CDS with Counterparty Risk in a Markov Chain Copula Model with Joint Defaults

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Abstract

We study counterparty risk on CDS in a Markov chain model of two reference credits, the firm underlying the CDS and the CDS protection seller. More specifically, we consider a Markov chain copula model in which wrong way risk is represented by the possibility of joint defaults between the counterpart and the firm. In this set-up we derive semi-explicit formulas for most quantities of interest with regard to CDS counterparty risk, like price, CVA, EPE or hedging strategies. Model calibration is made simple by the copula property of the model. Numerical results show adequation of the behavior of EPE and CVA in the model with stylized features.

Keywords: Counterparty Credit Risk, CDS, wrong way risk, CVA, EPE

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1 Introduction

Since the sub-prime crisis, counterparty risk is a crucial issue in connection with valuation and risk management of credit derivatives. Counterparty risk in general is ‘the risk that a party to an OTC derivative contract may fail to perform on its contractual obligations, causing losses to the other party’ (cf. Canabarro and Duffie [11]). A major issue in this regard is the so-called wrong way risk, namely the risk that the value of the contract be particularly high from the perspective of the other party at the moment of default of the counterparty. As classic examples of wrong way risk, one can mention the situations of selling a put option to a company on its own stock, or entering a forward contract in which oil is bought by an airline company (see Redon [20]).

Among papers dealing with general counterparty risk, one can mention, apart from the abovementioned references, Canabarro et al. [12], Zhu and Pykhtin [22], and the series of papers by Brigo et al. [7,8,9,10]. From the point of view of measurement and management of counterparty risk, two important notions emerge:

- The Credit Value Adjustment process (CVA), which measures the depreciation of a contract due to counterparty risk. So, in rough terms, \( \text{CVA}_t = P_t - \Pi_t \), where \( \Pi \) and \( P \) denote the price process of a contract depending on whether one accounts or not for counterparty risk.
- The Expected Positive Exposure process (EPE), where \( \text{EPE}_t \) is the risk-neutral expectation of the loss on a contract conditional on a default of the counterparty occurring at time \( t \).

Note that the CVA can be given an option-theoretic interpretation, so that counterparty risk can, in principle, be dynamically managed.

1.1 Counterparty Credit Risk

Wrong way risk is particularly important in the case of credit derivatives transactions, at least from the perspective of a credit protection buyer. Indeed, via economic cycle and default contagion effects, the time of default of a counterparty selling credit protection is typically a time of higher value of credit protection.

In a first attempt to deal with counterparty credit risk, we consider in this paper the problem of valuing and hedging a Credit Default Swap with counterparty risk (‘risky CDS’ in the sequel, as opposed to ‘risk-free CDS’, without counterparty risk). Note that this problem already received a lot of attention in the literature. This admittedly specific problem can thus be considered as a benchmark problem of counterparty credit risk. To quote but a few:

- Huge and Lando [14] propose a rating-based approach,
- Hull and White [15] study this problem in the set-up of a static copula model,
- Jarrow and Yu [16] use an intensity contagion model, further considered in Leung and Kwok [18],
- Brigo and Chourdakis [7] work in the set-up of their Gaussian copula and CIR++ intensity model, extended to the issue of bilateral counterparty credit risk in Brigo and Capponi [6],
1.2 A Markov Copula Approach

Here we consider a Markovian model of credit risk in which simultaneous defaults are possible. Wrong way risk is thus represented in the model by the fact that at the time of default of the counterparty, there is a positive probability that the firm on which the CDS is written defaults too, in which case the loss due to counterparty risk is the loss given default of the firm, that is a very large amount. Of course, this simple model should not be taken too literally. We are not claiming here that simultaneous defaults can happen in actual practice. The rationale and financial interpretation of our model is rather that at the time of default of the counterparty, there is a positive probability of a high defaults spreads environment, in which case, the value of the CDS for a protection buyer is close to the loss given default of the firm.

More specifically, we shall be considering a four-state Markov Chain model of two obligors, so that all the computations are straightforward, either that there are explicit formulas for all the quantities of interest, or, in case less elementary parameterizations of the model are used, that these quantities can be easily and quickly computed by solving numerically the related Kolmogorov ODEs.

This Markovian set-up makes it possible to address in a dynamic and consistent way the issues of valuing and hedging the CDS, and/or, if wished, the CVA, interpreted as an option as evoked above. To make this even more practical, we shall work in a Markovian copula set-up in the sense of Bielecki et al. [3], in which calibration of the model marginals to the related CDS curves is straightforward. The only really free model parameters are thus the few dependence parameters, which can be calibrated or estimated in ways that we shall explain in the paper.

1.3 Outline of the Paper

In Section 2 we first describe the mechanism and cash flows of a payer CDS with counterparty credit risk. We then state a few preliminary results about pricing and CVA of this CDS in a general set-up. In Section 3 we introduce our Markov chain copula model, in which we derive explicit formulas for most quantities of interest in regard to a risky CDS, like price, EPE, CVA or hedging strategies. Section 4 is about implementation of the model. Alternative model parameterizations and related calibration or estimation procedures are proposed and analyzed. Numerical results are presented and discussed, showing good agreement of model’s EPE and CVA with expected features. Section 5 recapitulates our model’s main properties and presents some directions for possible extensions of the previous results in terms of models and products, in view of integrating the CDS-CVA tool of this article into a real-life ‘cross-products and markets’ CVA engine.

2 General Set-Up

2.1 Cash Flows

As is well known, a CDS contract involves three entities: A reference credit (firm), a buyer of default protection on the firm, and a seller of default protection on the firm. The issue of counterparty risk on a CDS is:
• Primarily, the fact that the seller of protection may fail to pay the protection cash flows to the buyer in case of a default of the firm;
• Also, the symmetric concern that the buyer may fail to pay the contractual CDS spread to the seller.

We shall focus in this paper on the so-called unilateral counterparty credit risk involved in a payer CDS contract, namely the risk corresponding to the first bullet point above; however it should be noted that the approach of this paper could be extended to the issue of bilateral credit risk.

We shall refer to the buyer and the seller of protection on the firm as the risk-free investor and the defaultable counterpart, respectively. Indices 1 and 2 will refer to quantities related to the firm and to the counterpart, first of which, their default times $\tau_1$ and $\tau_2$.

Under a risky CDS (payer CDS with counterparty credit risk), the investor pays to the counterpart a stream of premia with spread $\kappa$, or Fees Cash Flows, from the inception date (time 0 henceforth) until the occurrence of a credit event (default of the counterpart or the firm) or the maturity $T$ of the contract, whichever comes first.

Let us denote by $R_1$ and $R_2$ the recovery of the firm and the counterpart, supposed to be adapted to the information available at time $\tau_1$ and $\tau_2$, respectively. If the firm defaults prior to the expiration of the contract, the Protection Cash Flows paid by the counterpart to the investor depends on the situation of the counterpart:
• If the counterpart is still alive, she can fully compensate the loss of investor, i.e., she pays $(1 - R_1)$ times the face value of the CDS to the investor;
• If the counterpart defaults at the same time as the firm (note that it is important to take this case into account in the perspective of the model with simultaneous defaults to be introduced later in this paper), she will only be able to pay to the investor a fraction of this amount, namely $R_2(1 - R_1)$ times the face value of the CDS.

Finally, there is a Close-Out Cash Flow which is associated to clearing the positions in the case of early default of the counterpart. As of today, CDSs are sold over-the-counter (OTC), meaning that the two parties have to negotiate and agree on the terms of the contract. In particular the two parties can agree on one of the following three possibilities to exit (unwind) a trade:
• Termination: The contract is stopped after a terminal cash flow (positive or negative) has been paid to the investor;
• Offsetting: The counterpart takes the opposite protection position. This new contract should have virtually the same terms as the original CDS except for the premium which is fixed at the prevailing market level, and for the tenor which is set at the remaining time to maturity of the original CDS. So the counterpart leaves the original transaction in place but effectively cancels out its economic effect;
• Novation (or Assignment): The original CDS is assigned to a new counterpart, settling the amount of gain or loss with him. In this assignment the original counterpart (or transferor), the new counterpart (transferee) and the investor agree to transfer all the rights and obligations of the transferor to transferee. So the transferor thereby ends his involvement in the contract and the investor thereafter deals with the default risk of the transferee.

In this paper we shall focus on termination. More precisely, if the counterpart defaults in the lifetime of the CDS while the firm is still alive, a ‘fair value’ $\chi(\tau_2)$ of the CDS is computed at time $\tau_2$ according to a methodology specified in the CDS contract at inception. If this value (from the perspective of the investor) is negative, $-\chi(\tau_2)$ is paid by the investor to the counterpart, whereas if
it is positive, the counterpart is assumed to pay to the investor a portion \( R_2 \) of \( \chi_{(\tau_2)} \).

**Remark 2.1** A typical specification is \( \chi_{(\tau_2)} = P_{\tau_2} \), where \( P_t \) is the value at time \( t \) of a risk-free CDS on the same reference name, with the same contractual maturity \( T \) and spread \( \kappa \) as the original risky CDS. The consistency of this rather standard way of specifying \( \chi_{(\tau_2)} \) is, in a sense, questionable. Given a pricing model accounting for the major risks in the product at hand, including, if appropriate, counterparty credit risk, with a related price process of the risky CDS denoted by \( \Pi \), it could be argued that a more consistent specification would be \( \chi_{(\tau_2)} = \Pi_{\tau_2} \) (or, more precisely, \( \chi_{(\tau_2)} = \Pi_{\tau_2}^- \), since \( \Pi_{\tau_2} = 0 \) in view of the usual conventions regarding the definition of ex-dividend prices). We shall see in section 4 that, at least in the specific model of this paper, adopting either convention makes little difference in practice.

### 2.2 Pricing

Let us be given a risk-neutral pricing model \((\Omega, \mathbb{F}, \mathbb{P})\), where \( \mathbb{F} = (\mathbb{F}_t)_{t \in [0,T]} \) is a given filtration making the \( \tau_i \)'s stopping times. In absence of further precision, all the processes, first of which, the *discount factor* process \( \beta \), are supposed to be \( \mathbb{F} \)-adapted, and all the *random variables* are assumed to be \( \mathbb{F} \)-measurable. The fair value \( \chi_{(\tau_2)} \) is supposed to be an \( \mathbb{F}_{\tau_2} \)-measurable random variable. The recoveries \( R_1 \) and \( R_2 \) are assumed to be \( \mathbb{F}_{\tau_1^-} \)- and \( \mathbb{F}_{\tau_2^-} \)-measurable random variables. Let \( \mathbb{E}_\tau \) stand for the conditional expectation under \( \mathbb{P} \) given \( \mathbb{F}_\tau \), for any stopping time \( \tau \).

We assume for simplicity that the face value of all the CDSs under consideration (risky or not) is equal to monetary unit and that the spreads are paid continuously in time. All the cash flows and prices are considered from the perspective of the investor. In accordance with the usual convention regarding the definition of ex-dividend prices, the integrals in this paper are taken open on the left and closed on the right of the interval of integration. In view of the description of the cash-flows in subsection 2.1, one then has,

**Definition 2.2** (i) The model *price* process of a risky CDS is given by \( \Pi_t = \mathbb{E}_t[\pi_T(t)] \), where \( \pi_T(t) \) corresponds to the *risky CDS cumulative discounted cash flows* on the time interval \((t, T]\), so,

\[
\beta_t \pi_T(t) = -\kappa \int_{t \wedge \tau_1 \wedge \tau_2 \wedge T} \beta_s ds + \beta_{\tau_1} (1 - R_1) 1_{t < \tau_1 \leq T} [1_{\tau_1 < \tau_2} + R_2 1_{\tau_1 = \tau_2}] + \beta_{\tau_2} 1_{t < \tau_2 \leq T} 1_{\tau_2 < \tau_1} [R_2 \chi_{(\tau_2)}^+ - \chi_{(\tau_2)}^-].
\]

(ii) The model *price* process of a risk-free CDS is given by \( P_t = \mathbb{E}_t[p_T(t)] \), where \( p_T(t) \) corresponds to the *risk-free CDS cumulative discounted cash flows* on the time interval \((t, T]\), so,

\[
\beta_t p_T(t) = -\kappa \int_{t \wedge \tau_1 \wedge T} \beta_s ds + (1 - R_1) \beta_{\tau_1} 1_{t < \tau_1 \leq T}.
\]

The first, second and third term on the right-hand side of (1) correspond to the fees, protection and close-out cash flows of a risky CDS, respectively. Note that there are no cash flows of any kind after \( \tau_1 \wedge \tau_2 \wedge T \) (in the case of the risky CDS) or \( \tau_1 \wedge T \) (in the case of the risk-free CDS), so \( \pi_T(t) = 0 \) for \( t \geq \tau_1 \wedge \tau_2 \wedge T \) and \( p_T(t) = 0 \) for \( t \geq \tau_1 \wedge T \).
Remark 2.3 In these definitions it is implicitly assumed that, consistently with the now standard theory of no-arbitrage [13], a primary market of financial instruments (along with the risk-free asset \( \beta^{-1} \)) has been defined, with price processes given as locally bounded \((\Omega, \mathbb{F}, \mathbb{P})\) – local martingales. No-arbitrage on the extended market consisting of the primary assets and a further CDS then motivates the previous definitions. Since the precise specification of the primary market is irrelevant until the question of hedging is dealt with, we postpone it to section 3.3.

Let the \( \mathbb{F}_{\tau_2} \)-measurable random variable \( \xi(\tau_2) \), interpreted as the loss incurred at \( \tau_2 \) due to counterparty risk, be given as

\[
\xi(\tau_2) = (1 - R_2) \times \begin{cases} 
(1 - R_1), & \tau_2 = \tau_1 \leq T, \\
\chi(\tau_2), & \tau_2 < \tau_1 \land T, \\
0, & \text{otherwise}
\end{cases}
\]  

(3)

where the subscript \( (\tau_2) \) is used again for emphasizing that \( \xi(\tau_2) \) is an \( \mathbb{F}_{\tau_2} \)-measurable random variable.

Definition 2.4 (i) The Credit Valuation Adjustment (CVA) is the process killed at \( \tau_1 \land \tau_2 \land T \) defined by, for \( t \in [0,T] \),

\[
\beta_t CVA_t = 1_{\{t<\tau_1 \land \tau_2\}} \mathbb{E}_t \left[ \beta_{\tau_2} \xi(\tau_2) \right].
\]  

(4)

(ii) The Expected Positive Exposure (EPE) is the function of time defined by, for \( t \in [0,T] \),

\[
EPE(t) = \mathbb{E} \left[ \xi(\tau_2) \mid \tau_2 = t \right].
\]  

(5)

Let \( \Pi^0 \) denote \( \Pi \) in case \( \chi(\tau_2) = P_{\tau_2} \) (see Remark 2.1). The following proposition justifies the name of Credit Valuation Adjustment which is used for the CVA process defined by (4). This is essentially the basic result that appears in the series of papers by Brigo et alii. Note that as opposed to Brigo et al., we do not exclude simultaneous defaults in our set-up, whence further terms in \( 1_{t<\tau_1 = \tau_2 \leq T} \) in the proof of Proposition 2.1.

Proposition 2.1 One has \( CVA_t = P_t - \Pi^0_t \) on \( \{ t < \tau_2 \} \).

Proof. If \( \tau_1 \leq t < \tau_2 \), then \( \Pi^0_t = P_t = CVA_t = 0 \) in view of (1), (2) and (4).

Assume \( t < \tau_1 \land \tau_2 \). Subtracting \( \pi_T(t) \) from \( p_T(t) \) yields,

\[
\beta_t (p_T(t) - \pi_T(t)) = -\kappa \int_{\tau_1 \land \tau_2 \land T} \beta_s ds + \beta_{\tau_1} (1 - R_1) 1_{t<\tau_1 \land T} 1_{\tau_1 \geq \tau_2} \\
- \beta_{\tau_1} R_2 (1 - R_1) 1_{t<\tau_1 \land T} 1_{\tau_1 = \tau_2} - \beta_{\tau_2} 1_{\tau_2 < \tau_1} 1_{\tau_2 \leq T} (R_2 \chi(\tau_2) - \chi^-). 
\]  

(6)

Moreover, in view of (2), one has,

\[
\beta_{\tau_2} p_T(\tau_2) 1_{\tau_2 < \tau_1} 1_{\tau_2 \leq T} = -\kappa \int_{\tau_1 \land \tau_2 \land T} \beta_s ds + (1 - R_1) \beta_{\tau_1} 1_{\tau_2 < \tau_1 \land T}. 
\]  

(7)
Now, using the following identity in the second term on the right-hand-side of (6):
\[
1_{t < \tau_1 \leq T} \mathbb{I}_{\tau_1 \geq \tau_2} = 1_{t < \tau_1 \leq T} \mathbb{I}_{\tau_2 < \tau_1} + 1_{t < \tau_1 = \tau_2 \leq T},
\]
and plugging (7) into (6), it comes (recall \( t < \tau_1 \wedge \tau_2 \)),
\[
\beta_t (p_T(t) - \pi_T(t)) = \beta_{\tau_2} \mathbb{I}_{\tau_2 < \tau_1} 1_{t < \tau_1 \leq T} p_T(\tau_2)
\]
\[
+ \beta_{\tau_2} \mathbb{I}_{\tau_2 = \tau_1} 1_{t < \tau_1 \leq T} (1 - R_2)(1 - R_1) - \beta_{\tau_2} \mathbb{I}_{\tau_2 < \tau_1} 1_{t < \tau_1 \leq T} R_2 \chi_{(\tau_2)}^+ - \chi_{(\tau_2)}^-. \]

In case \( \chi_{(\tau_2)} = P_{\tau_2} \) one can then proceed as follows:

- On the set \( \{ \tau_2 < \tau_1 \} \),
  \[
  \beta_t (p_T(t) - \pi_T(t)) = \beta_{\tau_2} p_T(\tau_2) - \beta_{\tau_2} (R_2 P_{\tau_2}^+ - P_{\tau_2})
  \]
As \( P_{\tau_2} = \mathbb{E}_{\tau_2}[p_T(\tau_2)] \), we then have, since \( R_2 \) is \( \mathbb{F}_{\tau_2} \)-measurable,
\[
\beta_t \mathbb{E}_{\tau_2}[p_T(t) - \pi_T(t)] = \beta_{\tau_2} P_{\tau_2}^+(1 - R_2). \tag{8}
\]

- On the set \( \{ \tau_1 = \tau_2 \} \),
  \[
  \beta_t (p_T(t) - \pi_T(t)) = \beta_{\tau_2} (1 - R_1)(1 - R_2)
  \]
and thus
\[
\beta_t \mathbb{E}_{\tau_2}[p_T(t) - \pi_T(t)] = \mathbb{E}_{\tau_2}[\beta_{\tau_2} (1 - R_1)(1 - R_2)]. \tag{9}
\]

Using the fact that \( \tau_2 < \tau_1 \) and \( \tau_2 = \tau_1 \) are \( \mathbb{F}_{\tau_2} \)-measurable, it follows,
\[
\beta_t P_t - \beta_t \Pi_t^0 = \beta_t \mathbb{E}_t[p_T(t)] - \beta_t \mathbb{E}_t[\pi_T(t)] = \beta_t \mathbb{E}_t[\mathbb{E}_{\tau_2}[p_T(t) - \pi_T(t)]]
\]
\[
= \beta_t \mathbb{E}_t \left[ \mathbb{E}_{\tau_2}[p_T(t) - \pi_T(t)] \mathbb{I}_{\tau_2 < \tau_1} + \mathbb{E}_{\tau_2}[p_T(t) - \pi_T(t)] \mathbb{I}_{\tau_2 = \tau_1} \right]
\]
\[
= \mathbb{E}_t \left[ \beta_{\tau_2} \chi_{(\tau_2)} \right] = \beta_t \text{CVA}_t.
\]

\[\Box\]

### 2.3 Special Case \( \mathbb{F} = \mathbb{H} \)

Let \( H = (H^1, H^2) \) denote the pair of the default indicator processes of the firm and the counterpart, so \( H^i_t = \mathbb{I}_{t \geq \tau_i} \). The following proposition gathers a few useful results that can be established in the special case of a model filtration \( \mathbb{F} \) given as
\[
\mathbb{F} = \mathbb{H} = (\mathbb{H}^1_t \vee \mathbb{H}^2_t)_{t \in [0, T]},
\]
with \( \mathbb{H}^i_t = \sigma(H^i_s; 0 \leq s \leq t) \).

**Proposition 2.2** (i) For \( t \in [0, T] \), any \( \mathbb{H}_t \)-measurable random variable \( Y_t \) can be written as
\[
Y_t = y_0(t) \mathbb{I}_{t < \tau_1 \wedge \tau_2} + y_1(t, \tau_1) \mathbb{I}_{\tau_1 \leq t < \tau_2} + y_2(t, \tau_2) \mathbb{I}_{\tau_2 \leq t < \tau_1} + y_3(t, \tau_1, \tau_2) \mathbb{I}_{\tau_2 \vee \tau_1 \leq t}
\]
where \( y_0(t), y_1(t, u), y_2(t, v), y_3(t, u, v) \) are deterministic functions.

**(ii)** For any integrable random variable \( Z \), one has,
\[
\mathbb{I}_{t < \tau_1 \land \tau_2} \mathbb{E}_t Z = \mathbb{I}_{t < \tau_1 \land \tau_2} \frac{\mathbb{E}(Z \mathbb{I}_{t < \tau_1 \land \tau_2})}{\mathbb{P}(t < \tau_1 \land \tau_2)}.
\]

**(iii)** The price process of the risky CDS is given by \( \Pi_t = \Pi(t, H_t) \), for a pricing function \( \Pi \) defined on \( \mathbb{R}_+ \times E_1 \times E_1 \) with \( E_1 = \{0, 1\} \), such that \( \Pi(t, e) = 0 \) for \( e \neq (0, 0) \). In particular, \( \Pi_t \) is given on the set \( \{t < \tau_1 \land \tau_2\} \) by the deterministic function
\[
\Pi(t, 0, 0) = u(t) := \frac{\mathbb{E}[\pi_T(t)]}{\mathbb{P}(\tau_1 \land \tau_2 > t)}.
\]

**(iv)** One has, for suitable functions \( \check{\chi}(-), \check{\xi}(-, -) \) and CVA(-),
\[
\mathbb{I}_{\{\tau_2 < \tau_1\}} \check{\chi}(\tau_2) = \mathbb{I}_{\{\tau_2 < \tau_1\}} \check{\chi}(\tau_2)
\]
\[
\check{\xi}(\tau_2) = \check{\xi}(\tau_1, \tau_2) := (1 - R_2) \left( (1 - R_1) \mathbb{I}_{\tau_2 = 1\land \tau_1 \leq T} + \mathbb{I}_{\tau_2 < 1\land T} \mathbb{X}^\dagger(\tau_2) \right)
\]
\[
\text{CVA}_t = \mathbb{I}_{t < \tau_1 \land \tau_2} \text{CVA}(t).
\]

**(v)** A function CVA(-) satisfying (14) is defined by, for \( t \in [0, T] \),
\[
\beta_t \text{CVA}(t) := \int_t^T \beta_s \text{EPE}(s) \frac{\mathbb{P}(\tau_2 \in ds)}{\mathbb{P}(t < \tau_1 \land \tau_2)}.
\]

**Proof.** (i) and (ii) are standard (see, e.g., [4]; (ii) in particular is the so-called **Key Lemma**).

(iii) Since there are no cash flows of a risky CDS beyond the first default (cf. [1]), one has \( \pi_T(t) = \pi_T(t) \mathbb{I}_{t < \tau_1 \land \tau_2} \). The Key Lemma then yields,
\[
\Pi_t = \mathbb{E}_t[\mathbb{I}_{t < \tau_1 \land \tau_2} \pi_T(t)] = (1 - H_t^1)(1 - H_t^2) \frac{\mathbb{E}[\pi_T(t)]}{\mathbb{P}(\tau_1 \land \tau_2 > t)}.
\]

Thus \( \Pi_t = \Pi(t, H_t^1, H_t^2) \), for a pricing function \( \Pi \) defined by
\[
\Pi(t, e_1, e_2) = (1 - e_1)(1 - e_2)u(t),
\]
where \( u(t) \) is defined by the right-hand-side of (11).

(iv) follows directly from part (i), given the definition of the random variables \( \check{\chi}(\tau_2), \check{\xi}(\tau_2) \) and of the CVA process.

(v) By (iv), one has, using (ii) again,
\[
\beta_t \mathbb{I}_{t < \tau_1 \land \tau_2} \text{CVA}_t = \mathbb{I}_{t < \tau_1 \land \tau_2} \mathbb{E}_t[\beta_2 \check{\xi}(\tau_2)] = \mathbb{I}_{t < \tau_1 \land \tau_2} \mathbb{E}_t[\beta_2 \check{\xi}(\tau_1, \tau_2)]
\]
\[
= \mathbb{I}_{t < \tau_1 \land \tau_2} \frac{\mathbb{E}[\beta_2 \check{\xi}(\tau_1, \tau_2) \mathbb{I}_{t < \tau_1 \land \tau_2}]}{\mathbb{P}(t < \tau_1 \land \tau_2)} = \mathbb{I}_{t < \tau_1 \land \tau_2} \frac{\mathbb{E}[\beta_2 \check{\xi}(\tau_1, \tau_2) \mathbb{I}_{t < \tau_2 \leq T}]}{\mathbb{P}(t < \tau_1 \land \tau_2)}
\]
\[
= \mathbb{I}_{t < \tau_1 \land \tau_2} \frac{\mathbb{E}[\beta_2 \text{EPE}(\tau_2) \mathbb{I}_{t < \tau_2 \leq T}]}{\mathbb{P}(t < \tau_1 \land \tau_2)} = \mathbb{I}_{t < \tau_1 \land \tau_2} \int_t^T \beta_s \text{EPE}(s) \frac{\mathbb{P}(\tau_2 \in ds)}{\mathbb{P}(t < \tau_1 \land \tau_2)},
\]
whence (v). \( \square \)


## 3 Markov Copula Factor Set-Up

### 3.1 Factor Process Model

We shall now introduce a suitable Markovian Copula Model for the pair of default indicator processes \( H = (H^1, H^2) \) of the firm and the counterpart. The name ‘Markovian Copula’ refers to the fact that the model will have prescribed marginals for the laws of \( H^1 \) and \( H^2 \), respectively (see Bielecki et al. [2, 3] for a general theory). The practical interest of a Markovian copula model is clear with respect to the task of model calibration, since the copula property allows one to decouple the calibration of the marginal and of the dependence parameters in the model (see again section 4.1). More fundamentally, the opinion developed in this paper is that it is also a virtue for a model to ‘take the right inputs to generate the right outputs’, namely taking as basic inputs the individual default probabilities (individual CDS curves), which correspond to the more reliable information on the market, and are then ‘coupled together’ in a suitable way (see section 4.1).

An apparent shortcoming of the Markov copula approach is that it is does not allow for default contagion effects in the usual sense (default of a name impacting the default intensities of the other ones). The way we shall introduce dependence between \( \tau_1 \) and \( \tau_2 \) is by relaxing the standard assumption of no simultaneous defaults. As we shall see, allowing for simultaneous defaults is a powerful way of modeling defaults dependence.

Specifically, we model the pair \( H = (H^1, H^2) \) as an inhomogeneous Markov chain relative to its own filtration \( \mathbb{H} \) on \((\Omega, \mathbb{H}, \mathbb{P})\), with state space \( E = \{(0, 0), (1, 0), (0, 1), (1, 1)\} \), and matrix-generator at time \( t \) given by the following \( 4 \times 4 \) matrix \( A(t) \), where the first to fourth rows (or columns) correspond to the four possible states \((0, 0), (1, 0), (0, 1) \) and \((1, 1)\) of \( H_t \):

\[
A(t) = \begin{bmatrix}
-l(t) & l_1(t) & l_2(t) & l_3(t) \\
0 & -q_2(t) & 0 & q_2(t) \\
0 & 0 & -q_1(t) & q_1(t) \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad (16)
\]

In (16) the \( l \)’s and \( q \)’s denote deterministic functions of time integrable over \([0, T]\), with in particular \( l(t) = l_1(t) + l_2(t) + l_3(t) \).

**Remark 3.1** The intuitive meaning of ‘(16) being the matrix-generator of \( H \)’ is the following (see, e.g., Rogers and Williams [21] for standard definitions and results on Markov Chains):

- **First line**: Conditional on the pair \( H_t = (H^1_t, H^2_t) \) being in state \((0, 0)\) (firm and counterpart still alive at time \( t \)), there is a probability \( l_1(t)dt \) (resp. \( l_2(t)dt \) resp. \( l_3(t)dt \)) of a default of the firm alone (resp. of the counterpart alone; resp. of a simultaneous default of the firm and the counterpart) in the infinitesimal time interval \((t, t + dt)\);
- **Second line**: Conditional on the pair \( H_t = (H^1_t, H^2_t) \) being in state \((1, 0)\) (firm defaulted but counterpart still alive at time \( t \)), there is a probability \( q_2(t)dt \) of a further default of the counterpart in the time interval \((t, t + dt)\);
- **Third line**: Conditional on the pair \( H_t = (H^1_t, H^2_t) \) being in state \((0, 1)\) (firm still alive but counterpart defaulted at time \( t \)), there is a probability \( q_1(t)dt \) of a further default of the firm in the time interval \((t, t + dt)\).
On each line the diagonal term is then set as minus the sum of the off-diagonal terms, so that the sum of the entries of each line be equal to zero, as should be for \( A(t) \) to represent the generator of a Markov process.

Moreover, for the sake of the desired Markov copula property (Proposition \( \text{[5.1]} \) iii) below), we impose the following relations between the \( l \)'s and the \( q \)'s.

**Assumption 3.2** \( q_1(t) = l_1(t) + l_3(t), \ q_2(t) = l_2(t) + l_3(t) \).

Observe that in virtue of these relations:

- Conditional on \( H^1_t \) being in state 0, and whatever the state of \( H^2_t \) may be (that is, in the state \((0,0)\) as in the state \((0,1)\) for \( H_t \)), there is a probability \( q_1(t)dt \) of a default of the firm (alone or jointly with the counterpart) in the next time interval \((t, t + dt)\);
- Conditional on \( H^2_t \) being in state 0, and whatever the state of \( H^1_t \) may be (that is, in the states \((0,0)\) or \((1,0)\) for \( H_t \)), there is a probability \( q_2(t)dt \) of a default of the counterpart (alone or jointly with the firm) in the next time interval \((t, t + dt)\).

In mathematical terms the default indicator processes \( H^1 \) and \( H^2 \) are \( \mathbb{H} \)-Markov processes on the state space \( E_I = \{0, 1\} \) with time \( t \) generators respectively given by

\[
A_1(t) = \begin{bmatrix} -q_1(t) & q_1(t) \\ 0 & 0 \end{bmatrix}, \ A_2(t) = \begin{bmatrix} -q_2(t) & q_2(t) \\ 0 & 0 \end{bmatrix}.
\]

To formalize the previous statements, and in view of the study of simultaneous jumps, let us further introduce the processes \( H^{1,1}, H^{1,2} \) and \( H^{1,2} \) standing for the indicator processes of a default of the firm alone, of the counterpart alone, and of a simultaneous default of the firm and the counterpart, respectively. So

\[
H^{1,2} = [H^1, H^2], \ H^{1,1} = H^1 - H^{1,2}, \ H^{1,2} = H^1 - H^{1,2}, \ H^{1,2} = H^1 - H^{1,2}, \]

where \([.,.]\) stands for the quadratic covariation. Equivalently, for \( t \in [0, T] \),

\[
H^{1,1}_t = \mathbb{1}_{\tau_1 \leq t, \tau_1 \neq \tau_2}, \ H^{1,2}_t = \mathbb{1}_{\tau_2 \leq t, \tau_1 \neq \tau_2}, \ H^{1,2}_t = \mathbb{1}_{\tau_1 = \tau_2 \leq t}.
\]

Note that the natural filtration of \((H^i)_{i \in I}\), with here and henceforth \( I = \{\{1\}, \{2\}, \{1, 2\}\} \), is equal to \( \mathbb{H} \). The proof of the following Proposition is deferred to Appendix [A].

**Proposition 3.1 (i)** The \( \mathbb{H} \)-intensity of \( H^i \) is of the form \( q_i(t, H_t) \) for a suitable function \( q_i(t, e) \) for every \( i \in I \), namely,

\[
q_{\{1\}}(t, e) = \mathbb{1}_{e_1 = 0}(\mathbb{1}_{e_2 = 0}l_1(t) + \mathbb{1}_{e_2 = 1}q_1(t))
\]

\[
q_{\{2\}}(t, e) = \mathbb{1}_{e_2 = 0}(\mathbb{1}_{e_1 = 0}l_2(t) + \mathbb{1}_{e_1 = 1}q_2(t))
\]

\[
q_{\{1,2\}}(t, e) = \mathbb{1}_{e = (0,0)}l_3(t).
\]

Put another way, the processes \( M^i \) defined by, for every \( i \in I \),

\[
M^i_t = H^i_t - \int_0^t q_i(s, H_s)ds,
\]

where...
Recall in particular (12), (14), (15). We use the notation of Proposition 2.2, which applies here since we are in the special case $F^3$. The discount factor writes $\beta_t = \exp(- \int_0^t r(s) ds)$, for a deterministic short-term interest-rate function $r$. The recovery rates $R_1$ and $R_2$ are constant.

Remark 3.3 In the Markov copula [3] terminology, the so-called consistency condition is satisfied ($H^1$ and $H^2$ are $\mathbb{H}$-Markov processes, see the Proposition 5.1 of [3]). The bi-variate model $H$ with generator $A$ is thus a Markovian copula model with marginal generators $A_1$ and $A_2$.

### 3.2 Pricing

We use the notation of Proposition 2.2, which applies here since we are in the special case $F = \mathbb{H}$. Recall in particular $\Pi_t = \Pi(t, H_t) = (1 - H^1_t)(1 - H^2_t)u(t)$, for a pricing function $\Pi(t, 0, 0) = u(t)$, as well as the identities (12), (14), (15).
Proposition 3.2 The pricing function \( u \) of the risky CDS is given by

\[
\beta_t u(t) = \int_t^T \beta_s e^{-\int_t^s l(u) \, du} \pi(s) \, ds \quad (25)
\]

with

\[
\pi(s) = (1 - R_1) \left[ l_1(s) + R_2 l_3(s) \right] + l_2(s) \left[ R_2 \tilde{\chi}(s)^+ - \tilde{\chi}(s)^- \right] - \kappa \quad (26)
\]

The function \( u \) satisfies the following ODE:

\[
\begin{aligned}
& \{ u(T) = 0 \\
& \frac{du}{dt}(t) - (r(t) + l(t))u(t) + \pi(t) = 0, \quad t \in [0, T) \}
\end{aligned} \quad (27)
\]

Proof. Recall (11):

\[
u(t) = \frac{\mathbb{E}[\pi_T(t)]}{P(\tau_1 \wedge \tau_2 > t)},
\]

where the denominator is given by Proposition 3.1(iii). For computing the numerator, one rewrites the expressions for the cumulative discounted Fee, Protection and Close-out cash flows in terms of integrals with respect to \( H^{(1)} \), \( H^{(2)} \) and \( H^{(1,2)} \), as follows:

Fees Cash Flow \( = \kappa \int_0^T \beta_s(1 - H^1_s)(1 - H^2_s)ds \)

Protection Cash Flow \( = (1 - R_1) \int_0^T \beta_s(1 - H^2_{s-})dH^{(1)}_s + R_2(1 - R_1) \int_0^T \beta_s dH^{(1,2)}_s \)

\( = (1 - R_1) \int_0^T \beta_s(1 - H^2_{s-})dM^{(1)}_s + (1 - R_1) \int_0^T \beta_s(1 - H^2_s)q^{(1)}(s, H_s)ds \)

\( + R_2(1 - R_1) \int_0^T \beta_s dM^{(1,2)}_s + R_2(1 - R_1) \int_0^T \beta_s q^{(1,2)}(s, H_s)ds \)

Close-out Cash Flow \( = \int_0^T \beta_s \left[ R_2 \tilde{\chi}(s)^+ - \tilde{\chi}(s)^- \right] (1 - H^1_{s-})dH^2_s \)

\( = \int_0^T \beta_s \left[ R_2 \tilde{\chi}(s)^+ - \tilde{\chi}(s)^- \right] (1 - H^1_{s-})dM^2_s \)

\( + \int_0^T \beta_s \left[ R_2 \tilde{\chi}(s)^+ - \tilde{\chi}(s)^- \right] (1 - H^1_{s-})q^{(2)}(s, H_s)ds \)

Taking care of the martingale property of \( M^{(1)}, M^{(2)} \) and \( M^{(1,2)} \) and of the fact that the integrals of bounded predictable processes with respect to these martingales are indeed martingales, it thus comes,

\[
\mathbb{E}(\pi_T(t)) = \mathbb{E}(\pi_T(t))
\]

(28)
with
\[ \beta_t \pi_T(t) = -\kappa \int_t^T \beta_s (1 - H^1_s)(1 - H^2_s) ds + (1 - R_1) \int_t^T \beta_s (1 - H^2_s) q_{(1)}(s, H_s) ds + R_2 (1 - R_1) \int_t^T \beta_s q_{(1,2)}(s, H_s) ds + \int_t^T \beta_s \left[ R_2 \chi(s) + - \chi(s) \right] (1 - H^1_s) q_{(2)}(s, H_s) ds. \]

Moreover, in view of the expressions for \( q_{(1)} \) and \( q_{(2)} \) in (20), one has
\[ (1 - H^2_s) q_{(1)}(s, H_s) = (1 - H^1_s)(1 - H^2_s) l_1(s), \]
\[ (1 - H^1_s) q_{(2)}(s, H_s) = (1 - H^1_s)(1 - H^2_s) l_2(s). \]

Plugging this into (28) and using (23), it comes,
\[ \beta_t E[\pi_T(t)] = \mathbb{E} \left[ \int_t^T \beta_s (1 - H^1_s)(1 - H^2_s) \pi(s) ds \right] \]
\[ = \int_t^T \beta_s \mathbb{E} \left[ (1 - H^1_s)(1 - H^2_s) \right] \pi(s) ds \]
\[ = \int_t^T \beta_s e^{-\int_0^s l(x) dx} \pi(s) ds \]

where \( \pi \) is given by (26). One can now check by inspection that the function \( u \) satisfies the ODE (27).

\[ \text{Remark 3.4} \] The equation (27) can also be interpreted as the Kolmogorov backward equation related to the valuation of a risky CDS in our set-up. This ODE can in fact be derived directly and independently by an application of the Itô formula to the martingale \( \Pi(t, H^1_t, H^2_t) \), which results in an alternative proof of Proposition 3.2.

\[ \text{Remark 3.5} \] In the set-up of the Markov chain copula model, the identity (whenever assumed) \( \chi(\tau_2) = \Pi_{\tau_2} \) (see Remark 2.1) is thus equivalent to
\[ \chi(\tau_2) = \Pi_{\tau_2} = \lim_{t \to \tau_2^-} u(t) = u(\tau_2), \]
by continuity of \( u \). This case thus corresponds to the case where the function \( \chi \) in Proposition (2.2)(iv) is in fact given by the function \( u \) (case \( \chi = u \)). In this case the negative and positive parts \( u^+ \) of \( u \) are sitting in the expression for \( \pi \) in (26). One thus deals with a non-linear valuation ODE (27), and the formula (25) is not explicit anymore, since \( u \) is ‘hidden’ in \( \pi \) in the right hand side of (25). However one can still compute \( u \) by numerical resolution of (27).
Proposition 3.3 \textit{The price of a risk-free CDS with spread} \(\kappa\) \textit{on the firm admits the representation:}

\[ P_t = P(t, H_t^1), \tag{30} \]

\textit{for a function} \(P\) \textit{of the form} \(P(t, e_1) = (1 - e_1)v(t)\). \textit{The pricing function} \(v\) \textit{is given by}

\[ \beta_t v(t) = \int_t^T \beta_s e^{-\int_t^s q_1(x)dx} p(s)ds \]

\textit{with}

\[ p(s) = (1 - R_1)q_1(s) - \kappa. \tag{31} \]

\textit{The pricing function} \(v\) \textit{thus solves the following pricing ODE:}

\[ \begin{cases} v(T) = 0 \\ \frac{dv}{dt}(t) - (r(t) + q_1(t))v(t) + p(t) = 0, \quad t \in [0, T]. \end{cases} \]

\textit{Proof.} \textit{One has,}

\[ \beta_t P_T(t) = -\kappa \int_t^T \beta_s (1 - H_s^1) ds + (1 - R_1) \int_t^T \beta_s dH_s^1 \]

\[ = -\kappa \int_t^T \beta_s (1 - H_s^1) ds + (1 - R_1) \int_t^T \beta_s dM_s^1 + (1 - R_1) \int_t^T \beta_s q_1(s)(1 - H_s^1) ds. \]

\textit{As} \(M^1\) \textit{is an} \(\mathbb{H}\)-martingale \textit{and} \(\beta\) \textit{a bounded continuous function, thus}

\[ \beta_t \mathbb{E}_t[P_T(t)] = \mathbb{E}_t \left[ \int_t^T \beta_s (1 - H_s^1) p(s) ds \right] = \int_t^T \beta_s \mathbb{E}_t[1 - H_s^1] p(s) ds, \tag{32} \]

\textit{with} \(p(t)\) \textit{defined by} (31), \textit{and where in virtue of Proposition 3.1(iii) and Proposition 2.2(ii) (Key Lemma), one has for} \(t < s,\)

\[ \mathbb{E}_t[1 - H_s^1] = \mathbb{E}[1 - H_s^1|H_t^1] = (1 - H_t^1) \frac{\mathbb{P}(\tau_1 > s)}{\mathbb{P}(\tau_1 > t)} = (1 - H_t^1)e^{-\int_t^s q_1(x)dx}. \]

\[ \square \]

Proposition 3.4 \textit{One has, for} \(t \in [0, T], \) \textit{(cf. (12), (14) (15)),}

\[ EPE(t) = (1 - R_2) \left( (1 - R_1) \frac{l_3(t)}{q_2(t)} + \overline{\chi}(t) \frac{l_2(t)}{q_2(t)} \right) e^{-\int_t^s t_1(x)dx} \tag{33} \]

\[ \text{CVA}(t) = \int_t^T (1 - R_2) \beta_s \left( (1 - R_1)l_3(s) + \overline{\chi}(s)l_2(s) \right) e^{-\int_t^s t_1(x)dx} ds. \tag{34} \]

\textit{Proof.} \textit{Set}

\[ \Phi(t_2) = \mathbb{E}(1_{\tau_1 = \tau_2 \leq T}), \]
which is characterized by, for every bounded Borel function \( \phi \),

\[
E(\Phi(t_2)\phi(t_2)) = E(\phi(t_2)1_{\tau_1 = \tau_2 \leq T}) .
\]  

(35)

Now, one has by application of the results of Proposition 3.1,

\[
E(\Phi(t_2)\phi(t_2)) = \int_0^\infty \Phi(t)\phi(t)q_2(t)e^{-\int_0^t q_2(s)ds}dt
\]

\[
E(\phi(t_2)1_{\tau_1 = \tau_2 \leq T}) = E(\int_0^T \phi(t)dH_t^{1,2})
\]

\[
= \int_0^T \phi(t)E((1 - H_t^1)(1 - H_t^2))l_3(t)dt = \int_0^T \phi(t)e^{-\int_0^t l(s)ds}l_3(t)dt .
\]

Hence (35) can be rewritten as

\[
\int_0^\infty \Phi(t)\phi(t)q_2(t)e^{-\int_0^t q_2(s)ds}dt = \int_0^T \phi(t)l_3(t)e^{-\int_0^t l(s)ds}dt ,
\]

for every bounded Borel function \( \phi \). So, for \( t \in [0, T] \),

\[
\Phi(t) = \frac{l_3(t)e^{-\int_0^t l(s)ds}}{q_2(t)}e^{\int_0^t q_2(s)ds} ,
\]

which gives the left term in (33).

As for the right term, one has likewise,

\[
E(1_{\tau_2 \leq \tau_1 \land T} | \tau_2) = \Psi(t_2) ,
\]

(36)

with for every bounded and measurable function \( \phi \),

\[
E(\Psi(t_2)\phi(t_2)) = E(\phi(t_2)1_{\tau_2 \leq \tau_1 \land T}) ,
\]

where by application of Proposition 3.1,

\[
E(\Psi(t_2)\phi(t_2)) = \int_0^\infty \Psi(t)\phi(t)q_2(t)e^{-\int_0^t q_2(s)ds}dt
\]

\[
E(\phi(t_2)1_{\tau_2 \leq \tau_1 \land T}) = E(\int_0^\infty \phi(t)1_{t \leq \tau_1 \land T}dH_t^{2}) = E\left( \int_0^\infty \phi(t)1_{t \leq \tau_1 \land T}q_2(t, H_t)dt \right)
\]

\[
= E\left( \int_0^\infty \phi(t)1_{t \leq \tau_1 \land T}(1 - H_t^1)(1 - H_t^2)l_2(t)dt \right) = \int_0^T \phi(t)e^{-\int_0^t l(s)ds}l_2(t)dt
\]

where the second identity in the second line uses that \( H_t^{2} \) does not jump at \( \tau_1 \). So (36) can be rewritten as

\[
\int_0^\infty \Psi(t)\phi(t)q_2(t)e^{-\int_0^t q_2(s)ds}dt = \int_0^T \phi(t)l_2(t)e^{-\int_0^t l(s)ds}dt ,
\]

for every bounded Borel function \( \phi \). Thus, for \( t \in [0, T] \),

\[
\Psi(t) = \frac{l_2(t)e^{-\int_0^t l(s)ds}}{q_2(t)}e^{\int_0^t q_2(s)ds} ,
\]
and (33) follows. Using (15), one then has for \( t \in [0, T] \),

\[
\beta_t \text{CVA}(t) = \int_t^T \beta_s \text{EPE}(s) e^{\int_0^s l(x)dx} e^{-\int_0^s q_2(x)dx} q_2(s) e^{-\int_t^s l(x)dx} ds
\]

\[
= \int_t^T \beta_s \text{EPE}(s) e^{\int_0^s l_1(x)dx} q_2(s) e^{-\int_t^s l(x)dx} ds.
\]

Hence (34) follows from (33).

\[ \square \]

**Remark 3.6** In view of the option-theoretic interpretation of the CVA, the CVA valuation formula (34) can also be established directly, without passing by the EPE, much like formula (25) in Proposition 3.2 above (using a probabilistic computation, or resorting to the related Kolmogorov pricing ODE).

### 3.3 Hedging

We now give another perspective on the counterparty credit risk of the risky CDS, by assessing to which extent the risky CDS could, in principle, be hedged by the risk-free CDS (CDS with the same characteristics, except for the counterparty credit risk).

#### 3.3.1 Price Dynamics

Let \( \hat{\Pi} \) denote the discounted cum-dividend price of the risky CDS, that is, the local martingale

\[
\hat{\Pi}_t = \beta_t \Pi_t + \pi_t(0).
\]

The Itô formula applied to \( \Pi_t = \Pi(t, H_t) \) yields, on \([0, \tau_1 \wedge \tau_2 \wedge T]\),

\[
d\hat{\Pi}_t = \beta_t (\delta \Pi_{\{1\}}(t) dM_{t}^{(1)} + \delta \Pi_{\{2\}}(t) dM_{t}^{(2)} + \delta \Pi_{\{1,2\}}(t) dM_{t}^{\{1,2\}})
\]

(37)

with

\[
\delta \Pi_{\{1\}}(t) = 1 - \bar{R}_1 - u(t), \quad \delta \Pi_{\{2\}}(t) = R_2 \bar{X}^+(t) - \bar{X}^-(t) - u(t), \quad \delta \Pi_{\{1,2\}}(t) = R_2 (1 - \bar{R}_1) - u(t).
\]

Similarly, setting

\[
\hat{P}_t = \beta_t P_t + p_t(0), \quad (38)
\]

it comes

\[
d\hat{P}_t = \beta_t \delta P_t(t) dM_t^1
\]

with

\[
\delta P_t(t) = 1 - \bar{R}_1 - v(t).
\]
3.3.2 Min-Variance Hedging

Let us denote by $\psi$ a (self-financing) strategy in the risk-free CDS with price process $P$ (and the savings account $\beta_1$) for tentatively hedging the risky CDS with price process $\Pi$.

Recall that $P$ is the risk neutral probability chosen by market. So the discounted cum-dividend price process $\hat{P}$ is a $P$-local martingale. As a result of the Galtchouk-Kunita-Watanabe decomposition, the min-variance hedging strategy $\psi_{va}$ is given by

$$\psi_{va} t = \frac{d\langle \hat{\Pi}, \hat{P} \rangle_t}{d\langle \hat{P} \rangle_t}.$$  

In view of the price dynamics (37)-(38), one has, for $t \leq \tau_1 \wedge \tau_2$,

$$d\langle \hat{\Pi}, \hat{P} \rangle_t = \frac{q_1(t)(\delta \Pi_1(t))(\delta P_1(t)) + q_{1,2}(t)(\delta \Pi_{1,2}(t))(\delta P_1(t))}{q_1(t)\delta P_1(t)^2}.$$ 

So

$$\psi_{va} t = \frac{l_1(t) 1 - R_1 - u(t)}{q_1(t) 1 - R_1 - v(t)} + \frac{l_3(t) R_2(1 - R_1) - u(t)}{q_1(t) 1 - R_1 - v(t)}$$

on $[0, \tau_1 \wedge \tau_2 \wedge T]$ (and $\psi_{va} = 0$ on $(\tau_1 \wedge \tau_2 \wedge T, T]$).

Remark 3.7 This min-variance hedging strategy can be easily extended to multi-instrument hedging schemes. In case three non-redundant hedging instruments are available, then, in view of (37), the risky CDS can be perfectly replicated.

4 Implementation

4.1 Affine Intensities Model Specification

Note that the Markov chain copula model primitives are the marginal pre-default intensity functions $q_1$ and $q_2$ as well as the ‘dependence intensity function’ $l_3$ in $A(t)$ (cf. (16)).

Let us specify, for constants $a$’s and $b$’s,

$$q_1(t) = a_i + b_i t , l_3(t) = a_3 + b_3 t ,$$  

(39)

with

$$a_3 = \alpha \min\{a_1, a_2\} , \quad b_3 = \alpha \min\{b_1, b_2\} ,$$

for a model dependence parameter $\alpha \in [0, 1]$ (for the sake of Assumption 3.2).

Remark 4.1 Such an affine specification of intensities was already used by Bielecki et. al. [2] in a context of CDO modeling.
It is immediate to check that under (39), the spread $\kappa_i$ of a risk-free CDS on name $i$ is given by (see Bielecki et. al. [2]),

$$\kappa_i = (1 - R_i) \int_0^T \beta_t (a_i + b_i t) \exp(-a_i t - \frac{b_i t^2}{2}) dt \int_0^T \beta_t \exp(-a_i t - \frac{b_i t^2}{2}) dt . \quad (40)$$

Also note that one has, by Proposition 3.1(v),

$$\rho := \rho(T) = \frac{e^{a_3 T + b_3 T^2 / 2} - 1}{\sqrt{(e^{a_1 T + b_1 T^2 / 2} - 1) (e^{a_2 T + b_2 T^2 / 2} - 1)}} , \quad (41)$$

or, equivalently,

$$\alpha = \ln \left( 1 + \rho \sqrt{(e^{a_1 T + b_1 T^2 / 2} - 1) (e^{a_2 T + b_2 T^2 / 2} - 1)} \right) aT + bT^2 / 2 \quad (42)$$

where $a = \min\{a_1, a_2\}$ and $b = \min\{b_1, b_2\}$.

### 4.1.1 Calibration Issues

Using (40), the $a_i$'s and $b_i$'s can be calibrated independently in a straightforward way to the market CDS curves of the firm and the counterpart, respectively. Note in this regard that market CDS curves can be considered as ‘risk-free CDS curves’.

As for the model dependence parameter $\alpha$, in case the market price of an instrument sensitive to the dependence structure of default times (basket credit instrument on the firm and the counterpart) is available, one can use it to calibrate $\alpha$. Admittedly however, this situation is an exception rather than the rule. It is thus important to devise a practical way of setting $\alpha$ in case such a market data is not available. A possible procedure thus consists in ‘calibrating’ $\alpha$ to a target value for the model probability $\mathbb{P}(H_1^T = H_2^T = 1)$ of joint default at the time horizon $T$. A target value for $\mathbb{P}(H_1^T = H_2^T = 1)$ can be obtained by plugging a standard static Gaussian copula correlation $\hat{\rho}$ into a bivariate normal distribution function. Regulatory capital requirements being based on the Vasicek formula, such a static copula correlation $\hat{\rho}$ can be retrieved from the Basel II correlations per asset class (cf. [1, pages 63 to 66]).

### 4.1.2 Special Case of Constant Intensities

We now look at a particular case in which $b_1 = b_2 = b_3 = 0$. This case will be referred to henceforth as the case of constant intensities, as opposed to the more general case of affine intensities introduced in subsection 4.1. In the case of constant intensities, one has,

$$q_1(t) = a_1 , \quad q_2(t) = a_2 , \quad l_3(t) = a_3 .$$

\[1\] We thank J.-P. Lardy for the suggestion of this procedure.
The correlation coefficient $\rho$ in (41) simplifies to

$$\rho = \frac{e^{a_3 T} - 1}{\sqrt{(e^{a_1 T} - 1) (e^{a_2 T} - 1)}}$$

from which $a_3$ can be calculated as

$$a_3 = \frac{1}{T} \ln \left( 1 + \rho \sqrt{(e^{a_1 T} - 1) (e^{a_2 T} - 1)} \right).$$

As is well known, the price of a risk-free CDS in a constant intensity model is null, i.e., $v(t) \equiv 0$ when $b_1 = 0$. So the EPE formula (33) simplifies to

$$\text{EPE}(t) = (1 - R_1)(1 - R_2) e^{a_3 T} e^{-a_1 t} = \frac{(1 - R_1)(1 - R_2)}{a_2 T} \left[ 1 + \rho \sqrt{(e^{a_1 T} - 1) (e^{a_2 T} - 1)} \right] e^{-a_1 t}$$

Also in this case the pricing formula (25) for the risky CDS reduces to (assuming here $r(t) = r$),

$$u(t) = -(1 - R_1)(1 - R_2) a_3 \frac{1 - e^{-(r + a_1 + a_2 - a_3)(T - t)}}{r + a_1 + a_2 - a_3}.$$

Finally, from Proposition 2.1 one gets,

$$\text{CVA}(t) = -u(t).$$

In particular, for low values of the coefficients,

$$\text{CVA}(0) \simeq (1 - R_1)(1 - R_2) a_3 T = (1 - R_1)(1 - R_2) \ln \left[ 1 + \rho \sqrt{(e^{a_1 T} - 1) (e^{a_2 T} - 1)} \right],$$

so, finally,

$$\text{CVA}(0) \simeq (1 - R_1)(1 - R_2) \sqrt{a_1 a_2 T \rho} \quad (43).$$

### 4.2 Numerical Results

Our aim is to assess by means of numerical experiments the impact of $\rho$ (the correlation coefficient of $H_1^T$ and $H_2^T$) on one hand, and of $\kappa_2$ (the risk-free CDS fair spread of the counterparty as of (40)) on the other hand, on the counterparty risk exposure of the investor.

Towards this end we fix the general data of Table 1 (case with affine intensities) or 3 (case with constant intensities, all $b$'s equal to 0), and we further consider twelve alternative sets of values for $a_2$, $b_2$, and $\rho$ given in columns one, two and seven of Table 2 (case with affine intensities), resp. for $a_2$ and $\rho$ given in columns one and five and seven of Table 4 (case with constant intensities).

In the case of affine intensities the corresponding spreads $\kappa_2$ at time 0 and model dependence parameters $\alpha$ are displayed respectively in the third and sixth column of Table 2, whereas the last column of Table 2 (which will be commented later in the text) gives the corresponding CVA’s at time 0. The risky and risk-free CDS pricing functions $u$ and $v$ corresponding to each of our twelve sets of parameters are displayed in Figure 1. On each graph three curves are represented (see Remark 3.5):
Table 1: Fixed Data — Affine Intensities.

<table>
<thead>
<tr>
<th>r</th>
<th>R₁</th>
<th>R₂</th>
<th>T</th>
<th>a₁</th>
<th>b₁</th>
<th>κ₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>40%</td>
<td>40%</td>
<td>10 years</td>
<td>.0095</td>
<td>.0010</td>
<td>84 bp</td>
</tr>
</tbody>
</table>

Table 2: Variable Data — Affine Intensities.

<table>
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<tr>
<th>a₂</th>
<th>b₂</th>
<th>κ₂</th>
<th>a₃</th>
<th>b₃</th>
<th>α</th>
<th>ρ</th>
<th>CVA(0)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>.0006</td>
<td>50 bp</td>
<td>.0008</td>
<td>.0001</td>
<td>.1368</td>
<td>10%</td>
<td>.0031</td>
</tr>
<tr>
<td>.0085</td>
<td>.0009</td>
<td>75 bp</td>
<td>.0009</td>
<td>.0001</td>
<td>.1124</td>
<td>10%</td>
<td>.0038</td>
</tr>
<tr>
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<td>.0010</td>
<td>100 bp</td>
<td>.0011</td>
<td>.0001</td>
<td>.1170</td>
<td>10%</td>
<td>.0044</td>
</tr>
<tr>
<td>.0189</td>
<td>.0014</td>
<td>150 bp</td>
<td>.0014</td>
<td>.0001</td>
<td>.1466</td>
<td>10%</td>
<td>.0054</td>
</tr>
<tr>
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<td>40%</td>
<td>.0144</td>
</tr>
<tr>
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<td>.0005</td>
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<td>40%</td>
<td>.0165</td>
</tr>
<tr>
<td>.0189</td>
<td>.0014</td>
<td>150 bp</td>
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<td>.0006</td>
<td>.5684</td>
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<td>.0199</td>
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<td>.0031</td>
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<tr>
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<td>.7543</td>
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<td>.0038</td>
</tr>
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<td>.0010</td>
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<td>70%</td>
<td>.0054</td>
</tr>
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</table>

Table 3: Fixed Data — Constant Intensities.

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<th>r</th>
<th>R₁</th>
<th>R₂</th>
<th>T</th>
<th>a₁</th>
<th>κ₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>40%</td>
<td>40%</td>
<td>10 years</td>
<td>.0140</td>
<td>84 bp</td>
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</table>

Table 4: Variable Data — Constant Intensities.

<table>
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<th>κ₂</th>
<th>a₃</th>
<th>α</th>
<th>ρ</th>
<th>CVA(0)</th>
</tr>
</thead>
<tbody>
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<td>.0008</td>
<td>.1366</td>
<td>10%</td>
<td>.0029</td>
</tr>
<tr>
<td>.0125</td>
<td>75 bp</td>
<td>.0009</td>
<td>.1124</td>
<td>10%</td>
<td>.0036</td>
</tr>
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<td>.0167</td>
<td>100 bp</td>
<td>.0011</td>
<td>.1171</td>
<td>10%</td>
<td>.0041</td>
</tr>
<tr>
<td>.0250</td>
<td>150 bp</td>
<td>.0014</td>
<td>.1461</td>
<td>10%</td>
<td>.0049</td>
</tr>
<tr>
<td>.0083</td>
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<td>.5374</td>
<td>40%</td>
<td>.0117</td>
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<td>40%</td>
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</tr>
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<td>.0163</td>
</tr>
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<td>40%</td>
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<td>.0204</td>
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<td>150 bp</td>
<td>.0135</td>
<td>.9648</td>
<td>70%</td>
<td>.0341</td>
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</table>
Figure 1: Pricing functions in the case of affine intensities — $v(t)$ (dashed blue curve), $u^0(t)$ (dotted red curve) and $u^1(t)$ (black curve).
• $v(t)$ (dashed blue curve),
• $u(t)$ with $\tilde{\chi} = v$ therein, denoted by $u^0(t)$ (doted red curve),
• $u(t)$ with $\tilde{\chi} = u$ therein, denoted by $u^1(t)$ (black curve).

The analogous results in the case of constant intensities are displayed in Table 4 and Figure 2. Note that on each graph in Figure 2, the function $v$ is equal to 0, as must be in the case of constant intensities.

In all the cases $u^0$ and $u^1$ are rather close to each other, and one can check numerically that using either one makes little difference regarding the related EPEs and CVAs. We present henceforth the results for $u = u^0$.

Figures 3, 4 and 5 show the graphs of the Expected Positive Exposure as a function of time, of the Credit valuation Adjustment as a function of time, and of the Credit valuation Adjustment at time 0 as a function of $\rho$, in the cases of affine (left graphs) or constant (right graphs) intensities.

One can see on Figure 3 the impact on the counterparty risk exposure of the investor of the default risk (as measured by the risk-free spread $\kappa_2$) of the counterpart. On each graph the correlation coefficient, $\rho$, is fixed, with from top to down $\rho = 10\%$, $40\%$ and $70\%$. The four curves on each graph of Figure 3 correspond to $EPE(t)$ for $\kappa_2 = 50$, 75, 100 and 150bps. Observe that as $\kappa_2$ decreases the counterparty risk exposure increases. This is in line with the stylized features and the financial intuition regarding the EPE: $EPE(t)$ is the expectation of the investor’s loss, given the default of the counterpart at time $t$. A default of a counterpart with a lower spread is interpreted by the markets as a worse news than a default of a counterpart with a higher spread. The related EPE is thus larger.

Figure 4 shows the graphs of the Credit Valuation Adjustment as a function of time, for affine (left column) or constant (right column) intensities. One can thus see the impact of $\kappa_2$ on the CVA. In each graph the correlation coefficient $\rho$ is fixed, with from top to down $\rho = 10\%$, $40\%$ and $70\%$. The four curves on each graph of Figure 4 correspond to $CVA(t)$ for $\kappa_2 = 50$, 75, 100 and 150bps. Observe that as opposed to the EPE, the CVA is increasing in $\kappa_2$, in line with stylized features. Also note that the CVA is a decreasing function of time, in accordance again with expected features: less time to maturity, less risk.

Finally Figure 5 represents the graphs of $CVA(0)$ as a function of $\rho$ for $\kappa_2 = 50$, 75, 100 and 150bps. One can see that $CVA(0)$ grows essentially linearly in $\rho$, as visible on the formula (43) in the case of constant coefficients.

5 Concluding Remarks and Perspectives

In this article we propose a model of CDS with counterparty credit risk, with the following desirable properties:

• Adequation of the behavior of EPE and CVA in the model with expected features (see Section 4.2),
• Wrong way risk (via joint defaults, specifically),
• Simplicity, since the model is a four-state Markov chain of two credit names, with one-name marginals automatically calibrated to the individual CDS curves,
• Fact, related to the previous one, that the model ‘takes the right inputs to generate the right
Figure 2: Pricing functions in the case of constant intensities — $v(t)$ (dashed blue curve), $u^0(t)$ (doted red curve) and $u^1(t)$ (black curve).
Figure 3: EPE(t) ($\tilde{\chi} = v, u = u^0$). In each graph $\rho$ is fixed. From top to down $\rho = 10\%$, $\rho = 40\%$ and $\rho = 70\%$. Left column: affine intensities. Right column: constant intensities.
Figure 4: CVA\((t)\) ($\tilde{\chi} = \nu$, $u = u^0$). In each graph $\rho$ is fixed. From top to down $\rho = 10\%$, $\rho = 40\%$ and $\rho = 70\%$. Left column: affine intensities. Right column: constant intensities.
outputs’, namely it takes as basic inputs the individual default probabilities (individual CDS curves), which correspond to the more reliable information on the market, which are then ‘coupled’ in a suitable way,

- Consistency, in the sense that it is a dynamic model with replication-based valuation and hedging arguments.

The present work might be extended in at least three directions.

First, for certain applications, it is important to model credit spread volatility. One may thus want to enrich the model by adding a reference filtration $\tilde{\mathbb{F}}$ so that the model filtration $\mathbb{F}$ be given as $\tilde{\mathbb{F}} \lor \mathbb{H}$, and the intensities $l, q$ are non-negative $\tilde{\mathbb{F}}$-adapted processes.

A second related issue is that of merging the CDS-CVA pricing tool of this paper into a more general, real-life CVA engine, including the following features:

- Netting, that is, aggregation in a suitable way of all the contracts (as opposed to only one CDS in this paper) relative to a given counterpart,
- Market (other than credit) risk factors,
- Margin agreements.

Finally at the stage of implementation such real-life CVA engines pose interesting challenges from the numerical point of view of Monte Carlo simulations (see, e.g., Zhu and Pykhtin [22]).

A Proof of Proposition 3.1

We shall need the following (essentially classic) Lemma.

Lemma A.1 Let $X$ be a right-continuous process with a finite state space $\mathcal{E}$ and adapted to some filtration $\mathbb{F}$. Condition (i), (ii) or (iii) below are necessary and sufficient conditions for $X$ to be an $\mathbb{F}$ – Markov chain with infinitesimal generator $A(t) = A_t = [A^{ij}_t]_{i, j \in \mathcal{E}}$: 

Figure 5: CVA(0) as a function of $\rho$ for $\kappa_2 = 50$ bp, 75 bp, 100 bp and 150 bp ($\tilde{\chi} = v, u = u^0$). Left: Affine intensities. Right: Constant intensities.
(i) For every function $h$ over $\mathcal{E}$,
\[ M_t^h = h(X_t) - \int_0^t (A_s h)(X_s) ds \]  
(44)
is an $\mathbb{F}$–local martingale;

(ii) For every $j \in \mathcal{E}$, the process $M^j$ defined by
\[ M_t^j = 1_{X_t = j} - \int_0^t A_s X_s^j ds \]
is an $\mathbb{F}$–local martingale;

(iii) For every $i, j \in \mathcal{E}$ the process $M^{i, j}$ given by
\[ M_t^{i, j} = 1_{X_t = i, X_s = j} - \int_0^t 1_{X_s = i} A_s^i j ds \]
is an $\mathbb{F}$–local martingale.

Proof.  (i) is the usual local martingale characterization of Markov chains (see, e.g., Chapter III, section 2 of Rogers and Williams [21]).

(ii) Since $\mathcal{E}$ is finite, the set of the indicator functions $1_{X_t = j}$ spans linearly the set of all functions over $\mathcal{E}$. The condition of part (ii) is thus equivalent to that of (i).

(iii) Necessity follows by combination of Proposition 11.2.2 and Lemma 11.2.3 in [4]. As for sufficiency, note that the $M^{i, j}$’s being $\mathbb{F}$–local martingales implies the same property for the $M^j$’s in (ii), by summation over $i$. We thus conclude by the sufficiency in part (ii).

Let us proceed with the proof of Proposition 3.1. First, note the processes $H^I_t$ can also be written as
\[ H_t^{(1)} = \sum_{0 < s \leq t} 1_{\Delta H_s = (1,0)}, \quad H_t^{(2)} = \sum_{0 < s \leq t} 1_{\Delta H_s = (0,1)}, \quad H_t^{(1,2)} = \sum_{0 < s \leq t} 1_{\Delta H_s = (1,1)}. \]

(i) Let us verify that the $M^I$’s in (19) are $\mathbb{H}$-martingales. For $I = \{1, 2\}$, one has,
\[ M_t^{(1,2)} = H_t^{(1,2)} - \int_0^t q_{\{1,2\}}(s, H_s) ds \]
\[ = \sum_{0 < s \leq t} 1_{\Delta H_s = (1,1)} - \int_0^t 1_{H_s = (0,0)} l_3(s) ds \]
\[ = \sum_{0 < s \leq t} 1_{H_s = (0,0), H_s = (1,1)} - \int_0^t 1_{H_s = (0,0)} l_3(s) ds . \]

Thus Lemma A.1 with $i = (0, 0)$ and $j = (1, 1)$, guarantees the martingale property of $M^{(1,2)}$. 
For $M^{(1)}$, one has,

$$M_t^{(1)} = H_t^{(1)} - \int_0^t q_{(1)}(s, H_s)ds$$

$$= \sum_{0 < s \leq t} 1_{\Delta H_s = (1,0)} - \int_0^t 1_{H_s^2 = 0} \left[ 1_{H_s^2 = 0} l_1(s) + 1_{H_s^2 = 1} q_1(s) \right] ds$$

$$= \left\{ \sum_{0 < s \leq t} 1_{H_s^2 = (0,0), H_s = (1,0)} - \int_0^t 1_{H_s = (0,0)} l_1(s)ds \right\}$$

$$+ \left\{ \sum_{0 < s \leq t} 1_{H_s^2 = (0,1), H_s = (1,1)} - \int_0^t 1_{H_s = (0,1)} q_1(s)ds \right\}.$$ 

Now we apply Lemma A.1 to the two terms in the last equation, with $i = (0,0)$ and $j = (1,0)$ for the first term and $i = (0,1)$ and $j = (1,1)$ for the second term. Thus $M^{(1)}$ being the sum of two $\mathbb{H}$-martingales is an $\mathbb{H}$-martingale. In the same way, $M^{(2)}$ is an $\mathbb{H}$-martingale.

(iii) Since the $M^{i}$’s are $\mathbb{H}$-martingales, this follows easily from the sufficiency in Lemma A.1(ii).

(iv) Formulas (23) follow directly from (22), in which we shall now show the first identity. One has for $t > s$ (see the end of the proof of Proposition 3.3),

$$\mathbb{P}(\tau_2 > t | \mathbb{H}_s) = (1-H^2_s)e^{-\int_s^t q_2(u)du}.$$ 

Thus,

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \mathbb{E}(1_{\tau_1 > s}\mathbb{E}(1_{\tau_2 > t} | \mathbb{H}_s))$$

$$= \mathbb{E}\left\{ (1-H^2_s)(1-H^2_s)e^{-\int_s^t q_2(u)du} \right\},$$

and the result follows.

(v) Since $H^2_t$ is a Bernoulli random variable with (cf. Proposition 3.1(iv))

$$\mathbb{P}(H^2_t = 1) = \mathbb{P}(\tau_1 \leq t) = 1 - \exp\left( -\int_0^t q_1(s)ds \right) =: F_t,$$

one has

$$\mathbb{V}ar(H^2_t) = F_t(1-F_t).$$

Also

$$\mathbb{C}ov(H^2_t, H^2_t) = \mathbb{C}ov(1-H^2_t, 1-H^2_t)$$

$$= \mathbb{E}[(1 - H^2_t)(1 - H^2_t)] - \mathbb{E}(1 - H^2_t)\mathbb{E}(1 - H^2_t)$$

$$= \mathbb{P}(\tau_1 > t, \tau_2 > t) - \mathbb{P}(\tau_1 > t)\mathbb{P}(\tau_2 > t)$$

$$= \exp\left( -\int_0^t l(s)ds \right) - \exp\left( -\int_0^t q_1(s)ds \right) \exp\left( -\int_0^t q_2(s)ds \right).$$
Thus, after some algebraic simplifications,

\[
\rho(t) = \frac{\text{Cov}(H_t^1, H_t^2)}{\sqrt{\text{Var}(H_t^1)\text{Var}(H_t^2)}} = \frac{\exp \left( \int_0^t l_3(s) \, ds \right) - 1}{\sqrt{\exp \left( \int_0^t q_1(s) \, ds \right) - 1} \left( \exp \left( \int_0^t q_2(s) \, ds \right) - 1 \right)}.
\]

References


