Counterparty Risk on a CDS in a Model with Joint Defaults and Stochastic Spreads

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Abstract

A Markov model with simultaneous defaults and stochastic spreads is devised for modeling the counterparty risk on a CDS. The ‘wrong way risk’ in this model is accounted for by the possibility of the common default of the reference name and the counterpart. Semi-explicit formulae are derived for most quantities of interest, such as Credit Valuation Adjustment (CVA) and Expected Positive Exposure (EPE). A dynamic copula property as well as appropriate specifications of the model make the calibration efficient. Numerical results are presented to show the adequation of the behavior of EPE and CVA in the model with stylized features. The spread risk in the model significantly increases the CVA in the cases of a low risk reference entity, or of a low correlation between the reference entity and the counterpart.

Keywords: Counterparty Credit Risk, CDS, CVA, EPE, Spread Risk.

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1 Introduction

The sub-prime crisis has highlighted the importance of counterparty risk in OTC derivative markets, particularly in the case of credit derivatives. We consider in this paper the case of a Credit Default Swap with counterparty risk (‘risky CDS’ in the sequel, as opposed to ‘risk-free CDS’, without counterparty risk). This topic, which corresponds to the emblematic case of CDSs between Lehman and AIG, already received a lot of attention in the literature. It can thus be considered as a benchmark problem of counterparty credit risk. To quote but a few:

- Huge and Lando [14] propose a rating-based approach,
- Hull and White [13] study this problem in the set-up of a static copula model,
- Jarrow and Yu [15] use an intensity contagion model, further considered in Leung and Kwok [18],
- Brigo and Chourdakis [7] work in the set-up of their Gaussian copula and CIR++ intensity model, extended to the issue of bilateral counterparty credit risk in Brigo and Capponi [6],

In this paper we shall work in a Markovian copula set-up in the sense of Bielecki et al. [4], in which calibration of the model marginals to the related CDS curves is straightforward. The so-called wrong way risk (see, e.g., Redon [19]) will thus be represented in the model by the stylized feature that at the time of default of the counterpart, there is a positive probability that the firm on which the CDS is written defaults too, in which case the investor occurs a very large loss (the loss given default of the firm, up to the recovery on the counterpart). Note that we are not claiming here that simultaneous defaults can happen in actual practice. The rationale and financial interpretation of our model is rather that at the time of default of the counterpart, there is a positive probability of a high defaults spreads environment, in which case, the value of the CDS for a protection buyer is close to the loss given default of the firm.

Besides, we shall account for spread risk via stochastic factors, since one expects that addition of CDS spread risk into the model increases CVA. In many cases this increase is rather limited because the main contributor to the CVA in our model is joint defaults, on which the impact of the spreads stochasticity is quite moderate. However we shall see that in the case of a low risk reference entity or a low correlation between the reference entity and the counterpart, the impact of the spreads stochasticity can be very significant.

Remark 1.1 A preparatory version of this model, without stochastic spreads, was studied in [9].

1.1 Outline of the Paper

Section 2 aims at recalling the basics of a payer CDS with counterparty credit risk. In Section 3 we present our Markov model with joint default and stochastic spreads. In Section 4 semi-explicit formulae for most quantities of interest in regard to a risky CDS, like EPE or CVA, are derived. In Section 5 we state an affine specification of the general model of
Section 3. A variant of the model using extended CIR processes will be proposed in section 6. Section 7 is dedicated to the numerical results.

2 Cash Flows and Pricing in a General Set-Up

In this section we briefly recall the basics of CDS counterparty risk, referring the reader to [9] for the details. Let us thus be given a payer CDS with maturity $T$ and contractual spread $\kappa$, as considered from the perspective of an investor assumed to be buyer of default protection on the reference firm of the CDS. Indices 1 and 2 will refer to quantities related to the firm and to the counterpart, first of which, their default times $\tau_1$ and $\tau_2$, and their recoveries upon default, $R_1$ and $R_2$. The default times $\tau_1$ and $\tau_2$ cannot occur at fixed times, but may occur simultaneously. The recovery rates $R_1$ and $R_2$ are assumed to be constant for simplicity. Finally one assumes a deterministic discount factor $\beta(t) = \exp(-\int_0^t r(s) \, ds)$, for a deterministic short-term interest-rate function $r$.

Given a risk-neutral pricing model $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is a given filtration for which the $\tau_i$s are stopping times, let $E_{\tau}$ stand for the conditional expectation under $\mathbb{P}$ given $\mathcal{F}_{\tau}$, for any stopping time $\tau$.

Definition 2.1 (i) The price process of the risk-free CDS is given by $P_t = E_t[p^t]$, where the risk-free CDS cumulative discounted cash flows $\beta(t)p^t$ on the time interval $(t,T)$ are given by,

$$\beta(t)p^t = -\kappa \int_t^{\tau_1 \wedge T} \beta(s) \, ds + (1 - R_1)\beta(\tau_1)1_{t < \tau_1 < T}.$$

(ii) The price process of the risky CDS is given by $\Pi_t = E_t[\pi^t]$, where the risky CDS cumulative discounted cash flows $\beta(t)\pi^t$ on the time interval $(t,T)$ are given by,

$$\beta(t)\pi^t = -\kappa \int_t^{\tau_1 \wedge \tau_2 \wedge T} \beta(s) \, ds + \beta(\tau_1)(1 - R_1)1_{t < \tau_1 < T} \left[1_{\tau_1 < \tau_2} + R_21_{\tau_1 = \tau_2}\right] + \beta(\tau_2)1_{t < \tau_2 < T}1_{\tau_2 < \tau_1} \left[R_2P^+_{\tau_2} - P^-_{\tau_2}\right],$$

in which $P^+$ stands for the positive part of $P$. In particular $p^t = 0$ for $t \geq \tau_1 \wedge T$, and $\pi^t = 0$ for $t \geq \tau_1 \wedge \tau_2 \wedge T$.

Regarding counterparty risk, the key concepts are:
- The Expected Positive Exposure function (EPE), where EPE$(t)$ is the risk-neutral expectation of the loss on a contract conditional on a default of the counterpart occurring at time $t$,
- The Effective Expected Positive Exposure function (efEPE), where efEPE$(t)$ is the risk-neutral expectation of the loss on a contract conditional on a default of the counterpart occurring before time $t$,
- The Credit Value Adjustment process (CVA), which measures the depreciation of a contract due to counterparty risk. So, in rough terms, CVA$_t = P_t - \Pi_t$.

In the case of a payer CDS, the EPE, the efEE and the CVA assume the following form, based on the so-called Potential Future Exposure (PFE).
Definition 2.2 (i) The Potential Future Exposure (PFE) is the $\mathcal{F}_{\tau_2}$-measurable random variable $\xi_{(\tau_2)}$ defined by,

$$\xi_{(\tau_2)} = (1 - R_2) \times \begin{cases} 
(1 - R_1), & \tau_2 = \tau_1 < T, \\
\mathbb{P}_t^1, & \tau_2 < \tau_1 \land T, \\
0, & \text{otherwise}.
\end{cases}$$

(ii) The Expected Positive Exposure (EPE) and the Effective EPE (efEPE) are the functions of time defined by, for $t \in [0, T]$,

$$\text{EPE}(t) = \mathbb{E} \left[ \xi_{(\tau_2)} | \tau_2 = t \right]$$

$$\text{efEPE}(t) = \mathbb{E} \left[ \xi_{(\tau_2)} | \tau_2 < t \right] = \int_0^t \text{EPE}(s) \mathbb{P}(\tau_2 \in ds) \mathbb{P}(\tau_2 < t).$$

(iii) The Credit Valuation Adjustment (CVA) is the process killed at $\tau_1 \land \tau_2 \land T$ defined by,

$$\beta(t) \text{CVA}_t = \mathbb{1}_{\{t < \tau_2\}} \mathbb{E}_t \left[ \beta(\tau_2) \xi_{(\tau_2)} \right].$$

So, in particular,

$$\text{CVA}_0 = \int_0^T \beta(s) \text{EPE}(s) \mathbb{P}(\tau_2 \in ds).$$

The following proposition (cf. [9] or [8]) justifies the name of Credit Valuation Adjustment which is used for the CVA process defined by (1).

Proposition 2.1 One has, $\text{CVA}_t = P_t - \Pi_t$ on $\{t < \tau_2\}$.

3 Model

We now introduce a Markovian model of credit risk with joint defaults and stochastic spreads. Let $H = (H^1, H^2)$ denote the pair of the default indicator processes, so $H^i_t = \mathbb{1}_{\tau_i \leq t}$.

Given a suitable factor process $X = (X_1, X_2)$ to be made precise below, we shall consider a Markovian model of the pair $(X, H)$ relative to its own filtration $\mathbb{F} = \mathbb{X} \lor \mathbb{H}$, with generator given by, for $u = u(t, x, e)$ with $t \in \mathbb{R}_+, x = (x_1, x_2) \in \mathbb{R}^2$, $e = (e_1, e_2) \in \{0, 1\}^2$:

$$\mathcal{A}u(t, x, e) = \partial_t u(t, x, e) + \sum_{1 \leq i \leq 2} l_i(t, x_i) \left( u(t, x, e^i) - u(t, x, e) \right) + l_3(t) \left( u(t, x, 1, 1) - u(t, x, e) \right)$$

$$+ \sum_{1 \leq i \leq 2} \left( b_i(t, x_i) \partial_{x_i} u(t, x, e) + \frac{1}{2} \sigma_i^2(t, x_i) \partial_{x_i}^2 u(t, x, e) \right) + \varrho \sigma_1(t, x_1) \sigma_2(t, x_2) \partial_{x_1, x_2} u(t, x, e),$$

where, for $i = 1, 2$:

\begin{itemize}
  \item $e^i$ denotes the vectors obtained from $e$ by replacing the component $i$, by number one,
  \item $b_i$ and $\sigma_i^2$ denote factor drift and variance functions, and $l_i$ is an individual default intensity function,
  \item $\varrho$ and $l_3(t)$ respectively stand for a factor correlation and a joint defaults intensity function.
\end{itemize}
Remark 3.1 By a convenient abuse of terminology we call here and henceforth generator of a process $\mathcal{X}$, what is strictly speaking the generator of the time-extended process $(t, \mathcal{X})$ (including a $\partial_t$ term).

The choice $\varrho = 0$ will thus correspond to independent factor processes $X^1$ and $X^2$, whereas it is also possible to consider a common factor process $X^1 = X^2 = X$ by letting $b_1 = b_2$, $\sigma_1 = \sigma_2$, $X_0^1 = X_0^2$ and $\varrho = 1$.

The $\mathcal{F}$–intensity matrix function of $H$ (see, e.g., Bielecki and Rutkowski [3]) is thus given by the following $4 \times 4$ matrix $A(t, x)$, where the first to fourth rows (or columns) correspond to the four possible states $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ of $H_t$:

$$A(t, x) = \begin{bmatrix} -l(t, x) & l_1(t, x_1) & l_2(t, x_2) & l_3(t) \\ 0 & -q_2(t, x_2) & 0 & q_2(t, x_2) \\ 0 & 0 & -q_1(t, x_1) & q_1(t, x_1) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with, for every $i = 1, 2$,

$$q_i(t, x_i) = l_i(t, x_i) + l_3(t) \quad (4)$$

and $l(t, x) = l_1(t, x_1) + l_2(t, x_2) + l_3(t)$. We assume standard regularity and growth assumptions on the coefficients of $A$ so as to ensure well-posedness of the related martingale problem (see, e.g., Ethier and Kurtz [12]).

Proposition 3.1 (i) For every $i = 1, 2$, the process $(X^i, H^i)$ is an $\mathcal{F}$-Markov process with generator given by, for $u_i = u_i(t, x_i, e_i)$, with $t \in \mathbb{R}_+, x_i \in \mathbb{R}, e_i \in \{0, 1\}$:

$$A_i u_i(t, x_i, e_i) = \partial_t u_i(t, x_i, e_i) + b_i(t, x_i) \partial_{x_i} u_i(t, x_i, e_i) + \frac{1}{2} \sigma_i^2(t, x_i) \partial_{x_i}^2 u_i(t, x_i, e_i) + q_i(t, x_i) \left(u_i(t, x_i, 1) - u_i(t, x_i, e_i)\right). \quad (5)$$

The $\mathcal{F}$–intensity matrix function of $H^i$ is thus given by

$$A_i(t, x_i) = \begin{bmatrix} -q_i(t, x_i) & q_i(t, x_i) \\ 0 & 0 \end{bmatrix}$$

In other words, the process $M^i$ defined by, for $i = 1, 2$,

$$M^i_t = H^i_t - \int_0^t (1 - H^i_s) q_i(s, X^i_s) ds, \quad (6)$$

is an $\mathcal{F}$-martingale.

(ii) One has,

$$\mathbb{P}(\tau_1 > t) = \mathbb{E} \exp \left( - \int_0^t q_1(s, X^i_s) du \right),$$

$$\mathbb{P}(\tau_1 \wedge \tau_2 > t) = \mathbb{E} \exp \left( - \int_0^t l(u, X_u) du \right). \quad (7)$$
Proof. (i) Applying the operator $A$ in (3) to $u(t, x, e) := u_i(t, x_i, e_i)$, one gets,

$$Au(t, x, e) = A_iu_i(t, x_i, e_i),$$

where $A_i$ is the operator defined in (5). In view of the Markov property of $(X, H)$, process $M^{u_i}$ defined by

$$M^{u_i} := u_i(t, X^i_t, H^i_t) - \int_0^t A_iu_i(s, X^i_s, H^i_s)ds = u(t, X_t, H_t) - \int_0^t Au(s, X_s, H_s)ds,$$

is an $\mathbb{F}$-martingale. By the martingale characterization of Markov processes, the process $(t, X^i_t, H^i_t)$ is thus $\mathbb{F}$-Markovian with generator $A_i$. In particular for $u_i(t, x_i, e_i) := e_i$, one has $A_iu_i(t, x_i, e_i) = q_i(t, x_i)(1 - e_i)$ and the martingale $M^{u_i}$ coincides with $M^i$ given in (6).

(ii) Since $\mathbb{P}(\tau_1 > t) = \mathbb{E} 1_{H^1_t = 0}$ and $\mathbb{P}(\tau_1 \wedge \tau_2 > t) = \mathbb{E} 1_{H^1_t = H^2_t = 0}$, and in view of the Markov properties of $(X^1_t, H^1_t)$ and $(X, H)$, identities (7) can be checked by verification in related Kolmogorov equations.

Remark 3.2 (i) In the terminology of [1], the model $(X, H)$ is a Markovian copula model with marginals $(X^i, H^i)$, (or, in the common factor case, $(X, H^i)$, with $X^1 = X^2 = X$), for $i = 1, 2$. The Markov property of the marginals is key to pricing risk-free CDSs on the firm or on the counterpart, as intensively done at the stage of model calibration.

(ii) One in fact has the following identity, which generalizes (7), for every $s, t > 0$ (see for instance Elouerkhaoui [11] for a proof):

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \mathbb{E} \exp \left( - \int_0^s l_1(u, X^1_u)du - \int_0^t l_2(u, X^2_u)du - \int_0^{s \vee t} l_3(u)du \right).$$

4 Main Results

Lemma 4.1 (i) For every $i = 1, 2$ and function $p = p(t, x_i)$, one has, for $t \in [0, T],$

$$\beta^{-1}(t)\mathbb{E}_t \int_t^T \beta(s)(1 - H^i_s)p(s, X^i_s)ds = (1 - H^i_t)v(t, X^i_t),$$

for a function $v = v(t, x_i)$ solving the following pricing PDE:

$$\begin{cases} v(T, x_i) = 0, & x_i \in \mathbb{R} \\ \left( \partial_t + b_i(t, x_i)\partial_x + \frac{1}{2}\sigma^2_i(t, x_i)\partial^2_x \right)v(t, x_i) - (r(t) + q_i(t, x_i))v(t, x_i) + p(t, x_i) = 0, & t \in [0, T), \ x_i \in \mathbb{R} \end{cases},$$

or, equivalently to [3],

$$v(t, x_i) = \mathbb{E} \left( \int_t^T e^{-\int_t^s (r(\zeta) + q_i(\zeta, X^i_s))d\zeta}p(s, X^i_s)ds \mid X^i_t = x_i \right).$$
(ii) For every function \( \pi_\tau = \pi(t, x) \), one has, for \( t \in [0, T] \),

\[
\beta^{-1}(t) \mathbb{E}_t \int_t^T \beta(s) (1 - H_{x}^1)(1 - H_{x}^2) \pi(s, X_s) \, ds = (1 - H_{x}^1)(1 - H_{x}^2) u(t, X_t),
\]

for a function \( u = u(t, x) \) solving the following pricing PDE:

\[
\begin{cases}
\partial_t u(t, x) + \sum_{1 \leq i \leq 2} (b_i(t, x_i) \partial_{x_i} + \frac{1}{2} \sigma_i^2(t, x_i) \partial_{x_i}^2) u(t, x) + q \sigma_1(t, x_1) \sigma_2(t, x_2) \partial_{x_1, x_2}^2 u(t, x) \\
- (r(t) + l(t, x)) u(t, x) + \pi(t, x) = 0, \quad t \in [0, T), \quad x \in \mathbb{R}^2,
\end{cases}
\]

or, equivalently to (11),

\[
u(t, x) = \mathbb{E}_t \left( \int_t^T e^{-\int_s^t (r(\zeta) + l(\zeta, X_\zeta))} \pi(s, X_s) \, ds \right | X_t = x).
\]

Proof. (i) The Markov property of \( (X^1, H^1) \) stated at Proposition 3.1(i) implies (8). Moreover, in view of the form (3) of the generator of \( (X^1, H^1) \), the function \( u \) has to satisfy (9). From Feynman-Kač formula, one then obtains (10).

(ii) The result follows as in point (i), using the form (3) of the generator of \( (X, H) \).

Let further \( H^{(1)} \), \( H^{(2)} \) and \( H^{(1,2)} \) stand for the indicator processes of a default of the firm alone, of the counterpart alone, and of a simultaneous default of the firm and the counterpart, respectively. So

\[
H^{(1,2)} = [H^1, H^2], \quad H^{(1)} = H^1 - H^{(1,2)}, \quad H^{(2)} = H^2 - H^{(1,2)},
\]

where \([H^1, H^2]\) stands for the quadratic covariation of the default indicator processes \( H^1 \) and \( H^2 \), so \([H^1, H^2]_t = \mathbb{1}_{\tau^1 = \tau^2 \leq t}\).

Lemma 4.2 The \( \mathbb{F} \)-intensity of \( H^{(i)} \) is of the form \( q_i(t, X_t, H_t) \) for a suitable function \( q_i(t, x, e) \) for every \( i \in I = \{1, 2, \{1, 2\}\} \), namely,

\[
q_{(1)}(t, x, e) = \mathbb{1}_{e_1=0} (\mathbb{1}_{e_2=0} l_1(t, x_1) + \mathbb{1}_{e_2=1} q_1(t, x_1))
\]

\[
q_{(2)}(t, x, e) = \mathbb{1}_{e_2=0} (\mathbb{1}_{e_1=0} l_2(t, x_2) + \mathbb{1}_{e_1=1} q_2(t, x_2))
\]

\[
q_{(1,2)}(t, x, e) = \mathbb{1}_{e=(0,0)} l_3(t).
\]

Put another way, for every \( i \in I \), the process \( M^i \) defined by,

\[
M^i_t = H^i_t - \int_0^t q_i(s, X_s, H_s) \, ds,
\]

is an \( \mathbb{F} \)-martingale, where the intensity processes \( q_i(t, X_t, H_t) \) are given by

\[
q_{(1)}(t, X_t, H_t) = (1 - H^1_t) ((1 - H^2_t) l_1(t, X_t^1) + H^2_t q_1(t, X_t^1))
\]

\[
q_{(2)}(t, X_t, H_t) = (1 - H^2_t) ((1 - H^1_t) l_2(t, X_t^2) + H^1_t q_2(t, X_t^2))
\]

\[
q_{(1,2)}(t, X_t, H_t) = (1 - H^1_t)(1 - H^2_t) l_3(t).
\]
Proof. An application of the $\mathcal{F}$-local martingale characterization of the $\mathcal{F}$-Markov process $(X,H)$ with generator $\mathcal{A}$ in (3) yields the $\mathcal{F}$-intensity $\gamma$ of process $H^1H^2$:

$$\gamma_t = (1 - H^1_t)H^2_t l_1(t, X^1_t) + (1 - H^2_t)H^1_t l_2(t, X^2_t) + (1 - H^1_t H^2_t) l_3(t).$$

Using Proposition 3.1(i), one easily deduces the desired expression for the $\mathcal{F}$-intensity process of $H_{\{1,2\}} = [H^1, H^2] = -\int_0^T \beta(s)(1 - H^1_s) ds + (1 - R_1)\int_0^T \beta(s) dM^1_s$.

We are now in a position to derive the risk-free and risky CDS pricing equations.

Proposition 4.3 (i) The price of the risk-free CDS admits the representation:

$$P_t = (1 - H^1_t) v(t, X^1_t),$$

for a pre-default pricing function $v = v(t, x_1)$ as of Lemma 4.1(i) with $i = 1$ and

$$p(t, x_1) = (1 - R_1)q_1(t, x_1) - \kappa$$ (12)

therein.

(ii) The price of the risky CDS admits the representation:

$$\Pi_t = (1 - H^1_t)(1 - H^2_t) u(t, X_t),$$

for a pre-default pricing function $u = u(t, x)$ as of Lemma 4.1(ii) with

$$\pi(t, x) = (1 - R_1) [l_1(t, x_1) + R_2 l_3(t)] + l_2(t, x_2) [R_2 v^+(t, x_1) - v^-(t, x_1)] - \kappa$$ (13)

therein.

Proof. (i) One has $P_t = \mathbb{E}_t (p^t)$, with

$$\beta(t)p^t = -\kappa \int_t^T \beta(s)(1 - H^1_s) ds + (1 - R_1) \int_t^T \beta(s) dH^1_s$$

$$= -\kappa \int_t^T \beta(s)(1 - H^1_s) ds + (1 - R_1) \int_t^T \beta(s) q_1(s, X^1_s)(1 - H^1_s) ds$$

$$+ (1 - R_1) \int_t^T \beta(s) dM^1_s$$

$$= \int_t^T \beta(s)(1 - H^1_s) p(s, X^1_s) ds + (1 - R_1) \int_t^T \beta(s) dM^1_s.$$
Proposition 4.4 One has, for \( t \in [0, T] \),

\[
\text{EPE}(t) = (1 - R_2) \left( (1 - R_1) - \frac{E\left( l_3(t) e^{-\int_0^t \lambda(s,X_s) ds} \right)}{E\left( q_2(t,X^2_t) e^{-\int_0^t \gamma_2(s,X^2_s) ds} \right)} + \frac{E\left( v^+(t,X^1_t) l_2(t,X^2_t) e^{-\int_0^t \lambda(s,X_s) ds} \right)}{E\left( q_2(t,X^2_t) e^{-\int_0^t \gamma_2(s,X^2_s) ds} \right)} \right) \tag{14}
\]

\[
\text{CVA}_0 = (1 - R_2) E \int_0^T \beta(t) \left( (1 - R_1) l_3(t) e^{-\int_0^t \lambda(s,X_s) ds} + v^+(t, X^1_t) l_2(t, X^2_t) e^{-\int_0^t \lambda(s,X_s) ds} \right) dt. \tag{15}
\]

Proof. In view of (2), (15) directly results from (14), that we now prove. Set

\[
\Phi(\tau_2) = E(1_{\tau_1 = \tau_2 < T | \tau_2}) \quad , \quad \Psi(\tau_2) = E(1_{\tau_2 < \tau_1 < T} v^+(\tau_2, X^1_{\tau_2}) | \tau_2),
\]

which are characterized by

\[
E(\Phi(\tau_2) \varphi(\tau_2)) = E(1_{\tau_1 = \tau_2 < T} \varphi(\tau_2)),
\]

\[
E(\Psi(\tau_2) \varphi(\tau_2)) = E(1_{\tau_2 < \tau_1 < T} v^+(\tau_2, X^1_{\tau_2}) \varphi(\tau_2)), \tag{16}
\]

with \( p \) defined by (12). Since \( E_t(\int_t^T \beta(s) dM^1_s) = 0 \), the result then follows by an application of Lemma 4.1(i).

(ii) One has \( \Pi_t = E_t(\pi^t) \), with

\[
\beta(t)\pi^t = -\kappa \int_t^{\tau_1 \land \tau_2 \land T} \beta(s) ds + \beta(\tau_1)(1 - R_1) 1_{t < \tau_1 < T} [1_{\tau_1 < \tau_2} + R_2 1_{\tau_1 = \tau_2}]
\]

\[+ \beta(\tau_2) 1_{t < \tau_2 < T} 1_{\tau_2 < \tau_1} [R_2 P^+_\tau_2 - P^-_{\tau_2}] \] 

\[= -\kappa \int_t^T \beta(s)(1 - H^1_s)(1 - H^2_s) ds \]

\[+ (1 - R_1) \int_t^T \beta(s)(1 - H^2_s) dH^1_s + R_2 (1 - R_1) \int_t^T \beta(s) dH^{1,2}_s \]

\[+ \int_t^T \beta(s) [R_2 v^+(s, X^1_s) - v^-(s, X^1_s)] (1 - H^1_s) dH^{2}_s , \]

where (i) was used in the last line. But in view of Lemma 4.2, this coincides, up to martingale terms, with

\[ -\kappa \int_t^T \beta(s)(1 - H^1_s)(1 - H^3_s) ds \]

\[+ (1 - R_1) \int_t^T \beta(s)(1 - H^1_s)(1 - H^2_s) l_1(s, X^1_s) ds + R_2 (1 - R_1) \int_t^T \beta(s)(1 - H^1_s)(1 - H^2_s) l_3(s) ds \]

\[+ \int_t^T \beta(s)(1 - H^1_s)(1 - H^2_s) l_2(s, X^2_s) [R_2 v^+(s, X^1_s) - v^-(s, X^1_s)] ds \]

\[= \int_t^T \beta(s)(1 - H^1_s)(1 - H^2_s) \pi(s, X_s) ds , \]

where \( \pi \) is given by (13). The result then follows by an application of Lemma 4.1(ii). \( \square \)

One can also derive semi-analytical formulas for the EPE function and for the CVA at time 0. So,

Proposition 4.4 One has, for \( t \in [0, T] \),
for every bounded Borel function \( \varphi \). Let us take \( \varphi(x) = 1_{x \leq t} \) for some \( t \in (0, T] \). Regarding the left-hand sides in (16), one has in view of the law (7) of \( \tau_2 \), for any function \( \Theta \),

\[
E(\Theta(\tau_2)1_{\tau_2 < t}) = \int_0^t \Theta(s)E(q_2(s, X_s) e^{-\int_0^s q_2(\zeta, X_\zeta) d\zeta}) ds .
\]

As for the right-hand-sides of (16), one has thanks to Lemma 4.2 and Proposition 3.1(ii),

\[
E(1_{\tau_2 < t}1_{\tau_1 = \tau_2 < t}) = E\left( \int_0^t dH_s^{(1,2)} \right)
= \int_0^t E((1 - H_s^1)(1 - H_s^2)) l_3(s) ds = \int_0^t E(e^{-\int_0^s l_i(\zeta, X_\zeta) d\zeta}) l_3(s) ds ,
\]

and

\[
E(1_{\tau_2 < t}1_{\tau_2 < \tau_1 \wedge T} v^+(\tau_2, X_{\tau_2}^1)) = E\left( \int_0^t 1_{s < \tau_1} v^+(s, X_s^1) dH_s^{(2)} \right)
= E\left( \int_0^t (1 - H_s^1)(1 - H_s^2) l_2(s, X_s^2) v^+(s, X_s^1) ds \right)
= E\left( \int_0^t e^{-\int_0^s l_i(\zeta, X_\zeta) d\zeta} l_2(s, X_s^2) v^+(s, X_s^1) ds \right)
\]

where Lemma 4.2 and Proposition 3.1(ii) were used in order to derive the last two equalities. Identities (16) can thus be rewritten as

\[
E\left( \int_0^t \Phi(s) q_2(s, X_s^2) e^{-\int_0^s q_2(\zeta, X_\zeta) d\zeta} ds \right) = E\left( \int_0^t l_3(s) e^{-\int_0^s l_i(\zeta, X_\zeta) d\zeta} ds \right) ,
\]

\[
E\left( \int_0^t \Psi(s) q_2(s, X_s^2) e^{-\int_0^s q_2(\zeta, X_\zeta) d\zeta} ds \right) = E\left( \int_0^t l_2(s, X_s^2) v^+(s, X_s^1) e^{-\int_0^s l_i(\zeta, X_\zeta) d\zeta} ds \right) .
\]

Taking derivative with respect to \( t \) in these equations, leads to

\[
\Phi(t) = \frac{E\left( l_3(t) e^{-\int_0^t l_i(\zeta, X_\zeta) d\zeta} \right)}{E\left( q_2(t, X_t^2) e^{-\int_0^t q_2(\zeta, X_\zeta) d\zeta} \right)} , \quad \Psi(t) = \frac{E\left( l_2(t, X_t^2) v^+(t, X_t^1) e^{-\int_0^t l_i(\zeta, X_\zeta) d\zeta} \right)}{E\left( q_2(t, X_t^2) e^{-\int_0^t q_2(\zeta, X_\zeta) d\zeta} \right)}
\]

and (14) follows. \( \square \)

5 Model Specification and Calibration

In view of the model generator (3), the model primitives are the factor coefficients \( b \) and \( \sigma \) and the intensity functions \( l_i \) for \( i = 1 \) to \( 3 \), or, equivalently to the latter via (4), the marginal intensity functions \( q_1 = q_1(t, x_1) \) and \( q_2 = q_2(t, x_2) \) and the joint defaults intensity function \( l_3 = l_3(t) \). In this section, following the lines of Brigo et al. [6, 7], we shall specify the factors in the form of CIR processes. Let thus the \( X \)'s be affine processes of the form

\[
dX_t^i = \eta(\mu_i - X_t^i) dt + \nu \sqrt{X_t^i} dW_t^i ,
\]
for non-negative coefficients $\eta, \mu_i$ and $\nu$. One then sets
\[ q_i(t, x_i) = f_i(t) + \delta x_i, \tag{17} \]
for functions $f_i(t)$ such that $f_i(t) \geq l_3(t)$ and $\delta \in \{0, 1\}$.

**Remark 5.1 (ii)** As in Brigo et al. [6, 7], we shall not restrict ourselves to the inaccessible origin case $2\eta \mu_i > \nu^2$, in order not to limit the range of the model CDS implied volatility.

(ii) The restriction $f_i(t) \geq l_3(t)$ is imposed to guarantee that, consistently with (4), $q_i(t, X^1_t)$ defined by (17) is never smaller than $l_3(t)$.

In the sequel, by $2F$, we mean the parametrization (17) with $\delta = 1$ and independent affine factors $X^1$ and $X^2$, that is two independent CIR++ factors. Also we denote by $0F$, the parametrization (17) with $\delta = 0$, that is without stochastic factors (case of time-deterministic, piecewise constant intensities).

5.1 Marginals

Under the model specification (17), one can derive a more explicit formula for the pricing function $v = v(t, x_1)$. Let $F_1(t) = \int_0^t f_1(s)ds$.

**Proposition 5.1** Assuming (17), one has
\[ \beta(t)v(t, x_1) = \int_t^T \beta(s)((1 - R_1)D_\delta(s, t, x_1) - \kappa)\mathcal{E}_\delta(s, t, x_1)ds, \]
where we set, for $s \geq t$,
\[
\mathcal{E}_\delta(s, t, x_1) = \mathbb{P}(\tau_1 > s \mid X^1_{s^+} = x_1) = \exp\left( -(F_1(s) - F_1(t)) + \delta \phi(s-t, 0)x_1 + \delta \xi(s-t, 0)\mu_1) \right),
\]
\[
\mathcal{D}_\delta(s, t, x_1) = \frac{\mathbb{P}(\tau_1 \in ds \mid X^1_{s^+} = x_1)}{\mathbb{P}(\tau_1 > s \mid X^1_{s^+} = x_1)} = f_1(s) + \delta \eta \mu_1 \phi(s-t, 0) + \delta \left( -\eta \phi(s-t, 0) - \frac{1}{2} \nu^2(\phi(s-t, 0))^2 + 1 \right) x_1,
\]
in which the functions $\phi$ and $\xi$ are those of Lemma 9.1.

**Proof.** Recall from Proposition 4.3 that
\[
\beta(t)v(t, x_1) = \int_t^T \beta(s)\mathbb{E}\left(e^{-\int_t^s q_1(\zeta, X^1_{\zeta})d\zeta}p(s, X^1_s)\mid X^1_t = x_1\right)ds
\]
with
\[
p(s, X^1_s) = (1 - R_1)q_1(s, X^1_s) - \kappa, \quad q_1(t, X^1_t) = f_1(t) + \delta X^1_t.
\]
For $\delta = 0$, the result follows immediately and for $\delta = 1$, it is obtained by an application of Lemma 9.1. \[\square\]

\[\text{\footnotesize{\textsuperscript{1}}Indeed in our numerical tests the calibrated parameters do not always satisfy $2\eta \mu_i > \nu^2$.} \]
In particular the model break-even spread at time 0 of a risk-free CDS of maturity $T$ on the firm, is given by

$$\kappa_0(T) = \left(1 - R_1\right) \frac{\int_0^T \beta(s) \mathcal{D}_\delta(s, 0, X_0^1) \mathcal{E}_\delta(s, 0, X_0^1) ds}{\int_0^T \beta(s) \mathcal{E}_\delta(s, 0, X_0^1) ds}.$$  

We denote by $p_1$ the cumulative distribution function (c.d.f. hereafter) of $\tau_1$, namely,

$$p_1(t) := \mathbb{P}(\tau_1 \leq t) = 1 - \mathcal{E}_\delta(t, 0, X_0^1). \quad (18)$$

Of course by symmetry analogous formulae hold for a risk-free CDS on the counterpart.

5.2 Joint Defaults

In case market prices of instrument sensitive to the dependence structure of default times are available (basket credit instrument on the firm and the counterpart), these can be used to calibrate $l_3$. Admittedly however, this situation is an exception rather than the rule. It is thus important to devise a practical way of calibrating $l_3$ in case such market data are not available.

Note that under parameterizations 0F and 2F, one has

$$\mathbb{P}(\tau_1 > t, \tau_2 > t) = \mathbb{P}(\tau_1 > t) \mathbb{P}(\tau_2 > t) e^{L_3(t)}, \quad (19)$$

for $L_3(t) = \int_0^t l_3(s) ds$. A possible procedure thus consists in ‘calibrating’ $l_3$ to target values for the model probabilities $p_{1,2}(t) = \mathbb{P}(\tau_1 < t, \tau_2 < t)$ of default of both name up to various time horizons $t$. More precisely, given a target for the function $p_{1,2}(t)$, one plugs it, together with the functions $p_1(t)$ and $p_2(t)$, into (19), to deduce $L_3(t)$.

**Remark 5.2** Regarding the derivation of a target for $p_{1,2}(t)$, note the following relation between $p_{1,2}(t)$ and a standard static Gaussian copula asset correlation $\rho(t)$ at the horizon $t$:

$$p_{1,2}(t) = \mathcal{N}_{2}^{\rho(t)} \left( \mathcal{N}_1^{-1}(p_1(t)), \mathcal{N}_1^{-1}(p_2(t)) \right), \quad (20)$$

where $\mathcal{N}_1$ denotes the standard Gaussian c.d.f., and $\mathcal{N}_{2}^{\rho(t)}$ denotes a bivariate centered Gaussian c.d.f. with one-factor Gaussian copula correlation matrix of parameter $\rho(t)$. A target value for $p_{1,2}(t)$ can thus be obtained by plugging values extracted from the market for $\rho(t)$, $p_1(t)$ and $p_2(t)$ into the RHS of (20). In particular a ‘market’ static Gaussian asset correlation $\rho(t)$ can be retrieved from the Basel II correlations per asset class (cf. [2], pages 63 to 66).

5.3 Calibration

We aim at calibrating the model to marginal CDS curves and to an asset correlation function $\rho(t)$ (see Remark 5.2). We assume that the functions $f_1$, $f_2$ and $l_3$ are piecewise constant functions of time.
We denote by \((T_1, ..., T_m)\) the term structure of the maturities of the market CDS between the counterparty and the reference entity, and we set \(\Delta_j = T_j - T_{j-1}\), with the convention \(T_0 = 0\).

One then proceeds in four steps as follows:

- One bootstraps the CDS curve for both names \(i\) into a piecewise constant c.d.f. \(p_i(\cdot)\), for \(i = 1, 2\), yielding \(p_i(t) = p_i(T_j)\) on \(T_{j-1} \leq t < T_j\).

- Next, given \(p_1(t)\), \(p_2(t)\) and \(\rho(t)\), one computes \(p_{1,2}(t) = P(\tau_1 < t, \tau_2 < t)\) via (20).

- The relation (19) yields a system of \(m\) linear equations in the \(m\) unknowns \(l_{3,1}, ..., l_{3,m}\).

\[
\begin{align*}
\Delta_1 l_{3,1} + \cdots + \Delta_j l_{3,j} &= -\ln \frac{P(\tau_1 > T_j, \tau_2 > T_j)}{P(\tau_1 > T_j) P(\tau_2 > T_j)} \\
\text{subject to } l_{3,j} \geq 0, j = 1, ..., m
\end{align*}
\]

- At last, formula (18) results in two systems of \(m\) linear equations in the \(m + 2\) unknowns \(X_0, \mu_i, f_{i,1}, ..., f_{i,m}\). That is, for \(i = 1, 2\),

\[
\begin{align*}
\delta \phi(T_j) X_0^i + \delta \xi(T_j) \mu_i + \Delta_1 f_{i,1} + \cdots + \Delta_j f_{i,j} &= -\ln \frac{P(\tau_i > T_j)}{P(\tau_1 > T_j)} \\
\text{subject to } X_0^i \geq 0, \mu_i \geq 0, f_{i,j} \geq l_{3,j}, j = 1, ..., m
\end{align*}
\]

In practice these equations are solved in the sense of mean-square minimization under the constraints.

### 6 A Variant of the Model with Extended CIR Factor processes

In this section we propose a variant of the general model of section 3 defined in terms of extended CIR factor processes. By comparison with (3), one thus chooses a specific, affine form of the factors, but one also lets the joint defaults intensity \(l_3\) be stochastic, via a ‘new’ factor \(X_3\). In particular one models the factors \(X^i\)’s as affine processes of the form

\[
dX^i_t = \eta(\mu_i(t) - X^i_t)dt + \nu \sqrt{X^i_t}dW^i_t,
\]

with \(W_1\) and \(W_2\) correlated at the level \(\rho\) and \(W_3\) independent from \(W_1\) and \(W_2\). Note in this regard that the factors \(X^i\)’s have the same coefficients but for \(\mu_i(t)\), to the effect that \(\hat{X}^i := X^i + X^3\), for \(i = 1, 2\), is again an extended CIR process, with parameters \(\eta, \bar{\mu}_i(t) = \mu_i(t) + \mu_3(t)\) and \(\nu\).

Let as before \(H = (H^1, H^2)\) and let now \(X = (X^1, X^2, X^3)\).

One thus considers a Markovian model of the pair \((X, H)\) relative to its natural filtration \(\mathbb{F}\), with generator of \((X, H)\) given by, for \(u = u(t, x, e)\) with \(t \in \mathbb{R}_+, x = (x_1, x_2, x_3) \in \mathbb{R}^3, e = \)
\[(e_1, e_2) \in \{0, 1\}^2:\]
\[
Au(t, x, e) = \partial u(t, x, e) + \sum_{1 \leq i \leq 2} l_i(t, x_i) (u(t, x, e^i) - u(t, x, e)) + l_3(t, x_3) (u(t, x, 1, 1) - u(t, x, e)) \\
+ \sum_{1 \leq i \leq 3} \left( \eta(\mu_i(t) - x_i) \partial_{x_i} u(t, x, e) + \frac{1}{2} \nu^2 x_i \partial_{x_i}^2 u(t, x, e) \right) + q \nu^2 \sqrt{x_1 x_2} \partial_{x_1, x_2}^2 u(t, x, e),
\]

where, for \(i = 1\) to \(3:\)

1. the default intensity function \(l_i\) is of the form
   \[
l_i(t, x_i) = x_i + g_i(t),
   \]
2. the coefficients \(\eta, \nu\) are non-negative constants and \(\mu_i(\cdot)\)s are non-negative functions of time.

The \(F\) – intensity matrix-function of \(H\) is now given by

\[
A(t, x) = \begin{bmatrix}
-l(t, x) & l_1(t, x_1) & l_2(t, x_2) & l_3(t, x_3) \\
0 & -q_2(t, x_2) & 0 & q_2(t, x_2) \\
0 & 0 & -q_1(t, x_1) & q_1(t, x_1) \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

with, for every \(i = 1, 2,\)
\[
\bar{x}_i = x_i + x_3 \\
q_i(t, \bar{x}_i) = l_i(t, x_i) + l_3(t, x_3) = \bar{x}_i + g_i(t) + g_3(t)
\]

and \(l = l_1 + l_2 + l_3\). Under standard regularity and growth assumptions on the coefficients of \(A\), one then has the following variant of Proposition 3.1.

**Proposition 6.1** (i) For every \(i = 1, 2\), the process \((\bar{X}^i, H^i)\) is an \(F\)-Markov process, with generator of \((\bar{X}^i, H^i)\) given by, for \(u_i = u_i(t, \bar{x}_i, e_i)\), with \(t \in \mathbb{R}_+, \bar{x}_i \in \mathbb{R}, e_i \in \{0, 1\}:\)

\[
A^i u_i(t, \bar{x}_i, e_i) = \partial_t u_i(t, \bar{x}_i, e_i) + \eta(\bar{\mu}_i(t) - \bar{x}_i) \partial_{\bar{x}_i} u_i(t, \bar{x}_i, e_i) + \frac{1}{2} \nu^2 \bar{x}_i \partial_{\bar{x}_i}^2 u_i(t, \bar{x}_i, e_i) + q_i(t, \bar{x}_i) (u_i(t, \bar{x}_i, 1) - u_i(t, \bar{x}_i, e_i)).
\]

The \(F\) – intensity matrix function of \(H^i\) is thus given by

\[
A_i(t, \bar{x}_i) = \begin{bmatrix}
-q_i(t, \bar{x}_i) & q_i(t, \bar{x}_i) \\
0 & 0
\end{bmatrix}
\]

In other words, the process \(M^i\) defined by, for \(i = 1, 2,\)
\[
M^i_t = H^i_t - \int_0^t (1 - H^i_s) q_i(s, \bar{X}_s^i) ds,
\]
is an \(F\)-martingale.
(ii) One has, for every \( t \geq 0 \),

\[
\mathbb{P}(\tau_i > t) = \mathbb{E} \exp \left( - \int_0^t q_i(u, \tilde{X}_i^1) du \right)
\]

\[
\mathbb{P}(\tau_1 \land \tau_2 > t) = \mathbb{E} \exp \left( - \int_0^t l(u, X_u) du \right)
\]

One thus gets in the terminology of \([4]\) a Markovian copula model \((X, H)\) with marginals \((\tilde{X}^i, H^i)\), for \( i = 1, 2 \) — or, in the 'common factor case' \( X^1 = X^2 = X \) (see Remark \(3.2\)), with marginals \((\tilde{X}, H^1)\), where we set \( \tilde{X} = X + X^3 \).

Let, for \( \tilde{x}_1 \in \mathbb{R}_+ \),

\[
p(t, \tilde{x}_1) = (1 - R_1)q_1(t, \tilde{x}_1) - \kappa.
\]

One then has much like in Propositions \(4.3(i)\) and \(4.4\) (an analog of Proposition \(4.3(ii)\) could be derived as well if wished),

**Proposition 6.2** (i) The price of the risk-free CDS admits the representation:

\[
P_t = (1 - H^1_t)v(t, \tilde{X}^1_t),
\]

for a pre-default pricing function \( v = v(t, \tilde{x}_1) \) solving the following pricing PDE:

\[
\begin{aligned}
&v(T, \tilde{x}_1) = 0, \quad \tilde{x}_1 \in \mathbb{R} \\
&\left( \partial_t + \eta(\bar{\mu}_1(t) - \tilde{x}_1) \partial_{\tilde{x}_1} + \frac{1}{2} \nu^2 \tilde{x}_1 \partial^2_{\tilde{x}_1} \right) v(t, \tilde{x}_1) - (r(t) + q_1(t, \tilde{x}_1)) v(t, \tilde{x}_1) + p(t, \tilde{x}_1) = 0,
\end{aligned}
\]

or, equivalently to \([9]\),

\[
v(t, \tilde{x}_1) = \mathbb{E} \left( \int_t^T e^{-\int_s^t (r(\zeta) + q_1(\zeta, \tilde{X}^1_{\zeta})) d\zeta} p(s, \tilde{X}^1_s) ds \bigg| \tilde{X}^1_t = \tilde{x}_1 \right); \tag{28}
\]

(ii) One has, for \( t \in [0, T] \),

\[
EPE(t) = (1 - R_2) \left( (1 - R_1) \frac{\mathbb{E} \left( l_3(t, X^3_t) e^{-\int_0^t l_0(t, X_s) ds} \right)}{\mathbb{E} \left( q_2(t, \tilde{X}_t^2) e^{-\int_0^t q_2(t, \tilde{X}_s^2) ds} \right)} + \frac{\mathbb{E} \left( v^+(t, \tilde{X}_t^1) l_2(t, X^2_t) e^{-\int_0^t l_0(t, X_s) ds} \right)}{\mathbb{E} \left( q_2(t, \tilde{X}_t^2) e^{-\int_0^t q_2(t, \tilde{X}_s^2) ds} \right)} \right) \tag{29}
\]

\[
CVA_0 = (1 - R_2) \mathbb{E} \int_0^T \beta(t) \left( (1 - R_1) l_3(t, X^3_t) + v^+(t, \tilde{X}_t^1) l_2(t, X^2_t) \right) e^{-\int_0^t l_0(t, X_s) ds} dt. \tag{30}
\]

### 6.1 Parametrization and Calibration

Let us consider the parametrization stated in \([22]\), with \( g_i = 0 \), therein. We assume that the \( \mu_i(\cdot) \)s are piecewise constant functions,

\[
\mu_i(t) = \mu_{i,j}, \quad \text{for } t \in [T_{j-1}, T_j).
\]
The marginal intensity processes \( q_i(t, \tilde{X}_i^t) \)‘s are then extended CIR processes (cf. (23)) with the following piecewise constant ‘long-term mean’ function \( \tilde{\mu}_i(\cdot) \),
\[
\tilde{\mu}_i(t) = \mu_{i,j} + \mu_{3,j}, \quad \text{for} \ t \in [T_{j-1}, T_j).
\]
We will refer to this model parametrization as 3F. Under this specification, one has the following proposition for the pricing function of a risk-free CDS on the firm. Let the functions \( \tilde{D}_1 \) and \( \tilde{E}_1 \) be defined as in Proposition 9.2 with \( \mu(\cdot) = \tilde{\mu}_1(\cdot) \) therein.

**Proposition 6.3** Assuming 3F, one has,
\[
\beta(t)v(t, \tilde{x}_1) = \int_t^T \beta(s)\left((1 - R_1)\tilde{D}(s, t, \tilde{x}_1) - \kappa\right)\tilde{E}(s, t, \tilde{x}_1, 0)ds.
\]

**Proof.** Recall from Proposition 6.2(i) that
\[
v(t, \tilde{x}_1) = \mathbb{E}\left(\int_t^T e^{-\int_t^s (r(\zeta) + q_1(\zeta, \tilde{X}_i^s))d\zeta} p(s, \tilde{X}_i^s)ds \mid \tilde{x}_1 = \tilde{x}_1\right), \quad (31)
\]
with
\[
p(s, \tilde{X}_i^s) = (1 - R_1)\tilde{X}_i^s - \kappa.
\]
The result thus follows by an application of Proposition 9.2. \( \square \)

Also, the spread at time 0 of a risk-free CDS of maturity \( T \) on the firm, is given by
\[
\kappa_0(T) = (1 - R_1)\frac{\int_0^T \beta(s)\tilde{D}_1(s, 0, X_0^1)\tilde{E}_1(s, 0, X_0^1, 0)ds}{\int_0^T \beta(s)\tilde{E}_1(s, 0, X_0^1, 0)ds}.
\]

As in the previous case, the input to the calibration is an asset correlation function \( \rho(t) \) and the piecewise constant marginal cumulative default probabilities obtained by bootstrapping from the related CDS curves. For simplicity of calibration, the volatility parameter \( \nu \) and the mean-reversion \( \eta \) are assumed to be given (as opposed to calibrated), whereas for each factor \( X^i \) the initial value \( X_i^0 \) and \( \mu_{i,1}, \ldots, \mu_{i,m} \) are calibrated.

Using identities (25) and Corollary 9.3, the following expressions follows for the marginal and joint survival probabilities:
\[
P(\tau_1 > T_j) = \mathbb{E}\left(\exp\left(-\int_0^{T_j} \tilde{X}_i^s ds\right)\right) = \exp\left(-a_{j,0}\tilde{X}_0^j - \sum_{k=1}^{j-1} \tilde{\mu}_{i,k}\xi(\Delta_k, a_{j,k})\right)
\]
and
\[
P(\tau_1 > T_j, \tau_2 > T_j) = \mathbb{E}\left(\exp\left(-\int_0^{T_j} (X_1^s + X_2^s + X_3^s) ds\right)\right) = P(\tau_1 > T_j)P(\tau_2 > T_j) \exp\left(a_{j,0}X_0^3 + \sum_{k=1}^{j} \mu_{3,k}\xi(\Delta_k, a_{j,k})\right).
\]
where the coefficients $a_{j,k}$ are given in (39).

One can then follow the same lines as in Subsection 5.3 to obtain the following three systems of linear equations with constraints. Each system consists of $m$ equations in $m + 1$ unknowns: For $j = 1, ..., m,$

$$
\begin{align*}
\begin{cases}
  a_{j,0}X_0^3 + \sum_{k=1}^{j} \xi(\Delta_k, a_{j,k})\mu_{3,k} = \ln \frac{\mathbb{P}(\tau_1 > T_j, \tau_2 > T_j)}{\mathbb{P}(\tau_1 > T_j)} \mathbb{P}(\tau_2 > T_j) \\
  X_0^3 \geq 0, \quad \mu_{3,k} \geq 0, \quad k = 1, ..., m.
\end{cases}
\end{align*}
$$

For $i = 1, 2,$

$$
\begin{align*}
\begin{cases}
  a_{j,0}\tilde{X}_0^i + \sum_{k=1}^{j} \xi(\Delta_k, a_{j,k})\tilde{\mu}_{i,k} = -\ln \mathbb{P}(\tau_i > T_j) \\
  \tilde{X}_0^i \geq X_0^3, \quad \tilde{\mu}_{i,k} \geq \mu_{3,k}, \quad k = 1, ..., m.
\end{cases}
\end{align*}
$$

In practice these equations are solved in the sense of mean-square minimization under the constraints.

### 7 Numerical Results

Our aim is to assess by means of numerical experiments the impact on the counterparty risk exposure of:

- $\rho(t) = \rho,$ a constant asset correlation between the firm and the counterpart,
- $p_2,$ the cumulative distribution function of the default time $\tau_2$ of the counterpart,
- $\nu,$ the volatility of the factors.

The numerical tests below have been done using the following model parameterizations:

**0F** No stochastic factor, as in (17) with $\delta = 0,$

**2F** Two independent CIR++ factors, as in (17) with $\delta = 1,$

**3F** Three independent extended CIR factors as of subsection 6.1.

The mean-reversion parameter $\eta$ is fixed to 10%. The recovery rates are set to 40% and the risk-free rate $r$ is taken constant and equal to 5%.

In the following example, we consider four CDSs written on the reference name UBS AG, with different counterparts: Gaz de France, Carrefour, AXA and Telecom Italia SpA, referred to in the sequel as CP1, CP2, CP3 and CP4. For each counterpart we consider six CDSs with maturities of one, two, three, five, seven and ten years, corresponding to data of March 30, 2008. Table 1 includes market CDS spreads on the five names in consideration and for the six different maturities. The bootstrapped piecewise constant c.d.f. of the five names are represented in Table 2. The counterparts are ordered from the least risky one, CP1, to the most risky one, CP4.
One can see on Figures 1 and 2 the impact of the default risk of the counterpart, on the counterparty risk exposure of the investor in the sense of EPE and efEPE, respectively. The graphs on the left, middle and right column of each figure correspond to the parameterizations 0F, 2F and 3F, respectively. On each graph the asset correlation $\rho$ is fixed, with from top to down $\rho = 10\%$, 40% and 70%. The four curves on each graph of Figure 1 (resp. 2) correspond to EPE($t$) (resp. efEPE($t$)) for CP1, CP2, CP3, CP4. Observe that the counterparty risk exposure (in both senses of EPE and efEPE) is decreasing in the default risk of the counterpart, in line with the stylized features and the financial intuition: EPE($t$) is the expectation of the investor’s loss, given the default of the counterpart at time $t$. A default of a counterpart with a lower spread is interpreted by the markets as a worse news than a default of a counterparty with a higher spread, so the related EPE and efEPE should be larger. Mathematically this is explained in our model by the presence of $q_2$ in the denominator of (14) and (29).

The discontinuities of the EPE profiles in cases 0F and 2F are due to the fact that one works in these cases with piecewise constant intensities. These discontinuities are smoothed out in the case of efEPE.

Figure 3 shows the Credit Valuation Adjustment at time 0 of a risky CDS on the reference name UBS AG, as a function of the volatility parameter $\nu$ of the CIR factors $X$’s.

The graphs on the left of this figure show the results obtained from the parametrization 2F while the graphs on the right correspond to the case of 3F. On each graph the asset correlation $\rho$ is fixed, with from top to down $\rho = 5\%$, 10%, 40% and 70%. The four curves on each graph of Figure 3 correspond to CVA(0) calculated for a risky CDS of maturity $T = 10$ years between Ref and CP1, CP2, CP3, CP4, respectively.

On this data set we observe that CVA(0) is:

---

### Table 1: Market spreads in bps for different time horizons on March 30, 2008.

<table>
<thead>
<tr>
<th>Ref</th>
<th>UBS AG</th>
<th>1 year</th>
<th>2 years</th>
<th>3 years</th>
<th>5 years</th>
<th>7 years</th>
<th>10 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>CP1</td>
<td>Gaz de France</td>
<td>27</td>
<td>35</td>
<td>42</td>
<td>53</td>
<td>57</td>
<td>61</td>
</tr>
<tr>
<td>CP2</td>
<td>Carrefour</td>
<td>34</td>
<td>42</td>
<td>53</td>
<td>67</td>
<td>71</td>
<td>76</td>
</tr>
<tr>
<td>CP3</td>
<td>AXA</td>
<td>72</td>
<td>83</td>
<td>105</td>
<td>128</td>
<td>129</td>
<td>128</td>
</tr>
<tr>
<td>CP4</td>
<td>Telecom Italia SpA</td>
<td>99</td>
<td>157</td>
<td>210</td>
<td>243</td>
<td>255</td>
<td>262</td>
</tr>
</tbody>
</table>

### Table 2: Default probabilities for different maturities.

<table>
<thead>
<tr>
<th>Ref</th>
<th>1 year</th>
<th>2 years</th>
<th>3 years</th>
<th>5 years</th>
<th>7 years</th>
<th>10 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>CP1</td>
<td>.0146</td>
<td>.0116</td>
<td>.0212</td>
<td>.0445</td>
<td>.0664</td>
<td>.1005</td>
</tr>
<tr>
<td>CP2</td>
<td>.0056</td>
<td>.0138</td>
<td>.0264</td>
<td>.0558</td>
<td>.0822</td>
<td>.1246</td>
</tr>
<tr>
<td>CP3</td>
<td>.0118</td>
<td>.0269</td>
<td>.0517</td>
<td>.1042</td>
<td>.1434</td>
<td>.1964</td>
</tr>
<tr>
<td>CP4</td>
<td>.0155</td>
<td>.0504</td>
<td>.1026</td>
<td>.1903</td>
<td>.2662</td>
<td>.3670</td>
</tr>
</tbody>
</table>

---
• increasing in the default risk of the counterpart,
• increasing in the asset correlation \( \rho \),
• slowly increasing in the volatility \( \nu \) of the common factor.

In Table 3 one can see the values of CVA(0) calculated within the parametrization 0F, that is with no stochastic factor. Note that for a CDS written on Ref, the risk-free value of the default leg is equal to DL0 = 0.1031.

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>CP1</th>
<th>CP2</th>
<th>CP3</th>
<th>CP4</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>0.0009</td>
<td>0.0011</td>
<td>0.0016</td>
<td>0.0025</td>
</tr>
<tr>
<td>10%</td>
<td>0.0018</td>
<td>0.0021</td>
<td>0.0030</td>
<td>0.0047</td>
</tr>
<tr>
<td>40%</td>
<td>0.0080</td>
<td>0.0093</td>
<td>0.0129</td>
<td>0.0186</td>
</tr>
<tr>
<td>70%</td>
<td>0.0163</td>
<td>0.0190</td>
<td>0.0262</td>
<td>0.0358</td>
</tr>
</tbody>
</table>

Table 3: CVA(0) for CDSs written on Ref in the case 0F.

Tables 4 and 5 represent the calibration error in basis points of 2F and 3F, respectively. Precisely, in each table, we consider

\[
er_i(t) = 10^4 \times \frac{|p_i(t) - \hat{p}_i(t)|}{p_i(t)} , \quad er_{1,2}(t) = 10^4 \times \frac{|p_{1,2}(t) - \hat{p}_{1,2}(t)|}{p_{1,2}(t)}
\]

where \( \hat{p}_1, \hat{p}_2, \hat{p}_{1,2} \) are obtained from equations (7) and (25) using the calibrated parameters. The corresponding errors in the case of 0F are 0.0000 bp.

The difference between market spreads and calibrated model spreads are represented in Tables 6, 7 and 8, respectively.

Table 9 displays the execution time of a calibration and of a computation of EPE(t) and CVA(0) for models 0F, 2F and 3F.

### 7.1 Case of a low-risk reference entity

In the previous example, except in the low \( \rho \) cases, the dependency of the CVA on \( \nu \) was rather limited (see Figure 3). For a low-risk reference entity, however, \( \nu \) is expected to have more impact on the CVA, including for larger \( \rho \)'s. To assess this numerically we thus now consider a low-risk obligor, referred to as Ref', whose piecewise constant c.d.f. is given in Table 10. For a CDS written on Ref', the risk-free value of the default leg is equal to DL'_0 = 0.0240.

On each graph of Figure 4 the asset correlation is fixed to \( \rho = 5\%, 10\%, 40\% \) or 70\%. One can see that CVA(0) is significantly sensitive to \( \nu \), and even extremely so in the case of low correlations \( \rho \). For comparison Table 11 shows the values of CVA(0) calculated within the parametrization 0F.
Table 4: Relative error in bps of the cumulative probabilities $p_1$, $p_2$ and $p_{1,2}$ in the case 2F with $\rho = 40\%$.

<table>
<thead>
<tr>
<th></th>
<th>Maturities</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>mean</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 year</td>
<td>2 years</td>
<td>3 years</td>
<td>5 years</td>
<td>7 years</td>
<td>10 years</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CP1</td>
<td>$e_{r1}$</td>
<td>0.2500</td>
<td>0.9630</td>
<td>0.8560</td>
<td>1.1390</td>
<td>2.1600</td>
<td>1.7600</td>
<td>1.1880</td>
<td>2.1600</td>
</tr>
<tr>
<td></td>
<td>$e_{r2}$</td>
<td>0.0191</td>
<td>0.0112</td>
<td>0.3790</td>
<td>0.0612</td>
<td>0.4027</td>
<td>0.0198</td>
<td>0.1488</td>
<td>0.4027</td>
</tr>
<tr>
<td></td>
<td>$e_{r1,2}$</td>
<td>0.1790</td>
<td>0.7700</td>
<td>0.5350</td>
<td>1.2040</td>
<td>1.1120</td>
<td>0.7610</td>
<td>0.7600</td>
<td>1.2040</td>
</tr>
<tr>
<td>CP2</td>
<td>$e_{r1}$</td>
<td>1.9240</td>
<td>1.8000</td>
<td>1.0500</td>
<td>1.3020</td>
<td>0.2590</td>
<td>2.1340</td>
<td>1.4110</td>
<td>2.1340</td>
</tr>
<tr>
<td></td>
<td>$e_{r2}$</td>
<td>0.3817</td>
<td>0.0726</td>
<td>0.0714</td>
<td>0.7904</td>
<td>0.3275</td>
<td>0.3992</td>
<td>0.3404</td>
<td>0.7904</td>
</tr>
<tr>
<td></td>
<td>$e_{r1,2}$</td>
<td>0.2471</td>
<td>0.1127</td>
<td>0.0644</td>
<td>0.0442</td>
<td>0.0990</td>
<td>0.0213</td>
<td>0.0981</td>
<td>0.2471</td>
</tr>
<tr>
<td>CP3</td>
<td>$e_{r1}$</td>
<td>0.4150</td>
<td>1.2380</td>
<td>0.4110</td>
<td>0.3430</td>
<td>1.3280</td>
<td>0.3410</td>
<td>0.6790</td>
<td>1.3280</td>
</tr>
<tr>
<td></td>
<td>$e_{r2}$</td>
<td>0.4730</td>
<td>0.8080</td>
<td>0.5390</td>
<td>0.6290</td>
<td>3.9800</td>
<td>0.8040</td>
<td>1.2050</td>
<td>3.9800</td>
</tr>
<tr>
<td></td>
<td>$e_{r1,2}$</td>
<td>0.0494</td>
<td>0.0437</td>
<td>0.0065</td>
<td>0.0111</td>
<td>0.0116</td>
<td>0.0029</td>
<td>0.0192</td>
<td>0.0494</td>
</tr>
<tr>
<td>CP4</td>
<td>$e_{r1}$</td>
<td>0.0364</td>
<td>0.1808</td>
<td>0.2382</td>
<td>0.1823</td>
<td>0.0988</td>
<td>0.3502</td>
<td>0.1811</td>
<td>0.3502</td>
</tr>
<tr>
<td></td>
<td>$e_{r2}$</td>
<td>0.0586</td>
<td>0.0167</td>
<td>0.0095</td>
<td>0.0376</td>
<td>0.0206</td>
<td>0.0285</td>
<td>0.0286</td>
<td>0.0586</td>
</tr>
<tr>
<td></td>
<td>$e_{r1,2}$</td>
<td>0.2097</td>
<td>0.3922</td>
<td>0.2680</td>
<td>0.0917</td>
<td>0.0815</td>
<td>0.1537</td>
<td>0.1994</td>
<td>0.3922</td>
</tr>
</tbody>
</table>

Table 5: Relative error in bps of the cumulative probabilities $p_1$, $p_2$ and $p_{1,2}$ in the case 3F with $\rho = 40\%$.

<table>
<thead>
<tr>
<th></th>
<th>Maturities</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>mean</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 year</td>
<td>2 years</td>
<td>3 years</td>
<td>5 years</td>
<td>7 years</td>
<td>10 years</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CP1</td>
<td>$e_{r1}$</td>
<td>5.0422</td>
<td>14.190</td>
<td>2.1420</td>
<td>15.050</td>
<td>15.920</td>
<td>0.4151</td>
<td>8.7940</td>
<td>15.920</td>
</tr>
<tr>
<td></td>
<td>$e_{r2}$</td>
<td>0.7888</td>
<td>1.6940</td>
<td>2.4450</td>
<td>0.0024</td>
<td>2.7510</td>
<td>1.0728</td>
<td>1.4591</td>
<td>2.7515</td>
</tr>
<tr>
<td></td>
<td>$e_{r1,2}$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>CP2</td>
<td>$e_{r1}$</td>
<td>0.5700</td>
<td>2.1950</td>
<td>2.6910</td>
<td>1.4200</td>
<td>0.4247</td>
<td>0.0426</td>
<td>1.2240</td>
<td>2.6910</td>
</tr>
<tr>
<td></td>
<td>$e_{r2}$</td>
<td>0.3025</td>
<td>0.1581</td>
<td>0.4489</td>
<td>0.4358</td>
<td>0.0444</td>
<td>0.4780</td>
<td>0.3113</td>
<td>0.4780</td>
</tr>
<tr>
<td></td>
<td>$e_{r1,2}$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>CP3</td>
<td>$e_{r1}$</td>
<td>0.0434</td>
<td>0.3486</td>
<td>0.4462</td>
<td>0.1976</td>
<td>0.1464</td>
<td>0.0282</td>
<td>0.2018</td>
<td>0.4462</td>
</tr>
<tr>
<td></td>
<td>$e_{r2}$</td>
<td>25.809</td>
<td>82.406</td>
<td>87.861</td>
<td>34.909</td>
<td>68.141</td>
<td>0.0325</td>
<td>49.860</td>
<td>87.861</td>
</tr>
<tr>
<td></td>
<td>$e_{r1,2}$</td>
<td>0.0001</td>
<td>0.0012</td>
<td>0.0020</td>
<td>0.0036</td>
<td>0.0063</td>
<td>0.0054</td>
<td>0.0032</td>
<td>0.0063</td>
</tr>
<tr>
<td>CP4</td>
<td>$e_{r1}$</td>
<td>1.5396</td>
<td>5.3363</td>
<td>24.188</td>
<td>59.728</td>
<td>40.733</td>
<td>0.9771</td>
<td>22.084</td>
<td>59.728</td>
</tr>
<tr>
<td></td>
<td>$e_{r2}$</td>
<td>0.0652</td>
<td>3.8962</td>
<td>4.6377</td>
<td>4.6433</td>
<td>3.8499</td>
<td>0.9716</td>
<td>3.0106</td>
<td>4.6432</td>
</tr>
<tr>
<td></td>
<td>$e_{r1,2}$</td>
<td>0.0002</td>
<td>0.0030</td>
<td>0.0102</td>
<td>0.0176</td>
<td>0.0263</td>
<td>0.0081</td>
<td>0.0109</td>
<td>0.0263</td>
</tr>
</tbody>
</table>
Table 6: bp-Differences between market spreads and calibrated spreads in the case of 0F with $\rho = 40\%$.

<table>
<thead>
<tr>
<th></th>
<th>1 year</th>
<th>2 years</th>
<th>3 years</th>
<th>5 years</th>
<th>7 years</th>
<th>10 years</th>
<th>mean</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref</td>
<td>0.2920</td>
<td>0.1622</td>
<td>0.3957</td>
<td>0.3296</td>
<td>0.2537</td>
<td>0.1964</td>
<td>0.2716</td>
<td>0.3957</td>
</tr>
<tr>
<td>CP1</td>
<td>0.1285</td>
<td>0.0693</td>
<td>0.1556</td>
<td>0.1080</td>
<td>0.0829</td>
<td>0.0639</td>
<td>0.1014</td>
<td>0.1556</td>
</tr>
<tr>
<td>CP2</td>
<td>0.1067</td>
<td>0.0576</td>
<td>0.1897</td>
<td>0.1370</td>
<td>0.1054</td>
<td>0.0819</td>
<td>0.1130</td>
<td>0.1897</td>
</tr>
<tr>
<td>CP3</td>
<td>0.0096</td>
<td>0.0052</td>
<td>0.3108</td>
<td>0.2665</td>
<td>0.2052</td>
<td>0.1586</td>
<td>0.1593</td>
<td>0.3108</td>
</tr>
<tr>
<td>CP4</td>
<td>0.0098</td>
<td>0.0060</td>
<td>0.4711</td>
<td>0.5125</td>
<td>0.4097</td>
<td>0.3310</td>
<td>0.2900</td>
<td>0.5125</td>
</tr>
</tbody>
</table>

Table 7: bp-Differences between market spreads and calibrated spreads in the case of 2F with $\rho = 40\%$.

<table>
<thead>
<tr>
<th></th>
<th>1 year</th>
<th>2 years</th>
<th>3 years</th>
<th>5 years</th>
<th>7 years</th>
<th>10 years</th>
<th>mean</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref</td>
<td>0.3343</td>
<td>0.2167</td>
<td>0.4315</td>
<td>0.3980</td>
<td>0.3120</td>
<td>0.3489</td>
<td>0.3402</td>
<td>0.4315</td>
</tr>
<tr>
<td>CP1</td>
<td>0.0719</td>
<td>0.0781</td>
<td>0.0131</td>
<td>0.0150</td>
<td>0.0305</td>
<td>0.0104</td>
<td>0.0521</td>
<td>0.1040</td>
</tr>
<tr>
<td>CP2</td>
<td>0.0345</td>
<td>0.0028</td>
<td>0.1352</td>
<td>0.0852</td>
<td>0.0598</td>
<td>0.0833</td>
<td>0.0668</td>
<td>0.1352</td>
</tr>
<tr>
<td>CP3</td>
<td>0.0203</td>
<td>0.0088</td>
<td>0.2876</td>
<td>0.2426</td>
<td>0.1461</td>
<td>0.1855</td>
<td>0.1485</td>
<td>0.2876</td>
</tr>
<tr>
<td>CP4</td>
<td>0.0698</td>
<td>0.0537</td>
<td>0.5219</td>
<td>0.5614</td>
<td>0.4584</td>
<td>0.3976</td>
<td>0.3438</td>
<td>0.5614</td>
</tr>
</tbody>
</table>

Table 8: bp-Differences between market spreads and calibrated spreads in the case of 3F with $\rho = 40\%$.

<table>
<thead>
<tr>
<th></th>
<th>1 year</th>
<th>2 years</th>
<th>3 years</th>
<th>5 years</th>
<th>7 years</th>
<th>10 years</th>
<th>mean</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref</td>
<td>1.8110</td>
<td>1.6440</td>
<td>0.6820</td>
<td>0.8820</td>
<td>0.4950</td>
<td>0.4790</td>
<td>0.9988</td>
<td>1.8110</td>
</tr>
<tr>
<td>CP1</td>
<td>0.7730</td>
<td>0.6560</td>
<td>0.4300</td>
<td>0.2370</td>
<td>0.2130</td>
<td>0.0750</td>
<td>0.3973</td>
<td>0.7730</td>
</tr>
<tr>
<td>CP2</td>
<td>0.7400</td>
<td>0.8190</td>
<td>0.4300</td>
<td>0.2030</td>
<td>0.2140</td>
<td>0.1250</td>
<td>0.4218</td>
<td>0.8190</td>
</tr>
<tr>
<td>CP3</td>
<td>1.1690</td>
<td>0.9320</td>
<td>1.4230</td>
<td>0.5160</td>
<td>0.6940</td>
<td>0.4710</td>
<td>0.8675</td>
<td>1.4230</td>
</tr>
<tr>
<td>CP4</td>
<td>5.6840</td>
<td>3.5300</td>
<td>1.7190</td>
<td>0.6740</td>
<td>0.5720</td>
<td>0.4910</td>
<td>2.1117</td>
<td>5.6840</td>
</tr>
</tbody>
</table>

Table 9: Execution time in seconds.

<table>
<thead>
<tr>
<th></th>
<th>0F</th>
<th>2F</th>
<th>3F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calibration</td>
<td>0.01</td>
<td>0.30</td>
<td>0.35</td>
</tr>
<tr>
<td>EPE(t)</td>
<td>0.015</td>
<td>5.1</td>
<td>12</td>
</tr>
<tr>
<td>CVA(0)</td>
<td>0.015</td>
<td>5.0</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 10: Default probabilities of Ref'.
8 Conclusions

In conclusion one can say that in case of a ‘risky enough’ reference entity and for a sufficient level of correlation between the counterpart and the reference entity, the time-deterministic specification of intensities does a good and quick job in estimating EPE and CVA.

In case of a low risk reference entity, or for a low level of correlation between the counterpart and the reference entity, the time-deterministic specification of intensities ‘misses’ a non-negligible component of EPE and CVA, due to spreads’ volatility. In this case, a stochastic specification of the intensities is preferred.

In a CIR++ specification of the intensities, marginal default intensities are given as sums of affine processes and deterministic functions of time. The joint defaults intensity in particular is time-deterministic, so that one might wonder whether a fully stochastic specification of the intensities would lead to even higher (possibly more realistic) CVAs.

We thus investigated a third specification of the intensities in the form of extended CIR processes with time-dependent parameters (and no deterministic component anymore). The levels of EPE and CVA happen to be quite similar to those got through the CIR++ specification. Moreover the calibration of the CIR++ specification is more robust and accurate than that of the extended CIR specification, and CIR++ computations are quicker than extended CIR ones. The CIR++ specification thus comes out as the method of choice for assessing counterparty risk on a CDS with a low risk reference entity, or for a low level of correlation between the counterpart and the reference entity.
Figure 1: EPE($t$). In each graph $\rho$ is fixed. From top to down $\rho = 10\%$, $\rho = 40\%$ and $\rho = 70\%$. The graphs on the left, middle and right column correspond to the case 0F, 2F and 3F, respectively ($\nu = 0.1$)
Figure 2: $e(eEPE(t))$. In each graph $\rho$ is fixed. From top to down $\rho = 10\%$, $\rho = 40\%$ and $\rho = 70\%$. The graphs on the left, middle and right column correspond to the case $0F$, $2F$ and $3F$, respectively ($\nu = 0.1$).
Figure 3: CVA₀ versus ν for a CDS written on Ref. The graphs on the left column correspond to the case 2F and those of the right column correspond to 3F. In each graph ρ is fixed. From top to down ρ = 5%, ρ = 10%, ρ = 40% and ρ = 70%.
Figure 4: CVA\textsubscript{0} versus \( \nu \) for a CDS written on Ref’ in the case 2F. In each graph \( \rho \) is fixed.

9 Appendix

Let \( X \) be an extended CIR process with dynamics
\[
dX_t = \eta (\mu (t) - X_t) dt + \nu \sqrt{X_t} dW_t
\]
where \( \eta \) and \( \nu \) are positive constants and \( \mu (\cdot) \) is a non-negative deterministic function.

The following lemma is a standard result in the affine processes literature (see for example [10]). Notice that (34) is obtained from (33) by derivation with respect to \( t \).

Lemma 9.1 Consider the process \( X \) in (32). If \( \mu (\cdot) \) is constant on \([t_0, t]\), then for every \( y \geq 0 \),
\[
E \left( e^{-\int_{t_0}^{t} X_s ds} - yX_t \big| X_{t_0} \right) = e^{-\phi(t-t_0; y)X_{t_0} - \xi(t-t_0; y)\mu} ,
\]
\[
E \left( X_t e^{-\int_{t_0}^{t} X_s ds} \big| X_{t_0} \right) = \left( \phi(t-t_0, 0)X_{t_0} + \xi(t-t_0, 0)\mu \right) e^{-\phi(t-t_0, 0)X_{t_0} - \xi(t-t_0, 0)\mu} ,
\]
where \( \phi \) and \( \xi \) satisfy the following system of ODE :
\[
\begin{align*}
\dot{\phi}(s, y) &= -\eta \phi(s, y) - \frac{\nu^2}{2} (\phi(s, y))^2 + 1 ; \quad \phi(0, y) = y \\
\dot{\xi}(s, y) &= \eta \phi(s, y) ; \quad \xi(0, y) = 0 .
\end{align*}
\]
Explicitly,
\[
\phi(s, y) = \frac{1 + D(y)e^{-A(y)s}}{B + C(y)e^{-A(y)s}},
\]
\[
\xi(s, y) = \frac{\eta \left\{ C(y) - BD(y) \right\}}{A(y)C(y)} \log \frac{B + C(y)e^{-A(y)s}}{B + C(y)} + s \}
\]
where \(A, B, C\) and \(D\) are given by
\[
B = \frac{1}{2} \left( \eta + \sqrt{\eta^2 + 2\nu^2} \right),
\]
\[
C(y) = (1 - By) \frac{\eta + \nu^2 y - \sqrt{\eta^2 + 2\nu^2}}{2\eta y + \nu^2 y - 2},
\]
\[
D(y) = (B + C(y))y - 1,
\]
\[
A(y) = -C(y)2B - \eta + D(y)(\nu^2 + \eta B).
\]

In the following proposition, we generalize Lemma 9.1 to the case of a piecewise constant function \(\mu(\cdot)\). We denote \(T_0 = 0\) and \(\Delta_j = T_j - T_{j-1}\). The functions \(\phi\) and \(\xi\) are those of Lemma 9.1.

**Proposition 9.2** Assume that \(\mu(\cdot)\) is a piecewise constant function : \(\mu(t) = \mu_k\) on \(t \in [T_{k-1}, T_k]\) for \(k = 1, \ldots, m\). For \(t < s\), let \(i \leq j\) such that \(t \in [T_i, T_j)\) and \(s \in (T_j, T_{j+1})\). Then

(i) For any \(x \geq 0\) and \(y \geq 0\),
\[
\tilde{E}(s, t, x, y) := \mathbb{E}\left( \exp \left( -\int_t^s X_u du - yX_s \right) | X_t = x \right) = \exp \left\{ -\mu_i \xi(T_i - t, y_i) - x\phi(T_i - t, y_i) - \sum_{k=i+1}^{j} \mu_k \xi(\Delta_k, y_k) - \mu_{j+1} \xi(s - T_j, y) \right\}
\]
with
\[
y_j = y_j(s) := \phi(s - T_j, y),
\]
\[
y_k = y_k(s) := \phi(\Delta_{k+1}, y_{k+1}(s)), \quad k < j.
\]

(ii) One has,
\[
\mathbb{E}\left( X_s \exp \left( -\int_t^s X_u du \right) | X_t \right) = \tilde{D}(s, t, x) \mathbb{E}\left( \exp \left( -\int_t^s X_u du \right) | X_t \right)
\]
where
\[
\tilde{D}(s, t, x) = \mu_i \frac{\partial \xi}{\partial y}(T_i - t, y_i) \frac{dy_i}{ds} + x \frac{\partial \phi}{\partial y}(T_i - t, y_i) \frac{dy_i}{ds}
\]
\[
+ \sum_{k=i+1}^{j} \mu_k \frac{\partial \xi}{\partial y}(\Delta_k, y_k) \frac{dy_k}{ds} + \mu_{j+1} \frac{\partial \xi}{\partial s}(s - T_j, 0),
\]
and the \(y_k\)s are as in (36) with \(y = 0\).
Proof. (i) By conditioning on $X_{T_j}$ and using Lemma 9.1 on the interval $[T_j, s]$, one gets

$$
\mathbb{E}\left( e^{-\int_t^s X_u du - yX_s} | X_t \right) = \mathbb{E}\left( e^{-\int_t^{T_j} X_u du} \mathbb{E}\left( e^{-\int_{T_j}^s X_u du - yX_s} | X_{T_j} \right) | X_t \right) = \mathbb{E}\left( e^{-\int_t^{T_j} X_u du} \phi(s-T_j,y) - \mu_{j+1} \xi(s-T_j,y) | X_t \right)
$$

Continuing in the same way,

$$
\mathbb{E}\left( e^{-\int_t^s X_u du - yX_s} | X_t \right) = e^{-\mu_{j+1} \xi(s-T_j,y) - \mu_j \xi(D_j,y_j) - \cdots - \mu_{i+1} \xi(D_{i+1},y_{i+1})} \mathbb{E}\left( e^{-\int_t^{T_i} X_u du - X_{T_i} y_i} | X_t \right).
$$

A final use of Lemma 9.1 on the interval $[t, T_i]$, yields the result.

(ii) By differentiating $\mathcal{E}(s, t, x, 0)$ with respect to $s$, one obtains (37) with

$$
\tilde{D}(s, t, x) = \frac{d}{ds} \{ \mu_i \xi(T_i - t, y_i) + x \phi(T_i - t, y_i) + \sum_{k=i+1}^j \mu_k \xi(D_k, y_k) + \mu_{j+1} \xi(s - T_j, 0) \}.
$$

Next, from the chain rule, one has

$$
\frac{d}{ds} \xi(D_k, y_k) = \frac{\partial}{\partial y} \xi(D_k, y_k) \frac{dy_k}{ds},
$$

$$
\frac{d}{ds} \phi(T_i - t, y_i) = \frac{\partial}{\partial y} \phi(T_i - t, y_i) \frac{dy_i}{ds},
$$

which completes the proof.

Setting $t = 0$ and $s = T_j$ in the first part of the above proposition one obtains:

**Corollary 9.3** One has

$$
\mathbb{E}\left( \exp\left( - \int_0^{T_j} X_u du \right) \right) = \exp\left( -a_{j,0} X_0 - \sum_{k=1}^j \mu_k \xi(D_k, a_{j,k}) \right)
$$

with

$$
a_{j,j} = 0, \\
a_{j,k} = \phi(D_{k+1}, a_{j,k+1}) \quad k < j.
$$

**References**


