Y}^{ij}_t$ is not defined outside the random interval introduced above). At the intuitive level, the process $\widehat{Y}^{ij}_t$ gives the value at time $t$ of the $i^{\text{th}}$ defaultable asset, provided that the first default has occurred at time $t$, and the $j^{\text{th}}$ entity is the first defaulting entity. Hence, $\widehat{Y}^{ij}_t$ is not a new process, but rather an additional notation introduced in order to simplify the formulae that follow.

**Remarks.** It is important to stress that the notion of a 'defaultable asset' should not be understood literally. For instance, if the case of the so-called *flight to quality* the price of a default-free bond is discontinuous, and it jumps at the moment $\tau$ associated with some 'default event' (see, e.g., Collin-Dufresne et al. (2003)). Thus, from the perspective of hedging a default-free bond may be formally classified as a 'defaultable asset'.

In order to find a replicating strategy $\phi$ for a first-to-default claim within the present setup, we need to impose the following $m$ conditions on its components $\phi^1, \dots, \phi^k$

$$\sum_{i=1,\, i\neq j}^{m} \phi^i_t \widehat{Y}^{ij}_t + \phi^j_t \bar{Y}^j_t + \sum_{i=m+1}^{k} \phi^i_t Y^i_t = Z^j_t, \quad \forall\, t \in [0,T],\ \forall\, j = 1, \dots, m, \tag{29}$$

where $Z^1, \dots, Z^m$ is a given family of recovery processes. Recall that $Z^j$ specifies the payoff received by the owner of a claim if the first default occurs prior to or at $T$, and the first defaulting entity is the $j^{\text{th}}$ entity.

For the sake of concreteness, assume that $Z^j_t = g_j(t, \widetilde{Y}^1_t, \dots, \widetilde{Y}^m_t, \bar{Y}^1_t, \dots, \bar{Y}^m_t, Y^{m+1}_t, \dots, Y^k_t)$ for some function $g : \mathbb{R}^{k+m+1} \to \mathbb{R}$. Under some additional assumptions, the system of equations (29) can be solved explicitly for $\phi^1, \dots, \phi^m$. In the second step, we need to choose processes $\phi^{m+1}, \dots, \phi^k$ in such a way that a strategy $\phi$ is self-financing prior to the first default, and thus also on the random interval $[\![0, \tau_{(1)} \wedge T]\!]$. Finally, the wealth of a strategy $\phi$ should match the promised payoff $X$ at time $T$ on the event $\{\tau_{(1)} > T\}$. Equivalently, the wealth of $\phi$ should coincide with the value of a considered claim prior to and at default, or up to time $T$ if there is no default in $[0, T]$. It is apparent that the problem of existence of a replicating strategy is non-trivial, but it can be solved in some circumstances. A detailed analysis of an explicit replication result for a particular example of a first-to-default claim is given in Section 6.2.3.

# 6 Vulnerable Claims and Credit Derivatives

In this section, we present a few examples of models and simple defaultable claims for which there exists explicit replicating strategy. We maintain our assumption that the default time $\tau$ admits a continuous hazard process $\Gamma$ with respect to $\mathbb{F}$ under $\mathbb{Q}^*$, where $\mathbb{F} = \mathbb{F}^{W^*}$ is generated by a Brownian motion $W^*$. Recall that $\Gamma$ is also assumed to be an increasing process.

## 6.1 Vulnerable Claims

Let us fix $T > 0$. We postulate that the $T$-maturity default-free bond and defaultable zero-coupon bond with zero recovery are also traded assets. As before, we assume that the risk-neutral dynamics of the discount default-free bond are

$$dB(t,T) = B(t,T)\big(r_t\,dt + b(t,T)\,dW^*_t\big)$$

for some $\mathbb{F}$-predictable volatility process $b(t,T)$.

### 6.1.1 Vulnerable Call Options

For a fixed $U > T$, we assume that the $U$-maturity default-free bond is also traded, and we consider a vulnerable European call option with the terminal payoff

$$\widehat{C}_T = \mathbb{1}_{\{\tau>T\}}(B(T,U)-K)^+ = \mathbb{1}_{\{\tau>T\}}X.$$

We thus deal with a defaultable claim $(X,0,0,\tau)$ with the promised payoff $X = (B(T,U)-K)^+$. The same method can be applied to an arbitrary $\mathcal{F}_T$-measurable promised payoff $X = g(B(T,U))$, where a function $g:\mathbb{R}\to\mathbb{R}$ satisfies usual technical assumptions.

We consider here the situation when one defaultable asset and two default-free assets are traded; we thus place ourselves within the framework of Corollary 5.2. Specifically, we take $Y^1_t = D^0(t,T)$, $Y^2_t = B(t,U)$ and $Y^3_t = B(t,T)$. Consider a strategy $\phi = (\phi^1,\phi^2,\phi^3)$ such that $V_t(\phi) = \phi^1_t \widetilde{D}^0(t,T)$ and $\phi^2_t B(t,U) + \phi^3_t B(t,T) = 0$ for every $t\in[0,T]$. Observe that in view of the definition of $\Gamma(t,T)$ (see Section 3.3) we have

$$\widetilde{Y}^{1,3}_t = \widetilde{D}^0(t,T)(B(t,T))^{-1} = \Gamma(t,T).$$

Moreover, $Y^{2,3}_t = F(t,U,T)$ and $\widehat{Y}^{2,3,1}_t = F(t,U,T)e^{-\widetilde{\alpha}^{2,3,1}_t}$, where $F(t,U,T) = B(t,U)(B(t,T))^{-1}$ and in view of formula (2)

$$\widetilde{\alpha}^{2,3,1}_t = \langle \ln F(\cdot,U,T), \ln\Gamma(\cdot,T)\rangle_t = \int_0^t \big(b(u,U)-b(u,T)\big)\beta(u,T)\,du.$$

Therefore, the dynamics of $\widehat{Y}^{2,3,1}$ under $\mathbb{Q}_T$ are

$$\begin{aligned} d\widehat{Y}^{2,3,1}_t &= \widehat{Y}^{2,3,1}_t\Big(\big(b(t,T)-b(t,U)\big)\beta(t,T)\,dt + \big(b(t,U)-b(t,T)\big)\,dW^T_t\Big)\\ &= \widehat{Y}^{2,3,1}_t\big(b(t,U)-b(t,T)\big)\,\big(dW^T_t - \beta(t,T)\,dt\big). \end{aligned}$$

Let $\widetilde{\mathbb{Q}}$ be a probability measure such that $\widehat{Y}^{2,3,1}$ is a martingale under $\widetilde{\mathbb{Q}}$. By virtue of Girsanov's theorem, it is clear that the process $\widetilde{W}$, given by the formula

$$\widetilde{W}_t = W^T_t - \int_0^t \beta(u,T)\,du, \quad \forall\, t\in[0,T],$$

is a Brownian motion under $\widetilde{\mathbb{Q}}$. Thus, the process $F(t,U,T)$ satisfies under $\widetilde{\mathbb{Q}}$

$$dF(t,U,T) = F(t,U,T)\big(b(t,U)-b(t,T)\big)\big(d\widetilde{W}_t + \beta(t,T)\,dt\big). \tag{30}$$

Since $\widetilde{D}^0(T,T) = 1$, equation (19) becomes

$$\widetilde{C}_0 + \int_0^T \widetilde{\phi}^{2,3,1}_u\,d\widehat{Y}^{2,3,1}_u = X = (F(T,U,T)-K)^+. \tag{31}$$

By a simple extension of (21), for any $t \in [0,T]$ the pre-default value of the option equals

$$\widetilde{C}_t = \widetilde{D}^0(t,T)\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\big((F(T,U,T)-K)^+ \,\big|\, \mathcal{F}_t\big), \tag{32}$$

provided that the integral in (31) is a $\widetilde{\mathbb{Q}}$-martingale, rather than a $\widetilde{\mathbb{Q}}$-local martingale. Let us denote

$$f(t) = \beta(t,T)(b(t,U)-b(t,T)), \quad \forall\, t \in [0,T], \tag{33}$$

and let us assume that $f$ is a deterministic function. Then we have the following result, which extends the valuation formula for a call option written on a default-free zero-coupon bond within the framework of the Gaussian HJM model.

**Proposition 6.1** *The pre-default price $\widetilde{C}_t$ of a vulnerable call option written on a default-free zero-coupon bond equals*

$$\widetilde{C}_t = \widetilde{D}^0(t,T)\Big(F(t,U,T)e^{\int_t^T f(u)\,du}N\big(h_+(t,U,T)\big) - KN\big(h_-(t,U,T)\big)\Big)$$

*where*

$$h_\pm(t,U,T) = \frac{\ln F(t,U,T) - \ln K + \int_t^T f(u)\,du \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

*and $v^2(t,T) = \int_t^T |b(u,U)-b(u,T)|^2\,du$. The replicating strategy $\phi = (\phi^1,\phi^2,\phi^3)$ for the option satisfies*

$$\phi^1_t = \widetilde{C}_t(\widetilde{D}^0(t,T))^{-1}, \quad \phi^2_t = e^{\widetilde{\alpha}^{2,3,1}_T - \widetilde{\alpha}^{2,3,1}_t}\Gamma(t,T)N\big(h_+(t,U,T)\big), \quad \phi^3_t = -\phi^2_t F(t,U,T).$$

*Proof.* Considering the Itô differential $d(\widetilde{C}_t/\widetilde{D}^0(t,T))$, and identifying terms in expression (31), we obtain that the process $\widetilde{\phi}^{2,3,1}$ in the integral representation (31) is given by the formula

$$\widetilde{\phi}^{2,3,1}_t = e^{\int_0^T f(u)\,du}N\big(h_+(t,U,T)\big) = e^{\widetilde{\alpha}^{2,3,1}_T}N\big(h_+(t,U,T)\big).$$

Consequently the valuation formula presented in the proposition is a rather straightforward consequence of (30) and (32). □

**Remarks.** Although we consider here the bond $B(t,U)$ as the underlying asset, it is apparent that the method (and thus also the result) can be applied to a much wider class of underlying assets. For instance, a zero-coupon bond can be substituted with a non-dividend paying stock with the price $S$ (this case was examined in Jeanblanc and Rutkowski (2003)). A suitable modification of formulae established in Proposition 6.1 can also be used to the valuation and hedging of vulnerable caplets, swaptions, and other vulnerable derivatives in lognormal market models of (non-defaultable) LIBORs and swap rates.

**Case of a deterministic hazard process.** Assume now that the $\mathbb{F}$-hazard process $\Gamma$ of $\tau$ is deterministic. Then $\beta(t,T) = 0$ for every $t \in [0,T]$, and thus $\widetilde{\alpha}^{2,3,1}_t = 0$ and $\widehat{Y}^{2,3,1}_t = F(t,U,T)$ for every $t \in [0,T]$. We thus obtain the following result.

**Corollary 6.1** *Let the $\mathbb{F}$-hazard process $\Gamma$ and the volatility $b(t,U)-b(t,T)$, $t \in [0,T]$, of the forward price $F(t,U,T)$ be deterministic. Then the pre-default price $\widetilde{C}_t$ of a vulnerable option satisfies $\widetilde{C}_t = \Gamma(t,T)C_t$, where $C_t$ is the price of an equivalent non-vulnerable option*

$$C_t = B(t,U)N\big(h_+(t,U,T)\big) - KB(t,T)N\big(h_-(t,U,T)\big)$$

*where*

$$h_\pm(t,U,T) = \frac{\ln F(t,U,T) - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

*and $v^2(t,T) = \int_t^T |b(u,U)-b(u,T)|^2\,du$. The replicating strategy $\phi = (\phi^1,\phi^2,\phi^3)$ is given by*

$$\phi^1_t = \widetilde{C}_t(\Gamma(t,T)B(t,T))^{-1}, \quad \phi^2_t = \Gamma(t,T)N\big(h_+(t,U,T)\big), \quad \phi^3_t = -\phi^2_t F(t,U,T).$$

### 6.1.2 Vulnerable Bonds

Let us consider the payoff of the form $\mathbb{1}_{\{\tau>T\}}$ which occurs at some date $U > T$. This payoff is, of course, equivalent to the payoff $B(T,U)\mathbb{1}_{\{\tau>T\}}$ at time $T$. We interpret this claim as a *vulnerable bond*; Vaillant (2001) proposes to term such a *delayed defaultable bond*. Although vulnerable bonds are not traded, under suitable assumptions one can show that they can be replicated by other liquid assets. Indeed, to replicate this claim within the framework of this section, it suffices to assume that default-free bonds with maturities $T$ and $U$, as well as the defaultable bond with maturity $T$ are among primary traded assets.

Specifically, we postulate that $\phi^2_t B(t,U) + \phi^3_t B(t,T) = 0$ for every $t \in [0,T]$ and thus the total wealth is invested in defaultable bonds of maturity $T$, so that $\phi^1_t \widetilde{D}^0(t,T) = \widetilde{U}_t(X)$ for every $t \in [0,T]$, where $X = B(T,U) = F(T,U,T)$. Let $\widetilde{D}^0(t,T,U)$ stand for the pre-default value of a vulnerable bond at time $t < T$. Then formulae (31) and (32) become

$$\widetilde{D}^0(0,T,U) + \int_0^T \widetilde{\phi}^{2,3,1}_u \, d\widehat{Y}^{2,3,1}_u = F(T,U,T)$$

and

$$\widetilde{D}^0(t,T,U) = \widetilde{D}^0(t,T)\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(F(T,U,T)\,|\,\mathcal{F}_t\big),$$

respectively. Using dynamics (30), we obtain

$$\widetilde{D}^0(t,T,U) = \widetilde{D}^0(t,T)F(t,T,U)e^{\int_t^T f(u)\,du} = \widetilde{D}^0(t,T)F(t,T,U)e^{\widetilde{\alpha}^{2,3,1}_T - \widetilde{\alpha}^{2,3,1}_t} \tag{34}$$

provided that $\widetilde{\alpha}^{2,3,1}$ is deterministic.

## 6.2 Credit Derivatives

The most widely traded credit derivatives are credit default swaps and swaptions, total rate of return swaps and credit linked notes. Furthermore, a large class of basket credit derivatives have a special feature of being linked to the default risk of several reference entities. We shall consider here only two examples: a credit default swap and a first-to-default contract. Before proceeding to the analysis of more complex contract, we shall first examine a standard (non-vulnerable) option written on a defaultable asset.

### 6.2.1 Options on a Defaultable Asset

We shall now consider a non-vulnerable call option written on a defaultable bond with maturity date $U$ and zero recovery. Let $T$ be the expiration date and let $K > 0$ stand for the strike. Formally, we deal with the terminal payoff $\bar{C}_T$ given by

$$\bar{C}_T = (D^0(T,U) - K)^+.$$

To replicate this option, we postulate that defaultable bonds of maturities $U$ and $T$ are primary assets. Notice also that

$$\bar{C}_T = \big(\mathbb{1}_{\{\tau>T\}}\widetilde{D}^0(T,U) - K\big)^+ = \mathbb{1}_{\{\tau>T\}}\big(\widetilde{D}^0(T,U) - K\big)^+ = \mathbb{1}_{\{\tau>T\}}X$$

where $X = (\widetilde{D}^0(T,U) - K)^+$, so that once again we deal with a defaultable claim of the form $(X, 0, 0, \tau)$. It should be stressed, however, that since the underlying asset is now defaultable, the valuation result will differ from Proposition 6.1.

We shall use two defaultable primary assets for replication. Specifically, we shall now apply Corollary 5.3, by choosing $Y^1_t = D^0(t,T)$ and $Y^2_t = D^0(t,U)$ as primary assets. As before, we denote by $\widetilde{C}_t$ the pre-default value of the option under consideration. By virtue of Corollary 5.3, it suffices to show that there exists a process $\phi^2$ such that

$$\widetilde{C}_0 + \int_0^T \phi_u^2 \, d\widetilde{Y}_u^{2,1} = X = (\widetilde{D}^0(T,U) - K)^+ = (\widetilde{Y}_T^{2,1} - K)^+ \tag{35}$$

where $\widetilde{Y}_t^{2,1} = \widetilde{D}^0(t,U)(\widetilde{D}^0(t,T))^{-1}$. Then the trading strategy $\phi = (\phi^1, \phi^2)$ where

$$\phi_t^1 = (\widetilde{C}_t - \phi_t^2 \widetilde{D}^0(t,U))(\widetilde{D}^0(t,T))^{-1}$$

is self-financing and it replicates the option. To derive the valuation formula, it suffices to find the probability measure $\widetilde{\mathbb{Q}}$ such that the process $\widetilde{Y}^{2,1}$ is a $\widetilde{\mathbb{Q}}$-martingale, and to use the generic representation

$$\widetilde{C}_t = \widetilde{D}^0(t,T) \, \mathbb{E}_{\widetilde{\mathbb{Q}}}\big((\widetilde{Y}_T^{2,1} - K)^+ \,\big|\, \mathcal{F}_t\big).$$

Recall that the price process $D^0(t,U)$ admits the representation $D^0(t,U) = \mathbb{1}_{\{\tau > t\}} \widetilde{D}^0(t,U)$ where $\widetilde{D}^0(t,U) = \Gamma(t,U)B(t,T)$. Assume that $\tau$ has a stochastic intensity $\gamma$. Then we have (see (3))

$$d\widetilde{D}^0(t,U) = \widetilde{D}^0(t,U)\Big(\big(r_t + \gamma_t + \beta(t,U)b(t,U)\big)\, dt + \big(\beta(t,U) + b(t,U)\big)\, dW_t^*\Big),$$

and the dynamics of $\widetilde{Y}_t^{2,1} = \widetilde{D}^0(t,U)(\widetilde{D}^0(t,T))^{-1}$ under $\mathbb{Q}^*$ are

$$d\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^{2,1}\Big(\big(r_t + \gamma_t + \beta(t,U)b(t,U)\big)\, dt + \big(\beta(t,U) + b(t,U) - b(t,T)\big)\,\big(dW_t^* - b(t,T)dt\big)\Big).$$

As we said above, it suffices to find the probability measure $\widetilde{\mathbb{Q}}$ such that the process $\widetilde{Y}^{2,1}$ is a $\widetilde{\mathbb{Q}}$-martingale. By applying standard Girsanov transformation, we can construct a measure $\widetilde{\mathbb{Q}}$ so that we have

$$d\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^{2,1}\big(\beta(t,U) + b(t,U) - b(t,T)\big)\, d\widetilde{W}_t.$$

where $\widetilde{W}$ is a Brownian motion under $\widetilde{\mathbb{Q}}$.

**Proposition 6.2** *Assume that $\beta(t,U) + b(t,U) - b(t,T)$, $t \in [0,T]$, is a deterministic function. Then the pre-default price $\widetilde{C}_t$ of a call option written on a $U$-maturity defaultable bond equals*

$$\widetilde{C}_t = \widetilde{D}^0(t,U)N\big(k_+(t,U,T)\big) - K\widetilde{D}^0(t,T)N\big(k_-(t,U,T)\big)$$

*where*

$$k_\pm(t,U,T) = \frac{\ln \widetilde{D}^0(t,U) - \ln \widetilde{D}^0(t,T) - \ln K \pm \frac{1}{2}\widetilde{v}^2(t,T)}{\widetilde{v}(t,T)}$$

*and $\widetilde{v}^2(t,T) = \int_t^T |\beta(u,U) + b(u,U) - b(u,T)|^2 \, du$. The replicating strategy $\phi = (\phi^1, \phi^2)$ for the option is given by*

$$\phi_t^1 = (\widetilde{C}_t - \phi_t^2 \widetilde{D}^0(t,U))(\widetilde{D}^0(t,T))^{-1}, \quad \phi_t^2 = N\big(k_+(t,U,T)\big).$$

**Case of a deterministic hazard process.** Assume that the $\mathbb{F}$-hazard process $\Gamma$ and the volatility $b(t,U) - b(t,T)$, $t \in [0,T]$, of the forward price $F(t,U,T)$ are deterministic.

**Corollary 6.2** *The pre-default price $\widetilde{C}_t$ of a call option written on a $U$-maturity defaultable bond equals*

$$\widetilde{C}_t = e^{-\int_t^U \gamma(u)\, du} B(t,U)N\big(k_+(t,U,T)\big) - Ke^{-\int_t^T \gamma(u)\, du} B(t,T)N\big(k_-(t,U,T)\big)$$

*where*

$$k_\pm(t,U,T) = \frac{\ln B(t,U) - \ln B(t,T) - \ln K - \int_T^U \gamma(u)\, du \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

*and $v^2(t,T) = \int_t^T |b(u,U) - b(u,T)|^2 \, du$. The replicating strategy $\phi = (\phi^1, \phi^2)$ for the option is given by*

$$\phi_t^1 = (\widetilde{C}_t - \phi_t^2 \widetilde{D}^0(t,U))(\widetilde{D}^0(t,T))^{-1}, \quad \phi_t^2 = N\big(k_+(t,U,T)\big).$$

Notice that this is exactly the same result as in the case of a call option written on a zero-coupon bond in a default-free term structure model with the interest rate $r_t$ substituted with the default-risk adjusted rate $r_t + \gamma(t)$.

### 6.2.2 Credit Default Swaps

A generic *credit default swap* (CDS, for short) is a derivative contract which allows to directly transfer the credit risk of the reference entity from one party (the risk seller) to another party (the risk buyer). The contingent payment is triggered by the pre-specified default event, provided that it happens before the maturity date $T$. The standard version of a credit default swap stipulates that the contract is settled at default time $\tau$ of the reference entity, and the recovery payoff equals $Z_\tau = 1 - \delta B(\tau, T)$ where $\delta$ represents the recovery rate at default of a reference entity. It is usually assumed that $0 \leq \delta < 1$ is non-random, and known in advance. This convention corresponds to the fractional recovery of Treasury value scheme for a defaultable bond issued by the reference entity. Otherwise, that is, in case of no default prior to or at $T$, the contract expires at time $T$ worthless. The following alternative market conventions are encountered in practice:

- The buyer pays a lump sum at inception, and the contract is termed a *default option*,
- The buyer pays annuities $\kappa$ at the predetermined dates $0 < T_1 < \ldots < T_{n-1} < T_n = T$ prior to $\tau$, so that the contract represents a plain-vanilla *default swap.*

In the former case, the (pre-default) value $\widetilde{U}_0(Z)$ at time 0 of the default option equals

$$\widetilde{U}_0(Z) = \mathbb{E}_{\mathbb{Q}^*}\Big(B_\tau^{-1}\left(1 - \delta B(\tau, T)\right)\mathbb{1}_{\{\tau \leq T\}}\Big). \tag{36}$$

In the latter case, the level of the annuity $\kappa$ should be chosen in such a way that the value of the contract at time 0 equals zero. The annuity $\kappa$ can thus be specified by solving the following equation

$$\widetilde{U}_0(Z) = \kappa\, \mathbb{E}_{\mathbb{Q}^*}\Big(\sum_{i=1}^{n} B_{T_i}^{-1}\, \mathbb{1}_{\{\tau > T_i\}}\Big)$$

where $\widetilde{U}_0(Z)$ is given by (36).

**Digital credit default swap.** The fixed leg of a CDS can be represented as the sequence of payoffs $c_i = \kappa \mathbb{1}_{\{\tau > T_i\}}$ at the dates $T_i$ for $i = 1, \ldots, n$. The fixed leg of a CDS can thus be seen as a portfolio of defaultable zero-coupon bonds with zero recovery, and thus the valuation of the fixed leg is rather straightforward. To simplify the valuation of the floating leg, we shall consider a digital CDS. Specifically, we postulate that the constant payoff $\delta$ is received at time $T_{i+1}$ if default occurs between $T_i$ and $T_{i+1}$. Therefore, the floating leg is represented by the following sequence of payoffs:

$$d_i = \delta \mathbb{1}_{\{T_i < \tau \leq T_{i+1}\}} = \delta \mathbb{1}_{\{\tau \leq T_{i+1}\}} - \delta \mathbb{1}_{\{\tau \leq T_i\}}$$

at the dates $T_{i+1}$ for $i = 1, \ldots, n-1$. Clearly

$$d_i = \delta(1 - \mathbb{1}_{\{\tau > T_{i+1}\}}) - \delta(1 - \mathbb{1}_{\{\tau > T_i\}}).$$

We conclude that in order to analyze the floating leg of a digital CDS, it suffices to focus on the valuation and replication of the payoff $\mathbb{1}_{\{\tau > T_i\}}$ that occurs at time $T_{i+1}$, that is, a vulnerable bond. The latter problem was already examined in Section 6.1.2, however (see, in particular, the valuation formula (34)).

### 6.2.3 First-to-Default Claims

We shall now focus on the issue of modeling dependent ("correlated") defaults, which arises in the context of *basket credit derivatives.* In order to model dependent default times, we shall employ

Kusuoka's (1999) setting with $n = 2$ default times (for related results, see Jarrow and Yu (2001) or Bielecki and Rutkowski (2003)). Our main goal is to show that the jump risk of a first-to-default claim can be perfectly hedged using the underlying defaultable zero-coupon bonds. Recovery schemes and the associated values of (deterministic) recovery rates should be specified a priori.

**Construction of dependent defaults.** Following Kusuoka (1999), we postulate that under the original probability $\mathbb{Q}$ the random times $\tau_i$, $i = 1, 2$, given on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, are assumed to be mutually independent random variables with exponential laws with parameters $\lambda_1$ and $\lambda_2$, resp. Let $\mathbb{F}$ be some reference filtration (generated by a Wiener process $W$, say) such that $\tau_1$ and $\tau_2$ are independent of $\mathbb{F}$ under $\mathbb{Q}$. We write $\mathbb{H}^i$ to denote the filtration generated by the process $H^i_t = \mathbb{1}_{\{\tau_i \le t\}}$ for $i = 1, 2$, and we set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2$. Notice that the process $M^i_t = H^i_t - \int_0^{t\wedge\tau_i} \lambda_i \, du = H^i_t - \lambda(\tau_i \wedge t)$ is a $\mathbb{G}$-martingale for $i = 1, 2$.

For a fixed $T > 0$, we define a probability measure $\mathbb{Q}^*$ on $(\Omega, \mathcal{G}_T)$ by setting

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \eta_T, \quad \mathbb{Q}\text{-a.s.}$$

where the Radon-Nikodým density process $\eta_t$, $t \in [0, T]$, satisfies

$$\eta_t = 1 + \sum_{i=1}^{2} \int_{]0,t]} \eta_{u-} \kappa^i_u \, dM^i_u$$

with auxiliary processes $\kappa^1, \kappa^2$ given by

$$\kappa^1_t = \mathbb{1}_{\{\tau_2 < t\}} \left( \frac{\alpha_1}{\lambda_1} - 1 \right), \quad \kappa^2_t = \mathbb{1}_{\{\tau_1 < t\}} \left( \frac{\alpha_2}{\lambda_2} - 1 \right).$$

Let $B(t, T)$ be the price of zero-coupon bond, and let $\mathbb{Q}_T$ be the forward martingale measure for the date $T$. It appears that the 'martingale intensities' under $\mathbb{Q}^*$ and under $\mathbb{Q}_T$ are

$$\lambda^1_t = \lambda_1 \mathbb{1}_{\{\tau_2 > t\}} + \alpha_1 \mathbb{1}_{\{\tau_2 \le t\}}, \quad \lambda^2_t = \lambda_2 \mathbb{1}_{\{\tau_1 > t\}} + \alpha_2 \mathbb{1}_{\{\tau_1 \le t\}}.$$

Specifically, the process $\bar{M}^i_t = H^i_t - \int_0^{t\wedge\tau_i} \lambda^i_u \, du$ is a $\mathbb{G}$-martingale under $\mathbb{Q}^*$ and under $\mathbb{Q}_T$ for $i = 1, 2$. Moreover, it is easily seen that the random times $\tau_1$ and $\tau_2$ are independent of the filtration $\mathbb{F}$ under $\mathbb{Q}^*$ and $\mathbb{Q}_T$. The following result shows that intensities $\lambda^1$ and $\lambda^2$ can be interpreted as *local intensities* of default with respect to the information available at time $t$. Therefore, the model can be reformulated as a two-dimensional Markov chain.

**Proposition 6.3** *For $i = 1, 2$ and every $t \in [0, T]$ we have*

$$\lambda_i = \lim_{h\downarrow 0} h^{-1} \, \mathbb{Q}_T\{t < \tau_i \le t + h \,|\, \mathcal{F}_t, \tau_1 > t, \, \tau_2 > t\}.$$

*Moreover*

$$\alpha_1 = \lim_{h\downarrow 0} h^{-1} \, \mathbb{Q}_T\{t < \tau_1 \le t + h \,|\, \mathcal{F}_t, \tau_1 > t, \, \tau_2 \le t\}$$

*and*

$$\alpha_2 = \lim_{h\downarrow 0} h^{-1} \, \mathbb{Q}_T\{t < \tau_2 \le t + h \,|\, \mathcal{F}_t, \tau_2 > t, \, \tau_1 \le t\}.$$

Assume that defaultable zero-coupon bonds are subject to zero recovery rule. Then the price of the bond issued by the $i^{\text{th}}$ entity is given by

$$D^0_i(t, T) = B(t, T) \, \mathbb{Q}_T\{\tau_i > T \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau_i > t\}} \widetilde{D}^0_i(t, T),$$

where, as usual, $\widetilde{D}^0_i(t, T)$ stands for the pre-default value of the bond. Let us denote $\lambda = \lambda_1 + \lambda_2$ and let us assume that $\lambda - \alpha_1 \ne 0$. Then straightforward calculations lead to an explicit formula for $\widetilde{D}^0_i(t, T)$ (for details, see Bielecki and Rutkowski (2003)). Of course, an analogous expression holds for the pre-default price $\widetilde{D}^0_2(t, T)$ provided that $\lambda - \alpha_2 \ne 0$.

**Proposition 6.4** *Assume that $\lambda - \alpha_1 \neq 0$. Then for every $t \in [0,T]$ the pre-default price $\widetilde{D}^0_1(t,T)$ equals*

$$\widetilde{D}^0_1(t,T) = \mathbb{1}_{\{\tau_2 > t\}}\, D^*_1(t,T) + \mathbb{1}_{\{\tau_2 \le t\}}\, \widehat{D}_1(t,T)$$

*where*

$$D^*_1(t,T) = \frac{B(t,T)}{\lambda - \alpha_1}\Big(\lambda_2 e^{-\alpha_1(T-t)} + (\lambda_1 - \alpha_1)e^{-\lambda(T-t)}\Big)$$

*represents the value of the bond prior to the first default, that is, on the random interval $[\![0, \tau_{(1)} \wedge T[\![$, and $\widehat{D}_1(t,T) = B(t,T)e^{-\alpha_1(T-t)}$ is the value of the bond after the default of the second entity, but prior to default of the issuer, that is, on $[\![\tau_2 \wedge T, \tau_1 \wedge T[\![$.*

Let $\tau_{(1)} = \tau_1 \wedge \tau_2$ be the date of the first default. Consider a first-to-default claim with the terminal payoff $X\mathbb{1}_{\{\tau_{(1)} > T\}}$, where $X$ is an $\mathcal{F}_T$-adapted random variable, and $\mathbb{F}$-predictable recovery processes $Z^1$ and $Z^2$. As primary traded assets, we take defaultable zero-coupon bonds $D^0_1(t,T)$ and $D^0_2(t,T)$ with respective default times $\tau_1$ and $\tau_2$, as well as the default-free zero-coupon bond $B(t,T)$.

In Section 5.4, we have examined the basic features of a replicating strategy for a first-to-default claim. Under the present assumptions, (29) becomes

$$\begin{aligned}
\phi^1_t B(t,T)e^{-\alpha_1(T-t)} + \phi^3_t B(t,T) &= Z^2_t,\\
\phi^2_t B(t,T)e^{-\alpha_2(T-t)} + \phi^3_t B(t,T) &= Z^1_t.
\end{aligned}$$

A strategy $\phi$ should be self-financing prior to the first default (and thus also on the random interval $[\![0, \tau_{(1)} \wedge T]\!]$). In other words, we are looking for $\phi$ such that the pre-default wealth process $\widetilde{V}(\phi)$, given by the formula

$$\widetilde{V}_t(\phi) = \phi^1_t D^*_1(t,T) + \phi^2_t D^*_2(t,T) + \phi^3_t B(t,T), \quad \forall\, t \in [0,T],$$

satisfies

$$d\widetilde{V}_t(\phi) = \phi^1_t\, dD^*_1(t,T) + \phi^2_t\, dD^*_2(t,T) + \phi^3_t\, dB(t,T). \tag{37}$$

Finally, at time $T$ the wealth of $\phi$ should coincide with the promised payoff $X$ on the event $\{\tau_{(1)} > T\}$. This means that the pre-default wealth needs to satisfy $\widetilde{V}_T(\phi) = X$, so that (37) becomes

$$\widetilde{V}_0(\phi) + \int_0^T \phi^1_t\, dD^*_1(t,T) + \int_0^T \phi^2_t\, dD^*_2(t,T) + \int_0^T \phi^3_t\, dB(t,T) = X.$$

Equivalently, the pre-default wealth should coincide with the pre-default value of a first-to-default claim on the random interval $[\![0, \tau_{(1)} \wedge T[\![$ and the jump of the wealth at default time $\tau_{(1)}$ should adequately reproduce the behavior at $\tau_{(1)}$ of a first-to-default claim.

**First-to-default credit swap.** For the sake of concreteness, let us consider a *first-to-default credit swap*. Specifically, we shall examine replication of a first-to-default claim with $X = 0$ and $Z^i_t = \delta B(t,T)$ for $i = 1,2$ where $0 \le \delta \le 1$. Let $U_t$ be the value of this claim at time $t \in [0,T]$. It can be shown that

$$\mathbb{Q}_T\{\tau_{(1)} > T \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau_{(1)} > t\}}\, e^{-\lambda(T-t)}.$$

Consequently, for every $t \in [0,T]$ we have

$$U_t = \mathbb{1}_{\{\tau_{(1)} > t\}}\, \delta\big(1 - e^{-\lambda(T-t)}\big)B(t,T) + \mathbb{1}_{\{\tau_{(1)} \le t\}}\, \delta B(t,T),$$

and thus the pre-default value equals $\widetilde{U}_t = \delta(1 - e^{-\lambda(T-t)})B(t,T)$. To find the replicating strategy $\phi$, we first observe that $\phi$ needs to satisfy, for every $t \in [0,T]$,

$$\phi^1_t e^{-\alpha_1(T-t)} + \phi^3_t = \delta, \quad \phi^2_t e^{-\alpha_2(T-t)} + \phi^3_t = \delta, \tag{38}$$

Moreover, the pre-default wealth process $\widetilde{V}(\phi)$, given by

$$\widetilde{V}_t(\phi) = \phi_t^1 D_1^*(t,T) + \phi_t^2 D_2^*(t,T) + \phi_t^3 B(t,T), \quad \forall\, t \in [0,T], \tag{39}$$

should satisfy $\widetilde{V}_t(\phi) = \widetilde{U}_t$ and

$$\widetilde{V}_t(\phi) = \widetilde{V}_0(\phi) + \int_0^t \phi_u^1 \, dD_1^*(u,T) + \int_0^t \phi_u^2 \, dD_2^*(u,T) + \int_0^t \phi_u^3 \, dB(u,T). \tag{40}$$

It is convenient to work with relative prices, by taking $B(t,T)$ as a numeraire, so that (39)-(40) become

$$\widetilde{V}_t^B(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 = \delta\big(1 - e^{-\lambda(T-t)}\big) \tag{41}$$

and

$$\widetilde{V}_t^B(\phi) = \widetilde{V}_0^B(\phi) + \int_0^t \phi_u^1 \, dY_u^1 + \int_0^t \phi_u^2 \, dY_u^2, \tag{42}$$

where $\widetilde{V}_t^B(\phi) = \widetilde{V}_t(\phi) B^{-1}(t,T)$ and

$$Y_t^1 = \frac{D_1^*(t,T)}{B(t,T)} = \frac{1}{\lambda - \alpha_1} \Big( \lambda_2 e^{-\alpha_1(T-t)} + (\lambda_1 - \alpha_1) e^{-\lambda(T-t)} \Big)$$

and

$$Y_t^2 = \frac{D_2^*(t,T)}{B(t,T)} = \frac{1}{\lambda - \alpha_2} \Big( \lambda_1 e^{-\alpha_2(T-t)} + (\lambda_2 - \alpha_2) e^{-\lambda(T-t)} \Big).$$

Of course, working with relative values is here equivalent to setting $B(t,T) = 1$ for every $t \in [0,T]$ in equations (39)-(40), as well as in the pricing formulae of Proposition 6.4.

¿From (38) it follows that $\phi^3$ equals

$$\phi_t^3 = \delta - \phi_t^1 e^{-\alpha_1(T-t)} = \delta - \phi_t^2 e^{-\alpha_2(T-t)}, \tag{43}$$

where $\phi^1$ and $\phi^2$ are related to each other through the formula

$$\phi_t^2 = \phi_t^1 e^{(\alpha_2 - \alpha_1)(T-t)}, \quad \forall\, t \in [0,T]. \tag{44}$$

By substituting the last equality in (41), we obtain the following expression for $\phi^1$

$$\phi_t^1 = -\delta e^{-\lambda(T-t)} \Big( Y_t^1 + Y_t^2 e^{(\alpha_2 - \alpha_1)(T-t)} - e^{-\alpha_1(T-t)} \Big)^{-1}$$

More explicitly,

$$\phi_t^1 = -\delta \xi_1 \xi_2 e^{-\xi_1(T-t)} (g(t))^{-1}, \tag{45}$$

where we denote $\xi_i = \lambda - \alpha_i$ for $i = 1,2$ and where $g(t)$ equals

$$g(t) = \lambda_2 \xi_2 + (\lambda_1 - \alpha_1)\xi_2 e^{-\xi_1(T-t)} + \lambda_1 \xi_1 + (\lambda_2 - \alpha_2)\xi_1 e^{-\xi_2(T-t)} - \xi_1 \xi_2.$$

To determine $\phi^2$ we may either use (44) with (45), or to observe that by the symmetry of the problem

$$\phi_t^2 = -\delta e^{-\lambda(T-t)} \Big( Y_t^2 + Y_t^1 e^{(\alpha_1 - \alpha_2)(T-t)} - e^{-\alpha_2(T-t)} \Big)^{-1}.$$

Of course, both methods yield, as expected, the same expression for $\phi^2$, namely,

$$\phi_t^2 = -\delta \xi_1 \xi_2 e^{-\xi_2(T-t)} (g(t))^{-1}.$$

Moreover, straightforward calculations show that for $\phi^1$, $\phi^2$ as above, we have

$$\phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2 = d\widetilde{V}_t^B(\phi) = -\delta \lambda e^{-\lambda(T-t)}.$$

Finally, the component $\phi^3$ can be found from (43), and thus the calculation of a replicating strategy for the considered example of first-to-default credit swap is completed.

**Remarks.** For the sake of simplicity, we have assumed that the recovery rate does not depend on the name of the first defaulting entity, that is, on $i$. The approach described above can also applied to the case when $Z_t^i = \delta_i B(t,T)$, however. In this case, the formula for $\widetilde{U}$ should be appropriately modified, and relationships (43)-(44) become

$$\phi_t^3 = \delta_1 - \phi_t^1 e^{-\alpha_1(T-t)} = \delta_2 - \phi_t^2 e^{-\alpha_2(T-t)}$$

and

$$\phi_t^2 = \phi_t^1 e^{(\alpha_2-\alpha_1)(T-t)} + (\delta_2 - \delta_1)e^{\alpha_2(T-t)}, \quad \forall\, t \in [0,T],$$

respectively. Closed-form expressions for $\phi^1, \phi^2$ (and thus also $\phi^3$) can be derived it this case as well.

## References

Arvanitis, A. and Laurent, J.-P. (1999) On the edge of completeness. *Risk,* October.

Bélanger, A., Shreve, S. and Wong, D. (2001) A unified model for credit derivatives. To appear in *Mathematical Finance.*

Bielecki, T.R., and Jeanblanc, M. (2003) Genuine mean-variance hedging of credit risk: a case study. Working paper.

Bielecki, T.R., Jeanblanc, M., and Rutkowski, M. (2004) Modeling and valuation of credit risk. In: *CIME-EMS Summer School on Stochastic Methods in Finance, Bressanone, July 6-12, 2003*, Springer-Verlag, Berlin Heidelberg New York.

Bielecki, T.R., and Rutkowski, M. (2002) *Credit Risk: Modelling, Valuation and Hedging.* Springer-Verlag, Berlin Heidelberg New York.

Bielecki, T.R., and Rutkowski, M. (2003) Dependent defaults and credit migrations. *Applicationes Mathematicae* 30, 121-145.

Blanchet-Scalliet, C. and Jeanblanc, M. (2004) Hazard rate for credit risk and hedging defaultable contingent claims. *Finance and Stochastics* 8, 145–159.

Collin-Dufresne, P. and Hugonnier, J.-N. (2002) On the pricing and hedging of contingent claims in the presence of extraneous risks. Preprint, Carnegie Mellon University.

Collin-Dufresne, P, Goldstein, R.S, and Hugonnier, J.-N. (2003) A general formula for valuing defaultable securities. Preprint.

Duffie, D. (2003) *Dynamic Asset Pricing Theory.* 3rd ed. Princeton University Press, Princeton.

Greenfield, Y. (2000) Hedging of credit risk embedded in derivative transactions. Thesis, Carnegie Mellon University.

Jamshidian, F. (2004) Valuation of credit default swap and swaptions. To appear in *Finance and Stochastics.*

Jarrow, R.A. and Yu, F. (2001) Counterparty risk and the pricing of defaultable securities. *Journal of Finance* 56, 1756–1799.

Jeanblanc, M. and Rutkowski, M. (2002) Default risk and hazard processes. In: *Mathematical Finance – Bachelier Congress 2000*, H. Geman, D. Madan, S.R. Pliska and T. Vorst, editors, Springer-Verlag, Berlin Heidelberg New York, pp. 281–312.

Jeanblanc, M. and Rutkowski, M. (2003) Modelling and hedging of default risk. In: *Credit Derivatives: The Definitive Guide*, J. Gregory, editor, Risk Books, 2003, 385–416.

Lando, D. (1998) On Cox processes and credit-risky securities. *Review of Derivatives Research* 2, 99–120.

Musiela, M. and Rutkowski, M. (1997) *Martingale Methods in Financial Modelling.* Springer-Verlag, Berlin Heidelberg New York.

Schönbucher, P.J. (1998) Pricing credit risk derivatives. Working paper, University of Bonn.

Schönbucher, P.J. (2003) *Credit Derivatives Pricing Models.* J.Wiley, Chichester.

Vaillant, N. (2001) A beginner's guide to credit derivatives. Working paper, Nomura International.